Topological K-theory, Lecture 1

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1 Motivation: Hopf invariant one

A **division algebra** structure on $\mathbf{R}^n$ is a (continuous) "multiplication" map $\mu : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ which is

- bilinear
- has no zero divisors – for any pair of non-zero vectors $0 \neq v, w \in \mathbf{R}^n$, $\mu(v, w) \neq 0$.

Example 1.

- The real numbers $\mathbf{R}$ with ordinary multiplication.
- The plane $\mathbf{R}^2 \cong \mathbf{C} = \{a + bi | a, b \in \mathbf{R}\}$ with multiplication given by multiplication of complex numbers.
- The four dimensional space $\mathbf{R}^4$, presented as the so called "Cayley numbers" (or: Quaternions) $\mathbf{R}^4 \cong \mathbf{H} = \{a + bi + cj + dk | a, b, c, d \in \mathbf{R}\}$ with multiplication analogous to the one of complex numbers, governed by the relations $i^2 = j^2 = k^2 = -1, \;\; ij = k, \;\; ji = -k$.
- The eight dimensional space $\mathbf{R}^8$ can be given the structure of a division algebra, presented as the so-called **Octonions** $\mathbf{O} = \{a + bi + cj + dk + el + fm + gn + ho\}$ with $i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = -1, o^2 = 1, ...$

Remark 1.1. Note that we did not require our multiplication $\mathbf{R} \otimes \mathbf{R} \longrightarrow \mathbf{R}$ to be associative. The Quaternions in fact associative division algebra, but the Octonions are not.

Question 1.2. Are there more? in which dimensions can we multiply vectors?.

Theorem 1.3. *(Adams, Atiyah) The space $\mathbf{R}^n$ admits a structure of a division algebra, iff $n = 1, 2, 4, 8$.*

Adams' proof was the first one. It consisted of 80 pages, accessible only for a handful of experts. Using topological K-theory, Atiyah gave a very short and elegant proof for Adams theorem. To demonstrate it, he wrote it on a postcard and mailed it to a colleague!

In this course we will study define and study topological K-theory. We will first develop the tools of topological K-theory and once these will be sufficiently developed, we'll see Atiyah's proof, among other interesting applications.

2 Fiber bundles

We restrict our attention to compactly generated topological spaces. The main feature to have in mind is the exponential law: $\mathrm{map}(X, \mathrm{map}(Y, Z)) \cong \mathrm{map}(X \times Y, Z)$.

Let B be a connected space.

Definition 2.1. A map $p : E \longrightarrow B$ is called a fiber bundle with fiber F if

- it is surjective.
- for every $b \in B$, there exists an open neighbourhood U_b and an isomorphism of spaces, called a **trivialization** $\Psi_{U_b} : p^{-1}(U_b) \xrightarrow{\cong} U_b \times F$, compatible with the map p in that the triangle

$$\begin{array}{ccc} p^{-1}(U_b) & \xrightarrow{\cong} & U_b \times F \\ & \searrow \quad \swarrow & \\ & U_b. & \end{array}$$

Remark 2.2. Thus, for any $b \in B$, $\Psi|_{p^{-1}(b)} : p^{-1}(b) \xrightarrow{\cong} \{b\} \times F$.

Remark 2.3. You saw in the previous course that any fiber bundle is a Serre fibration.

Example 2.
1. The projection map $B \times F \longrightarrow B$. This is called the **trivial bundle**.
2. Let $S^1 \subseteq \mathbf{C}$ be the unit circle. The map $p_n : S^1 \longrightarrow S^1$ given by $z \mapsto z^n$ is a fiber bundle with the fiber over $1 \in S^1$ given by the set of n-th roots of unity.
3. The map $exp : \mathbf{R} \longrightarrow S^1$ given by $exp(t) = e^{2\pi i t} \in S^1$ is a fiber bundle with fiber $\mathbf{Z}$.
4. Recall that the n-dimensional real projective space is defined by $\mathbf{R}P^n = S^n/$ where $x \quad -x \in S^n \subseteq \mathbf{R}^{n+1}$. Then, the quotient map $S^n \longrightarrow \mathbf{R}P^n$ is a fiber bundle with fiber $\{1, -1\}$.
5. Let $S^{2n+1} \subseteq \mathbf{C}^{n+1}$ and let $\mathbf{C}P^n = S^{2n+1}/$ where $x \quad ux$ for any $u \in S^1$. Then the quotient map $S^{2n+1} \longrightarrow \mathbf{C}P^n$ is a fiber bundle with fiber S^1.
6. Consider the Moebeus band $M = [0,1] \times [0,1]/$ where $(t, 0) \quad (1-t, 1)$ and consider the "center circle" $C = (1/2, s) \in M$. The projection map $M \longrightarrow C$ given by $(t, s) \mapsto (1/2, s)$ is a fiber bundle with fiber $[0,1]$.

Definition 2.4. Let $p_1 : E_1 \longrightarrow B_1$ and $p_2 : E_2 \longrightarrow B_2$ be fiber bundles. A map of fiber bundles is a commutative square

$$\begin{array}{ccc} E_1 & \xrightarrow{\overline{\varphi}} & E_2 \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow[\varphi]{} & B_2 \end{array}$$

Note: such a map induces, for each $b \in B$, a map between the fibers

$$(E_1)_b \longrightarrow (E_2)_{\varphi(b)}.$$

We have thus defined the category of fiber bundles.

Observe 2.5. A map $p : E \longrightarrow B$ is a covering space iff it's a fiber bundle with discrete fiber.

3 Vector bundles

Let $\mathbf{k}$ be either of the (topological) fields $\mathbf{R}$ or $\mathbf{C}$. We will restrict attention to finite dim'l vector spaces over $\mathbf{k}$. Note that such a vector space V is always assumed to be a topological vector space, in the sense that addition of vectors and multiplication by a scalar define continuous maps $V \times V \longrightarrow V$ and $k \times V \longrightarrow V$.

Definition 3.1. Let V be an n-dim'l vector space over $\mathbf{k}$, and let B be a connected space. An n-dim'l vector bundle with fiber V is a fiber bundle $p : E \longrightarrow B$ with the structure of a vector space on each fiber $p^{-1}(b) = E_b$ such that, for each $b \in B$, the maps $\Psi_{U_b} : p^{-1}(U_b) \longrightarrow U_b \times V$ restrict to k-linear maps (hence isomorphisms)

$$\Psi_{U_b}|_{p^{-1}(b)} : p^{-1}(b) \xrightarrow{\cong} \{b\} \times V$$

on each fiber.

A map of vector bundles is a map of fiber bundles which is k-linear on each fiber. The category of vector bundles is denoted VB and that of vector bundles over a fixed space B is denoted VB/B.

Remark 3.2. We assume throughout that our base space B is **connected**. If $B = \coprod_\alpha B_\alpha$ is a disjoint union of path components, then a vector bundle E over B is by definition a collection of vector bundles E_α over each B_α and the rank of each E_α may be different. We will assume all our base spaces are connected in order to simplify the discussion. All the arguments could be extended to the case of non-connected base in a straightforward way.

Example 3.

Given an n-dim'l k-vector space V, the projection $B \times V \longrightarrow B$ is a vector bundle, called the **trivial vector bundle**.

The **Moebeus line bundle** is given as follows. Let $E = [0,1] \times \mathbf{R}/$ where $(0,t)$ $(1,-t)$. Let C be the middle circle $C = \{(s,0) \in E\}$. Then the projection $E \longrightarrow C$, $(s,t) \mapsto (s,0)$ is a vector bundle with fiber $\mathbf{R}$.

Define the **canonical line bundle** over the projective space $\mathbf{R}P^n$ as follows. The space $\mathbf{R}P^n$ may be thought of as the space of lines ℓ through the origin in $\mathbf{R}^{n+1}$. Let $E = (\ell, v)|\ell \in \mathbf{R}P^n,\ v \in \ell$ and define $E \longrightarrow \mathbf{R}P^n$ by setting $(\ell, v) \longrightarrow \ell$.

Proposition 3.3. *Let*

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

be a map of vector bundles. Then φ is an isomorphism iff $\varphi|_{p^{-1}(b)} : p^{-1}(b) \longrightarrow q^{-1}(b)$ is an isomorphism for each $b \in B$.

Proof. Clearly, if φ has a (categorical) inverse φ^{-1}, it restricts to an isomorphism on each fiber. Conversely, suppose $E = B \times V$ and $F = B \times W$ are trivial vector bundles and that $\varphi : E \longrightarrow F$ restricts to an isomorphism on each fiber. By the exponential law for spaces, we have homeomorphism of spaces (with respect to the compact-open topology)

$$\mathrm{map}_{/B}(B \times V, B \times W) \cong \mathrm{map}(B \times V, W) \cong \mathrm{map}(B, \mathrm{map}(V, W)) \qquad (1)$$

where the left-hand side denotes maps over B. When we restrict attention to vector bundle maps on the left-hand side, we get a homeomorphism

$$\mathrm{VB}/B(B \times V, B \times W) \cong \mathrm{map}(B, \mathrm{Hom}(V, W))$$

where $\mathrm{Hom}(V, W)$ is the space of k-linear maps with the obvious topology.

The (vector bundle) map $\varphi : E \longrightarrow F$ thus corresponds to a map $\Phi : B \longrightarrow \mathrm{Hom}(V, W)$ which is in fact a map $\Phi : B \longrightarrow \mathrm{Iso}(V, W)$ by our assumption on φ. If we denote the (continuous) inversion map by $i : \mathrm{Iso}(V, W) \longrightarrow \mathrm{Iso}(W, V)$ then we get the composite $\Psi = i \circ \Phi : B \longrightarrow \mathrm{Iso}(W, V)$ which by 1 (with the roles of V and W interchanged) corresponds to a vector bundle map $\psi : F \longrightarrow E$. The map ψ is clearly an inverse to φ since it is such on each fiber.

Thus, the statement is true locally. If now $\varphi : E \longrightarrow F$ is a map of (arbitrary) vector bundles which is an isomorphism on each fiber, then φ is one-to-one and onto, and we need to show that its set-theoretical inverse φ^{-1} is continuous. But φ^{-1} coincides with ψ on each piece of an open cover and we have shown that ψ is continuous so φ^{-1} must be continuous.

□

4 Sections

A **section** of a vector bundle $p : E \longrightarrow B$ is a map $s : B \longrightarrow E$ such that $ps = \mathrm{id}_B$. Thus, a section is a continuous correspondence $b \mapsto v_b$ of a vector $v_b \in \mathcal{E}_b$ to each point $b \in B$. For example, we see that every vector bundle has at least one section – the zero section $b \mapsto 0_{E_b}$.

Proposition 4.1. *An n-dim'l vector bundle is trivial iff it admits n linearly independent sections, i.e. sections $\{s_1, ..., s_n\}$ s.t. $\{s_1(b), ..., s_n(b)\}$ are linearly independent for each $b \in B$.*

Proof. Clearly, $B \times \mathbf{k}^n$ has such sections, and any vector bundle isomorphism takes linearly independent sections to linearly independent sections. Conversely, if $s_1, ..., s_n$ are linearly independent sections of $p : E \longrightarrow B$ then the map

$$\varphi : B \times k^n \longrightarrow E$$

given by $\varphi(b, \lambda_1, ..., \lambda_n) = \Sigma \lambda_i s_i(b)$ is an isomorphism on each fiber and hence an isomorphism of vector bundles. □

5 Pullbacks

Let $p : E \longrightarrow B$ be a vector bundle and $B' \longrightarrow B$ any map.

Observe 5.1. There is an induced vector bundle structure on the pullback $p' : E' := E \times_B B' \longrightarrow B'$.

6 Direct sums

Given vector bundles $p_1 : E_1 \longrightarrow B$ and $p_2 : E_2 \longrightarrow B$, their **direct sum** is $E_1 \oplus E_2 := E_1 \times_B E_2$ together with the projection map $p : E_1 \oplus E_2 \longrightarrow B$. Note that $p^{-1}(b) = p_1^{-1}(b) \oplus p_2^{-1}(b)$ so that the name is reasonable.

Proposition 6.1. *The projection $E_1 \oplus E_2 \longrightarrow B$ is a vector bundle.*

Proof. Given two vector bundles $p_1 : E_1 \longrightarrow B_1$ and $p_2 : E_2 \longrightarrow B_2$ the product $p_1 \times p_2 : E_1 \times E_2 \longrightarrow B_1 \times B_2$ is a vector bundle, for if $\varphi_1 : p_1^{-1}(U_{b_1}) \xrightarrow{\cong} U_{b_1} \times V$ and $\varphi_2 : p_2^{-1}(U_{b_2}) \xrightarrow{\cong} U_{b_2} \times W$ are trivializations, then $\varphi_1 \times \varphi_2 : p_1^{-1}(U_{b_1}) \times p_2^{-1}(U_{b_2}) \longrightarrow U_{b_1} \times U_{b_2} \times V \times W$ is a trivialization for $E_1 \times E_2$.

In our case, $p_1 \times p_2 : E_1 \times E_2 \longrightarrow B \times B$ is a vector bundle, and its pullback along the diagonal $\delta : B \longrightarrow B \times B$ is precisely $E_1 \oplus E_2$ which is therefore a vector bundle itself. □

References

[Ati] M. F. Atiyah *K-theory*, New York: WA Benjamin (1967).

[Hat] A. Hatcher, *Vector bundles and K-theory.* Author's website (2009).