Topological K-theory, Lecture 3

Matan Prasma

March 2, 2015

Recall that we assume throughout our base space B is connected.

1 Classification of vector bundles – continued

Last time we showed that any n-dimensional vector bundle $p : E \longrightarrow B$ is isomorphic to a vector bundle obtained via pulling back the bundle
$\gamma_n : E_n \longrightarrow G_n$ as depicted below

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[dr, "p"'] & f^*(E_n(\mathbf{k}^\infty)) \arrow[r, "\tilde{f}"] \arrow[d] & E_n(\mathbf{k}^\infty)\mathbf{k} \arrow[r, "\pi"] \arrow[d, "\gamma_n"] & \mathbf{k}^\infty \\ & B \arrow[r, "f"] & G_n(\mathbf{k}^\infty). & \end{tikzcd}$$

Furthermore, in such a setting, one can see that the map $\pi\tilde{f}\iota_0$ is a linear injection on each fiber.

The following Theorem will be a key in the second part of the classification:

Theorem 1.1. *Let B be paracompact and let $p : E \longrightarrow B \times I$ be a vector bundle. Then $E|_{X\times\{0\}} \cong E|_{X\times\{1\}}$.*

We first prove a couple of lemmas

Lemma 1.2. *Let B be paracompact. A vector bundle $p : E \longrightarrow B \times I$ whose restrictions over $B \times [0, t]$ and over $B \times [t, 1]$ are trivial is trivial as well.*

Proof. Let $h_0 : E_0 := E|_{B\times[0,t] \xrightarrow{\cong} B\times[0,t]\times V}$ and $h_1 : E_1 := E|_{B\times[t,1]} \xrightarrow{\cong} B\times[t,1]\times V$ be isomorphisms to trivial bundles. The maps h_0, h_1 may not agree on $E|_{B\times\{t\}}$ so we cannot yet glue them. Define an isomorphism $h_{01} : B \times [t, 1] \times V \longrightarrow B \times [t, 1] \times V$ by duplicating the map $h_0 h_1^{-1} : B \times \{t\} \times V \longrightarrow B \times \{t\} \times V$ on each slice $B \times \{s\} \times V$ for $t \leq s \leq 1$ and set $\overline{h_1} := h_{01} h_1$. Then $\overline{h_1}$ is an isomorphism of bundles and agrees with h_0 on $E|_{B\times\{t\}}$. we can now glue together h_0 and h_1 to get the desired. $\square$

Lemma 1.3. *For every vector bundle $p : E \longrightarrow B \times I$ there is an open cover $\{U_\alpha\}_\alpha$ such that each restriction $E|_{U_\alpha\times I} \longrightarrow U_\alpha \times I$ is trivial.*

Proof. For each $b \in B$, take open neighbourhoods U_b with $0 = t_0 < t_1 < ... < t_k = 1$ such that $E|_{U_b \times [t_{i-1}, t_i]} \longrightarrow U_b \times [t_{i-1}, t_i]$ is trivial. This is possible because for each (b, t) we can find an open neighbourhood of the form $U_b \times J_t$ (where J_t is an open interval) over which E is trivial; if we then fix b then the collection $\{J_t\}$ covers I and we can take a finite subcover $J_1, ..., J_{k+1}$ and choose $t_i \in J_i \cap J_{i+1}$; this way E remains trivial over $U_b \times [t_{i-1}, t_i]$. Now, by Lemma 1.2, E is trivial over $U_b \times I$. $\square$

Proof of Theorem 1.1. By Lemma 1.3, take an open cover $\{U_\alpha\}_\alpha$ of B such that $E|_{U_\alpha \times I}$ is trivial. Assume first that B is compact. Then we can take a cover of the form $\{U_i\}_{i=1}^n$. Take a partition of unity $\{h_i : B \longrightarrow I\}_{i=1}^n$ subordinated to $\{U_i\}$. For $i \geq 0$, set $g_i = h_1 + ... + h_i$ ($g_0 = 0$, $g_n = 1$), let $B_i = \mathrm{Gr}(g_i) \subseteq B \times I$ be the graph of g_i and let $p_i : E_i \longrightarrow B_i$ be the restriction of E to B_i. The map $B_i \longrightarrow B_{i-1}$ given by $(b, g_i(b)) \mapsto (b, g_{i-1}(b))$ is a homeomorphism, and since $E|_{U_i \times I}$ is trivial, the dotted isomorphism in the diagram below exists:

$$\begin{tikzcd}
E|_{B_i \cap (U_i \times I)} \arrow[r, dotted, "\cong"] \arrow[d] & E|_{B_{i-1} \cap (U_i \times I)} \arrow[d] \\
B_i \cap (U_i \times I) \arrow[r, "\cong"] & B_{i-1} \cap (U_i \times I)
\end{tikzcd}$$

(specifically: a restriction of a trivial bundle is trivial, and trivial bundles over homeomorphic bases are isomorphic.) Since outside U_i, $h_i = 0$, $E|_{B_i \cap U_i^c} = E|_{B_{i-1} \cap U_i^c}$, we obtain an isomorphism of vector bundles (over different bases) $f_i : E|_{B_i} \xrightarrow{\cong} E|_{B_{i-1}}$. The composition $f = f_1 \circ ... \circ f_n$ is then an isomorphism from $E|_{B_n} = E|_{B \times \{1\}}$ to $E|_{B_0} = E|_{B \times \{0\}}$.

Assume now B is paracompact. Take a countable cover $\{V_i\}_i$ such that each V_i is a disjoint union of opens, each contained in some U_α. This means that E is trivial over each $V_i \times I$. Let $\{h_i : B \longrightarrow I\}$ be a partition of unity subordinated to $\{V_i\}_i$ and set as before $g_i := h_1 + ... + h_i$ and $p_i : E_i \longrightarrow B_i := \mathrm{Gr}(g_i)$ the restriction. As before, we obtain isomorphisms $f_i : E_i \xrightarrow{\cong} E_{i+1}$. The infinite composition $f = f_1 f_2 ...$ is well-defined since for every point, almost all f_i's are the identity. As before, f is an isomorphism from $E|_{B \times \{1\}}$ to $E|_{B \times \{0\}}$. $\square$

In other words, Theorem 1.1 tells us that homotopic maps induce isomorphic pullback bundles. Let $\mathrm{VBun}^n_{\mathbf{k}}(B)$ be the set of isomorphism classes of rank n $\mathbf{k}$-vector bundles over B.

Corollary 1.4. *A homotopy equivalence of paracompact spaces $f : A \longrightarrow B$ induces a bijection $f^* : \mathrm{VBun}^n_{\mathbf{k}}(B) \xrightarrow{\cong} \mathrm{VBun}^n_{\mathbf{k}}(A)$. In particular, any vector bundle over a contractible paracompact space is trivial.*

Proof. If g is a homotopy inverse of f, then $f^* g^* = \mathrm{id}^* = \mathrm{id}$ and $g^* f^* = \mathrm{id}^* = \mathrm{id}$. $\square$

We are ready to state and prove the classification theorem.

Theorem 1.5. *Let B be paracompact. Then pullback along $\gamma_n : E_n(\mathbf{k}^\infty) \longrightarrow G_n(\mathbf{k}^\infty)$ induces a bijection*

$$[B, G_n(\mathbf{k}^\infty)] \xrightarrow{\cong} \mathrm{VBun}^n_{\mathbf{k}}(B) \tag{1}$$
$$[f] \mapsto f^* E_n$$

Proof. The map is well-defined since two homotopic maps give two isomorphic pullback vector bundles. Proposition 5.1 of Lecture 2 gives surjectivity of 1 so we are left with injectivity.

Assume we have two maps $f_0, f_1 : B \longrightarrow G_n(\mathbf{k}^\infty)$ which induce isomorphic bundles upon pullback. It would be convenient to assume we are given a vector bundle $p : E \longrightarrow B$ and a couple of isomorphisms of bundles $i_0 : E \xrightarrow{\cong} f_0^* E_n$ and $i_1 : E \xrightarrow{\cong} f_1^* E_n$ so that we have the following commutative diagram

$$\begin{tikzcd}
f_1^*E_n \arrow[drrr, bend left, "\overline{f_1}"] \arrow[dddr, bend right] & & & & \\
& E \arrow[ul, "\cong"', "i_1"] \arrow[r, "\cong", "i_0"'] \arrow[dr, "p"] \arrow[dd, "p"'] & f_0^*E_n \arrow[r, "\overline{f_0}"] \arrow[d] & E_n(\mathbf{k}^\infty) \arrow[r, "\pi"] \arrow[d, "\gamma_n"] & \mathbf{k}^\infty \\
& & B \arrow[r, "f_0"] & G_n(\mathbf{k}^\infty) & \\
& B \arrow[ur, equal] \arrow[urr, "f_1"'] & & &
\end{tikzcd} \tag{2}$$

in which the maps $\overline{f_0}$ and $\overline{f_1}$ are obtained as pullbacks of f_0 and f_1 respectively.

Define $g_0 := \pi \overline{f_0} i_0$ and $g_1 := \pi \overline{f_1} i_1$. Then $g_0, g_1 : E \longrightarrow \mathbf{k}^\infty$ are linear injections on each fiber and satisfy $f_0(b) = g_0(E_b)$ and $f_1(b) = g_1(E_b)$. It will thus be enough to find a homotopy $\{g_t\}$ from g_0 to g_1 in which all maps g_t are linear injections on each fiber since we could then define $f_t(b) = g_t(E_b) \in G_n(\mathbf{k}^\infty)$ to obtain a homotopy from f_0 to f_1.

Composing g_0 with the maps $L_t : \mathbf{k}^\infty \longrightarrow \mathbf{k}^\infty$ given by

$$(v_1, v_2, ...) \mapsto (1-t)(v_1, v_2, ...) + t(v_1, 0, v_2, 0, ...)$$

gives a homotopy from g_0 to a map $\overline{g_0}$ through maps which are linear injections on each fiber. The image of $\overline{g_0}$ lies in the subspace of $\mathbf{k}^\infty$ consisting of vectors with non-zero components only in the odd coordinates. Similarly, we can replace g_1 by a map $\overline{g_1} : E \longrightarrow \mathbf{k}^\infty$ whose image lies in the subspace of $\mathbf{k}^\infty$ consisting of vectors with non-zero components only in the even coordinates.Clearly, it is enough to construct a homotopy $\{\overline{g_t}\}$ from $\overline{g_0}$ to $\overline{g_1}$ through maps which are linear injections on each fiber. But this is easy now: we set $\overline{g_t} := (1-t)\overline{g_0} + t\overline{g_1}$ and finish the argument.

$\square$

Theorem 1.5 justifies the following terminology.

Definition 1.6. The bundle $\gamma_n : E_n(\mathbf{k}^\infty) \longrightarrow G_n(\mathbf{k}^\infty)$ is called the **universal rank-n vector bundle**.

2 Applications of the classification theorem

Let us see how the classification theorem can be used.

Example 1. The bundle $\gamma_n : E_n(\mathbf{k}^\infty) \longrightarrow G_n(\mathbf{k}^\infty)$ admits an inner product, induced from an inner product on $\mathbf{k}^\infty$. Since every rank-n vector bundle is obtained as a pullback along γ_n, we deduce that any vector bundle admits an inner product – that obtained by pulling back the one on γ_n. This is a shortened proof to what you already showed in the exercise.

Example 2. Let us compute the Picard group of complex projective spaces. By the classification theorem we have

$$\mathrm{VBun}^1_{\mathbf{C}}(\mathbf{C}P^n) \cong [\mathbf{C}P^n, G_1(\mathbf{C}^\infty)] = [\mathbf{C}P^n, \mathbf{C}P^\infty].$$

You have shown in the exercise that $V_1(\mathbf{C}^\infty) \longrightarrow G_1(\mathbf{C}^\infty)$ is a fiber bundle with fiber $\mathrm{GL}_1(\mathbf{C})$. By a theorem you proved in Algebraic topology I, any fiber bundle is a Serre fibration, so that we have a fibration sequence

$$\mathrm{GL}_1(\mathbf{C}) \longrightarrow V_1(\mathbf{C}^\infty) \longrightarrow G_1(\mathbf{C}^\infty). \tag{3}$$

You have seen in the exercise class that the spaces $V_n(\mathbf{k}^\infty)$ are contractible and it is easy to see that $\mathrm{GL}_1(\mathbf{C}) \simeq S^1$ (this is so since $\mathrm{GL}_1(\mathbf{C}) = \mathbf{C} \setminus \{0\}$). Thus, the long exact sequence for the fibration sequence 3 implies that $G_1(\mathbf{C}^\infty) = \mathbf{C}P^\infty$ is a $K(\mathbf{Z}, 2)$. Now, an application of Brown's representability theorem (which you proved in Algebraic Topology I) implies that $[\mathbf{C}P^n, \mathbf{C}P^\infty] = [\mathbf{C}P^n, K(\mathbf{Z}, 2)] \cong H^2(\mathbf{C}P^n; \mathbf{Z})$ – i.e. the second cohomology group of $\mathbf{C}P^n$. Using cellular cohomology (this is an elementary way of calculation, given in any first course in cohomology) we deduce from the cell structure of $\mathbf{C}P^n$ (one cell in each even dimension and no others) that $\mathrm{Pic}(\mathbf{C}P^n) = \mathrm{VBun}^1_{\mathbf{C}}(\mathbf{C}P^n) \cong H^2(\mathbf{C}P^n; \mathbf{Z}) \cong \mathbf{Z}$. In fact, it follows from what you showed in the exercise that group structure is given by the tensor product. Thus, there is a line bundle ζ on $\mathbf{C}P^n$ (corresponding to $1 \in \mathbf{Z}$) such that $\zeta \otimes \ldots \otimes \zeta$ (n-times) correspond to $n \in \mathbf{Z}$ – this is the canonical line bundle introduced in Lecture 1!