THE VOLUME OF HYPERBOLIC COXETER POLYTOPES OF EVEN DIMENSION

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1. INTRODUCTION.

Let $H^n$ denote hyperbolic space of dimension $n$, and let $S$ be an index set for a finite collection of open half spaces $H_s^+$ in $H^n$ bounded by codimension one hyperplanes $H_s$. We assume that for all distinct $s, t \in S$ either $H_s \cap H_t$ is not empty and the (interior) dihedral angle of $H_s^+ \cap H_t^+$ along $H_s \cap H_t$ has size $\frac{\pi}{m_{st}}$ for certain integers $m_{st} = m_{ts} \geq 2$, or $H_s \cap H_t$ is empty while $H_s^+ \cap H_t^+$ is not empty. In the latter case we put $m_{st} = m_{ts} = \infty$ and we also put $m_{ss} = 1$. Under these assumptions the intersection $C = \bigcap_s H_s^+$ is not empty, and its closure $D$ is called a hyperbolic Coxeter polytope.

By abuse of notation let $s \in S$ also denote the reflection of $H^n$ in the hyperplane $H_s$. Now the group $W$ of motions of $H^n$ generated by the reflections $s \in S$ is discrete, and $D$ is a strict fundamental domain for the action of $W$ on $H^n$. Moreover $(W, S)$ is a Coxeter group with Coxeter matrix $M = (m_{st})$, i.e. $W$ has a presentation with generators $s \in S$ and relations $(st)^{m_{st}} = 1$ for $s, t \in S$. Let $\ell(w)$ denote the length of $w \in W$ with respect to the generating set $S$, and let $P_W(t) \in \mathbb{Z}[[t]]$ be the Poincaré series of $W$ defined by $P_W(t) = \sum_w t^{\ell(w)}$.

**Theorem:** If $D$ has finite hyperbolic volume then we have the relation

$$\frac{1}{P_W(1)} = \begin{cases} \frac{(-1)^{\#2vol_n(D)}}{vol_n(S^n)} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$
For $D$ compact this can be derived from the work by Serre on the cohomology of discrete groups [Se]. Here we obtain the result as a consequence of the differential volume formula of Schl"afli. This method was inspired by a recent paper of Kellerhals where $\text{vol}_{2n}(D)$ was computed in case $D$ is a (possibly simply or doubly truncated) orthoscheme [Ke1, IH].

The above theorem is essentially just a specialization of the Gauss-Bonnet theorem to the present situation [Ho, Fe, AW, Ch, Sa]. Nevertheless I have found it worthwhile to write these things up in some details in order to emphasize the elementary nature of this approach. For partial results on the computation of $\text{vol}_n(D)$ for $n$ odd one is referred to [Ke2, Ke3] and the references mentioned there.

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2. THE DIFFERENTIAL VOLUME FORMULA OF SCHLÄFLI AND SOME CONSEQUENCES.

Let $D$ be a spherical or a hyperbolic simplex of dimension $n$. The codimension one faces of $D$ are labeled $D_s$ for $s \in S$ an index set of cardinality $n+1$. The faces of $D$ are of the form $D_J = \bigcap_{s \in J} D_s$ with $J$ a proper subset of $S$. Clearly $D_J$ has codimension $|J|$. The interior angle of $D$ along $D_J$ is denoted by $\alpha_J$. Clearly $D_J$ is a simplicial cone in a euclidean space of dimension $|J|$, and it also determines a spherical simplex $D^J \cap S^{|J|−1}$ of dimension $|J|−1$. Note that the simplex $D$ is determined up to motions by its dihedral angles $\alpha_J := \text{vol}_1(D^J \cap S^1)$ with $J \subset S$ and $|J| = 2$.

**Theorem (Differential Volume Formula of Schl"afli):** For $J \subset S$ with $|J| = 2$ we have

$$\frac{\partial}{\partial \alpha_J}(\text{vol}_n(D)) = \frac{\varepsilon}{n-1} \text{vol}_{n-2}(D_J)$$

where $\varepsilon = 1$ if $D$ is a spherical simplex and $\varepsilon = -1$ if $D$ is a hyperbolic simplex.

In the spherical case this formula was found by Schl"afli in 1852 [Sc]. The three dimensional hyperbolic version goes back to Lobachevsky [Co]. A nice and simple proof of this formula (valid in both spherical and hyperbolic case) was given by Kneser [Kn, BH].

**Corollary:** Renormalize $\text{vol}_n(D)$ by putting $G_n(D) = \frac{\text{vol}_n(D)}{\text{vol}_n(S^n)}$. For $J \subset S$ with
\(|J| = 2\) we have
\[
\frac{\partial G_n(D)}{\partial G_1(D^J \cap S^1)} = \varepsilon G_{n-2}(D_J).
\]

Proof: This is just a reformulation of the differential volume formula using that vol\(_n(S^n) = 2\pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)^{-1}\).

QED.

**THEOREM (REDUCTION FORMULA):** With the convention \(G_{-1}(\cdot) = 1\) we have
\[
\varepsilon \frac{n}{\pi} (1 + (-1)^n) G_n(D) = \sum_{I \subseteq S} (-1)^{|I|} G_{|I| - 1}(D^I \cap S^{|I| - 1}).
\]

Proof: By induction on the dimension \(n\) of \(D\). The case \(n = 1\) is trivial. In case \(n = 2\) and \(D\) is a triangle with angles \(\alpha, \beta, \gamma\) the equality of the left hand side \(2\pi \frac{n}{\pi} \operatorname{vol}_2(D)\) and the right hand side \((1 - \frac{3}{2} + \frac{1}{2\pi}(\alpha + \beta + \gamma))\) is a familiar formula. Now suppose \(n \geq 3\). Suppose \(J \subset S\) with \(|J| = 2\). We will check that the derivatives of both sides with respect to the renormalized dihedral angle \(G_1(D^J \cap S^1)\) of \(D\) along \(D_J\) are equal. This implies that the formula is correct up to an additive constant. Indeed for the left hand side we get
\[
\varepsilon \frac{n}{\pi} (1 + (-1)^n) \frac{\partial G_n(D)}{\partial G_1(D^J \cap S^1)} = \varepsilon \frac{n+2}{\pi} (1 + (-1)^{n-2}) G_{n-2}(D_J),
\]
and for the right hand side we get
\[
\sum_{J \subset I \subset S} (-1)^{|I|} G_{|I| - 1}(D^I \cap S^{|I| - 1}) = \sum_{K \supseteq S \setminus J} (-1)^{|K|} G_{|K| - 1}(D_J^K \cap S^{|K| - 1}).
\]

Here we have used that for \(J \subset I \subseteq S\) we have \((D^I)_J = (D_J)^I \setminus J\). Hence we arrive at the reduction formula for the face \(D_J\). It remains to check the constant. In the spherical case we take \(D\) a simplex with all dihedral angles equal to \(\frac{\pi}{2}\). Hence \(G_n(D) = 2^{-n-1}\) and the reduction formula reduces in this case to the correct identity \((1 + (-1)^n)2^{-n-1} = \sum_{k=0}^n (\binom{n+1}{k})(-\frac{1}{2})^k\). This proves the reduction formula for \(D\) a spherical simplex. Taking a shrinking sequence of spherical simplices it follows that the angle sum on the right hand side of the reduction formula vanishes for a euclidean simplex \(D\). In turn this also shows that the constant matches for \(D\) a hyperbolic simplex.

QED.

For spherical simplices the reduction formula is due to Schläfli. Unaware of Schläfli’s work the reduction formula was rediscovered by Poincaré with a different and elegant
proof [Po]. The extension from a spherical to a hyperbolic simplex was made by Hopf [Ho].

**COROLLARY**: Suppose $D$ is a convex hyperbolic polytope with finite volume and of dimension $n$. Denote by $F(D)$ the collection of faces of $D$, and for $F$ a face of $D$ of codimension $|F|$ write $D^F$ for the interior angle (in $\mathbb{R}^{|F|}$) of $D$ along $F$. Then the following reduction formula holds

$$2 \cos \left( \frac{n \pi}{2} \right) G_n(D) = \sum_{F \in F(D)} (-1)^{|F|} G_{|F|-1}(D^F \cap S^{|F|-1}).$$

**Proof**: If $D$ is unbounded but with finite volume then some vertices of $D$ lie on the boundary of $H^n$. At such a cusp like vertex the size of the interior angle of $D$ equals zero. Hence by continuity we may assume that $D$ is bounded. For $D = \bigcup D_i$ a simplicial subdivision of $D$ we get

$$2 \cos \left( \frac{n \pi}{2} \right) G_n(D) = \sum_i 2 \cos \left( \frac{n \pi}{2} \right) G_n(D_i) = \sum_i \sum_{I \subseteq S_i} (-1)^{|I|} G_{|I|-1}(D^I_i \cap S^{|I|-1}) = \sum_{F \subseteq S} \sum_{(i, I) \sim F} (-1)^{|I|} G_{|I|-1}(D^I_i \cap S^{|I|-1})$$

where $F$ runs over the faces of $D$, and we write $(i, I) \sim F$ if the relative interior of $D_{i,I}$ is contained in the relative interior of $F$. Since for fixed $(i, I) \sim F$ the interior angles $D^I_j$ with $D_{j,I} = D_{i,I}$ make up an interior angle $D^F \times \mathbb{R}^{|I|-|F|}$ we conclude that

$$\sum_{(i, I) \sim F} (-1)^{|I|} G_{|I|-1}(D^I_i \cap S^{|I|-1}) = (-1)^{|F|} G_{|F|-1}(D^F \cap S^{|F|-1}),$$

because the euler characteristic of the relative interior of $F$ is equal to $(-1)^{\dim(F)}$. QED.

A direct consequence of this corollary is the Gauss-Bonnet formula for hyperbolic space forms originally derived by Hopf along these lines.

**COROLLARY**: For $\Gamma$ a group acting discretely on $H^{2n}$ with a smooth compact oriented quotient $\Gamma \backslash H^{2n}$ the euler characteristic $\chi(\Gamma \backslash H^{2n})$ of $\Gamma \backslash H^{2n}$ is given by

$$\chi(\Gamma \backslash H^{2n}) \text{vol}_{2n}(S^{2n}) = (-1)^n 2 \text{vol}_{2n}(\Gamma \backslash H^{2n}).$$
3. HYPERBOLIC COXETER GROUPS.

Let $M = (m_{st})$ be a Coxeter matrix, i.e. $m_{ss} = 1$ for all $s \in S$ and $m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}$ for all $s, t \in S$. Let $G = (g_{st})$ with $g_{st} = -2 \cos \frac{\pi}{m_{st}}$ if $m_{st}$ is finite, and if $m_{st} = \infty$ let $g_{st} = -2c_{st}$ with $c_{st} = c_{ts} \geq 1$ an additional parameter. Let $V$ be a real vector space with basis $\{\alpha_s; s \in S\}$, and equip $V$ with a symmetric bilinear form by $(\alpha_s, \alpha_t) = g_{st}$. For $\alpha \in V$ with $(\alpha, \alpha) = 2$ let $r_\alpha \in GL(V)$ be the orthogonal reflection in the hyperplane perpendicular to $\alpha$: $r_\alpha(\lambda) = \lambda - (\lambda, \alpha)\alpha$ for $\lambda \in V$. Let $(W, S)$ be the Coxeter system corresponding to the matrix $M$. The homomorphism $\sigma : W \to GL(V)$ defined by $\sigma(s) = r_s$ for $s \in S$ ($r_s$ is short for $r_{\alpha_s}$) is the (possibly deformed) geometric representation. The theory as developed for example in [Hu, Chapter 5] for the ordinary (i.e. $c_{st} = 1$ if $m_{st} = \infty$) geometric representation goes thru verbatim in the present situation.

Let $V^*$ be the dual vector space of $V$ and $\{\xi_s; s \in S\}$ the basis of $V^*$ dual to $\{\alpha_s; s \in S\}$. Hence $(\xi_s, \alpha_t) = \delta_{st}$ for all $s, t \in S$ where $(, ,)$ also denotes the pairing between $V^*$ and $V$. For $J \subset S$ we put

$$C_J := \left\{ \sum_s x_s \xi_s; x_s = 0 \text{ if } s \in J, \ x_s > 0 \text{ if } s \notin J \right\}.$$  

Clearly $C_S = \{0\}$ and $C := C_\emptyset$ is an open simplicial cone. The closure $D$ of $C$ admits a partition $D = \cup_J C_J$ and $C_J$ is a face of $D$ of codimension $|J|$. For $w \in W$ and $\xi \in V^*$ write $w(\xi)$ for $\sigma^*(w)(\xi)$. The Tits cone

$$U := \bigcup_w w(D)$$

is a convex cone in $V^*$. Moreover $C_J \cap w(C_J)$ is not empty for $I, J \subset S$ and $w \in W$ if and only if $I = J$ and $w \in W_J$. Here $W_J$ is the (parabolic) subgroup of $W$ generated by $J$.

Let $V'$ be the orthocomplement in $V^*$ of the kernel $K$ of the symmetric bilinear form $(, ,)$ on $V$. Clearly $V/K$ inherits a canonical non-degenerate symmetric bilinear form from $V$, and since $V/K$ and $V'$ are dual vector spaces this form can be transfered to $V'$. By abuse of notation we denote this form again by $(, ,)$. For $J \subset S$ let $G_J$ denote the submatrix of $G$ with indices taken from $J$. 

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PROPOSITION: Suppose the matrix $G$ is indecomposable and has smallest eigenvalue $< 0$. Let $J \subset S$ such that $G_J$ is positive definite. Then there exists a vector $\xi_J \in C_J \cap V'$ with $(\xi_J, \xi_J) < 0$, and $C_J \cap V'$ is a face of the polyhedral cone $D \cap V'$ of codimension $|J|$.

Proof. Let $J \subset S$ such that $G_J$ is positive definite. Let $1_J$ denote the matrix with 1 on the places $ss$ for $s \notin J$ and 0 elsewhere. For $t \in \mathbb{R}$ sufficiently large the matrix $G + t1_J$ is positive definite, and let $t_J$ be the infimum of these $t$. Clearly $t_J > 0$ and the matrix $G + t_J 1_J$ is positive semidefinite with nonzero kernel. By the Perron-Frobenius lemma [Hu, Section 2.6] the kernel is one dimensional and spanned by a vector $x_J$ with all coordinates $x_{J,s} > 0$ for $s \in S$. Now put

$$\alpha_J := \sum_{s \in S} x_{J,s} \alpha_s \in V, \quad \xi_J := \sum_{s \notin J} x_{J,s} \xi_s \in V^*.$$ 

Then we have on the one hand (the brackets denote the bilinear form on $V$)

$$(\alpha_J, \alpha_s) = 0 \quad \text{for} \quad s \in J$$

$$(\alpha_J, \alpha_s) = -t_J x_{J,s} \quad \text{for} \quad s \notin J,$$

and on the other hand (the brackets denote the pairing between $V^*$ and $V$)

$$(\xi_J, \alpha_s) = 0 \quad \text{for} \quad s \in J$$

$$(\xi_J, \alpha_s) = x_{J,s} \quad \text{for} \quad s \notin J.$$ 

Hence $(\alpha_J, \alpha) + (t_J \xi_J, \alpha) = 0$ for all $\alpha \in V$. In turn this implies $\xi_J \in V'$ and $(\xi_J, \xi_J) = -t_J^{-1} (\alpha_J, \xi_J) = -t_J^{-1} \sum_{s \notin J} x_{J,s}^2 < 0$. Finally the codimension of $C_J$ as face of $D$ and the codimension of $C_J \cap V'$ as face of $D \cap V'$ is equal, because the intersection $C_J \cap V'$ is transversal (immediate by induction on $|J|$).

QED.

REMARK: Suppose the matrix $G$ is indecomposable and has smallest eigenvalue $< 0$. If $J \subset S$ such that $G_J$ is positive semidefinite then it may happen that $C_J \cap V'$ is empty. However it can be shown that there exist a proper subset $I$ of $S$ containing $J$ and a vector $\xi_I \in C_I \cap V'$ with $(\xi_I, \xi_I) \leq 0$.

DEFINITION: The matrix $G$ is called hyperbolic if $G$ is indecomposable, and the smallest eigenvalue of $G$ is $< 0$, and all remaining eigenvalues of $G$ are $\geq 0$. The (irreducible) Coxeter group $(W, S)$ with Coxeter matrix $M$ is called hyperbolic if there exists a hyperbolic matrix $G$ compatible with $M$. 

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From now on assume that the matrix $G$ is hyperbolic. The set $\{\xi \in V'; (\xi, \xi) < 0\}$ consists of two connected components, and the one containing the point $\xi_0$ is denoted by $V'_-$. 

**Proposition**: The open cone $V'_-$ is contained in $U \cap V'$.

*Proof:* Let $R = \{w(\alpha_s); w \in W, s \in S\}$ be the (normalized) root system in $V$, and let $R' \subset V'$ be the “restriction” of $R$ to $V'$. It is not hard to show (and for this $G$ need not be hyperbolic) that $R'$ is a discrete subset of $\{\xi \in V'; (\xi, \xi) = 2\}$. In turn this implies that the reflection hyperplanes $H_\alpha = \{\xi \in V'; (\xi, \alpha) = 0\}$ for $\alpha \in R$ are locally finite on $V'_-$. Now for $\xi \in V'$ we have the familiar criterium: $\xi \in U$ if and only if the segment $[\xi_0, \xi]$ intersects only finitely many reflection hyperplanes $H_\alpha$ for $\alpha \in R$. Hence $V'_-$ is contained in $U \cap V'$. QED.

**Theorem**: The intersection $C_J \cap V'_-$ is not empty if and only if the matrix $G_J$ is positive definite, and in that case $C_J \cap V'_-$ is a face of $D \cap V'_-$ of codimension $|J|$.

*Proof:* The stabilizer of $\xi \in V'_-$ in the Lorentz group $O(V') = \{g \in GL(V'); g \text{ preserves } (.,.)\}$ is compact, and hence the stabilizer of $\xi \in V'_-$ in $W$ is finite (as the intersection of a compact with a discrete set). Hence if $C_J \cap V'_-$ is not empty then $W_J$ is finite, which is equivalent with $G_J$ being positive definite. The converse and the remaining part of the theorem follows from the first proposition of this section. QED.

Now let $H = \{\xi \in V'; (\xi, \xi) = -1\}$ be hyperbolic space. The hyperbolic Coxeter polytope $D \cap H$ is a fundamental domain for the action of the group $W$ on $H$. Moreover each action of an irreducible reflection group on hyperbolic space arises in this way.

**Conclusion**: The Coxeter polytope $D \cap H$ is compact if and only if $C_J \cap V'$ is empty for all $J \subsetneq S$ with $G_J$ not positive definite. Also $D \cap H$ has finite hyperbolic volume if and only if $C_J \cap V'$ is empty for all $J \subsetneq S$ with $G_J$ indefinite.

In some examples it can be cumbersome to check the above conditions. The results of this section are essentially due to Vinberg, and we refer to the nice survey paper [Vi] for a discussion of examples.

4. **Proof of the Theorem.**

Let $(W, S)$ be an arbitrary Coxeter group, and write $P_W(t) = \sum_w t^{\ell(w)}$ for the Poincaré series of $(W, S)$. The following formula due to Steinberg [St] gives an effective way of computing $P_W(t)$ by induction on $|S|$. 

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**PROPOSITION:** The Poincaré series $P_W(t)$ is a rational function of $t$ satisfying

$$\frac{1}{P_W(t^{-1})} = \sum_{J \subset S, W_J \text{ finite}} (-1)^{|J|} \frac{1}{P_{W_J}(t)}.$$ 

**Proof:** For $X \subset W$ write $P_X(t) = \sum_{w \in X} t^{\ell(w)}$. If for $J \subset S$ we write $W^J := \{ w \in W; \ell(ws) > \ell(w) \forall s \in J \}$ for the minimal length representatives for the left cosets of $W_J$ then $P_W(t) = P_{W_J}(t)P_{W^J}(t)$. For $J \subset S$ with $W_J$ finite let $N(J)$ be the length of the longest element $w_0(J)$ in $W_J$. If $J(w) := \{ s \in S; \ell(ws) < \ell(w) \}$ for $w \in W$ then $w \in W^Jw_0(J)$ for some $J \subset S$ with $W_J$ finite precisely when $J \subset J(w)$. We claim that

$$\sum_{J \subset S, W_J \text{ finite}} (-1)^{|J|} P_{W^Jw_0(J)}(t) = 1.$$ 

Indeed the contribution of $w \in W$ to the sum on the left hand side equals $\sum_{J \subset J(w)} (-1)^{|J|}$, which equals 0 unless $J(w)$ is empty. But $J(w)$ is empty precisely when $w = 1$ and the contribution becomes 1. Now we have

$$P_{W^Jw_0(J)}(t) = t^{N(J)}P_{W^J}(t) = t^{N(J)} \frac{P_W(t)}{P_{W_J}(t)} = \frac{P_W(t)}{P_{W_J}(t^{-1})},$$

and the desired formula

$$\sum_{J \subset S, W_J \text{ finite}} (-1)^{|J|} \frac{1}{P_{W_J}(t^{-1})} = \frac{1}{P_W(t)}$$

follows. QED.

The theorem of the introduction follows by applying the reduction formula of Section 2 to the Coxeter polytope with finite hyperbolic volume. Combining the theorem of Vinberg of Section 3 with the above formula of Steinberg (evaluated at $t = 1$) indeed proves the desired formula.

5. **FINAL REMARKS.**

Suppose $G$ is a discrete cocompact group of isometries of hyperbolic space $H^n$. Fix a generic point $x \in H^n$ with trivial stabilizer in $G$, and put

$$D = \{ y \in H^n; d(y, x) \leq d(y, gx) \forall g \in G \}$$
with $d$ the hyperbolic distance. The compact convex polytope $D$ is a fundamental domain for the action of $G$ on $H^n$, and the set

$$S = \{ g \in G; g(D) \cap D \text{ has codimension one} \}$$

is a finite set of generators for $G$. Let $\ell = \ell_S$ denote the length function on $G$ with respect to $S$. It was shown by Cannon that the growth series

$$P_{G,S}(t) = \sum_{g \in G} t^{\ell(g)}$$

is the power series around $t = 0$ of a rational function in $t$ [Ca]. Now it is a natural question whether the theorem from the introduction remains valid in the present situation. Although this seems to be quite often the case, there are counterexamples for dimension $n = 2$ [Pa, FP]. We refer to the latter paper for a further discussion of this problem.

**REFERENCES.**


