

# An Odd Presentation for $W(E_6)$

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May 11, 2016

## Abstract

The Weyl group  $W(E_6)$  has an odd presentation due to Christopher Simons as factor group of the Coxeter group on the Petersen graph by deflation of the free hexagons. The goal of this paper is to give a geometric meaning for this presentation, coming from the action of  $W(E_6)$  on the moduli space of marked maximally real cubic surfaces and its natural tessellation as seen through the period map of Allcock, Carlson and Toledo.

## 1 Introduction

We denote by  $\mathcal{M}(1^n)$  the moduli space of  $n$  ordered mutually distinct points on the complex projective line. If  $n = n_1 + \cdots + n_r$  is a partition of  $n$  with  $r \geq 4$  parts we denote by  $\mathcal{M}(n_1 \cdots n_r)$  the moduli space of  $r$  points on the complex projective line with weights  $n_1, \dots, n_r$  respectively, and to be viewed as part of a suitable compactification of  $\mathcal{M}(1^n)$  by collisions according to the given partition.

The case of 4 points is classical and very well known. If  $z = (z_1, z_2, z_3, z_4)$  represents a point of  $\mathcal{M}(1^4)$  then we consider for the elliptic curve

$$E(z) : y^2 = \prod (x - z_i)$$

with periods (say  $z_i$  are all real with  $z_1 < z_2 < z_3 < z_4$ )

$$\pi_i(z) = \int_{z_i}^{z_{i+1}} \frac{dx}{y}$$

resulting in a coarse period isomorphism (by taking the ratio of two consecutive periods)

$$\mathcal{M}(1^4)/S_4 \longrightarrow \mathbb{H}/\Gamma$$

of orbifolds. Here  $S_n$  is the symmetric group on  $n$  objects and  $\Gamma$  is the modular group  $\mathrm{PSL}_2(\mathbb{Z})$  acting on the upper half plane  $\mathbb{H} = \{\tau \in \mathbb{C}; \Im\tau > 0\}$  by fractional linear transformations. The Klein four-group  $V_4 \triangleleft S_4$  acts trivially on  $\mathcal{M}(1^4)$  and the above period map lifts to a fine period isomorphism

$$\mathcal{M}(1^4) \longrightarrow \mathbb{H}/\Gamma(2)$$

with  $\Gamma(n)$  the principal congruence subgroup of  $\Gamma$  of level  $n$ . Taking the quotient on the left by  $S_4/V_4 \cong S_3$  and on the right by  $\Gamma/\Gamma(2) \cong S_3$  turns this fine period isomorphism into the previous coarse one.

There are two different real loci: either all 4 points are real, or 2 points are real and 2 form a complex conjugate pair. Indeed, 2 complex conjugate pairs always lie on a circle, so this case reduces to the first locus. This first component is called the maximal real locus. Under the coarse period isomorphism the maximal real locus corresponds to the imaginary axis in  $\mathbb{H}$  since  $\pi_{i+1}/\pi_i$  is purely imaginary, while the other real locus corresponds to the unit circle in  $\mathbb{H}$ . The group  $\Gamma(2)$  has 3 cusps and is of genus 0 meaning that the compactification  $\overline{\mathbb{H}}/\Gamma(2)$  by filling in the cusps is isomorphic to the complex projective line  $\mathbb{C} \sqcup \infty$ . Taking for the 3 cusps in  $\mathbb{C} \sqcup \infty$  the cube roots of unity  $\{1, \omega, \omega^2\}$ , the action of  $S_3$  on  $\mathbb{C} \sqcup \infty$  is given by multiplication  $z \mapsto \omega^j z$  with a cube roots of unity, possibly composed with  $z \mapsto 1/z$ . The maximal real locus in  $\mathbb{C} \sqcup \infty$  corresponds to the unit circle, while the other real locus corresponds to the 3 lines  $\mathbb{R}\omega^j$ . The orbit  $\{-\omega^j\}$  of  $S_3$  in  $\mathbb{C} \sqcup \infty$  corresponds to the Gauss elliptic curve (with  $\tau = i \in \mathbb{H}$ , or equivalently with the 4 points  $\{0, \pm 1, \infty\} \cong \{\pm 1, \pm i\}$  in  $\mathcal{M}(1^4)/S_4$ ) and lies in both real components, while the orbit  $\{0, \infty\}$  in  $\mathbb{C} \sqcup \infty$  corresponds to the Eisenstein elliptic curve in the other real locus (with  $\tau = \omega \in \mathbb{H}$ , or equivalently with the 4 points  $\{0, 1, \omega, \omega^2\}$  in  $\mathcal{M}(1^4)/S_4$ ).

This classical picture allows a beautiful generalization. If  $z = (z_1, \dots, z_6)$  represents a point of  $\mathcal{M}(1^6)$  then we consider the curve

$$C(z) : y^3 = \prod (x - z_i)$$

which is of genus 4 by the Hurwitz formula. The Jacobian  $J(C(z))$  is a principally polarized Abelian variety of dimension 4 with an endomorphism structure by the group ring  $\mathbb{Z}[C_3]$  of the cyclic group of order 3. The PEL theory of Shimura [17], [18], [4] gives that these Jacobians in the full moduli space  $\mathcal{A}_4 = \mathbb{H}_4/\mathrm{Sp}_8(\mathbb{Z})$  form an open dense part of a ball quotient  $\mathbb{B}/\Gamma$  of dimension 3. More precisely and thanks to the work of Deligne and Mostow [8] and of Terada [20] we have a coarse period isomorphism

$$\mathcal{M}(1^6)/S_6 \longrightarrow \mathbb{B}^\circ/\Gamma$$

with  $\mathbb{B}^\circ/\Gamma$  the complement of a Heegner divisor in a ball quotient  $\mathbb{B}/\Gamma$ . More explicitly, let  $\mathcal{E} = \mathbb{Z} + \mathbb{Z}\omega$  with  $\omega = (-1 + i\sqrt{3})/2$  be the ring of Eisenstein integers and let

$$L = \mathcal{E} \otimes \mathbb{Z}^{3,1}$$

be the Lorentzian lattice over  $\mathcal{E}$  then it turns out that the automorphism group  $U(L)$  is a group generated by the hexaflections (order 6 complex reflections) in norm one vectors. If  $e \in L$  is a norm one vector then the hexaflection with root  $e$  is defined by  $h_e(l) = l + \omega\langle l, e \rangle e$ , with  $\langle \cdot, \cdot \rangle$  the sesquilinear form on  $L$  of Lorentzian signature. We denote the complement of the mirrors of all these hexaflections by  $\mathbb{B}^\circ$ . The main result of Deligne and Mostow in this particular case can be rephrased by the commutative diagram

$$\begin{array}{ccccc} \mathcal{M}^\circ & \longrightarrow & \mathcal{M} & \longrightarrow & \overline{\mathcal{M}}^{\text{HM}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{B}^\circ/\Gamma & \longrightarrow & \mathbb{B}/\Gamma & \longrightarrow & \overline{\mathbb{B}}^{\text{BB}}/\Gamma \end{array}$$

with  $\mathcal{M}^\circ$  short for  $\mathcal{M}(1^6)/S_6$ . The horizontal maps are injective and the vertical maps are isomorphisms from the top horizontal line (the geometric side) to the bottom horizontal line (the arithmetic side). The moduli space

$$\overline{\mathcal{M}}^{\text{HM}} = \text{Proj} \left( S(S^6\mathbb{C}^2)^{\text{SL}_2(\mathbb{C})} \right)$$

is the Hilbert–Mumford compactification of  $\mathcal{M}^\circ$  through GIT of degree 6 binary forms, which consists of the open stable locus  $\mathcal{M}$  with at most double collisions and the polystable (also called strictly semistable) locus, a point with two triple collisions. In the bottom line we have the ball quotient  $\mathbb{B}/\Gamma$  with  $\Gamma = \text{PU}(L)$  and its Baily–Borel compactification

$$\overline{\mathbb{B}}^{\text{BB}}/\Gamma = \text{Proj} \left( \mathcal{A}(\mathbb{L}^\times)^{U(L)} \right)$$

with  $\mathbb{L}^\times = \{v \in \mathbb{C} \otimes \mathbb{Z}^{3,1}; \langle v, v \rangle < 0\} \longrightarrow \mathbb{B} = \mathbb{P}(\mathbb{L})$  the natural  $\mathbb{C}^\times$ -bundle and  $\mathcal{A}(\mathbb{L}^\times)^{U(L)}$  the algebra of modular forms, graded by weight (minus the degree, or maybe better by minus degree/3 in order to match with the degree on the geometric side: the center of  $\text{SL}_2(\mathbb{C})$  has order 2 while the center of  $U(L)$  has order 6).

A similar commutative diagram also holds in the case of ordered points, so with  $\mathcal{M}^\circ = \mathcal{M}(1^6)/S_6$  replaced by  $\mathcal{M}_m^\circ = \mathcal{M}(1^6)$  and  $U(L)$  replaced by the principal congruence subgroup  $U(L)(1 - \omega)$ . The subindex  $m$  stands for marking. This latter group is generated by all triflections in norm one

vectors, namely by the squares of the previous hexaflections. Then we have according to Deligne and Mostow [8] a commutative diagram

$$\begin{array}{ccccc} \mathcal{M}_m^\circ & \longrightarrow & \mathcal{M}_m & \longrightarrow & \overline{\mathcal{M}}_m^{\text{HM}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{B}^\circ/\Gamma(1-\omega) & \longrightarrow & \mathbb{B}/\Gamma(1-\omega) & \longrightarrow & \overline{\mathbb{B}}^{\text{BB}}/\Gamma(1-\omega) \end{array}$$

The group isomorphism  $\Gamma/\Gamma(1-\omega) \cong S_6$  explains that the quotient of this commutative diagram by this finite group gives back the former commutative diagram.

The real locus in the space  $\mathcal{M}(1^6)/S_6$  of degree 6 binary forms with nonzero discriminant has 4 connected components. There are  $k$  complex conjugate pairs and the remaining points  $6 - 2k$  points are real for  $k = 0, 1, 2, 3$  respectively. All 6 points real is called the maximal real locus, and will be denoted  $\mathcal{M}_r^\circ = \mathcal{M}_r(1^6)/S_6$ . It was shown by Yoshida [23] that we have a similar commutative diagram

$$\begin{array}{ccccc} \mathcal{M}_r^\circ & \longrightarrow & \mathcal{M}_r & \longrightarrow & \overline{\mathcal{M}}_r^{\text{HM}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{B}_r^\circ/\Gamma & \longrightarrow & \mathbb{B}_r/\Gamma & \longrightarrow & \overline{\mathbb{B}}_r^{\text{BB}}/\Gamma \end{array}$$

with the bar in the upper horizontal line denoting the real Zariski closure of the maximal real locus in the GIT compactification, and the bar in the lower horizontal line denoting the Baily–Borel compactification of  $\mathbb{B}_r$ . Here  $\mathbb{B}_r$  is the real hyperbolic ball associated to the Lorentzian lattice  $\mathbb{Z}^{3,1}$ . Likewise  $\mathbb{B}_r^\circ$  is the complement of the mirrors in norm one roots in  $\mathbb{Z}^{3,1}$  and  $\Gamma = \text{O}^+(\mathbb{Z}^{3,1})$ .

Likewise we have a marked version in the real case with commutative diagram

$$\begin{array}{ccccc} \mathcal{M}_{rm}^\circ & \longrightarrow & \mathcal{M}_{rm} & \longrightarrow & \overline{\mathcal{M}}_{rm}^{\text{HM}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{B}_r^\circ/\Gamma(3) & \longrightarrow & \mathbb{B}_r/\Gamma(3) & \longrightarrow & \overline{\mathbb{B}}_r^{\text{BB}}/\Gamma(3) \end{array}$$

with  $\mathcal{M}_{rm}^\circ = \mathcal{M}_r(1^6)$  the moduli space of 6 distinct ordered real points and  $\Gamma(3)$  the principal congruence subgroup of  $\Gamma = \text{O}^+(\mathbb{Z}^{3,1})$  of level 3. The group isomorphism  $\Gamma/\Gamma(3) = \text{PGO}_4(3) \cong S_6$  shows that the quotient of this commutative diagram by  $S_6$  gives the previous commutative diagram just as in the complex case.

Deliberately we have suppressed the index  $n = 3$  of the Lorentzian lattice  $\mathbb{Z}^{n,1}$  in the above diagrams, because there are similar stories to tell for  $n = 2, 3, 4$ . The case  $n = 2$  corresponds to  $\mathcal{M}^\circ = \mathcal{M}(21^4)/S_4$  and  $\mathcal{M}_m^\circ = \mathcal{M}(21^4)$ , and is also due to Deligne and Mostow. The case  $n = 4$  corresponds to  $\mathcal{M}^\circ = \mathcal{M}(cs)$ , the moduli space of smooth cubic surfaces, and is due to Allcock, Carlson and Toledo [1].

A smooth cubic surface  $S$  can be obtained by blowing up 6 points in the projective plane (in general position: no three on a line, no six on a conic). Hence  $H_2(S, \mathbb{Z})$  with its intersection form is isomorphic to the lattice  $\mathbb{Z}^{1,6}$  with standard basis  $l, e_1, \dots, e_6$  given by a line and the exceptional curves with

$$l^2 = 1, l \cdot e_i = 0, e_i \cdot e_j = -\delta_{ij}.$$

The anticanonical class  $k$  corresponds to  $3l - \sum e_i$  and has norm 3. Such an isomorphism  $H_2(S, \mathbb{Z}) \cong \mathbb{Z}^{1,6}$  with  $k \cong (3l - \sum e_i)$  is called a marking of the cubic surface  $S$ . Any two markings of  $S$  are conjugated in a simply transitive manner by the stabilizer group in  $O(\mathbb{Z}^{1,6})$  of  $3l - \sum e_i$ , which by Vinberg's theorem (Theorem 2.1) is just equal to the Weyl group  $W(E_6)$ . This group is also equal to the automorphism group of the configuration of the 27 lines on  $S$ , which can be identified with  $\{e \in H_2(S, \mathbb{Z}); k \cdot e = 1, e^2 = -1\}$ . We denote by  $\mathcal{M}_m^\circ = \mathcal{M}_m(cs)$  the moduli space of marked smooth cubic surfaces, which is a Galois cover of  $\mathcal{M}^\circ = \mathcal{M}(cs)$  with Galois group  $W(E_6)$ .

The maximal real locus  $\mathcal{M}_r^\circ = \mathcal{M}_r(cs)$  is by definition the moduli space of smooth real cubic surfaces with 27 real lines, and likewise we denote  $\mathcal{M}_{rm}^\circ = \mathcal{M}_{rm}(cs)$  for the marked covering. All four commutative diagrams remain valid in case  $n = 4$ . The group isomorphism  $\Gamma/\Gamma(3) = \text{PGO}_5(3) \cong W(E_6)$  shows that the quotient of the commutative diagram in the marked case becomes the commutative diagram in the unmarked case.

Consider the following commutative diagram

$$\begin{array}{ccccc} \mathcal{M}_{rm}^\circ & \longrightarrow & \overline{\mathcal{M}}_{rm} & \longrightarrow & \overline{\mathcal{M}}_r \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{B}_r^\circ/\Gamma(3) & \longrightarrow & \overline{\mathbb{B}}_r/\Gamma(3) & \longrightarrow & \overline{\mathbb{B}}_r/\Gamma \end{array}$$

with  $\Gamma/\Gamma(3) = \text{PGO}_{n+1}(3)$  the Weyl group of type  $A_3, A_5, E_6$  for  $n = 2, 3, 4$  respectively. The two left horizontal arrows are inclusions and the two right horizontal maps are quotient maps for the action of  $\Gamma/\Gamma(3)$ . In fact we shall for the moment only consider the bottom horizontal line for all  $2 \leq n \leq 7$ , independently of the modular interpretations for  $n \leq 4$ .

Fix a connected component of the mirror complement  $\mathbb{B}_r^\circ$  of norm one roots in  $\mathbb{Z}^{n,1}$  and denote by  $P$  its closure in  $\overline{\mathbb{B}}_r$ . It is a fundamental domain for the action on  $\overline{\mathbb{B}}_r$  of the subgroup  $\Gamma_1$  of  $\Gamma = \mathrm{O}^+(\mathbb{Z}^{n,1})$  generated by the reflections in norm one roots. Clearly  $\Gamma_1$  is a subgroup of the principal congruence subgroup  $\Gamma(2)$  of level 2. It was shown by Everitt, Ratcliffe and Tschantz [9] that  $\Gamma_1 = \Gamma(2)$  if and only if  $n \leq 7$ , which will be assumed from now on. The polytope  $P$  will be called the Gosset polytope, by analogy with the terminology of Coxeter [7] in case  $n = 6$ . The symmetry group  $\Gamma_0$  of  $P$  in  $\Gamma$  is the Coxeter group of type  $E_n$ , with  $E_5 = D_5$ ,  $E_4 = A_4$ ,  $E_3 = A_1 \sqcup A_2$  and  $E_2 = A_1$ . For  $n \geq 3$  it permutes the faces of  $P$  transitively, and a face of  $P^n$  is equal to  $P^{n-1}$ . The ball quotient  $\overline{\mathbb{B}}_r/\Gamma(3)$  inherits a regular tessellation by polytopes  $\gamma P$  with  $\gamma \in \Gamma/\Gamma(3)\Gamma_0$ . The cardinality of the factor space  $\Gamma/\Gamma(3)\Gamma_0$  is equal to 12, 60, 432 for  $n = 2, 3, 4$  respectively in accordance with the discussion by Yoshida [23], [24], who gives a description of this tessellation on the geometric side.

Two walls of  $P$  are either orthogonal (with nonempty intersection in  $\mathbb{B}_r$ ) or parallel (with only intersection at an ideal point of  $\overline{\mathbb{B}}_r$ ), and so  $P$  is a right angled polytope. Equivalently, the Coxeter diagram of the chamber  $P$  of the Coxeter group  $\Gamma_1$  has only edges with mark  $\infty$ . This Coxeter diagram (after deletion of all marks  $\infty$ ) is of type  $A_3, \tilde{A}_5$  for  $n = 2, 3$  respectively, while for  $n = 4$  it is the Peterson graph, which we denote by  $P_{10}$ .

Since  $\Gamma/\Gamma(6) \cong \Gamma/\Gamma(2) \times \Gamma/\Gamma(3)$  we have  $\Gamma(2)/\Gamma(6) \cong \Gamma/\Gamma(3)$ , and so the group  $\Gamma/\Gamma(3)$  is generated by the cosets modulo  $\Gamma(3)$  of a set of generators of  $\Gamma(2)$ . Since  $\Gamma(2) = \Gamma_1$  is a Coxeter group we take  $r_i$  the reflections in the walls of  $P$  as Coxeter generators for  $\Gamma(2)$  and hence  $t_i = r_i\Gamma(3)$  are generators for  $\Gamma/\Gamma(3)$ . Because the  $r_i$  are reflections the  $t_i$  remain involutions in  $\Gamma/\Gamma(3)$ . Likewise if  $r_i$  and  $r_j$  commute so do  $t_i$  and  $t_j$  commute. The relations between the  $t_i$  in dimension  $n$  are also valid in dimension  $n+1$ . In dimension  $n = 2$  it is easy to check that  $t_i t_j t_i = t_j t_i t_j$  if the corresponding walls are parallel. Hence we recover the Coxeter presentation of  $S_4$ . In all dimensions  $2 \leq n \leq 7$  the group  $\Gamma/\Gamma(3)$  becomes a factor group of the Coxeter group of the simply laced Coxeter diagram obtained from that of  $P$  by deletion of the marks  $\infty$ . For  $n = 3$  this Coxeter diagram is the affine diagram of type  $\tilde{A}_5$  and it is easy to check that the translation lattice dies in  $\Gamma/\Gamma(3)$ . This relation is also called deflation of the free hexagon. We can now state the main result, which will be proven in the next section as Theorem 2.4.

**Theorem 1.1.** *For  $2 \leq n \leq 7$  the group  $\Gamma/\Gamma(3)$  is a factor group of  $W/N$ . Here  $W$  is the Coxeter group of the simply laced Coxeter diagram associated*

with  $P$  as above and  $W/N$  is the quotient by deflation of the free hexagons. For  $n \leq 4$  we have in fact equality  $\Gamma/\Gamma(3) = W/N$  and for  $n = 4$  we recover a presentation for  $W(E_6)$  found by Simons [19].

The fact that for  $n = 4$  these are a complete set of relations is an easy exercise with the Petersen graph. The essential point of the theorem is to explain that this presentation has a natural geometric meaning from the action of  $W(E_6)$  on the moduli space  $\overline{\mathcal{M}}_{rm}(cs)$  of marked maximally real cubic surfaces with its natural equivariant tessellation as seen on the arithmetic side.

We do not know whether for  $n = 5, 6, 7$  the generators and relations given in the theorem for  $\Gamma/\Gamma(3)$  suffice to give a presentation. However this presentation for  $W(E_6)$  was found by Simons by analogy with similar presentations for the orthogonal group  $\text{PGO}_8^-(2)$  and the bimonster group  $M \wr 2$  as factor group of the Coxeter group on the incidence graph of the projective plane over a field of 2 and 3 elements by deflation of the free octagons and dodecagons respectively. This presentation of the bimonster was found by Conway and Simons [6] as a variation of the Ivanov–Norton theorem, which gives the bimonster group as a factor group of the Coxeter group  $W(Y_{555})$  modulo the spider relation [12], [14]. This presentation for  $\text{PGO}_8^-(2)$  and some of its subgroups (for example the Weyl group  $W(E_7)$ ) can be given a similar geometric meaning. We would like to thank Masaaki Yoshida for comments on an earlier version of this paper. We are also grateful to the referee for sharing his insightful comments.

## 2 The odd unimodular lattice $\mathbb{Z}^{n,1}$

The odd unimodular lattice  $\mathbb{Z}^{n,1}$  has basis  $e_i$  for  $0 \leq i \leq n$  with scalar product  $(e_i, e_j) = \delta_{ij}$  for all  $i, j$  except for  $i = j = 0$  in which case  $e_0^2 = -1$ . The open set

$$\mathbb{L}_r^\times = \{v \in \mathbb{R}^{n,1}; v^2 < 0\}$$

has two connected components, and the component containing  $e_0$  is denoted by  $\mathbb{L}_r^+$ . The quotient space

$$\mathbb{B}_r = \mathbb{L}_r^\times / \mathbb{R}^\times = \mathbb{L}_r^+ / \mathbb{R}^+$$

is the real hyperbolic ball. The forward Lorentz group  $O^+(\mathbb{R}^{n,1})$  is the index two subgroup of the full Lorentz group  $O(\mathbb{R}^{n,1})$  preserving the component  $\mathbb{L}_r^+$  and it acts faithfully on the ball  $\mathbb{B}_r$ . In addition

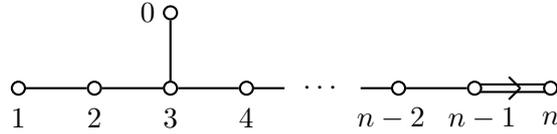
$$\Gamma = O^+(\mathbb{Z}^{n,1}) = O^+(\mathbb{R}^{n,1}) \cap O(\mathbb{Z}^{n,1})$$

is a discrete subgroup of  $O^+(\mathbb{R}^{n,1})$  acting on  $\mathbb{B}_r$  properly discontinuously with cofinite volume. It contains reflections

$$s_\alpha(\lambda) = \lambda - 2(\lambda, \alpha)\alpha/\alpha^2$$

in roots  $\alpha \in \mathbb{Z}^{n,1}$  of norm 1 or norm 2. Our notation is  $\alpha^2 = (\alpha, \alpha)$  for the norm of  $\alpha \in \mathbb{Z}^{n,1}$ . The next theorem is a (special case of a more general) result due to Vinberg [22] and for a pedestrian exposition of the proof we refer to the lecture notes on Coxeter groups by one of us [11].

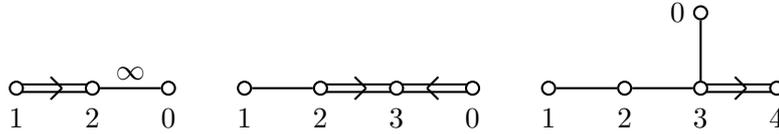
**Theorem 2.1.** *For  $2 \leq n \leq 9$  the group  $\Gamma = O^+(\mathbb{Z}^{n,1})$  is generated by reflections  $s_\alpha$  in roots  $\alpha \in \mathbb{Z}^{n,1}$  of norm 1 or norm 2. Moreover the Coxeter diagram of this reflection group  $\Gamma$  is given by*



with simple roots

$$\alpha_0 = e_0 - e_1 - e_2 - e_3, \alpha_1 = e_1 - e_2, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n.$$

For  $n = 2, 3, 4$  the Coxeter diagrams become



with  $\alpha_0 = e_0 - e_1 - e_2$  a norm 1 vector in case  $n = 2$ .

The vertices of the closed fundamental chamber  $D$  in  $\overline{\mathbb{B}}_r$  are represented by the vectors (for  $j = 3, \dots, n$ )

$$v_0 = e_0, v_1 = e_0 - e_1, v_2 = 2e_0 - e_1 - e_2, v_j = 3e_0 - e_1 - e_2 - \dots - e_j$$

as (anti)dual basis of the basis of simple roots. Let  $D_0$  be the face of  $D$  cut out by the long simple roots. Hence  $D_0$  is the edge of the triangle  $D$  with vertices represented by  $v_0, v_2$  for  $n = 2$ , while  $D_0$  is the vertex of the simplex  $D$  represented by  $v_n$  for  $3 \leq n \leq 9$ . Let  $\Gamma_0$  be the subgroup of  $\Gamma$  generated by the long simple roots, and so  $\Gamma_0$  is the stabilizer of the face  $D_0$ . Clearly

the group  $\Gamma_0$  is a finite Coxeter group (of type  $A_1, A_1 \sqcup A_2, A_4, D_5, E_6, E_7, E_8$  respectively) for  $2 \leq n \leq 8$ , which will be assumed from now on.

The convex polytope  $P$  defined by

$$P = \cup_{w \in \Gamma_0} wD$$

is the star of  $D_0$ , and will be called the Gosset polytope. The walls of  $D$  which do not meet the relative interior of  $D_0$  are cut out by the mirrors of the short simple roots. For  $n = 2$  there are 2 such edges of  $D$  and for  $3 \leq n \leq 8$  there is just a unique such wall of  $D$ . Hence the interior of  $P$  is just a connected component of the complement of all mirrors in norm 1 roots, and  $P$  is a fundamental chamber for the normal subgroup  $\Gamma_1$  of  $\Gamma$  generated by the reflections in norm 1 roots. Note that  $\Gamma_1$  is in fact a subgroup of the principal congruence subgroup  $\Gamma(2)$  of  $\Gamma$  of level 2. Because  $\Gamma_0 = \{w \in \Gamma; wP = P\}$  and the reflection group  $\Gamma_1$  is normal in  $\Gamma$  and has  $P$  as fundamental chamber we have the semidirect product decomposition  $\Gamma = \Gamma_1 \rtimes \Gamma_0$ .

For  $3 \leq n \leq 8$  all walls of  $P^n$  are congruent and of the form  $P^{n-1}$ . By induction on the dimension it can be shown that the set of vertices of  $P$  consists of two orbits under  $\Gamma_0$ . One orbit  $\Gamma_0 v_0$  are the actual vertices and the other orbit  $\Gamma_0 v_1$  are the ideal vertices of  $P$ . In turn this shows by a local analysis at  $v_0$  and  $v_1$  that all dihedral angles of  $P$  inside  $\mathbb{B}_r$  are  $\pi/2$ , and so  $P$  is a right-angled polytope. Of course, at ideal vertices of  $P$  the dihedral angle of intersecting walls can be 0 as well. In other words, the Coxeter diagram of the group  $\Gamma_1$  generated by reflections in the norm 1 roots with fundamental chamber  $P$  has only edges with mark  $\infty$ . The next result is due to Everitt, Ratcliffe and Tschantz [9].

**Theorem 2.2.** *For  $2 \leq n \leq 7$  the group  $\Gamma(2)$  is generated by the reflections in norm 1 roots, while for  $n = 8$  the subgroup of  $\Gamma(2)$  generated by the reflections in norm 1 roots has index 2.*

*Proof.* Since  $\Gamma = \Gamma_1 \rtimes \Gamma_0$  we have to show that  $\Gamma_0 \cap \Gamma(2)$  is the trivial group for  $2 \leq n \leq 7$  and has order 2 for  $n = 8$ . For  $n = 2$  the sublattice  $L_0 = \mathbb{Z}v_0 + \mathbb{Z}v_2$  has discriminant  $d = 2$  while for  $3 \leq n \leq 7$  the sublattice  $L_0 = \mathbb{Z}v_n$  has discriminant  $d = 9 - n$ . Hence the orthogonal complement  $Q_0$  of  $L_0$  in  $\mathbb{Z}^{n,1}$  is just the root lattice of the finite Coxeter group  $\Gamma_0$  (of type  $A_1, A_1 \sqcup A_2, A_4, D_5, E_6, E_7, E_8$  respectively). Indeed, that root lattice is contained in  $Q_0$  and has the correct discriminant  $d$ . The corresponding (rational) weight lattice  $P_0$ , by definition the dual lattice of  $Q_0$ , is the orthogonal projection of  $\mathbb{Z}^{n,1}$  on  $\mathbb{Q} \otimes Q_0$ .

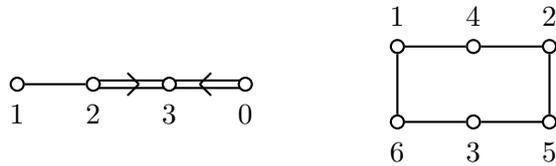
Now  $w \in \Gamma_0$  also lies in  $\Gamma(2)$  if and only if  $w\lambda - \lambda \in 2Q_0$  for all  $\lambda \in P_0$ . It is well known that for  $2 \leq n \leq 7$  the set  $\{\lambda \in P_0; \lambda^2 < 2\}$  is nonempty and spans  $P_0$ . For all these  $\lambda$  the norm  $(w\lambda - \lambda)^2$  is smaller than 8 by the triangle inequality. But the only vector in  $2Q_0$  of norm smaller than 8 is the null vector. Hence  $w = 1$  and so  $\Gamma_0 \cap \Gamma(2)$  is the trivial group. For  $n = 8$  the elements of minimal positive norm in the lattice  $P_0 = Q_0$  of type  $E_8$  form the root system  $R(E_8)$  of type  $E_8$  of vectors of norm 2. If  $(w - 1)\alpha \in 2Q_0$  for  $w \in \Gamma_0$  and  $\alpha \in R(E_8)$  then either  $(w - 1)\alpha$  has norm smaller than 8 and  $w\alpha = \alpha$ , or  $(w - 1)\alpha$  has norm 8 and  $w\alpha = -\alpha$ . If  $w\alpha = \pm\alpha$  for all  $\alpha \in R(E_8)$  then one easily concludes that  $w = \pm 1$ . Hence  $\Gamma_0 \cap \Gamma(2) = \{\pm 1\}$  has order 2 for  $n = 8$ .  $\square$

For  $n = 2, 3, 4$  the Coxeter diagram of the reflection group  $\Gamma_1 = \Gamma(2)$  has the following explicit description.

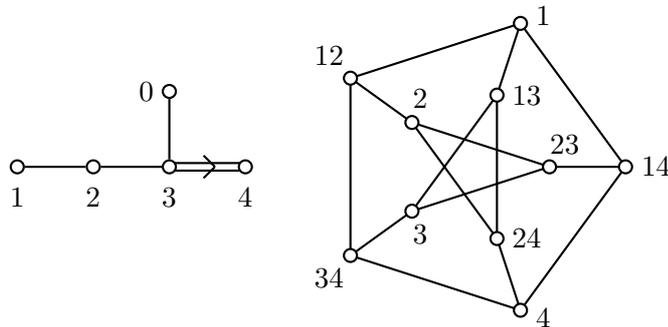
**Theorem 2.3.** *The Coxeter diagrams of  $\Gamma$  on the left and of  $\Gamma(2)$  on the right are given by*



for  $n = 2$ , and



for  $n = 3$ , and



for  $n = 4$  respectively. All edges of the Coxeter diagrams of  $\Gamma(2)$  have mark  $\infty$ , but for simplicity and because of the next theorem these are left out in the drawn diagrams. The last diagram for  $n = 4$  with 10 nodes is the so called Petersen graph and will be denoted  $P_{10}$ . The automorphism groups  $\Gamma_0 \cong \Gamma/\Gamma(2)$  of these Coxeter diagrams of  $\Gamma(2)$  are equal to  $S_2, S_2 \times S_3, S_5$  as the Weyl groups of type  $A_1, A_1 \sqcup A_2, A_4$  respectively.

*Proof.* Let  $s_i$  for  $i = 0, 1, \dots, n$  be the simple reflections of the group  $\Gamma$  as numbered in Theorem 2.1. We shall treat the cases  $n = 2, 3, 4$  separately.

For  $n = 2$  the fundamental domain  $D$  is a hyperbolic triangle with angles  $\{\pi/4, 0, \pi/2\}$  at the vertices  $v_0, v_1, v_2$  respectively. The Gosset polytope  $P = D \cup s_1 D$  is a hyperbolic triangle with angles  $\{\pi/2, 0, 0\}$  at the vertices  $v_0, v_1, s_1 v_1$ . It is a fundamental domain for the action of the Coxeter group  $\Gamma(2)$  with simple generators

$$r_1 = s_1 s_2 s_1, r_2 = s_2, r_3 = s_0$$

whose Coxeter diagram is the  $A_3$  diagram with marks  $\infty$  on the edges rather than the usual mark 3.

For  $n = 3$  the Gosset polytope  $P$  is a double tetrahedron  $P = T \cup s_0 T$  with hyperbolic tetrahedron  $T$  the union over  $wD$  with  $w \in S_3 = \langle s_1, s_2 \rangle$  and  $\{v_0, v_1, s_1 v_1, s_2 s_1 v_1\}$  as the set of vertices. The Coxeter diagram of  $T$  is the  $D_4$  diagram with marks 4 on the edges rather than the usual mark 3. The reflection  $s_0$  corresponds to the central node, and the reflections

$$r_1 = s_1 r_2 s_1, r_2 = s_2 s_3 s_2, r_3 = s_3$$

correspond to the three extremal nodes. The polytope  $P$  is the fundamental domain for the action of the Coxeter group  $\Gamma(2)$  with simple generators

$$r_1 = s_1 r_2 s_1, r_2 = s_2 s_3 s_2, r_3 = s_3, r_4 = s_0 r_3 s_0, r_5 = s_0 r_1 s_0, r_6 = s_0 r_2 s_0$$

whose Coxeter diagram is the  $\tilde{A}_5$  diagram with marks  $\infty$  on the edges rather than the usual mark 3.

For  $n = 4$  the Gosset polytope  $P$  is the union  $\cup_w wD$  over  $w \in \Gamma_0$  with  $\Gamma_0 = S_5$  the group generated by the reflections  $s_0, s_1, s_2, s_3$  in the long simple roots. The vertex  $v_4$  of  $D$  is interior point of  $P$  and  $\Gamma_0$  is the symmetry group of  $P$  generated by the reflections in the mirrors through  $v_4$ . The group  $\Gamma(2)$  is generated by the simple reflections

$$r_i = w s_4 w^{-1}$$

with  $w \in S_5$  and  $i \in I = S_5/(S_2 \times S_3)$  the left coset of  $w$  for the centralizer of  $s_4$  in  $S_5$ , which is just generated by  $s_0, s_1, s_2$ . The cardinality of  $I$  is equal to 10 and the Coxeter diagram of  $P$  is the Petersen graph  $P_{10}$ , but with the edges marked  $\infty$  rather than 3. Indeed, by Theorem 2.1

$$\alpha_0 = e_0 - e_1 - e_2 - e_3, \alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4, \alpha_4 = e_4$$

is the basis of simple roots for  $D$ . Hence both  $\beta_3 = s_3(\alpha_4) = e_3$  and  $\beta_{12} = s_0(\beta_3) = e_0 - e_1 - e_2$  are simple roots for  $P$ . Using the action of  $\langle s_1, s_2, s_3 \rangle$  we see that

$$\beta_i = e_i, \beta_{jk} = e_0 - e_j - e_k$$

are simple roots of  $P$  for  $1 \leq i \leq 4$  and  $1 \leq j < k \leq 4$ . Because  $P$  has 10 simple roots these are all simple roots of  $P$ . The Gosset polytope  $P$  has 5 actual vertices, which are the transforms under  $\Gamma_0$  of  $v_0$ . Likewise it has 5 ideal vertices, which are the transforms under  $\Gamma_0$  of the cusp  $v_1$  of  $D$ .  $\square$

The Petersen graph was described by Petersen in 1898 [15], but was in fact discovered before in 1886 by Kempe [13].

**Theorem 2.4.** *Let  $\Gamma = O^+(\mathbb{Z}^{n,1})$  and let  $\Gamma(2)$  and  $\Gamma(3)$  be the principal congruence subgroups of level 2 and level 3 respectively for  $n = 2, 3, 4$ . Then the group  $\Gamma/\Gamma(3)$  is equal to*

$$\text{PGO}_3(3) = S_4 = W(A_3), \text{PGO}_4(3) = S_6 = W(A_5), \text{PGO}_5(3) = W(E_6)$$

respectively. If we denote by  $r_i$  the Coxeter generators of  $\Gamma(2)$  in the notation of Theorem 2.3 then  $t_i = r_i\Gamma(3)$  are generators for  $\Gamma/\Gamma(3)$ . In fact  $\Gamma/\Gamma(3)$  has a presentation with generators the involutions  $t_i$  and with braid and deflation relations. The braid relations amount to

$$t_i t_j = t_j t_i, t_i t_j t_i = t_j t_i t_j$$

if the nodes with index  $i$  and  $j$  are disconnected and connected respectively, and so  $\Gamma/\Gamma(3)$  is a factor group of the Coxeter group associated to the simply laced Coxeter diagrams  $A_3, \tilde{A}_5, P_{10}$  of Theorem 2.3. The deflation relations mean that for each subdiagram of type  $\tilde{A}_5$ , also called a free hexagon, the translation lattice of the affine Coxeter group  $W(\tilde{A}_5)$  dies in  $\Gamma/\Gamma(3)$ .

*Proof.* It is well known that  $\text{PGO}_{n+1}(3)$  is equal to  $W(A_3), W(A_5), W(E_6)$  for  $n = 2, 3, 4$  respectively [5]. Clearly  $\Gamma/\Gamma(3) \cong \Gamma(2)/\Gamma(6)$ , and so  $\Gamma/\Gamma(3)$

is a factor group of the Coxeter group  $\Gamma(2)$  with Coxeter diagram given by Theorem 2.3 with all edges marked  $\infty$ .

If  $\alpha, \beta \in \mathbb{Z}^{n,1}$  are norm 1 roots with  $(\alpha, \beta) = -1$  then a straightforward computation yields

$$(s_\beta s_\alpha s_\beta - s_\alpha s_\beta s_\alpha)\lambda = 6(\lambda, \alpha)\alpha - 6(\lambda, \beta)\beta$$

for all  $\lambda \in \mathbb{Z}^{n,1}$ , which in turn implies  $s_\beta s_\alpha s_\beta \equiv s_\alpha s_\beta s_\alpha$  modulo  $\Gamma(3)$ . Hence  $\Gamma/\Gamma(3)$  is a factor group of the Coxeter group with the simply laced Coxeter diagrams of Theorem 2.3, because the marks  $\infty$  become a 3 and are deleted. For  $n = 2$  we recover the Coxeter presentation of  $S_4 = W(A_3)$ .

For  $n = 3$  the group  $\Gamma/\Gamma(3) = S_6$  is the factor group of the affine Coxeter group  $W(\tilde{A}_5)$  by its translation lattice. Indeed, in the notation of Theorem 2.3 and its proof we have

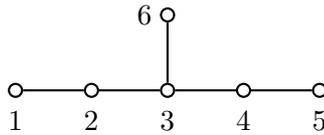
$$r_1 = s_{e_1}, r_2 = s_{e_2}, r_3 = s_{e_3}, r_4 = s_{e_0 - e_1 - e_2}, r_5 = s_{e_0 - e_2 - e_3}, r_6 = s_{e_0 - e_1 - e_3}$$

and the relation

$$t_1 t_4 t_2 t_5 t_3 t_6 t_3 t_5 t_2 t_4 = 1$$

in  $\Gamma/\Gamma(3)$  follows by direct inspection. Since the element on the left side in the affine Coxeter group  $W(\tilde{A}_5)$  is a translation over a coroot this shows that the translation lattice dies in  $\Gamma/\Gamma(3)$ . This relation is also called deflation of the free hexagon.

For  $n = 4$  we recover a presentation for the group  $W(E_6)$  as found by Christopher Simons [19]. It is the factor group of the Coxeter group  $W(P_{10})$  of the Petersen graph  $P_{10}$  by deflation of all free hexagons. This somewhat odd presentation for  $W(E_6)$  can be seen in the usual  $E_6$  diagram



as follows. The group generated by the simple reflections  $s_i$  for  $1 \leq i \leq 5$  is the symmetric group  $S_6$ . The orbit under the symmetric group  $S_5$  generated by  $s_i$  for  $1 \leq i \leq 4$  of the root  $\alpha_6$  has cardinality 10 and the reflections in these 10 roots generate the Weyl group  $W(D_5)$  generated by the reflections  $s_1, s_2, s_3, s_4, s_6$ . However  $S_6$  has an outer automorphism [21], and the image of  $S_5$  under this automorphism is denoted  $\tilde{S}_5$ . The orbit under the twisted  $\tilde{S}_5$  of the root  $\alpha_6$  has again cardinality 10, and the Gram matrix of this set

of 10 roots is the incidence matrix of the Petersen graph, so  $(\alpha, \beta) = 0, 1, 2$  if  $\alpha$  and  $\beta$  are disconnected, or are connected by an edge, or are equal respectively.

An explicit way of understanding that a set of 10 vectors with such a Gram matrix exists in the root system  $R(E_6)$  goes as follows. Denote by  $\{\alpha_j\}$  the basis of simple roots of  $R(E_6)$  numbered as in the above diagram. Then we take

$$\beta_{13} = -\alpha_1, \beta_1 = \alpha_2, \beta_{14} = -\alpha_3, \beta_4 = \alpha_4, \beta_{34} = -\alpha_5, \beta_{23} = \alpha_6$$

in the numbering of nodes of  $P_{10}$  as in Theorem 2.3. In turn this implies

$$\begin{aligned} \beta_3 &= -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 \\ \beta_{24} &= \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \\ \beta_2 &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 \\ \beta_{12} &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_6 \end{aligned}$$

by looking for suitable free hexagons, as the alternating sum of the roots of a free hexagon vanishes. Hence we recover the presentation of Simons for the Weyl group  $W(E_6)$  as the quotient of the Coxeter group  $W(P_{10})$  by deflation of all free hexagons.  $\square$

**Remark 2.5.** *The automorphism group  $S_5$  of the Petersen graph can be identified with the group of geometric automorphisms of the Clebsch diagonal surface*

$$u + v + w + x + y = 0, \quad u^3 + v^3 + w^3 + x^3 + y^3 = 0$$

*in projective three space. Via the period map this surface corresponds to the central point  $v_4 = 3e_0 - e_1 - e_2 - e_3 - e_4$  of the Gosset polytope  $P$  for  $n = 4$ . In this way  $S_5$  becomes a subgroup of  $W(E_6)$  as symmetry group of the configuration of the 27 lines on the Clebsch diagonal surface. This monomorphism  $S_5 \hookrightarrow W(E_6)$ , as described in the above proof, was already discussed by Segre [16].*

*Likewise the dihedral group  $D_6$  of order 12 as automorphism group of the free hexagon can be identified with the group of geometric automorphisms of the degree 6 binary form  $u^6 + v^6$ , which corresponds via the period map to the central point  $v_3 = 3e_0 - e_1 - e_2 - e_3$  of the Gosset polytope  $P$  for  $n = 3$ . In this way  $D_6 \hookrightarrow S_6$  and up to conjugation by (inner and outer) automorphisms of  $S_6$  there is a unique monomorphism  $D_6 \hookrightarrow S_6$ .*

*The symmetric group  $S_2$  as automorphism group of the Coxeter diagram  $A_3$  can be identified with the group of geometric automorphisms of the one*

parameter family of degree 6 binary forms  $(u+v)^2(u^4 + tu^2v^2 + v^4)$  with  $-2 < t < 2$  via  $(u, v) \mapsto (v, u)$ , which corresponds via the period map to the central line segment between the vertices  $v_0$  and  $v_2$  inside the Gosset polytope  $P$  for  $n = 2$ . In this way  $S_2 \hookrightarrow V_4 \hookrightarrow S_4$  and up to conjugation there is a unique such monomorphism.

Via the period map isomorphism  $\overline{\mathcal{M}}_{rm} \rightarrow \overline{\mathbb{B}}_r/\Gamma(3)$  we get a tessellation of the moduli space  $\overline{\mathcal{M}}_{rm}$  of marked maximally real objects by congruent copies  $\gamma P$  of the Gosset polytope with  $\gamma$  in the factor space  $\Gamma/\Gamma(3)\Gamma_0$  and  $\Gamma_0 = \text{Aut}(P) \hookrightarrow \Gamma/\Gamma(3)$  the natural monomorphism. The glue prescription is given by

$$\overline{\mathbb{B}}_r/\Gamma(3) = \{\sqcup_{\gamma} \gamma P\} / \sim$$

with

$$\gamma P \supset \gamma F_i \sim (\gamma t_i) F_i \subset (\gamma t_i) P$$

and  $F_i$  the wall of  $P$  fixed by  $r_i$  in the notation of Theorem 2.4. The glue prescription was discussed in geometric terms by Yoshida [23],[24]. This paper grew out of an attempt to understand his work.

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