Stable Localizations of Module Spectra

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Contents

Introduction

The aim of this work is to start a systematic study of idempotent functors in the stable homotopy category and to describe certain structures that are preserved under such functors. We translate into stable homotopy some of the basic ideas of localization with respect to a map, which have been developed in unstable homotopy during the past decade. In order to do so, we take off from suggestions raised by Bousfield in a recent article. We study the preservation of ring spectra structures and module spectra structures under localizations. For this, we use general properties of idempotent functors in combination with suitable models for the stable homotopy category. We describe all possible localizations of the integral Eilenberg–Mac Lane spectrum $H\mathbb{Z}$ and, more generally, we discuss properties of localizations of $H\mathbb{Z}$-modules, motivated by a similar approach recently undertaken in unstable homotopy.

Precedents

Localization with respect to a continuous map f is a universal construction in the homotopy category of spaces (CW-complexes or simplicial sets) that turns f into a homotopy equivalence. This general form of localization specializes into previously known transformations, such as homological localizations, plus constructions, Postnikov sections, or localization at sets of primes. A full-fledged theory was developed in the past decade by Bousfield [Bou94, Bou97], Farjoun [DF], and others. This theory had implications in unstable chromatic towers, K-theory localizations of spaces, Bousfield classes, and other periodic phenomena in unstable homotopy.

In a recent article by Bousfield [Bou96], the language of localization with respect to a map is brought back to stable homotopy, in an attempt to relate in a clear way stable periodicity with unstable periodicity. As a result, a large amount of new idempotent functors become available, as generalizations of stable homological localizations and other classical constructions. In fact, it is possible that all idempotent functors in stable homotopy can be represented as localizations with respect to suitable maps. The analogous question in unstable homotopy was answered by Casacuberta, Scevenels and Smith in [CSS].

There are many algebraic structures that are preserved by localizations in unstable homotopy and in other categories. For example, the class of GEMs (products of abelian Eilenberg–Mac Lane spaces) is closed under localization with respect to any map. A recent survey on preservation of structures by idempotent functors was offered in [Cas00]. On the other hand, it was observed in [EKMM97] that the classes of (strict) ring spectra and (strict) module spectra are preserved by homological localizations. It seems thus natural to believe that these and other structures are preserved by more general localizations in stable homotopy. This consideration and some of its consequences served as motivation for our work. Modules over the ring spectrum $H\mathbb{Z}$ of ordinary homology are the precise analog of GEMs in stable homotopy.

Structure of Contents

This work is divided in six chapters. In the first chapter, an introduction if offered to the classical stable homotopy category of Adams [Ada74], in view of its use in the next chapters.

In the second chapter, we define stable localization with respect to a map, following Bousfield [Bou96] and emphasizing the analogy with the unstable case. We explain the key role of connective covers of function spectra in this context. We show that, as in unstable homotopy, f-localizations are idempotent functors and hence share many well-known properties.

Several examples of stable f-localizations are given in the third chapter, including nullifications, homological localizations, and localization at sets of primes. Postnikov sections are important examples of f-localizations that do not commute with suspension. Our reference for many fundamental facts of stable homotopy is Rudyak's book [Rud]. Using idempotence properties of f-localizations, we have improved certain results of Rudyak.

The fourth chapter gives a brief presentation of model category structures for stable homotopy. We refer to the old, useful Bousfield–Freidlander model [BF79] and to the very recent models consisting of symmetric spectra [HSS], S-modules [EKMM], or simplicial functors [Lyd].

The fifth chapter deals with localizations of Eilenbeg–Mac Lane spectra. In this chapter, we present new results on preservation of ring spectra structures and module spectra structures under f-localizations. We recall the definition of a stable GEM and prove that stable GEMs are preserved under f-localizations. We show that the localization of any Eilenberg–Mac Lane spectrum is a product of at most two Eilenberg–Mac Lane spectra. In the case of the spectrum $H\mathbb{Z}$ of ordinary homology, each of its localizations has at most one nonzero homotopy group. We characterize this single homotopy group following the approach of [CRT].

A detailed proof of the existence of f-localizations is proved in the Appendix, for simplicial model categories satisfying certain assumptions. The argument is essentially Quillen's small object argument. It is thoroughly discussed in Hirschhorn's manuscript [Hirsch].

Future Prospects

Our results about preservation of stable structures under localizations use the classical (homotopical) notions of ring spectra and module spectra. That is, the structure diagrams for the multiplication and the unit commute up to homotopy. However, we believe that localizations also preserve strict ring spectra and strict module spectra (in the sense allowed by the symmetric monoidal category structures of spectra developed in [HSS] or [EKMM]). There are two possible approaches to prove this claim. Namely, one could extend the result given for homological localizations in [EKMM97] to arbitrary f-localizations, or one could show that f-localizations preserve A_∞ and E_∞ structures (both in stable and unstable homotopy). This is not carried out in this work, but it is one of the natural directions for further progress. After this is done, one could use the equivalence between the homotopy category of strict HR-module spectra (where R is any ring) and the homotopy category of unbounded chain complexes of R-modules, in order to sharpen the results contained in the present work.

Another future aim is the characterization of the rings that arise as homotopy groups of stable homological localizations of the integral Eilenberg–Mac Lane spectrum, along the lines marked in the unstable case in [Bou82]. A more ambitious project is to study f-localizations of the sphere spectrum.

Agradecimientos

En primer lugar me gustaría expresar mi más sincero agradecimiento a mi director, Carles Casacuberta, por su continua dedicación y paciencia y por haberme introducido en el mundo de la homotopía estable. Sin su inestimable ayuda la realización de este trabajo no hubiera sido posible.

Quisiera agradecer también a todas las personas que han colaborado de una forma u otra a la consecución de este trabajo. A John Greenlees y Neil Strickland por sus observaciones y nuevos enfoques. A Bob Bruner y W. Chacholski por las conversaciones mantenidas, las cuales me han aportado interesantes ideas. Y también a A.K. Bousfield por sus valiosas sugerencias.

Una mención especial merece la colaboración de Gustavo Granja, del que me gustaría destacar tanto su calidad científica como personal. Sus aportaciones al trabajo han sido de gran importancia.

Agradecer el apoyo en todo momento de todos los mienbros del Grupo de Topología de la Universitat Autònoma de Barcelona, en especial de Ramón y Gemma.

No podría olvidarme de mis profesores de Logroño: Ignacio, Tere y Luis Javier. Gracias a ellos he tenido la oportunidad de venir a Barcelona a realizar mis estudios de doctorado.

Por último dar las gracias a mi familia y a Elena por sus constantes animos y apoyo

Chapter 1

The Stable Category of Spectra

In this chapter we present some preliminaries from stable homotopy theory. There are different categories of spectra proposed by different authors. Here we describe the category of spectra proposed by Adams. The basic reference is [Ada74]. We recall the basic properties of spectra, generalized homology and cohomology theories and smash product of spectra.

1.1 Basic Properties of Spectra

The objects of the stable homotopy category are called spectra.

Definition 1.1.1. A *spectrum* E is a sequence $(E_n, \varepsilon_n)_{n\in\mathbb{Z}}$ of pointed CW-complexes E_n and CW-embeddings $\varepsilon_n \colon \Sigma E_n \longrightarrow E_{n+1}$. Here ΣE_n means reduced suspension, i.e., $\Sigma E_n = S^1 \wedge E_n$.

A *subspectrum* of a spectrum E is a spectrum $(F_n, \tau_n)_{n\in\mathbb{Z}}$ such that F_n is a pointed CW-subcomplex of E_n and $\tau_n \colon \Sigma F_{n+1} \longrightarrow F_n$ is the restriction of ε_n.

If the maps ε_n are weak equivalences for n sufficiently large, then the spectrum is called a Σ-*spectrum* or suspension spectrum. If the adjoint maps $\varepsilon_n' \colon E_n \longrightarrow \Omega E_{n+1}$ are weak equivalences for $n \geq 0$, then E is called an Ω-*spectrum*.

Example 1.1.1. Given any CW-complex X, we can define a spectrum $\Sigma^\infty X$ in the following way

$$(\Sigma^\infty X)_n = \begin{cases} \Sigma^n X & \text{if } n \geq 0 \\ * & \text{if } n < 0 \end{cases}$$

with ε_n the obvious maps. The spectrum $\Sigma^\infty X$ is called the *suspension spectrum* of X. As a particular case we have the sphere spectrum $\Sigma^\infty S^0$, which we denote simply by S. This is an example of a Σ-spectrum.

Example 1.1.2. Given an abelian group A we can construct an Ω-spectrum HA which we call an Eilenberg–Mac Lane spectrum,

$$(HA)_n = \begin{cases} K(A,n)X & \text{if } n \geq 0 \\ * & \text{if } n < 0. \end{cases}$$

The maps ε'_n are always weak equivalences for $n \geq 0$ because $\Omega K(A, n+1) \simeq K(A,n)$.

Given a spectrum E and an integer k, we define a new spectrum $\Sigma^k E$ by setting $(\Sigma^k E)_n = E_{n+k}$ where the map $\Sigma(\Sigma^k E)_n \longrightarrow (\Sigma^k E)_{n+1}$ is ε_{n+k}. So, we can either suspend or desuspend a spectrum arbitrarily.

We now define the homotopy groups of a spectrum. These are really stable homotopy groups. Given a spectrum E we have the following homomorphisms

$$\pi_{n+r}(E_n) \longrightarrow \pi_{n+r+1}(\Sigma E_n) \xrightarrow{(\varepsilon_n)_*} \pi_{n+r+1}(E_{n+1}) \longrightarrow \ldots$$

We define $\pi_r(E)$ as the direct limit of the above direct system

$$\pi_r(E) = \lim_{n \to +\infty} \pi_{n+r}(E_n).$$

Example 1.1.3. For the Eilenberg–Mac Lane spectrum HA, we have that

$$\pi_r(HA) = \lim_{n \to +\infty} \pi_{n+r}(K(A,n)) = \begin{cases} A & \text{if } r = 0 \\ 0 & \text{if } r \neq 0. \end{cases}$$

In order to define the category of spectra we need to specify what are the maps between spectra.

Definition 1.1.2. Let (E_n, ε_n) and (F_n, τ_n) be spectra. A function $f\colon E \longrightarrow F$ is a family of pointed cellular maps $f_n\colon E_n \longrightarrow F_n$ such that the diagram

$$\begin{tikzcd} \Sigma E_n \arrow[r, "\Sigma f_n"] \arrow[d, "\varepsilon_n"'] & \Sigma F_n \arrow[d, "\tau_n"] \\ E_{n+1} \arrow[r, "f_n"'] & F_{n+1} \end{tikzcd}$$

commutes for all $n \in \mathbb{Z}$.

Definition 1.1.3. A *cell* of a spectrum E is a sequence $(e, \Sigma e, \ldots, \Sigma^k e, \ldots)$ where e is a cell of any E_n such that e is not the suspension of any cell of E_{n-1}.

A subspectrum F of a spectrum E is cofinal in E if every cell of E is eventually in F, i.e., for every cell of E_n there exists an m such that $\Sigma^m e$ belongs to F_{n+m}.

Definition 1.1.4. Let E and F be two spectra. We consider the set S of all pairs (E', f') such that $E' \subset E$ is a cofinal subspectrum and $f'\colon E' \longrightarrow F$ is a function. Consider the equivalence relation $\sim$ on S such that $(f', E') \sim (f'', E'')$ if and only if there is a pair (E''', f''') with $E''' \subset E' \cap E''$, E''' cofinal and $f' \mid E'' = f''' = f'' \mid E''$. Every such equivalence class is called a *map* from E to F.

If (E_n, ε_n) is a spectrum and X is a CW-complex, we define the spectrum $E \wedge X$ as $(E \wedge X)_n = E_n \wedge X$. We can now define homotopies between maps of spectra.

Definition 1.1.5. We say that two maps of spectra $f_0, f_1\colon E \longrightarrow F$ are *homotopic* $f_0 \sim f_1$, if there exists a map $H\colon E \wedge I^+ \longrightarrow F$ with $h \circ i_0 = f_0$ and $h \circ i_1 = f_1$, where i_0 and i_1 are the maps $i_0, i_1\colon E \longrightarrow E \wedge I^+$ induced by the inclusions of 0 and 1 in I^+, respectively.

Homotopy is an equivalence relation, so we define $[E, F]$ to be the set of equivalence classes of maps $f\colon E \longrightarrow F$. The fundamental property of stable homotopy is that suspension yields a natural bijection $[E, F] \cong [\Sigma E, \Sigma F]$. This implies, among other things, that $[E, F]$ is an abelian group for all E, F.

We can give an alternative description of the homotopy groups of a spectrum by taking $\pi_r(E) = [\Sigma^r S, E]$.

Definition 1.1.6. A spectrum E is called *connective* if $\pi_i(E) = 0$ for $i < 0$.

Definition 1.1.7. For any map $f\colon E \longrightarrow F$ of spectra we call the sequence

$$E \xrightarrow{f} F \xrightarrow{i} Cf,$$

where Cf is the cone of f, a *strict cofiber sequence.* A sequence

$$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$$

is called a *cofiber sequence* if there exists a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & Y \arrow[r, "\psi"] \arrow[d, "\beta"'] & Z \arrow[d, "\gamma"'] \\ E \arrow[r, "f"'] & F \arrow[r, "i"'] & Cf \end{tikzcd}$$

where α, β and γ are homotopy equivalences.

Every cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ yields a long cofiber sequence

$$\ldots \longrightarrow \Sigma^{-1}Y \xrightarrow{\Sigma^{-1}g} \Sigma^{-1}Z \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow \ldots$$

Theorem 1.1.4. *For every spectrum E, the long cofiber sequence*

$$\ldots \longrightarrow \Sigma^{-1}Y \xrightarrow{\Sigma^{-1}g} \Sigma^{-1}Z \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y \longrightarrow \ldots$$

yields exact sequences of abelian groups

$$\ldots \longleftarrow [\Sigma^{-1}Z, E] \longleftarrow [X, E] \xleftarrow{f^*} [Y, E] \xleftarrow{g^*} [Z, E] \longleftarrow [\Sigma X, E] \longleftarrow \ldots$$
$$\ldots \longrightarrow [E, \Sigma^{-1}Z] \longrightarrow [E, X] \xrightarrow{f_*} [E, Y] \xrightarrow{g_*} [Z, E] \longrightarrow [E, \Sigma X] \longrightarrow \ldots$$

Remark 1.1.1. The first of these sequences is similar to the sequence associated with a cofibration $f\colon X \longrightarrow Y$ with cofiber Z, and the second is similar to the sequence associated with a fibration $f\colon X \longrightarrow Y$ with fiber $\Sigma^{-1}Z$. In the stable category, the distiction between fibrations and cofibrations disappears.

1.2 The Smash Product of Spectra

We have defined the smash product of a spectrum E and a CW-complex X as $(E \wedge X)_n = E_n \wedge X$ with the obvious maps. It is possible to construct a smash product of spectra as a generalization of this case. We do not describe the construction here. A detailed definition of the smash product is given by Adams in [Ada74]. We only list the properties of this smash product of spectra.

For every three spectra E, F and G we have the following properties of the smash product:

- It is a covariant functor of each of its components.
- There are natural homotopy equivalences:

$$\begin{aligned} a_{(E,F,G)} \colon (E \wedge F) \wedge G &\longrightarrow E \wedge (F \wedge G) \\ \tau_{(E,F)} \colon E \wedge F &\longrightarrow F \wedge E \\ l \colon S \wedge E &\longrightarrow E \\ r \colon E \wedge S &\longrightarrow E \\ \Sigma_{(E,F)} \colon \Sigma E \wedge F &\longrightarrow \Sigma(E \wedge F) \end{aligned}$$

  The map τ is called the *twist map*.

- Let $\{E_\alpha\}$ be a family of spectra and let $i_\alpha \colon E_\alpha \longrightarrow \vee_\alpha E_\alpha$ be the inclusions. Then the map

$$\{i_\alpha \wedge 1\} \colon \vee_\alpha (E_\alpha \wedge F) \longrightarrow (\vee_\alpha E_\alpha) \wedge F$$

  is a homotopy equivalence.

- The following diagrams that relate the natural homotopy equivalences a, τ, l, and r commute up to homotopy:

$$\begin{tikzcd} ((E \wedge F) \wedge G) \wedge H \arrow[r, "a"] \arrow[d, equal] & (E \wedge F) \wedge (G \wedge H) \arrow[r, "a"] & E \wedge (F \wedge (G \wedge H)) \\ ((E \wedge F) \wedge G) \wedge H \arrow[r, "a \wedge 1"] & (E \wedge (F \wedge G)) \wedge H \arrow[r, "a"] & E \wedge ((F \wedge G) \wedge H) \arrow[u, "1 \wedge a"'] \end{tikzcd}$$

$$
\begin{tikzcd}
(E \wedge F) \wedge G \arrow[r, "\tau\wedge 1"] \arrow[d, "a"'] & (F \wedge E) \wedge G \arrow[r, "a"] & F \wedge (E \wedge G) \arrow[d, "1\wedge\tau"] \\
E \wedge (F \wedge G) \arrow[r, "\tau"] & (F \wedge G) \wedge E \arrow[r, "a"] & F \wedge (G \wedge E)
\end{tikzcd}
$$

$$
\begin{tikzcd}
E \wedge F \arrow[r, equal] \arrow[d, "\tau"'] & E \wedge F \\
F \wedge E \arrow[r, equal] & F \wedge E \arrow[u, "\tau"']
\end{tikzcd}
\qquad
\begin{tikzcd}
S \wedge S \arrow[r, "1"] \arrow[d, equal] & S \wedge S \arrow[d, equal] \\
S \wedge S \arrow[r, "\tau"] & S \wedge S
\end{tikzcd}
$$

$$
\begin{tikzcd}
(S \wedge E) \wedge F \arrow[r, "a"] \arrow[d, "l\wedge 1"'] & S \wedge (E \wedge F) \arrow[d, "l"] \\
E \wedge F \arrow[r, equal] & E \wedge F
\end{tikzcd}
\qquad
\begin{tikzcd}
(E \wedge F) \wedge S \arrow[r, "a"] \arrow[d, "r"'] & E \wedge (F \wedge S) \arrow[d, "l\wedge r"] \\
E \wedge F \arrow[r, equal] & E \wedge F
\end{tikzcd}
$$

$$
\begin{tikzcd}
(E \wedge S) \wedge F \arrow[r, "a"] \arrow[d, "r\wedge 1"'] & E \wedge (S \wedge F) \arrow[d, "1\wedge l"] \\
E \wedge F \arrow[r, equal] & E \wedge F
\end{tikzcd}
\qquad
\begin{tikzcd}
S \wedge E \arrow[r, "\tau"] \arrow[d, "l"'] & E \wedge S \arrow[d, "r"] \\
E \arrow[r, equal] & E
\end{tikzcd}
$$

A complete proof of all these properties can be found in [Ada74].

Given two spectra Y and Z, the functor $[- \wedge Y, Z]$ is representable by Brown's representability theorem. Hence, there exists a spectrum $F(Y, Z)$ called function spectrum obtained by right adjunction of the smash product

$$[X \wedge Y, Z] \cong [X, F(Y, Z)].$$

1.3 Homology and Cohomology

Let E be a spectrum. We define the E-homology and the E-cohomology of another spectrum X as follows

$$E_n(X) = \pi_n(E \wedge X) = [\Sigma^n S, E \wedge X],$$
$$E^n(X) = [X, \Sigma^n E].$$

The Eilenberg–Mac Lane spectrum HG yields ordinary homology and cohomology in the case $G = \mathbb{Z}$. For any other abelian group G, the spectrum HG yields ordinary homology and cohomology with coefficients in G.

For any abelian group G, there exists a spectrum MG, unique up to homotopy, with the following properties:

- $\pi_i(MG) = 0$ for $i < 0$.
- $\pi_0(MG) \cong G \cong (H\mathbb{Z})_0(MG)$.
- $(H\mathbb{Z})_i(MG) = 0$ for $i \neq 0$.

We call MG the *Moore spectrum* of the abelian group G.

Theorem 1.3.1. *Let $\alpha\colon S \longrightarrow E$ be a map inducing isomorphism in π_0, and let X be a spectrum. If $\pi_i(X) = 0$ for $i < n$, then $E_i(X) = 0$ for $i < n$ and the map*

$$\alpha_* = (\alpha \wedge X)_*\colon \pi_k(X) \longrightarrow E_k(X)$$

is an isomorphism for $k = n$ and an epimorphism for $k = n + 1$.

For every spectrum X we have a homomorphism

$$h = (\iota \wedge X)_*\colon \pi_k(X) \longrightarrow (H\mathbb{Z})_k(X),$$

where $\iota\colon S \longrightarrow H\mathbb{Z}$ yields the unit in $\pi_0(H\mathbb{Z}) = \mathbb{Z}$. We call h the Hurewicz homomorphism.

We have a Universal Coefficient Theorem that relates ordinary homology and cohomology with coefficients with ordinary integral homology.

Theorem 1.3.2. *For every spectrum E and every abelian group G there are exact sequences*

$$0 \longrightarrow \mathrm{Ext}((H\mathbb{Z})_{n-1}(E), G) \longrightarrow (HG)^n(E) \longrightarrow \mathrm{Hom}((H\mathbb{Z})_n(E), G) \longrightarrow 0$$

and

$$0 \longrightarrow (H\mathbb{Z})_n(E) \otimes G \longrightarrow (HG)_n(E) \longrightarrow \mathrm{Tor}((H\mathbb{Z})_{n-1}(E), G) \longrightarrow 0.$$

In particular, $\pi_0(HB \wedge HA) = (HB)_0(HA) \cong A \otimes B$ and $[HA, HB] = (HB)^0(HA) \cong \mathrm{Hom}(A, B)$ for any two abelian groups A, B.

Chapter 2

Localization of Spectra

2.1 Introduction

Localizations exist in simplicial model categories that satisfy suitable hypotheses (see Appendix A for details). We will use the Bousfield–Friedlander model [BF78] that is in fact a proper simplicial model category and satisfies the required conditions to assure that f-localization exists in stable homotopy theory.

In the Bousfield-Friedlander model, $\mathrm{HOM}(X, Y)$ is the simplicial set whose n-simplices are the maps $X \wedge \Delta[n]_+ \longrightarrow Y$ of spectra, and in the homotopy category, one has

$$\begin{aligned} \pi_n(\mathrm{HOM}(X,Y)) &= [S^n, \mathrm{HOM}(X,Y)] = [X \wedge S^n, Y] \\ &= [\Sigma^n X, Y] = \pi_n(F(X,Y)) \qquad n \geq 0, \end{aligned}$$

where $F(X, Y)$ is the natural function spectrum obtained by right adjunction of the smash product $[X \wedge Y, Z] \simeq [X, F(Y, Z)]$. Thus, $[X, Y] = [X \wedge S, Y] = [S, F(X, Y)] = \pi_0(F(X, Y))$ for all spectra X and Y. So the simplicial set $\mathrm{HOM}(X, Y)$ and the spectrum $F^c(X, Y)$ have the same homotopy groups, where $F^c(X, Y)$ is the connective cover of $F(X, Y)$, i.e., it has the same homotopy groups as $F(X, Y)$ for $n \geq 0$, and $\pi_n(F^c(X, Y)) = 0$ for $n < 0$.

2.2 Basic Definitions and Properties

We will work in the 'standard' stable homotopy category Ho^s (as described in Chapter 1). This is equivalent to the homotopy category derived from any of the model categories ([BF], [EKMM], [LYD], [HSS]) described in Chapter 4. We begin with the definition of f-localization of spectra.

Definition 2.2.1. Let $f\colon A \longrightarrow B$ be a map of spectra.

(i) A spectrum X is an f-local spectrum if f induces a homotopy equivalence $F^c(B, X) \simeq F^c(A, X)$, where $F^c(-, -)$ denotes the connective cover of the function spectrum.

(ii) A map $g\colon X \longrightarrow Y$ in Ho^s is an f-equivalence if g induces a homotopy equivalence $F^c(Y, Z) \simeq F^c(X, Z)$ for all f-local spectra Z.

(iii) An f-localization of a spectrum X is a map $l_X\colon X \longrightarrow \widehat{X}$ where $\widehat{X}$ is f-local and l_X is an f-equivalence.

As we recall in Chapter 4, we can endow the category of spectra with a simplicial model structure whose homotopy category is the ordinary stable homotopy category. Therefore, as shown in the Appendix, every spectrum X has an f-localization $L_f X$ for every map f, and L_f can be constructed as a homotopy functor on the category of spectra. As we next explain, $L_f X$ is unique up to homotopy equivalence.

The following discussion uses ideas from [Ada] or [CPP]. The objects X in Ho^s such that $X \simeq L_f X$ are precisely the f-local objects. The maps $X \longrightarrow Y$ such that $L_f X \longrightarrow L_f Y$ is a homotopy equivalence are precisely the f-equivalences.

If we denote by Ho^s_l the full subcategory of Ho^s of f-local objects, then the localization functor L_f is left adjoint to the inclusion functor $i\colon Ho^s_l \longrightarrow Ho^s$. Moreover, the localization map l_X has two characteristic properties:

- It is initial among maps from X to f-local objects in Ho^s, i.e., if $g\colon X \longrightarrow Y$ is a map where Y is f-local, then there exists a unique

$h\colon L_fX \longrightarrow Y$ such that $g \simeq l_X \circ h$,

$$\begin{tikzcd} X \arrow[r, "l_X"] \arrow[d, "g"'] & L_fX \arrow[dl, "\exists ! h"] \\ Y & \end{tikzcd}$$

- It is terminal among f-equivalences with domain X, i.e., if $g\colon X \longrightarrow Y$ is an f-equivalence, then there exists a unique $h\colon Y \longrightarrow L_fX$ such that $l_X \simeq h \circ g$,

$$\begin{tikzcd} X \arrow[r, "l_X"] \arrow[d, "f"'] & L_fX \\ Y \arrow[ur, "\exists ! g"'] & \end{tikzcd}$$

Either one of these two universal properties ensures that the localization functor L_f is unique up to homotopy, that is, if we have a map $X \longrightarrow Y$ which is an f-equivalence and where Y is f-local, then $Y \simeq L_fX$.

The first universal property is proved as follows. If Y is f-local, then using (ii) from Definition 2.2.1,

$$\begin{aligned}[X,Y] &\cong \pi_0(F(X,Y)) \cong \pi_0(F^c(X,Y)) \cong \pi_0(F^c(L_fX,Y)) \\ &\cong \pi_0(F(L_fX,Y)) \cong [L_fX,Y]. \end{aligned} \tag{2.1}$$

The second universal property is proved similarly. If $X \longrightarrow Y$ is an f-equivalence, then

$$\begin{aligned}[X,L_fX] &\cong \pi_0(F(X,L_fX)) \cong \pi_0(F^c(X,L_fX)) \cong \pi_0(F^c(Y,L_fX)) \\ &\cong \pi_0(F(Y,L_fX)) \cong [Y,L_fX]. \end{aligned}$$

A functor L on a category $\mathcal{C}$ is called *idempotent* if it is equipped with a natural transformation $l\colon \mathrm{Id} \longrightarrow L$ such that $Ll = lL$ and $Ll\colon L \longrightarrow L^2$ is an isomorphism.

Proposition 2.2.1. *The localization functor L_f is idempotent in the homotopy category Ho^s.*

Proof. The localization map $l_X : X \longrightarrow L_f X$ defines a natural transformation $l : \mathrm{Id} \longrightarrow L_f$. Therefore, the following diagram commutes up to homotopy by naturality

$$\begin{tikzcd} X \arrow[r, "l_X"] \arrow[d, "l_X"'] & L_f X \arrow[d, "L_f l_X"] \\ L_f X \arrow[r, "l_{L_f X}"'] & L_f L_f X, \end{tikzcd}$$

i.e., $l_{L_f X} \circ l_X \simeq L_f l_X \circ l_X$ for all spectra X. Using the universal property 2.1 of the localization functor this implies that $l_{L_f X} \simeq L_f l_X$. For each spectrum X, take the identity map of $L_f X$ and let μ_X be the unique map that closes the diagram

$$\begin{tikzcd} L_f X \arrow[r, "l_{L_f X}"] \arrow[d, "\mathrm{Id}_{L_f X}"'] & L_f L_f X \arrow[dl, dotted, "\mu_X"] \\ L_f X. & \end{tikzcd}$$

We have that $\mu_X \circ l_{L_f X}$ is the identity of $L_f X$. The universal property (2.1) also implies that $l_{L_f X} \circ \mu_X$ is the identity of $L_f L_f X$. Hence, $l_{L_f X}$ is an isomorphism in Ho^s, so the functor L_f is idempotent. □

We have defined f-localization using the connective cover of the function spectrum $F(X, Y)$ because the simplicial set $\mathrm{HOM}(X, Y)$ has homotopy groups defined only in positive dimensions. However, there are special cases in which localization can be characterized using the function spectrum $F(X, Y)$ instead of its connective cover.

Lemma 2.2.2. *Let $f : A \longrightarrow B$ a map of spectra. Then*

(i) The class of f-local spectra is closed under Σ^{-1}.

(ii) The class of f-equivalences is closed under Σ.

Proof. If Z is an f-local spectrum, then $F^c(B, Z) \simeq F^c(A, Z)$. Hence,

$$[\Sigma^n S, F(B, Z)] \cong [\Sigma^n S, F(A, Z)] \quad \text{for all } n \geq 0.$$

We know that $F(X, \Sigma^{-1}Y) \simeq \Sigma^{-1}F(X,Y)$ for every spectra X and Y, so for all $n \geq 0$,

$$\begin{aligned}
&[\Sigma^n S, F(B, \Sigma^{-1}Z)] \cong [\Sigma^n S, \Sigma^{-1}F(B,Z)] \cong [\Sigma^{n+1}S, F(B,Z)] \\
&\cong [\Sigma^{n+1}S, F(A,Z)] \cong [\Sigma^n S, \Sigma^{-1}F(A,Z)] \cong [\Sigma^n S, F(A, \Sigma^{-1}Z)].
\end{aligned}$$

Therefore $\Sigma^{-1}Z$ is f-local.

If $g\colon X \longrightarrow Y$ is an f-equivalence, then $F^c(Y,Z) \simeq F^c(X,Z)$ for every f-local spectrum Z. Hence,

$$[\Sigma^n S, F(Y,Z)] \cong [\Sigma^n S, F(X,Z)] \quad \text{for all } n \geq 0.$$

We know that $F(\Sigma A, B) \simeq \Sigma^{-1}F(A,B)$ for every spectra A and B, so for all $n \geq 0$,

$$\begin{aligned}
&[\Sigma^n S, F(\Sigma Y, Z)] \cong [\Sigma^n S, \Sigma^{-1}F(Y,Z)] \cong [\Sigma^{n+1}S, F(Y,Z)] \\
&\cong [\Sigma^{n+1}S, F(X,Z)] \cong [\Sigma^n S, \Sigma^{-1}F(X,Z)] \cong [\Sigma^n S, F(\Sigma X, Z)].
\end{aligned}$$

Therefore, Σg is an f-equivalence. □

Theorem 2.2.3. *Let $f\colon A \longrightarrow B$ be a map of spectra. Then the following statements are equivalent:*

(i) $\Sigma L_f X \simeq L_f \Sigma X$ *for every spectrum X.*

(ii) *The class of f-local spectra is closed under Σ.*

(iii) *The class of f-equivalences is closed under Σ^{-1}.*

(iv) *The natural map $\Sigma^k L_f X \longrightarrow L_f \Sigma^k X$ is a homotopy equivalence for every spectrum X and for all $k \in \mathbb{Z}$.*

(v) $\Sigma^k L_f X \simeq L_f \Sigma^k X$ *for every spectrum X and for all $k \in \mathbb{Z}$.*

(vi) *The map f induces a homotopy equivalence $F(B,Z) \simeq F(A,Z)$ for every f-local spectrum Z.*

Proof. It is clear that (i) implies (ii) because $X \simeq L_f X$.

Let $g\colon X \longrightarrow Y$ be an f-equivalence and suppose that the class of f-local spectra is closed under Σ. Then

$$\begin{aligned} F^c(\Sigma^{-1}Y, Z) \simeq F^c(\Sigma^{-1}Y, \Sigma^{-1}\Sigma Z) \simeq F^c(Y, \Sigma Z) \simeq F^c(X, \Sigma Z) \\ \simeq F^c(\Sigma^{-1}X, \Sigma^{-1}\Sigma Z) \simeq F^c(\Sigma^{-1}X, Z) \end{aligned}$$

for every f-local spectrum Z, because if Z is f-local, then ΣZ is also f-local. Hence, (ii) implies (iii).

To see that (iii) implies (iv), suppose now that the class of f-equivalences is closed under Σ^{-1}. The map $l_{\Sigma^k X}\colon \Sigma^k X \longrightarrow L_f \Sigma^k X$ is an f-equivalence, so the same happens for the map $\Sigma^{-k} l_{\Sigma^k X}$ by hypothesis or using Lemma 2.2.2 depending on whether k is positive or negative. We have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Sigma^{-k} l_{\Sigma^k X}"] \arrow[d, "l"'] & \Sigma^{-k} L_f \Sigma^k X \\ L_f X \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

The map g exists because $\Sigma^{-k} L_f \Sigma^k X$ is f-local and is a homotopy equivalence. Therefore the natural map

$$\Sigma^k L_f X \xrightarrow{\Sigma^k g} L_f \Sigma^k X$$

is also a homotopy equivalence.

The implications (iv)$\Rightarrow$(v) and (v)$\Rightarrow$(i) are obvious.

Now we will prove that conditions (v) and (vi) are equivalent. First suppose that $\Sigma^k L_f X \simeq L_f \Sigma^k X$ for all $k \in \mathbb{Z}$. Since the object $L_f \Sigma^k X$ is f-local,

$$F^c(B, L_f \Sigma^k X) \simeq F^c(A, L_f \Sigma^k X) \quad \text{for all } k \in \mathbb{Z}.$$

Thus,

$$\pi_n(F(B, L_f \Sigma^k X)) \cong \pi_n(F(A, L_f \Sigma^k X)) \quad \text{for all } n \geq 0,\ k \in \mathbb{Z}.$$

By assumption, this is the same as

$$\pi_n(F(B, \Sigma^k L_f X)) \cong \pi_n(F(A, \Sigma^k L_f X)) \quad \text{for all } n \geq 0,\ k \in \mathbb{Z}.$$

Now $F(B, \Sigma^k L_f X) \simeq \Sigma^k F(B, L_f B)$ for all $k \in \mathbb{Z}$, from which we infer that

$$\pi_{n-k}(F(B, L_f X)) \cong \pi_{n-k}(F(A, L_f X)) \quad \text{for all } n \geq 0,\ k \in \mathbb{Z},$$

so $F(B, L_f X) \simeq F(A, L_f X)$.

Conversely, suppose now that X is f-local. To prove that $L_f \Sigma^k X \simeq \Sigma^k L_f X$ for all $k \in \mathbb{Z}$, we have to see that $\Sigma^k L_f X$ is f-local and $\Sigma^k X \longrightarrow \Sigma^k L_f X$ is an f-equivalence.

- $\Sigma^k L_f X$ is f-local if $F(B, \Sigma^k L_f X) \simeq F(A, \Sigma^k L_f X)$. This is equivalent to $\Sigma^k F(B, L_f X) \simeq \Sigma^k F(A, L_f X)$ for all $k \in \mathbb{Z}$, and this is true because $L_f X$ is f-local.
- The map $\Sigma^k X \longrightarrow \Sigma^k L_f X$ is an f-equivalence if $F(\Sigma^k L_f X, Z) \simeq F(\Sigma^k X, Z)$ for every f-local spectrum Z. This is equivalent to say that $\Sigma^{-k} F(L_f X, Z) \simeq \Sigma^{-k} F(X, Z)$ for all $k \in \mathbb{Z}$, and this is true because the map $X \longrightarrow L_f X$ is an f-equivalence.

□

We say that the functor L_f commutes with suspension if these equivalent conditions are fulfilled. We will see examples of f-localization functors that do not commute with suspension in Chapter 3.

Chapter 3

Examples of f-localizations

In this chapter we describe some classical examples of localizations in the stable homotopy category. These examples include nullifications, homological localizations and localizations at sets of primes. Each of these examples is an f-localization for a suitable map f and we will explicitly give this map in each case.

3.1 Nullifications

When the map f has the form $f\colon A \longrightarrow *$ where $*$ is the 'point-spectrum', we get an important case of f-localizations called nullifications. In this case we denote the functor L_f as P_A and call it *A-nullification.*

Definition 3.1.1. Let A be a spectrum.

- A spectrum X is called *A-null* if $F^c(A, X) \simeq *$.
- A map $g\colon X \longrightarrow Y$ is an *A-equivalence* if g induces a homotopy equivalence
$$F^c(Y, Z) \simeq F^c(X, Z)$$
for each A-null spectrum Z.

A special case of nullification is when we localize with respect to the map $f\colon \Sigma^k S \longrightarrow *$ where S is the sphere spectrum and $k \in \mathbb{Z}$. In the case of nullifications of spaces with respect to the map $S^{n+1} \longrightarrow *$ one obtains that $P_{S^{n+1}}X$ is the n-th Postnikov section P_nX (see [DF]). For spectra the result is similar but first of all we have to define Postnikov towers in stable homotopy.

Definition 3.1.2. A *Postnikov tower* of a spectrum E is a commutative diagram of spectra

$$\begin{tikzcd}
\cdots \arrow[r] & E_{(n+2)} \arrow[r] & E_{(n+1)} \arrow[r] & E_{(n)} \arrow[r] & E_{(n-1)} \arrow[r] & E_{(n-2)} \arrow[r] & \cdots \\
 & & & E \arrow[ull, "\tau_{n+2}"] \arrow[ul, "\tau_{n+1}"] \arrow[u, "\tau_n"] \arrow[ur, "\tau_{n-1}"'] \arrow[urr, "\tau_{n-2}"'] & & &
\end{tikzcd}$$

such that, for every n,

- $\pi_i(E_{(n)}) = 0$ for $i > n$.
- $(\tau_n)_* \colon \pi_i(E) \longrightarrow \pi_i(E_{(n)})$ is an isomorphism for $i \leq n$.

The spectrum $E_{(n)}$ is called the n-th Postnikov section of E.

Theorem 3.1.1. *Given a spectrum E, the nullification of E with respect to $\Sigma^{k+1}S$ is the k-th Postnikov section of E, i.e. $P_{\Sigma^{k+1}S} \simeq E_{(k)}$.*

Proof. It is enough to prove that $E_{(k)}$ is $(\Sigma^{k+1}S)$-null and $\tau_k \colon E \longrightarrow E_{(k)}$ is a $(\Sigma^{k+1}S)$-equivalence (see the universal properties of L_f in Section 2.2).

- $E_{(k)}$ is $(\Sigma^{k+1}S)$-null because

  $$\pi_n(F^c(\Sigma^{k+1}S, E_{(k)})) \cong \pi_n(F(\Sigma^{k+1}S, E_{(k)})) \quad \text{for all } n \geq 0,$$

  and

  $$\pi_n(F(\Sigma^{k+1}S, E_{(k)})) \cong \pi_{n+k+1}(E_{(k)}) = 0 \quad \text{for all } n \geq 0.$$

- $\tau_k \colon E \longrightarrow E_{(k)}$ is a $(\Sigma^{k+1}S)$-equivalence. We have to prove that the induced map $F^c(E_{(k)}, Z) \longrightarrow F^c(E, Z)$ is a homotopy equivalence for every spectrum Z that is $(\Sigma^{k+1}S)$-null. A spectrum Z is $(\Sigma^{k+1}S)$-null if $F^c(\Sigma^{k+1}S, Z) \simeq *$ and this is equivalent to say that $\pi_i(Z) = 0$ if $i > k$. Now we have a cofiber sequence

  $$F \longrightarrow E \xrightarrow{\tau_k} E_{(k)}$$

where F is the fiber of τ_k and and it is characterized by $\pi_i(F) = 0$ if $i \leq k$. This cofiber sequence yields an exact sequence

$$\cdots \longleftarrow [F, Z] \longleftarrow [E, Z] \longleftarrow [E_{(k)}, Z] \longleftarrow [\Sigma F, Z] \longleftarrow \cdots$$

The term $[F, Z]$ is zero because the obstructions for $[F, Z]$ to be non-zero are elements $\theta_k \in (H\pi_k(F))^k(E)$, but $\pi_i(F) = 0$ for $i \leq k$ and $\pi_i(Z) = 0$ for $i > k$ by hypothesis. So $\theta_k = 0$ for all k and then $[F, Z] = 0$. For the same reason $[\Sigma^j F, Z] = 0$ for $j > 0$. We get an isomorphism $[\Sigma^j E, Z] \cong [\Sigma^j E_{(k)}, Z]$ for $j \geq 0$ and so a weak equivalence $F^c(E_{(k)}, Z) \xrightarrow{\simeq} F^c(E, Z)$.

□

Remark 3.1.1. Postnikov sections are an easy example of localization functors that do not commute with suspension. Consider for example a connective spectrum E and the nullification functor $P_{\Sigma^n S}$ where $n > 0$. Then

$$\pi_i(\Sigma P_{\Sigma^n S} E) = \begin{cases} \pi_{i-1}(E) & i \leq n+1 \\ 0 & i > n+1 \end{cases}$$

$$\pi_i(P_{\Sigma^n S} \Sigma E) = \begin{cases} \pi_{i-1}(E) & i \leq n \\ 0 & i > n \end{cases}$$

Therefore, $P_{\Sigma^n S} \Sigma E \not\simeq \Sigma P_{\Sigma^n S} E$ if $\pi_n(E) \neq 0$.

Now we will see that although they may seem very different, there is a close relation between f-localization functors and nullification functors. An L_f functor that commutes with suspension is the same as the nullification functor with respect to the cofiber of f.

Proposition 3.1.2. *Given a cofiber sequence of spectra $X \xrightarrow{f} Y \longrightarrow Z$, then L_f commutes with suspension if and only if P_Z commutes with suspension.*

Proof. If $X \xrightarrow{f} Y \longrightarrow Z$ is a cofiber sequence, then

$$F(X, E) \longleftarrow F(Y, E) \longleftarrow F(Z, E)$$

is another cofiber sequence for every spectrum E. This new sequence yields an exact sequence of homotopy groups

$$\begin{aligned} \ldots \longrightarrow \pi_{n-1}(\Sigma^{-1}F(Y,E)) \longrightarrow \pi_{n-1}(\Sigma^{-1}F(X,E)) \longrightarrow \pi_n(F(Z,E)) \\ \longrightarrow \pi_n(F(Y,E)) \longrightarrow \pi_n(F(X,E)) \longrightarrow \pi_{n+1}(\Sigma F(Z,E)) \longrightarrow \ldots \end{aligned}$$

The functor L_f commutes with suspension if and only if A is f-local if $F(Y,A) \simeq F(X,A)$ (by Theorem 2.2.3). Now, we have that $\pi_n(F(Y,A)) \cong \pi_n(F(X,A))$ for all $n \in \mathbb{Z}$. Using the homotopy exact sequence we get

$$0 = \pi_{n+i}(\Sigma^i F(Z,A)) \cong \pi_n(F(Z,A)$$

This is equivalent to $F(Z,A) \simeq *$ and this happens if and only if P_Z commutes with suspension by Theorem 2.2.3. $\square$

Proposition 3.1.3. *Given a cofiber sequence of spectra $X \xrightarrow{f} Y \longrightarrow Z$, if the functors L_f or P_Z commute with suspension, then for every spectrum E, we have $L_f E \simeq P_Z E$.*

Proof. If L_f commutes with suspension, then P_Z also does, and conversely by Proposition 3.1.2. It is enough to prove that $P_Z E$ is f-local and $E \longrightarrow P_Z E$ is an f-equivalence.

- $P_Z E$ is f-local if $F(Y, P_Z E) \simeq F(X, P_Z E)$ and this is true because of the homotopy exact sequence of the cofiber sequence for the function spectrum and the fact that $F(Z, P_Z E) \simeq *$, because $P_Z E$ is Z-null by definition.

- $E \longrightarrow P_Z E$ is an f-equivalence if $F(P_Z E, A) \simeq F(E, A)$ for every A that is f-local. But if A is f local, then A is Z-null (again use the homotopy exact sequence) and $F(P_Z E, A) \simeq F(E, A)$ since $E \longrightarrow P_Z E$ is a Z-equivalence.

$\square$

3.2 Homological Localizations

The theory of localizations of spectra with respect to homology is described by Bousfield in [Bou74]. This is a localization functor that commutes with suspension. We recall the definition of homological localizations of spectra. As we know any spectrum E yields a homology theory on spectra defined as $E_n(X) = \pi_n(E \wedge X)$.

Definition 3.2.1. Let E be any spectrum.

(i) A spectrum X is called *E-acyclic* if $E_k(X) = 0$ for all $k \in \mathbb{Z}$ or, equivalently, $E \wedge X \simeq *$.

(ii) A map of spectra $f\colon X \longrightarrow Y$ is an *E-equivalence* if it induces an isomorphism in E-homology, i.e., if the map $f_*\colon E_k(X) \longrightarrow E_k(Y)$ is an isomorphism for all $k \in \mathbb{Z}$.

(iii) A spectrum Z is E-local if each E-equivalence $f\colon X \longrightarrow Y$ induces a homotopy equivalence $F(Y, Z) \simeq F(X, Z)$ or, equivalently, if the function spectrum $F(X, Z)$ is contractible for each E-acyclic spectrum X.

Definition 3.2.2. An E-localization of a spectrum X is a map $\eta_X\colon X \longrightarrow L_E X$ where $L_E X$ is an E-local spectrum, and η_X is an E-equivalence.

Assigning to each spectrum X its localization $L_E X$ defines a functor. We call it an E-localization functor. Moreover, this functor is unique up to homotopy and idempotent. The existence of L_E for every spectrum E is proved by Bousfield in [Bou74].

E-localization is actually an f-localization for a suitable map f. In fact, it is a nullification.

Theorem 3.2.1. *Let E be any spectrum. Then $L_E X \simeq L_f X$ for every spectrum X for a suitable map f.*

Proof. We sketch the construction of the map f. For full details see [Bou74] or [Nee01].

Let X be a spectrum. We denote by $loc(X)$ the localizing subcategory generated by X, i.e., the smallest class of spectra that contains X and is closed by cofiber sequences and arbitrary wedges. Then for any spectrum E there exists an E-acyclic spectrum Z such that $loc(Z)$ equals the class of all E-acyclic spectra. The spectrum Z is constructed as a wedge of E-acyclic spectra, such that Z contains, up to equivalence, each E-acyclic spectrum bellow a certain cardinality. If we take f as the map $f\colon Z \longrightarrow *$, then $L_E X \simeq L_f X = P_Z X$ for every spectrum X. $\square$

3.3 Localization at Sets of Primes

The localization theory of groups and spaces at sets of primes has been widely studied (see [BK] and [HMR]). For a given family P of primes, a simply connected space is P-local if all its homotopy groups are P-local, i.e., $\mathbb{Z}_{(P)}$-modules, where $\mathbb{Z}_{(P)}$ denotes the integers localized at P. For every simply connected space X, there is a (homotopy unique) map $X \longrightarrow X_{(P)}$ that induces the natural homomorphism $\pi_k(X) \longrightarrow \pi_k(X) \otimes \mathbb{Z}_{(P)}$ for all k. Here we will recall the corresponding theory for spectra.

Let Λ be any subring of the rationals. Thus $\Lambda = \mathbb{Z}_{(P)}$ for some set of primes P, possibly empty.

Definition 3.3.1. Let G be an abelian group. The Λ-localization of G is the group homomorphism

$$\begin{aligned} l_G \ : \ G &\longrightarrow G \otimes \Lambda \\ g &\longrightarrow g \otimes 1 \end{aligned}$$

The group G is called Λ-local if l_G is an isomorphism. A group homomorphism $f\colon G \longrightarrow H$ Λ-localizes G if there exists an isomorphism $g\colon G \otimes \Lambda \longrightarrow H$ such that the following diagram commutes

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "l_G"'] & H \\ G \otimes \Lambda. \arrow[ur, "g"'] & \end{tikzcd}$$

The homomorphism $l_\Lambda\colon \Lambda \longrightarrow \Lambda \otimes \Lambda$ is an isomorphism, so $G \otimes \Lambda$ is Λ-local for every abelian group G (since $(G \otimes \Lambda) \otimes \Lambda \cong G \otimes \Lambda$).

In order to give the definition of Λ-localization for spectra, which is analogous to the definition given for groups, we use the smash product instead of the tensor product, and we replace the subring Λ by its associated Moore spectrum $M\Lambda$. Recall from Section 1.3 that given an abelian group A, its associated Moore spectrum MA is a spectrum such that

- $H_0(MA) = \pi_0(MA) = A$
- $\pi_i(MA) = 0$ if $i < 0$

- $H_i(MA) = 0$ if $i \neq 0$

Using the unit in $\Lambda = \pi_0(M\Lambda)$, we get a morphism $\mathbf{1}\colon S \longrightarrow M\Lambda$.

Definition 3.3.2. For every spectrum E, we call Λ-localization of E the map of spectra

$$l_E\colon E \simeq E \wedge S \xrightarrow{\mathsf{Id}\wedge 1} E \wedge M\Lambda\,.$$

A spectrum E is called Λ-local if l_E is an equivalence. We set $E_\Lambda = E\wedge M\Lambda$. A morphism $f\colon E \longrightarrow F$ of spectra Λ-localizes E if there exists a homotopy equivalence $g\colon E_\Lambda \longrightarrow F$ such that $g \circ l_E \simeq f$.

As it happened in the case of simply connected spaces, Λ-localization of spectra Λ-localizes all the homotopy groups. This is even more general because any spectrum yields a homology and a cohomology theory in the stable homotopy category, and Λ-localization Λ-localizes both homology and cohomology groups. (For a detailed proof of the following theorem see [Rudyak].)

Theorem 3.3.1. *Given any spectrum E, there are isomorphisms*

$$(E_\Lambda)_*(X) \cong E_*(X) \otimes \Lambda$$

$$(E_\Lambda)^*(X) \cong E^*(X) \otimes \Lambda$$

for every spectrum X. This isomorphism are natural with respect to E and X, and the maps $(l_E)_*$ and $(l_E)^*$, Λ-localize $E_*(X)$ and $E^*(X)$ respectively. As a particular case of this theorem, we have that l_E Λ-localizes homotopy groups of spectra.

Given a map $f\colon E \longrightarrow F$ one defines $f_\Lambda\colon E_\Lambda \longrightarrow F_\Lambda$ as $f_\Lambda = f \wedge \mathsf{Id}$. Then one has $(g \circ f)_\Lambda = g_\Lambda \circ f_\Lambda$, so $(-)_\Lambda$ is a functor.

For every Λ-local group G, the spectra MG and HG (where HG means the Eilenberg–Mac Lane spectrum of G) are Λ-local spectra.

We have defined the localization of spectra at a set of primes P by smashing with the $\mathbb{Z}_{(P)}$ Moore spectrum. This localization is indeed an f-localization for certain map f of spectra. We will explicitly disply map f. As we know, $\Sigma(E \wedge M\mathbb{Z}_{(P)}) \simeq \Sigma E \wedge M\mathbb{Z}_{(P)}$, so prime localization commutes with suspension and we can use the function spectrum instead of its connective cover to define f-localization (see Theorem 2.2.3).

Suppose we want to localize a spectrum E at a set of primes P. For each prime q that is not in P, we have a map $S \longrightarrow S$ inducing multiplication by q in $\pi_0(S) \cong \mathbb{Z}$. Let f be the wedge of all these maps for every prime q not in P,

$$f\colon \bigvee_{q \notin P} S \longrightarrow \bigvee_{q \notin P} S.$$

For each of these maps $S \longrightarrow S$, we have that the induced map

$$F(S, E \wedge M\mathbb{Z}_{(P)}) \longrightarrow F(S, E \wedge M\mathbb{Z}_{(P)})$$

is a homotopy equivalence because the corresponding map of homotopy groups

$$\pi_n(E \wedge M\mathbb{Z}_{(P)}) \longrightarrow \pi_n(E \wedge M\mathbb{Z}_{(P)})$$

is an isomorphism for all n. Recall that $\pi_n(E \wedge M\mathbb{Z}_{(P)}) \cong \pi_n(E) \otimes \mathbb{Z}_{(P)}$ by Theorem 3.3.1 and multiplication by q is an isomorphism in $\mathbb{Z}_{(P)}$. This is equivalent to say that $\pi_n(E \wedge M\mathbb{Z}_{(P)})$ is a $\mathbb{Z}_{(P)}$-module. So the map

$$F(\vee_{q \notin P} S, E \wedge M\mathbb{Z}_{(P)}) \longrightarrow F(\vee_{q \notin P} S, E \wedge M\mathbb{Z}_{(P)})$$

is a homotopy equivalence, and $E \wedge M\mathbb{Z}_{(P)}$ is an f-local spectrum. Indeed, f-local spectra are precisely those spectra Z whose homotopy groups $\pi_n(Z)$ are $\mathbb{Z}_{(P)}$-modules, because if Z is f-local, then the map

$$F(\vee_{q \notin P} S, Z) \longrightarrow F(\vee_{q \notin P} S, Z)$$

is a homotopy equivalence or equivalently, multiplication by all primes $q \notin P$ is an isomorphism on $\pi_n(Z)$. This is the same as claiming that $\pi_n(Z)$ is a $\mathbb{Z}_{(P)}$-module.

We also have that the map $S \xrightarrow{\eta} M\mathbb{Z}_{(P)}$ is an f-equivalence. To see this, we need to check that the map

$$F(M\mathbb{Z}_{(P)}, Z) \longrightarrow F(S, Z)$$

is a homotopy equivalence, or equivalently that the map

$$[\Sigma^i M\mathbb{Z}_{(P)}, Z] \longrightarrow \pi_i(Z)$$

is an isomorphism for all $i \in \mathbb{Z}$ and for f-local spectra Z. Taking a free presentation of $\mathbb{Z}_{(P)}$

$$0 \longrightarrow \oplus_\alpha \mathbb{Z} \longrightarrow \oplus_\beta \mathbb{Z} \longrightarrow \mathbb{Z}_{(P)} \longrightarrow 0$$

we have a cofiber sequence

$$\vee_\alpha \Sigma^i S \longrightarrow \vee_\beta \Sigma^i S \longrightarrow \Sigma^i M\mathbb{Z}_{(P)} \longrightarrow \vee_\alpha \Sigma^{i+1} S \longrightarrow \vee_\beta \Sigma^{i+1} S$$

which yields an exact sequence

$$0 \longrightarrow \mathrm{Ext}(\mathbb{Z}_{(P)}, \pi_{i-1} Z) \longrightarrow [\Sigma^i M\mathbb{Z}_{(P)}, Z] \longrightarrow \mathrm{Hom}(\mathbb{Z}_{(P)}, \pi_i(Z)) \longrightarrow 0.$$

Now, $\mathrm{Hom}(\mathbb{Z}_{(P)}, \pi_i(Z)) \cong \pi_i(Z)$ because $\pi_i(Z)$ is a $\mathbb{Z}_{(P)}$-module and

$$\mathrm{Ext}(\mathbb{Z}_{(P)}, \pi_{i-1}(Z)) \cong \mathop{\mathrm{Ext}}_{\mathbb{Z}_{(P)}}(\mathbb{Z}_{(P)}, \pi_{i-1}(Z)) = 0$$

because $\mathbb{Z}_{(P)}$ is free over $\mathbb{Z}_{(P)}$. So, the map $[\Sigma^i M\mathbb{Z}_{(P)}, Z] \longrightarrow \pi_i(Z)$ is an isomorphism for all Z that are f-local. The spectrum $M\mathbb{Z}_{(P)}$ is f-local because $\pi_i(M\mathbb{Z}_{(P)}) \cong \pi_i(S) \otimes \mathbb{Z}_{(P)}$ is a $\mathbb{Z}_{(P)}$-module for all $i \in \mathbb{Z}$, so $L_f S \simeq M\mathbb{Z}_{(P)}$ and the map $S \xrightarrow{\eta} M\mathbb{Z}_{(P)}$ is the localization map for S.

If we smash η with the identity map on a spectrum E, then we have another f-equivalence

$$E \wedge S \simeq E \xrightarrow{1 \wedge \eta} E \wedge M\mathbb{Z}_{(P)}$$

because the localization functor commutes with suspension. We have obtained the following

Theorem 3.3.2. *Let P be a set of primes and let $f\colon \vee_{q \notin P} S \longrightarrow \vee_{q \notin P} S$ be a wedge of maps inducing multiplication by q in $\pi_0(S)$ for each prime q not in P. Then, $L_f E \simeq E \wedge M\mathbb{Z}_{(P)}$ for every spectrum E.* □

Remark 3.3.1. This theorem tells us that, in order to determine, the prime localization of a spectrum E, it is enough to know the prime localization of the sphere spectrum S, because $L_f E \simeq E \wedge L_f S$. This property holds for a special class of localization functors called smashing localizations.

Since the functor $(-)_\Lambda$ is L_f for a map f, it is idempotent in the sense of Section 2.2. The idempotence of $(-)_\Lambda$ gives a theorem that improves one in [Rud], in which there are connectivity conditions on spectra that can in fact be removed. Although this theorem follows immediately from the idempotence of $(-)_\Lambda$, we give here another proof using only prime localization machinery.

Theorem 3.3.3. *If F is a Λ-local spectrum, then, for every morphism $f\colon E \longrightarrow F$ there exists a unique morphism $g\colon E_\Lambda \longrightarrow F$ such that $g \circ l_E = f$*

$$\begin{tikzcd} E \arrow[r, "l_E"] \arrow[d, "f"'] & E_\Lambda \arrow[dl, dotted, "\exists! g"] \\ F & \end{tikzcd} \qquad [E, F] \cong [E_\Lambda, F]$$

Proof. Let $f\colon E \longrightarrow F$. F is Λ-local, so l_F is an isomorphism; we can take $g = (l_F)^{-1} \circ f_\Lambda$. Now we have to proof the uniquenes of g. Suppose we have another morphism $h\colon E_\Lambda \longrightarrow F$ such that $h \circ l_E = f$. Naturality gives us the commutativity of the following diagram,

$$\begin{tikzcd} E_\Lambda \arrow[r, "h"] \arrow[d, "l_{E_\Lambda}"'] & F \arrow[d, "l_F"] \\ (E_\Lambda)_\Lambda \arrow[r, "h_\Lambda"] & F_\Lambda \end{tikzcd}$$

Now,

$$l_F \circ h = h_\Lambda \circ l_{E_\Lambda} = h_\Lambda \circ (l_E)_\Lambda = f_\Lambda$$

because $(-)_\Lambda$ is idempotent. Then $h = (l_F)^{-1} \circ f_\Lambda^{-1} \circ f_\Lambda = g$. □

Λ-localization also preserves multiplicative structures.

Lemma 3.3.4. *For every two spectra E and F there exists an equivalence*

$$\phi\colon (E \wedge F)_\Lambda \longrightarrow E_\Lambda \wedge F_\Lambda$$

Proof. See [Rudyak]. □

Theorem 3.3.5. *Let (E, μ, η) be a ring spectrum and (F, m) a module spectrum over E. Then*

(i) E_Λ admits a unique (up to homotopy) ring structure $(E_\Lambda, \overline{\mu}, \overline{\eta})$ and the localization map $l_E\colon E \longrightarrow E_\Lambda$ is a ring map.

(ii) F_Λ admits a unique (up to homotopy) module structure $(F_\Lambda, \overline{m})$ and the localization map $l_F\colon F \longrightarrow F_\Lambda$ is an E-module map.

Proof. This is a common property of all the f-localization functors that commutes with suspension. For a general proof, see Theorem 5.3.1. □

Chapter 4

Model Category Structures for Stable Homotopy

4.1 Introduction

In this chapter we describe some model structures for the stable homotopy category. The first attempt to endow the stable homotopy category a model category structure appears in the paper [BF78] by Bousfield and Friedlander. This category is not symmetric monoidal. At present, highly structured models for the category of spectra are available which admits a symmetric, associative and homotopically well behaved smash product. Examples of these categories are the symmetric spectra of Hovey, Shipley and Smith [HSS], the S-modules of Elmendorf, Kriz, Mandell and May [EKMM] or the simplicial functors of Lydakis [Lyd].

4.2 The Bousfield–Friedlander Model

This is probably the simplest model category of spectra.

An object X of this category is a sequence of simplicial sets X^n and morphisms (of simplicial sets)

$$\sigma^n \colon S^1 \wedge X^n \longrightarrow X^{n+1}$$

where $S^1 = \Delta[1]/\partial\Delta[1]$. Let us call *spectra* these objects. A morphism of spectra $f\colon X \longrightarrow Y$ consists of morphisms $f^n\colon X^n \longrightarrow Y^n$ such that the following diagram commutes

$$\begin{tikzcd} S^1 \wedge X^n \arrow[r, "1\wedge f^n"] \arrow[d] & S^1 \wedge Y^n \arrow[d] \\ X^{n+1} \arrow[r, "f^{n+1}"] & Y^{n+1} \end{tikzcd}$$

$\mathrm{HOM}(X,Y)$ = simplicial set whose n-simplices are morphisms $X \wedge \Delta[n]_+ \longrightarrow Y$ of spectra.

If K is a simplicial set:

- $X \wedge K$ is a spectrum, $(X \wedge K)^n = X^n \wedge K$
- $\hom(K,X)$ is a spectrum, $(\hom(K,X))^n = \hom(K,X^n)$

We have a mapping space functor

$$\mathrm{HOM}(-,-)\colon \mathit{Spect}^{op} \times \mathit{Spect} \longrightarrow \mathcal{SS}$$

where *Spect* is the category of spectra, and $\mathcal{SS}$ is the category of simplicial sets. The functor $\mathrm{HOM}(-,Y)\colon \mathit{Spect}^{op} \longrightarrow \mathcal{SS}$ has a left adjoint

$$\hom(-,Y)\colon \mathcal{SS} \longrightarrow \mathit{Spect}$$

and the functor $\mathrm{HOM}(X,-)\colon \mathit{Spect} \longrightarrow \mathcal{SS}$ has a right adjoint

$$X \wedge -\colon \mathcal{SS} \longrightarrow \mathit{Spect}.$$

We define the following classes of morphisms in the category *Spect*:

- **Weak equivalences**: Given a morphism $f\colon X \longrightarrow Y$ in *Spect*, f is a weak equivalence if
$$f_*\colon \pi_*(X) \longrightarrow \pi_*(Y)$$
is an isomorphism of groups, where $\pi_*(X) = \varinjlim \pi_{*+n}(X^n)$

- **Cofibrations**: A morphism $f\colon X \longrightarrow Y$ is a cofibration if $f^0\colon X^0 \longrightarrow Y^0$ and each induced map (by the pushout)

$$X^{n+1} \coprod_{S^1\wedge X^n} S^1 \wedge Y^n \longrightarrow Y^{n+1}$$

  are cofibrations of simplicial sets.

- **Fibrations**: A morphism $f\colon X \longrightarrow Y$ is a fibration if it has the RLP with respect to all trivial cofibrations.

Theorem 4.2.1. *Spect has a simplicial model category structure and its homotopy category Ho(Spect) is equivalent to the usual stable homotopy category.*

Proof. For a detailed proof see [BF78]. □

4.3 Symmetric Spectra

The category of symmetric spectra by Hovey, Shipley and Smith, provides the most elementary construction of a category of spectra with a strictly commutative and associative smash product before passing to the homotopy category. This category has a proper, monoidal model category structure and is Quillen equivalent to the Bousfield-Friedlander category.

A *symmetric spectrum* is a sequence X_0, $X_1, \ldots$, X_n of pointed simplicial sets with a pointed map

$$\sigma\colon S^1 \wedge X_n \longrightarrow X_{n+1},$$

for each $n \geq 0$ and a basepoint preserving left action of Σ_n on X_n such that the composition

$$\sigma^p = \sigma \circ (S^1 \wedge \sigma) \circ \ldots \circ (S^{p-1} \wedge \sigma)\colon S^p \wedge X_n \longrightarrow X_{n+p}$$

of the maps $S^i \wedge S^1 \wedge X_{n+p-i+1} \xrightarrow{S^i\wedge\sigma} S^i \wedge X_{n+p-i}$ is $\Sigma_p \times \Sigma_n$-equivariant for $p \geq 1$ and $n \geq 0$.

A map of symmetric spectra $f\colon X \longrightarrow Y$ is a sequence of pointed $f_n\colon X_n \longrightarrow Y_n$ such that f_n is Σ_n-equivariant and the diagram

$$\begin{tikzcd} S^1 \wedge X_n \arrow[r, "\sigma"] \arrow[d, "S^1\wedge f_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\ S^1 \wedge Y_n \arrow[r, "\sigma"] & Y_{n+1} \end{tikzcd}$$

is commutative for each $n \geq 0$. Let Sp^{Σ} denote the category of symmetric spectra. Define

$$Map_{Sp^{\Sigma}}(X,Y) = Sp^{\Sigma}(X \wedge \Delta[-]_+, Y),$$

$\hom(-,Y)$ and $X \wedge -$ are obtained by prolongation (see [HSS] p.8) of the $\mathcal{S}_*$-functors $(-)^K\colon \mathcal{SS}_* \longrightarrow \mathcal{SS}_*$ and $-\wedge K\colon \mathcal{SS}_* \longrightarrow \mathcal{SS}_*$ respectively.

We define the following classes of morphisms in Sp^{Σ}:

- **Stable equivalences**: A map of symmetric spectra $f\colon X \longrightarrow Y$ is a stable equivalence if

  $$E^0 f\colon E^0 Y \longrightarrow E^0 X$$

  is an isomorphism for every injective Ω-spectrum E (see [HSS] p.20).

- **Level (trivial) cofibration**: If each map $f_n\colon X_n \longrightarrow Y_n$ is a (trivial) cofibration of simplicial sets.

- **Level (trivial) fibration**: If each map $f_n\colon X_n \longrightarrow Y_n$ is a (trivial) fibration of simplicial sets.

- **Stable cofibration**: $f\colon X \longrightarrow Y$ is a stable cofibration if it has the LLP with respect to every level trivial fibration. (Stable trivial cofibration = Stable cofibration + Stable equivalence).

- **Stable fibration**: $f\colon X \longrightarrow Y$ is a stable fibration if it has the RLP with respect to every map that is a stable trivial cofibration. (Stable trivial fibration = Stable stable fibration + Stable equivalence).

Theorem 4.3.1. *The category of symmetric spectra Sp^{Σ} with the class of stable equivalences, stable cofibrations and stable fibrations is a (proper) model category that is Quillen equivalent to the [BF78] category.*

4.4 Other Models

The category of S-modules of Elmendorf et al was the first model category for stable homotopy with a true smash product. In the categories of spectra considered previously to this one, the smash product was neither commutative nor associative and one had to pass to the stable homotopy category in order to obtain commutativity and associativity. The homotopy category of this category, whose objects whose objects are called S-modules, is equivalent to the stable homotopy category and the equivalence preserve smash products. For a complete description of the category see [EKMM].

Lydakis' simplicial functors are another solution to the problem of finding a model category Quillen equivalent to the model category of spectra, and which has a symmetric monoidal product corresponding to the smash product of spectra. With the smash product of simplicial functors, one obtains more descriptive constructions of the model structures on spectra. As it happens with the other categories, the stable model structure on simplicial functors is Quillen equivalent to the model structure of [BF78]. For a complete description of the category see [Lyd].

Chapter 5

Localizations of Eilenberg–Mac Lane Spectra

5.1 Ring Spectra and Module Spectra

In this section we recall the definition of a ring spectrum and a module spectrum. Our sources of reference are [Bou], [Rud] These are spectra equipped with a multiplication and a unit rendering certain diagrams homotopy commutative.

Definition 5.1.1. A spectrum E is called a *ring spectrum* if it is equipped with two maps of spectra $\mu\colon E\wedge E \longrightarrow E$ (called the product) and $\eta\colon S \longrightarrow E$ (called the unity) such that the following diagrams commute up to homotopy

- Associativity:

$$\begin{tikzcd} E\wedge E\wedge E \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] & E\wedge E \arrow[d, "\mu"] \\ E\wedge E \arrow[r, "\mu"'] & E \end{tikzcd}$$

- Unity:

$$\begin{tikzcd} S\wedge E \arrow[r, "\eta\wedge 1"] \arrow[dr] & E\wedge E \arrow[d, "\mu"] & E\wedge S \arrow[l, "1\wedge\eta"'] \arrow[dl] \\ & E & \end{tikzcd}$$

where S is the sphere spectrum, and $S \wedge E \longrightarrow E$ is the natural homotopy equivalence described in Section 1.2. The pair (μ, η) is called a ring structure on E.

A ring spectrum is called *commutative* if the following diagram commutes up to homotopy

$$\begin{tikzcd} E \wedge E \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & E \wedge E \arrow[dl, "\mu"] \\ & E & \end{tikzcd}$$

where τ is the twist map (see Section 1.2).

A ring map $\varphi\colon (E, \mu, \eta) \longrightarrow (E', \mu', \eta')$ between ring spectra is a map $\varphi\colon E \longrightarrow E'$ that is compatible with the ring structure on E and E', i.e., the following diagrams commute up to homotopy:

$$\begin{tikzcd} E \wedge E \arrow[r, "\varphi\wedge\varphi"] \arrow[d, "\mu"'] & E' \wedge E' \arrow[d, "\mu'"] \\ E \arrow[r, "\varphi"'] & E' \end{tikzcd} \qquad \begin{tikzcd} & S \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\ E \arrow[rr, "\varphi"'] & & E \end{tikzcd}$$

Definition 5.1.2. A spectrum M is called a *module* spectrum over a ring spectrum (E, μ, η) or an E-*module* spectrum if it is equipped with a map of spectra $m\colon E \wedge M \longrightarrow M$ such that the following diagrams commute up to homotopy:

- Associativity:

$$\begin{tikzcd} E \wedge E \wedge M \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge m"'] & E \wedge M \arrow[d, "m"] \\ E \wedge M \arrow[r, "m"'] & M \end{tikzcd}$$

- Unity:

$$\begin{tikzcd} S \wedge M \arrow[rr, "\eta\wedge 1"] \arrow[dr] & & E \wedge M \arrow[dl, "m"] \\ & M & \end{tikzcd}$$

An E-module map $\psi\colon M \longrightarrow N$ of E-module spectra is a map that is compatible with the map m, i.e., the following diagram commutes up to homotopy to homotopy

$$
\begin{tikzcd}
E \wedge M \arrow[r, "1\wedge\psi"] \arrow[d, "m"'] & E \wedge N \arrow[d, "m'"] \\
M \arrow[r, "\psi"] & N
\end{tikzcd}
$$

Every ring spectrum E is an E-module spectrum with $m = \mu$.

Simple examples of ring spectra and module spectra are Eilenberg–Mac Lane spectra. If R is a ring, the multiplication map in HR comes from the product in R, as we next make explicit, cf.[Rud].

Proposition 5.1.1. *If R is a ring and M is an R-module, then*

(i) The spectrum HR is a ring spectrum.

(ii) The spectrum HM is a module spectrum over HR.

Proof. For the first part, we know that HR and $HR \wedge HR$ are (-1)-connected and using the Hurewicz homomorphism

$$
\begin{aligned}
(H\mathbb{Z})_0(HR \wedge HR) &\cong \pi_0(HR \wedge HR) = (H\mathbb{Z})_0(HR; R) \\
&= (H\mathbb{Z})_0(HR) \otimes R \cong \pi_0(HR) \otimes R = R \otimes R,
\end{aligned}
$$

The product in the ring R gives us a map $\mu\colon HR \wedge HR \longrightarrow HR$, since because

$$
\begin{aligned}
[HR \wedge HR, HR] &= (H\mathbb{Z})^0(HR \wedge HR; R) \\
&\cong \mathrm{Hom}((H\mathbb{Z})_0(HR \wedge HR), R) = \mathrm{Hom}(R \otimes R, R).
\end{aligned}
$$

The unit of the ring R is an element in $\pi_0(HR)$ and therefore it yields a unit $\eta\colon S \longrightarrow HR$ on HR. Then (HR, μ, η) is a ring spectrum. The diagrams commute because of the natural isomorphism

$$[X, HR] \cong \mathrm{Hom}(H_0(X), H_0(HR)),$$

if X is (-1)-connected (by Universal Coefficient Theorem; see Theorem 1.3.2)

The second part is proved in the same way. The R-module structure of M yields a map $m\colon HR \wedge HM \longrightarrow HM$ because

$$[HR \wedge HM, HM] = (H\mathbb{Z})^0(HR \wedge HM; M)$$
$$\cong \mathrm{Hom}((H\mathbb{Z})_0(HR \wedge HM), M) = \mathrm{Hom}(R \otimes M, M).$$

Thus, (HM, m) is an HR-module spectrum. □

Corollary 5.1.2. *For every ring R, the spectrum HR is an $H\mathbb{Z}$-module spectrum.* □

Remark 5.1.1. If M is an E-module spectrum, then for every spectrum X, the abelian group $[X, M]$ is a $\pi_0(E)$-module. For every map $\alpha \in \pi_0(E)$ and every map $f \in [X, M]$ we obtain a map in $[X, M]$ smashing α with f and composing with the E-module structure map

$$X \simeq S \wedge X \xrightarrow{\alpha \wedge f} E \wedge M \xrightarrow{m} M.$$

In particular, if M is an HR-module spectrum, then $\pi_n(M)$ is an R-module for all n.

5.2 Modules over Eilenberg–Mac Lane Spectra

In this section we recall a characterization of module spectra over the ring spectrum $H\mathbb{Z}$. We define stable GEMs as an analog of the classical unstable GEMs and see that $H\mathbb{Z}$-modules are the same as stable GEMs. Our proof includes ideas from [Rud]. Let R be a ring with unit.

Definition 5.2.1. A spectrum E is called a stable R-GEM if it is homotopically equivalent to a wedge of suspensions of Eilenberg–Mac Lane spectra, i.e.

$$E \simeq \bigvee_{n \in \mathbb{Z}} \Sigma^n HA_n,$$

where each A_n is an R-module (hence, each HA_n is an HR-module spectrum). If $R = \mathbb{Z}$, then stable $\mathbb{Z}$-GEMs are called stable GEMs. Note that the natural map

$$\bigvee_{n \in \mathbb{Z}} \Sigma^n HA_n \longrightarrow \prod_{n \in \mathbb{Z}} \Sigma^n HA_n$$

is a homotopy equivalence, because exceptionally, $\pi_i(\vee_{n \in \mathbb{Z}} \Sigma^n HA_n) \cong A_i$.

If M and N are E-module spectra, $[M,N]_{E\text{-mod}} \subset [M,N]$ denotes the set of E-module maps in the stable homotopy category.

If E is a ring spectrum then $E \wedge X$ has a natural E-module structure given by $E \wedge E \wedge X \xrightarrow{\mu\wedge 1} E \wedge X$ for every spectrum X.

Lemma 5.2.1. *Let E be any ring spectrum. Let M be an E-module, X a spectrum and $f\colon X \longrightarrow M$ any map. Then there exists a unique E-module map $\tilde{f}\colon E \wedge X \longrightarrow M$ such that the diagram*

$$\begin{tikzcd} S \wedge X \simeq X \arrow[r, "f"] \arrow[d, "\eta\wedge 1"'] & M \\ E \wedge X \arrow[ur, dotted, "\tilde{f}"'] & \end{tikzcd}$$

commutes up to homotopy. Thus, $E\wedge X$ is the 'free E-module generated by X'. This gives a natural bijection

$$[X,M] \cong [E \wedge X, M]_{E\text{-}mod}$$

which relates homotopy classes of maps with homotopy classes of E-module maps.

Proof. We give the proof in three steps:

(i) Construction of $\tilde{f}$. The following diagram

$$\begin{tikzcd} S \wedge X \arrow[r, "1\wedge f"] \arrow[d, "\eta\wedge 1"'] & S \wedge M \arrow[d, "\eta\wedge 1"] \arrow[dr, "\simeq"] & \\ E \wedge X \arrow[r, "1\wedge f"'] & E \wedge M \arrow[r, "m_M"'] & M \end{tikzcd}$$

commutes. The triangle on the right commutes because M is an E-module. So, if we define $\tilde{f} = m_M \circ (1 \wedge f)$, the diagram

$$\begin{tikzcd} S \wedge X \simeq X \arrow[r, "f"] \arrow[d, "\eta\wedge 1"'] & M \\ E \wedge X \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

commutes.

(ii) The map $\tilde{f}$ is a map of E-modules. We have to check the commutativity of the diagram

$$\begin{tikzcd}
E \wedge (E \wedge X) \arrow[r, "1\wedge(1\wedge f)"] \arrow[d, equal, "\simeq"'] & E \wedge (E \wedge M) \arrow[r, "1\wedge m_M"] \arrow[d, equal, "\simeq"'] & E \wedge M \arrow[d, equal, "\simeq"'] \\
(E \wedge E) \wedge X \arrow[r, "(1\wedge 1)\wedge f"] \arrow[d, "\mu_E\wedge 1"'] & (E \wedge E) \wedge M \arrow[r, "\mu_E\wedge 1"] \arrow[d, "\mu_E"'] & E \wedge M \arrow[d, "m_M"'] \\
E \wedge X \arrow[r, "1\wedge f"'] & E \wedge M \arrow[r, "m_M"'] & M.
\end{tikzcd}$$

The square on the lower left commutes by functoriality of the smash product and the square on the lower right commutes because M is an E-module.

(iii) There is a unique choice of $\tilde{f}$. Let g be a map of E-modules such that the diagram

$$\begin{tikzcd}
S \wedge X \simeq X \arrow[r, "f"] \arrow[d, "\eta\wedge 1"'] & M \\
E \wedge X \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes. Then in the diagram

$$\begin{tikzcd}
E \wedge (S \wedge X) \arrow[r, "\simeq"] \arrow[d, "1\wedge(\eta\wedge 1)"'] & E \wedge X \arrow[d, "1\wedge f"] \arrow[ddl, bend right, "1"'] \\
E \wedge (E \wedge X) \arrow[r, "1\wedge g"] \arrow[d, "1\wedge\mu_E"'] & E \wedge M \arrow[d, "m_M"] \\
E \wedge X \arrow[r, "g"] & M
\end{tikzcd}$$

the upper square commutes by hypothesis and the lower square commutes becase g is a map of E-modules, so $g \simeq m_M \circ (1 \wedge f) \simeq \tilde{f}$.

□

Lemma 5.2.2. *Let MG be the Moore spectrum associated with an abelian group G. Then, there is a natural exact sequence*

$$0 \longrightarrow \mathrm{Ext}(G, \pi_1 X) \longrightarrow [MG, X] \longrightarrow \mathrm{Hom}(G, \pi_0 X) \longrightarrow 0$$

for each spectrum X.

Proof. Let $\oplus_\alpha \mathbb{Z} \longrightarrow \oplus_\beta \mathbb{Z} \longrightarrow G$ be a free presentation of the group G. There is a cofibration

$$\vee_\alpha S \longrightarrow \vee_\beta S \longrightarrow MG \longrightarrow \vee_\alpha \Sigma S \longrightarrow \vee_\beta \Sigma S$$

which gives a long exact sequence

$$\begin{aligned} \mathrm{Hom}(\oplus_\beta \mathbb{Z}, \pi_1 X) \longrightarrow \mathrm{Hom}(\oplus_\alpha \mathbb{Z}, \pi_1 X) \longrightarrow [MG, X] \\ \longrightarrow \mathrm{Hom}(\oplus_\beta \mathbb{Z}, \pi_0 X) \longrightarrow \mathrm{Hom}(\oplus_\alpha \mathbb{Z}, \pi_0 X). \end{aligned}$$

□

Now, let M be an $H\mathbb{Z}$-module. Let $G_i = \pi_i M$ and for each $i \in \mathbb{Z}$ take one map $\alpha_i \in [\Sigma^i MG_i, M]$ corresponding to the identity in $\mathrm{Hom}(G_i, G_i)$ where MG_i is the Moore spectrum corresponding to G_i. Consider the map

$$\vee_i \Sigma^i MG_i \xrightarrow{\vee_i \alpha_i} M.$$

Proposition 5.2.3. *For every $H\mathbb{Z}$-module M, there is a map of $H\mathbb{Z}$-modules $H\mathbb{Z} \wedge (\vee_i \Sigma^i MG_i) \xrightarrow{\vee_i \tilde{\alpha}_i} M$ which is also a homotopy equivalence.*

Proof. For each $\alpha_i \in [\Sigma^i MG_i, M]$ constructed as before we have a unique $H\mathbb{Z}$-module map

$$\tilde{\alpha}_i \colon H\mathbb{Z} \wedge \Sigma^i MG_i \longrightarrow M \qquad \tilde{\alpha}_i = m_M \circ (1 \wedge \alpha_i)$$

by lemma 5.2.1. It is enough to prove that this map induces an isomorphism on π_i, since $H\mathbb{Z} \wedge \Sigma^i MG_i = \Sigma^i HG_i$. This follows from the commutativity of the diagram

$$\begin{tikzcd} S \wedge \Sigma^i MG_i \arrow[r, "1\wedge\alpha_i"] \arrow[d, "\eta\wedge 1"'] & S \wedge M \arrow[d, "\eta\wedge 1"'] \arrow[dr, "\simeq"] & \\ H\mathbb{Z} \wedge \Sigma^i MG_i \arrow[r, "1\wedge\alpha_i"'] & H\mathbb{Z} \wedge M \arrow[r, "m_M"'] & M \end{tikzcd}$$

and the fact that $\eta \wedge 1$ and $1 \wedge \alpha_i$ are isomorphisms on π_i by construction. So $\vee_i \tilde{\alpha}_i$ is a map of $H\mathbb{Z}$-modules and a homotopy equivalence. □

Therefore every $H\mathbb{Z}$-module is homotopically equivalent to a free one.

Corollary 5.2.4. $[HA, \Sigma^i HB]_{H\mathbb{Z}\text{-mod}} = 0$ *unless* $i = 0$ *or* $i = 1$.

Proof. We know that $HA \simeq H\mathbb{Z} \wedge MA$. By Lemma 5.2.1 there is a natural bijection

$$[MA, \Sigma^i HB] \cong [H\mathbb{Z} \wedge MA, \Sigma^i HB]_{H\mathbb{Z}\text{-mod}}$$

and the result follows directly from Lemma 5.2.2 because the homotopy groups of $\Sigma^i HB$ are zero except in dimension i. □

Proposition 5.2.3 tells us that $H\mathbb{Z}$-module spectra are exactly the stable GEMs, because $HG \simeq H\mathbb{Z} \wedge MG$. Similarly, the stable HR-modules are exactly the stable R-GEMs because each HR-module spectrum is an $H\mathbb{Z}$-module spectrum and the homotopy groups of HR-module spectra are R-modules (see Remark 5.1.1). But it is important to notice that the equivalence given by Proposition 5.2.3 is need not be an HR-module map in general.

5.3 Localizations of Ring Spectra and Module Spectra

Now we will study the interaction of f-localization functors with ring and module structures. In the case of f-localization functors that commute with suspension, the localization of a ring or module spectrum has a natural ring or module structure, as we next explain.

Theorem 5.3.1. *Let $f\colon A \longrightarrow B$ a map of spectra. If the f-localization functor commutes with suspension, then:*

- *If E is a ring spectrum, then the spectrum $L_f E$ has a unique ring spectrum structure such that the localization map $l_E\colon E \longrightarrow L_f E$ is a ring map.*

- *If M is a module spectrum over the ring spectrum E, then the spectrum $L_f M$ has a unique E-module structure such that the localization map $l_M\colon M \longrightarrow L_f M$ is an E-module map. Morover, $L_f M$ admits a unique $L_f E$-module structure extending the E-module structure.*

Proof. For the first part we need to construct a product $\overline{\mu}$ and a unit $\overline{\eta}$ on L_fE. Let μ and η be the product and unit of the ring spectrum E, respectively. We have an equivalence $F(E, L_fE) \simeq F(L_fE, L_fE)$ because L_fE is f-local and the functor L_f commutes with suspension by hypothesis. Then,

$$\begin{aligned}[E \wedge E, L_fE] &\cong [E, F(E, L_fE)] \cong [E, F(L_fE, L_fE)] \\ &\cong [E \wedge L_fE, L_fE] \cong [L_fE, F(E, L_fE)] \cong [L_fE, F(L_fE, L_fE)] \\ &\cong [L_fE \wedge L_fE, L_fE]\end{aligned}$$

Hence, the product $\mu\colon E \wedge E \longrightarrow E$ extends to a unique map $\overline{\mu}\colon L_fE \wedge L_fE \longrightarrow L_fE$ rendering commutative the diagram

$$\begin{tikzcd} E \wedge E \arrow[r, "\mu"] \arrow[d, "l_E \wedge l_E"'] & E \arrow[d, "l_E"] \\ L_fE \wedge L_fE \arrow[r, dotted, "\overline{\mu}"'] & L_fE \end{tikzcd} \qquad l_E \circ \mu \simeq \overline{\mu} \circ (l_E \wedge l_E)$$

We define the unit $\overline{\eta}$ as the composition $l_E \circ \eta$

$$\begin{tikzcd} E \arrow[r, "l_E"] & L_fE \\ S \arrow[u, "\eta"] \arrow[ur, dotted, "\overline{\eta}"'] & \end{tikzcd} \qquad \overline{\eta} \simeq l_E \circ \eta$$

Now, $(L_fE, \overline{\mu}, \overline{\eta})$ is a ring spectrum. The commutativity of the diagrams for $\overline{\mu}$ and $\overline{\eta}$ follows from the commutativity of the diagrams for μ and η and the construction of $\overline{\mu}$ and $\overline{\eta}$.

The second part is proved exactly in the same way. We need a map $\overline{m}\colon E \wedge L_fM \longrightarrow L_fM$ endowing L_fM the structure of an E-module spectrum. Let $m\colon E \wedge M \longrightarrow M$ be the E-module structure map on M. Then

$$\begin{aligned}[E \wedge M, L_fM] &\cong [E, F(M, L_fM)] \cong [E, F(L_fM, L_fM)] \\ &\cong [E \wedge L_fM, L_fM] \end{aligned} \tag{5.1}$$

Hence, the map $m\colon E \wedge M \longrightarrow M$ extends to a unique map $\overline{m}\colon E \wedge L_fM \longrightarrow L_fM$ rendering commutative the diagram

$$\begin{tikzcd} E \wedge M \arrow[r, "m"] \arrow[d, "1 \wedge l_M"'] & M \arrow[d, "l_M"] \\ E \wedge L_fM \arrow[r, dotted, "\overline{m}"'] & L_fM \end{tikzcd} \qquad l_M \circ \mu \simeq \overline{m} \circ (1 \wedge l_M)$$

and therefore $(L_fM, \overline{m})$ is an E-module spectrum. As before, the commutativity of the diagrams for $\overline{m}$ follows easily from the commutativity of the diagrams for m and the construction of $\overline{m}$.

Recall from the first part of the Theorem that if E is a ring spectrum then L_fE is also a ring spectrum. To give L_fM the structure of an L_fE-module we only have to go a step further in the bijections (5.1). We have

$$\begin{aligned}[E \wedge M, L_fM] &\cong [E \wedge L_fM, L_fM] \cong [L_fM, F(E, L_fM)] \\ &\cong [L_fM, F(L_fE, L_fM)] \cong [L_fE \wedge L_fM, L_fM]\end{aligned}$$

and the map $m\colon E \wedge M \longrightarrow L_fM$ also extends to a unique map $\widetilde{m}\colon L_fE \wedge L_fM \longrightarrow L_fM$.

$$\begin{tikzcd} E \wedge M \arrow[r, "m"] \arrow[d, "l_E \wedge l_M"'] & M \arrow[d, "l_M"] \\ L_fE \wedge L_fM \arrow[r, dotted, "\widetilde{m}"'] & L_fM \end{tikzcd} \qquad l_M \circ \mu \simeq \widetilde{m} \circ (l_E \wedge l_M)$$

giving L_fM the structure of an L_fE-module. □

For more general f-localization functors, it is not true that they preserve ring or module structures in general. The following example, in which we can see that the f-localization of a non-connective ring spectrum need not be a ring spectrum, is due to Rudyak [Rud].

Example 5.3.2. For every ring spectrum E we have a homomorphism

$$\begin{aligned}\pi_0(S) \otimes (H\mathbb{Z}/p)_n(E) &\xrightarrow{\eta_* \otimes 1} \pi_0(E) \otimes (H\mathbb{Z}/p)_n(E) \\ &\xrightarrow{h \otimes 1} (H\mathbb{Z}/p)_0(E) \otimes (H\mathbb{Z}/p)_n(E) \xrightarrow{\mu_*} (H\mathbb{Z}/p)_n(E) \quad (5.2)\end{aligned}$$

where η_* is induced by the unit η_E of the ring spectrum E, h is the Hurewicz homomorphism induced by the unit in the ring spectrum $H\mathbb{Z}/p$ and the map μ_* is defined as follows. An element in $(H\mathbb{Z}/p)_0(E) \otimes (H\mathbb{Z}/p)_n(E)$ has the form $\alpha \otimes \beta$, where

$$S \xrightarrow{\alpha} E \wedge H\mathbb{Z}/p \text{ and } \Sigma^n S \xrightarrow{\beta} E \wedge H\mathbb{Z}/p.$$

Recall that if E and F are ring spectra, then the smash product $E \wedge F$ is also a ring spectrum, so applying μ_* to $\alpha \otimes \beta$ is to compose with the product in $E \wedge H\mathbb{Z}/p$, $\mu_*(\alpha \otimes \beta) = \mu_{E \wedge H\mathbb{Z}/p} \circ (\alpha \wedge \beta)$,

$$S \simeq S \wedge \Sigma^n S \xrightarrow{\alpha \wedge \beta} (E \wedge H\mathbb{Z}/p) \wedge (E \wedge H\mathbb{Z}/p) \xrightarrow{\mu_{E \wedge H\mathbb{Z}/p}} E \wedge H\mathbb{Z}/p.$$

The homomorphism (5.2) is an isomorphism because if η_E and $\eta_{H\mathbb{Z}/p}$ are the units of E and $H\mathbb{Z}/p$ respectively, then $\eta_E \wedge \eta_{H\mathbb{Z}/p}$ is the unit of the ring spectrum $E \wedge H\mathbb{Z}/p$. Because of this isomorphism, if E is a ring spectrum with $(H\mathbb{Z}/p)_0(E) = 0$, then $(H\mathbb{Z}/p)_n(E) = 0$ for all n.

Now, given a natural number n and a fixed prime p, let $K(n)$ be the ring spectrum corresponding to n-th Morava K-theory. This is a non-connective spectrum. If we take its nullification $P_{\Sigma S}K(n)$, i.e., f-localization with respect to the map $f\colon \Sigma S \longrightarrow *$, we have that (see [Rudyak]):

$$(H\mathbb{Z}/p)_k(P_{\Sigma S}K(n)) = \begin{cases} 0 & \text{if } n = 0 \\ \neq 0 & \text{otherwise.} \end{cases}$$

So, $P_{\Sigma S}K(n)$ cannot be a ring spectrum, although $K(n)$ is a ring spectrum.

However, under some connectivity conditions, we find that the functor L_f preserves ring structures and module structures.

Theorem 5.3.3. *Let $f\colon A \longrightarrow B$ be a map of spectra. Then:*

- *If E is a connective ring spectrum and L_fE is connective, then the spectrum L_fE has a unique ring structure such that the localization map $l_E\colon E \longrightarrow L_fE$ is a ring map.*

- *If M is an E-module, where E is a connective ring spectrum, then L_fM has a unique E-module structure such that the localization map $l_M\colon M \longrightarrow L_fM$ is an E-module map. Morover, if L_fE is connective, then L_fM also admits a unique L_fE-module structure extending the E-module structure.*

Proof. The proof is exactly the same as in Theorem 5.3.1, but using the fact that, if E is connective, then we have an equivalence

$$F^c(E, F^c(X,Y)) \simeq F^c(E \wedge X, Y)$$

that gives a bijection

$$[E, F^c(X,Y)] \cong [E \wedge X, Y].$$

$\square$

Using these two theorems we find that the localization functor restricts to a localization functor in the following subcategories:

- In the subcategory of ring spectra and ring maps, if the localization functor commutes with suspension.
- In the subcategory of E-module spectra and E-module maps, if the localization functor commutes with suspension.
- In the subcategory of connective ring spectra and ring maps, if L_fE is connective for every connective ring spectra E.
- In the subcategory of E-module spectra and E-module maps, if E is a connective ring spectrum.

To prove this, we only have to check that if $g\colon E \longrightarrow F$ is a morphism (a ring map or an E-module map) in the corresponding subcategory, then $L_fg\colon L_fE \longrightarrow L_fF$ is also a morphism in the subcategory. We only show the argument for the first subcategory, as for the other three it is exactly the same. Suppose that $g\colon E \longrightarrow F$ is a ring map between two ring spectra (E, μ_E, η_E) and (F, μ_F, η_F). We have the following diagram

$$
\begin{tikzcd}
& E \wedge E \arrow[rr, "g\wedge g"] \arrow[dl, "l_E\wedge l_E"'] \arrow[dd, "\mu_E"] & & F \wedge F \arrow[dl, "l_F\wedge l_F"'] \arrow[dd, "\mu_F"] \\
L_fE \wedge L_fE \arrow[rr, "L_fg\wedge L_fg"] \arrow[dd, "\overline{\mu}_E"'] & & L_fF \wedge L_fF \arrow[dd, "\overline{\mu}_F"] & \\
& E \arrow[rr, "g"] \arrow[dl, "l_E"'] & & F \arrow[dl, "l_F"] \\
L_fE \arrow[rr, "L_fg"] & & L_fF &
\end{tikzcd}
$$

In order to prove that L_fg is a ring map, we have to show that the front side of the cube commutes. But the front side commutes because the other five sides do. The left and right sides commute because L_fE and L_fF are ring spectra; the back side commutes because g is a ring map; and the top and bottom sides commute because of the naturality of l.

We can summarize in the following table all the results we have obtained about preservations of ring and module structures under f-localizations.

$$\text{Localization functor}\begin{cases}\text{Commutes with suspension}\begin{cases}\text{Sends ring spectra to ring spectra}\\[1ex] \text{Sends } E\text{-module spectra to } E\text{-module spectra}\end{cases}\\[4ex] \text{Does not commute with suspension}\begin{cases}\text{Sends connective ring spectra to connective ring spectra if } L_fE \text{ is connective for every connective ring spectrum } E\\[1ex] \text{Sends } E\text{-module spectra to } E\text{-module spectra if } E \text{ is connective}\end{cases}\end{cases}$$

5.4 Localizations of Stable GEMs

In this section we will compute the localization of stable R-GEMs in some particular cases. As we already know from Proposition 5.2.3, the $H\mathbb{Z}$-modules are precisely the stable GEMs. Using the results on f-localizations of E-modules of Section 5.3 we have the following.

Theorem 5.4.1. *If E is a stable GEM, then L_fE is also a stable GEM and the localization map $l_E\colon E \longrightarrow L_fE$ is an $H\mathbb{Z}$-module map.*

Proof. A stable GEM is the same as an $H\mathbb{Z}$-module and $H\mathbb{Z}$ is a connective spectrum. Apply now Theorem 5.3.3. □

Remark 5.4.1. This theorem is also true if we replace stable GEM by stable R-GEM for a ring R, with the exception that the localization map need not be an HR-module map.

We have obtained that if a spectrum E is homotopy equivalent to $\vee_{i\in\mathbb{Z}}\Sigma^i HA_i$ where A_i is an R-module for each $i \in \mathbb{Z}$, then $L_fE \simeq \vee_{i\in\mathbb{Z}}\Sigma^i HG_i$ where each G_i is an R-module.

Next, we are going to study the case when the spectrum E is a suspension of an Eilenberg–Mac Lane spectrum, i.e., $E \simeq \Sigma^n HG$ where G is an R-module. By Theorem 5.4.1 we know that $L_f\Sigma^n HG \simeq \vee_{i\in\mathbb{Z}}\Sigma^i HG_i$ with each G_i an R-module, because $\Sigma^n HG$ is a stable R-GEM. In fact, most of the R-modules G_i are zero. Consider the following commutative diagram, where p_i is the projection onto the i-th factor

$$\begin{tikzcd} \Sigma^n HG \arrow[r, "l_{\Sigma^n HG}"] \arrow[dr] & L_f\Sigma^n HG \simeq \vee_{i\in\mathbb{Z}}\Sigma^i HG_i \arrow[d, "p_i"] \\ & \Sigma^i HG_i \end{tikzcd}$$

The spectra $\Sigma^n HG$ and $\vee_{i\in\mathbb{Z}}\Sigma^i HG_i$ are stable R-GEMs. In particular they are stable GEMs or, equivalently, $H\mathbb{Z}$-modules, and the maps $l_{\Sigma^n HG}$ and p_i are $H\mathbb{Z}$-module maps. By Corollary 5.2.4, $[\Sigma^n HG, \Sigma^i HG_i]_{H\mathbb{Z}-\mathrm{mod}} = 0$ unless $i = n$ or $i = n+1$. So, the universal property (2.1) and the fact that $\Sigma^i HG_i$ is f-local (it is a retract of $\vee_{i\in\mathbb{Z}}\Sigma^i HG_i$) tells us that $G_i = 0$ if $i \neq n$ or $i \neq n+1$.

What we get is that the localization of any suspension of an Eilenberg–Mac Lane spectrum has at most two nonzero homotopy groups.

Theorem 5.4.2. *Let $f\colon A \longrightarrow B$ be a map of spectra. Let G be any abelian group and $n \in \mathbb{Z}$. Then $L_f\Sigma^n HG \simeq \Sigma^n HG_1 \vee \Sigma^{n+1}HG_2$ for some groups G_1, G_2.* □

There are some special cases in which the localization of an Eilenberg–Mac Lane spectrum is a single Eilenberg–Mac Lane spectrum. This is the case of localizations of the Eilenberg–Mac Lane spectrum $H\mathbb{Z}$ for ordinary homology.

From Theorem 5.4.2, we know that $L_f H\mathbb{Z}$ has at most two nonzero homotopy groups. Actually, it has at most one homotopy group. Consider the following commutative diagram

$$\begin{tikzcd} H\mathbb{Z} \arrow[r, "l_{H\mathbb{Z}}"] \arrow[dr, "p_2\circ l_{H\mathbb{Z}}"'] & HA \vee \Sigma HB \arrow[d, "p_2"] \\ & \Sigma HB \end{tikzcd}$$

where p_2 is the projection onto the second factor and $l_{H\mathbb{Z}}$ the localization map. Since the maps $l_{H\mathbb{Z}}$ and p_2 are $H\mathbb{Z}$-module maps, so is also the composite $p_2 \circ l_{H\mathbb{Z}}$. By Lemma 5.2.1, $[H\mathbb{Z}, \Sigma HB]_{H\mathbb{Z}_{\mathrm{mod}}} \cong [M\mathbb{Z}, \Sigma HB]$ and $[M\mathbb{Z}, \Sigma HB] = 0$ because $\mathrm{Hom}(\mathbb{Z}, \pi_0(\Sigma HB)) = 0$ and $\mathrm{Ext}(\mathbb{Z}, \pi_1(\Sigma HB)) = 0$. The map p_2 is nullhomotopic as seen from the fact that ΣHB is f-local (because it is a retract of $HA \wedge \Sigma HB$, which is f-local) and the universal property (2.1). Hence, $B = 0$.

We have proved that the localization of $H\mathbb{Z}$ has at most one homotopy group. Considering now the same commutative diagram as before but now projecting onto the first factor

$$\begin{tikzcd} H\mathbb{Z} \arrow[r, "l_{H\mathbb{Z}}"] \arrow[dr, "p_1 \circ l_{H\mathbb{Z}}"'] & HA \vee \Sigma HB \arrow[d, "p_1"] \\ & HA \end{tikzcd}$$

we can obtain information about this homotopy group A. We have a bijection of abelian groups

$$[HA, HA] \times [\Sigma HB, HA] \cong [H\mathbb{Z}, HA].$$

But $[\Sigma HB, HA] = 0$ because

$$\begin{aligned} [\Sigma HB, HA] \cong (H\mathbb{Z})^0(\Sigma HB; A) &\cong \mathrm{Hom}((H\mathbb{Z})_0(\Sigma HB), A) \\ &\cong \mathrm{Hom}(\pi_0(\Sigma HB), A) = 0. \end{aligned}$$

Hence we get

$$\mathrm{Hom}(A, A) \cong [HA, HA] \cong [H\mathbb{Z}, HA] \cong \mathrm{Hom}(\mathbb{Z}, A) \cong A.$$

So we can characterize the homotopy group that appears in f-localizations of $H\mathbb{Z}$ by the property $\mathrm{Hom}(A, A) \cong A$. Therefore, if A is nonzero, then it admits a ring structure.

Definition 5.4.1. A ring A with unit satisfying the property $\mathrm{Hom}(A, A) \cong A$ is called a *rigid ring* (see [CRT]).

Examples of rigid rings are $\mathbb{Z}$, $\mathbb{Q}$, $\hat{\mathbb{Z}}_p$, $\mathbb{Z}/p$. All solid rings in the sense of [BK] are rigid rings. However, there are rigid rings of arbitrarily large cardinality (see [CRT]).

We can summarize the results obtained for f-localizations of $H\mathbb{Z}$ in the following theorem.

Theorem 5.4.3. *Let $f\colon A \longrightarrow B$ be a map of spectra. Then the f-localization of the spectrum $H\mathbb{Z}$ has at most one nonzero homotopy group, i.e., $L_f H\mathbb{Z} \simeq HG$. Moreover, the group G has a rigid ring structure.* $\square$

Remark 5.4.2. The same can be done if we replace $H\mathbb{Z}$ by any suspension $\Sigma^k H\mathbb{Z}$. What we get in this case is that $L_f \Sigma^k H\mathbb{Z} \simeq \Sigma^k HA$. Again the group A has a rigid ring structure.

We give an interesting corollary. A spectrum E is *smashing* if the homological localization L_E with respect to E_* satisfies

$$L_E X \simeq X \wedge L_E S \quad \text{for all spectra } X,$$

where S is the sphere spectrum. For example, the spectrum K of (complex) K-theory and the Johnson–Wilson spectra $E(n)$ are smashing for all n.

Theorem 5.4.4. *If E is smashing, then $(H\mathbb{Z})_n(L_E S) = 0$ if $n \neq 0$, and it is a rigid ring if $n = 0$.*

Proof. We have that

$$(H\mathbb{Z})_n(L_E S) = \pi_n(H\mathbb{Z} \wedge L_E S) \cong \pi_n(L_E H\mathbb{Z}) \cong \pi_n(HA)$$

for some rigid ring A, by □

Appendix A

Localization in simplicial model categories

In this chapter we will prove the existence of a localization functor L_f for a simplicial model category under certain hypothesis. One of these hypothesis is that our category must be cofibrantly generated. This fact allows us to use *Quillen's small object argument* to construct a localization. We begin with the basic definitions of model categories (see [Qui] or [Hov])

A.1 Simplicial Model Categories

Definition A.1.1. A closed category is a category $\mathcal{C}$ together with three classes of maps: cofibrations, fibrations and weak equivalences, that satisfies the properties

(CM.1) $\mathcal{C}$ is closed under all finite limits and colimits

(CM.2) Suppose that the following diagram commutes in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "h"'] & Y \arrow[dl, "f"] \\ Z & \end{tikzcd}$$

If any two of f, g and h are weak equivalences then so is the third.

(CM.3) Given a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "i"] \arrow[d, "f"] & X \arrow[r, "r"] \arrow[d, "g"] & X' \arrow[d, "f"] \\ Y' \arrow[r, "i'"] & Y \arrow[r, "r'"] & Y' \end{tikzcd}$$

in which $r \circ i = \mathrm{id}_{X'}$, $r' \circ i' = \mathrm{id}_{Y'}$ (we say f is a retract of g). If g is a weak equivalence, fibration or cofibration, then so is f.

(CM.4) Suppose that we are given a commutative solid arrow diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ V \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where i is a cofibration and p is a fibration. Then the dotted arrow exists making the diagram commute, if either i or p is also a weak equivalence.

(CM.5) Any map $f\colon X \longrightarrow Y$ may be factored

- $f = p \circ i$ where p is a fibration and i a trivial cofibration.
- $f = q \circ j$ where q is trivial fibration and j a cofibration.

A trivial fibration (cofibration) is a map that is both a fibration (cofibration) and a weak equivalence.

An object X is fibrant (cofibrant) if the map $X \longrightarrow *$ ($* \longrightarrow X$) is a fibration (cofibration) in $\mathcal{C}$.

Definition A.1.2. A map $f\colon X \longrightarrow Y$ in $\mathcal{C}$ has the right lifting property (or RLP) with respect to a class of morphisms S if in every solid diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

with $i \in S$ the dotted arrow exists making the diagram commute. Similarly i has the left lifting property (or LLP) with respect to S if the dotted arrow exists where f is in S.

Cofibrations and fibrations determine each other via these lifting properties.

Lemma A.1.1. *Let $\mathcal{C}$ be a closed model category. Then we have the following:*

- *A map $i\colon A \longrightarrow B$ of $\mathcal{C}$ is a cofibration if and only if it has the LLP with respect to all trivial fibrations.*
- *The map i is a trivial cofibration if and only if it has the LLP with respect to all fibrations.*
- *A map $f\colon X \longrightarrow Y$ of $\mathcal{C}$ is a fibration if and only if it has the RLP with respect to all trivial cofibrations.*
- *The map f is a trivial fibration if and only if it has the RLP with respect to all cofibrations.*

Definition A.1.3. Let $\mathcal{SS}$ denote the category of simplicial sets. A simplicial category is a category $\mathcal{C}$ and a functor

$$\mathrm{HOM}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{SS}$$

satisfying

(SC.1) $\mathrm{HOM}_{\mathcal{C}}(A,B)_0 = \mathcal{C}(A,B)$.

(SC.2) $\mathrm{HOM}_{\mathcal{C}}(A,-)$ has a left adjoint $A \otimes -\colon \mathcal{SS} \longrightarrow \mathcal{C}$ and there is an isomorphism

$$A \otimes (K \times L) \cong (A \otimes K) \otimes L \qquad A \in \mathcal{C} \text{ and } K, L \in \mathcal{SS}$$

natural in A, K and L.

(SC.3) $\mathrm{HOM}_{\mathcal{C}}(-,B)$ has a left adjoint $Hom_{\mathcal{C}}(-,B)\colon \mathcal{SS} \longrightarrow \mathcal{C}^{op}$

Definition A.1.4. A category $\mathcal{C}$ is a simplicial model category if it is a closed model category, a simplicial category and satisfies the ‘simplicial model axiom’

(SM.7) If $j\colon A \longrightarrow B$ is a cofibration and $q\colon X \longrightarrow Y$ is a fibration, then

$$\mathrm{HOM}_{\mathcal{C}}(B,X) \longrightarrow \mathrm{HOM}_{\mathcal{C}}(A,X) \times_{\mathrm{HOM}_{\mathcal{C}}(A,Y)} \mathrm{HOM}_{\mathcal{C}}(B,Y)$$

is a fibration of simplicial sets, which is trivial if j or q is trivial.

Definition A.1.5. A proper model category $\mathcal{C}$ is a model category satisfying

(P.1) Given a pullback diagram

$$\begin{tikzcd} X \arrow[r, "g'"] \arrow[d] & Y \arrow[d, "p"] \\ Z \arrow[r, "g"'] & W \end{tikzcd}$$

of $\mathcal{C}$ with p a fibration, if g is a weak equivalence the so is g'.

(P.2) Given a pushout diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d] \\ Z \arrow[r, "f'"'] & W \end{tikzcd}$$

with i a cofibration, if f is a weak equivalence then so is f'.

A.2 Small Objects in Model Categories

The main tool to prove the existence of localization functors is the *small object argument* (see [Qui], [Hov]), which tells us how to construct functorial factorizations in model categories. We will need some results on infinite compositions.

Let $\mathcal{C}$ be a model category which has all limits and colimits. Fix an infinite cardinal number λ and let $Seq(\lambda)$ denote the smallest ordinal number of cardinality λ. Recall that an ordinal is the well-ordered set of all smaller ordinals. We can think of an ordinal as a category where there is a unique map from α to β if and only if $\alpha \leq \beta$.

A λ-diagram is a functor $E\colon Seq(\lambda) \longrightarrow \mathcal{C}$. E is a λ-diagram of cofibrations if each of the morphisms $E_t \longrightarrow E_s$ is a cofibration in $\mathcal{C}$.

Definition A.2.1. Let γ be an infinite cardinal. An object $X \in \mathcal{C}$ is γ-small if for all λ-diagrams of cofibrations E in $\mathcal{C}$ with $\lambda \geq \gamma$ the natural map of sets

$$\operatorname*{colim}_{\beta<\lambda} \mathcal{C}(X, E_\beta) \longrightarrow \mathcal{C}(X, \operatorname*{colim}_{\beta<\lambda} E_\beta)$$

is an isomorphism.

An object X is small is there exists γ such that X is γ-small

For an object to X to be small means that every map from X to $\operatorname{colim}_{\beta<\lambda} E_\beta$ factors through a E_α for some α.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\exists"'] & \operatorname{colim}_{\beta<\lambda} E_\beta \\ E_\alpha \arrow[ur, "p_\alpha"'] & \end{tikzcd}$$

A.3 Construction of Localization Functors

Let $\mathcal{C}$ be a simplicial proper model category. Let $f\colon A \longrightarrow B$ be a cofibration in $\mathcal{C}$ between cofibrant objects. First of all we have to assume some hypothesis on the category $\mathcal{C}$:

- $\mathcal{C}$ is cocomplete.
- A, B, $\Delta[n] \otimes A \coprod_{\partial\Delta[n]\otimes A} \partial\Delta[n] \otimes B$, $\Delta[n] \otimes B$ must be small objects.
- $\mathcal{C}$ is *cofibrantly generated* i.e. there is a set K of trivial cofibrations in $\mathcal{C}$ such that a map is a fibration if it has the right lifting property (RLP) for all members of K. Morover all domains and codomains of members of K must be small.

Now we will define what is a localization funtor in a simplicial model category.

Definition A.3.1 (f-local objects and f-equivalences). Let X be an object in $\mathcal{C}$, and let $f\colon A \longrightarrow B$ be a cofibration between cofibrant objects. Then

- X is f-local is X is fibrant and the map $\mathrm{HOM}(B, X) \longrightarrow \mathrm{HOM}(A, X)$ induced by f is a weak equivalence of simplicial sets.

- $g\colon X \longrightarrow Y$ is an f-equivalence if there is a cofibrant approximation $\widetilde{g}\colon \widetilde{X} \longrightarrow \widetilde{Y}$ to g such that $\widetilde{g}^*\colon \mathrm{HOM}(\widetilde{Y}, Z) \longrightarrow \mathrm{HOM}(\widetilde{X}, Z)$ is a weak equivalence of simplicial sets for all f-local objects Z.

Definition A.3.2 (f-localization). An f-localization is a morphism $X \longrightarrow \widehat{X}$ that is an f-equivalence and where $\widehat{X}$ is f-local.

Our main goal is to prove that f-localization exists in any simplicial model category, for every cofibration map $f\colon A \longrightarrow B$ between cofibrant objects.

We want to construct an f-local object $\widehat{X}$ together with a natural f-equivalence $X \longrightarrow \widehat{X}$. The object $\widehat{X}$ must be f-local so first of all it must be a fibrant object, so the map $\widehat{X} \longrightarrow *$ must have the RLP with respect to all members of K.

If $\widehat{X}$ is fibrant, then $f^*\colon \mathrm{HOM}(B, \widehat{X}) \longrightarrow \mathrm{HOM}(A, \widehat{X})$ is a fibration (see for example [GJ99], p. 89). Thus, if $\widehat{X}$ is fibrant, $\widehat{X}$ is f-local if and only if f^* is a trivial fibration of simplicial sets, i.e. has the RLP with respect to the inclusions $\partial\Delta[n] \longrightarrow \Delta[n]$, $n \geq 0$.

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r] \arrow[d] & \mathrm{HOM}(B, \widehat{X}) \arrow[d] \\
\Delta[n] \arrow[r] \arrow[ur, dotted, "\exists"] & \mathrm{HOM}(A, \widehat{X})
\end{tikzcd}$$

The adjunction $\mathcal{SS}(X, \mathrm{HOM}(Y, Z)) \cong \mathcal{C}(X \otimes Y, Z)$, for all $X \in \mathcal{SS}$ and for all $Y, Z \in \mathcal{C}$ implies that this is true if and only if the dotted arrow exists in every diagram of the form

$$\begin{tikzcd}
\Delta[n] \otimes A \coprod_{\partial\Delta[n]\otimes A} \partial\Delta[n] \otimes B \arrow[r] \arrow[d] & \widehat{X} \arrow[d] \\
\Delta[n] \otimes B \arrow[r] \arrow[ur, dotted, "\exists"] & *
\end{tikzcd}$$

Therefore,

Proposition A.3.1. *An object $\widehat{X} \in \mathcal{C}$ is f-local if and only if the map $\widehat{X} \longrightarrow *$ has the right lifting property with respect to the following families of maps:*

- *Members of K;*
- $\Delta[n] \otimes A \coprod_{\partial\Delta[n]\otimes A} \partial\Delta[n] \otimes B \longrightarrow \Delta[n] \otimes B$. □

We call $Hor(f) = \{\Delta[n] \otimes A \coprod_{\partial\Delta[n]\otimes A} \partial\Delta[n] \otimes B \longrightarrow \Delta[n] \otimes B \mid n \geq 0\}$ the set of horns on f and $\overline{Hor}(f) = Hor(f) \cup K$ the set of augmented horns on f. Let

$$Hor_n(A, B) = \{\Delta[n] \otimes A \coprod_{\partial\Delta[n]\otimes A} \partial\Delta[n] \otimes B\}$$

$$\overline{Hor}_n(A, B) = Hor_n(A, B) \cup \{\text{Domains of } K\}$$

Let λ be an infinite cardinal such that A, B, $\Delta[n] \otimes A \coprod_{\partial\Delta[n]\otimes A} \partial\Delta[n] \otimes B$, $\Delta[n] \otimes B$ and all domains and codomains of morphisms in K are λ-small. We want to construct a λ-sequence of cofibrations

$$X = E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_\beta \longrightarrow \cdots \quad (\beta < \lambda)$$

and let $\widehat{X} = \operatorname{colim}_{\beta<\lambda} E_\beta$. Set $E_0 = X$, take all the morphisms

$$\overline{Hor}_n(A, B) \longrightarrow X$$

and form the pushout of the following diagram,

$$\begin{tikzcd}
\coprod \overline{Hor}_n(A, B) \arrow[r] \arrow[d] & E_0 = X \arrow[d] \\
(\coprod \Delta[n] \otimes B) \coprod (\text{Codomains of } K) \arrow[r] & E_1
\end{tikzcd}$$

In this way we obtain E_1. Now we proceed inductively; to get E_n, take E_{n-1} and form the pushout

$$\begin{tikzcd}
\coprod \overline{Hor}_n(A, B) \arrow[r] \arrow[d] & E_{n-1} \arrow[d] \\
(\coprod \Delta[n] \otimes B) \coprod (\text{Codomains of } K) \arrow[r] & E_n
\end{tikzcd}$$

Let $\widehat{X} = \operatorname{colim}_{\beta<\lambda} E_\beta$.

Lemma A.3.2. *The object $\widehat{X}$ is f-local.*

Proof. For every map $h\colon \overline{Hor}_n(A,B) \longrightarrow \widehat{X}$ there is an ordinal α such that h factors trough $E_\alpha \longrightarrow \widehat{X}$ because elements in $\overline{Hor}_n(A,B)$ are small. Thus, if the map $C \longrightarrow D$ is in $\overline{Hor}(f)$, then the dotted arrow exists in every solid arrow diagram of the form

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \widehat{X} \\ D \arrow[ur, dotted] & \end{tikzcd}$$

and so the map $\widehat{X} \longrightarrow *$ has the RLP with respect to every element in $\overline{Hor}_n(f)$. ☐

It only remains to proof that $X \longrightarrow \widehat{X}$ is a f-equivalence.

Lemma A.3.3. *Every map in $\overline{Hor}_n(f)$ is a cofibration.*

Proof. For $h \in K$ is true by definition of the set K. For all $n \geq 0$ the map

$$\Delta[n] \otimes A \coprod_{\partial\Delta[n]\otimes A} \partial\Delta[n] \otimes B \longrightarrow \Delta[n] \otimes B$$

is a cofibration by **(SM.7)** and the adjunction $\mathcal{SS}(X, \mathrm{HOM}(Y,Z)) \cong \mathcal{C}(X \otimes Y, Z)$. ☐

Lemma A.3.4. *Cofibrations are closed under both pushouts and transfinite compositions.*

Proof. See [Hirsch] Proposition 12.2.19 ☐

Corollary A.3.5. *The map $X \longrightarrow \widehat{X}$ is a cofibration.*

Proof. The map $X \longrightarrow \widehat{X}$ is constructed as a transfinite composition of pushouts of elements in $\overline{Hor}(f)$, so the result follows from lemma A.3.3 and lemma A.3.4. ☐

Now we have proved $X \longrightarrow \widehat{X}$ is a cofibration we must show that $\mathrm{HOM}(\widehat{X}, W) \simeq \mathrm{HOM}(X, W)$ for all f-local objects W.

Lemma A.3.6. *Every map in $\overline{Hor(f)}$ is an f-equivalence.*

Proof. Let g be a map in $\overline{Hor(f)}$.

- If $g \in K$, then $g\colon C \longrightarrow D$ is a trivial cofibration. Now, g induces a trivial fibration of simplicial sets $\mathrm{HOM}(D, Z) \longrightarrow \mathrm{HOM}(C, Z)$ if Z is fibrant, and this is true for all Z that are f-local.

- If g is a map of the form

$$\Delta[n] \otimes A \coprod_{\partial\Delta[n]\otimes A} \partial\Delta[n] \otimes B \longrightarrow \Delta[n] \otimes B$$

then we have to check that there exists a map

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & \mathrm{HOM}(\Delta[n] \otimes B, Z) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & \mathrm{HOM}(\Delta[n] \otimes A \coprod_{\partial\Delta[n]\otimes A} \partial\Delta[n] \otimes B, Z) \end{tikzcd}$$

rendering commutativity of the diagram. We can prove the existence of this map using the adjunction $\mathcal{SS}(X, \mathrm{HOM}(Y, Z)) \cong \mathcal{C}(X \otimes Y, Z)$ and the fact that if there exists a map rendering commutativity of the diagram

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & \mathrm{HOM}(B, X) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dotted, "\exists"] & \mathrm{HOM}(A, X), \end{tikzcd}$$

then the same happens in the diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d] & \mathrm{HOM}(B, X) \arrow[d] \\ K \arrow[r] \arrow[ur, dotted, "\exists"] & \mathrm{HOM}(A, X), \end{tikzcd}$$

for any simplicial pair (K, L). For full details see [Hirsch] Proposition 10.3.3 and Proposition 10.3.10.

□

Lemma A.3.7. *Every map $E_\beta \longrightarrow E_{\beta+1}$ is an f-equivalence for all β.*

Proof. Each map $E_\beta \longrightarrow E_{\beta+1}$ is a pushout of both f-local equivalences and cofibration maps. The result follows from [Hirsch] Proposition 1.2.16. $\square$

Lemma A.3.8. *If we have a diagram $E_0 \longrightarrow E_1 \longrightarrow \ldots \longrightarrow E_\beta \longrightarrow \ldots$ in which each map $E_\beta \longrightarrow E_{\beta+1}$ is a f-equivalence then the natural map $E_0 \longrightarrow \operatorname{colim}_\beta E_\beta$ is also an f-equivalence.*

Proof. If W is f-local, then W is fibrant so we have trivial fibrations $\mathrm{HOM}(E_\beta, W) \longleftarrow \mathrm{HOM}(E_{\beta+1}, W)$ because $E_\beta \longrightarrow E_{\beta+1}$ is a cofibration (see [GJ] p.89) and a f-equivalence. We have

$$\mathrm{HOM}(E_0, W) \xleftarrow{\simeq} \mathrm{HOM}(E_1, W) \xleftarrow{\simeq} \cdots \xleftarrow{\simeq} \mathrm{HOM}(E_\beta, W) \xleftarrow{\simeq} \cdots$$

so

$$\mathrm{HOM}(E_0, W) \simeq \lim_\beta \mathrm{HOM}(E_\beta, W) \simeq \mathrm{HOM}(\operatorname*{colim}_\beta E_\beta, W)$$

$\square$

With all these ingredients we can now state the existence theorem for f-localizations.

Theorem A.3.9. *Let $\mathcal{C}$ be a proper simplicial model category, then for every cofibration map $f\colon A \longrightarrow B$ between cofibrant objects and every object $X \in \mathcal{C}$, there exists a f-localization of X.*

Proof. $\widehat{X}$ is f-local by lemma A.3.2 and the map $X \longrightarrow \widehat{X}$ is a f-equivalence by lemma A.3.8, so the map $X \longrightarrow \widehat{X}$ is a f-localization of X. $\square$

Remark A.3.1. This construction is functorial, so f-localization is a functor.

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