A model structure for operads in symmetric spectra

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SECA V

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Outline of the talk

Motivation

- Operads in a monoidal model category $\mathcal{E}$ carry a Quillen model structure under some conditions on $\mathcal{E}$ [Berger-Moerdijk, 2007].

- Use the "transfer principle"

$$F : Coll(\mathcal{E}) \rightleftarrows Oper(\mathcal{E}) : U.$$

- Conditions on $\mathcal{E}$:

  (i) Cofibrant unit.
  (ii) Symmetric monoidal fibrant replacement functor.
  (iii) Extra conditions (coalgebra interval,...)

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- Topological spaces, simplicial sets.
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- For a fixed set of colours C, construct a coloured operads whose algebras are C-coloured operads.
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Coloured operads

- Let $\mathcal{E}$ be a cocomplete closed symmetric monoidal category. Let C be a set, whose elements will be called **colours**. A C-**coloured collection** is a set P of objects $P(c_1, \ldots, c_n; c)$ in $\mathcal{E}$ for every $n \geq 0$ and each tuple $(c_1, \ldots, c_n; c)$ of colours, together with maps

$$\sigma^* \colon P(c_1, \ldots, c_n; c) \longrightarrow P(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; c)$$

  for all permutations $\sigma \in \Sigma_n$, yielding together a right action.
- A C-**coloured operad** is a C-coloured collection P equipped with unit maps $I \to P(c; c)$ and *composition product* maps

$$P(c_1, \ldots, c_n; c) \otimes P(a_{1,1}, \ldots, a_{1,k_1}; c_1) \otimes \cdots \otimes P(a_{n,1}, \ldots, a_{n,k_n}; c_n)$$
$$\longrightarrow P(a_{1,1}, \ldots, a_{1,k_1}, a_{2,1}, \ldots, a_{2,k_2}, \ldots, a_{n,1}, \ldots, a_{n,k_n}; c)$$

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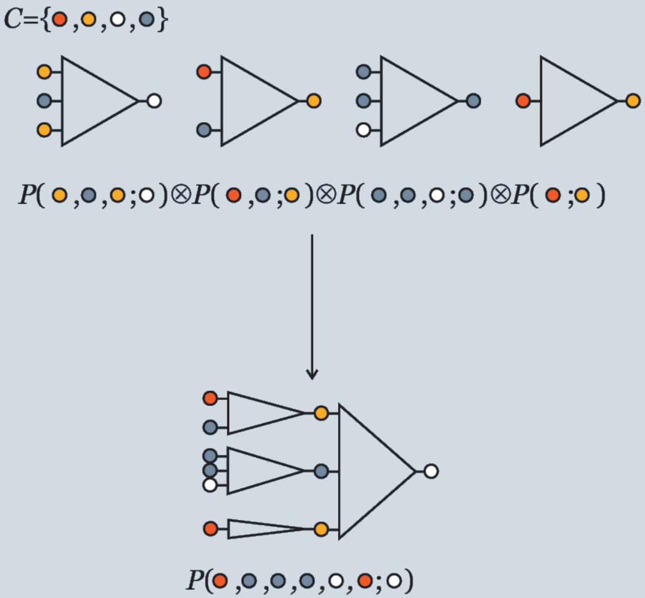

Algebras over coloured operads

- If P is a C-coloured operad, a P**-algebra** is an object $\mathbf{X} = (X(c))_{c\in C}$ in $\mathcal{E}^C$ together with a morphism of C-coloured operads

$$P \longrightarrow \mathrm{End}(\mathbf{X}),$$

where the C-coloured operad $\mathrm{End}(\mathbf{X})$ is defined as

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The coloured operad S^C

Let C be a set of colours. We define a coloured operad S^C in *Sets* whose algebras are C-coloured operads in *Sets*.

$$col(S^C) = \{(c_1, \ldots, c_n; c) \,|\, c_i, c \in C,\ n \geq 0\}.$$

We will use the following notation, $\overline{c}_i = (c_{i,1}, \ldots, c_{i,k_i}; c_i)$ and $\overline{a} = (a_1, \ldots, a_m; a)$. The elements of $S^C(\overline{c}_1, \ldots, \overline{c}_n; \overline{a})$ are equivalence classes of triples (T, σ, τ) where:

- T is a planar rooted C-coloured tree with m input edges coloured by $a_1, \ldots, a_m$, a root edge coloured by a and n vertices.
- σ is a bijection $\sigma\colon \{1, \ldots, n\} \longrightarrow V(T)$ with the property that $\sigma(i)$ has k_i input edges coloured from left to right by $c_{i,1}, \ldots, c_{i,k_i}$ and one output edge coloured by c_i.
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Two such triples (T, σ, τ), $(T'\sigma', \tau')$ are equivalent if and only if there is a planar isomorphism $\varphi\colon T \longrightarrow T'$ such that $\varphi \circ \sigma = \sigma'$ and $\varphi \circ \tau = \tau'$.

Example

If $C = \{a, b, c\}$, then an element (T, σ, τ) of

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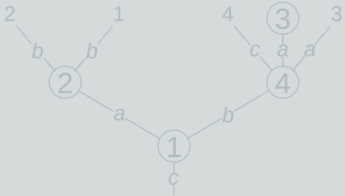

- Any element in α in Σ_n induces a map

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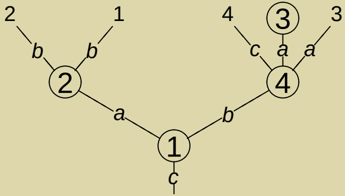

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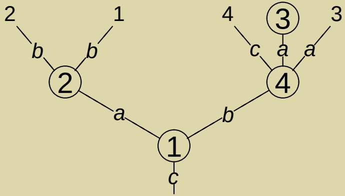

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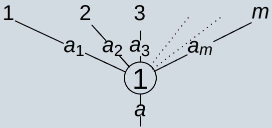

- The composition product on S^C is defined as follows. Given an element (T,σ,τ) of $S^C(\overline{c}_1,\ldots,\overline{c}_n;\overline{a})$ and n elements $(T_1,\sigma_1,\tau_1),\ldots,(T_n,\sigma_n,\tau_n)$ of

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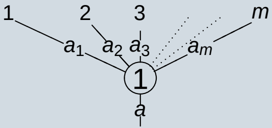

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(i) T' is obtained by replacing the vertex $\sigma(i)$ of T by the tree T_i for every i. This is done by identifying the input edges of $\sigma(i)$ in T with the input edges T_i via the bijection τ_i. The $c_{i,j}$-coloured input edge of $\sigma(i)$ is matched with the $c_{i,j}$-coloured input edge $\tau_i(j)$ of T_i. (Note that the colours of the input edges and the output of $\sigma(i)$ coincide with the colours of the input edges and the root of T_i.)

(ii) The vertices of T' are numbered following the order, i.e., first number the subtree T_1 in T' ordered by σ_1, then T_2 ordered by σ_2 and so on.

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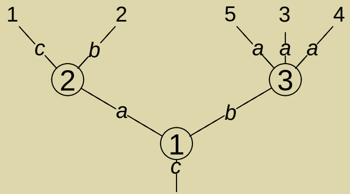
represented by the tree



Example (cont.)

and T_1, T_2 and T_3 be elements of

$$S^C((a,b;c),(c;c);(a,b;c)),\ S^C((b,b;a),(c;b);(c,b;a))$$

$$\text{and } S^C((a,a;c),(a,c;b);(a,a,a;b))$$

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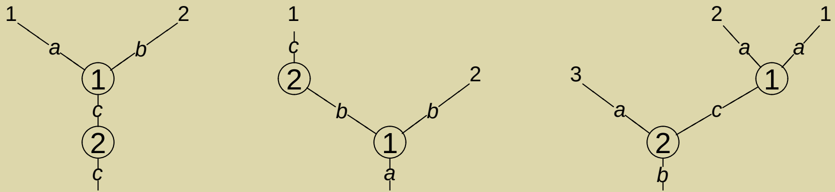

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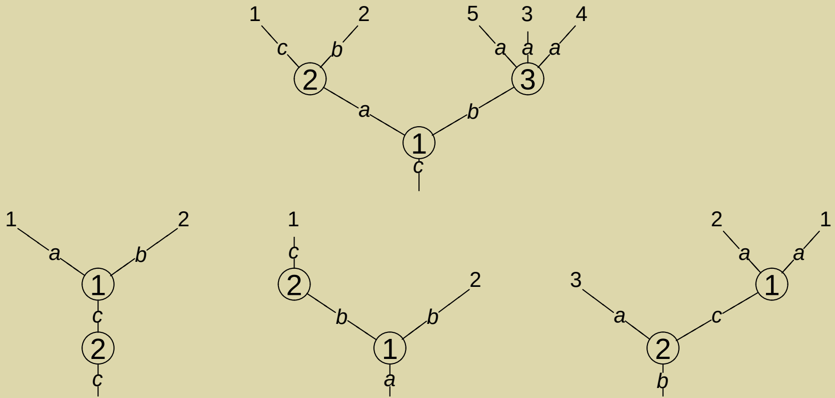
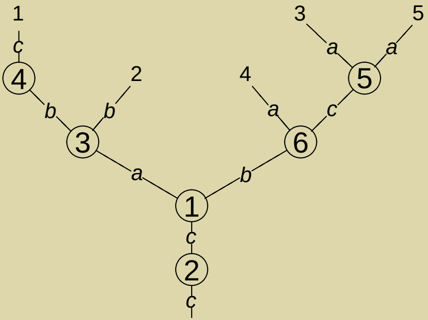


Example (cont.)

By the composition product, we get an element in

$$S^C((a,b;c),(c;c),(b,b;a),(c;b),(a,c;b),(a,a;c);(c,b,a,a,a;c))$$

that will be represented by the following tree



- The above composition product endows the collection S^C with a coloured operad structure. An algebra over S^C is a C-coloured operads in *Sets* and conversely.
- The strong symmetric monoidal functor $(-)_{\mathcal{E}}\colon Sets \longrightarrow \mathcal{E}$ defined as $X_{\mathcal{E}} = \coprod_{x\in X} I$ sends coloured operads to coloured operads. Hence $S^C_{\mathcal{E}}$ is a coloured operad in $\mathcal{E}$ whose algebras are C-coloured operads in $\mathcal{E}$.
- More generally, if $\mathcal{E}$ is a closed symmetric monoidal category enriched over a closed symmetric monoidal category $\mathcal{V}$, then coloured operads in $\mathcal{V}$ act on $\mathcal{E}$. Thus, $S^C_{\mathcal{V}}$ is a coloured operad in $\mathcal{V}$ whose algebras (when acting on $\mathcal{E}$) are C-coloured operads in $\mathcal{E}$.

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Main result

Theorem (G, 2008)

Let C be any set of colours. Then the category of C-coloured operads with values in symmetric spectra admits a model structure where a map of C-coloured operads $f : P \longrightarrow Q$ is a weak equivalence (resp. fibration) if for every $(c_1, \ldots c_n; c)$ the induced map

$$P(c_1, \ldots, c_n; c) \longrightarrow Q(c_1, \ldots, c_n; c)$$

is a weak equivalence (resp. fibration) of symmetric spectra with the positive model structure.

- The result is also true for any cofibrantly generated simplicial monoidal model category satisfying that every relative FJ-cell complex is a weak equivalence, where $F : Coll_C(\mathcal{E}) \longrightarrow Oper_C(\mathcal{E})$ and J is the set of generating trivial cofibrations.

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