Cellularization of structures in triangulated categories

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Precedents

- Cellularization functors were introduced by Farjoun in 1996 in the category of topological spaces.
- Given A and X two pointed topological spaces, $Cell_A X$ contains the information on X that can be built up from A.
- X is called *A-cellular* if $Cell_A X \simeq X$ and it is the smallest class that contains A and it is closed under weak equivalences and homotopy colimits.
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- Examples: n-connective covers, universal covers.
- Cellularization for groups and modules has been studied by Farjoun-Göbel-Segev-Shelah and Rodríguez-Strüngmann .

Objectives

- Describe the formal properties of cellularization functors in triangulated categories.
- Study the algebraic structures preserved by these functors.

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Cellular and null objects

Let $(\mathcal{T}, \Sigma, [-,-])$ be a triangulated category with arbitrary coproducts and a set of generators.

Definition

Let A be any object of $\mathcal{T}$.

i) A map $f\colon X \longrightarrow Y$ in $\mathcal{T}$ is an *A-cellular equivalence* if the induced map

$$[\Sigma^k A, X] \xrightarrow{g_*} [\Sigma^k A, Y]$$

is an isomorphism of abelian groups for all $k \geq 0$.

ii) An object Z of $\mathcal{T}$ is *A-cellular* if the induced map

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is an isomorphism for every A-cellular equivalence $f\colon X \longrightarrow Y$ and for all $k \geq 0$.

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Cellular and null objects

Definition

Let A be any object of $\mathcal{T}$.

i) An object X is *A-null* if $[\Sigma^k A, X] = 0$ for every $k \geq 0$.

ii) A map $g\colon X \longrightarrow Y$ is an *A-null equivalence* if the induced map

$$[\Sigma^k Y, Z] \cong [\Sigma^k X, Z]$$

is an isomorphism of abelian groups for $k \geq 0$.

- An *A-cellularization functor* is a colocalization functor $(Cell_A, c)$ such that for every object X of $\mathcal{T}$, the map $c_X\colon Cell_A X \longrightarrow X$ is an A-cellular equivalence and $Cell_A X$ is A-cellular.
- An *A-nullification functor* is a localization functor (P_A, l) such that for every object X of $\mathcal{T}$, the map $l_X\colon X \longrightarrow P_A X$ is an A-null equivalence and $P_A X$ is A-null.

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Cellular and null objects

- We say that P_A or $Cell_A$ are *exact* if they are triangulated functors.

Existence

- Assume that there is a stable model category $\mathcal{M}$ such that $\mathcal{T} = Ho(\mathcal{M})$. Cellularization and nullfication functors always exist if $\mathcal{M}$ is a proper combinatorial model category.
- Examples to keep in mind: Spectra, $\mathcal{D}(R)$, E-local spectra, $\mathcal{D}(shv/X)$,...

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Cellular and null objects

Closure properties

Let $X \longrightarrow Y \longrightarrow Z$ be an exact triangle in $\mathcal{T}$

i) If Y and Z are A-null then X is A-null.

ii) If X and Z are A-null then Y is A-null.

iii) If X and Y are A-cellular then Z is A-cellular.

iv) If X and Z are A-cellular then Y is *not* A-cellular in general.

v) The class of A-null objects and the class of A-cellular equivalences are closed under desuspensions.

vi) The class of A-cellular objects and the class of A-null equivalences are closed under suspensions.

vii) If P_A and $Cell_A$ are exact the above classes are closed under suspensions and desuspensions.

Colocalization functors satisfying the analog of condition ii) are called *quasiexact*.

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Exact triangles

There are natural maps

$$Cell_A X \longrightarrow X \longrightarrow P_A X.$$

This is *not* an exact triangle in general.

Theorem

Let A and X be two objects of $\mathcal{T}$.

i) *There is an exact triangle $Cell_A X \longrightarrow X \longrightarrow P_A X$ if and only if the morphism of abelian groups $[\Sigma^{-1}A, Cell_A X] \longrightarrow [\Sigma^{-1}A, X]$ is injective (e.g. if $[\Sigma^{-1}A, Cell_A X] = 0$).*

ii) *There is an exact triangle $Cell_A X \longrightarrow X \longrightarrow P_{\Sigma A} X$ if and only if $[A, X] \longrightarrow [A, P_{\Sigma A} X]$ is the zero map (e.g. if $[A, X] = 0$).*

iii) *If $Cell_A$ or P_A are exact, then $Cell_A X \longrightarrow X \longrightarrow P_A X$ is an exact triangle.*

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Example

Not every nullification and cellularization functor fiting into an exact triangle are exact.

If $\mathcal{T}$ is the stable homotopy category of spectra and S is the sphere spectrum, then we have an exact triangle

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Colocalizations associated to nullifications

Let $F_A X$ be the fiber of the map $X \longrightarrow P_A X$

$$F_A X \longrightarrow X \longrightarrow P_A X$$

The universal property of P_A and the fact that P_A is quasiexact make F_A a colocalization functor (augmented and idempotent).

Moreover

- F_A is quasiexact
- F_A-colocal objects are closed under suspensions
- $[F_A X, P_A Y] = 0$ for every X and Y in $\mathcal{T}$.

Under Vopěnka's principle $F_A = Cell_E$ for some E [Chorny, 2008]. A construction of E in pointed spaces is possible not relying on Vopěnka's principle [Chacholski-Parent-Stanley, 2004].

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Colocalizations associated to nullifications

Let F_AX be the fiber of the map $X \longrightarrow P_AX$

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t-structures

Definition

A *t-structure* on $\mathcal{T}$ is a pair of full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ such that, denoting $\mathcal{T}^{\leq n} = \Sigma^{-n}\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq n} = \Sigma^{-n}\mathcal{T}^{\geq 0}$, the following hold:

i) For every object X of $\mathcal{T}^{\leq 0}$ and every object Y of $\mathcal{T}^{\geq 1}$, $[X, Y] = 0$.
ii) $\mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 1} \subset \mathcal{T}^{\geq 0}$.
iii) For every object X of $\mathcal{T}$, there is an exact triangle

$$U \longrightarrow X \longrightarrow V,$$

where U is an object of $\mathcal{T}^{\leq 0}$ and V is an object of $\mathcal{T}^{\geq 1}$.

The *core* of the t-structure is the full subcategory $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$. The core is always an abelian subcategory of $\mathcal{T}$.

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For any object A in $\mathcal{T}$ the full subcategory of ΣA-null objects and the full subcategory of F_A-colocal objects define a t-structure on $\mathcal{T}$.

- If $Cell_A$ and P_A fit into an exact triangle, then the t-structure is given by the A-cellular objects and the ΣA-null objects
- If $Cell_A$ and P_A are exact, then the associated t-structure is trivial.

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Example

Let $\mathcal{T}$ be a monogenic stable homotopy category with unit S, such that $[\Sigma^k S, S] = 0$ for every $k < 0$. Let R denote the ring $[S, S]$. Then the functors $Cell_{\Sigma^k S}$ and $P_{\Sigma^k S}$ are the k-th connective cover functor and the k-th Postnikov section functor respectively:

$$[\Sigma^n S, Cell_{\Sigma^k S} X] = \begin{cases} 0 & \text{if } n < k \\ [\Sigma^n S, X] & \text{if } n \geq k \end{cases}$$

$$[\Sigma^n S, P_{\Sigma^k S} X] = \begin{cases} 0 & \text{if } n \geq k \\ [\Sigma^n S, X] & \text{if } n < k \end{cases}$$

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The core of the associated t-structure is the full subcategory of $\mathcal{T}$ with objects X such that such that $[\Sigma^n S, X] = 0$ if $n \neq k$ and it is equivalent to the category of R-modules. The objects in the core are called *Eilenberg-Mac Lane objects*.

Note that $Cell_{\Sigma^k S}$ is not an exact functor. For example, if $[\Sigma^{k-1} S, X] \neq 0$, then $Cell_{\Sigma^k S}\Sigma X \neq \Sigma Cell_{\Sigma^k S} X$.

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Cellularization of modules and rings

Let $\mathcal{T}$ be a monoidal triangulated category with tensor product $\otimes$, unit S and internal hom $F(-,-)$, such that

- $\mathcal{T}$ is monogenic.
- $\mathcal{T}$ is connective, i.e., $[\Sigma^k S, S] = 0$ for $k < 0$.

An object X is called *connective* if $Cell_S X \simeq X$ and if X is connective, then

$$[X, Cell_S F(Y,Z)] \cong [X, F(Y,Z)] \cong [X \otimes Y, Z].$$

A ring R in $\mathcal{T}$ is a monoid object and an R-module in $\mathcal{T}$ is a monoid over the monoid R.

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Let $\mathcal{T}$ be a monoidal triangulated category with tensor product $\otimes$, unit S and internal hom $F(-,-)$, such that

- $\mathcal{T}$ is monogenic.
- $\mathcal{T}$ is connective, i.e., $[\Sigma^k S, S] = 0$ for $k < 0$.

An object X is called *connective* if $Cell_S X \simeq X$ and if X is connective, then

$$[X, Cell_S F(Y,Z)] \cong [X, F(Y,Z)] \cong [X \otimes Y, Z].$$

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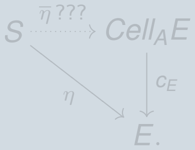
Theorem

If E is a connective ring object and M is an E-module, then for any object A, the object $Cell_A M$ has an E-module structure such that the cellularization map $Cell_A M \longrightarrow M$ is a map of E-modules. If $Cell_A$ is exact, we can avoid the connectivity condition.

The case for rings is more involved. If R is a ring,then $Cell_A R$ will not be a ring in general, *even if $Cell_A$ is exact!*



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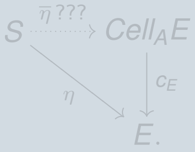

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$$\begin{array}{ccc} S & \overset{\overline{\eta}\,???}{\dashrightarrow} & Cell_A E \\ & {\scriptstyle \eta} \searrow & \downarrow {\scriptstyle c_E} \\ & & E. \end{array}$$

Cellularization of modules and rings

Let C be the cofiber of $Cell_A E \longrightarrow E$

$$Cell_A E \longrightarrow E \longrightarrow C$$

Theorem

Let E be a ring object. Assume that either one of the following holds:

i) *$Cell_A$ commutes with suspension, the morphism $\pi_1(E) \twoheadrightarrow \pi_1(C)$ is surjective and the morphism $\pi_0(C) \rightarrowtail \pi_{-1}(Cell_A E)$ is in jective or*

ii) *$Cell_A E$ is connective, $Cell_A$ is of the form F_B for some B, the morphism $\pi_1(E) \twoheadrightarrow \pi_1(P_B E)$ is surjective and $\pi_0(P_B E) = 0$.*

Then $Cell_A E$ has a unique ring structure such that the cellularization map is a map of rings.

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Cellularization of modules and rings

Example

Let $A = S$, then $Cell_A E$ is the connective cover of E. There is an exact triangle

$$Cell_S E \longrightarrow E \longrightarrow P_S E$$

where P_S is the Postnikov section functor, i.e., it kills all the homotopy groups in dimensions bigger or equal to zero. So $\pi_1 P_S E = \pi_0 P_S E = 0$ and by part ii) of the previous theorem we have that if E is a ring object, then so is its connective cover $Cell_S E$.

Some computations

How to compute $Cell_A\Sigma^k HG$ for any abelian group G.

Theorem

Let G be any abelian group, $n \in \mathbb{Z}$ and A be any object in $\mathcal{T}$. Then

$$Cell_A\Sigma^n HG \simeq \Sigma^{n-1} HB \vee \Sigma^n HC$$

for some abelian groups B and C. Moreover

i) $Hom(B, B) \oplus Ext(B, C) \cong Ext(B, G)$.

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For every object A in $\mathcal{T}$ and any interger m, we have that

$$Cell_A \Sigma^m H\mathbb{Z}/p^n \simeq \Sigma^m H\mathbb{Z}/p^j,$$

where $1 \leq j \leq n$.

If $A = \Sigma^m H\mathbb{Z}/p^k$, then

$$Cell_A \Sigma^m H\mathbb{Z}/p^n \simeq \begin{cases} \Sigma^m H\mathbb{Z}/p^k \text{ if } n \geq k \\ \Sigma^m H\mathbb{Z}/p^n \text{ if } n < k. \end{cases}$$

This shows that $Cell_{H\mathbb{Z}/p}$ is not quasiexact, since $H\mathbb{Z}/p$ is A-cellular but $H\mathbb{Z}/p^2$ is not.

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Let $\mathcal{T}$ be the homotopy category of spectra. Let E be any spectrum and let L_E be homological localization with respect to E. Bousfield proved that there is another spectrum A, such that $L_E X \simeq P_A X$ for every X. Since L_E commutes with suspension there is an exact triangle

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where $Cell_A$ is the *E-acyclization* functor (in Bousfield languaje).

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The cellularization $Cell_A H\mathbb{Z}$ is either zero or one of the following three possibilities

$$H\mathbb{Z}, \qquad \Sigma^{-1} H(\oplus_{p\in P}\mathbb{Z}/p^\infty), \qquad \Sigma^{-1} H((\prod_{p\in P}\widehat{\mathbb{Z}}_p)/\mathbb{Z}).$$

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