Matrix valued orthogonal polynomials related to compact Gel'fand pairs of rank one

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Chapter 1

Introduction

In this dissertation we present a construction of matrix valued polynomials in one variable, with special properties, out of matrix coefficients on certain compact groups. The construction generalizes the theory that relates Jacobi polynomials to certain matrix coefficients on compact groups.

In the present chapter we motivate the research in this dissertation and we give an outline of the results. To this end, we discuss in Sections 1.1 and 1.2 the notion of matrix valued orthogonal polynomials and how their simplest examples, the scalar valued orthogonal polynomials, are related to matrix coefficients on compact Lie groups. Once the definitions and notations are fixed we can formulate the research goal. In Section 1.3 we discuss a spectral problem in the theory of Lie groups. In Section 1.4 we indicate that for the solutions of the spectral problem there is a general framework of certain functions, whose structure is determined by a family of matrix valued orthogonal polynomials. In Section 1.5 we discuss the use of this construction and some of the open questions. We close this chapter with an overview of the contents of the various chapters in Section 1.6.

1.1 Orthogonal polynomials

1.1.1. Let $I=(a,b)\subset\mathbb{R}$ be an interval, possibly unbounded. Let $w:I\to\mathbb{R}$ be a non-negative integrable function satisfying $\int_I w(x)dx>0$, where dx is the Lebesgue measure and suppose that the moments are finite, i.e. $\int_I |x^n|w(x)dx<\infty$ for all $n\in\mathbb{N}$. Define the inner product $\langle\cdot,\cdot\rangle_w$ on the space of complex valued continuous functions on I by

$$\langle f,h\rangle_w=\int_I \overline{f(x)}h(x)w(x)dx.$$

A sequence of orthogonal polynomials on I with respect to $\langle\cdot,\cdot\rangle_w$ is a sequence $\{p_d : d\in\mathbb{N}\}$ with $p_d(x)\in\mathbb{C}[x]$ of degree d satisfying $\langle p_d,p_{d'}\rangle_w=c_d\delta_{d,d'}$, with $c_d\in\mathbb{R}$ a positive number. A family of orthogonal polynomials satisfies a three term recurrence relation, i.e. there are sequences of complex numbers $\{a_d : d \in \mathbb{N}\}, \{b_d : d \in \mathbb{N}\}$ and $\{c_d : d \in \mathbb{N}\}$ with the $a_d \neq 0$ for which the functional equation

$$xp_d(x) = a_d p_{d+1}(x) + b_d p_d(x) + c_d p_{d-1}(x) \tag{1.1}$$

holds on I. Conversely, given a three term recurrence relation (1.1) one can construct a sequence of polynomials $\{p_d(x) : d \in \mathbb{N}\}$. Favard's theorem gives sufficient conditions on the coefficients a_d, b_d and c_d to guarantee the existence of a positive Borel measure μ so that $\{p_d(x) : d \in \mathbb{N}\}$ is a sequence of orthogonal polynomials with respect to μ. See e.g. [Chi78] for an introduction to the theory of orthogonal polynomials.

The classical orthogonal polynomials of Hermite, Laguerre and Jacobi are also eigenfunctions of a second order differential operator that is symmetric with respect to $\langle\cdot,\cdot\rangle_w$ and Bochner showed that this additional property characterizes them among all the families of orthogonal polynomials.

1.1.2. Among the generalizations of the theory of orthogonal polynomials is the theory of matrix valued orthogonal polynomials in one variable. Matrix valued polynomials are elements in $\mathbb{C}[x] \otimes \mathrm{End}(V)$ where V is a finite dimensional complex vector space. The space $\mathbb{C}[x] \otimes \mathrm{End}(V)$ is a bimodule over the matrix algebra $\mathrm{End}(V)$. The Hermitian adjoint of $A \in \mathrm{End}(V)$ is denoted by A^*.

Let $I = (a, b) \subset \mathbb{R}$ be an interval, possibly unbounded. A matrix weight W on I is an $\mathrm{End}(V)$-valued function for which all its matrix entries are integrable functions such that $W(x)$ is positive definite almost everywhere on I. Suppose that W has finite moments, i.e., that $\int_I |x|^n |W_{i,j}(x)| dx < \infty$ for all i, j and all $n \in \mathbb{N}$, where the integration is entry wise. Define the pairing

$$\langle\cdot,\cdot\rangle_W : \mathbb{C}[x] \otimes \mathrm{End}(V) \times \mathbb{C}[x] \otimes \mathrm{End}(V) \to \mathrm{End}(V)$$

by $\langle P, Q\rangle_W = \int_I P(x)^* W(x) Q(x) dx$. The pairing $\langle\cdot,\cdot\rangle_W$ is called an $\mathrm{End}(V)$-valued inner product, i.e. it has the following properties.

- $\langle P, QS + RT\rangle_W = \langle P, Q\rangle_W S + \langle P, R\rangle_W T$ for all $S, T \in \mathrm{End}(V)$ and all $P, Q, R \in \mathbb{C}[x] \otimes \mathrm{End}(V)$,
- $\langle P, Q\rangle_W^* = \langle Q, P\rangle_W$ for all $P, Q \in \mathbb{C}[x] \otimes \mathrm{End}(V)$,
- $\langle P, P\rangle_W \geq 0$ for all $P \in \mathbb{C}[x] \otimes \mathrm{End}(V)$, i.e. $\langle P, P\rangle_W$ is positive semi-definite. If $\langle P, P\rangle_W = 0$ then $P = 0$.

The right $\mathrm{End}(V)$-module $\mathbb{C}[x] \otimes \mathrm{End}(V)$ with the pairing $\langle\cdot,\cdot\rangle_W$ is called a pre-Hilbert module, see e.g. [Lan95] and [RW98]. A family $\{P_d : d \in \mathbb{N}\}$ with $P_d \in \mathbb{C}[x] \otimes \mathrm{End}(V)$ is called a family of matrix valued orthogonal polynomials for $\langle\cdot,\cdot\rangle_W$ if (i) $\deg P_d = d$, (ii) the leading coefficient of P_d is non-singular and (iii) $\langle P_d, P_{d'}\rangle_W = S_d \delta_{d,d'}$ with $S_d \in \mathrm{End}(V)$. Note that S_d is positive definite. The existence of a family of matrix valued orthogonal polynomials for a given matrix weight $W(x)$ is proved in e.g. [GT07, Prop. 2.4], by a generalization of the Gram-Schmidt orthogonalization.

An important property of a family of matrix valued orthogonal polynomials is that they satisfy a three term recurrence relation, i.e. there are sequences $\{A_d : d \in \mathbb{N}\}, \{B_d : d \in \mathbb{N}\}$ and $\{C_d : d \in \mathbb{N}\}$ in $\mathrm{End}(V)$ with the A_d invertible, such that

$$xP_d(x) = P_{d+1}(x)A_d + P_d(x)B_d + P_{d-1}(x)C_d$$

holds on I. However, for matrix valued orthogonal polynomials with explicit expressions, the explicit computation of the coefficients may be difficult.

The weight matrix W may be conjugated by an invertible matrix $U \in \mathrm{End}(V)$ to obtain a new matrix weight UWU^*. We say that the matrix weights W and UWU^* are similar. If W is similar to a matrix weight of blocks, i.e. a matrix weight of the form $\mathrm{diag}(W_1, W_2)$ with W_1 and W_2 both matrix weights, then we say that W is decomposable. Otherwise we say that the matrix weight W is indecomposable. If the commutator

$$\{W(x) : x \in I\}' = \{J \in \mathrm{End}(V) | \forall x \in I : W(x)J = JW(x)\}$$

is one-dimensional then the weight is indecomposable.

1.1.3. Assume that $I \subset \mathbb{R}$ is an open interval. A differential operator of order n for the $\mathrm{End}(V)$-valued functions on I is given by an expression

$$\sum_{i=0}^{n} A_i(x) \frac{d^i}{dx^i},$$

with A_i an $\mathrm{End}(V)$-valued function that acts by multiplication on the left. Let us consider a second order $\mathrm{End}(V)$-linear differential operator D for which there is a family $\{P_d : d \in \mathbb{N}\}$ of matrix valued orthogonal polynomials that are eigenfunctions of D, i.e. there are elements $\Lambda_d(D) \in \mathrm{End}(V)$ such that

$$(DP_d)(x) = P_d(x)\Lambda_d(D). \tag{1.2}$$

The existence of a matrix weight W together with a second order differential operator D that has a family of matrix valued orthogonal polynomial as its eigenfunctions, has been studied by Durán [Dur97]. He shows that if the eigenvalue Λ_d acts on the same side as the $\mathrm{End}(V)$-valued functions of the differential equations, then the matrix weight W is similar to a diagonal matrix. Hence the interesting examples are those in which the eigenvalue acts on the other side, as in (1.2). At the time there were no examples available but a few years later plenty examples of these families of polynomials were found after analyzing the representation theory of the pair of Lie groups $(\mathrm{SU}(3), \mathrm{U}(2))$, see [GPT01, GPT02, GPT03, GPT04]. A pair (W, D) consisting of a matrix weight W and a second order differential operator D that has a family of matrix valued orthogonal polynomials as eigenfunctions is called a classical pair. The notion of a matrix valued classical pair was first introduced in [GPT03] and examples of classical pairs with a non-diagonalizable weight are given there. Other examples of classical pairs are given in e.g. [DG05a].

Given a matrix weight W and a family $\{P_d : d \in \mathbb{N}\}$ of matrix valued orthogonal polynomials we define $\mathcal{D}(W)$ as the algebra of differential operators that have the family of matrix valued orthogonal polynomials as eigenfunctions. This algebra is not always commutative, see e.g. Chapter 4. See also [GT07] for more results on $\mathcal{D}(W)$.

Finally, we note that the convention on left and right that we use in this dissertation is opposite to the conventional one in the literature on matrix valued orthogonal polynomials. To pass from one choice to the other one has to take Hermitian adjoints in the appropriate places. We have two arguments in favor of our choice. The first is that the examples of matrix valued polynomials that we present in this dissertation are in fact vector valued polynomials that are put carefully in a matrix. The differential operators that act on the matrix valued polynomials, actually act on the columns from the left. Moreover, the vector valued polynomials in the columns are eigenfunctions, with scalar eigenvalues, which translates to a diagonal eigenvalue for the matrix valued polynomials that acts on the right. A second argument in favor of our choice is that the theory of matrix valued orthogonal polynomials can be expressed in terms of Hilbert C*-modules, in which it is customary to have the C*-algebra-valued inner product linear in the second variable.

1.2 Realization as matrix coefficients

1.2.1. The family of Jacobi polynomials $\{P_d^{(\alpha,\beta)} : d \in \mathbb{N}\}$ in one variable is an example of a family of (scalar valued) classical orthogonal polynomials. The Jacobi polynomials have two real parameters $\alpha > -1$ and $\beta > -1$ and they are defined as follows. On $(-1,1)$ the function $w(x) = (1-x)^\alpha(1+x)^\beta$ is positive, so $\langle p, q\rangle = \int_{-1}^1 p(x)q(x)w(x)dx$ defines an inner product on the space of real-valued polynomials in one variable. Using the Gram-Schmidt process on the basis of polynomials $\{1, x, x^2, \ldots\}$, we obtain a family of orthogonal polynomials whose members are explicitly given by

$$P_d^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_d}{d!}\,{}_2F_1\left(\begin{matrix}-d, d+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right),$$

where ${}_2F_1$ is the hypergeometric function defined by

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k,$$

where $(a)_k = a\cdot(a+1)\cdots(a+k-1)$ is the Pochhammer symbol. The Jacobi polynomials satisfy a three term recurrence relation

$$xP_d^{(\alpha,\beta)}(x) = a_d P_{d+1}^{(\alpha,\beta)}(x) + b_d P_d^{(\alpha,\beta)}(x) + c_d P_{d-1}^{(\alpha,\beta)}(x)$$

of which the coefficients can be expressed in α, β and d. Moreover, the $P_d^{(\alpha,\beta)}(x)$ are eigenfunctions of the differential operator

$$(1-x)^2\frac{d^2}{dx^2} + (\beta - \alpha - (\alpha+\beta+2)x)\frac{d}{dx} \tag{1.3}$$

with eigenvalue $-d(d+\alpha+\beta+1)$.

1.2.2. For the special choice $\alpha=\beta=\frac{1}{2}$ and a renormalization, we obtain the Chebyshev polynomials of the second kind,

$$U_d(x)=(d+1)\,{}_2F_1\left(\begin{matrix}-d,d+2\\ \frac{3}{2}\end{matrix};\frac{1-x}{2}\right).$$

The orthogonality is given by $\langle U_d,U_{d'}\rangle=\frac{\pi}{2}\delta_{dd'}$ and the three term recurrence relation is given by $2xU_d(x)=U_{d+1}(x)+U_{d-1}(x)$ with starting values $U_0(x)=1$ and $U_1(x)=2x$. Moreover, $U_d(\cos(t))=\sin((d+1)t)/\sin(t)$, which is Weyl's character formula for the compact Lie group SU(2). The group SU(2) consists of unitary 2×2 matrices with determinant one and we study SU(2) via its irreducible representations: homomorphisms $\pi_H:\mathrm{SU}(2)\to\mathrm{GL}(H)$ where H is a finite dimensional complex vector space without non-trivial SU(2)-invariant subspaces. The subgroup of diagonal matrices $u_t=\mathrm{diag}(e^{it},e^{-it})$ in SU(2) is a circle that we denote by T. The irreducible representations of SU(2) are parametrized by $\ell\in\frac{1}{2}\mathbb{N}$ and the corresponding representation spaces H^ℓ are $2\ell+1$-dimensional. Moreover, the space H^ℓ has an SU(2)-invariant Hermitian inner product $\langle\cdot,\cdot\rangle$. Let us denote an irreducible representation by τ_ℓ. Such a representation is completely determined by the restriction of its character $\chi_\ell:T\to\mathbb{C}:u_t\mapsto\mathrm{tr}(\tau_\ell(u_t))$. We have $\chi_\ell(u_t)=U_{2\ell}(\cos(t))$, which shows that the characters of SU(2) are Chebyshev polynomials (in the coordinate $x=\cos(t)$).

1.2.3. The restricted characters of SU(2) can also be obtained in another way. Let us denote $G=\mathrm{SU}(2)\times\mathrm{SU}(2)$ with subgroup $K\cong\mathrm{SU}(2)$ embedded via $i:K\to G:k\mapsto(k,k)$. The irreducible representations of G are parametrized by two half integers (ℓ_1,ℓ_2). The pair (ℓ_1,ℓ_2) corresponds to the representation $\pi_{\ell_1,\ell_2}=\tau_{\ell_1}\otimes\tau_{\ell_2}$ acting on the space $H^{\ell_1,\ell_2}=H^{\ell_1}\otimes H^{\ell_2}$ and $\langle\cdot,\cdot\rangle$ denotes a G-invariant Hermitian inner product on H^{ℓ_1,ℓ_2}. The representation $\pi_{\ell_1,\ell_2}\circ i:K\to\mathrm{GL}(H^{\ell_1,\ell_2})$ is not irreducible in general, but it is isomorphic to the direct sum of irreducible representations of K, given by the familiar Clebsch-Gordan rule

$$H^{\ell_1,\ell_2}\cong\bigoplus_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2}H^\ell. \tag{1.4}$$

The trivial representation $K\to\mathrm{GL}(\mathbb{C})$ corresponds to $\ell=0$. The trivial representation occurs in the decomposition (1.4) if and only if $\ell_1=\ell_2$. In this case, with $\ell_1=\ell_2=\ell$, there is a vector $v_0\in H^{\ell,\ell}$ of length one with the property that $\pi_{\ell,\ell}(k)v_0=v_0$ for all $k\in K$. Consider the matrix coefficient

$$m^{\ell,\ell}_{v_0,v_0}:G\to\mathbb{C}:g\mapsto\langle v_0,\pi_{\ell,\ell}(g)v_0\rangle,$$

which is K-bi-invariant. It is called an elementary zonal spherical function associated to $\pi_{\ell,\ell}$ and it is related to the character χ_ℓ of SU(2) as follows. Let $\psi:G\to\mathrm{SU}(2)$ be defined by $\psi(k_1,k_2)=k_1k_2^{-1}$. The fiber of ψ at the identity is K and we obtain a diffeomorphism $\overline{\psi} : G/K \to \mathrm{SU}(2)$. We have $\chi_\ell \circ \overline{\psi} = (2\ell+1)m^{\ell,\ell}_{v_0,v_0}$. In fact, the inverse image A of T under the map $\overline{\psi}$ is a one-dimensional torus. The group G admits a decomposition $G = KAK$, i.e. we can write every element $g \in G$ as a product $g = k_1ak_2$ for some $k_1, k_2 \in K$ and $a \in A$. This implies that $m^{\ell,\ell}_{v_0,v_0}$ is completely determined by its restriction to A. The Chebyshev polynomials are now related to the elementary zonal spherical functions of the pair (G, K).

1.2.4. In 1.2.3 we have seen that the Chebyshev polynomials of the second kind can be realized as matrix coefficients on a compact group, restricted to a suitable torus. It can be shown that the second order differential operator for which the Chebyshev polynomials are eigenfunctions comes from a differential operator, the Casimir operator, that acts on functions on $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Koornwinder [Koo85] observed that in a similar fashion, certain matrix coefficients of $\mathrm{SU}(2)\times\mathrm{SU}(2)$, carefully arranged in a matrix, when restricted to a suitable torus, give matrix valued polynomials (in coordinate $x = \cos(t)$). In Chapters 4 and 5, the polynomials of Koornwinder from [Koo85] are further developed into a family of matrix valued orthogonal polynomials satisfying a second order differential operator. In [GPT02] and [PTZ12] families of matrix valued orthogonal polynomials are found by means of solving differential equations associated to the pair $(\mathrm{SU}(3), \mathrm{U}(2))$ and $(\mathrm{SO}(4), \mathrm{SO}(3))$ respectively.

1.2.5. The goal of this dissertation is to present a uniform construction of a family of matrix valued orthogonal polynomials $\{P_d : d \in \mathbb{N}\}$ together with a commutative algebra $\mathbb{D}$ of differential operators, whose elements have the polynomials P_d as joint eigenfunctions, with the eigenvalues being diagonal matrices acting on the right. The construction generalizes the theory that relates Jacobi polynomials in one variable to certain matrix coefficients on compact groups. Moreover, the examples that we discussed in 1.2.4 fit into our construction.

1.3 The spectral problem

1.3.1. Our construction is based on the analysis of a spectral problem in the theory of compact Lie groups. Given a compact connected Lie group we identify the irreducible representations by their highest weights. From this point on we assume that the reader is more or less familiar with these notions.

The main ingredient of our construction is a triple (G, K, μ) consisting of a compact connected Lie group G, a closed connected subgroup $K \subset G$ and an irreducible K-representation τ of highest weight μ with the property that the induced G-representation $\mathrm{ind}^G_K(\tau)$ is a multiplicity free direct sum of irreducible representations of G. Such a triple is called a multiplicity free triple. The only examples that we know have the following additional property: μ lies in a face F of the positive integral weights with the property that for every element $\mu' \in F$ the triple (G, K, μ') is a multiplicity free triple. Such a triple (G, K, F) is called a multiplicity free system.

G		K	λ_{sph}	faces F
$\mathrm{SU}(n+1)$	$n \geq 1$	$\mathrm{U}(n)$	$\varpi_1 + \varpi_n$	any
$\mathrm{SO}(2n)$	$n \geq 2$	$\mathrm{SO}(2n-1)$	ϖ_1	any
$\mathrm{SO}(2n+1)$	$n \geq 2$	$\mathrm{SO}(2n)$	ϖ_1	any
$\mathrm{Sp}(2n)$	$n \geq 3$	$\mathrm{Sp}(2n-2) \times \mathrm{Sp}(2)$	ϖ_2	$\dim F \leq 2$
F_4		$\mathrm{Spin}(9)$	ϖ_1	$\dim F \leq 1$ or $F = \mathbb{N}\omega_1 + \mathbb{N}\omega_2$
$\mathrm{Spin}(7)$		G_2	ϖ_3	$\dim F \leq 1$
G_2		$\mathrm{SU}(3)$	ϖ_1	$\dim F \leq 1$

Table 1.1: Compact multiplicity free systems of rank one. In the third column we have given the highest weight $\lambda_{\mathrm{sph}} \in P_G^+$ of the fundamental zonal spherical representation in the notation for root systems of Bourbaki [Bou68, Planches], except for the case $(G, K) = (\mathrm{SO}_4(\mathbb{C}), \mathrm{SO}_3(\mathbb{C}))$ where G is not simple and $\lambda_{\mathrm{sph}} = \varpi_1 + \varpi_2 \in P_G^+ = \mathbb{N}\varpi_1 + \mathbb{N}\varpi_2$.

Examples of multiplicity free systems are $(G, K, 0)$ with (G, K) a Gel'fand pair. Indeed, the definition of a Gel'fand pair is that the trivial K-representation τ_0 occurs with multiplicity at most one in the restriction of any irreducible G-representation to K. Using Frobenius reciprocity we see that this is the same as $\mathrm{ind}_K^G(\tau_0)$ being a multiplicity free G-representation. The rank of a Gel'fand pair is the dimension of G/K minus the dimension of the maximal K-orbit in G/K.

The spectral problem is to find all multiplicity free systems modulo a suitable equivalence relation among them. We have solved this problem successfully for the triples (G, K, F) with (G, K) a rank one Gel'fand pair. The classification is presented in Table 1.1. In [HNOO12] a classification of multiplicity free systems (G, K, F) is presented with (G, K) a symmetric pair of arbitrary rank.

1.3.2. We have used two techniques to prove the classification of multiplicity free systems (G, K, F) with (G, K) a rank one Gel'fand pair. The first one amounts to translating the problem in terms of complex algebraic groups. The notion of Gel'fand pairs of compact groups translates to that of spherical pairs of reductive groups. A pair $(G_{\mathbb{C}}, K_{\mathbb{C}})$ of reductive algebraic groups is called a spherical pair if a Borel subgroup $B \subset G_{\mathbb{C}}$ has an open orbit in the quotient $G_{\mathbb{C}}/K_{\mathbb{C}}$. A face F of the dominant integral weights of $K_{\mathbb{C}}$ corresponds to a parabolic subgroup $P \subset K_{\mathbb{C}}$. The problem of showing that some triple (G, K, F) is or is not a multiplicity free system boils down to show the (non)-existence of an open orbit of a Borel subgroup of $G_{\mathbb{C}}$ in the space $G_{\mathbb{C}}/P$. We used this technique to prove the statement on the multiplicity free systems involving the symplectic groups.

The second technique is to analyze the branching rules in a sophisticated manner. The compact Gel'fand pairs of rank one have the property that there is a one-dimensional torus $A \subset G$ such that there is a decomposition $G = KAK$. This means that for each $g \in G$ there are elements $k_1, k_2 \in K$ and $a \in A$ with $g = k_1 a k_2$. Let $M = Z_K(A)$, the centralizer of A in K. Under the hypothesis that (G,K) is a rank one Gel'fand pair we show that (G,K,F) is a multiplicity free system if and only if the restriction to M of every irreducible K-representation of highest weight $\mu \in F$ decomposes multiplicity free. We use this technique to prove the statements on the multiplicity free systems involving exceptional groups.

1.3.3. The determination of the multiplicity free systems (G,K,F) with (G,K) a rank one Gel'fand pair is not enough for our purposes. We need more precise information. Given such a system (G,K,F) we want to determine, for a fixed $\mu \in F$, the irreducible G-representations whose restriction to K contains an irreducible representation of highest weight μ. We denote the dominant integral weights of G and K by P_G^+ and P_K^+ respectively. The highest weights of the G-representations that contain a K-representation of given weight μ is denoted by $P_G^+(\mu)$.

A branching rule for a pair (G,K) of compact groups describes the irreducible K-representations that occur in the restriction of any irreducible G-representation. Determination of $P_G^+(\mu)$ amounts to inverting the branching rules for the pair (G,K).

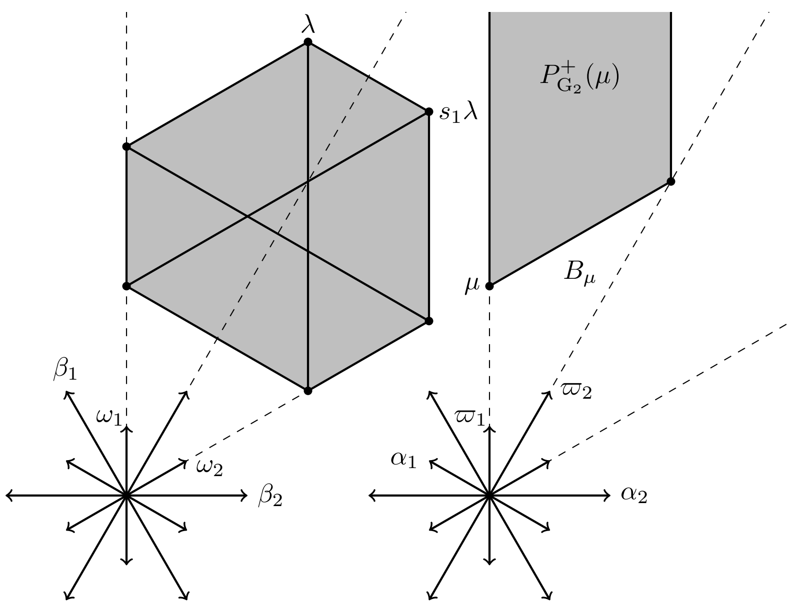

Figure 1.1: Branching from G_2 to $\mathrm{SL}_3(\mathbb{C})$ on the left and the μ-well on the right.

1.3.4. Consider the pair $(\mathrm{G}_2, \mathrm{SU}(3))$, which is a Gel'fand pair of rank one. The irreducible representations of G_2 and $\mathrm{SU}(3)$ are parametrized by $P_{G_2}^+$ and $P_{\mathrm{SU}(3)}^+$ which are monoids generated over $\mathbb{N}$ by $\{\varpi_1, \varpi_2\}$ and $\{\omega_1, \omega_2\}$ respectively, which are depicted in Figure 1.1. The picture on the left in Figure 1.1 shows in the gray area all the SU(3)-representations that occur in the restriction of a G_2-representation of highest weight λ. More precisely, we should only look at the integral points in the gray area. There are also multiplicities involved: the multiplicities are one on the outer hexagon, and increase by one on each inner shell hexagon, until the hexagon becomes a triangle and then multiplicities stabilize. From this picture we see that the two faces F that yield multiplicity free systems, are $\omega_1\mathbb{N}$ and $\omega_2\mathbb{N}$. The picture on the right in Figure 1.1 also shows what the $P^+_{\mathrm{SU}(3)}(\mu)$ should be for the indicated μ.

1.3.5. We close this section with an observation that is very important for the construction of matrix valued orthogonal polynomials. For $\mu = 0$, the set $P^+_G(0)$ is a monoid over $\mathbb{N}$ whose generator λ_{sph} is called the spherical weight. This means that the restriction to K of the irreducible G-representation of highest weight λ_{sph} contains the trivial K-representation. Moreover, λ_{sph} is the smallest weight with this property. We denote this representation by π_{sph}.

We show that if $\lambda \in P^+_G(\mu)$, then $\lambda + \lambda_{\mathrm{sph}} \in P^+_G(\mu)$. For every $\lambda \in P^+_G(\mu)$ there is a minimal element $\nu \in P^+_G(\mu)$ with the property that $\lambda = \nu + d\lambda_{\mathrm{sph}}$ and $\nu - \lambda_{\mathrm{sph}} \notin P^+_G(\mu)$. The set of these elements ν is finite, and we denote it by B_μ.

The set $P^+_G(\mu)$ is of the form $\mathbb{N} \times B_\mu$, i.e. there is an isomorphism of sets

$$\lambda : \mathbb{N} \times B_\mu \to P^+_G(\mu) : (d, \nu) \mapsto \nu + d\lambda_{\mathrm{sph}}.$$

See the picture on the right in Figure 1.1 for an illustration of the set $P^+_G(\mu)$. We call $P^+_G(\mu)$ the μ-well and we call B_μ the bottom of the μ-well.

We say that an element $\lambda(d, \nu) \in P^+_G(\mu)$ is of degree d. The finite set $B_\mu \subset P^+_G$ inherits the standard partial ordering of P^+_G, and together with the standard ordering on $\mathbb{N}$, we obtain a partial ordering $\preceq_\mu$ on $P^+_G(\mu)$:

$$\lambda(d_1, \nu_1) \preceq_\mu \lambda(d_2, \nu_2) \Leftrightarrow d_1 < d_2 \vee (d_1 = d_2 \wedge \nu_1 \preceq \nu_2).$$

This ordering first looks at the degree and then at the partial ordering on the bottom B_μ. Every μ-well that we encounter in Table 1.1, except[1] for the case $G = \mathrm{F}_4$, is of this shape and we are able to determine all the bottoms explicitly.

Let (G, K, F) be a multiplicity free system with (G, K) a Gel'fand pair of rank one, with G other than F_4, and let $\mu \in F$. Let $\lambda \in P^+_G(\mu)$ and let π be an irreducible G-representation of highest weight λ. Consider the tensor product $\pi \otimes \pi_{\mathrm{sph}}$. Suppose that π' is an irreducible representation of G of highest weight λ' that occurs in the decomposition of $\pi \otimes \pi_{\mathrm{sph}}$. If $\lambda' \in P^+_G(\mu)$, then $\lambda - \lambda_{\mathrm{sph}} \preceq_\mu \lambda' \preceq_\mu \lambda + \lambda_{\mathrm{sph}}$, which we prove in Theorem 2.4.2.

[1] A few days before printing we discovered that there are good faces other than $\{0\}$ in this case. Unfortunately there was no time left to analyze the μ-wells in these cases.

1.4 The general construction in a nutshell

In this section we fix a multiplicity free system (G, K, F) with (G, K) a rank one Gel'fand pair and G not of type F_4. We also fix an element $\mu \in F$ and an irreducible K-representation τ of highest weight μ. Let V denote the representation space of τ.

1.4.1. Let $R(G)$ denote the algebra of matrix coefficients on G and consider the action of $K \times K$ on $R(G) \otimes \mathrm{End}(V)$ given by

$$((k_1, k_2)\Phi)(g) = \tau(k_1)\Phi(k_1^{-1} g k_2)\tau(k_2)^{-1},$$

where Φ is a map $G \to \mathrm{End}(V)$. The functions that are fixed under this action are called spherical functions of type μ. Let E^μ denote the space of spherical functions of type μ. Typically, a spherical function of type μ is obtained as follows. Let $\lambda \in P_G^+(\mu)$ and let π be the corresponding representation of G in the space V_λ. There is a K-equivariant embedding $b : V \to V_\lambda$ and its adjoint $b^* : V_\lambda \to V$ is also K-equivariant. Define $\Phi_\mu^\lambda : G \to \mathrm{End}(V)$ by

$$\Phi_\lambda^\mu(g) = b^* \circ \pi(g) \circ b.$$

The function Φ_λ^μ is called an elementary spherical function of type μ associated to $\lambda \in P_G^+(\mu)$. The elementary spherical functions of type μ constitute a basis of E^μ. The partial ordering $\preceq_\mu$ on $P_G^+(\mu)$ induces a grading on E^μ.

A special instance of an elementary μ-spherical function is $\Phi_{\lambda_{\mathrm{sph}}}^0$, which we call the fundamental zonal elementary spherical function. We denote this (scalar) valued function by ϕ. Note that ϕ is K-bi-invariant. The space E^0 of spherical functions of type 0 is called the space of zonal spherical functions. E^0 is the space of K-bi-invariant functions in $R(G)$ and it is isomorphic to $\mathbb{C}[\phi]$.

1.4.2. Let $\lambda \in P_G^+(\mu)$. Then the function $g \mapsto \phi(g)\Phi_\lambda^\mu(g)$ is in E^μ. In fact, we show that it is a linear combination of functions $\Phi_{\lambda'}^\mu$ with $\lambda' \in P_G^+(\mu)$ satisfying $\lambda - \lambda_{\mathrm{sph}} \preceq_\mu \lambda' \preceq_\mu \lambda + \lambda_{\mathrm{sph}}$. This implies that E^μ is an E^0-module. Moreover, multiplication with ϕ respects the grading. We deduce that E^μ is a free E^0-module with $|B_\mu|$ generators, B_μ being the bottom of the μ-well. Hence we can express an elementary spherical function of type μ as an E^0-linear combination of the $|B_\mu|$ elementary spherical functions of degree zero.

1.4.3. The final step in the construction is restricting the μ-spherical functions to the torus $A \subset G$ that we have discussed before. The torus A is one-dimensional and we have a decomposition $G = KAK$. In view of their transformation behavior, the spherical functions of type μ are completely determined by their restrictions to A. We show that the restricted elementary spherical functions of type μ of degree zero are linearly independent in each point of a dense subset $A_{\mu-\mathrm{reg}}$ of A. Applying the base change in each point to Φ_λ^μ gives a family of vector valued polynomials functions on A that we denote by Q_λ^μ. The functions Q_λ^μ are polynomial in ϕ and if $\lambda = \lambda(d, \nu)$ is of degree d then there is exactly one entry of Q_λ^μ that is a polynomial in ϕ of degree d.

1.4.4. The matrix valued polynomials Q^{μ}_{d} are obtained by arranging the $|B_{\mu}|$ vector valued functions Q^{μ}_{λ} of the same degree in a matrix. This yields a matrix valued function whose entries are polynomials in ϕ. Upon writing $Q^{\mu}_{d}(a) = P^{\mu}_{d}(x)$ for $x = \phi(a)$ we obtain a family of matrix valued polynomials whose leading coefficient (that of x^{d}) is invertible.

Let $I \subset \mathbb{R}$ denote the image of A under the fundamental zonal spherical function ϕ. We show that (possibly after rescaling $I = [-1, 1]$), $\{P^{\mu}_{d} : d \in \mathbb{N}\}$ is a family of matrix valued orthogonal polynomials with respect to a matrix weight $W^{\mu}(x)$ that is of the form $(1-x)^{\alpha}(1+x)^{\beta}\widetilde{W}^{\mu}(x)$ with suitable values for α and β and with $\widetilde{W}^{\mu}(x)$ a self-adjoint matrix valued polynomial. Moreover, we show that there is a commutative algebra of differential operators $\mathbb{D}(W^{\mu})$ for which the polynomials P^{μ}_{d} are simultaneous eigenfunctions, i.e. $DP_d = P_d\Lambda_d(D)$ for a diagonal matrix $\Gamma_d(D)$ whose entries are polynomial in d. This algebra is a quotient of the algebra $U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{k}_{\mathbb{C}}}$, the subalgebra of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, that centralizes $\mathfrak{k}_{\mathbb{C}}$. The map to $\mathbb{D}(W^{\mu})$ is given by taking radial parts, a conjugation by the base change and a substitution of variables. Note that the Casimir operator is in $U(\mathfrak{g}_{\mathbb{C}})^{\mathfrak{k}_{\mathbb{C}}}$. We denote its image in $\mathbb{D}(W^{\mu})$ by $D_{\Omega,\mu}$. We have constructed in a uniform way a family of matrix valued orthogonal polynomials. Moreover, the pairs $(W^{\mu}, D_{\Omega,\mu})$ are classical.

1.5 Applications and further studies

1.5.1. The construction of matrix valued orthogonal polynomials that we presented in Section 1.4 boils down to the construction of the particular Jacobi polynomials that we discussed in 1.2.3, if we take $\mu = 0$. Moreover, we show that the matrix valued orthogonal polynomials of Grünbaum et. al. [GPT02] are closely related to the ones constructed in this dissertation, for the corresponding multiplicity free triple.

1.5.2. In the particular examples [GPT02] and Chapters 4 and 5 of this dissertation, the results are very explicit, i.e. one can obtain very explicit expressions for the coefficients that are involved. The coefficients that are involved are mostly Clebsch-Gordan coefficients for the various tensor product decompositions and it is not likely that we can be as specific for the other multiplicity free triples.

1.5.3. It would be desirable to have a better understanding of the following aspects of the weight functions. The first is that we would like to know whether or not $W^{\mu}(x)$ is indecomposable. In the case $(\mathrm{SU}(2)\times\mathrm{SU}(2), \mathrm{SU}(2))$ that we studied in Chapter 4 we know that the weight decomposes into two indecomposable blocks. This decomposition is closely related to the Cartan-involution for the symmetric pair $(\mathrm{SU}(2)\times\mathrm{SU}(2), \mathrm{SU}(2))$ and we expect that the only decomposable matrix weights W^{μ} come from multiplicity free triples $(\mathrm{SO}(2n), \mathrm{SO}(2n-1), \mu)$. Moreover, we expect that they are decomposable into no more than two blocks.

We are also interested in the points where the determinant of W^{μ} is zero. We expect that the critical values of ϕ are the only points where this happens. Indeed, if there are more singularities then we expect that the conjugated differential operators would detect them. However, there seem to be no other singularities. It would be interesting to learn more about this matter.

1.5.4. The examples that we have constructed have many hidden properties, see e.g. all the very particular results in Chapter 5 on decompositions of the weight. We found for example the LDU-decomposition of the weight W, i.e. a decomposition $W(x) = L(x)D(x)L(x)^*$ with $L(x)$ a lower triangular matrix valued polynomial and D a diagonal matrix valued function. It would be interesting to see what the LDU-decomposition of the weight means on the level of the Lie groups. Having a good control over these examples may be fruitful if we want to understand the general matrix valued polynomials of Jacobi type, introduced by Grünbaum and Durán in [DG05a].

1.5.5. The algebras $\mathbb{D}(W^\mu)$ and $\mathcal{D}(W)$ are not yet understood in sufficient detail. It would be interesting to learn about a global description of these algebras and what the precise role is of the elements other than the images of the Casimir operator.

1.6 This dissertation

This dissertation consists of two parts. In Part I we make the results that we discussed in Sections 1.3 and 1.4 precise. This material originates from collaborating with Gert Heckman. A joint article is in preparation.

Part II consists of two articles [KvPR11, KvPR12] that are written in collaboration with Erik Koelink and Pablo Román. The article in Chapter 4 has been accepted for publication by International Mathematical Research Notices with the title "Matrix-valued orthogonal polynomials related to $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$". The article in Chapter 5 has been submitted with the title "Matrix-valued orthogonal polynomials related to $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$, II". Time constraints forced the author to put the articles integrally in this dissertation, instead of redirecting them into one new chapter.

- **Chapter 2: Multiplicity Free Systems.** In this chapter we define multiplicity free triples (G, K, μ) and multiplicity free systems (G, K, F). We classify the multiplicity free systems (G, K, F) with (G, K) a rank one Gel'fand pair. Moreover, we describe the spectrum $P_G^+(\mu)$ that we associate to a multiplicity free triple (G, K, F) with G other than F_4 and we equip it with a partial ordering. We show that this partial ordering behaves well with respect to taking the tensor product with π_{sph}, the fundamental spherical representation.

- **Chapter 3: Matrix Valued Polynomials associated to Multiplicity Free Systems.** Given a multiplicity free triple (G, K, μ) we introduce spherical functions of type μ. We explain the construction of a family of matrix valued orthogonal polynomials starting from a multiplicity free system (G, K, F), with (G, K) a rank one Gel'fand pair with G other than F_4, and an element $\mu \in F$. We obtain some explicit results about the matrix weight and we obtain a commutative algebra of differential operators for which the matrix valued polynomials are simultaneous eigenfunctions.

- **Chapter 4: Matrix Valued Orthogonal Polynomials related to** $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$**, I.** We continue the work of Koornwinder by constructing families of matrix valued orthogonal polynomials out of the vector valued polynomials in [Koo85]. We show that Koornwinder's polynomials are the same as the ones we construct, but our construction in this chapter is closely related to the construction in Chapter 3. Using Magma we obtain expressions for second order differential operators that have the matrix valued orthogonal polynomials as eigenfunctions. In fact, we find non-commuting differential operators and differential operators of order one among them. We also show that the matrix weight decomposes into no more than two blocks of matrix weights

- **Chapter 5: Matrix Valued Orthogonal Polynomials related to** $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$**, II.** We continue study of Chapter 4 by deducing the differential operators from the group theory. This amounts to calculating radial parts as in [CM82], a base change and a substitution of variables. We explain where the differential operator of order one comes from. Moreover, we find a remarkable LDU-decomposition for the matrix weight. This decomposition easily implies our previous conjecture about the determinant of the matrix weight. The LDU-decomposition also provides a link to the (scalar valued) Gegenbauer and Racah polynomials. Moreover, we express the matrix valued polynomials as matrix valued hypergeometric functions in the sense of Tirao [Tir03].

Part I

The General Framework

Chapter 2

Multiplicity Free Systems

2.1 Introduction

2.1.1. Let (G, K) be a Gel'fand pair of algebraic reductive groups over $\mathbb{C}$, i.e. the restriction of any irreducible rational representation $\pi : G \to \mathrm{GL}(V)$ decomposes into a direct sum of irreducible rational representations of K and the multiplicity of the trivial K-representation is at most one. The algebra of matrix coefficients of G that are K-bi-invariant is isomorphic to a commutative algebra of Krull-dimension r, the rank of the pair (G, K). If we assume furthermore that (G, K) is a symmetric pair with G simply connected and both G and K connected, then the algebra of matrix coefficients of G that are K-bi-invariant is isomorphic to a polynomial algebra with r generators. Under this isomorphism the matrix coefficients can be identified with Jacobi polynomials. The Jacobi polynomials in one variable enjoy many special properties: they satisfy a three term recurrence relation, they are eigenfunctions of a second order differential operator and if we consider them on the compact form $G_0 \subset G$ then they lead to a system of orthogonal polynomials. Moreover, all the explicit expressions are known, see [KLS10, Ch. 9.8]. In Chapter 3 we generalize the construction of these Jacobi polynomials in one variable to a construction of vector and matrix valued polynomials that enjoy properties similar to those of the Jacobi polynomials. In the present chapter we investigate the ingredients for this construction which are multiplicity free triples (Definition 2.1.8). A multiplicity free triple consists of a pair of connected reductive complex algebraic groups $K \subset G$ and an irreducible rational representation of K that plays the role of the trivial representation in a Gel'fand pair: it occurs with multiplicity at most one in the decomposition of the restriction of any irreducible rational representation of G to K. The multiplicity free triples that we know have the property that we may vary the K-type in a monoid that is contained in a face of a Weyl chamber of K, without losing the multiplicity property. This defines what we call a multiplicity free system (Definition 2.2.1) and we study these systems by means of parabolic subgroups of K. We classify the multiplicity free systems of rank one and we obtain the explicit data concerning the involved representations that we need in Chapter 3.

All algebraic groups and varieties in this section are over the complex numbers. References for notions in the theory of algebraic groups are [Bor91, Hum81, Spr09].

Definition 2.1.2. *Let G be a connected reductive algebraic group and let X be a homogeneous G-variety. Then X is called a spherical G-variety if X admits an open B orbit for some Borel subgroup $B \subset G$. Let $H \subset G$ be a closed connected subgroup. The pair (G, H) is called a spherical pair if the quotient G/H is a spherical G-variety. We call a spherical pair (G, K) reductive if both G and K are reductive.*

Proposition 2.1.3. *Let (G, H) be a spherical pair. If $H' \subset G$ is a subgroup with $H \subset H'$ then (G, H') is also a spherical pair.*

Proof. There exists a Borel $B \subset G$ such that B has an open orbit in G/H. Equivalently, the action of $B \times H$ on G given by $(b, g, h) \mapsto bgh^{-1}$ has an open orbit in G which implies that the action of $B \times H'$ on G also has an open orbit. Hence (G, H') is a spherical pair. □

2.1.4. Let (G, K) be a reductive spherical pair and let $X = G/K$, which is an affine variety with coordinate ring $\mathbb{C}[X] = \mathbb{C}[G]^K$, the algebra of matrix coefficients that are invariant for the right regular representation of K. As a G-module, $\mathbb{C}[X]$ is isomorphic to a direct sum of irreducible rational G-representations. The fact that X is a G-spherical variety is equivalent to $\mathbb{C}[X]$ being a multiplicity free G-module, i.e. the multiplicity of any irreducible representation of G in the decomposition of $\mathbb{C}[X]$ is at most one. The irreducible representations that occur in the decomposition of $\mathbb{C}[X]$ are precisely those with a vector v in the representation space V that is fixed by K. It follows that a pair (G, K) of connected reductive groups is a Gel'fand pair if and only if it is a reductive spherical pair. In general a G-homogeneous variety X is a spherical variety if and only if for every G-homogeneous line bundle $L \to X$ the space of global sections $\Gamma(X, L)$ decomposes multiplicity free as a G-module.

2.1.5. An isogeny $c : G \to G'$ is a surjective group homomorphism with finite kernel. Two groups G, G' are called isogenous if there is an isogeny in one of the directions and this generates an equivalence relation. A pair (G, K) with $K \subset G$ is isogenous to a pair (G', K') if G is isogenous to G' and restriction to K and K' yields an isogeny. If (G_1, K_1) and (G_2, K_2) are spherical pairs then so is $(G_1 \times G_2, K_1 \times K_2)$. A spherical pair of the form $(G_1 \times G_2, K_1 \times K_2)$ is called decomposable. A spherical pair that is not isogenous to a decomposable spherical pair is called indecomposable.

Mikityuk [Mik86] and Brion [Bri87] classified (independently) the indecomposable reductive spherical pairs modulo isogenies. One part of the classifications had already been done by Krämer [Krä79] who classified the spherical pairs (G, K) with G simple and K reductive. The list of indecomposable reductive spherical pairs consists of 22 families (G_n, K_n) of which 15 have G_n simple, and 21 pairs (G, K) of which 20 have G simple.

2.1.6. We fix notations for the representations of a connected reductive group G over $\mathbb{C}$ of semisimple rank n. Let $B \subset G$ be a Borel subset and let $T \subset B$ be a maximal torus. The Lie algebra of G is denoted by $\mathfrak{g}$ and the Lie algebra of T by $\mathfrak{t}$. Let $\mathfrak{z}_{\mathfrak{g}}$ denote the center of $\mathfrak{g}$ and denote $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. We have $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}} \oplus \mathfrak{g}'$. The roots of G are denoted by R_G. The choice for B fixes a notion of positivity and we get a system of simple roots Π_G and a set of positive roots $R_G^+ \subset R_G$. The Weyl group of the root system is denoted by W_G. The fundamental weights that correspond to the roots $\alpha_i \in R_G^+$ are denoted by ϖ_i. The dual $\mathfrak{z}_{\mathfrak{g}}^\vee$ of $\mathfrak{z}_{\mathfrak{g}}$ is isomorphic to a number of $\dim \mathfrak{z}_{\mathfrak{g}}$ copies of $\mathbb{C}$. The weight lattice P_G is a full sublattice of $\mathbb{Z}^{\dim \mathfrak{z}_{\mathfrak{g}}} \oplus \mathrm{span}_{\mathbb{Z}}\{\varpi_i : i = 1, \ldots, n\}$ because there is a finite covering of G by $\widetilde{G}$, the direct product of the center of G and the simply connected covering $\widetilde{G'}$ of G', the semisimple part of G. Denote $V = P_G \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\langle \cdot, \cdot \rangle$ denote the pairing on V dual to the Killing form. The choice for B also determines a positive Weyl chamber $\mathcal{C}_G^+$ and a set of positive weights P_G^+. A relatively open face $f \subset \overline{\mathcal{C}_G^+}$ gives a subset $F = f \cap P_G^+$ of P_G^+ which we call a (relatively open) face of P_G^+. The elements in P_G^+ correspond to irreducible rational representations of G via the theorem of the highest weight. If G is semisimple we denote by $\lambda \preceq \lambda'$ the partial ordering on P_G defined by $\lambda' - \lambda$ being an $\mathbb{N}$-linear combination of positive roots.

2.1.7. A reductive spherical pair (G, K) gives an affine G-homogeneous space $X = G/K$ and the G-module $\mathbb{C}[X]$ decomposes multiplicity free. The highest weights of the irreducible representations that occur in the decomposition constitute a subset $P_G^+(0) \subset P_G^+$ where the zero refers to the highest weight of the trivial K-representation. The set $P_G^+(0)$ is called the spherical monoid (over $\mathbb{N}$) because it is closed under addition and $0 \in P_G^+(0)$. The number of generators of $P_G^+(0)$ is in general $\geq r$, the rank of the pair (G, K). However, the dimension of the cone generated by $P_G^+(0)$ is always r. Krämer [Krä79] provides the generators of $P_G^+(0)$ for the reductive spherical pairs (G, K) with G simple for the cases in which $P_G^+(0)$ is generated by r elements. In the two other cases he provides the general expression for the elements in $P_G^+(0)$.

Let (G, K) be a reductive pair, let $\mu \in P_K^+$, $\lambda \in P_G^+$ and let τ_μ and π_λ be irreducible representations of K and G of highest weights μ and λ respectively. We denote $m_\lambda^{G,K}(\mu) = [\pi_\lambda|_K : \tau_\mu]$. the number of copies of τ_μ in the restriction of π_λ to K. The following definition generalizes the notion of a Gel'fand pair.

Definition 2.1.8. *A triple (G, K, μ) consisting of a connected reductive algebraic group G, a closed connected reductive subgroup $K \subset G$ and a weight $\mu \in P_K^+$ is called a multiplicity free triple if $m_\lambda^{G,K}(\mu) \leq 1$ for all $\lambda \in P_G^+$.*

Example 2.1.9. Consider the spherical pair $(\mathrm{SL}_{n+1}(\mathbb{C}), \mathrm{GL}_n(\mathbb{C}))$ where the embedding $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{SL}_{n+1}(\mathbb{C})$ is given by $x \mapsto \mathrm{diag}(x, \det x^{-1})$. Let π be any irreducible rational representation of $\mathrm{SL}_{n+1}(\mathbb{C})$. From the classical branching rules (see e.g. [Kna02, Ch. IX]) it follows that the restriction of π to $\mathrm{GL}_n(\mathbb{C})$ decomposes multiplicity free. In particular, $m_\lambda^{\mathrm{SL}_{n+1}(\mathbb{C}),\mathrm{GL}_n(\mathbb{C})}(\mu) \leq 1$ for all $\lambda \in P^+_{\mathrm{SL}_n(\mathbb{C})}$ and all $\mu \in P^+_{\mathrm{GL}_n(\mathbb{C})}$. This implies that

$((\mathrm{SL}_{n+1}(\mathbb{C}), \mathrm{GL}_n(\mathbb{C})), \mu)$ is a multiplicity free triple for every $\mu \in P^+_{\mathrm{GL}_n(\mathbb{C})}$. Similarly, $(\mathrm{SO}_n(\mathbb{C}), \mathrm{SO}_{n-1}(\mathbb{C}), \mu)$ is a multiplicity free triple for every $\mu \in P^+_{\mathrm{SO}_{n-1}(\mathbb{C})}$.

A multiplicity free triple has no geometrical counterpart in the sense of Gel'fand pairs versus spherical pairs. In Section 2.2 we define multiplicity free systems by imposing further conditions on the K-type μ. This notion does have a geometrical counterpart. Before we come to this point we discuss some results in representation theory.

2.1.10. Let (G, H) be a pair of connected algebraic groups with $H \subset G$ a closed subgroup and G reductive and let $\tau : H \to \mathrm{GL}(V)$ be a rational finite dimensional H-representation. The group H acts on the space $G \times V$ by $h(g, v) = (gh^{-1}, \tau(h)v)$. The quotient $G \times_\tau V$ is a G-homogeneous vector bundle over the homogeneous space G/H. Any G-homogeneous vector bundle $E \to G/H$ is of this form where the H-representation is given by the representation of H in the the fiber of $E \to G/H$ over the point eH, for $e \in G$ the neutral element.

The global sections of the bundle $G \times_\tau V$ correspond to elements in $\mathbb{C}[G] \otimes V$ that are invariant for the H-action defined by $(h \cdot f)(g) = \tau(h)f(gh^{-1})$. We denote this space by $\mathrm{ind}_H^G(V)$. The left regular representation of G on $\mathrm{ind}_H^G(V)$ is denoted by $\mathrm{ind}_H^G(\tau)$ and it is called the induced representation of τ to G. The induction procedure is a functor ind_H^G from the category of rational H-representations to the category of rational G-representations. Given G-representations π, π' in V and V' respectively, we denote $\dim(\mathrm{Hom}_G(V, V'))$ by $[\pi : \pi']$. If π is irreducible then $[\pi : \pi']$ is the number of copies of π in the decomposition of V' as a direct sum of irreducible G-representations. The restriction functor res_H^G, which associates to a rational G-representation its restriction to H, is a left adjoint for ind_H^G. This implies Frobenius reciprocity: we have $[\mathrm{ind}_H^G(\tau) : \pi] = [\mathrm{res}_H^G(\pi) : \tau]$. See [Jan03, Ch. 3].

A special instance of induced representations is the realization of irreducible rational representations of reductive connected algebraic groups. See e.g. [DK00, Ch. 4].

Theorem 2.1.11 (Borel-Weil)**.** *Let G be a connected algebraic group, $B \subset G$ a Borel subgroup, $T \subset B$ a maximal torus and let $\lambda \in P_G^+$. Let $\chi_{-\lambda} : T \to \mathbb{C}^\times$ be the character of weight $-\lambda$. Let $P \subset G$ be the standard parabolic subgroup associated to the subset of simple roots $\alpha \in \Pi_G$ with the property that $\langle \lambda, \alpha \rangle \geq 0$. The extension of $\chi_{-\lambda}$ to a character of B by letting it be trivial on the unipotent part is denoted by $\chi_{-\lambda} : B \to \mathbb{C}^\times$. We denote the similar extension of $\chi_{-\lambda}$ to a character of P by $\chi_{-\lambda} : P \to \mathbb{C}^\times$.*

- *The rational G-representation $\mathrm{ind}_B^G(\chi_{-\lambda})$ is irreducible of highest weight $-s_G(\lambda)$, where $s_G \in W_G$ is the longest Weyl group element.*
- *The rational G-representation $\mathrm{ind}_P^G(\chi_{-\lambda})$ is irreducible of highest weight $-s_G(\lambda)$. The map $G/B \to G/P$ induces an isomorphism $\mathrm{ind}_P^G(\chi_{-\lambda}) \to \mathrm{ind}_B^G(\chi_{-\lambda})$ which is G-equivariant.*

2.1.12. The irreducible representation of highest weight $-s_G\lambda$ is dual to the irreducible representation of highest weight λ. In Theorem 2.1.11 we associated to an element $\lambda \in P_G^+$

a unique parabolic subgroup P. Note that the elements in an open face $F \subset P_G^+$ all yield the same parabolic subgroup.

2.2 Multiplicity free systems

Definition 2.2.1. *A multiplicity free system is a triple (G, K, F) with G a connected reductive algebraic group, $K \subset G$ a closed connected reductive subgroup and $F \subset P_K^+$ a non-empty relatively open face of P_K^+ such that for every weight $\mu \in F$ the triple (G, K, μ) is a multiplicity free triple.*

Example 2.2.2. Example 2.1.9 shows that (G, K, P_K^+) is a multiplicity free system for $(G, K) = (\mathrm{SL}_{n+1}(\mathbb{C}), \mathrm{GL}_n(\mathbb{C}))$ and $(\mathrm{SO}_{n+1}(\mathbb{C}), \mathrm{SO}_n(\mathbb{C}))$.

Theorem 2.2.3. *Let (G, K) be a pair of connected reductive groups with $K \subset G$ a closed subgroup and let $F \subset P_K^+$ be a relatively open face. Denote by $P \subset K$ the parabolic subgroup that is associated to the face $-s_K(F)$ (as in Theorem 2.1.11). Then (G, K, F) is a multiplicity free system if and only if (G, P) is a spherical pair.*

Proof. Suppose that (G, K, F) is a multiplicity free system and let $L \to G/P$ be a G-homogeneous line bundle. By 2.1.4 we need to show that the G-module $\Gamma(G/P, L)$ decomposes multiplicity free. We may assume that $\Gamma(G/P, L) = \mathrm{ind}_P^G \chi$ for some character χ of P. The character χ is of the form $\chi = \chi_\mu$ for some $\mu \in P_K$. The functoriality of ind implies that $\mathrm{ind}_P^G \chi_\mu = \mathrm{ind}_K^G(\mathrm{ind}_P^K(\chi_\mu))$ as a G-representation. Hence, if $-\mu \notin P_K^+$ then $L \to G/P$ has no nonzero global sections and the claim is vacuously true. We may assume that $-\mu \in P_K^+$ and in fact $\mu \in s_K(F)$. Theorem 2.1.11 implies that $\mathrm{ind}_P^K(\chi_\mu)$ is an irreducible K-representation of highest weight $s_K(\mu) \in F$. Let π be an irreducible G-representation of highest weight λ. Since (G, K, F) is a multiplicity free triple we have $m_\lambda^{G,K}(-s_K(\mu)) \leq 1$. Frobenius reciprocity implies $m_\lambda^{G,K}(-s_K(\mu)) = [\mathrm{ind}_P^G(\chi_\mu) : \pi]$ and hence $\Gamma(G/P, L)$ decomposes multiplicity free. Conversely, if (G, P) is a spherical pair then every irreducible G-representation occurs with multiplicity at most one in the decomposition of any G-representation $\mathrm{ind}_P^G(\chi_{-s_K(\mu)})$ with $\mu \in F$. Frobenius reciprocity implies that $m_\lambda^{G,K}(\mu) \leq 1$ for every $\lambda \in P_G^+$. □

Lemma 2.2.4. *Let (G, K, F) be a multiplicity free system, let $f \subset \mathcal{C}_K^+$ be the relatively open face such that $F = f \cap P_K^+$ and let $F' \subset P_G^+$ be a face with $F' \subset \overline{f}$. Then (G, K, F') is a multiplicity free system. In particular, if (G, K, F) is a multiplicity free system then (G, K) is a spherical pair.*

Proof. Let P and P' denote the parabolic subgroups in K that correspond to the faces F and F' respectively. The condition $F' \subset \overline{f}$ implies that $P \subset P'$. It follows from Proposition 2.1.3 that the pair (G, P') is G-spherical and Proposition 2.2.3 implies in turn that (G, K, F') is a multiplicity free system. Since $F \neq \emptyset$ we have $0 \in \overline{f}$ and we see that (G, K) is spherical. □

G		K	λ_{sph}	faces F
$\mathrm{SL}_{n+1}(\mathbb{C})$	$n \geq 1$	$\mathrm{GL}_n(\mathbb{C})$	$\varpi_1 + \varpi_n$	any
$\mathrm{SO}_{2n}(\mathbb{C})$	$n \geq 2$	$\mathrm{SO}_{2n-1}(\mathbb{C})$	ϖ_1	any
$\mathrm{SO}_{2n+1}(\mathbb{C})$	$n \geq 2$	$\mathrm{SO}_{2n}(\mathbb{C})$	ϖ_1	any
$\mathrm{Sp}_{2n}(\mathbb{C})$	$n \geq 3$	$\mathrm{Sp}_{2n-2}(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C})$	ϖ_2	$\dim F \leq 2$
F_4		$\mathrm{Spin}_9(\mathbb{C})$	ϖ_1	$\dim F \leq 1$ or $F = \mathbb{N}\omega_1 + \mathbb{N}\omega_2$
$\mathrm{SO}_7(\mathbb{C})$		G_2	ϖ_3	$\dim F \leq 1$
G_2		$\mathrm{SL}_3(\mathbb{C})$	ϖ_1	$\dim F \leq 1$

Table 2.1: Multiplicity free systems with (G, K) a spherical pair of rank one. In the third column we have given the highest weight $\lambda_{\mathrm{sph}} \in P_G^+$ of the fundamental zonal spherical representation in the notation for root systems of Bourbaki [Bou68, Planches], except for the case $(G, K) = (\mathrm{SO}_4(\mathbb{C}), \mathrm{SO}_3(\mathbb{C}))$ where G is not simple and $\lambda_{\mathrm{sph}} = \varpi_1 + \varpi_2 \in P_G^+ = \mathbb{N}\varpi_1 + \mathbb{N}\varpi_2$.

2.2.5. Let (G, K, F) be a multiplicity free system and let (G', K') be a spherical pair that is isogenous to (G, K). We have $F = f \cap P_K^+$ for some relatively open face $f \subset \mathfrak{t}_K = \mathfrak{t}_{K'}$. Define $F' = f \cap P_{K'}^+$. Then (G', K', F') is a multiplicity free system. We say that (G, K, F) and (G', K', F') are isogenous multiplicity free triples and this relation generates an equivalence relation.

Let (G, K, F) be a multiplicity free system with (G, K) a spherical pair of rank one. Let $G_0 \subset G$ and $K_0 \subset K$ denote compact Lie groups with $K_0 \subset G_0$ and whose complexifications are G and K. We claim that (G_0, K_0) is a compact two-point-homogeneous space, i.e. given two pairs of points (p, q) and (p', q') in G_0/K_0 for which the distances $d(p, q)$ and $d(p', q')$ are the same, there is an element $g \in G$ with $p' = gp$ and $q' = gp$. To see this note that this property is equivalent to the property that K_0 acts transitively on the unit sphere in $T_{eK_0}(G_0/K_0)$ (see e.g [Hel01, p. 535]), which we prove in Proposition 3.2.3. This is in turn equivalent to the fact that the convolution algebra of K_0-bi-invariant matrix coefficients on G_0 is generated by one element, hence the claim. Compact two-point-homogeneous spaces have been classified by Wang in [Wan52].

In Table 2.1 we have listed the indecomposable spherical pairs (G, K) of rank one modulo isogeny. In the third column we have put the fundamental spherical weights as indicated in [Krä79], see Example 2.3.2. In the fourth column we have indicated the faces F which lead to multiplicity free systems (G, K, F). In the remainder of this section we prove that this is a complete list of multiplicity free systems with (G, K) a spherical pair of rank one and we briefly discuss examples of higher rank.

The Lie algebras associated to the spherical pairs (G, K) of rank one that are not symmetric admit decompositions similar to the Iwasawa decomposition that we have for symmetric pairs.

2.2.6. Let $\mathrm{G}_{2,0}$ and $\mathrm{Spin}(7)$ denote the compact Lie groups whose complexifications are G_2 and $\mathrm{Spin}_7(\mathbb{C})$. We want to find a copy of $\mathrm{G}_{2,0}$ in $\mathrm{Spin}(7)$. First we find a copy of $\mathrm{Spin}(7)$ in $\mathrm{SO}(8)$. The group $\mathrm{SO}_8(\mathbb{C})$ is defined by $\{x \in \mathrm{Mat}_8(\mathbb{C}) | \forall v, w \in \mathbb{C}^8 : Q(xv, xw) = Q(v, w), \det x = 1\}$ where the bilinear form Q is given by $Q(v, w) = v^t S w$ with

$$S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Let $\{v_{\pm e_i}, i = 1, 2, 3, 4\}$ be a weight basis of $\mathbb{C}^8$. Let $V \subset \mathbb{C}^8$ denote the real subspace with respect to the conjugation $cv_{\mp e_i} \mapsto \bar{c} v_{\pm e_i}$. Then $\mathrm{SO}(8) = \{x \in \mathrm{Mat}_8(\mathbb{R}) | \forall v, w \in V : Q(xv, xw) = Q(v, w), \det(x) = 1\}$.

The maximal torus $\mathfrak{t}$ of the Lie algebra $\mathfrak{so}_7(\mathbb{C})$ is spanned by the elements $H_i = E_{i,i} - E_{3+i,3+i} \in \mathfrak{gl}_7(\mathbb{C})$ and we denote by ϵ_i the elements in $\mathfrak{t}^\vee$ defined by $\epsilon_i(H_j) = \delta_{ij}$. The roots of $(\mathfrak{so}_7(\mathbb{C}), \mathfrak{t})$ are given by $R = \{\pm\epsilon_i \pm \epsilon_j : 1 \leq i < j \leq 3\} \cup \{\pm\epsilon_i : 1 \leq i \leq 3\}$, see Figure 2.1.

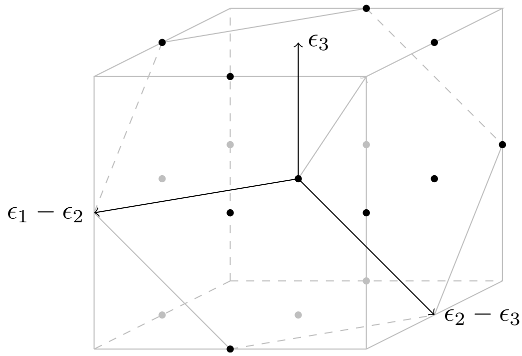

Figure 2.1: Roots of B_3.

The simple roots are $\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3\}$. The spherical weight for the pair $(\mathfrak{so}_7(\mathbb{C}), \mathfrak{g}_2)$ is $\lambda_{\mathrm{sph}} = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3)$ which is the highest weight of the spin representation $\mathrm{Spin}_7(\mathbb{C}) \to \mathrm{SO}_8(\mathbb{C})$. This realizes $\mathrm{Spin}_7(\mathbb{C}) \subset \mathrm{SO}_8(\mathbb{C})$. The weights of this representation are $\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3)$, as depicted in Figure 2.2.

Let v_1 and v_8 be the highest and lowest weight vectors in $V_{\lambda_{\mathrm{sph}}}$. We have $\mathrm{Spin}(7) = \mathrm{Spin}_7(\mathbb{C}) \cap \mathrm{SO}(8)$ and $G_{2,0}$ is the stabilizer in $\mathrm{Spin}(7)$ of the vector $v_1 + v_8 \in V$. Note that the stabilizer of $v_1 + v_8$ in the (standard) representation of $\mathrm{SO}_8(\mathbb{C})$ is a copy of $\mathrm{SO}_7(\mathbb{C})$. Hence we have $\mathrm{SO}_7(\mathbb{C}) \cap \mathrm{Spin}_7(\mathbb{C}) = \mathrm{G}_2$ and similarly for the compact subgroups.

Let $\mathfrak{a} \subset \mathfrak{so}_7(\mathbb{C})$ be spanned by $H_A = H_1 + H_2 + H_3$. Let $A \subset \mathrm{Spin}_7(\mathbb{C})$ be the torus with Lie algebra $\mathfrak{a}$ and write $M = Z_{\mathrm{G}_2}(A)$. Then $M \cong \mathrm{SL}_3(\mathbb{C})$ which can be seen from Figure 2.1, as the long roots of G_2 are in $\lambda_{\mathrm{sph}}^\perp$ (with respect to the Killing form). To determine $A \cap M$ we look at the coroot lattices $Q^\vee_{\mathrm{Spin}_7(\mathbb{C})} = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 + n_2 + n_3 \in 2\mathbb{Z}\}$ and $Q^\vee_M = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 + n_2 + n_3 = 0\}$. We have $\lambda^\vee_{\mathrm{sph}} = (1, 1, 1)$ and thus $\frac{1}{3}\lambda^\vee_{\mathrm{sph}}$ is in $Q^\vee_M$ modulo $Q^\vee_{\mathrm{Spin}_7(\mathbb{C})}$. Hence $A \cap M$ has three elements.

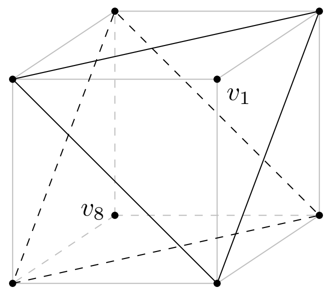

Figure 2.2: Weight diagram of $\mathrm{Spin}_7(\mathbb{C}) \to \mathrm{SO}_8(\mathbb{C})$ with orbits of M.

Let $\alpha : \mathfrak{a} \to \mathbb{C}$ be defined by $\alpha(sH_A) = s$. Then the adjoint action of $\mathfrak{a}$ on $\mathfrak{so}_7(\mathbb{C})$ gives a decomposition of $\mathfrak{a}$-weight spaces

$$\mathfrak{so}_7(\mathbb{C}) = \mathfrak{n}^- \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+,$$

where $\mathfrak{m}$ is the Lie algebra of M, $\mathfrak{n}^- = \mathfrak{so}_7(\mathbb{C})_{-\alpha} \oplus \mathfrak{so}_7(\mathbb{C})_{-2\alpha}$, $\mathfrak{n}^+ = \mathfrak{so}_7(\mathbb{C})_{\alpha} \oplus \mathfrak{so}_7(\mathbb{C})_{2\alpha}$ and where $\mathfrak{so}_7(\mathbb{C})_{\pm\alpha}$ and $\mathfrak{so}_7(\mathbb{C})_{\pm 2\alpha}$ are the direct sums of the appropriate root spaces of $\mathfrak{so}_7(\mathbb{C})$.

A short root of $\mathfrak{g}_2$ is contained in some direct sum $\mathfrak{so}_7(\mathbb{C})_{\epsilon_i} \oplus \mathfrak{so}_7(\mathbb{C})_{-\epsilon_j-\epsilon_k}$ but not in one of the summands. It follows that there is a map $\theta : \mathfrak{n}^+ \to \mathfrak{n}^-$ with the property $(\theta + I)\mathfrak{n}^+ = \mathfrak{m}^\perp \subset \mathfrak{g}_2$. Hence we have $\mathfrak{so}_7(\mathbb{C}) = \mathfrak{g}_2 \oplus \mathfrak{a} \oplus \mathfrak{n}^+$.

2.2.7. The spherical weight for $(\mathrm{G}_2, \mathrm{SL}_3(\mathbb{C}))$ is ϖ_1 which corresponds precisely to the embedding $\mathrm{G}_2 \to \mathrm{SO}_7(\mathbb{C})$ that we discussed in 2.2.6. The roots of $\mathfrak{g}_2$ are $\{\pm\alpha_i | i = 1, \ldots, 6\}$, see Figure 2.3.

The weight ϖ_1 is a root α_4 of $\mathfrak{g}_2$. Let $\mathfrak{s} \subset \mathfrak{g}_2$ be an $\mathfrak{sl}_2(\mathbb{C})$-triple with root ϖ_1. The root spaces of $\mathfrak{s}$ are perpendicular to $\mathfrak{sl}_3(\mathbb{C})$ with respect to the Killing form. Let $g \in \mathrm{G}_2$ correspond to the element

$$g' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

in the subgroup $\mathrm{SL}_2(\mathbb{C}) \subset \mathrm{G}_2$ whose Lie algebra is $\mathfrak{s}$. Let $H_A = \mathrm{Ad}(g)H_1$, where $H_1 \in \mathfrak{t}_{\mathrm{G}_2}$ is the element in $\mathfrak{t}_{\mathrm{G}_2}$ that corresponds to $\varpi_1^\vee$. Note that $H_A \in H_1^\perp \subset \mathfrak{s}$. Let $\mathfrak{a} \subset \mathfrak{g}_2$ denote the subspace spanned by H_A. Let $A \subset \mathrm{G}_2$ denote the torus whose Lie algebra is $\mathfrak{a}$. Define $M = Z_{\mathrm{SL}_3(\mathbb{C})}(A)$ which is a copy of $\mathrm{SL}_2(\mathbb{C}) \subset \mathrm{G}_2$ whose roots are the long roots perpendicular to ϖ_1. Let $\mathfrak{t}_M$ be the torus spanned by the orthocomplement of H_1. Then $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}_M$ is a Cartan subalgebra of $\mathfrak{g}_2$ whose root spaces are $\mathrm{Ad}(g)(\mathfrak{g}_{2,\pm\alpha_i})$. Let $\alpha : \mathfrak{a} \to \mathbb{C}$ be defined by $\alpha(sH_A) = s$. The adjoint action of $\mathfrak{a}$ on $\mathfrak{g}_2$ gives a decomposition of $\mathfrak{a}$-weight spaces

$$\mathfrak{g}_2 = \mathfrak{n}^- \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{a} \oplus \mathfrak{n}^+,$$

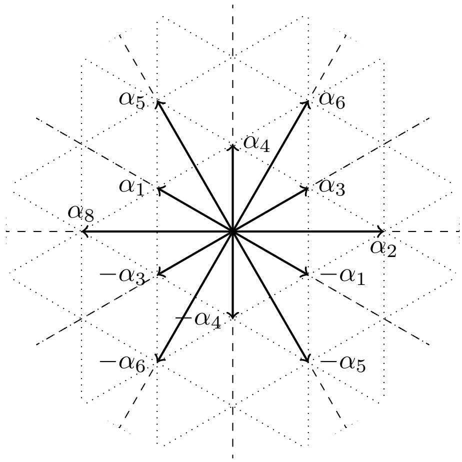

Figure 2.3: Roots for G_2.

where $\mathfrak{m}$ is the Lie algebra of M, $\mathfrak{n}^- = \mathfrak{g}_{2,-\alpha/2} \oplus \mathfrak{g}_{2,-\alpha} \oplus \mathfrak{g}_{2,-3\alpha/2}$, $\mathfrak{n}^+ = \mathfrak{g}_{2,\alpha/2} \oplus \mathfrak{g}_{2,\alpha} \oplus \mathfrak{g}_{2,3\alpha/2}$ and where $\mathfrak{g}_{2,\pm\alpha/2}, \mathfrak{g}_{2,\pm\alpha}$ and $\mathfrak{g}_{2,\pm 3\alpha/2}$ are the direct sums of the appropriate root spaces of $\mathfrak{g}_2$.

The space $\mathfrak{g}_{2,\alpha_5} \oplus \mathfrak{g}_{2,\alpha_1} \oplus \mathfrak{g}_{2,-\alpha_3} \oplus \mathfrak{g}_{2,-\alpha_6}$ is an irreducible $\mathfrak{s}$-representation. The representations of $\mathrm{SL}_2(\mathbb{C})$ are parametrized by $\frac{1}{2}\mathbb{N}$ and given $\ell \in \frac{1}{2}\mathbb{N}$ we realize the corresponding representation T^ℓ in the space $\mathbb{C}[x,y]_{2\ell}$ of homogeneous polynomials of degree 2ℓ. The $\mathfrak{s}$-representation under consideration corresponds to $\ell = \frac{3}{2}$ and the weight vectors are x^3, x^2y, xy^2 and y^3. The highest and lowest weight vectors correspond to root vectors of M. A small calculation shows that $T^\ell(g')x^3 + 3T^\ell(g')xy^2$ is in the direct sum of the highest and lowest weight spaces and similarly for $T^\ell(g')y^3 + 3T^\ell(g')x^2y$. This gives rise to a map $\theta : \mathfrak{n}^+ \to \mathfrak{n}^-$ with the property that $(\theta + I)\mathfrak{n}^+ = \mathfrak{m}^\perp \subset \mathfrak{k}$. Finally we note that $A \cap M$ has two elements because $\frac{1}{2}\alpha_4^\vee - \alpha_6^\vee \in \mathfrak{t}_M^\vee$.

Lemma 2.2.8. *Let (G,K) be a spherical pair of rank one. Let $\mathfrak{a} \subset \mathfrak{k}^\perp$ be a line consisting of semisimple elements and let $A \subset G$ denote the one-dimensional torus with Lie algebra $\mathfrak{a}$. Define $M = Z_K(A)$ with Lie algebra $\mathfrak{m}$ and Cartan subalgebra $\mathfrak{t}_M$, the Lie algebra of T_M. Then $\mathfrak{a} \oplus \mathfrak{t}_M$ is a Cartan subalgebra of $\mathfrak{g}$. Fix notions of positivity on $\mathfrak{a}$ and $\mathfrak{a} \oplus \mathfrak{t}_M$ that are compatible and let $\mathfrak{n}^+$ denote the direct sum of root spaces for the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$. Then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$. The group $W_A = N_K(A)/M$ has two elements where the non-trivial element acts on $\mathfrak{a}$ by reflection in the origin. There is a linear map $\theta : \mathfrak{n}^+ \to \mathfrak{n}^-$ with the property that $(\theta + I)\mathfrak{n}^+ = \mathfrak{m}^\perp \subset \mathfrak{k}$.*

Proof. If (G,K) is a symmetric pair then this is part of the usual structure theory with θ the corresponding involution. In the other two cases we have proved all statements for a specific choice of $\mathfrak{a}$ in 2.2.6 and 2.2.7. The restriction that $\mathfrak{a}$ has semisimple elements implies that it is the complexification of a line in $\mathfrak{k}_0^\perp$ in the compact picture. The compact

(G,K)	M
$(\mathrm{SL}_{n+1}(\mathbb{C}),\mathrm{GL}_n(\mathbb{C}))$	$\mathrm{S}(\mathrm{diag}(\mathrm{GL}_1(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C}))\times\mathrm{GL}_{n-1}(\mathbb{C}))$
$(\mathrm{SO}_{2n}(\mathbb{C}),\mathrm{SO}_{2n-1}(\mathbb{C}))$	$\mathrm{SO}_{2n-2}(\mathbb{C})$
$(\mathrm{SO}_{2n+1}(\mathbb{C}),\mathrm{SO}_{2n}(\mathbb{C}))$	$\mathrm{SO}_{2n-1}(\mathbb{C})$
$(\mathrm{Sp}_{2n}(\mathbb{C}),\mathrm{Sp}_{2n-2}(\mathbb{C})\times\mathrm{Sp}_2(\mathbb{C}))$	$\mathrm{Sp}_2(\mathbb{C})\times\mathrm{Sp}_{2n-4}(\mathbb{C})$
$(\mathrm{F}_4,\mathrm{Spin}_9(\mathbb{C}))$	$\mathrm{Spin}_7(\mathbb{C})$
$(\mathrm{Spin}_7(\mathbb{C}),\mathrm{G}_2)$	$\mathrm{SL}_3(\mathbb{C})$
$(\mathrm{G}_2,\mathrm{SL}_3(\mathbb{C}))$	$\mathrm{SL}_2(\mathbb{C})$

Table 2.2: Spherical pairs of rank one with $M = Z_K(A)$ modulo conjugation with K. Note that the embedding $\mathrm{Spin}_7(\mathbb{C}) \subset \mathrm{Spin}_9(\mathbb{C})$ is the standard embedding $\mathrm{Spin}_7(\mathbb{C}) \subset \mathrm{Spin}_8(\mathbb{C})$ followed by the embedding $\mathrm{Spin}_8(\mathbb{C}) \subset \mathrm{Spin}_9(\mathbb{C})$ that is twisted by the automorphism $\epsilon_1 - \epsilon_2 \leftrightarrow \epsilon_3 - \epsilon_4$, see [BS79, §6].

group K_0 acts transitively on these lines (see Proposition 3.2.3) and we see that the results in 2.2.6 and 2.2.7 are independent of the choice of $\mathfrak{a}$. The action of K_0 is actually transitive on the sphere in $\mathfrak{k}^\perp$ which implies that the only element in K that normalizes $\mathfrak{a}$ non-trivially is the reflection in the origin. □

Proposition 2.2.9. *Let (G,K) be a spherical pair of rank one. Let A, M be as in Lemma 2.2.8. Let $P \subset K$ be a parabolic subgroup. Then G/P is G-spherical if and only if K/P is M-spherical.*

Proof. Let $N^+ \subset G$ be the closed unipotent subgroup with Lie algebra $\mathfrak{n}^+$ and let $B_M \subset M$ be a Borel subgroup. Define $B = B_M A N^+$ which is a Borel subgroup of G and consider the map $c : G/P \to G/K$. The open orbit of B in G/K is $BK/K \cong B/(B\cap K)$. Note that $B \cap K$ contains B_M with finite index. We observe that G/P has an open B-orbit if and only if $c^{-1}(BK/K)$ has an open B-orbit, because $c^{-1}(BK/K)$ is an open B-stable subset of G/P. The latter holds if and only if $B \cap K$ has an open orbit in the fiber $c^{-1}(K/K) = K/P$ which is in turn equivalent to K/P being M-spherical. □

Lemma 2.2.10. *Let* F_4 *denote the connected (simply connected) reductive algebraic group of type* F_4. *The triple* $(\mathrm{F}_4, \mathrm{Spin}_9(\mathbb{C}), F)$ *is a multiplicity free system if and only if* $\dim F \leq 1$ *or* $F = \mathbb{N}\omega_1 + \mathbb{N}\omega_2$.

Proof. The branching from K to M via the twisted embedding $\mathrm{Spin}_8(\mathbb{C}) \subset \mathrm{Spin}_9(\mathbb{C})$ is described in [BS79, §6] and goes as follows. The restriction of an irreducible representation of highest weight μ to the standard $\mathrm{Spin}_8(\mathbb{C})$ is known and the representation of the twisted embedding are obtained by interchanging the roots $\epsilon_1 - \epsilon_2$ and $\epsilon_3 - \epsilon_4$. Each of these representations is then restricted to the standard $\mathrm{Spin}_7(\mathbb{C}) \subset \mathrm{Spin}_8(\mathbb{C})$. One easily checks that the indicated representations decompose multiplicity free. To exclude the other faces we only need to exclude the other two-dimensional faces. By looking at the Dynkin diagram of B_4 one calculates the dimensions of the corresponding parabolic subgroups P. The dimensions are 22 or 23. Hence, $\dim \mathrm{F}_4/P$ is equal to 29 or 30 whereas $\dim B_{\mathrm{F}_4} = 28$. It follows that there cannot be an open orbit for these faces. □

Lemma 2.2.11. *The triple* $(\mathrm{G}_2, \mathrm{SL}_3, F)$ *is a multiplicity free system if and only if F is of dimension* ≤ 1.

Proof. The branching rules for the pair $(\mathrm{G}_2, \mathrm{SL}_3)$ are known, see e.g. [KQ78]. It follows that an irreducible $\mathrm{SL}_3(\mathbb{C})$-representation occurs with multiplicity ≤ 1 if and only if its highest weight is on a face of dimension ≤ 1. The result follows from Proposition 2.2.9. □

Lemma 2.2.12. *The triple* $(\mathrm{Spin}_7, \mathrm{G}_2, F)$ *is a multiplicity free system if and only if F is of dimension* ≤ 1.

Proof. We have seen that $M = Z_K(A) = \mathrm{SL}_3(\mathbb{C})$. The result follows from Lemma 2.2.11 and Proposition 2.2.9. □

We have discussed all rows in Table 2.1 except for the symplectic groups. We need the following result of Brion [Bri87, Prop. 3.1].

Proposition 2.2.13. *Let G be a connected reductive algebraic group and let $H \subset G$ be an algebraic subgroup. Let $H = H^r H^u$ be a Levi decomposition of H. Let $Q \subset G$ be a parabolic subgroup of G with Levi decomposition $Q = Q^r Q^u$ such that $H^r \subset Q^r$ and $H^u \subset Q^u$. Then the following are equivalent.*

- *(G, H) is a spherical pair,*
- *(Q^r, H^r) is a spherical pair and if $B_{Q^r} \subset Q^r$ is a Borel subgroup opposite to H^r, i.e. $B_{Q^r} H^r \subset Q^r$ is open, then $B_{Q^r} \cap H^r$ has an open orbit in Q^u/H^u.*

2.2.14. Let $(G, K) = (\mathrm{SL}_{n+1}(\mathbb{C}), \mathrm{GL}_n(\mathbb{C}))$ and let $B_G \subset G$ and $B_K \subset K$ denote the standard Borel subgroups consisting of the upper triangular matrices. We claim that (G, B_K) is a spherical pair. Proposition 2.2.3 then implies that (G, K, P_K^+) is a multiplicity free system. This statement is already clear form the classical branching rules, but the present proof serves as an alternative. The claim follows from Proposition 2.2.13. Indeed, in this case $H = B_K$ and $Q = B_G$ and $H^r = Q^r = T \subset B_G$, the standard torus consisting of diagonal matrices. The quotient B_G/B_K is a vector space isomorphic to $\oplus \mathfrak{g}_\alpha$ where $\alpha \in R_G^+ \backslash R_K^+$. There is an open orbit of the action of the torus $T \subset B$ on $\oplus \mathfrak{g}_\alpha$, hence the claim. A similar argument for this claim may also be found in [VK78, Rk. 7].

Lemma 2.2.15. *The triple* $(\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{Sp}_{2n-2}(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C}), F)$ *is a multiplicity free system if and only if* $\dim F \leq 2$.

Proof. Let $G = \mathrm{Sp}_{2n}(\mathbb{C})$ and $K = \mathrm{Sp}_{2n-2}(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C})$. We proceed in five steps. In the first step we parametrize the standard parabolic subgroups $P \subset K$ that correspond to the faces of dimension two. In the second step we determine the parabolic subgroups $Q \subset G$ with the properties (I) $P^r \subset Q^r$ and (II) $P^u \subset Q^u$. In the third step we show that the pairs (Q^r, P^r) are spherical and we determine a Borel subgroup $B_{Q^r} \subset Q^r$ with the property that $B_{Q^r}P^r \subset Q^r$ is open. In the fourth step we show that $B_{Q^r} \cap P^r$ has a dense orbit in Q^u/P^u. In view of Proposition 2.2.13 we conclude that the pairs (G, P) are spherical pairs. In the fifth step we show that for a face F of dimension three, (G, K, F) is not spherical.

(1). We introduce a system of simple roots for K,

$$\Pi_K = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-2} - \epsilon_{n-1}, 2\epsilon_{n-1}, 2\epsilon_n\} \tag{2.1}$$

A standard parabolic subgroup $P_\Gamma \subset K$ that corresponds to a two dimensional face is determined by a subset $\Gamma \subset \Pi_K$ consisting of $n-2$ elements. We divide the various possibilities in four cases.

(i) $\{2\epsilon_{n-1}, 2\epsilon_n\} \cap \Gamma = \emptyset$,

(ii) $\{2\epsilon_{n-1}, 2\epsilon_n\} \cap \Gamma = \{2\epsilon_n\}$,

(iii) $\{2\epsilon_{n-1}, 2\epsilon_n\} \cap \Gamma = \{2\epsilon_{n-1}\}$,

(vi) $\{2\epsilon_{n-1}, 2\epsilon_n\} \cap \Gamma = \{2\epsilon_{n-1}, 2\epsilon_n\}$.

(2). We introduce two systems of simple roots for G,

$$\Pi_G = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\} \tag{2.2}$$
$$\tilde{\Pi}_G = \{\epsilon_n - \epsilon_1, \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-2} - \epsilon_{n-1}, 2\epsilon_{n-1}\}. \tag{2.3}$$

For each Γ in step one we determine a set of simple roots Γ' such that the corresponding standard parabolic subgroup $Q_{\Gamma'} \subset G$ satisfies properties (I) and (II). In cases (i) and (ii) we have $\Gamma \subset \Pi_G$ and we take $\Gamma' = \Gamma \subset \Pi_G$. In case (iii) we switch to the system $\tilde{\Pi}_G$ of simple roots. We observe that $\Gamma \subset \tilde{\Pi}_G$ and therefore we take $\Gamma' = \Gamma \subset \tilde{\Pi}_G$. In case (iv) we take $\Gamma' = (\Gamma \cap \Pi_G) \cup \{\epsilon_{n-1} - \epsilon_n\} \subset \Pi_G$. This is the smallest system of simple roots in Π_G that generates Γ.

Claim. The standard parabolic subgroups $Q_{\Gamma'} \subset G$ satisfy (I) and (II).

Proof of claim. In each case we write $P = P_\Gamma$ and $Q = Q_{\Gamma'}$. It is clear that $P^r \subset Q^r$ by the choice of Q. In fact, we have $P^r = Q^r$ in cases (i)-(iii). The more difficult part lies in showing that $P^u \subset Q^u$. To show this inclusion we look at the Lie algebras of P^u and Q^u. The Lie algebra of P^u is spanned by the root spaces $\mathfrak{k}_\beta$ with $\beta \in R_K^+ \backslash R_{P^r}^+$. Similarly for Q^u but we have to be aware of the different sets Γ'. In cases (i) and (ii) we have $P^r = Q^r$ and hence $R_K^+ \backslash R_{P^r}^+ \subset R_G^+ \backslash R_{Q^r}^+$. In case (iii) the set of positive roots $R_G'^+$ is different from the standard one. The Lie algebra of Q^u is spanned by the root spaces $\mathfrak{g}_\alpha$ with $\alpha \in R_G'^+ \backslash R_{Q^r}^+$. By inspection of $R_G'^+$ we see that $R_K^+ \backslash R_{P^r}^+ \subset R_G'^+ \backslash R_{Q^r}^+$. In case (iv) we have $P^r \subsetneq Q^r$ but again $R_K^+ \backslash R_{P^r}^+ \subset R_G^+ \backslash R_{Q^r}^+$ by inspection. This settles the claim.

(3). The groups P^r and Q^r can be read from the Dynkin diagrams of K and G by deleting the appropriate nodes. We have seen that $P^r = Q^r$ in cases (i)-(iii) from which it follows that the pairs (Q^r, P^r) are spherical in these cases. The groups P^r are as follows.

(i) $P^r = \mathbb{C}^\times \times \mathrm{SL}_{n-1}(\mathbb{C}) \times \mathbb{C}^\times$ where the Lie algebra of the copies of $\mathbb{C}^\times$ are spanned by $H_1 + \cdots + H_{n-1}$ and H_n.

(ii) $P^r = \mathrm{SL}_p(\mathbb{C}) \times \mathbb{C}^\times \times \mathrm{SL}_{n-p-1}(\mathbb{C}) \times \mathbb{C}^\times \times \mathrm{Sp}_2(\mathbb{C})$ for some $1 \leq p \leq n-2$. The Lie algebras of the copies of $\mathbb{C}^\times$ are spanned by $H_1 + \cdots + H_p$ and $H_{p+1} + \cdots + H_{n-1}$.

(iii) $P^r = \mathrm{SL}_p(\mathbb{C}) \times \mathbb{C}^\times \times \mathrm{Sp}_{2(n-p-1)}(\mathbb{C}) \times \mathbb{C}^\times$ for some $1 \leq p \leq n-2$. The Lie algebras of the copies of $\mathbb{C}^\times$ are spanned by $H_1 + \cdots + H_p$ and H_n.

(iv) $P^r = \mathrm{SL}_p(\mathbb{C}) \times \mathbb{C}^\times \times \mathrm{SL}_q(\mathbb{C}) \times \mathbb{C}^\times \times \mathrm{Sp}_{2(n-p-q-1)}(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C})$ for some $p, q \in \mathbb{N}_{\geq 1}$ with $2 \leq p + q \leq n - 2$. The Lie algebras of the copies of $\mathbb{C}^\times$ are spanned by $H_1 + \cdots + H_p$ and $H_{p+1} + \cdots + H_{p+q}$.

In case (iv) we have $Q^r = \mathrm{SL}_p(\mathbb{C}) \times \mathbb{C}^\times \times \mathrm{SL}_q(\mathbb{C}) \times \mathbb{C}^\times \times \mathrm{Sp}_{2(n-p-q)}(\mathbb{C})$. It is clear that (Q^r, P^r) is a spherical pair in case (iv) too. For any Borel subgroup $B \subset Q^r$ the quotient Q^r/P^r has an open orbit. In all the cases we choose the Borel subgroup $B_{Q^r} \subset Q^r$ to be the product of the standard Borel subgroups in each factor.

(4). The groups $B_{Q^r} \cap P^r = B_{P^r}$ are products of standard Borel subgroups for each factor of P^r. We need to determine Q^u/P^u. In all cases there is a set $n_\Gamma \subset R_G$ such that quotient Q^u/P^u is isomorphic to the direct sum of root spaces $\mathfrak{g}_\alpha$ with $\alpha \in n_\Gamma$. In cases (i) and (ii) we have $n_\Gamma = R_G^+ \backslash R_K^+$ and in case (iii) $n_\Gamma = R_G'^+ \backslash R_K^+$. In case (iv) we have

$$n_\Gamma = R_G^+ \backslash (R_K^+ \cup R^+_{\mathrm{Sp}_{2(n-p-q)}(\mathbb{C})}) = \{\epsilon_i \pm \epsilon_n : i = 1 \ldots, p+q\}.$$

The actions of P^r on the vector spaces Q^u/P^u in the four cases are given as follows.

(i) $Q^u/P^u \cong \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^{n-1})$ and P^r acts by

$$(s, x, t)M = sx \circ M \circ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

(ii) $Q^u/P^u \cong \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^p \oplus \mathbb{C}^{n-p-1})$ and P^r acts by

$$(x, s, y, t, z)M = (sx \oplus ty) \circ M \circ z^{-1}.$$

(iii) $Q^u/P^u \cong \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^p) \oplus \mathbb{C}^{2(n-p-1)}$ and P^r acts by

$$(x, s, y, t)(M, V) = \left(sx \circ M \circ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, yV \right).$$

(iv) $Q^u/P^u \cong \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^p \oplus \mathbb{C}^q)$ and P^r acts by

$$(x, s, y, t, u, v)M = (tx \oplus sy) \circ M \circ v^{-1}.$$

In each case B_{P^r} has an open orbit. Indeed, checking this boils down to verify that the actions

$$\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}^\times \times \mathrm{End}(\mathbb{C}^2) \to \mathrm{End}(\mathbb{C}^2) : (x,t)M = x \circ M \circ \mathrm{diag}(t, t^{-1}) \tag{2.4}$$

and

$$\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C}) \times \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4) \to \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4), \\ (x,y,z)M = (x \oplus y) \circ M \circ z^{-1} \tag{2.5}$$

are spherical for the cases (i, iii) and (ii, iv) respectively. We claim that (1) $B \times \mathbb{C}^\times$ has an open orbit in $\mathrm{End}(\mathbb{C}^2)$ for the action (2.4), where $B \subset \mathrm{GL}_2(\mathbb{C})$ is the Borel subgroup consisting of upper triangular matrices and (2) that $B_1 \times B_2 \times B_3$ has an open orbit for the action (2.5), where $B_i \subset \mathrm{GL}_2(\mathbb{C})$ is the Borel subgroup consisting of upper triangular matrices for $i = 1, 2$ and $B_3 \subset \mathrm{Sp}_2(\mathbb{C})$ is the Borel subgroup consisting of upper triangular matrices. This is achieved by considering the orbits of

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

respectively.

(5). The spherical actions (2.4, 2.5) are neat in the sense that replacing (one of the copies of) $\mathrm{GL}_2(\mathbb{C})$ or $\mathrm{Sp}_2(\mathbb{C})$ by its maximal torus, no longer yields a spherical action. It follows that if we remove a root from Γ in cases (i), (ii) and (iii), the action B_{P^r} no longer has an open orbit, since Q^u/P^u remains the same, but correspondingly, in (2.4, 2.5) we must replace (one of the copies of) $\mathrm{GL}_2(\mathbb{C})$ by its maximal torus. Similarly for removing a root other than $2\epsilon_{n-1}$ or $2\epsilon_n$ from Γ in case (iv). However, if we remove one of the roots $2\epsilon_{n-1}$ or $2\epsilon_n$ we end up in one of the cases (i)-(iii) with one root removed. Hence, in this case, we also cannot have an open orbit of B_{P^r} in Q^u/P^u. □

Theorem 2.2.16. *For each row in Table 2.1 we can choose one of the indicated faces in the last column to obtain a triple (G, K, F). Modulo isogenies these are all the multiplicity free systems with (G, K) a spherical pair of rank one.*

Proof. In Example 2.2.2 we argued that the first three rows yield multiplicity free systems. The other rows are discussed in Lemmas 2.2.11, 2.2.12 and 2.2.15. □

2.2.17. We know that there are more multiplicity free systems.

- (G, K, F) with (G, K) a spherical pair of rank > 1 and F the face that parametrizes the one-dimensional representations. K has one-dimensional representations only if it has a center. The examples in the list of Krämer are
  - $(\mathrm{SO}_{2n+1}(\mathbb{C}), \mathrm{GL}_n(\mathbb{C}))$,

- $(\mathrm{SL}_{2n+1}(\mathbb{C}), \mathrm{GL}_1(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C}))$,
- $(\mathrm{SO}_{10}(\mathbb{C}), \mathrm{SO}_2(\mathbb{C}) \times \mathrm{Spin}_7(\mathbb{C}))$,
- $(\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{GL}_1(\mathbb{C}) \times \mathrm{Sp}_{2n-2}(\mathbb{C}))$,
- The compact Hermitian pairs, see e.g. [Hel01, Ch. 10]. For an analysis of the spherical functions for these pairs see [HS94, Part I, Ch. 5].

- Let $G = \mathrm{SL}_{n+1}(\mathbb{C}) \times \mathrm{SL}_{n+1}(\mathbb{C})$, $K = \mathrm{diag}(G)$ and $F = \mathbb{N}\omega_1$ or $F = \mathbb{N}\omega_n$. The pair (G, K) is a symmetric pair and the Cartan involution θ is the flip. Let $T \subset \mathrm{SL}_{n+1}(\mathbb{C})$ be the standard torus and denote by A the image of the map $T \to G : t \mapsto (t, t^{-1})$. Then A is an n-dimensional torus in G for which $G = KAK$. Denote $M = Z_K(A)$, which is equal to the diagonal embedding of T in G. Then (G, K, F) is a multiplicity free system and the rank of the pair (G, K) is n. Indeed, the restrictions to M of the K-representations of highest weight $\mu \in F$ decompose multiplicity free.

- Let $G = \mathrm{Spin}_9(\mathbb{C})$, $K = \mathrm{Spin}_7(\mathbb{C})$ and $F = \mathbb{N}\omega_1$. Then (G, K, F) is a multiplicity free system with (G, K) a spherical pair of rank two. Let $A \subset G$ be the two-dimensional torus for which $G = KAK$ and let $M = Z_K(A)$. In the next section we show that an irreducible representation of $\mathrm{Spin}_7(\mathbb{C})$ of highest weight $\mu \in F$ restricts to an irreducible representation of G_2, which in turn decomposes multiplicity free upon restriction to M, which is isomorphic to $\mathrm{SL}_3(\mathbb{C})$.

In [HNOO12] a classification of multiplicity free systems (G, K, F) is presented with (G, K) a symmetric pair of arbitrary rank.

2.3 Inverting the branching rule

Definition 2.3.1. *Let (G, K, F) be a multiplicity free system with (G, K) a spherical pair of rank one and let $\mu \in F$. Define the set $P_G^+(\mu) = \{\lambda \in P_G^+ : m_\lambda^{G,K}(\mu) = 1\}$. This set is called the μ-well. Furthermore, we define the set $P_M^+(\mu) = \{\nu \in P_M^+ : m_\mu^{K,M}(\nu) = 1\}$.*

Example 2.3.2. The 0-well for a spherical pair (G, K) of rank one is a monoid (over $\mathbb{N}$) that is generated by the spherical weight λ_{sph}. The generators for the spherical pairs of rank one are indicated in Table 2.1. If (G, K) is a symmetric pair (of any rank) then the generators of $P_G^+(0)$ can be calculated by means of the Cartan-Helgason Theorem, see [Kna02, Thm. 8.49].

Proposition 2.3.3. *Let (G, K, F) be a multiplicity free system from Table 2.1, let $\mu \in F$ and let $\lambda \in P_G^+(\mu)$. Let V_λ denote the representation space of the irreducible G-representation of highest weight λ. Let $V_\lambda^{\mathfrak{n}^+} = \{v \in V_\lambda : \mathfrak{n}^+ v = 0\}$. Then M acts on $V_\lambda^{\mathfrak{n}^+}$ as an irreducible representation. Moreover, any non-zero vector $v \in V_\lambda^{\mathfrak{n}^+}$ is K-cyclic.*

Proof. The subgroup $MAN^+ \subset G$ is parabolic because it contains the Borel $B = B_M AN^+$ of Proposition 2.2.9. The first part of the statement is a reformulation of

Theorem 5.104 in [Kna02]. This leaves us to show the last statement. Let $v \in V_\lambda^{\mathfrak{n}^+}$ be non-zero and let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. We have $U(\mathfrak{g})v = V_\lambda$ because the representation is irreducible. By Lemma 2.2.8 we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ which gives a decomposition $U(\mathfrak{g}) = U(\mathfrak{k}) \otimes U(\mathfrak{a}) \otimes U(\mathfrak{n}^+)$. Since $U(\mathfrak{n}^+)v = 0$ and $U(\mathfrak{a})v \subset \mathbb{C} \cdot v$ we see that $U(\mathfrak{k})v = V_\lambda$. □

2.3.4. Let $\mathfrak{t}_M \subset \mathfrak{m}$ be a Cartan subalgebra and define $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}_M$. Fix compatible notions of positivity on $\mathfrak{a}$ and $\mathfrak{h}$ so that positive roots with respect to $(\mathfrak{g}, \mathfrak{h})$ restrict to non-negative functionals on $\mathfrak{a}$. Let $P_G'^+$ be the set of dominant weights with respect to this ordering. Let $\lambda \in P_G'^+$ and let π be an irreducible representation of highest weight λ. The highest weight of the irreducible representation of M on V_λ^+ is $\lambda|_{\mathfrak{t}_M}$. The vectors in $v \in V_\lambda^{\mathfrak{n}^+}$ transform according to $\pi(H)v = \lambda(H)v$ for all $H \in \mathfrak{a}$.

Proposition 2.3.5. *Let (G, K, F) be a multiplicity free system with (G, K) a spherical pair of rank one, let $\mu \in F$ and let $\lambda \in P_G^+(\mu)$. Let ν be the highest weight of the representation of M on $V_\lambda^{\mathfrak{n}^+}$. Then $\nu \in P_M^+(\mu)$.*

Proof. Let $W \subset V_\lambda$ be a K-invariant subspace of positive dimension. Let $G_0 \subset G$ and $K_0 \subset K$ be a compact forms and let $(\cdot, \cdot)$ be a symmetric non-degenerate G_0-invariant pairing on V_λ. Let $v \in V_\lambda^{\mathfrak{n}^+}$ be non-zero. Then $(v, w) \neq 0$ for some $w \in W$. Indeed, if this were not true then $(V_\lambda, W) = 0$, because v is K-cyclic, which would imply that $W = \{0\}$, contradicting $\dim W > 0$. One of the K-invariant subspaces is a copy of V_μ. We have shown that there is a vector $v' \in V_\mu \subset V_\lambda$ with $(v, v') \neq 0$. Schur's Lemma implies $m_\mu^{K,M}(\nu) \geq 1$ and hence $m_\mu^{K,M}(\nu) = 1$ by Proposition 2.2.9. □

Proposition 2.3.6. *Let (G, K, μ) be a multiplicity free triple. Let $\lambda \in P_G^+(\mu)$. Then $\lambda + \lambda_{\mathrm{sph}} \in P_G^+(\mu)$.*

Proof. Let $W = b_\lambda(V_\mu) \otimes b_{\lambda_{\mathrm{sph}}}(V_0) \subset V_\lambda \otimes V_{\lambda_{\mathrm{sph}}}$. Since K acts trivially on $V_0 = \mathbb{C}$ we see that the space W is isomorphic to V_μ as K-representation. Let $a : V_\lambda \otimes V_{\lambda_{\mathrm{sph}}} \to \oplus_{\lambda'} V_{\lambda'}$ be the G-equivariant intertwiner that governs the decomposition of $V_\lambda \otimes V_{\lambda_{\mathrm{sph}}}$. We show that the image of W of the projection $\mathrm{pr}_{\lambda+\lambda_{\mathrm{sph}}} \circ a : V_\lambda \otimes V_{\lambda_{\mathrm{sph}}} \to V_{\lambda+\lambda_{\mathrm{sph}}}$ is non-zero which implies the result because $\mathrm{pr}_{\lambda+\lambda_{\mathrm{sph}}}$ is G-equivariant.

We realize the spaces V_λ and $V_{\lambda_{\mathrm{sph}}}$ as global sections of line bundles L_λ and $L_{\lambda_{\mathrm{sph}}}$ over G/B where $B \subset G$ is a Borel subgroup. The (G-equivariant) projection $\mathrm{pr}_{\lambda+\lambda_{\mathrm{sph}}} \circ a : V_\lambda \otimes V_{\lambda_{\mathrm{sph}}} \to V_{\lambda+\lambda_{\mathrm{sph}}}$ corresponds to the bilinear map $\Gamma(G/B, L_\lambda) \times \Gamma(G/B, L_{\lambda_{\mathrm{sph}}}) \to \Gamma(G/B, L_{\lambda+\lambda_{\mathrm{sph}}}) : (s_v, s_{v'}) \mapsto s_v \cdot s_{v'}$. Hence the image under $\mathrm{pr}_{\lambda+\lambda_{\mathrm{sph}}}$ of a non-zero vector $v \otimes v' \in W$ corresponds to a section $s_v s_{v'}$ of $L_{\lambda+\lambda_{\mathrm{sph}}}$. Since both s_v and $s_{v'}$ are non-zero and holomorphic, so is their product. Hence $\mathrm{pr}_{\lambda+\lambda_{\mathrm{sph}}}(a(W)) \neq \{0\}$. □

2.3.7. It follows from Proposition 2.3.5 that there is a map $p_\mu : P_G^+(\mu) \to P_M^+(\mu)$ that assigns to an element λ the highest weight of the M-representation in $V_\lambda^{\mathfrak{n}^+}$. From Proposition 2.3.6 we see that there is an asymptotic (or spherical) direction. The nature of the sets $P_G^+(\mu)$ has also been noted by Camporesi in [Cam05a] and [Cam05b]. Our proof of Proposition 2.3.5 is based on Remark 1 in [Cam05a, p.106].

The projection map $p_\mu : P_G^+(\mu) \to P_M^+(\mu)$ is surjective. For the symmetric pairs this follows from the identity

$$\lim_{m\mapsto\infty} m^{G,K}_{b_\mu(\nu)+m\cdot\lambda_{\mathrm{sph}}}(\mu) = m^{K,M}_\mu(\nu),$$

see [Wal73, Cor. 8.5.15], and for the other two pairs it follows from the explicit branching rules. The number of elements in $P_M^+(\mu)$ is denoted by $d_\mu(M)$. This number is equal to the dimension of $\mathrm{End}_M(V_\mu)$ and to the number of elements in the bottom of the μ-well.

The fibers of $p_\mu : P_G^+(\mu) \to P_M^+(\mu)$ are affine copies of $\mathbb{N}\lambda_{\mathrm{sph}}$. To see this we note that (1) if $p_\mu(\lambda) = \nu$ then $p_\mu(\lambda+\lambda_{\mathrm{sph}}) = \nu$, which follows from 2.3.4, and (2) λ_{sph} is primitive, i.e. $\lambda_{\mathrm{sph}}/n \notin P_G^+$ for $n \in \mathbb{N}$ with $n \geq 2$. The latter is clear from the descriptions of λ_{sph} in Table 2.1. It follows that a fiber of p_μ has a minimal element.

Definition 2.3.8. *Let (G,K,F) be a multiplicity free system with (G,K) a spherical pair of rank one and let $\mu \in F$. We denote the minimal element (in the ordering discussed in 2.3.7) of the fiber $p_\mu^{-1}(\nu)$ by $b_\mu(\nu)$. The set $B_\mu := \{b_\mu(\nu) : \nu \in P_M^+(\mu)\}$ is called the μ-bottom or the bottom of the μ-well.*

Proposition 2.3.9. *Let (G,K,F) be a multiplicity free system from Table 2.1 and let $\mu \in F$. We have an isomorphism of sets $B_\mu \times \mathbb{N} \to P_G^+(\mu)$ given by $(b_\mu(\nu), d) \mapsto b_\mu(\nu) + d\cdot\lambda_{\mathrm{sph}}$.*

Proof. Propositions 2.3.5 and 2.3.6 imply that the map λ is injective. To see that it is surjective we note that λ_{sph} is primitive. □

2.3.10. Camporesi calculated the bottoms of the μ-wells for the first three triples in Table 2.1 in [Cam05a]. He obtains partial results for the bottoms in the symplectic case. The determination of the sets $P_G^+(\mu)$ is what we call an inversion of the branching rules. The most important part is to determine the bottoms of the μ-wells. Indeed, if we know the bottoms B_μ then we can reconstruct the μ-wells by means of Proposition 2.3.9.

For our purposes we need to reformulate the descriptions of the μ-bottoms of Camporesi. Also, we need to invert the branching rules for the remaining cases.

2.3.11. In the remainder of this section we give descriptions of the μ-wells for multiplicity free triples (G,K,F) with (G,K) a spherical pair of rank one and G simply connected and other[1] than F_4. The μ-wells for other multiplicity free triples can be deduced from them by looking at the appropriate sublattices.

Inverting the branching rules for $(\mathrm{SL}_{n+1}(\mathbb{C}), \mathrm{GL}_n(\mathbb{C}))$

Let $G = \mathrm{SL}_{n+1}(\mathbb{C})$ and $K \subset G$ the image of $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{SL}_{n+1}(\mathbb{C}) : x \mapsto \mathrm{diag}(x, \det x^{-1})$. We use the choices and notations for roots and weights as in [Bou68]. The roots and weights of G are denoted by $\alpha_1, \ldots, \alpha_n$ and $\varpi_1, \ldots, \varpi_n$. The roots and weights of K are denoted by $\beta_1, \ldots, \beta_{n-1}$ and $\omega_1, \ldots, \omega_{n-1}$. See Figure 2.4 for a picture in rank two.

[1] A few days before printing we discovered that there are good faces other than $\{0\}$ in this case. Unfortunately there was no time left to analyze the μ-wells in these cases.

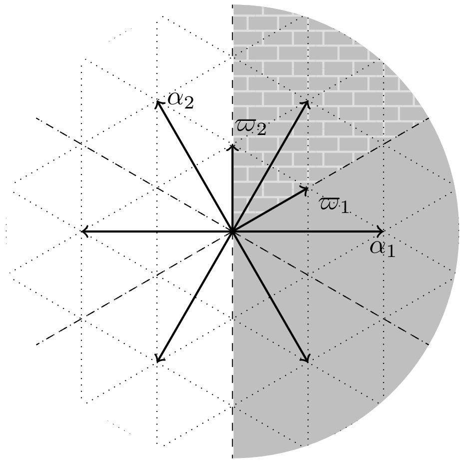

Figure 2.4: Roots and fundamental weights for $\mathfrak{sl}_3(\mathbb{C})$. The positive Weyl chamber of $\mathfrak{gl}_2(\mathbb{C})$ is light gray and it contains the positive Weyl chamber of $\mathfrak{sl}_3(\mathbb{C})$ in which we have laid some bricks.

The spherical weight $\lambda_{\text{sph}} = \epsilon_1 - \epsilon_{n+1}$ is a root of G but not of K. In the $\mathrm{SL}_2(\mathbb{C})$-copy with root λ_{sph} we can rotate λ_{sph} to a semisimple element in $\mathfrak{k}^{\perp}$. This gives a torus $A \subset G$. The group $M = Z_K(A)$ equals $M = \{(x, y, x) \in \mathrm{S}(\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_{n-1}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}))\}$. The map $M \to \mathrm{GL}_{n-1}(\mathbb{C}) : (x, y, x) \mapsto y$ is a double cover. The weight lattices of G, K and M parametrize the respective irreducible rational representations and by means of the Killing form we can realize P_K and P_M in P_G. Define $\widetilde{\varpi}_i = \varpi_i - \epsilon_i + \frac{1}{2}(\epsilon_1 + \epsilon_{n+1})$ for $i = 1, \ldots, n$. We obtain

$$P_G^+ = \varpi_1\mathbb{N} \oplus \cdots \oplus \varpi_{n-1}\mathbb{N} \oplus \varpi_n\mathbb{N}, \quad (2.6)$$
$$P_K^+ = \varpi_1\mathbb{N} \oplus \cdots \oplus \varpi_{n-1}\mathbb{N} \oplus \varpi_n\mathbb{Z}, \quad (2.7)$$
$$P_M^+ = \widetilde{\varpi}_1\mathbb{Z} \oplus \widetilde{\varpi}_2\mathbb{N} \oplus \cdots \oplus \widetilde{\varpi}_{n-1}\mathbb{N}. \quad (2.8)$$

Note that $P_M^+ \subset (\epsilon_1 - \epsilon_{n+1})^{\perp}$. We formulate two branching rules which follow from the classical branching rules, see also [Cam05a, p.13].

Theorem 2.3.12. *Let $\lambda \in P_G^+$ and $\mu \in P_K^+$ and write*

$$\lambda = \sum_{i=1}^{n+1} a_i\epsilon_i, \quad \mu = \sum_{i=1}^{n+1} b_i\epsilon_i. \quad (2.9)$$

Then $m_\lambda(\mu) = 1$ if and only if (i) $a_i - b_i \in \mathbb{Z}$ *and* (ii) $a_i \geq b_i$ *and* $b_i \geq a_{i+1}$ *for* $1 \leq i \leq n$.

Theorem 2.3.13. *Let $\mu \in P_K^+$ and $\nu \in P_M^+$ and write*

$$\mu = \sum_{i=1}^{n+1} b_i\epsilon_i, \quad \nu = \sum_{i=1}^{n+1} c_i\epsilon_i. \quad (2.10)$$

Then $m_\mu(\nu) = 1$ *if and only if* (i) $b_i - c_i \in \mathbb{Z}$ *for* $2 \leq i \leq n$ *and* (ii) $b_i \geq c_{i+1}$ *and* $c_{i+1} \geq b_{i+1}$ *for* $1 \leq i \leq n-1$.

Let $\mu \in P_K^+$. To determine the μ-well $P_G^+(\mu)$ we need to describe the μ-bottom B_μ. See Figure 2.5 for a picture of a μ-well in the rank two case. Note that the bottom is not linear as a function of the M-types.

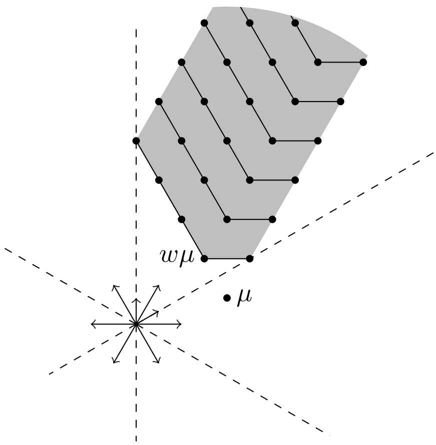

Figure 2.5: The μ-well for $(\mathfrak{sl}_3, \mathfrak{gl}_2)$ and $\mu = 4\omega_1 - \omega_2$.

Lemma 2.3.14. *Let* $w \in W_G$ *be a Weyl group element such that* $w\mu \in P_G^+$. *Then* $w\mu \in B_\mu$.

PROOF. Write $\mu = \sum_{i=1}^{n+1} b_i\epsilon_i$. $W_G = \mathfrak{S}_{n+1}$, the permutation group of $n+1$ objects and w is such that $b_{w^{-1}} \geq \cdots \geq b_{w^{-1}(n+1)}$. It follows from Theorem 2.3.12 that $m_{w\mu}(\mu) = 1$. Note that $w^{-1}1 = 1$ or $w^{-1}(n+1) = n$. It follows that $w\mu - \lambda_{\text{sph}} \notin B_\mu$ because $b_{w^{-1}1} - 1 \geq b_1$ or $b_n \leq b_{w^{-1}(n+1)} + 1$ will be violated. □

The Weyl translation $w\mu$ plays an important role. We noted already that $P_M^+ \subset (\epsilon_1 - \epsilon_{n+1})^\perp$. Let $p : P_G^+ \to P_M^+$ denote the orthogonal projection. M acts irreducibly on $V_{w\mu}^{\mathfrak{n}^+}$ with a representation of highest weight $p(w\mu)$ which we denote by $\nu^* = p(w\mu)$.

Lemma 2.3.15. *Let* $\nu \in P_M^+(\mu)$. *Then there is a unique* n*-tuple* $(m_1, \ldots, m_n)$ *with* $m_n \in \mathbb{N}$ *and with at least one* $m_i = 0$, *such that*

$$\nu - \nu^* = \sum_{i=1}^{n} m_i p(\alpha_i).$$

PROOF. Let $\lambda \in P_G^+$ with $p(\lambda) = \nu$. We claim that $\lambda - w\mu$ is in the root lattice Q_G. We may assume that $\lambda \in P_G^+(\mu)$ by adding a multiple of λ_{sph} that is large enough. This implies that $\lambda - \mu \in Q_G$. Since $\mu - w\mu \in Q_G$ we have $\lambda - w\mu \in Q_G$, hence the claim.

Write $\lambda - w\mu = \sum_{i=1}^n r_i\alpha_i$ for some integers r_i and let $r = \min r_i$. The r_i are unique because the simple roots span Q_G. Then $\lambda - w\mu = \sum_i (r_i - r)\alpha_1 + r(\epsilon_1 - \epsilon_{n+1})$ because $\sum_{i=1}^n \alpha_i = \lambda_{\mathrm{sph}}$. Application of p yields $\nu - \nu^* = \sum_i (r_i - r)p(\alpha_i)$ as desired. □

Theorem 2.3.16. *Let $\mu \in P_K^+$ and let $\nu \in P_M^+(\mu)$. Let $(m_1, \ldots, m_n)$ denote the unique n-tuple from Lemma 2.3.15. Then the bottom element $b_\mu(\nu) \in B_\mu$ satisfies*

$$b_\mu(\nu) = w\mu + \sum_{i=1}^n m_i\alpha_i. \tag{2.11}$$

Proof. Define $\lambda = w\mu + \sum_{i=1}^n m_i\alpha_i$ and write $\lambda = \sum_{i=1}^{n+1} a_i\epsilon_i$. Camporesi has shown [Cam05a, §3] that $\lambda = b_\mu(\nu)$ if and only if $\lambda = \nu + (b_1 - b_n + |b_1 + b_n - a_1 - a_{n+1}|)\lambda_{\mathrm{sph}}$. We show that λ is of this form, thereby proving the theorem. We proceed in three steps. In step one we give an expression for $\lambda - w\mu$. In step two we calculate which conditions the coefficients of $\lambda - w\mu$ should satisfy. In step three we show that the conditions in step two are satisfied for the various possibilities of μ.

(1). The Weyl group element satisfies $w(n+1) = k$ for some $k \in \{1, 2, \ldots, n+1\}$. We have

$$\lambda - \mu = \sum_{i=1}^n \left(\sum_{j=1}^i (a_i - b_i) \right) \alpha_i \tag{2.12}$$

and the integers $\sum_{j=1}^i (a_i - b_i)$ are all non-negative. Secondly, we have

$$w\mu - \mu = \sum_{i=k}^n (b_{n+1} - b_i)\alpha_i. \tag{2.13}$$

In this case the coefficients are all non-negative because $b_{n+1} \geq b_i$ for $i \geq k$. Finally, we have

$$\lambda - w\mu = \sum_{i=1}^{k-1} \left(\sum_{j=1}^i (a_j - b_j) \right) \alpha_i + \sum_{i=k}^n \left(\sum_{j=1}^i (a_j - b_j) - (b_{n+1} - b_i) \right) \alpha_i.$$

The coefficients for $i = 1, \ldots, k-1$ are non-negative integers. If $i \geq k$ then we have

$$\sum_{j=1}^i (a_j - b_j) - (b_{n+1} - b_i) = \sum_{j=1}^{i-1} (a_j - b_j) + a_i - b_{n+1} = \sum_{j=i}^n (b_j - a_{j+1}) \geq 0.$$

(2). Write $\lambda - w\mu = \sum_{i=1}^n r_i\alpha_i$ for positive integers r_i. Let $\langle \cdot, \cdot \rangle$ denote inner product on $\mathbb{R} \otimes_{\mathbb{Z}} P_G^+$ dual to the Killing form. We have

$$\langle \lambda, \epsilon_1 - \epsilon_{n+1} \rangle = b_{w^{-1}1} - b_{w^{-1}(n+1)} + r_1 + r_{n+1} - 2r.$$

Hence we must have $b_{w^{-1}1} - b_{w^{-1}(n+1)} + r_1 + r_{n+1} - 2r = (b_1 - b_n + |b_1 + b_n - a_1 - a_{n+1}|)$.

(3). We distinguish three cases, according to in which Weyl chamber for G the element μ lies. The three cases are (i) $w(n+1) = n+1$, (ii) $w(n+1) = 1$ and (iii) $2 \leq w(n+1) \leq n$.

(i). $w(n+1) = n+1$. In this case $r_1 = a_1 - b_1$, $r_n = \sum_{i=1}^{n}(a_i - b_i)$ and $r = r_1$. Hence

$$b_{w1} - b_{w(n+1)} + r_1 + r_n - 2r = \\ b_1 - b_{n+1} + \sum_{i=2}^{n}(a_i - b_i) = b_1 - b_n + \left(b_1 + b_n + \sum_{i=2}^{n} a_i\right). \quad (2.14)$$

Note that $b_1 + b_n + \sum_{i=2}^{n} a_i = \sum_{i=2}^{n-1}(a_i - b_i) + a_n - b_{n+1}$ and $a_n - b_{n+1} \geq a_n - b_n \geq 0$. Hence $\lambda = b_\mu(\nu)$.

(ii). $w(1) = n+1$. Then $r_i = \sum_{j=i}^{n}(b_j - a_{j+1})$ and $r = r_n$. Hence

$$b_{w^{-1}1} - b_{w^{-1}(n+1)} + r_1 + r_n - 2r = b_1 - b_n - \left(b_1 + b_n + \sum_{i=2}^{n} a_i\right). \quad (2.15)$$

In this case we have $b_1 + b_n + a_2 + \ldots + a_n = \sum_{j=2}^{n-1} a_{j+1} - b_j + a_2 - b_{n+1}$. But $a_2 - b_{n+1} \leq a_2 - b_1 \leq 0$ which implies $b_1 + b_n - a_1 - a_{n+1} \leq 0$. Hence $\lambda = b_\mu(\nu)$.

(iii). $w(k) = n+1, 1 < k < n+1$. We have $r_i = \sum_{j=1}^{i}(a_j - b_j)$ if $i = 1, \ldots, k-1$ and $r_i = \sum_{j=i}^{n}(b_j - a_{j+1})$ if $i = k, \ldots, n$. Note that $r_1 \leq r_2 \cdots \leq r_{k-1}$ and $r_k \geq r_{k+1} \geq \cdots \geq r_n$. Hence $r_1 + r_n - 2r = r_1 - r_n$ if $r_n \leq r_1$ and $r_1 + r_n - 2r = r_n - r_1$ if $r_1 \leq r_n$. Suppose $r_1 \leq r_n$. We find

$$b_{w^{-1}1} - b_{w^{-1}(n+1)} + r_1 + r_n - 2r = b_1 - b_n + r_n - r_1 \quad (2.16)$$

and $0 \leq r_n - r_1 = b_n - a_{n+1} - a_1 + b_1$. Suppose $r_n \leq r_1$. We find

$$b_{w1} - b_{w(n+1)} + r_1 + r_n - 2r = b_1 - b_n + r_1 - r_n \quad (2.17)$$

and $0 \leq r_1 - r_n = a_1 - b_1 - b_n + a_{n+1}$. In both cases we conclude that $\lambda = b_\mu(\nu)$. □

Inverting the branching rules for $(\mathrm{Spin}_d(\mathbb{C}), \mathrm{Spin}_{d-1}(\mathbb{C}))$

The inversion of the branching law is taken from [Cam05a]. We treat the odd and even case simultaneously. Let $G = \mathrm{Spin}_d(\mathbb{C})$ and $K = \mathrm{Spin}_{d-1}(\mathbb{C})$ with $d \geq 3$ and consider the representation of G of highest weight ϖ_1. The fundamental weights of G are denoted by ϖ_i and those of K by ω_i. See Figure 2.6 for a picture in rank two.

The Killing form allows us to realize P_K in P_G as follows.

$$P_{\mathrm{Spin}_{2n}} \to P_{\mathrm{Spin}_{2n+1}} : \omega_i \mapsto \varpi_i, \quad (2.18)$$
$$P_{\mathrm{Spin}_{2n-1}} \to P_{\mathrm{Spin}_{2n}} : \omega_i \mapsto \varpi_{i+1}. \quad (2.19)$$

The spherical weight $\lambda_{\mathrm{sph}} = \epsilon_1$ can be rotated by an element in a copy of $\mathrm{Spin}_3(\mathbb{C})$ to a vector in $\mathfrak{k}^\perp$, which gives a torus $A \subset G$. $M = Z_K(A)$ is isomorphic to $\mathrm{Spin}_{d-2}(\mathbb{C})$.

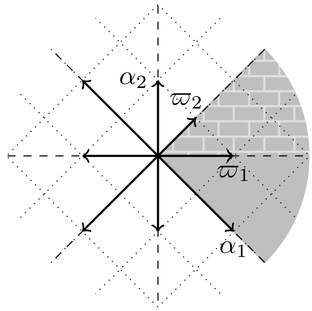

Figure 2.6: Roots and fundamental weights for $\mathfrak{so}_5$. The positive Weyl chamber of $\mathfrak{so}_4$ is light gray and it contains the positive Weyl chamber for $\mathfrak{so}_5$ in which we have laid some bricks.

Let $\lambda \in P_G^+$, $\mu \in P_K^+$ and $\nu \in P_M^+$. Write $\lambda = \sum_i a_i\epsilon_i$, $\mu = \sum_i b_i\epsilon_i$ and $\nu = \sum_i c_i\epsilon_i$. All the differences $a_i - a_j$, $b_i - b_j$ and $c_i - c_j$ are integers and

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq 0, \quad (2.20)$$
$$b_1 \geq b_2 \geq \cdots \geq |b_n| \geq 0, \quad (2.21)$$
$$c_2 \geq \cdots \geq c_n \geq 0 \quad (2.22)$$

if $d = 2n+1$ and

$$a_1 \geq a_2 \geq \cdots \geq |a_n| \geq 0, \quad (2.23)$$
$$b_2 \geq \cdots \geq b_n \geq 0, \quad (2.24)$$
$$c_2 \geq \cdots \geq |c_n| \geq 0 \quad (2.25)$$

if $d = 2n$. The branching rules from G to K and from K to M are classical, see e.g. [Kna02, Thm. 9.16].

Theorem 2.3.17. *Let $d = 2n+1$ and let λ, μ and ν be as above.*

- $m_\lambda(\mu) = 1$ *if and only if* $a_i - b_i \in \mathbb{Z}$, *and*

$$a_1 \geq b_1 \geq a_2 \geq \cdots \geq a_n \geq |b_n|.$$

- $m_\mu(\nu) = 1$ *if and only if* $b_i - c_i \in \mathbb{Z}$, *and*

$$b_1 \geq c_2 \geq b_2 \geq \cdots \geq c_n \geq |b_n|.$$

Theorem 2.3.18. *Let $d = 2n$ and let λ, μ and ν be as above.*

- $m_\lambda(\mu) = 1$ *if and only if* $a_i - b_i \in \mathbb{Z}$, *and*

$$a_1 \geq b_2 \geq a_2 \geq \cdots \geq b_n \geq |a_n|.$$

- *$m_\mu(\nu) = 1$ if and only if $b_i - c_i \in \mathbb{Z}$, and*

$$b_2 \geq c_2 \geq b_2 \geq \cdots \geq b_n \geq |c_n|.$$

Note that the projection $p : P_G^+ \to P_M^+$ is given by projection along ϵ_1, i.e. by putting $a_1 = 0$. We can obtain the inversion of the branching rule from G to K as follows. Given a K-type μ we determine the set $P_M^+(\mu)$ using the appropriate branching rule. The elements $\nu \in P_M^+$ can be lifted to P_G^+ by adding a number of $\lambda_{\mathrm{sph}} = \epsilon_1$. Adding the right vector $h \cdot \lambda_{\mathrm{sph}}$ lifts ν to the μ-well. In Figure 2.7 we have depicted a μ-well for $(\mathrm{Spin}_5(\mathbb{C}), \mathrm{Spin}_4(\mathbb{C}))$.

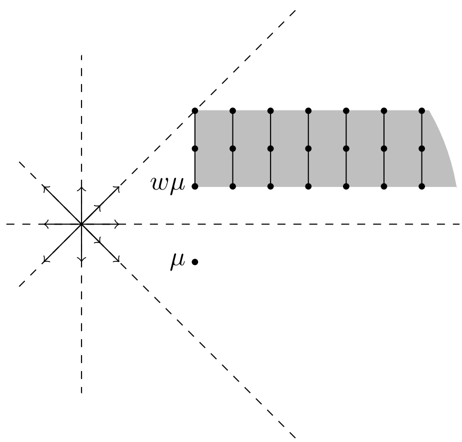

Figure 2.7: The μ-well for $(\mathfrak{so}_5, \mathfrak{so}_4)$ and $\mu = 4\omega_1 + 2\omega_2$.

Theorem 2.3.19. *Let $\mu \in P_K^+$ and write $\mu = \sum_i b_i\epsilon_i$.*

- *Let $d = 2n+1$. The map $P_M^+(\mu) \to B_\mu : \nu \mapsto \nu + b_1\lambda_{\mathrm{sph}}$ is a bijection.*
- *Let $d = 2n$. The map $P_M^+(\mu) \to B_\mu : \nu \mapsto \nu + b_2\lambda_{\mathrm{sph}}$ is a bijection.*

PROOF. The images of both maps are in the μ-well $P_G^+(\mu)$, which follows from the branching laws. Since the first inequalities $a_1 \geq b_1$ and $a_1 \geq b_2$ are sharp the result follows. □

Inverting the branching rules for $(\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{Sp}_{2n-2}(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C}))$

Let $G = \mathrm{Sp}_{2n}(\mathbb{C})$ and $K = \mathrm{Sp}_{2n-2}(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C})$ with $n \geq 3$. The weight lattice P_G is spanned by $\varpi_i = \sum_{j=1}^{i} \epsilon_j$ for $i = 1, \ldots, n$. The weight lattice P_K is spanned by $\omega_i = \sum_{j=1}^{i} \epsilon_j$ for $i = 1, \ldots, n-1$ and $\omega_n = \epsilon_n$. We choose the standard notion of positivity: P_G^+ and P_K^+ are generated by the ϖ_i and ω_i respectively. The spherical weight

$\lambda_{\mathrm{sph}} = \epsilon_1 + \epsilon_2$ lies in P_K^+. We choose another Weyl chamber for G for a moment so that λ_{sph} translates to $\lambda'_{\mathrm{sph}} = \epsilon_1 - \epsilon_n$ which is not a root of K. Inside the $\mathrm{SL}_2(\mathbb{C})$ copy in G with root λ'_{sph} we can rotate λ'_{sph} to a semisimple element in $\mathfrak{k}^{\perp}$. This gives a one-dimensional torus $A \subset G$. Define $M = Z_K(A)$. Then $M \cong \mathrm{Sp}_2(\mathbb{C}) \times \mathrm{Sp}_{2n-4}(\mathbb{C})$ and the embedding in K is as follows. Let $K_1 = \mathrm{Sp}_2(\mathbb{C}) \times \mathrm{Sp}_{2n-4}(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C}) \subset K$. Then $M \to K_1$ is given by $(x, y) \mapsto (x, y, x)$. The weight lattice of M is spanned by $\frac{1}{2}(\epsilon_1 + \epsilon_n)$ and the $\varpi_i - \epsilon_1$ for $2 \le i \le n-1$.

We write $\lambda \in P_G^+$ as a vector $\lambda = (a_1, \ldots, a_n)$ of integers that satisfy

$$a_1 \ge a_2 \ge \ldots \ge a_n \ge 0.$$

Similarly we write $P_K^+ \ni \mu = (b_1, \ldots, b_n)$ where the integers b_i satisfy

$$b_1 \ge b_2 \ge \ldots \ge b_{n-1} \ge 0, \quad b_n \ge 0$$

and $P_M^+ \ni \nu = (c_1, \ldots, c_n)$ where the integers c_i satisfy

$$c_1 = c_n, \quad 2c_1 \in \mathbb{N}, \quad c_2 \ge c_3 \ge \ldots \ge c_{n-1} \ge 0.$$

The branching rule for G to K is in the next theorem due to Lepowsky [Lep71], see [Kna02, Thm. 9.50] for a proof.

Theorem 2.3.20. *Let $\lambda = (a_1, \ldots, a_n) \in P_G^+$ and $\mu = (b_1, \ldots, b_n) \in P_K^+$. Define*

- $A_1 = a_1 - \max(a_2, b_1)$,
- $A_k = \min(a_k, b_{k-1}) - \max(a_{k+1}, b_k)$ *for* $2 \le k \le n-1$
- $A_n = \min(a_n, b_{n-1})$.

The multiplicity $m_\lambda(\mu) = 0$ unless all $A_i \ge 0$ and $b_1 + \sum_{i=1}^n A_i \in 2\mathbb{Z}$. In this case the multiplicity is given by

$$m_\lambda(\mu) = p_\Sigma(A_1\epsilon_1 + A_2\epsilon_2 + \cdots + (A_n - b_n)\epsilon_n) - p_\Sigma(A_1\epsilon_1 + A_2\epsilon_2 + \cdots + (A_n + b_n + 2)\epsilon_n)$$

where p_Σ is the multiplicity function for the set $\Sigma = \{\epsilon_i \pm \epsilon_n : 1 \le i \le n-1\}$.

The branching rule for K to M is due to Baldoni-Silva [BS79]. It is an application of Lepowsky's branching rule that uses the auxiliary group K_1. We do not need the generality of this formula. We rather need another formulation of Lepowsky's branching rule.

Theorem 2.3.21. *Let $G' = \mathrm{Sp}_{2n-2}(\mathbb{C})$ and let $K' = \mathrm{Sp}_2(\mathbb{C}) \times \mathrm{Sp}_{2n-4}(\mathbb{C})$. Let $\mu' = (b_1, \ldots, b_{n-1}) \in P_G^+$ and $\nu' = (c_1, \ldots, c_{n-1}) \in P_K^+$. Define*

- $C_1 = b_1 - \max(b_2, c_2)$,

- $C_k = \min(b_k, c_k) - \max(b_{k+1}, c_{k+1})$ *for* $2 \le k \le n-2$,
- $C_{n-1} = \min(b_{n-1}, c_{n-1})$.

The multiplicity $m_{\mu'}(\nu') = 0$ *unless all* $C_i \ge 0$ *and* $c_1 + \sum_{i=1}^{n-1} C_i \in 2\mathbb{Z}$. *In this case the multiplicity is given by*

$$m_{\mu'}(\nu') = p_\Xi((C_1 - c_1)\epsilon_1 + C_2\epsilon_2 + \cdots + C_n\epsilon_n) - p_\Xi((C_1 + c_1 + 2)\epsilon_1 + C_2\epsilon_2 + \cdots + C_{n-1}\epsilon_n)$$

where p_Ξ *is the multiplicity function for the set* $\Xi = \{\epsilon_i \pm \epsilon_1 : 2 \le i \le n-1\}$.

Because $\mu = x\omega_i + y\omega_j$ is special we have control over the multiplicity formulas p_Σ and p_Ξ.

Lemma 2.3.22. *(1). Let* $\mu = x\omega_i + y\omega_j$ *with* $1 \le i \le j \le n-1$. *If* $\lambda \in P_G^+(\mu)$ *then* $A_k = 0$ *unless* $k \in \{1, i+1, j+1\}$. *(2). Let* $\mu = x\omega_i + y\omega_n$ *with* $1 \le i \le n-2$. *If* $\lambda \in P_G^+$ *then* $A_k = 0$ *unless* $k \in \{1, i+1\}$.

Proof. Denote $\mu = (c_1, c_2, \ldots, c_n)$. (1). We may assume that $i < j$ for if $i = j$ then we are in case (2) with $y = 0$. If $2 \le k \le i$ or $i+2 \le k \le j$ or $j+2 \le k \le n-1$ then $A_k = \min(a_k, c_{k-1}) - \max(a_{k+1}, c_k) = c_{k-1} - c_k = 0$ because $a_k \ge c_k = c_{k-1}$ and $a_{k+1} \le c_{k-1} = c_k$. Furthermore $A_n = \min(a_n, c_{n-1}) = 0$ if $j \le n-2$. (2). Let $\mu = x\omega_i + y\omega_n$ with $1 \le i \le n-1$. If $2 \le k \le i$ or $i+2 \le k \le n-1$ then $c_{k-1} = c_k$ which implies $A_k = 0$. □

Lemma 2.3.23.
- *Let* $A_i \in \mathbb{N}$ *for* $i = 1, 2, 3$ *and define* $A = \sum_{i=1}^3 A_i/2$. *Let* $\Sigma = \{\epsilon_i \pm \epsilon_4 : 1 \le i \le 3\}$. *Then* $p_\Sigma(\sum_{i=1}^3 A_i\epsilon_i) - p_\Sigma(\sum_{i=1}^3 A_i\epsilon_i + 2\epsilon_4) = 1$ *if and only if* $\max(A_i) \le A$ *and* $A \in \mathbb{Z}$.
- *Let* $A_1, A_2 \in \mathbb{N}$ *and define* $A = (A_1 + A_2)/2$. *Let* $y \in \mathbb{N}$. *Let* $\Sigma = \{\epsilon_i \pm \epsilon_3 : 1 \le i \le 2\}$. *Then* $p_\Sigma(A_1\epsilon_1 + A_2\epsilon_2 - y\epsilon_3) - p_\Sigma(A_1\epsilon_1 + A_2\epsilon_2 + (y+2)\epsilon_3) = 1$ *if and only if* $A - y/2 \in \mathbb{Z}$ *and* $y \le A_1 + A_2$, $A_1 \le A_2 + y$ *and* $A_2 \le A_1 + y$.
- *Let* $C_1, C_2, c_1 \in \mathbb{N}$ *and let* $\Xi = \{\epsilon_2 \pm \epsilon_1\}$. *Then* $p_\Xi((C_1 - c_1)\epsilon_1 + C_2\epsilon_2) - p_\Xi((C_1 + c_1 + 2)\epsilon_1 + C_2\epsilon_2) = 1$ *if and only if* $C_1 + C_2 - c_1$ *is even and* $C_1 + C_2 - c_1 \ge 0$ *and* $C_2 - C_1 + c_1 \ge 0$.

Proof. We only prove the first statement. The others are proved in a similar but simpler fashion. We have

$$\sum_{k=1}^{3} A_k\epsilon_k = \sum_{k=1}^{3} B_k(\epsilon_k + \epsilon_4) + \sum_{k=1}^{3}(A_k - B_k)(\epsilon_k - \epsilon_4)$$

if and only if $\sum_{i=1}^3 B_k = A$. It follows that

$$\mathcal{P}_{\Sigma'}(A_1\epsilon_1 + A_2\epsilon_2 + A_3\epsilon_3) = \#\{(B_1, B_2, B_3) \in \mathbb{N}^3 : \sum_{k=1}^{3} B_k = A \text{ and } B_k \le A_k\}$$

and similarly

$$p_\Sigma(A_1\epsilon_1 + A_2\epsilon_2 + A_3\epsilon_3 + 2\epsilon_4) = \\ \#\{(B_1, B_2, B_3) \in \mathbb{N}^3 : \sum_{k=1}^{3} B_k = A+1 \text{ and } B_k \le A_k\}. \quad (2.26)$$

Assume that $A_1 \ge A_2 \ge A_3$. We distinguish two possibilities: (1) $A_1 \le A$ and (2) $A_1 > A$. In case (1) we have

$$p_\Sigma(\sum_{i=1}^{3} A_i\epsilon_i) = \#\{\text{lattice points in hexagon indicated in Figure 2.8}\}$$

which is given by

$$p_\Sigma(\sum_{i=1}^{3} A_i\epsilon_i) = (A+1)(A+2)/2 - \sum_{i=1}^{3}(A-A_i)(A-A_i+1)/2.$$

Similarly

$$p_\Sigma(\sum_{i=1}^{3} A_i\epsilon_i + 2\epsilon_4) = (A+2)(A+3)/2 - \sum_{i=1}^{3}(A+1-A_i)(A-A_i+2)/2$$

and the difference is one, as was to be shown.

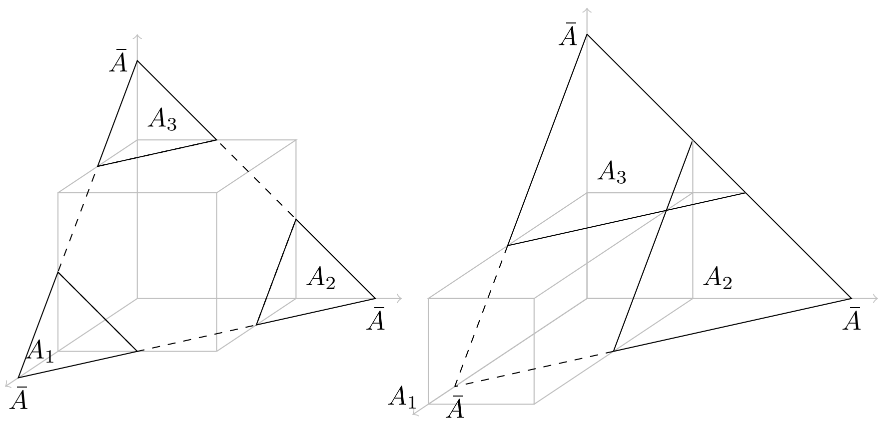

Figure 2.8: Counting integral points.

In case (2) where $A_1 > A$ we have

$$p_\Sigma(\sum_{i=1}^{3} A_i\epsilon_i) = \#\{\text{lattice points in parallelogram in Figure 2.8}\}$$

which is given by A_2A_3. Similarly $p_\Sigma(\sum_{i=1}^3 A_i\epsilon_i + 2\epsilon_4) = A_2A_3$ and hence the difference is zero. ☐

2.3.24. The projection in P_G^+ along the direction λ_{sph} is denoted by p_{sph}. Let $\lambda = (a_1, \ldots, a_n)$. Then $p_{\mathrm{sph}}(\lambda)$ is the form $(c_1, -c_1, a_3, \ldots, a_n)$. Application of a suitable Weyl group element $w \in W_G$ gives $w(p_{\mathrm{sph}}(\lambda)) = (c_1, c_3, \ldots, c_{n-1}, c_1) \in P_M^+$. This element corresponds to the projection of $w\lambda$ along $\lambda'_{\mathrm{sph}} = \epsilon_1 - \epsilon_n$. Hence this is the highest weight of the M-representation in $V_\lambda^{\mathfrak{n}^+}$ In order to apply Theorem 2.3.21 in the proof of the next result we need to make use of this translation.

Theorem 2.3.25. *Denote $P_M^+ \ni \nu = (c_1, \ldots, c_n)$.*

(i) Let $\mu = x\omega_i + y\omega_j$ with $2 \le i \le j \le n-1$. Then the map $b_\mu : P_M^+(\mu) \to B_\mu$ is given by

$$b_\mu(\nu) = (c_1, -c_1, c_2, \ldots, c_{n-1}) + (c_1 + x + y)\lambda_{\mathrm{sph}}.$$

(ii) Let $\mu = x\omega_i + y\omega_n$ with $2 \le i \le n-1$. Then the map $b_\mu : P_M^+(\mu) \to B_\mu$ is given by

$$b_\mu(\nu) = (c_1, -c_1, c_2, \ldots, c_{n-1}) + (c_1 + x)\lambda_{\mathrm{sph}}.$$

(iii) Let $\mu = x\omega_1 + y\omega_j$ with $2 \le j \le n-1$. Then the map $b_\mu : P_M^+(\mu) \to B_\mu$ is given by

$$b_\mu(\nu) = (c_1, -c_1, c_2, \ldots, c_{n-1}) + \frac{1}{2}(x + y + C_j + \max(c_2, y))\lambda_{\mathrm{sph}}.$$

(iv) Let $\mu = x\omega_1 + y\omega_n$. Then the map $b_\mu : P_M^+(\mu) \to B_\mu$ is given by

$$b_\mu(\nu) = (c_1, -c_1, c_2, \ldots, c_{n-1}) + \frac{1}{2}(c_2 + x + y)\lambda_{\mathrm{sph}}.$$

PROOF. Write $\mu = (b_1, \ldots, b_n)$ and let $\lambda = (a_1, \ldots, a_n) \in P_G^+(\mu)$. Let A_i and C_i be as in Theorems 2.3.20 and 2.3.21. Write $p_{\mathrm{sph}}(\lambda) = (b, -b, a_3, \ldots, a_n)$.

(i). Lemma 2.3.22 shows that $A_k = 0$ unless $k \in \{1, i+1, j+1\}$. Lemma 2.3.23 implies that $\max(A_k) \le \frac{1}{2}(A_1 + A_{i+1} + A_{j+1})$. Since $A_1 = a_1 - a_2$, this inequality is invariant for adding $\mathbb{Z}$-multiples of λ_{sph} to λ. Hence the inequalities $A_k \ge 0$ determine whether the multiplicity is zero or one. The smallest value $z \in \frac{1}{2}\mathbb{N}$ for which $p_{\mathrm{sph}}(\lambda) + z\lambda_{\mathrm{sph}} \in P_G^+(\mu)$ is $z = b + x + y$. The proof of (ii) is completely analogous.

(iii) Lemma 2.3.22 shows that $A_k = 0$ unless $k \in \{1, 2, j+1\}$. Lemma 2.3.23 implies that $\max(A_k) \le \frac{1}{2}(A_1 + A_2 + A_{j+1})$. Two inequalities are invariant for adding $\mathbb{Z}$-multiples of λ_{sph} to λ. The third is $A_1 + A_2 \ge A_{j+1}$ which implies $a_1 + a_2 \ge \frac{1}{2}(\max(a_3, y) + A_{j+1} + x + y)$. Together with $A_1 \ge$ and $A_2 \ge 0$ this implies that the smallest $z \in \frac{1}{2}\mathbb{N}$ for which $p_{\mathrm{sph}}(\lambda) + z\lambda_{\mathrm{sph}} \in P_G^+(\mu)$ is

$$z = \max(x + y - b, y + b, \frac{1}{2}(\max(a_3, y) + A_{j+1} + x + y)).$$

The element $p_{\rm sph}(\lambda)$ corresponds to $\nu \in P_M^+(\mu)$ via $\nu = (b, a_2, \ldots, a_n, b)$. In order to determine z we need an estimate on b. To this end we consider the branching of the irreducible representation of G' of highest weight μ upon restriction to K'. We use Theorem 2.3.21. In this case $C_k = 0$ unless $k = 1$ or $k = j$. The inequalities in the third statement of Lemma 2.3.23 with $2 = j$ imply, after a small computation, that $z = \frac{1}{2}(\max(a_3, y) + A_{j+1} + x + y)$.

(iv). Lemma 2.3.22 shows that $A_k = 0$ unless $k \in \{1, 2\}$. Lemma 2.3.23 implies that $\min(A_k) \le \frac{1}{2}(A_1 + A_2 + A_{j+1})$. Two inequalities are invariant for adding $\mathbb{Z}$-multiples of $\lambda_{\rm sph}$ to λ. The third is $A_1 + A_2 \ge y$ which implies $a_1 + a_2 = \frac{1}{2}(x + y + a_3)$. Together with $A_1 \ge 0$ and $A_2 \ge 0$ this implies that the smallest $z \in \frac{1}{2}\mathbb{N}$ for which $p_{\rm sph}(\lambda) + z\lambda_{\rm sph} \in P_G^+(\mu)$ is

$$z = \max(x - b, b, \frac{1}{2}(a_3 + x + y)).$$

The element $p_{\rm sph}(\lambda)$ corresponds to $\nu \in P_M^+(\mu)$ via $\nu = (b, a_3, \ldots, a_n, b)$. In order to determine z we need an estimate on b and a_3. To this end we consider the branching of the irreducible representation of G' of highest weight $\mu - y\epsilon_n = x\epsilon_1$ upon restriction to K'. We use Theorem 2.3.21. In this case $C_k = 0$ unless $k = 1$. The inequalities in the third statement of Lemma 2.3.23 imply that $c_1' + c_2' = x$ whenever $\nu = (c_1', c_2', \ldots, c_{n-1}')$ occurs in the decomposition of $x\epsilon_1$. The diagonal embedding of the factor $\mathrm{Sp}_2(\mathbb{C})$ of M in K_1 implies that the weight (ν', y) decomposes into M types $(c_1, c_2', \ldots, c_{n-1}', c_1)$ with c_1 running from $\frac{1}{2}|c_1' - y|$ to $\frac{1}{2}(c_1' + y)$. Since $c_2' = a_3$ we find, after a small computation, that $z = \frac{1}{2}(x + y + a_3)$. □

In Figure 2.9 we have depicted a μ-well in the rank three case. Note that the bottom is linear and that the affine rank of the bottom is at most six. Indeed, with at most three numbers A_i non-zero we can have only 6 parameters a_i varying.

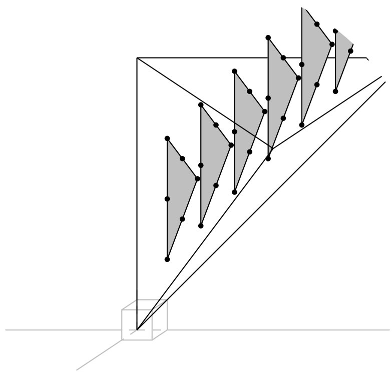

Figure 2.9: The μ-well for $(\mathfrak{sp}_6(\mathbb{C}), \mathfrak{sp}_4(\mathbb{C}) \oplus \mathfrak{sp}_2(\mathbb{C}))$ and $\mu = 2\omega_2 + 3\omega_3$.

Inverting the branching rules for $(\mathrm{Spin}_7(\mathbb{C}), \mathrm{G}_2)$

In this subsection we take $G = \mathrm{Spin}_7(\mathbb{C})$ with Lie algebra $\mathfrak{g}$ of type B_3. Let $\mathfrak{t}_G \cong \mathbb{C}^3$ be a Cartan subalgebra with positive roots R_G^+ given by

$$e_i - e_j,\ e_i + e_j,\ e_i$$

for $1 \leq i < j \leq 3$, and basis of simple roots $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3$. The fundamental weights $\varpi_1 = e_1, \varpi_2 = e_1 + e_2, \varpi_3 = (e_1 + e_2 + e_3)/2$ are a basis over $\mathbb{N}$ for the set P_G^+ of dominant weights. As the Cartan subalgebra $\mathfrak{t}_K$ for $K = \mathrm{G}_2$ we shall take the orthogonal complement of $h = (-e_1 + e_2 + e_3)$. The elements $e_1 + e_3, e_1 + e_2, e_2 - e_3$ are the long positive roots in R_K^+, while

$$\epsilon_1 = (2e_1 + e_2 + e_3)/3,\ \epsilon_2 = (e_1 + 2e_2 - e_3)/3,\ \epsilon_3 = (e_1 - e_2 + 2e_3)/3$$

are the short positive roots in R_K^+. The natural projection $q : R_G^+ \to P_K^+$ is a bijection onto the long roots and two to one onto the short roots in R_K^+. Note that $\epsilon_i = q(e_i)$ for $i = 1, 2, 3$. The simple roots in R_K^+ are $\{\beta_1 = \epsilon_3, \beta_2 = \epsilon_2 - \epsilon_3\}$ with corresponding fundamental weights $\{\omega_1 = \epsilon_1, \omega_2 = \epsilon_1 + \epsilon_2\}$. Observe that $\omega_1 = q(\varpi_1) = q(\varpi_3)$ and $\omega_2 = q(\varpi_2)$, and hence $q : P_G^+ \to P_K^+$ is a surjection. Note that the natural projection $q : P_G \to P_K$ is equivariant for the action of the Weyl group $W_M \cong \mathfrak{S}_3$ of the centralizer $M = \mathrm{SL}_3(\mathbb{C})$ in K of h. The Weyl group W_G is the semidirect product of $C_2 \times C_2 \times C_2$ acting by sign changes on the three coordinates and the permutation group $\mathfrak{S}_3$.

As a set with multiplicities we have

$$A = q(R_G^+) - R_K^+ = \{\epsilon_1, \epsilon_2, \epsilon_3\}$$

whose partition function p_A enters in the formula for the branching from B_3 to G_2. Note that $p_A(k\epsilon_1 + l\epsilon_2) = p_A(k\epsilon_1 + m\epsilon_3) = k + 1$ for $k, l, m \in \mathbb{N}$ and $p_A(\mu) = 0$ otherwise.

Lemma 2.3.26. *For $\lambda \in P_G^+$ and $\mu \in P_K^+$ the multiplicity $m_\lambda^{G,K}(\mu) \in \mathbb{N}$ with which an irreducible representation of K with highest weight μ occurs in the restriction to K of an irreducible representation of G with highest weight λ is given by*

$$m_\lambda^{G,K}(\mu) = \sum_{w \in W_G} \det(w) p_A(q(w(\lambda + \rho_G) - \rho_G) - \mu)$$

and if we extend $m_\lambda^{G,K}(\mu) \in \mathbb{Z}$ by this formula for all $\lambda \in P_G$ and $\mu \in P_K$ then

$$m_{w(\lambda+\rho_G)-\rho_G}^{G,K}(v(\mu + \rho_K) - \rho_K) = \det(w)\det(v) m_\lambda^{G,K}(\mu)$$

for all $w \in W_G$ and $v \in W_K$. Here ρ_G and ρ_K are the Weyl vectors of R_G^+ and R_K^+ respectively, i.e. half the sum of the positive roots.

This lemma was obtained in Heckman [Hec82] as a direct application of the Weyl character formula. The above type formula, valid for any pair $K < G$ of connected compact Lie groups [Hec82], might be cumbersome for practical computations of the multiplicities, because of the (possibly large) alternating sum over a Weyl group W_G and the piecewise polynomial behaviour of the partition function. However in the present (fairly small) example one can proceed as follows.

If $\lambda = k\varpi_1 + l\varpi_2 + m\varpi_3 = klm = (x, y, z)$ with

$$x = k + l + m/2, y = l + m/2, z = m/2 \Leftrightarrow k = x - y, l = y - z, m = 2z$$

then λ is dominant if $k, l, m \geq 0$ or equivalently if $x \geq y \geq z \geq 0$. We tabulate the 8 elements $w_1, \cdots, w_8 \in W_G$ such that the projection $q(w_i\lambda) \in \mathbb{N}\epsilon_1 + \mathbb{N}\epsilon_2$ is dominant for R_M^+ for all λ which are dominant for R_G^+. Clearly the projection of (x, y, z) is given by

$$q(x, y, z) = x\epsilon_1 + y\epsilon_2 + z\epsilon_3 = (x + z)\epsilon_1 + (y - z)\epsilon_2$$

and $\rho_G = \varpi_1 + \varpi_2 + \varpi_3 = (2\frac{1}{2}, 1\frac{1}{2}, \frac{1}{2})$ is the Weyl vector for R_G^+.

i	$\det(w_i)$	$w_i\lambda$	$q(w_i\lambda)$	$q(w_i\rho_G - \rho_G)$
1	$+$	(x, y, z)	$(x + z)\epsilon_1 + (y - z)\epsilon_2$	0
2	$-$	$(x, y, -z)$	$(x - z)\epsilon_1 + (y + z)\epsilon_2$	$-\epsilon_3$
3	$+$	$(x, z, -y)$	$(x - y)\epsilon_1 + (y + z)\epsilon_2$	$-\epsilon_1 - \epsilon_3$
4	$-$	$(x, -z, -y)$	$(x - y)\epsilon_1 + (y - z)\epsilon_2$	$-\epsilon_1 - \epsilon_2 - \epsilon_3$
5	$-$	(y, x, z)	$(y + z)\epsilon_1 + (x - z)\epsilon_2$	$-\epsilon_3 + 0$
6	$+$	$(y, x, -z)$	$(y - z)\epsilon_1 + (x + z)\epsilon_2$	$-\epsilon_3 - \epsilon_3$
7	$+$	(z, x, y)	$(y + z)\epsilon_1 + (x - y)\epsilon_2$	$-\epsilon_3 - \epsilon_2$
8	$-$	$(-z, x, y)$	$(y - z)\epsilon_1 + (x - y)\epsilon_2$	$-\epsilon_3 - \epsilon_1 - \epsilon_2$

In Figure 2.10, the location of the points $q(w_i\lambda) \in P_M^+$, indicated by the number i, with the sign of $\det(w_i)$ attached, is drawn. Observe that $q(w_1\lambda) = (k+m)\omega_1 + l\omega_2 \in P_K^+$ for all $\lambda = klm \in P_G^+$.

Let us denote $a = (k+l+m)\epsilon_1$ and $b = (k+l)\epsilon_1$, and so these two points together with the four points $q(w_i\lambda)$ for $i = 1, 2, 3, 4$ form the vertices of a hexagon with three pairs of parallel sides. In the picture we have drawn all six vertices in P_K^+, which happens if and only if $q(w_3\lambda) = k\epsilon_1 + (l+m)\epsilon_2 \in P_K^+$, or equivalently if $k \geq (l+m)$. But in general some of the $q(w_i\lambda) \in P_M^+$ for $i = 2, 3, 4$ might lie outside P_K^+. Indeed $q(w_2\lambda) = (k+l)\epsilon_1 + (l+m)\epsilon_2$ lies outside P_K^+ if $k < m$, and $q(w_4\lambda) = k\epsilon_1 + l\epsilon_2$ lies outside P_K^+ if $k < l$.

For fixed $\lambda \in P_G^+$, the sum $m_\lambda(\mu)$ of the following six partition functions as a function of $\mu \in P_K$

$$\sum_1^4 \det(w)p_A(q(w_i(\lambda + \rho_G) - \rho_G) - \mu) - p_A(a - \epsilon_2 - \mu) + p_A(b - \epsilon_1 - \epsilon_2 - \mu)$$

is just the familiar multiplicity function for the weight multiplicities of the root system A_2. It vanishes outside the hexagon with vertices a, b and $q(w_i\lambda)$ for $i = 1, 2, 3, 4$. On the outer shell hexagon it is equal to 1, and it steadily increases by 1 for each inner shell

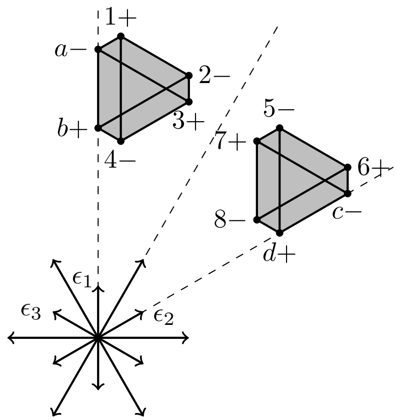

Figure 2.10: Projection of $W_G\lambda$ onto P_M^+.

hexagon, until the hexagon becomes a triangle, and from that moment on it stabilizes on the inner triangle. The two partition functions we have added corresponding to the points a and b are invariant as a function of μ for the action $\mu \mapsto s_2(\mu+\rho_K)-\rho_K$ of the simple reflection $s_2 \in W_K$ with mirror $\mathbb{R}\omega_1$, because $s_2(A)=A$. In order to obtain the final multiplicity function

$$\mu \mapsto m_\lambda^{G,K}(\mu) = \sum_{v\in W_K} \det(v) m_\lambda(v(\mu+\rho_K)-\rho_K)$$

for the branching from G to K we have to antisymmetrize for the shifted by ρ_K action of W_K. Note that the two additional partition functions and their transforms under W_K all cancel because of their symmetry and the antisymmetrization. For $\mu \in P_K^+$ the only terms in the sum over $v \in W_K$ that have a non-zero contribution are those for $v=e$ the identity element and $v=s_1$ the reflection with mirror $\mathbb{R}\omega_2$, and we arrive at the following result.

Theorem 2.3.27. *For $\lambda \in P_G^+$ and $\mu \in P_K^+$ the branching multiplicity from $G = Spin_7(\mathbb{C})$ to $K = \mathrm{G}_2$ is given by*

$$m_\lambda^{G,K}(\mu) = m_\lambda(\mu) - m_\lambda(s_1\mu - \epsilon_3)$$

with m_λ the weight multiplicity function of type A_2 as given by the above alternating sum of the six partition functions.

Indeed, we have $s_1(\mu+\rho_K)-\rho_K = s_1\mu - \epsilon_3$. As before, we denote $klm = k\varpi_1 + l\varpi_2 + m\varpi_3$ and $kl = k\omega_1 + l\omega_2$ with $k,l,m \in \mathbb{N}$ for the highest weight of irreducible representations of G and K respectively. For $\mu \in \mathbb{N}\omega_1$ the multiplicities $m_\lambda^{G,K}(\mu)$ are only governed by the first term with $v=e$ and so are equal to 1 for $\mu = n0$ with

$n = (k+l), \cdots, (k+l+m)$ and 0 elsewhere. Indeed $\mu = n0$ has multiplicity one if and only if it is contained in the segment from $b = (k+l)\epsilon_1$ to $a = (k+l+m)\epsilon_1$.

Corollary 2.3.28. *The fundamental representation of G with highest weight $\lambda = 001$ is the spin representation of dimension 8 with K-types $\mu = 10$ and $\mu = 00$. It is the fundamental spherical representation for the Gel'fand pair (G, K). The irreducible spherical representations of G have highest weights $00m$ with K-spectrum the set $\{n0; 0 \leq n \leq m\}$.*

Corollary 2.3.29. *For any irreducible representation of G with highest weight $\lambda = klm$ all K-types with highest weight $\mu \in F_1 = \mathbb{N}\omega_1$ are multiplicity free, and the K-type with highest weight $\mu = n0$ has multiplicity one if and only if $(k+l) \leq n \leq (k+l+m)$. The domain of those $\lambda = klm$ for which the K-type $\mu = n0$ occurs has a well shape $W_{n0} = B_{n0} + \mathbb{N}001$ with bottom*

$$B_{n0} = \{klm \in P_G^+; k+l+m = n\}$$

given by a single linear relation.

Indeed, if we denote by W_{n0} the set of $\lambda = klm \in P_G^+$ for which $\mu = n0$ occurs in the corresponding K-spectrum, then $klm \in W_{n0}$ implies that $kl(m+1) \in W_{n0}$. If we denote by B_{n0} the set of those $klm \in W_{n0}$ for which $kl(m-1) \notin W_{n0}$, then $klm \in B_{n0}$ implies $n = (k+l+m)$ by the first part of the above corollary. This ends our discussion that $(G, K, F_1 = \mathbb{N}\omega_1)$ is a multiplicity free triple. In order to show that $(G, K, F_2 = \mathbb{N}\omega_2)$ is also a multiplicity free triple we shall carry out a similar analysis.

Corollary 2.3.30. *For an irreducible representation of G with highest weight $\lambda = klm$ all K-types with highest weight $\mu \in F_2 = \mathbb{N}\omega_2$ are multiplicity free, and the K-type with highest weight $\mu = 0n$ has multiplicity one if and only if $\max(k, l) \leq n \leq \min(k+l, l+m)$. The domain of those $\lambda = klm$ for which the K-type $\mu = 0n$ occurs has a well shape $W_{0n} = B_{0n} + \mathbb{N}001$ with bottom*

$$B_{0n} = \{klm \in P_G^+; m \leq k \leq n, l+m = n\}$$

given by a single linear relation and inequalities.

As before let W_{0n} the set of $klm \in P_G^+$ for which $\mu = 0n$ occurs in the corresponding K-spectrum. Under the assumption of the first part of this proposition $klm \in W_{0n}$ implies that $kl(m+1) \in W_{0n}$, and the bottom B_{0n} of those $klm \in W_{0n}$ for which $kl(m-1) \notin W_{0n}$ contains klm if and only if $n = l + m$ and $k \geq m$. It remains to show the first part of the proposition.

In order to determine the K-spectrum associated to the highest weight $\lambda = klm \in \mathbb{N}^3$ for G observe that

$$q(w_3\lambda) = k\epsilon_1 + (l+m)\epsilon_2$$

and so the K-spectrum on $\mathbb{N}\omega_2$ is empty for $k > (l+m)$, while for $k = (l+m)$ the K-spectrum has a unique point $k\omega_2$ on $\mathbb{N}\omega_2$. If $k < (l+m)$, the point $q(w_3\lambda)$ moves out of

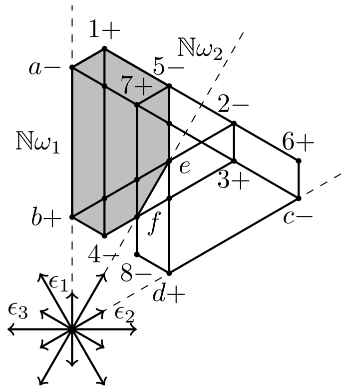

Figure 2.11: Support of the multiplicity function $\mu \mapsto m_\lambda(\mu)$.

the dominant set P_K^+ into $P_M^+ - P_K^+$, and the support of the function $P_K^+ \ni \mu \mapsto m_\lambda^{G,K}(\mu)$ consists of (the integral points of) a heptagon with an additional side on $\mathbb{N}\omega_2$ from e to f as in Figure 2.11. On the outer shell heptagon the multiplicity is one, and the multiplicities increase by one for each inner shell heptagon, until the heptagon becomes a triangle or quadrangle, and it stabilizes. This follows from Theorem 2.3.27 in a straightforward way.

Depending on whether the vertex

$$q(w_2\lambda) = (k+l)\epsilon_1 + (l+m)\epsilon_2$$

lies in P_K^+ (for $k \geq m$) or in $P_M^+ - P_K^+$ (for $k < m$) we get $e = (l+m)\omega_2$ or $e = (k+l)\omega_2$ respectively. Hence we find

$$e = \min(k+l, l+m)\omega_2, \ f = \max(k,l)\omega_2$$

by a similar consideration for

$$q(w_4\lambda) = k\epsilon_1 + l\epsilon_2$$

as before ($f = k$ for $k \geq l$ and $f = l$ for $k < l$). This finishes the proof of Corollary 2.3.30.

Our choice of positive roots for $G = B_3$ and $K = \mathrm{G}_2$ was made in such a way that the dominant set P_K^+ for K was contained in the dominant set P_G^+ for G. In turn this implies that the set

$$A = q(R_G^+) - R_K^+ = \{\epsilon_1, \epsilon_1, \epsilon_3\}$$

lies in an open half plane, which was required for the application of the branching rule of Lemma 2.3.26.

However, we now switch to a different positive system in R_G, or rather we keep R_G^+ fixed as before, but take the Lie algebra $\mathfrak{k}$ of G_2 to be perpendicular to the spherical direction $\varpi_3 = (e_1 + e_2 + e_3)/2$ instead. Under this assumption the positive roots R_M^+ form a parabolic subsystem in R_G^+, and so the simple roots $\{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3\}$ of R_M^+ are also simple roots in R_G^+.

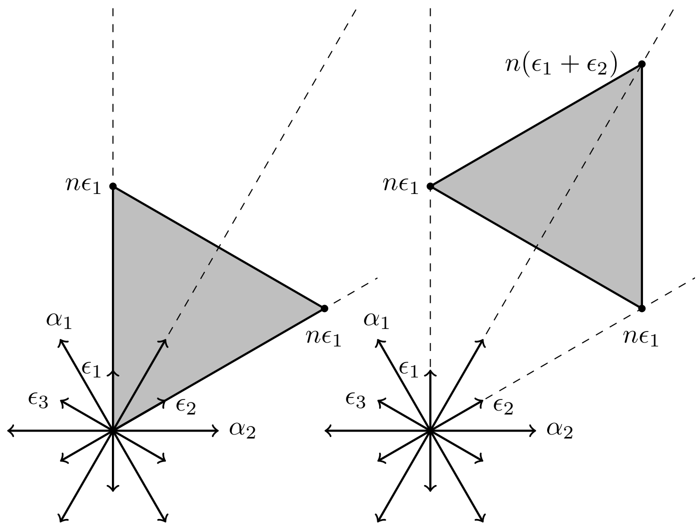

Figure 2.12: Projections of the bottoms B_{n0} and B_{0n}.

Let $p : P_G \to P_M = P_K$ be the orthogonal projection along the spherical direction. By abuse of notation we denote (with $p(\varpi_3) = 0$)

$$\epsilon_1 = p(\varpi_1) = (2, -1, -1)/3 \ , \ \epsilon_2 = p(\varpi_2) = (1, 1, -2)/3$$

for the fundamental weights for $P_M^+ = p(P_G^+)$. It is now easy to check that this projection

$$p : B_{n0} \to p(B_{n0}) \ , \ p : B_{0n} \to p(B_{0n})$$

is a bijection from the bottom onto its image in P_M^+. In Figure 2.12 we have drawn the projections

$$p(B_{n0}) = \{k\epsilon_1 + l\epsilon_2; k + l \leq n\} \ , \ p(B_{0n}) = \{k\epsilon_1 + l\epsilon_2; k, l \leq n, k + l \geq n\}$$

on the left and the right side respectively.

In fact, it follows from general principles that the orthogonal projection p along the spherical direction is a bijection from the bottom B_μ of the induced G-spectrum W_μ onto its image $p(B_\mu)$ in P_M^+ for $\mu \in P_K^+ \cap F$ and (G, K, F) any multiplicity free triple. Moreover the image $p(B_\mu)$ is just the M-spectrum of the irreducible representation of K with highest weight μ. In our example this is clear from the familiar branching rule from G_2 to A_2, which we recall in the next section.

Inverting the branching rules for $(\mathrm{G}_2, \mathrm{SL}_3(\mathbb{C}))$

In this subsection we take G of type G_2 and $K = \mathrm{SL}_3(\mathbb{C})$ a subgroup of type A_2. Having the same rank we take $\mathfrak{t}_G = \mathfrak{t}_K$ with simple roots $\{\alpha_1, \alpha_2\}$ in R_G^+ and $\{\beta_1, \beta_2\}$ in R_K^+ as in the Figure 2.13 and $P_G^+ = \mathbb{N}\varpi_1 + \mathbb{N}\varpi_2$ is contained in $P_K^+ = \mathbb{N}\omega_1 + \mathbb{N}\omega_2$.

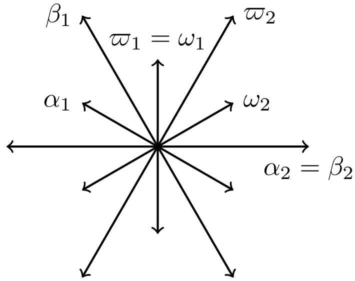

Figure 2.13: Roots and weights for G_2 and $\mathrm{SL}_3(\mathbb{C})$.

The branching rule from G to K is well known, see for example [Hec82]. In Figure 2.14 $s_1 \in W_G$ is the orthogonal reflection in the mirror $\mathbb{R}\varpi_2$.

The multiplicities are one on the outer hexagon, and increase by one on each inner shell hexagon, until the hexagon becomes a triangle and they stabilize. Hence the K-spectrum of any irreducible representation of G with highest weight $\lambda \in P_G^+$ is multiplicity free on the two faces $\mathbb{N}\omega_1$ and $\mathbb{N}\omega_2$ of the dominant set P_K^+. In other words, the triples $(\mathrm{G}_2, \mathrm{SL}_3(\mathbb{C}), F_i = \mathbb{N}\omega_i)$ are multiplicity free for $i = 1, 2$.

The irreducible spherical representations of G have highest weight in $\mathbb{N}\varpi_1$. Given $\mu = n\omega_1 \in F_1$ (and likewise $\mu = n\omega_2 \in F_2$) the corresponding induced G-spectrum is multiplicity free by Frobenius reciprocity, and by inversion of the branching rule has multiplicity one on the well shaped region

$$W_\mu = B_\mu + \mathbb{N}\varpi_1 \ , \ B_\mu = \{k\varpi_1 + l\varpi_2 ; k + l = n\}$$

with bottom B_μ. As in the previous section the bottom is given by a single linear relation. If we take M the $\mathrm{SL}_2(\mathbb{C})$ corresponding to the roots $\{\pm\alpha_2\}$ and denote by $p : P_G^+ \to P_M^+ = \mathbb{N}(\frac{1}{2}\alpha_2)$ the natural projection along the spherical direction ϖ_1, then p is a bijection from the bottom B_μ onto the image $p(B_\mu)$, which is just the M-spectrum of the irreducible representation of K with highest weight μ, as should.

2.4 Module structure

2.4.1. Let (G, K, F) be a multiplicity free system with (G, K) a spherical pair of rank one and let $\mu \in F$. In Section 2.3 we have calculated the μ-wells. We have seen that the

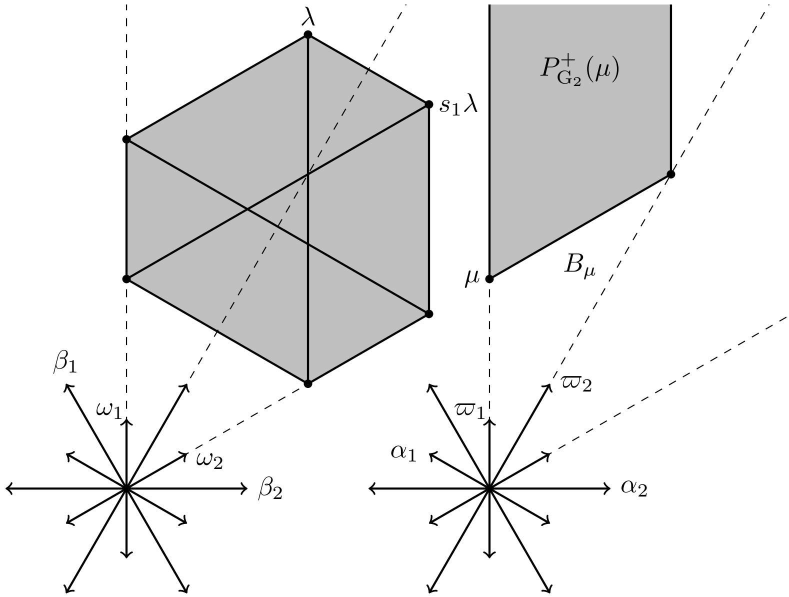

Figure 2.14: Branching from G_2 to $SL_3(\mathbb{C})$ on the left and the μ-well on the right.

projection $p_\mu : P^+_G(\mu) \to P^+_M(\mu)$ from 2.3.7 is surjective. Proposition 2.3.9 implies that the map

$$\lambda : \mathbb{N} \times P^+_M(\mu) \to P^+_G : (d, \nu) \mapsto b_\nu(\mu) + d\lambda_{\text{sph}} \tag{2.27}$$

is a bijection of sets. The degree of $\lambda = \lambda(d, \nu)$ is defined to be d. The μ-bottom $B_\mu \subset P^+_G$ inherits the standard partial ordering $\preceq$ which in turn induces a partial ordering on $P^+_M(\mu)$ that we denote by $\leq_\mu$. Together with the standard ordering on $\mathbb{N}$ this gives the lexicographic ordering $(\leq, \leq_\mu)$ on $\mathbb{N} \times P^+_M(\mu)$. Via the isomorphism $\lambda : \mathbb{N} \times P^+_M(\mu) \to P^+_G$ we push $(\leq, \leq_\mu)$ to a partial ordering on $P^+_G(\mu)$ that we denote by $\preceq_\mu$.

Let $\lambda = \lambda(d, \nu) \in P^+_G(\mu)$ and let π_λ be an irreducible representation of highest weight λ. Let π_{sph} denote the irreducible representation of highest weight λ_{sph}. The decomposition of the representation $\pi_\lambda \otimes \pi_{\text{sph}}$ gives a set of weights (with multiplicities) and among them are weights λ' that contain the irreducible representation of highest weight μ upon restriction to K. For example the weight $\lambda + \lambda_{\text{sph}} = \lambda(d+1, \nu)$ occurs in this decomposition. For later purposes we want to know the maximal and the minimal elements $\lambda \in P^+_G(\mu)$ with respect to $\preceq_\mu$ that occur as highest weight in the decomposition of $\pi_\lambda \otimes \pi_{\text{sph}}$, whenever they exist. Note that weights may also fall out of the μ-well, as depicted in Figure 2.16.

Theorem 2.4.2. *Let (G, K, F) be a multiplicity free system with (G, K) a spherical pair of rank one and let $\mu \in F$. Let $\lambda, \lambda' \in P^+_G(\mu)$ and let $\pi_\lambda, \pi_{\lambda'}$ be irreducible representations* *of G of highest weight λ and λ' respectively. Then $[\pi_\lambda \otimes \pi_{\mathrm{sph}} : \pi_{\lambda'}] \geq 1$ implies $\lambda - \lambda_{\mathrm{sph}} \preceq_\mu \lambda' \preceq_\mu \lambda + \lambda_{\mathrm{sph}}$ if $\lambda - \lambda_{\mathrm{sph}} \in P_G^+(\mu)$ and $\lambda' \preceq_\mu \lambda + \lambda_{\mathrm{sph}}$ otherwise.*

Proof. We prove the statement case by case for the multiplicity free systems (G, K, F) with G simply connected. The statement for other multiplicity free systems follows from these results. The case $(\mathrm{F}_4, \mathrm{Spin}_9(\mathbb{C}), \{0\})$ is clear. There are five other cases.

(1). Let $(G, K) = (\mathrm{SL}_{n+1}(\mathbb{C}), \mathrm{GL}_n(\mathbb{C}))$. The spherical weight $\lambda_{\mathrm{sph}} = \varpi_1 + \varpi_n$ is the highest weight of the adjoint representation $\mathrm{Ad} : \mathrm{SL}_{n+1} \to \mathrm{GL}(\mathfrak{sl}_{n+1})$. Let $\lambda = \lambda(d, \nu)$ for some $\nu \in P_M^+(\mu)$. According to the proof of Lemma 2.3.15 we may write $\lambda = w\mu + \sum_{i=1}^n r_i\alpha_i + d\lambda_{\mathrm{sph}}$ with $r_i \in \mathbb{N}$ and for at least one i we have $r_i = 0$. Consider $\lambda' = \lambda + \epsilon_i - \epsilon_j$ with $i < j$. If $\lambda' \in P_G^+(\mu)$ and the degree of λ' is $d+1$ then $\lambda' = \lambda + \lambda_{\mathrm{sph}} - \sum_k \alpha_k$ where $1 \leq k < i$ and $j < k \leq n$ because $\epsilon_i - \epsilon_j = \alpha_i + \ldots + \alpha_j$. This shows that the r_k with $1 \leq k < i$ and $j < k \leq n$ must have been ≥ 1 in the first place. If this were not the case then the degree could not have been raised by adding the root $\epsilon_i - \epsilon_j$. In either case we find $\lambda' \preceq_\mu \lambda + \lambda_{\mathrm{sph}}$. Similarly, adding negative roots yields $\lambda' \succeq_\mu \lambda - \lambda_{\mathrm{sph}}$ and we are done. See Figure 2.15 for an illustration.

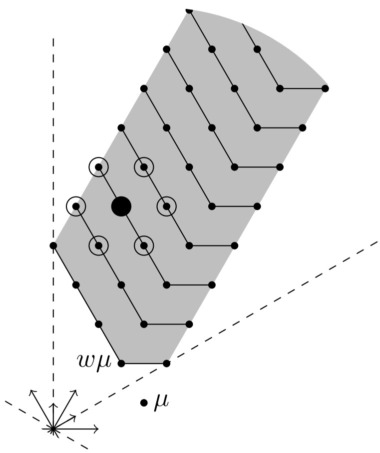

Figure 2.15: The ordering $\preceq_\mu$ in case $(\mathfrak{sl}_3, \mathfrak{gl}_2)$ behaves well under taking the tensor product with λ_{sph}.

(2). Let $(G, K) = (\mathrm{Spin}_d(\mathbb{C}), \mathrm{Spin}_{d-1}(\mathbb{C}))$ with $d \geq 3$. The spherical weight $\lambda = \varpi_i$ is the highest weight of the fundamental representation $\mathrm{SO}_d \to \mathrm{GL}(\mathbb{C}^d)$ whose non-zero weights are the short roots. The only weights that influence the degree are $\pm\epsilon_1$. Hence $\lambda - \lambda_{\mathrm{sph}} \preceq_\mu \lambda' \preceq_\mu \lambda + \lambda_{\mathrm{sph}}$, as illustrated in Figure 2.15.

(3). Let $(G, K) = (\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{Sp}_{2n-2}(\mathbb{C}) \times \mathrm{Sp}_2(\mathbb{C}))$. The spherical weight is $\lambda_{\mathrm{sph}} = \epsilon_1 + \epsilon_2$ and the irreducible representation of highest weight λ_{sph} has weights $\pm\epsilon_i \pm \epsilon_j$ for $1 \leq i < j \leq n$. In view of the description of the μ-wells in Theorem 2.3.25 for the various μ we see that the degree of λ can be raised by at most one upon adding a positive weight

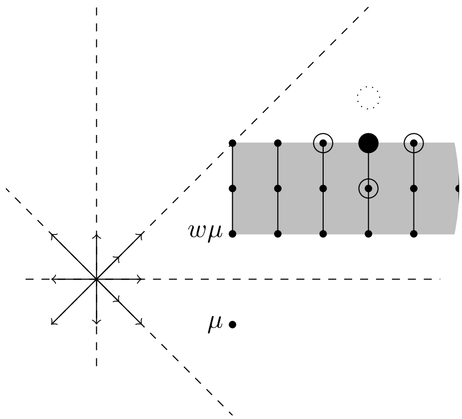

Figure 2.16: Ordering $\preceq_\mu$ in case $(\mathfrak{so}_5, \mathfrak{so}_4)$ behaves well under taking the tensor product with λ_{sph}.

$\epsilon_i \pm \epsilon_j$. If this happens we have $\lambda + \epsilon_i \pm \epsilon_j = \lambda' \preceq_\mu \lambda + \lambda_{\mathrm{sph}}$ because $\lambda_{\mathrm{sph}} - (\epsilon_i \pm \epsilon_j) \in Q_G^+$.

(4). Let $(G, K) = (\mathrm{Spin}_7(\mathbb{C}), \mathrm{G}_2)$. The spherical representation $\lambda_{\mathrm{sph}} = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3)$ is the highest weight of the spin representation $\mathrm{Spin}_7(\mathbb{C}) \to \mathrm{SO}_8(\mathbb{C})$. The non-negative weights are $\varpi_3, -\varpi_1 + \varpi_3, \varpi_1 - \varpi_2 + \varpi_3$ and $\varpi_2 - \varpi_3$. The degree is raised only by adding ϖ_3 or $\varpi_1 - \varpi_2 + \varpi_3$. The latter is equal to $\varpi_3 - \alpha_2 - \alpha_3$ which shows that $\lambda' \preceq_\mu \lambda + \lambda_{\mathrm{sph}}$. Similarly, adding negative weights shows $\lambda' \succeq_\mu \lambda - \lambda_{\mathrm{sph}}$ and we are done.

(5). Let $(G, K) = (\mathrm{G}_2, \mathrm{SL}_3(\mathbb{C}))$. The spherical weight ϖ_1 is the highest weight of the representation $\mathrm{G}_2 \to \mathrm{SO}_7(\mathbb{C})$ whose weights are the short roots and the zero root. The inequalities $\lambda - \lambda_{\mathrm{sph}} \preceq_\mu \lambda' \preceq_\mu \lambda + \lambda_{\mathrm{sph}}$ follow by inspection. We have illustrated the good behavior in Figure 2.17.

□

2.4.3. Theorem 2.4.2 plays a key role for the computation of matrix valued orthogonal polynomials in Chapter 3. The proof uses the classification of multiplicity free systems and explicit case by case knowledge of the branching rules. It would be desirable to have a more conceptual proof.

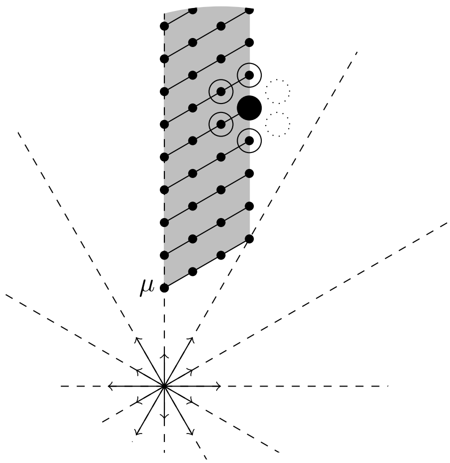

Figure 2.17: The ordering $\preceq_\mu$ in case $(\mathfrak{g}_2, \mathfrak{sl}_3)$ behaves well under taking the tensor product with λ_{sph}.

Chapter 3

Matrix Valued Polynomials associated to Multiplicity Free Systems

3.1 Introduction

In Chapter 2 we have classified the multiplicity free systems of rank one. In the present chapter we turn our attention to compact groups and to the analogue of multiplicity free systems for compact groups. Given a compact multiplicity free system (Definition 3.2.1) we construct families of matrix valued orthogonal polynomials on a compact interval. The construction is divided into two steps. In the first step we generalize the construction of Jacobi polynomials on rank one symmetric pairs to a construction of vector valued polynomials for compact multiplicity free systems of rank one. Along the way we keep track of the orthogonality and recurrence relations and differential equations. The second step is to arrange the vector valued polynomials into matrices to obtain matrix valued polynomials. We obtain families of matrix valued orthogonal polynomials which are simultaneous eigenfunctions for a commutative algebra of differential operators. In the final subsection of this chapter we discuss briefly how the matrix valued polynomials relate to the ones obtained by Grünbaum et al. [GPT02].

In this chapter we use different notations for groups. The compact Lie groups are now denoted by roman capitals G, K and their complexifications are denoted by $G_{\mathbb{C}}, K_{\mathbb{C}}$. This is the price we pay to get rid of the indices of the compact groups that we would have had to introduce otherwise.

G		K	λ_{sph}	faces F
$\mathrm{SU}(n+1)$	$n \geq 1$	$\mathrm{U}(n)$	$\varpi_1 + \varpi_n$	any
$\mathrm{SO}(2n)$	$n \geq 2$	$\mathrm{SO}(2n-1)$	ϖ_1	any
$\mathrm{SO}(2n+1)$	$n \geq 2$	$\mathrm{SO}(2n)$	ϖ_1	any
$\mathrm{Sp}(2n)$	$n \geq 3$	$\mathrm{Sp}(2n-2) \times \mathrm{Sp}(2)$	ϖ_2	$\dim F \leq 2$
F_4		$\mathrm{Spin}(9)$	ϖ_1	$\dim F \leq 1$ or $F = \mathbb{N}\omega_1 + \mathbb{N}\omega_2$
$\mathrm{Spin}(7)$		G_2	ϖ_3	$\dim F \leq 1$
G_2		$\mathrm{SU}(3)$	ϖ_1	$\dim F \leq 1$

Table 3.1: Compact multiplicity free systems of rank one. In the third column we indicated the spherical weight. In the third column we have given the highest weight $\lambda_{\mathrm{sph}} \in P_G^+$ of the fundamental zonal spherical representation in the notation for root systems of Bourbaki [Bou68, Planches], except for the case $(G, K) = (\mathrm{SO}(4), \mathrm{SO}(3))$ where G is not simple and $\lambda_{\mathrm{sph}} = \varpi_1 + \varpi_2 \in P_G^+ = \mathbb{N}\varpi_1 + \mathbb{N}\varpi_2$.

3.2 Multiplicity free triples

Let G be a compact connected Lie group, $K \subset G$ a closed connected subgroup and let $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$ denote the complexifications of G and K. Weyl's unitary trick provides a correspondence between the rational irreducible representations of $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$ and the unitary irreducible representations of G and K. In 2.1.6 we have fixed the notations of the roots and weights for the algebraic groups $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$. In this chapter we use the same notations. Once we have chosen a maximal torus in G and a notion of positivity we denote by R_G^+ the set of positive roots, by P_G^+ the set of dominant integral weights and by $\mathcal{C}_G^+$ the positive Weyl chamber and similarly for K. By a relatively open face $F \subset P_K^+$ we mean the intersection $P_K^+ \cap f$ for $f \subset \mathcal{C}_K^+$ a relatively open face of the positive Weyl chamber.

Definition 3.2.1. *Let G, K be compact Lie groups and let $G_{\mathbb{C}}, K_{\mathbb{C}}$ denote their complexifications. Let $F \subset P_K^+$ be a relatively open face and let $\mu \in P_K^+$. A triple (G, K, μ) is called a compact multiplicity free triple if $(G_{\mathbb{C}}, K_{\mathbb{C}}, \mu)$ is a multiplicity free triple. A triple (G, K, F) is called a compact multiplicity free system if $(G_{\mathbb{C}}, K_{\mathbb{C}}, F)$ is a multiplicity free system.*

The pairs (G, K) in Table 3.1 are the compact Lie groups whose complexifications $(G_{\mathbb{C}}, K_{\mathbb{C}})$ appear in Table 2.1. In particular we denote $\mathrm{Sp}(2n) = \mathrm{U}(2n) \cap \mathrm{Sp}_{2n}(\mathbb{C})$ where some other authors may use $\mathrm{Sp}(n)$ for the same compact Lie group. The rank of a compact pair (G, K) is defined by the rank of the pair $(G_{\mathbb{C}}, K_{\mathbb{C}})$ that we discussed in 2.1.7.

Recall from 2.1.5 that an isogeny is a finite covering homomorphism and that two Lie groups G and G' are called isogenous if there is an isogeny in one of the directions. Let $c : G' \to G$ be an isogeny of compact Lie groups. Then the complexified map

$c_\mathbb{C} : G'_\mathbb{C} \to G_\mathbb{C}$ is also an isogeny. Conversely, an isogeny $c_\mathbb{C} : G'_\mathbb{C} \to G_\mathbb{C}$ of the complex reductive groups $G'_\mathbb{C}, G_\mathbb{C}$ restricts to an isogeny of the compact subgroups G' and G.

Definition 3.2.2. *Let $c : G' \to G$ be an isogeny of compact groups, let $K' \subset G'$ be a closed subgroup and denote $K = c(K')$. Then c is called an isogeny of the pairs (G, K) and (G', K'). Two pairs (G, K) and (G', K') are called isogenous is there is an isogeny between (G, K) and (G', K') in one of the directions.*

In view of Theorem 2.2.16 we see that modulo isogenies, Table 3.1 comprises all the multiplicity free systems with (G, K) of rank one. The corresponding spaces G/K are precisely the two-point-homogeneous spaces as classified by Wang in [Wan52].

Proposition 3.2.3. *Let (G, K, F) be a compact multiplicity free system of rank one. Let $X = G/K$ and let $x_0 = eK \in X$. Identify the tangent space $T_eX = \mathfrak{k}^\perp$ where $\perp$ is with respect to the Killing form on $\mathfrak{g}$. The action of K on T_eX is transitive on lines. Let $\mathfrak{a} \subset \mathfrak{k}^\perp$ be a line. Let $A \subset G$ be the torus with $\mathfrak{a}$ as Lie algebra. Then we have a decomposition $G = KAK$.*

Proof. We only have to check this for the two non-symmetric pairs since for the rank one symmetric pairs the statement is clear, see [Kna02, Thm. 6.51]. Spin(7) acts transitively on S^7 and G_2 acts transitively on S^6, see e.g. [MS43]. We can be more precise, see [Ada96, Cor. 5.4, Thm. 5.5]. The stabilizer of a point $s \in S^7$ of the Spin(7)-action is (isomorphic to) G_2 and the corresponding action of G_2 on $T_sS^7 \cong \mathbb{R}^7$ is transitive on lines. The stabilizer of a point $t \in S^6$ of the G_2 action is (isomorphic to) SU(3) the corresponding action of SU(3) on $\mathbb{C}^3$ is transitive on $S^5 \subset T_tS^6$. Hence SU(3) acts transitively on lines in T_tS^6. We conclude that the orbits of K in G/K are parametrized by A and hence $G = KAK$. □

3.2.4. Let (G, K, F) be a compact multiplicity free system of rank one. The Lie algebras of G and K are denoted by $\mathfrak{g}$ and $\mathfrak{k}$ and their complexifications by $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{k}_\mathbb{C}$. We have seen in Lemma 2.2.8 that $\mathfrak{g}_\mathbb{C}$ admits a decomposition $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{a}_\mathbb{C} \oplus \mathfrak{n}^+_\mathbb{C}$ where $\mathfrak{a}_\mathbb{C}$ is a one-dimensional torus that consists of semisimple elements. Denote $\mathfrak{a} = \mathfrak{g} \cap \mathfrak{a}_\mathbb{C}$. The one-dimensional torus $\mathfrak{a}$ is the Lie algebra of a torus $A \subset G$ with the property that $G = KAK$. Denote $M = Z_K(A)$. In Proposition 2.2.9 we have seen that an irreducible K-representation τ of highest weight $\mu \in F$ decomposes multiplicity free upon restriction to M. In Table 3.2 we have indicated the subgroups M modulo conjugation by K.

3.2.5. The μ-well $P^+_G(\mu) = \{\lambda \in P^+_G : m_\lambda(\mu) = 1\}$ is isomorphic to $\mathbb{N} \times B_\mu$ where the bottom B_μ is a finite set that is isomorphic to $P^+_M(\mu)$, see Proposition 2.3.9. The number of elements in B_μ is $d_\mu(M) = |P^+_M(\mu)|$. In 2.27 we have defined the isomorphism

$$\lambda : \mathbb{N} \times P^+_M(\mu) \to P^+_G(\mu) : (d, \nu) \mapsto b_\mu(\nu) + d \cdot \lambda_{\mathrm{sph}}. \tag{3.1}$$

The bottom B_μ has been determined case by case (see Subsection 2.3) and λ_{sph} generates the spherical monoid $P^+_G(0)$ over $\mathbb{N}$. λ_{sph} is called the fundamental spherical weight.

G	K	M
$\mathrm{SU}(n+1)$, $n \geq 1$	$\mathrm{U}(n)$	$\mathrm{S}(\mathrm{diag}(\mathrm{U}(1) \times \mathrm{U}(1)) \times \mathrm{U}(n-1))$
$\mathrm{SO}(2n)$, $n \geq 2$	$\mathrm{SO}(2n-1)$	$\mathrm{SO}(2n-2)$
$\mathrm{SO}(2n+1)$, $n \geq 2$	$\mathrm{SO}(2n)$	$\mathrm{SO}(2n-1)$
$\mathrm{Sp}(2n)$, $n \geq 3$	$\mathrm{Sp}(2n-2) \times \mathrm{Sp}(2)$	$\mathrm{Sp}(2) \times \mathrm{Sp}(2n-4)$
F_4	$\mathrm{Spin}(9)$	$\mathrm{Spin}(7)$
$\mathrm{Spin}(7)$	G_2	$\mathrm{SU}(3)$
G_2	$\mathrm{SU}(3)$	$\mathrm{SU}(2)$

Table 3.2: Spherical pairs of rank one with $M = Z_K(A)$ modulo conjugation with K. Note that the embedding $\mathrm{Spin}_7(\mathbb{C}) \subset \mathrm{Spin}_9(\mathbb{C})$ is the standard embedding $\mathrm{Spin}_7(\mathbb{C}) \subset \mathrm{Spin}_8(\mathbb{C})$ followed by the embedding $\mathrm{Spin}_8(\mathbb{C}) \subset \mathrm{Spin}_9(\mathbb{C})$ that is twisted by the automorphism $\epsilon_1 - \epsilon_2 \leftrightarrow \epsilon_3 - \epsilon_4$, see [BS79, §6].

$P_G^+(\mu)$ is endowed with the partial ordering $\preceq_\mu$ that we defined in 2.4.1. The partial ordering $\preceq_\mu$ behaves well with respect to taking the tensor product with an irreducible representations of highest weight λ_{sph}. Indeed, let $\lambda, \lambda' \in P_G^+(\mu)$ and let $\pi_\lambda, \pi_{\lambda'}$ be irreducible representation of highest weights λ and λ' in the vector spaces $V_\lambda, V_{\lambda'}$. Let $\pi_{\lambda_{\mathrm{sph}}}$ be an irreducible representation of highest weight λ_{sph} in $V_{\lambda_{\mathrm{sph}}}$. We have shown in Theorem 2.4.2 that $[\pi_\lambda \otimes \pi_{\lambda_{\mathrm{sph}}} : \pi_{\lambda'}] \geq 1$ implies

$$\lambda - \lambda_{\mathrm{sph}} \preceq_\mu \lambda' \preceq_\mu \lambda + \lambda_{\mathrm{sph}}. \tag{3.2}$$

Moreover, we have shown in Proposition 2.3.6 that $\lambda + \lambda_{\mathrm{sph}} \in P_G^+(\mu)$ and that the projection $V_\lambda \otimes V_{\lambda_{\mathrm{sph}}} \to V_{\lambda+\lambda_{\mathrm{sph}}}$ is onto.

3.3 Spherical functions

From now on (G, K, F) is a compact multiplicity free system with (G, K) a compact Gel'fand pair of rank one and with G other[1] than F_4.

3.3.1 Spherical functions and representations

3.3.1. The irreducible unitary representations of K of highest weight $\mu \in P_K^+$ are all equivalent. We fix an irreducible representation for every $\mu \in P_K^+$ and we denote the representation space with V_μ. Similarly for G where we fix for every $\lambda \in P_G^+$ a unitary irreducible representation of highest weight λ in the representation space that we denote by V_λ. Note that our choices imply that the spaces V_μ and V_λ are endowed with Hermitian inner products $\langle\cdot,\cdot\rangle_\mu$ and $\langle\cdot,\cdot\rangle_\lambda$. We use the convention that a Hermitian inner product is complex linear in the second variable.

[1] A few days before printing we discovered that there are good faces other than $\{0\}$ in this case. Unfortunately there was no time left to analyze the μ-wells in these cases.

Let $\lambda \in P_G^+$ and let π denote the corresponding irreducible representation in V_λ. The matrix coefficients of λ are denoted by $m^\lambda_{v_1,v_2}(g) = \langle v_1, \pi(g)v_2\rangle_\lambda$.

Definition 3.3.2. *Let $\mu \in F$ and let τ be the K-representation of highest weight μ in V_μ. A spherical function on G of type μ is a matrix valued function $\Phi : G \to \mathrm{End}(V_\mu)$ such that*

$$\Phi(k_1 g k_2) = \tau(k_1)\Phi(g)\tau(k_2) \quad \textit{for all } k_1, k_2 \in K \textit{ and } g \in G. \tag{3.3}$$

3.3.3. Let (G, K, μ) be a multiplicity free triple and let $\lambda \in P_G^+(\mu)$. The representation space V_λ decomposes uniquely as a direct sum $V_\lambda = V_\mu \oplus V_\mu^\perp$.

Definition 3.3.4. *Let $\mu \in F$ and let π be the representation of G in V_λ of highest weight $\lambda \in P_G^+(\mu)$. Let $b : V_\mu \to V_\lambda$ be a unitary K-equivariant embedding and let $b^* : V_\lambda \to V_\mu$ be its Hermitian adjoint. The spherical function of type μ associated to λ is defined by*

$$\Phi^\mu_\lambda : G \to \mathrm{End}(V_\mu) : g \mapsto b^* \circ \pi(g) \circ b. \tag{3.4}$$

An elementary spherical function of type μ is a spherical function of type μ associated to some $\lambda \in P_G^+(\mu)$.

3.3.5. Note that the spherical function Φ^μ_λ only depends on the weights μ and λ as long as we take the K-equivariant embedding $b : V_\mu \to V_\lambda$ unitary.

3.3.6. Let $\mu \in F$ and let $\lambda \in P_G^+(\mu)$. Fix a basis $\{v_i : i = 1, \ldots, \dim(\mu)\}$ for V_μ. Then

$$\langle v_i, \Phi^\mu_\lambda(g)v_j\rangle_\mu = \langle b_\mu(v_i), \pi_\lambda(g)b_\mu(v_j)\rangle_\lambda, \tag{3.5}$$

from which we see that the matrix entries of the elementary spherical functions are matrix coefficients of the irreducible representation of G.

3.3.7. Definition 3.3.4 applies only to multiplicity free triples (G, K, μ). There are more general definitions of a spherical function available. Indeed, for triples (G, K, τ) where G is a locally compact group, K a compact subgroup and τ an irreducible representation of K, for which the multiplicity $[\pi|_K : \tau]$ may be greater than one for some irreducible representation π of G, one can also define elementary spherical functions, see for example [War72a], [Tir77], [GV88]. However, at this moment we cannot work in this generality because for the construction of the matrix valued polynomials we need the structure of compact Lie groups. Moreover, we need the elementary spherical functions to take their values in one and the same matrix algebra. The latter will not be the case if we allow multiplicities $[\pi|_K : \tau] > 1$.

3.3.8. Let $\mu \in F$ and let $\lambda \in P_G^+(\mu)$. Let τ be a unitary irreducible K-representation of highest weight μ and let $b : V_\mu \to V_\lambda$ be a unitary K-equivariant embedding. Let $\chi_\mu = \dim(V_\mu)\mathrm{tr}(\tau)$. Define the convolution algebra $C_\mu(G) = \mathrm{span}\{m^\lambda_{v,w} : \lambda \in P_G^+(\mu), v, w \in b(V_\mu) \subset V_\lambda\}$. Proposition 3.3.9 and Corollary 3.3.10 give different characterizations of elementary spherical functions. We present the results here because several authors use Corollary 3.3.10 as a definition of elementary spherical functions, see e.g. [Tir77], [GPT02], [RT06]. The proof of the Proposition 3.3.9 is given in [GV88, Thm. 1.3.5]. The proof of Corollary 3.3.10 is included in [GV88, Lem. 1.3.4].

Proposition 3.3.9. *Let Φ be a spherical function of type $\mu \in F$. Define the linear map*

$$L_\Phi : C_\mu(G) \to \mathrm{End}(V_\mu) : f \mapsto \int_G f(g)\Phi(g)dg. \tag{3.6}$$

If Φ is an elementary spherical function of type μ then L_Φ is an irreducible representation of the convolution algebra $C_\mu(G)$. Conversely, if L is an irreducible representation of $C_\mu(G)$ of the form $L = L_\Phi$, with Φ a μ-spherical function, then Φ is an elementary spherical function of type μ.

Corollary 3.3.10. *Let $\mu \in F$ and let $\Phi : G \to \mathrm{End}(V_\mu)$ be a continuous function. The following are equivalent.*

- *Φ is an elementary spherical function of type μ,*
- *$\Phi(e) = I$ and for all $x, y \in G$ we have*

$$\Phi(x)\Phi(y) = \int_K \chi_\mu(k^{-1})\Phi(xky)dk. \tag{3.7}$$

3.3.11. Let $\mu \in F$ and let $\lambda \in P_G^+(\mu)$. Let $U(\mathfrak{k}_\mathbb{C})$ and $U(\mathfrak{g}_\mathbb{C})$ denote the universal enveloping algebras of $\mathfrak{k}_\mathbb{C}$ and $\mathfrak{g}_\mathbb{C}$ respectively. Let τ and π be unitary irreducible representations of K and G in V_μ and V_λ of highest weight μ and λ respectively. The representations τ and π induce representations $\dot\tau : U(\mathfrak{k}_\mathbb{C}) \to \mathrm{End}(V_\mu)$ and $\dot\pi : U(\mathfrak{g}_\mathbb{C}) \to \mathrm{End}(V_\lambda)$. Let $I^\mu \subset U(\mathfrak{k}_\mathbb{C})$ denote the kernel of $\dot\tau$. Let $U(\mathfrak{g}_\mathbb{C})^{\mathfrak{k}_\mathbb{C}}$ be the centralizer of $\mathfrak{k}_\mathbb{C}$ in $U(\mathfrak{g}_\mathbb{C})$. The restricted representation $\dot\pi : U(\mathfrak{g}_\mathbb{C})^{\mathfrak{k}_\mathbb{C}} \to \mathrm{End}(V_\lambda)$ is not irreducible, an operator $\dot\pi(X)$ commutes with the projections on the isotypical K-types. Let $b : V_\mu \to V_\lambda$ be a unitary K-equivariant embedding. We obtain a representation

$$\widetilde{\omega}_{\lambda,\mu} : U(\mathfrak{g}_\mathbb{C})^{\mathfrak{k}_\mathbb{C}} \to \mathrm{End}_K(V_\mu) : X \mapsto b^*\dot\pi(X)b. \tag{3.8}$$

We identify $\mathrm{End}_K(V_\mu) \cong \mathbb{C}$ to obtain a representation $\omega_{\lambda,\mu} : U(\mathfrak{g}_\mathbb{C})^{\mathfrak{k}_\mathbb{C}} \to \mathbb{C}$. Define $\mathbb{D}^\mu = U(\mathfrak{g}^\mathbb{C})^{\mathfrak{k}^\mathbb{C}}/(U(\mathfrak{g}^\mathbb{C})^{\mathfrak{k}^\mathbb{C}} \cap U(\mathfrak{g}^\mathbb{C})I^\mu)$. The space $\mathbb{D}^\mu$ is an algebra [Dix96, Prop. 9.1.10 (ii)]. The representation $\omega_{\lambda,\mu} : U(\mathfrak{g}^\mathbb{C})^{\mathfrak{k}^\mathbb{C}} \to \mathbb{C}$ factors through the quotient $U(\mathfrak{g}^\mathbb{C})^{\mathfrak{k}^\mathbb{C}} \to \mathbb{D}^\mu$ and the induced map, which we denote by $\kappa_{\lambda,\mu} : \mathbb{D}^\mu \to \mathbb{C}$, is an irreducible representation.

Proposition 3.3.12. *Let $\mu \in F$ and let Φ be a spherical function of type μ. Define the linear map*

$$\kappa_\Phi : \mathbb{D}^\mu \to \mathrm{End}_K(V_\mu) : D \mapsto (D\Phi)(e). \tag{3.9}$$

If Φ is an elementary spherical function of type μ then κ_Φ is an irreducible representation of the algebra $\mathbb{D}^\mu$. Conversely, if κ is an irreducible representation of $\mathbb{D}^\mu$ of the form $\kappa = \kappa_\Phi$ for a μ-spherical function Φ, then Φ is an elementary spherical function of type μ.

3.3.13. The result in Proposition 3.3.12 is proved in e.g. [GV88, Thm. 1.4.5], [Tir77, Prop. 4.5]. Proposition 3.3.12 implies that a spherical function Φ of type μ is elementary if and only if Φ is a simultaneous eigenfunction for the algebra $\mathbb{D}^{\mu}$ and $\Phi(e) = I$. To see this let $D \in \mathbb{D}^{\mu}$ and for $g \in G$ let L_g be the map $G \to G$ defined by $L_g(h) = gh$. Since D is left invariant, i.e. $L_g^* D = D$, we have

$$D(\Phi^{\mu}_{\lambda})(g) = L_g^*(D(\Phi^{\mu}_{\lambda}))(e) = D(L_g^*\Phi^{\lambda}_{\mu})(e) = \Phi(g)D(\Phi^{\lambda}_{\mu})(e).$$

For the converse, if Φ is a simultaneous eigenfunction for $\mathbb{D}^{\mu}$ with $\Phi(e) = I$ then $D \mapsto D(\Phi)(e)$ is a representation of $\mathbb{D}^{\mu}$. Because Φ is μ-spherical we have $D(\Phi)(e) \in \mathrm{End}_K(V_{\mu})$ and $\mathrm{End}_K(V_{\mu}) \cong \mathbb{C}$ from which we see that $D \mapsto D(\Phi)(e)$ is an irreducible representation. Proposition 3.3.12 implies that Φ is an elementary spherical function of type μ.

Proposition 3.3.14. *The algebra $\mathbb{D}^{\mu}$ is commutative.*

Proof. In [Dix96, 9.2.10] Dixmier argues that there is an injective algebra anti homomorphism $\mathbb{D}^{\mu} \to U(\mathfrak{a}_{\mathbb{C}}) \otimes \mathrm{End}_M(V_{\mu})$. Since $\mathrm{End}_M(V_{\mu})$ is commutative, so are the algebras $U(\mathfrak{a}_{\mathbb{C}}) \otimes \mathrm{End}_M(V_{\mu})$ and $\mathbb{D}^{\mu}$. □

3.3.2 Recurrence relations for the spherical functions

Definition 3.3.15. *Let $\mu \in F$ and let $\lambda_{\mathrm{sph}} \in P_G^+$ denote the fundamental spherical weight (see 3.2.5). The spherical functions of type $\mu = 0$ are called zonal spherical functions. The elementary zonal spherical function that is associated to $d\cdot\lambda_{\mathrm{sph}}$ is denoted by $\phi_d = \Phi^0_{d\cdot\lambda_{\mathrm{sph}}}$. The elementary zonal spherical function associated to λ_{sph} is denoted by $\phi = \phi_1$ and it is called the fundamental zonal spherical function.*

3.3.16. Note that the zonal spherical functions are K-bi-invariant. We use the partial ordering $\preceq_{\mu}$ on the μ-well $P_G^+(\mu)$ that we have discussed in 3.2.5 to obtain a recurrence relation for the elementary μ-spherical functions. The recurrence is obtained by multiplication with ϕ.

Proposition 3.3.17. *Let $\mu \in F$ and $\lambda \in P_G^+(\mu)$. Then there are coefficients $a^{\mu}_{\lambda}(\lambda') \in \mathbb{C}$ such that*

$$\phi\Phi^{\mu}_{\lambda} = \sum_{\lambda-\lambda_{\mathrm{sph}}\preceq_{\mu}\lambda'\preceq_{\mu}\lambda+\lambda_{\mathrm{sph}}} a^{\mu}_{\lambda}(\lambda')\Phi^{\mu}_{\lambda'}, \quad \lambda' \in P_G^+(\mu). \tag{3.10}$$

Moreover, $a^{\mu}_{\lambda}(\lambda + \lambda_{\mathrm{sph}}) \neq 0$ and $\sum_{\lambda-\lambda_{\mathrm{sph}}\preceq_{\mu}\lambda'\preceq_{\mu}\lambda+\lambda_{\mathrm{sph}}} a^{\mu}_{\lambda}(\lambda') = 1$.

Proof. Let π_{λ} be a unitary representation of highest weight λ. Denote $V = \oplus_{\lambda'} V_{\lambda'}$ where is sum is taken over all λ' that occur as highest weights of irreducible representations in the decomposition of $\pi_{\lambda} \otimes \pi_{\lambda_{\mathrm{sph}}}$. The sum is finite but there may be repetitions if $\pi_{\lambda'}$ occurs in $\pi_{\lambda} \otimes \pi_{\lambda_{\mathrm{sph}}}$ with multiplicity greater than one. Let $\langle\cdot,\cdot\rangle$ denote a Hermitian inner product on V for which the G representation is unitary. There is a unitary G-intertwining isomorphism $a : V_{\lambda} \otimes V_{\lambda_{\mathrm{sph}}} \to V$. Let $b_{\lambda} : V_{\mu} \to V_{\lambda}$ be the K-equivariant map from Definition 3.3.4. We use these maps to find copies of V_μ in both $V_\lambda \otimes V_{\lambda_{\mathrm{sph}}}$ and V as follows. Define $b_1 : V_\mu \to V_\lambda \otimes V_{\lambda_{\mathrm{sph}}} : v \mapsto b_\lambda(v) \otimes v_{\mathrm{sph}}$ where $v_{\mathrm{sph}} \in V_{\lambda_{\mathrm{sph}}}$ is a K-fixed vector of length one. The map b_1 is a unitary K-equivariant linear map, uniquely determined up to a scalar of length one. The maps $b_{\lambda'} : V_\mu \to V_{\lambda'}$ determine a unique map $b_2 : V_\mu \to V$ such that for every projection $\mathrm{pr}_{\lambda''} : V \to V_{\lambda''}$ we have $\mathrm{pr}_{\lambda''} \circ b_2 = b_{\lambda''}$. Let $v \in V_\mu$. It follows from the K-invariance of the maps $b_1, b_{\lambda'}$ and Schur's Lemma that there are coefficients $\widetilde{a}^\mu_\lambda(\lambda') \in \mathbb{C}$ such that

$$ab_1(v) = \sum_{\lambda'} \widetilde{a}^\mu_\lambda(\lambda') b_{\lambda'}(v). \tag{3.11}$$

We have $\langle ab_1(v), \oplus\pi_{\lambda'}(g)ab_1(w)\rangle = \langle v, \Phi^\mu_\lambda(g)w\rangle_\mu \phi(g)$ which is equal to

$$\sum_{\lambda'} |\widetilde{a}^\mu_\lambda(\lambda')|^2 \langle v, \Phi^\mu_{\lambda'}(g)w\rangle_\mu$$

by (3.11). Put $a^\mu_\lambda(\lambda') = |\widetilde{a}^\mu_\lambda(\lambda')|^2$. If $\lambda' \notin P^+_G(\mu)$ then $a^\mu_\lambda(\lambda') = 0$. We find

$$\phi\Phi^\mu_\lambda = \sum_{\lambda' \in P^+_G} a^\mu_\lambda(\lambda)\Phi^\mu_\lambda.$$

From 3.2.5 it follows that we only need to sum over $\lambda' \in P^+_G(\mu)$ with $\lambda - \lambda_{\mathrm{sph}} \preceq_\mu \lambda' \preceq_\mu \lambda + \lambda_{\mathrm{sph}}$. We have $||\mathrm{pr}_{\lambda'} ab_1(v)||^2 = a^\mu_\lambda(\lambda')||v||^2$. In 3.2.5 we have seen that $\mathrm{pr}_{\lambda+\lambda_{\mathrm{sph}}}(ab_1(v)) \neq 0$. Hence $a^\mu_\lambda(\lambda + \lambda_{\mathrm{sph}}) \neq 0$. The last statement is immediate from (3.11). Note that the coefficients $a^\mu_\lambda(\lambda')$ are independent of the unitary intertwiner a. □

Corollary 3.3.18. *Let $\mu \in F$ and let Φ^μ_λ be an elementary spherical function with $\lambda = \lambda(d, \nu)$. Then there are $d_\mu(M)$ polynomials $q^\mu_{\lambda,\nu} \in \mathbb{C}[\phi]$, $\nu \in P^+_M(\mu)$ such that*

$$\Phi^\mu_\lambda = \sum_{\nu \in P^+_M(\mu)} q^\mu_{\lambda,\nu}(\phi)\Phi^\mu_{\lambda(0,\nu)}.$$

The polynomials are uniquely determined and their degrees are $\leq d$.

Proof. We argue by induction on the set $P^+_G(\mu)$. For $\lambda = \lambda(0, \nu)$ the statement is clear. Let $\lambda(d, \nu) \in P^+_G(\mu)$ with $d > 0$ and suppose that for all $\lambda' = \lambda(d', \nu')$ with $\lambda' \preceq_\mu \lambda$ we can express $\Phi^\mu_{\lambda'}$ as a $\mathbb{C}[\phi]$-linear combination of $\Phi^\mu_{\lambda(0,\nu)}$, $\nu \in P^+_M(\mu)$ with coefficients of degree $\leq d'$. Consider $\lambda - \lambda_{\mathrm{sph}} = \lambda(d-1, \nu)$. It follows from (3.10) together with the fact that the coefficient of the highest degree $a^\mu_{\lambda(d-1,\nu)}(\lambda) \neq 0$ that

$$\Phi^\mu_\lambda = \frac{1}{a^\mu_{\lambda(d-1,\nu)}(\lambda)} \left(\phi\Phi^\mu_{\lambda(d-1,\nu)} - \sum_{\lambda - 2\lambda_{\mathrm{sph}} \preceq_\mu \lambda' \prec_\mu \lambda} a^\mu_{\lambda(d-1,\nu)}(\lambda')\Phi^\mu_{\lambda'} \right), \tag{3.12}$$

from which the result follows, because $d' \leq d$. □

3.3.19. In the case $\mu = 0$ we get a three term recurrence relation for the elementary zonal spherical functions ϕ_d. This means that we can express ϕ_d as a polynomial of degree d in ϕ. The polynomials that we get are, up to an affine transformation, Jacobi polynomials. In Table 3.3 we provide the possibilities in the cases of our interest.

3.3.3 The space of spherical functions

3.3.20. Let $\mu \in F$ and let τ be an irreducible K-representation of highest weight μ. Let $C(G)$ denote the space of continuous functions on G and let $R(G) \subset C(G)$ denote the space of representative functions, i.e. the subspace spanned by the matrix coefficients $m^{\lambda}_{v,w}$ with $\lambda \in P^+_G$. Consider the space of $\mathrm{End}(V_\mu)$-valued functions $R_\mu(G) = R(G) \otimes \mathrm{End}(V_\mu)$. Define the action of $K \times K$ on $R_\mu(G)$ by

$$(k_1, k_2)\Phi(g) = \tau(k_1)\Phi(k_1^{-1} g k_2)\tau(k_2)^{-1}, \quad \text{for } \Phi \in R_\mu(G). \tag{3.13}$$

Definition 3.3.21. *Define the complex vector space $E^\mu = R_\mu(G)^{K\times K}$. E^μ is called the space of μ-spherical functions.*

Proposition 3.3.22. *The complex vector space E^μ is generated by the elementary spherical functions of type μ.*

Proof. It is clear that the elementary spherical functions are contained in E^μ. For the converse we use the Peter-Weyl decomposition for $R_\mu(G)$. Keeping track of the $K \times K$-action shows that

$$E^\mu = \bigoplus_{\lambda \in P^+_G} \mathrm{Hom}_K(V_\mu, V_\lambda) \otimes \mathrm{Hom}_K(V_\lambda, V_\mu). \tag{3.14}$$

The space $\mathrm{Hom}_K(V_\mu, V_\lambda)$ is one-dimensional if and only if $\lambda \in P^+_G(\mu)$. In this case the unitary embedding $b : V_\mu \to V_\lambda$ is in $\mathrm{Hom}_K(V_\mu, V_\lambda)$. The function in E^μ that corresponds to $b \otimes b^*$ is Φ^μ_λ. This shows that the elementary spherical functions span E^μ. □

3.3.23. We have already argued in 3.3.19 that E^0, the space of K-bi-invariant matrix coefficients, is a polynomial algebra, i.e. $E^0 = \mathbb{C}[\phi]$. We investigate the algebraic structure of general E^μ. Let $f \in E^0$ and $\Phi \in E^\mu$. Then the function $x \mapsto f(x)\Phi(x)$ is contained in E^μ because of Proposition 3.3.22 and (3.10). This observation shows that E^μ has an E^0-module structure. In fact, E^μ is a finitely generated E^0-module, see e.g. [Kra85, II.3.2]. Since E^μ is torsion free and E^0 is a polynomial algebra in one variable, E^μ is a free E^0-module, see e.g. [Lan02, III.§7].

3.3.24. The space $\mathrm{End}(V_\mu)$ is equipped with a Hermitian inner product $\langle A, B\rangle = \mathrm{tr}(A^*B)$, where A^* is the Hermitian adjoint. This induces a Hermitian inner product on the space of spherical functions of type μ as follows. Define the pairing $\langle\cdot,\cdot\rangle_{\mu,G} : E^\mu \times E^\mu \to \mathbb{C}$

$$\langle \Phi, \Phi' \rangle_{\mu,G} := \int_G \mathrm{tr}(\Phi(g)^*\Phi'(g))dg, \tag{3.15}$$

where dg denotes the Haar measure on G normalized by $\int_G dg = 1$. The pairing is linear in the second variable and it does not depend on the basis of V_μ in which we express the functions Φ, Φ'. Furthermore, the function $g \mapsto \mathrm{tr}(\Phi(g)^*\Phi'(g))$ is in E^0 because it is K-bi-invariant and it is the sum of matrix coefficients.

Proposition 3.3.25. *The pairing (3.15) is an inner product. The elementary spherical functions of type μ form an orthogonal system with respect to $\langle\cdot,\cdot\rangle_{\mu,G}$. More precisely:*

$$\langle\Phi^{\mu}_{\lambda},\Phi^{\mu}_{\lambda'}\rangle_{\mu,G}=\frac{(\dim\mu)^2}{\dim\lambda}\delta_{\lambda,\lambda'}. \tag{3.16}$$

Proof. By Schur orthogonality we have

$$\int_G \overline{m^{\lambda}_{v,w}(g)}m^{\lambda'}_{v',w'}(g)dg=\dim(\lambda)^{-1}\delta_{\lambda,\lambda'}\langle v,v'\rangle\langle w,w'\rangle.$$

Fix an orthonormal basis $\{v_1,\ldots,v_r\}$ for V_μ. Let $\lambda,\lambda'\in P^+_G(\mu)$ and let $b_\lambda:V_\mu\to V_\lambda$ and $b_{\lambda'}:V_\mu\to V_{\lambda'}$ be unitary K-equivariant embeddings. The entries of the elementary μ-spherical functions are matrix coefficients $m^{\lambda}_{b_\lambda(v_i),b_\lambda(v_j)}$ by 3.3.6. We find

$$\langle\Phi^{\mu}_{\lambda},\Phi^{\mu}_{\lambda'}\rangle_{\mu,G}=\sum_{j=1}^{r}\sum_{i=1}^{r}\int_G\overline{m^{\lambda}_{b_\lambda(v_i),b_\lambda(v_j)}(g)}m^{\lambda'}_{b_{\lambda'}(v_i),b_{\lambda'}(v_j)}(g)dg=\frac{\dim(\nu)^2}{\dim(\lambda)}\delta_{\lambda,\lambda'},$$

as was to be shown. □

3.3.26. Spherical functions Φ can also be studied by considering their traces $\mathrm{tr}\Phi$, see e.g. [God52], [War72a], [GV88], [Cam97], [Ped97]. We indicate briefly the correspondence between the spherical functions and their traces.

Let (G,K,μ) be a multiplicity free triple and let $b_\lambda:V_\mu\to V_\lambda$ be a unitary K-equivariant embedding for $\lambda\in P^+_G(\mu)$. We rewrite (3.14) to

$$\begin{aligned}E^{\mu}&=\bigoplus_{\lambda\in P^+_G(\mu)}\mathrm{Hom}_K(V_\mu,b_\lambda(V_\mu))\otimes\mathrm{Hom}_K(b_\lambda(V_\mu),V_\mu)\\&=\bigoplus_{\lambda\in P^+_G(\mu)}\left(\mathrm{End}(b_\lambda(V_\mu))\otimes\mathrm{End}(V_\mu)\right)^{K\times K}.\end{aligned} \tag{3.17}$$

Taking traces gives a map $\mathrm{tr}:E^{\mu}\to R(G):\Phi\mapsto\mathrm{tr}\Phi$. In view of the identification (3.17) this amounts to the linear isomorphism

$$\left(\mathrm{End}(b_\lambda(V_\mu))\otimes\mathrm{End}(V_\mu)\right)^{K\times K}\to\mathrm{End}_K(b_\lambda(V_\mu)):A\otimes B\mapsto\mathrm{tr}(B)A \tag{3.18}$$

for every $\lambda\in P^+_G(\mu)$. Let τ be a unitary irreducible K-representation of highest weight μ. The inverse of (3.18) is given by $C\mapsto\int_K\tau(k)^{-1}C\otimes\tau(k)dk$. In view of the identification (3.17) we obtain the identity for spherical functions of type μ,

$$\Phi(g)=\int_K\mathrm{tr}(\Phi(gk^{-1}))\tau(k)dk,$$

which shows how to set up the correspondence between the spherical functions and their traces. Although this point of view on μ-spherical functions simplifies matters in the sense that the values are no longer matrices but scalars, this is not the way we want to understand the space of spherical functions E^{μ}. Instead, we study E^{μ} on the one hand via the spherical functions restricted to a suitable torus and on the other hand via the recurrence relations, because we want to construct matrix valued polynomials.

3.3.27. Finally we note that the differential operators in $\mathbb{D}^\mu$ act on E^μ by scalars, see Proposition 3.3.12. We will keep track of what happens to the differential operators if we simplify the space E^μ by restricting its elements to A or by using the recurrence relations to simplify E^μ.

3.4 Spherical functions restricted to A

3.4.1 Transformation behavior

3.4.1. Let $\mu \in F$ and let τ be a unitary irreducible K-representation of highest weight μ. Recall from Proposition 3.2.3 that there is a one-dimensional torus $A \subset G$ such that $G = KAK$ and $\mathfrak{a} \perp \mathfrak{k}$. We fix such a torus $A \subset G$. In view of the transformation behavior (3.3) a μ-spherical function $\Phi \in E^\mu$ is completely determined by its restriction $\Phi|_A$.

Denote the space of continuous functions on A by $C(A)$. Let $R(A) \subset C(A)$ denote the space of representative functions, i.e. the subspace of $C(A)$ spanned by the matrix coefficients of A. Since $A \cong S^1$ we have $R(A) = \mathbb{C}[e^{it}, e^{-it}]$.

3.4.2. We recall some facts and notations from 2.3.4. Let $M = Z_K(A)$ and let $T_M \subset M$ be a maximal torus with Lie algebra $\mathfrak{t}_M$. The torus $T_M A \subset G$ is a maximal torus and we consider the root system R'_G associated to $(\mathfrak{g}_\mathbb{C}, \mathfrak{a}_\mathbb{C} \oplus \mathfrak{t}_{M,\mathbb{C}})$. A choice for a lexicographic ordering on $\mathfrak{a}_\mathbb{C} \oplus \mathfrak{t}_{M,\mathbb{C}}$ where $\mathfrak{a}_\mathbb{C}$ comes first defines a notion of positivity on R'_G. The positive roots are denoted by R'^{+}_G and the set of dominant integral weights is denoted by P'^{+}_G. The root systems R'_G are different from the root systems R_G in [Bou68, Planches] that we usually have in mind. However, the two systems differ only by a conjugation. The spherical weight λ_{sph} in the corresponds to $\lambda'_{\mathrm{sph}} \in P'^{+}_G$. We denote by $R'(\mathfrak{a})$ the set of restricted roots $\alpha|_\mathfrak{a}$, $\alpha \in R'_G$. Note that $R'_G(\mathfrak{a})$ need not be a root system, for in the case $(\mathrm{G}_2, \mathrm{SU}(3))$ we find three different lengths of restricted roots.

3.4.3. M acts on the space $\mathrm{End}(V_\mu)$ by conjugation and the invariant elements for this action are the M-equivariant endomorphisms $\mathrm{End}_M(V_\mu)$. The elements in $\mathrm{End}_M(V_\mu)$ can be simultaneously diagonalized because the restriction to M of τ of highest weight μ decomposes multiplicity free (see 3.2.4).

Lemma 3.4.4. *Let (G, K, F) be a multiplicity free system from Table 3.1, let $\mu \in F$ and let $\Phi \in E^\mu$. Then $\Phi|_A \in R(A) \otimes \mathrm{End}_M(V_\mu)$.*

Proof. Since $M = Z_K(A)$ we have $\tau(m)\Phi(a) = \Phi(ma) = \Phi(am) = \Phi(a)\tau(m)$ for all $a \in A$ and $m \in M$. □

3.4.5. Let $W = N_K(A)/M$ which is a group of order two. The non-trivial element in W can be represented by $n \in N_K(A)\backslash M$. Define actions of W and $M \cap A$ on $R(A) \otimes \mathrm{End}_M(V_\mu)$ by

$$(m \cdot \Phi)(a) = \tau(m)\Phi(m^{-1}a), \quad m \in A \cap M, \tag{3.19}$$

$$(w \cdot \Phi)(a) = \tau(n)\Phi(n^{-1}an)\tau(n)^{-1}, \quad w \in W. \tag{3.20}$$

In the symmetric case the actions of $A \cap M$ and W commute because then the elements in $M \cap A$ are of order two. In general $N_K(A)$ normalizes $A \cap M$ so we obtain an action of $W \ltimes (A \cap M)$ on $R(A) \otimes \mathrm{End}_M(V_\mu)$. The fixed points for this action are denoted by

$$(R(A) \otimes \mathrm{End}_M(V_\mu))^{W \ltimes (A \cap M)}.$$

Define $E^\mu_A = \{\Phi|_A : \Phi \in E^\mu\}$. We observe that

$$E^\mu_A \subset (R(A) \otimes \mathrm{End}_M(V_\mu))^{W \ltimes (A \cap M)},$$

but in general we do not have equality. Indeed, suppose that $A \cap M$ acts trivially on V_μ and consider the constant function $A \to \mathrm{End}_M(V_\mu) : a \mapsto I$ which is in

$$(R(A) \otimes \mathrm{End}_M(V_\mu))^{W \ltimes (M \cap A)}.$$

This function is not the restriction of a non-constant $\Phi \in E^\mu$ because the elementary spherical functions are (real) analytic, as their entries are matrix coefficients (see 3.3.6). Hence the function $a \mapsto I$ is in E^μ_A if and only if $0 \in P^+_G(\mu)$. The latter is the case if and only if $\mu = 0$. In fact, for a compact symmetric space one can give a global description of the zonal spherical functions via the isomorphism $E^0 \cong E^0_A = R(A)^{W \times (A \cap M)}$.

Proposition 3.4.6. *The restriction map*

$$\mathrm{res}_A : E^\mu \to E^\mu_A : \Phi \mapsto \Phi|_A \tag{3.21}$$

is an isomorphism of vector spaces.

Proof. The $G = KAK$ decomposition implies that, in view of the transformation behavior (3.3), a spherical function is uniquely determined by its restriction to A. This shows that res_A is injective. It is surjective by definition. □

3.4.2 Orthogonality and recurrence relations

3.4.7. Let $\mu \in F$. Consider the space of restricted spherical functions E^μ_A. If we restrict the functions in (3.10) then we obtain a recurrence relation for the restricted spherical functions,

$$\phi(a)\Phi^\mu_{\lambda(d,\nu)}(a) = \sum_{\lambda - \lambda_{\mathrm{sph}} \preceq_\mu \lambda' \preceq_\mu \lambda + \lambda_{\mathrm{sph}}} a^\mu_\lambda(\lambda')\Phi^\mu_{\lambda'}(a), \tag{3.22}$$

for all $a \in A$ and with $a^\mu_\lambda(\lambda + \lambda_{\mathrm{sph}}) \neq 0$. This shows that E^μ_A is an E^0_A-module. The restriction map $E^\mu \to E^\mu_A$ is an isomorphism of vector spaces that respects the module structures.

3.4.8. In view of the $G = KAK$ decomposition (Proposition 3.2.3) we want to understand an integration of K-bi-invariant functions over G as an integration of their restrictions to A over the torus A.

G	K	$\phi(a_t)$	(α,β)
$\mathrm{SU}(n+1)$	$\mathrm{U}(n)$	$\frac{(n+1)\cos^2(t)-1}{n}$	$(n-1,0)$
$\mathrm{SO}(2n)$	$\mathrm{SO}(2n-1)$	$\cos(t)$	$(n-\frac{3}{2},n-\frac{3}{2})$
$\mathrm{SO}(2n+1)$	$\mathrm{SO}(2n)$	$\cos(t)$	$(n-1,n-1)$
$\mathrm{Sp}(2n)$	$\mathrm{Sp}(2n-2)\times\mathrm{Sp}(2)$	$\frac{n\cos^2(t)-1}{n-1}$	$(2n-3,1)$
F_4	$\mathrm{Spin}(9)$	$\cos(2t)$	$(7,3)$
$\mathrm{Spin}(7)$	G_2	$\cos(3t)$	$(\frac{5}{2},\frac{5}{2})$
G_2	$\mathrm{SU}(3)$	$\cos(2t)$	$(2,2)$

Table 3.3: The fundamental spherical functions restricted to $A\cong S^1$ in coordinate $a_t\leftrightarrow e^{it}$.

Proposition 3.4.9. *There is a weight function $D:A\to\mathbb{C}$ such that the following holds. If $f:G\to\mathbb{C}$ is K-bi-invariant then $\int_G f(g)dg=\int_A f(a)|D(a)|da$.*

Proof. The statement is classical in case (G,K) is a symmetric pair, see e.g. [Hel62, Thm. X.1.19]. In the two cases $(\mathrm{Spin}(7),\mathrm{G}_2)$ and $(\mathrm{G}_2,\mathrm{SU}(3))$ the quotient space $X=G/K$ can also be written as $X=G'/K'$ with (G',K') the symmetric pair $(\mathrm{SO}(8),\mathrm{SO}(7))$ and $(\mathrm{SO}(7)/\mathrm{SO}(6))$ respectively. Since the orbits of K and K' in X are the same we see that the weight function for $K\backslash G/K$ is equal to the weight function for $K'\backslash G'/K'$. □

3.4.10. The fundamental spherical functions in Table 3.3 correspond to the Jacobi polynomials of degree one with parameters (α,β). One can also calculate the fundamental zonal spherical function by explicitly calculating the matrix coefficient that corresponds to the K-fixed vector in $V_{\lambda_{\mathrm{sph}}}$. In the two non-symmetric cases we know that the fundamental zonal spherical function on the quotient $A/(A\cap M)$ is equal to $\cos(t)$. To lift it back to A we need to triple or double the periods since $A\cap M$ has three or two elements respectively.

Theorem 3.4.11. *Let $\Phi,\Phi'\in E^\mu_A$. Define the pairing $\langle\cdot,\cdot\rangle_{\mu,A}:E^\mu_A\times E^\mu_A\to\mathbb{C}$ by*

$$\langle\Phi,\Phi'\rangle_{\mu,A}=\int_A \mathrm{tr}((\Phi(a))^*\Phi'(a))|D(a)|da.$$

Then $\langle\cdot,\cdot\rangle_{\mu,A}$ is an inner product on E^μ_A. The restriction map $\mathrm{res}_A:E^\mu\to E^\mu_A$ is unitary with respect to the inner products $\langle\cdot,\cdot\rangle_{\mu,G}$ (see (3.15)) and $\langle\cdot,\cdot\rangle_{\mu,A}$. This shows that the basis $\{\Phi^\mu_\lambda|_A:\lambda\in P^+_G(\mu)\}$ of E^μ_A is orthogonal for $\langle\cdot,\cdot\rangle_{\mu,A}$.

Proof. By 3.3.24 and Proposition 3.4.9 we see that $\langle\Phi,\Phi'\rangle_{\mu,G}=\langle\Phi|_A,\Phi'|_A\rangle_{\mu,A}$ for all $\Phi,\Phi'\in E^\mu$. □

3.4.12. The vector space $\mathrm{End}_M(V_\mu)$ is isomorphic to $\oplus_{\nu\in P^+_M(\mu)}\mathrm{End}_M(V_\nu)$, see Definition 2.3.1. The isomorphism is given as follows. Let $b_\nu:V_\nu\to V_\mu$ be a unitary M-equivariant embedding and let b_ν^* denote its Hermitian adjoint. Define the maps

$$\beta_\nu : \mathrm{End}_M(V_\mu) \to \mathrm{End}_M(V_\nu) : T \mapsto \mathrm{tr}(b_\nu^* \circ T \circ b_\nu),$$

which are surjective for $\nu \in P_M^+(\mu)$. Together the maps $\{\beta_\nu : \nu \in P_M^+(\mu)\}$ give a linear map $\beta : \mathrm{End}_M(V_\mu) \to \oplus_{\nu \in P_M^+(\mu)} \mathrm{End}_M(V_\nu)$ which is surjective by construction and injective by dimension count. Recall that $d_\mu(M) = \dim \mathrm{End}_M(V_\mu)$ is the number of elements in the bottom B_μ of the μ-well (see Definition 2.3.8). Parametrize the standard basis vectors of $\mathbb{C}^{d_\mu(M)}$ by $P_M^+(\mu)$. This fixes an isomorphism

$$\oplus_{\nu \in P_M^+(\mu)} \mathrm{End}_M(V_\nu) \to \mathbb{C}^{d_\mu(M)}.$$

The elements in $\mathrm{End}_M(V_\mu)$ are diagonal matrices that consist of blocks. For later purposes we want to view these diagonal matrices as vectors without repetition. To this end, define the map $(\cdot)^{\mathrm{up}}$ by the following commutative diagram.

$$\begin{array}{ccc} \mathrm{End}_M(V_\mu) & \xrightarrow{(\cdot)^{\mathrm{up}}} & \mathbb{C}^{d_\mu(M)} \\ \Big\downarrow \beta & \nearrow & \\ \oplus_{\nu \in P_M^+(\mu)} \mathrm{End}_M(V_\nu) & & \end{array}$$

The inner product $(T, S) \mapsto \mathrm{tr}(T^* S)$ that we discussed in 3.3.24 corresponds via $(\cdot)^{\mathrm{up}}$ to the standard inner product on $\mathbb{C}^{d_\mu(M)}$ that is $\mathbb{C}$-linear in the second variable.

Let $\Phi \in E_A^\mu$. We denote by $\Phi^{\mathrm{up}} : A \to \mathbb{C}^{d_\mu(M)}$ the composition of Φ and $(\cdot)^{\mathrm{up}}$. The space of $\mathbb{C}^{d_\mu(M)}$-valued functions on A is endowed with the inner product $\langle F, F' \rangle_{\mu,\mathrm{up}} = \int_A \langle F(a), F'(a) \rangle |D(a)| da$ where $\{a \mapsto D(a)\}$ is the weight function from Proposition 3.4.9. We have $\langle \Phi, \Phi' \rangle_{\mu,A} = \langle \Phi^{\mathrm{up}}, \Phi'^{\mathrm{up}} \rangle_{\mu,\mathrm{up}}$. Hence the map $E_A^\mu \to (E_A^\mu)^{\mathrm{up}}$ given by composition with $(\cdot)^{\mathrm{up}}$ is a unitary isomorphism of vector spaces. It is clear that $(\cdot)^{\mathrm{up}}$ respects the module structures.

3.4.13. Finally we say something about the degrees of the restricted spherical functions. We have seen in Lemma 3.4.4 that elementary spherical functions Φ_λ^μ restrict to $\mathrm{End}_M(V_\mu)$-valued Fourier polynomials $\Phi_\lambda^\mu|_A$. In particular the entries of the $\mathbb{C}^{d_\mu(M)}$-valued functions $(\Phi_\lambda^\mu)^{\mathrm{up}}$ are Fourier polynomials. The degrees of the polynomials are determined as follows. Let $H_A \in \mathfrak{a}$ denote the smallest non-zero element such that $\exp H_A = 1$. Let $\lambda' \in P_G'^+$ (the alternative set of dominant integral weights we discussed in 3.4.2) and consider a matrix element $m_{v,w}^{\lambda'}$. Then the restriction $m_{v,w}^{\lambda'}|_A$ is an element in $R(A)$ of degree $\leq |\lambda'(H_A)|$.

Proposition 3.4.14. *Let $\mu \in F$, let $\lambda = \lambda(d, \nu) \in P_G^+(\mu)$ for some $\nu \in P_M^+(\mu)$ and $d \in \mathbb{N}$. Let $\lambda' \in P_G'^+(\mu)$ denote the corresponding weight in the (alternative) set of dominant integral weights that we discussed in 3.4.2. The entries of $\Phi_\lambda^\mu|_A$ are of degree $< |\lambda'(H_A)|$ except for the entry that corresponds to the M-type ν. The degree of the latter entry, $a \mapsto m_{b_\mu(v),b_\mu(v)}^{\lambda'}(a)$, is $|\lambda'(H_A)|$.*

PROOF. Let $\nu' \in P_M^+(\mu)$ and let $v \in b_{\lambda'}(b_\nu(V_{\nu'}))$ be non-zero. By Proposition 2.3.3 M acts irreducibly of type ν on the space $V_{\lambda'}^{\mathfrak{n}^+}$. Write v as a sum of $\mathfrak{a}_\mathbb{C}$-weight vectors, $v = \sum v_\eta$. The degree of $a \mapsto m_{v,v}^{\lambda'}(a)$ is $|\lambda'(H_A)|$ if and only if $v_{\lambda'|_\mathfrak{a}} \neq 0$, i.e. $(v, V_{\lambda'}^{\mathfrak{n}^+}) \neq 0$. So we need to show that $(v, V_{\lambda'}^{\mathfrak{n}^+}) \neq 0$ if and only if $\nu = \nu'$.

If $(v, V_{\lambda'}^{\mathfrak{n}^+}) \neq 0$ then $\nu = \nu'$. For the converse suppose that $\nu = \nu'$ and $(v, V_{\lambda'}^{\mathfrak{n}^+}) = 0$. Since any non-zero vector $w \in V_{\lambda'}^{\mathfrak{n}^+}$ is K-cyclic by Proposition 2.3.3 we have $(w, b_{\lambda'}(V_\mu)) \neq 0$. But w only pairs non-trivially against vectors from the M-isotypical subspace of type ν, $b_{\lambda'}(b_\nu(V_{\nu'}))$. Hence $(v, w) \neq 0$ for some $w \in V_{\lambda'}^{\mathfrak{n}^+}$, a contradiction. □

3.4.3 Differential operators

3.4.15. In this subsection we study how the differential operators in $\mathbb{D}^\mu$ relate to differential operators acting on the restricted spherical functions. More precisely, we calculate the radial parts of the operators in $\mathbb{D}^\mu$. We use the results in [CM82] in which Casselman and Miličić calculate the radial parts for μ-spherical functions for symmetric pairs of any rank. It turns out that, using Lemma 3.4.17, for the rank one cases that we study, the proofs in [CM82] carry over mutatis mutandis to the non-symmetric examples.

3.4.16. We use the notations from 3.4.2. The eigenspace for an element $\gamma \in R_G'^+(\mathfrak{a})$ is denoted by $(\mathfrak{g}_\mathbb{C})_\gamma$. An element γ gives rise to a character of A that we denote by e^γ. Let $\mu \in F$ and let $\tau \in \widehat{K}$ be of highest weight μ. Let $a \in A$ and $X \in U(\mathfrak{g}_\mathbb{C})$. We define $X^a = \mathrm{Ad}(a^{-1})X$. Define the trilinear map $B_a : \mathrm{U}(\mathfrak{a}_\mathbb{C}) \times U(\mathfrak{k}_\mathbb{C}) \times U(\mathfrak{k}_\mathbb{C}) \to U(\mathfrak{g}_\mathbb{C})$ by

$$B_a(H, X, Y) = X^a HY.$$

For $Z \in U(\mathfrak{m}_\mathbb{C})$, we have $B_a(H, XZ, Y) = B_a(H, X, ZY)$, so B_a induces a linear map

$$\Gamma_a : \mathrm{U}(\mathfrak{a}_\mathbb{C}) \otimes U(\mathfrak{k}_\mathbb{C}) \otimes_{U(\mathfrak{m}_\mathbb{C})} U(\mathfrak{k}_\mathbb{C}) \to U(\mathfrak{g}_\mathbb{C}) : H \otimes X \otimes Y \mapsto X^a HY.$$

We denote $U(\mathfrak{a}_\mathbb{C}) \otimes U(\mathfrak{k}_\mathbb{C}) \otimes_{U(\mathfrak{m}_\mathbb{C})} U(\mathfrak{k}_\mathbb{C}) = \mathcal{A}$. Our first aim is to prove that Γ_a is an isomorphism of vector spaces for $a \in A_{\mathrm{reg}} = \{a \in A | \forall \gamma \in R_G'^+(\mathfrak{a}) : e^\gamma(a) \neq 1\}$.

Lemma 3.4.17. *Let $\gamma \in R_G'^+(\mathfrak{a})$ and let $Z \in (\mathfrak{g}_\mathbb{C})_\gamma$. Then there is a unique root vector $Z' \in (\mathfrak{g}_\mathbb{C})_{\gamma'}$ with $\gamma' \in R_G'^-(\mathfrak{a})$ such that $Z + Z' = U \in \mathfrak{k}_\mathbb{C}$. Moreover, we have*

$$Z = \frac{1}{e^{-\gamma'}(a) - e^{-\gamma}(a)} \left(U^a - e^{-\gamma}(a)U\right) \tag{3.23}$$

$$Z' = \frac{1}{e^{-\gamma}(a) - e^{-\gamma'}(a)} \left(U^a - e^{-\gamma'}(a)U\right). \tag{3.24}$$

PROOF. This follows from Lemma 2.2.8. The proof of the equations (3.23, 3.24) follows then from $U^a = e^{-\gamma}(a)Z + e^{-\gamma'}(a)Z'$. □

3.4.18. Recall from Lemma 2.2.8 that we denote the association $\mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C} : Z \mapsto Z'$ of Lemma 3.4.17 by θ. The image of $(I + \theta) : \mathfrak{n}_\mathbb{C}^+ \to \mathfrak{k}_\mathbb{C}$ is the orthogonal complement of

$\mathfrak{m}_{\mathbb{C}}$ with respect to the Killing form and we denote it by $\mathfrak{q}_{\mathbb{C}}$. For any $a \in A_{\text{reg}}$ we have $\mathfrak{g}_{\mathbb{C}} = \mathfrak{q}_{\mathbb{C}}^a \oplus \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}}$. Indeed, by Lemma 3.4.17 we have $\mathfrak{g}_{\mathbb{C}} = \mathfrak{q}_{\mathbb{C}}^a + \mathfrak{a}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}$ and counting the dimensions shows that the sum is direct.

Theorem 3.4.19. *For $a \in A_{\text{reg}}$, $\Gamma_a : \mathcal{A} \to U(\mathfrak{g}_{\mathbb{C}})$ is an isomorphism of vector spaces.*

Proof. This is clear from the theorem of Poincaré-Birkhoff-Witt applied to the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{q}_{\mathbb{C}}^a \oplus \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}}$ from 3.4.18. □

3.4.20. Let $\mathcal{R}$ denote the algebra of functions on A that is generated by $\{e^{\gamma}, (1-e^{2\gamma})^{-1} : \gamma \in R_G'^{+}(\mathfrak{a})\}$. Fix $a \in A_{\text{reg}}$ and define

$$\mathcal{R} \otimes \mathcal{A} \to U(\mathfrak{g}_{\mathbb{C}}) : f \otimes X \mapsto f(a)\Gamma_a(X).$$

We denote this map also by Γ_a. We have the following result (see [CM82, Thm. 2.4]).

Theorem 3.4.21. *For each $X \in U(\mathfrak{g}_{\mathbb{C}})$ there exists a unique $\Pi(X) \in \mathcal{R} \otimes \mathcal{A}$ such that $\Gamma_a(\Pi(X)) = X$ for every $a \in A_{\text{reg}}$.*

3.4.22. The map $\Pi : U(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{R} \otimes \mathcal{A}$ is crucial in calculating the radial parts of the differential operators in $\mathbb{D}^{\mu}$. Let us first explain what we mean by the radial part. In subsection 3.3.3 we studied the space E^{μ} that consists of certain $\text{End}(V_{\mu})$-valued functions with transformation behavior (3.3). Let $\mathcal{E}^{\mu}$ denote the space of smooth functions $G \to \text{End}(V_{\mu})$ that satisfy (3.3). Let $\mathcal{E}^{\mu}_A$ denote the space of functions $\Phi|_A$ with $\Phi \in \mathcal{E}^{\mu}$. Then $E^{\mu} \subset \mathcal{E}^{\mu}$, $E^{\mu}_A \subset \mathcal{E}^{\mu}_A$ and $(U(\mathfrak{g}_{\mathbb{C}}))^{\mathfrak{k}_{\mathbb{C}}}$ acts on $\mathcal{E}^{\mu}$ as differential operators. The radial part of a differential operator $X \in (U(\mathfrak{g}_{\mathbb{C}}))^{\mathfrak{k}_{\mathbb{C}}}$ is the operator $\mathcal{E}^{\mu}_A \to \mathcal{E}^{\mu}_A : \Phi|_A \mapsto (X\Phi)|_A$.

3.4.23. Let $\iota : U(\mathfrak{g}_{\mathbb{C}}) \to U(\mathfrak{g}_{\mathbb{C}})$ denote the anti-automorphism induced by $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}} : X \mapsto -X$. Define

$$\xi_{\mu} : U(\mathfrak{k}_{\mathbb{C}}) \otimes U(\mathfrak{k}_{\mathbb{C}}) \to \text{Hom}(\text{End}_M(V_{\mu}), \text{End}(V_{\mu})) :$$
$$\xi_{\mu}(X \otimes Y)(T) = \tau(X) \circ T \circ \tau(\iota(Y)). \quad (3.25)$$

Then $\xi_{\mu}(XZ \otimes Y) = \xi_{\mu}(X \otimes ZY)$ for all $X, Y \in U(\mathfrak{k}_{\mathbb{C}})$ and $Z \in U(\mathfrak{m}_{\mathbb{C}})$ which implies that ξ_{μ} induces a linear map

$$\xi_{\mu} : U(\mathfrak{k}_{\mathbb{C}}) \otimes_{U(\mathfrak{m}_{\mathbb{C}})} U(\mathfrak{k}_{\mathbb{C}}) \to \text{Hom}(\text{End}_M(V_{\mu}), \text{End}(V_{\mu}))$$

that we also denote by ξ_{μ}. Finally, define the linear map $\eta_{\mu} = 1 \otimes 1 \otimes \xi_{\mu}$,

$$\eta_{\mu} : \mathcal{R} \otimes U(\mathfrak{a}_{\mathbb{C}}) \otimes U(\mathfrak{k}_{\mathbb{C}}) \otimes_{U(\mathfrak{m}_{\mathbb{C}})} U(\mathfrak{k}_{\mathbb{C}}) \to \mathcal{R} \otimes U(\mathfrak{a}_{\mathbb{C}}) \otimes \text{Hom}(\text{End}_M(V_{\mu}), \text{End}(V_{\mu}))$$

and put $\Pi_{\mu} = \eta_{\mu} \circ \Pi : U(\mathfrak{g}_{\mathbb{C}}) \to \mathcal{R} \otimes U(\mathfrak{a}_{\mathbb{C}}) \otimes \text{Hom}(\text{End}_M(V_{\mu}), \text{End}(V_{\mu}))$. The elements in $\mathcal{R} \otimes U(\mathfrak{a}_{\mathbb{C}}) \otimes \text{Hom}(\text{End}_M(V_{\mu}), \text{End}(V_{\mu}))$ act as differential operators, transforming smooth $\text{End}_M(V_{\mu})$-valued functions on A_{reg} into $\text{End}(V_{\mu})$-valued functions on A_{reg} by the rule

$$(f \otimes H \otimes T)(F) = f \cdot H(TF).$$

We call $\Pi_{\mu}(X)$ the μ-radial part of $X \in (U(\mathfrak{g}_{\mathbb{C}}))^{\mathfrak{k}_{\mathbb{C}}}$, which is justified by the following result.

Theorem 3.4.24. *For every $\Phi \in \mathcal{E}^\mu$ and $X \in (U(\mathfrak{g}_\mathbb{C}))^{\mathfrak{k}_\mathbb{C}}$ we have*

$$(X\Phi)|_A = \Pi_\mu(X)\Phi|_A.$$

Moreover, the map $\Pi_\mu : (U(\mathfrak{g}_\mathbb{C}))^{\mathfrak{k}_\mathbb{C}} \to \mathcal{R} \otimes \mathcal{A} \otimes \mathrm{End}(\mathrm{End}_M(V_\mu))$ is a homomorphism that factors through the quotient $(U(\mathfrak{g}_\mathbb{C}))^{\mathfrak{k}_\mathbb{C}} \to \mathbb{D}^\mu$ and we obtain an injective homomorphism $\mathbb{D}^\mu \to \mathcal{R} \otimes U(\mathfrak{a}_\mathbb{C}) \otimes \mathrm{End}(\mathrm{End}_M(V_\mu))$.

Proof. The first part of the statement is [CM82, Thm 3.1 and Thm. 3.3] and the proofs go through for the two non-symmetric pairs mutatis mutandis. It is left to show that the map Π_μ has the ideal $(U(\mathfrak{g}_\mathbb{C}))^{\mathfrak{k}_\mathbb{C}} \cap U(\mathfrak{g}_\mathbb{C})I^\mu$ as its kernel. Let $X \in (U(\mathfrak{g}_\mathbb{C}))^{\mathfrak{k}_\mathbb{C}}$ and suppose that $\Pi_\mu(X) = 0$. This implies that $X\Phi^\mu_\lambda(e) = 0$ for all $\lambda \in P^+_G(\mu)$ and hence that $X \in (U(\mathfrak{g}_\mathbb{C}))^{\mathfrak{k}_\mathbb{C}} \cap U(\mathfrak{g}_\mathbb{C})I^\mu$. Conversely, if $X \in (U(\mathfrak{g}_\mathbb{C}))^{\mathfrak{k}_\mathbb{C}} \cap U(\mathfrak{g}_\mathbb{C})I^\mu$ and $\Phi \in \mathcal{E}^\mu$ then $X\Phi = 0$ because $\mathcal{E}^\mu \subset L^2(E^\mu)$ is dense, where $L^2(E^\mu)$ is the Hilbert space completion of E^μ with respect to $\langle \cdot, \cdot \rangle_{\mu,G}$. It follows that $\Pi_\mu(X)\Phi|_A = 0$ and this finishes the proof. □

3.4.25. The image of the injective homomorphism from Theorem 3.4.24 is an algebra that we denote by

$$\mathbb{D}^\mu_A = \Pi_\mu(\mathbb{D}^\mu) \subset \mathcal{R} \otimes U(\mathfrak{a}_\mathbb{C}) \otimes \mathrm{End}(\mathrm{End}_M(V_\mu)).$$

We denote the induced isomorphism $\Pi_\mu : \mathbb{D}^\mu \to \mathbb{D}^\mu_A$. In the symmetric case with $\mu = 0$ we know that $\mathbb{D}^0 \cong U(\mathfrak{a}_\mathbb{C})^W$ via the Harish-Chandra homomorphism, where W is the Weyl group of $R'_G(\mathfrak{a})$. In the more general case of Table 3.1 we do not know of a global description of $\mathbb{D}^\mu$ other than an embedding $\mathbb{D}^\mu \subset U(\mathfrak{a}_\mathbb{C}) \otimes \mathrm{End}_M(V_\mu)$, see [Dix96, Ch. 9].

3.4.26. The group $W = N_K(A)/M$ acts on $\mathrm{End}_M(V_\mu)$ via $W \times \mathrm{End}_M(V_\mu) \to \mathrm{End}_M(V_\mu)$: $w \cdot S = \tau(n)S\tau(n)^{-1}$ where $n \in N_K(A)\backslash M$. Now we have an action of W on $\mathcal{R} \otimes U(\mathfrak{a}_\mathbb{C}) \otimes \mathrm{End}(\mathrm{End}_M(V_\mu))$ defined point wise by

$$w \cdot (f \otimes H \otimes T)(a)(S) = f(nan^{-1}) \otimes \mathrm{Ad}(n)H \otimes \tau(n) \circ T(\tau(n)^{-1}S\tau(n)) \circ \tau(n)^{-1}.$$

Consider the actions of $M \cap A$ on $\mathcal{R}$ and $\mathrm{End}(\mathrm{End}_M(V_\mu))$ given by $m \cdot f(a) = f(m^{-1}a)$ and $(m \cdot T)(S) = T(\tau(m)^{-1} \circ S)$. These extend to an action of $M \cap A$ on $\mathcal{R} \otimes U(\mathfrak{a}_\mathbb{C}) \otimes \mathrm{End}(\mathrm{End}_M(V_\mu))$ by letting $M \cap A$ act trivially on the second tensor factor. Note that the natural action of M on the second factor is trivial anyway. We obtain an action of $W \ltimes (M \cap A)$ on $\mathcal{R} \otimes U(\mathfrak{a}_\mathbb{C}) \otimes \mathrm{End}(\mathrm{End}_M(V_\mu))$.

Proposition 3.4.27. $\mathbb{D}^\mu_A \subset (\mathcal{R} \otimes U(\mathfrak{a}_\mathbb{C}) \otimes \mathrm{End}(\mathrm{End}_M(V_\mu)))^{W \ltimes (M \cap A)}$.

Proof. The algebra $\mathbb{D}^\mu_A$ acts faithfully on E^μ_A so it is sufficient to show that $(\sigma \cdot D)\Phi = D\Phi$ for all $\Phi \in E^\mu_A$ and all $\sigma \in W \ltimes (M \cap A)$. As $W \ltimes (M \cap A)$ acts on E^μ_A it induces an action on $\mathbb{D}^\mu_A$ as follows. For $\sigma \in W \ltimes (M \cap A)$ we have $(\sigma \star D)\Phi = \sigma \cdot (D(\sigma^{-1} \cdot \Phi))$ where $\sigma \cdot \Phi$ is the usual action. Upon writing $D = \sum f \otimes H \otimes T$ it is easily checked that indeed $\sigma \cdot D = \sigma \star D$. Since $(\sigma \star D)(\Phi^\mu_\lambda|_A) = D(\Phi^\mu_\lambda|_A)$ for $\lambda \in P^+_G(\mu)$ we have $\sigma \cdot D = D$. □

3.4.28. We close this subsection by noting that equations (3.23) and (3.24) can be used to compute the radial part effectively. In fact, in [War72b, Prop. 9.2.1.11] we find an expression for the μ-radial part of the Casimir operator for the symmetric space cases. We use the notation of 3.4.2. Let $\Omega \in \mathfrak{Z}_{\mathfrak{g}_{\mathbb{C}}}$ be the Casimir operator of order two normalized as follows. Let $\mathfrak{h}_0 \subset \mathfrak{h}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{t}_{M,\mathbb{C}}$ denote the real form on which all roots in R'_G take real values. So $\mathfrak{h}_0 = i\mathfrak{a} \oplus i\mathfrak{t}_M$. Choose root vectors $E_\alpha \in (\mathfrak{g}_{\mathbb{C}})_\alpha$ such that $B(E_\alpha, E_{-\alpha}) = 1$ for all $\alpha \in R'_G$. Let $\{H_1, \ldots, H_n\}$ be an orthonormal basis with respect to the Killing form B and such that $H_1 \in i\mathfrak{a}$. Then the Casimir operator Ω is of the form

$$\Omega = \sum_{i=1}^{n} H_i^2 + \sum_{\alpha \in R'_G} E_\alpha E_{-\alpha}$$

and the action of Ω on an irreducible representation of highest weight $\lambda \in P'^{+}_G$ is given by the scalar $\langle \lambda, \lambda \rangle + \langle \lambda, \rho_G \rangle$ (see e.g. [Kna02, Prop. 5.28]), where $\langle \cdot, \cdot \rangle$ is the pairing on $\mathfrak{h}_0^\vee$ dual to the Killing form. Let $\Delta_{nc} \subset R'_G$ denote the roots that are not perpendicular to $\mathfrak{t}_{M,\mathbb{C}}$ with $\Delta_{nc}^+ \subset \Delta_{nc}$ the positive roots. Let $\Omega_{\mathfrak{m}}$ denote the Casimir operator for M. Then we have

$$\Omega = H_1^2 + \Omega_{\mathfrak{m}} + \sum_{\alpha \in \Delta_{nc}} E_\alpha E_{-\alpha}$$

and since $E_\alpha \in \mathfrak{n}_{\mathbb{C}}^+$ for $\alpha \in \Delta_{nc}$ we can use (3.23), (3.24) to calculate the radial part. For $\alpha \in \Delta_{nc}$ let $H_\alpha \in \mathfrak{a}_{\mathbb{C}}$ denote the element such that $\alpha(H_1) = B(H_\alpha, H_1)$. Moreover, let Y_α denote the unique element in $\mathfrak{k}_{\mathbb{C}}$ such that $E_\alpha - Y_\alpha \in (\mathfrak{g}_{\mathbb{C}})_{-\alpha}$. We get

$$\begin{aligned} \Pi_\mu(\Omega) = H_1^2 + \tau(\Omega_{\mathfrak{m}}) + \sum_{\alpha \in \Delta_{nc}^+} \frac{e^\alpha + e^{-\alpha}}{e^\alpha - e^{-\alpha}} H_\alpha \\ + 8 \sum_{\alpha \in \Delta_{nc}^+} \frac{1}{(e^\alpha - e^{-\alpha})^2} \left(\bullet \tau(Y_\alpha)\tau(Y_{-\alpha}) + \tau(Y_\alpha)\tau(Y_{-\alpha}) \bullet \right) \\ - 8 \sum_{\alpha \in \Delta_{nc}^+} \frac{(e^\alpha + e^{-\alpha})}{(e^\alpha - e^{-\alpha})^2} \left(\tau(Y_\alpha) \bullet \tau(Y_{-\alpha}) \right) \end{aligned} \tag{3.26}$$

where the bullet $\bullet$ indicates where to put the function $\Phi \in \mathcal{E}^\mu_A$.

3.5 Spherical polynomials

3.5.1 Spherical polynomials on G

3.5.1. Let $\mu \in F$. We have seen in Proposition 3.3.17 that the elementary spherical functions Φ^μ_λ satisfy recurrence relations. Given an elementary μ-spherical function Φ^μ_λ there exist unique polynomials $q^\mu_{\lambda,\nu} \in \mathbb{C}[\phi]$ such that

$$\Phi^\mu_\lambda = \sum_{\nu \in P^+_M(\mu)} q^\mu_{\lambda,\nu}(\phi) \Phi^\mu_{\lambda(0,\nu)},$$

see Corollary 3.3.18. This defines the map

$$\mathrm{rec} : E^{\mu} \to E^0 \otimes \mathbb{C}^{d_\mu(M)} : \Phi^{\mu}_{\lambda} \mapsto (q^{\mu}_{\lambda,\nu})_{\nu\in P^+_M(\mu)}. \tag{3.27}$$

where the name "rec" refers to the recurrence relations. The image of an elementary spherical function of type μ under the map rec is denoted by $\mathrm{rec}(\Phi^{\mu}_{\lambda}) = Q^{\mu}_{\lambda}$. We investigate the orthogonality measure for the space $\mathrm{rec}(E^{\mu})$.

Definition 3.5.2. *Define* $V^{\mu} \in E^0 \otimes \mathrm{End}(\mathbb{C}^{d_\mu(M)})$ *by*

$$(V^{\mu}(g))_{\nu,\nu'} = \mathrm{tr}\left((\Phi^{\mu}_{\lambda(0,\nu)}(g))^*\Phi^{\mu}_{\lambda(0,\nu')}(g)\right).$$

3.5.3. The entries of $(V^{\mu})_{\nu,\nu'}$ are indeed in E^0 by 3.3.23. Define the pairing $\langle\cdot,\cdot\rangle_{\mu,q,G} : E^0 \otimes \mathbb{C}^{d_\mu(M)} \times E^0 \otimes \mathbb{C}^{d_\mu(M)} \to \mathbb{C}$ by

$$\langle Q, Q'\rangle_{\mu,q,G} = \int_G (Q(g))^* V^{\mu}(g) Q'(g) dg. \tag{3.28}$$

Proposition 3.5.4. *The map* $\mathrm{rec} : E^{\mu} \to E^0 \otimes \mathbb{C}^{d_\mu(M)}$ *is an isomorphism of vector spaces that respects the* E^0*-module structure. The pairing* $\langle\cdot,\cdot\rangle_{\mu,q,G}$ *defines a Hermitian inner product on the space* $E^0 \otimes \mathrm{End}(\mathbb{C}^{d_\mu(M)})$*. Moreover, the map* rec *is unitary for* $\langle\cdot,\cdot\rangle_{\mu,G}$ *and* $\langle\cdot,\cdot\rangle_{\mu,q,G}$.

Proof. The map rec is linear and it respects the module structures by definition. To show that rec is injective, let $\Phi \in E^{\mu}$ and suppose that $\mathrm{rec}(\Phi) = 0$. We can express Φ as a finite sum of elementary spherical functions. These spherical functions are linearly dependent by the assumption $\mathrm{rec}(\Phi) = 0$. Since no finite set of elementary spherical functions is linearly dependent (because these correspond to characters of a commutative algebra) we must have $\Phi = 0$. The map rec is surjective because it is E^0-linear and the minimal elementary spherical functions $\Phi^{\mu}_{\lambda(0,\nu)}$ are mapped to the constant functions that generate $E^0 \otimes \mathbb{C}^{d_\mu(M)}$ as an E^0-module. Note that

$$\begin{aligned}\langle\Phi^{\mu}_{\lambda}, \Phi^{\mu}_{\lambda'}\rangle_{\mu,G} &= \int_G \mathrm{tr}\left(\sum_{\nu,\nu'} \overline{q^{\mu}_{\lambda,\nu}}(\phi(g))\Phi^{\mu}_{\lambda}(g)^*\Phi^{\mu}_{\lambda(0,\nu')}(g) q^{\mu}_{\lambda',\nu'}(\phi(g))\right) dg \\ &= \sum_{\nu,\nu'}\int_G \overline{q^{\mu}_{\lambda,\nu}}(\phi(g))\mathrm{tr}\left(\Phi^{\mu}_{\lambda(0,\nu)}(g)^*\Phi^{\mu}_{\lambda(0,\nu')}(g)\right) q^{\mu}_{\lambda',\nu'}(\phi(g))dg,\end{aligned}$$

which is equal to $\langle Q^{\mu}_{\lambda}, Q^{\mu}_{\lambda'}\rangle_{\mu,q,G}$. It follows that rec is unitary. □

3.5.5. The space $E^0 \otimes \mathbb{C}^{d_\mu(M)}$ is called the space of μ-spherical polynomials. By Proposition 3.5.4 the space of μ-spherical polynomials is isomorphic as E^0-module to the space of μ-spherical functions. The Hermitian structure on $E^0 \otimes \mathbb{C}^{d_\mu(M)}$ is governed by the function $g \mapsto V^{\mu}(g)$. The function V^{μ} is called the matrix valued μ-weight function. Note that the family $\{Q^{\mu}_{\lambda} : \lambda \in P^+_G(\mu)\}$ is an orthogonal family for $\langle\cdot,\cdot\rangle_{\mu,q,G}$.

3.5.6. The weight functions may be decomposed in blocks by conjugating with a certain constant matrix. In the case $(\mathrm{SU}(2)\times\mathrm{SU}(2),\mathrm{diag})$, see 4.5.5, where the commutant $\{V^{\mu}(g):g\in G\}'$ is spanned by two elements. In the cases $(\mathrm{SO}(2n),\mathrm{SO}(2n-1))$ the commutant $\{V^{\mu}(g):g\in G\}'$ will in general not be one-dimensional because the theta-involution interchanges the spin components of the occurring M-types. It is yet unclear what the commutants are in the generality of Table 3.1.

3.5.2 Spherical polynomials restricted to A

In this subsection we study the restriction map $E^0\otimes\mathbb{C}^{d_\mu(M)}\to E^0_A\otimes\mathbb{C}^{d_\mu(M)}$. The space $E^0_A\otimes\mathbb{C}^{d_\mu(M)}$ is an E^0_A-module and the restriction map $E^0\to E^0_A$ is an isomorphism of vector spaces that respects the module structures, see Proposition 3.4.6 and Theorem 3.4.11. Hence the restriction map

$$\mathrm{res}_A: E^0\otimes\mathbb{C}^{d_\mu(M)}\to E^0_A\otimes\mathbb{C}^{d_\mu(M)} \tag{3.29}$$

is an isomorphism of vector spaces that respects the module structures. Define the pairing $\langle\cdot,\cdot,\rangle_{\mu,q,A}:E^0_A\otimes\mathbb{C}^{d_\mu(M)}\times E^0_A\otimes\mathbb{C}^{d_\mu(M)}\to\mathbb{C}$ by

$$\langle Q,Q'\rangle_{\mu,q,A}=\int_A (Q(a))^*V^{\mu}(a)Q'(a)|D(a)|da. \tag{3.30}$$

Lemma 3.5.7. *The pairing $\langle\cdot,\cdot\rangle_{\mu,q,A}$ defines a Hermitian inner product on $E^0_A\otimes\mathbb{C}^{d_\mu(M)}$. The map res_A is unitary with respect to $\langle\cdot,\cdot\rangle_{\mu,q,G}$ and $\langle\cdot,\cdot\rangle_{\mu,q,A}$.*

Proof. It is sufficient to check the identity $\langle Q^{\mu}_{\lambda},Q^{\mu}_{\lambda'}\rangle_{\mu,q,G}=\langle Q^{\mu}_{\lambda}|_A,Q^{\mu}_{\lambda'}|_A\rangle_{\mu,q,A}$ which follows from a straightforward calculation. □

3.5.8. Another way of studying the space $E^0_A\otimes\mathbb{C}^{d_\mu(M)}$ is by means of the recurrence relations to E^{μ}_A. In 3.4.7 we have seen that the restricted elementary spherical functions of type μ also satisfy recurrence relations. Hence we can express a restricted elementary spherical function $\Phi^{\mu}_{\lambda}|_A$ as an E^0_A-linear combination of the $d_\mu(M)$ elementary spherical functions $\Phi^{\mu}_{\lambda(0,\nu)}$, $\nu\in P^+_M(\mu)$. The coefficients that we obtain are $q^{\mu}_{\lambda,\nu}|_A$ with $q^{\mu}_{\lambda,\nu}$ defined as in Corollary 3.3.18. This defines the map

$$\mathrm{rec}_A: E^{\mu}_A\to E^0_A\otimes\mathbb{C}^{d_\mu(M)}:\Phi^{\mu}_{\lambda}|_A\mapsto(q^{\mu}_{\lambda,\nu}|_A)_{\nu\in P^+_M(\mu)}. \tag{3.31}$$

Proposition 3.5.9. *The map $\mathrm{rec}_A: E^{\mu}_A\to E^0_A\otimes\mathbb{C}^{d_\mu(M)}$ is an isomorphism of vector spaces. It respects the module structures and it is unitary for the inner products given by (3.28) and (3.30).*

Proof. The proof of rec_A being an isomorphism of vector spaces is similar to the proof of rec being an isomorphism, see Proposition 3.5.4. The module structures are clearly respected by rec_A. To see that rec is unitary we have to show the identity $\langle\Phi^{\mu}_{\lambda}|_A,\Phi^{\mu}_{\lambda'}|_A\rangle_{\mu,A}=\langle Q^{\mu}_{\lambda}|_A,Q^{\mu}_{\lambda'}|_A\rangle_{\mu,q,A}$ which is apparent from the definitions. □

3.5.10. The restriction maps (3.21) and (3.29) and the maps given by the recurrence relations (3.27) and (3.31) fit in the following commutative diagram.

$$\begin{array}{ccc} E^{\mu} & \xrightarrow{\text{rec}} & E^{0}\otimes\mathbb{C}^{d_{\tau}(M)} \\ \downarrow{\scriptstyle \text{res}_A} & & \downarrow{\scriptstyle \text{res}_A} \\ E^{\mu}_{A} & \xrightarrow[\text{rec}_A]{} & E^{0}_{A}\otimes\mathbb{C}^{d_{\tau}(M)} \end{array}$$

The recurrence map rec_A can be understood in a different way. The recurrence relations imply that a restricted spherical function $\Phi^{\mu}_{\lambda}|_A$ can be written as a E^0_A-linear combination of the $d_{\mu}(M)$ elementary spherical functions $\Phi^{\mu}_{\lambda(0,\nu)}$ with $\nu \in P^{+}_{M}(\mu)$. In 3.4.12 we have seen that E^{μ}_{A} is isomorphic to $(E^{\mu}_{A})^{\text{up}}$ via composition with $(\cdot)^{\text{up}}$. The isomorphism is unitary and it respects the module structures. The map $\Psi^{*}_{\mu} : E^0_A \otimes \mathbb{C}^{d_{\mu}(M)} \to (E^{\mu}_{A})^{\text{up}}$ is defined by the following commutative diagram.

$$\begin{array}{ccc} E^{\mu}_{A} & \xrightarrow{\text{rec}_A} & E^{0}_{A}\otimes\mathbb{C}^{d_{\mu}(M)} \\ \downarrow{\scriptstyle (\cdot)^{\text{up}}} & \swarrow{\scriptstyle \Psi^{*}_{\mu}} & \\ (E^{\mu}_{A})^{\text{up}} & & \end{array}$$

The map Ψ^{*}_{μ} is unitary and it respects the module structures. The notation suggests that Ψ^{*}_{μ} is the pull back of a diffeomorphism. Theorem 3.5.12 shows that this is the case on a dense open subset of A.

Definition 3.5.11. *Define $A_{\mu-\text{reg}} = \{a \in A : \det(V^{\mu}(a)) \neq 0\}$. The set $A_{\mu-\text{reg}}$ is called the μ-regular part of A.*

Theorem 3.5.12. *The set $A_{\mu-\text{reg}}$ is open and dense in A. The restriction*

$$\Psi^{*}_{\mu} : (E^{\mu}_{A_{\mu-\text{reg}}})^{\text{up}} \to E^{0}_{A_{\mu-\text{reg}}} \otimes \mathbb{C}^{d^{\mu}(M)} \tag{3.32}$$

is given by pointwise multiplication by the matrix $\Psi_{\mu}(a)$ which has the vectors $\Phi^{\mu}_{\lambda(0,\nu)}(a)^{\text{up}}$, $\nu \in P^{+}_{M}(\mu)$ as its columns. In particular (3.32) is induced by a diffeomorphism of the (trivial) vector bundle $A_{\mu-\text{reg}} \times \mathbb{C}^{d_{\mu}(M)}$ to itself.

Proof. Define $\Psi_{\mu} : A \to \text{End}(\mathbb{C}^{d_{\mu}(M)})$ point wise by the matrix $\Psi_{\mu}(a)$ whose column vectors are $(\Phi^{\mu}_{\lambda(0,\nu)})^{\text{up}}(a)$ with $\nu \in P^{+}_{M}(\mu)$. The definition of the weight function V^{μ} implies that $V^{\mu}(a) = (\Psi_{\mu}(a))^{*}\Psi_{\mu}(a)$. The determinant of $V^{\mu}(a)$ is a polynomial in $\phi(a)$ of positive degree by Proposition 3.4.14. As $\phi : A \to \mathbb{C}$ has only finitely many zeros on A we see that $A_{\mu-\text{reg}}$ is open and dense in A. The construction of Ψ_{μ} implies that on $A_{\mu-\text{reg}}$ we have the identity

$$\Psi_{\mu}(a)Q^{\mu}_{\lambda}(a) = (\Phi^{\mu}_{\lambda})^{\text{up}}(a).$$

It follows that rec_A restricted to $(E^{\mu}_{A_{\mu-\text{reg}}})^{\text{up}}$ is given by

$$Q^{\mu}_{\lambda}(a) = \Psi_{\mu}(a)^{-1}(\Phi^{\mu}_{\lambda})^{\text{up}}(a),$$

which finishes the proof. □

3.5.13. The differential operators $D \in \mathbb{D}_A^\mu$ acting on E_A^μ give rise to differential operators acting on $(E_A^\mu)^{\mathrm{up}}$. We denote the corresponding algebra by $\mathbb{D}_A^{\mu,\mathrm{up}}$. Let $D^{\mathrm{up}} \in \mathbb{D}^{\mu,\mathrm{up}}$ and define

$$(\Psi_\mu^* D^{\mathrm{up}})Q = (\Psi_\mu^*)^{-1}(D^{\mathrm{up}}(\Psi_\mu^*(Q))).$$

We obtain a homomorphism $\Psi_\mu^* : \mathbb{D}_A^{\mu,\mathrm{up}} \to \mathcal{R} \otimes U(\mathfrak{a}_\mathbb{C}) \otimes \mathrm{End}(\mathbb{C}^{d_\mu(M)})$. The algebra $\mathcal{R} \otimes U(\mathfrak{a}_\mathbb{C}) \otimes \mathrm{End}(\mathbb{C}^{d_\mu(M)})$ admits an action of $W \ltimes (M \cap A)$ by letting it act trivially on the third tensor factor and in the usual way on the first two tensor factors. A small calculation shows that

$$\Psi_\mu^* \mathbb{D}_A^{\mu,\mathrm{up}} \subset (\mathcal{R} \otimes U(\mathfrak{a}_\mathbb{C}))^{W \ltimes (M \cap A)} \otimes \mathrm{End}(\mathbb{C}^{d_\mu(M)}). \tag{3.33}$$

3.6 Full spherical polynomials

3.6.1 Construction

3.6.1. Let $\mu \in F$ and consider the isomorphism $\lambda : \mathbb{N} \times P_M^+(\mu) \to P_G^+(\mu)$, see (3.1). The number of elements in $P_M^+(\mu)$ is $d_\mu(M)$ and this is precisely the dimension of the vector space in which the polynomials $\{Q_\lambda^\mu\}_{\lambda \in P_G^+}$ take their values. It follows that the number of polynomials Q_λ^μ of degree d is $d_\mu(M)$.

Definition 3.6.2. *The matrix valued function* $Q_d^\mu : A \to \mathrm{End}(\mathbb{C}^{d_\mu(M)})$, $d \in \mathbb{N}$ *is defined point wise by the matrix* $Q_d^\mu(a)$ *whose column with index* $\nu \in P_M^+(\mu)$ *is* $Q_{\lambda(d,\nu)}^\mu(a)$.

Note that $\{Q_d^\mu\}_{d \in \mathbb{N}}$ is a family of matrix valued functions on A for which the entries are polynomials in ϕ. To get a family of matrix valued polynomials on a compact interval we have to perform a final operation.

Definition 3.6.3. *Let* $\mu \in F$ *and let* $I = \phi(A)$ *denote the image of the fundamental zonal spherical function. The matrix valued function* Q_d^μ *satisfies* $Q_d^\mu(a) = P_d^\mu(\phi(a))$ *where* $P_d^\mu : A \to \mathrm{End}(\mathbb{C}^{d_\mu(M)})$ *is a uniquely determined matrix valued polynomial. We call* $\{P_d^\mu\}_{d \in \mathbb{N}}$ *the family of matrix valued polynomials associated to the multiplicity free triple* (G, K, μ).

3.6.2 Properties

3.6.4. The family of matrix valued polynomials associated to a triple (G, K, μ) has several nice properties. In particular, the members of the family satisfy a three term recurrence relation, they are orthogonal with respect to a matrix valued inner product and they are simultaneous eigenfunctions for a commutative algebra of differential operators with matrices as eigenvalues. We address all these properties in this section.

Proposition 3.6.5. *The matrix valued polynomials satisfy a three term recurrence relation. More precisely, for every $d \in \mathbb{N}$ there are three matrices $A_d, B_d, C_d \in \mathrm{End}(\mathbb{C}^{d_\mu(M)})$ such that*

$$xP^\mu_d = P^\mu_{d+1}A_d + P^\mu_d B_d + P^\mu_{d-1}C_d \tag{3.34}$$

holds for $x \in I$. Moreover, the matrix A_d is invertible.

Proof. We prove the equivalent recurrence relation

$$\phi Q^\mu_d = Q^\mu_{d+1}A_d + Q^\mu_d B_d + Q^\mu_{d-1}C_d \tag{3.35}$$

which in turn is equivalent to a recurrence relation for the columns of the functions $\{Q^\mu_\lambda\}_{\lambda\in P^+_G(\mu)}$. We have such a recurrence relation for the functions Φ^μ_λ, namely (3.22). Applying the isomorphism Ψ^*_A gives the desired recurrence relation for the functions Q^μ_λ. To see that A_d is invertible note that $a^\mu_\lambda(\lambda+\lambda_{\mathrm{sph}}) \neq 0$. Moreover, if $\lambda' \succ_\mu \lambda+\lambda_{\mathrm{sph}}$ then $a^\mu_\lambda(\lambda) = 0$. Hence the matrices A_d are similar to lower triangular matrices with non-zero entries on the diagonal. □

3.6.6. If $p \in \mathbb{C}[\phi]$ then $\int_A p(a)|D(a)|da = \int_I p(x)w(x)dx$ for some function $w(x)$. If we change the variable using a linear transformation $c : I \to [-1,1]$ then $w(a) = (x-1)^\alpha(x+1)^\beta$ with α, β functions of the root multiplicities. We have indicated α and β in Table 3.3.

Definition 3.6.7. *The entries of the weight function V^μ (see 3.5.2) are polynomials in ϕ. Define $W^\mu \in \mathbb{C}[x]\otimes\mathrm{End}(\mathbb{C}^{d_\mu(M)})$ point wise by $W^\mu(\phi(a)) = V^\mu(a)w(\phi(a))$. Define*

$$\langle\cdot,\cdot\rangle_{W^\mu} : \mathrm{End}(\mathbb{C}^{d_\mu(M)})[x]\times\mathrm{End}(\mathbb{C}^{d_\mu(M)})[x] \to \mathrm{End}(\mathbb{C}^{d_\mu(M)}),$$
$$\langle P, P'\rangle_{W^\mu} = \int_I P(x)^* W^\mu(x)P'(x)dx. \tag{3.36}$$

Proposition 3.6.8. *The pairing $\langle\cdot,\cdot\rangle_{W^\mu}$ is a matrix valued inner product in the sense of 1.1.2. The family $\{P^\mu_d\}_{d\in\mathbb{N}}$ is a family of matrix valued orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle_{W^\mu}$.*

Proof. The weight function V^μ is positive definite on $A_{\mu-\mathrm{reg}}$, which implies that W^μ is positive definite on a dense subset of I. This shows that $\langle\cdot,\cdot\rangle_{W^\mu}$ is a matrix valued inner product. The orthogonality of the P^μ_d is clear from the definitions. □

3.6.9. The matrix valued function $x \mapsto W^\mu(x)$ is called the weight function for the family of matrix valued polynomials $\{P^\mu_d\}_{d\in\mathbb{N}}$. If $\mu = 0$ then the weight function reduces to $w(x)$. The variable x runs over the image of $\phi|_A$ and if we rescale to the interval $[-1,1]$ we find the Jacobi weights $(y+1)^\alpha(y-1)^\beta$. The parameters α and β only depend on the pair (G,K) and we have indicated them in Table 3.3 for the various cases.

3.6.10. Finally, we discuss the differential operators. Recall from 3.5.13 that the vector valued functions Q^μ_λ are simultaneous eigenfunctions of the algebra $\Psi^*_\mu\mathbb{D}^{\mu,\mathrm{up}}_A$.

Theorem 3.6.11. *The algebra $\Psi_\mu^*\mathbb{D}_A^{\mu,\mathrm{up}}$ can be pushed forward by ϕ. Upon writing $\phi(a) = x$, we obtain a homomorphism*

$$\phi_* : \Psi_\mu^*\mathbb{D}_A^\mu \to \mathbb{C}[x] \otimes \mathbb{C}[\partial_x] \otimes \mathrm{End}(\mathbb{C}^{d_\mu(M)}).$$

We denote the image $\phi_*(\Psi_\mu^*\mathbb{D}_A^\mu) = \mathbb{D}(W_\mu)$.

Proof. The map ϕ_* exists because $\Psi_\mu^*\mathbb{D}_A^{\mu,\mathrm{up}}$ is $W \ltimes (M \cap A)$-invariant. Pushing forward a priori gives a subalgebra of $\mathbb{C}(x) \otimes \mathbb{C}[\partial_x] \otimes \mathrm{End}(\mathbb{C}^{d_\mu(M)})$. But the algebra acts faithfully on the matrix valued polynomials P_d^μ from which it follows that the coefficients are polynomials in x, i.e. $\mathbb{D}(W_\mu) \subset \mathbb{C}[x] \otimes \mathbb{C}[\partial_x] \otimes \mathrm{End}(\mathbb{C}^{d_\mu(M)})$. □

3.6.12. The algebra $\mathbb{D}(W^\mu)$ acts on $\mathrm{End}(\mathbb{C}^{d_\mu(M)})$-valued polynomials column wise. The polynomials P_d^μ are simultaneous eigenfunctions for $\mathbb{D}(W^\mu)$, whose eigenvalues are diagonal matrices that act on the right, i.e. for all $D \in \mathbb{D}^{W^\mu}$ and $d \in \mathbb{N}$ there is a diagonal matrix $\Lambda_{D,d} \in \mathrm{End}(\mathbb{C}^{d_\mu(M)})$ such that $DP_d^\mu = P_d^\mu \Lambda_{D,d}$.

3.6.13. The algebra $\mathcal{D}(W^\mu)$ is defined as the algebra of differential operators that have the polynomials $\{P_d^\mu\}$ as eigenfunctions, see 1.1.3. This means that for $D \in \mathcal{D}(W^\mu)$ there are matrices $\{\Lambda_{D,d} : d \in \mathbb{N}\}$ such that $DP_d^\mu = P_d^\mu \Lambda_{D,d}$. Clearly $\mathbb{D}(W^\mu) \subset \mathcal{D}(W^\mu)$. However, in general we do not have equality. Indeed, in Proposition 4.8.2 it is shown that $\mathcal{D}(W^\mu)$ contains non-commuting elements. A differential operator $D \in \mathcal{D}(W^\mu)$ whose eigenvalue is a diagonal matrix seems to be in $\mathbb{D}(W^\mu)$. We cannot prove this because we do not have control over the algebra $\Psi_\mu^*\mathbb{D}_A^\mu$.

3.6.14. Let $D \in \mathbb{D}(W_\mu)$ and write $D = \sum_{i=0}^r a_i \partial_x^i$. Then we see that $\deg a_i \leq i$ because D has the sequence $\{P_d\}_{d\in\mathbb{N}}$ as eigenfunctions. In Theorem 5.7.15 we see that the Casimir element $\Omega \in (U(\mathfrak{g}_\mathbb{C}))^{\mathfrak{k}_\mathbb{C}}$ of order two gives $\phi_*\Psi_A^*\Pi_\mu\Omega$, a second order differential operator of hypergeometric type in the sense of Tirao [Tir03]. It would be interesting to see whether explicit expressions can be calculated in the other cases. This could help to calculate the polynomials P_d^μ explicitly by describing their columns as vector valued hypergeometric functions.

3.6.15. We can describe the restricted spherical functions globally on A as vector valued functions that are polynomial in ϕ. Locally on $A_{\mu-\mathrm{reg}}$ this amounts to applying a diffeomorphism. But because we end up with smooth functions in ϕ we expect that Ψ_μ^* is regular on A_{reg}. This means that we expect $\Psi_\mu(a)$ to be invertible on A_{reg}, i.e. that the zero set of the determinant $\det \Psi_\mu(a)$ is the same as the set of critical points of ϕ. In the case $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$ we have observed this fact, see Corollary 5.2.3. It is important to have a better understanding of this matter.

3.6.3 Comparison to other constructions

3.6.16. One of the motivations of studying the construction of vector and matrix valued polynomials is that we wanted to understand the results by Grünbaum, Pacharoni and

Tirao in [GPT02] as a generalization of classical Jacobi polynomials. In [GPT02] the spherical functions associated to the compact multiplicity free system $(\mathrm{SU}(3), \mathrm{U}(2), P^+_{\mathrm{U}(2)})$ are studied. In this subsection we indicate how the matrix valued orthogonal polynomials $\{P^\mu_\lambda : \lambda \in P^+_G(\mu)\}$ compare to polynomials that are obtained by Grünbaum, Pacharoni and Tirao.

Let $G = \mathrm{SU}(3)$, $K = \mathrm{U}(2)$ and fix $\mu \in P^+_K$. Let τ be an irreducible unitary K-representation of highest weight μ.

Definition 3.6.17. *Denote by* $\varkappa : \mathrm{End}(\mathbb{C}^3) \to \mathrm{End}(\mathbb{C}^2)$ *the map*

$$\varkappa \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

and define $\Phi_\mu : G \to \mathrm{End}(V_\mu) : g \mapsto \tau(\varkappa(g))$, *where* $\tau : \mathrm{End}(\mathbb{C}^2) \to \mathrm{End}(V_\mu)$ *is the unique holomorphic extension of* $\tau : \mathrm{GL}_2 \to \mathrm{GL}(V_\mu)$*.*

3.6.18. The function Φ_μ is an auxiliary function that is used to neutralize the right K-action as follows. Φ_μ is μ-spherical and in fact, if $\mu \in P^+_G$ then Φ_μ is the elementary spherical function of type μ, see [GPT02, Thm. 2.10]. Let $\mathcal{G} = \{g \in G : \det(\varkappa(g)) \neq 0\}$. Note that $\mathcal{G}$ is a dense open subset of G and $\det(\Phi_\mu(g)) \neq 0$ for $g \in \mathcal{G}$. Define

$$H^\mu_\lambda : \mathcal{G} \to \mathrm{End}(V_\mu) : g \mapsto \Phi^\mu_\lambda(g)\Phi_\mu(g)^{-1}, \quad \lambda \in P^+_G(\mu).$$

The functions H^μ_λ satisfy

$$H^\mu_\lambda(gk) = H^\mu_\lambda(g), \tag{3.37}$$

$$H^\mu_\lambda(kg) = \tau(k)H^\mu_\lambda(g)\tau(k)^{-1} \tag{3.38}$$

for all $g \in \mathcal{G}$ and $k \in K$. This shows, in view of $G = KAK$, that H^μ_λ is determined by its restriction $H^\mu_{\lambda,A}$ to $A \subset G$. Note that $H^\mu_\lambda(A) \subset \mathrm{End}_M(V_\mu)$.

The algebra $U(\mathfrak{g}_\mathbb{C})^{\mathfrak{k}_\mathbb{C}}$ is isomorphic to $\mathfrak{Z}_{\mathfrak{k}_\mathbb{C}} \otimes \mathfrak{Z}_{\mathfrak{g}_\mathbb{C}}$, see [Kno90]. The algebra $\mathfrak{Z}_{\mathfrak{g}_\mathbb{C}}$ is generated by the elements Δ_2 and Δ_3 that are specified in [GPT02, Prop. 3.1]. The operators Δ_2, Δ_3 correspond to differential operators $\widetilde{\Delta}_2, \widetilde{\Delta}_3$ that have the functions $H^\mu_{\lambda,A}$ as simultaneous eigenfunctions. The system of differential equations

$$\{\widetilde{\Delta}_i H^\mu_{\lambda,A} = c^\mu_{i,\lambda} \cdot H^\mu_{\lambda,A} : i = 1, 2 \text{ and } \lambda \in P^+_G(\mu)\}$$

is solved using a circle of ingenious ideas. The solutions $H^\mu_{\lambda,A}$ turn out to be polynomials in ϕ.

To see how the functions $H^\mu_{\lambda,A}$ compare to the function Q^μ_λ we denote $H^{\mu,\mathrm{up}}_\lambda = (H^\mu_{\lambda,A})^{\mathrm{up}}$. The group M is a one-dimensional torus. In particular $\dim(V_\nu) = 1$ for all $\nu \in P^+_M(\mu)$. Identify $\mathrm{End}_M(V_\mu) = V_\mu$ via $\mathrm{End}_M(V_\nu) \leftrightarrow V_\nu$. This seems rather unnatural, but in this way we get the identity of functions

$$\Phi_\mu H^{\mu,\mathrm{up}}_\lambda = \Psi_\mu Q^\mu_\lambda \tag{3.39}$$

on A. In principle, (3.39) indicates how to transfer the properties (orthogonality and recurrence relations, differential equations) of the functions Q^{μ}_{λ} to the functions H^{μ}_{λ}. The calculation of the spherical functions of Grünbaum, Pacharoni and Tirao in [GPT02] by solving a system of differential equations is now connected to our construction which is mainly based on the recurrence relations, i.e. the decomposition of tensor products of certain representations.

Part II

The Example $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$

Chapter 4

Matrix valued orthogonal polynomials related to $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$

Abstract

The matrix valued spherical functions for the pair $(K \times K, K)$, $K = \mathrm{SU}(2)$, are studied. By restriction to the subgroup A the matrix valued spherical functions are diagonal. For suitable set of representations we take these diagonals into a matrix valued function, which are the full spherical functions. Their orthogonality is a consequence of the Schur orthogonality relations. From the full spherical functions we obtain matrix valued orthogonal polynomials of arbitrary size, and they satisfy a three-term recurrence relation which follows by considering tensor product decompositions. An explicit expression for the weight and the complete block-diagonalization of the matrix valued orthogonal polynomials is obtained. From the explicit expression we obtain right-hand sided differential operators of first and second order for which the matrix valued orthogonal polynomials are eigenfunctions. We study the low-dimensional cases explicitly, and for these cases additional results, such as the Rodrigues' formula and being eigenfunctions to first order differential-difference and second order differential operators, are obtained.

4.1 Introduction

The connection between special functions and representation theory of Lie groups is a very fruitful one, see e.g. [Vil68], [VK93]. For the special case of the group $\mathrm{SU}(2)$ we know that the matrix elements of the irreducible finite-dimensional representations are explicitly expressible in terms of Jacobi polynomials, and in this way many of the properties of the Jacobi polynomials can be obtained from the group theoretic interpretation. In particular, the spherical functions with respect to the subgroup $S(U(1) \times U(1))$ are the Legendre polynomials, and using this interpretation one obtains product formula, addition formula, integral formula, etc. for the Legendre polynomials, see e.g. [GV88], [HS94], [Hel00], [Vil68], [VK93] for more information on spherical functions.

In the development of spherical functions for a symmetric pair (G, K) the emphasis has been on spherical functions with respect to one-dimensional representations of K, and in particular the trivial representation of K. Godement [God52] considered the case of higher-dimensional representations of K, see also [GV88], [Tir77] for the general theory. Examples studied are [Cam00], [vDP99], [GPT02], [Koo85], [Ped98]. However, the focus is usually not on obtaining explicit expressions for the matrix valued spherical functions, see Section 4.2 for the definition, except for [GPT02] and [Koo85]. In [GPT02] the matrix valued spherical functions are studied for the case $(U, K) = (\mathrm{SU}(3), \mathrm{U}(2))$, and the calculations revolve around the study of the algebra of differential operators for which these matrix valued orthogonal polynomials are eigenfunctions. See also [GPT01], [GPT03] and [GPT04]. The approach in this paper is different.

In our case the paper [Koo85] by Koornwinder is relevant. Koornwinder studies the case of the compact symmetric pair $(U, K) = (\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$ where the subgroup is diagonally embedded, and he calculates explicitly vector-valued orthogonal polynomials. The goal of this paper is to study this example in more detail and to study the matrix valued orthogonal polynomials arising from this example. The spherical functions in this case are the characters of $\mathrm{SU}(2)$, which are the Chebychev polynomials of the second kind corresponding to the Weyl character formula. So the matrix valued orthogonal polynomials can be considered as analogues of the Chebychev polynomials. Koornwinder [Koo85] introduces the vector-valued orthogonal polynomials which coincide with rows in the matrix of the matrix valued orthogonal polynomials in this paper. We provide some of Koornwinder's results with new proofs. The matrix valued spherical functions can be given explicitly in terms of the Clebsch-Gordan coefficients, or $3-j$-symbols, of $\mathrm{SU}(2)$. Moreover, we find many more properties of these matrix valued orthogonal polynomials. In particular, we give an explicit expression for the weight, i.e. the matrix valued orthogonality measure, in terms of Chebychev polynomials by using an expansion in terms of spherical functions of the matrix elements and explicit knowledge of Clebsch-Gordan coefficient. This gives some strange identities for sums of hypergeometric functions in Appendix 4.A. Another important result is the explicit three-term recurrence relation which is obtained by considering tensor product decompositions. Also, using the explicit expression for the weight function we can obtain differential operators for which these matrix valued orthogonal polynomials are eigenfunctions.

Matrix valued orthogonal polynomials arose in the work of Krein [Kre71], [Kre49] and have been studied from an analytic point of view by Durán and others, see [Dur97, DG04, DG05a, DG05b, DLR04, Grü03, DPS08, MPY01, vA07, LR99, Ger81, Ger82] and references given there. As far as we know, the matrix valued orthogonal polynomials that we obtain have not been considered before. Also 2×2-matrix valued orthogonal polynomials occur in the approach of the non-commutative oscillator, see [IW07] for more references. A group theoretic interpretation of this oscillator in general seems to be lacking.

The results of this paper can be generalized in various ways. First of all, the approach can be generalized to pairs (U,K) with $(U\mathfrak{g})^{\mathfrak{k}}$ abelian, but this is rather restrictive [Kno90]. Given a pair (U,K) and a representation δ of K such that $[\pi|K:\delta]\leq 1$ for all representations π of G and $\delta|_M$ is multiplicity free, we can perform the same construction to get matrix valued orthogonal polynomials. Needless to say, in general it might be difficult to be able to give an explicit expression of the weight function. Another option is to generalize to $(K\times K,K)$ to obtain matrix valued orthogonal polynomials generalizing Weyl's character formula for other root systems, see e.g. [HS94].

We now discuss the contents of the paper. In Section 4.2 we introduce the matrix valued spherical functions for this pair taking values in the matrices of size $(2\ell+1)\times(2\ell+1)$, $\ell\in\frac{1}{2}\mathbb{N}$. In Section 4.3 we prove the recurrence relation for the matrix valued spherical functions using a tensor product decomposition. This result gives us the opportunity to introduce polynomials, and this coincides with results of Koornwinder [Koo85]. In Section 4.4 we introduce the full spherical functions on the subgroup A, corresponding to the Cartan decomposition $U=KAK$, by putting the restriction to A of the matrix valued spherical function into a suitable matrix. In Section 4.5 we discuss the explicit form and the symmetries of the weight. Moreover, we calculate the commutant explicitly and this gives rise to a decomposition of the full spherical functions, the matrix valued orthogonal polynomials and the weight function in a 2×2-block diagonal matrix, which cannot be reduced further. After a brief review of generalities of matrix valued orthogonal polynomials in Section 4.6, we discuss the even and odd-dimensional cases separately. In the even dimensional case an interesting relation between the two blocks occur. In Section 4.7 we discuss the right hand sided differential operators, and we show that the matrix valued orthogonal polynomials associated to the full spherical function are eigenfunctions to a first order differential operator as well as to a second order differential operator. Section 4.8 discusses explicit low-dimensional examples, and gives some additional information such as the Rodrigues' formula for these matrix valued orthogonal polynomials and more differential operators. Finally, in the appendices we give somewhat more technical proofs of two results.

4.2 Spherical Functions of the pair $(\mathrm{SU}(2)\times\mathrm{SU}(2),\mathrm{diag})$

Let $K=\mathrm{SU}(2)$, $U=K\times K$ and $K_*\subset U$ the diagonal subgroup. An element in K is of the form

$$k(\alpha,\beta)=\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix},\quad |\alpha|^2+|\beta|^2=1,\quad \alpha,\beta\in\mathbb{C}. \tag{4.1}$$

Let $m_t:=k(e^{it/2},0)$ and let $T\subset K$ be the subgroup consisting of the m_t. T is the (standard) maximal torus of K. The subgroup $T\times T\subset U$ is a maximal torus of U.

Define

$$A_* = \{(m_t, m_{-t}) : 0 \leq t < 4\pi\} \quad \text{and} \quad M = \{(m_t, m_t) : 0 \leq t < 4\pi\}.$$

We write $a_t = (m_t, m_{-t})$ and $b_t = (m_t, m_t)$. We have $M = Z_{K_*}(A_*)$ and the decomposition $U = K_* A_* K_*$. Note that M is the standard maximal torus of K_*.

The equivalence classes of the unitary irreducible representations of K are paramatrized by $\widehat{K} = \frac{1}{2}\mathbb{N}$. An element $\ell \in \frac{1}{2}\mathbb{N}$ determines the space

$$H^\ell := \mathbb{C}[x, y]_{2\ell},$$

the space of homogeneous polynomials of degree 2ℓ in the variables x and y. We view this space as a subspace of the function space $C(\mathbb{C}^2, \mathbb{C})$ and as such, K acts naturally on it via

$$k : p \mapsto p \circ k^t,$$

where k^t is the transposed. Let

$$\psi_j^\ell : (x, y) \mapsto \binom{2\ell}{\ell - j}^{\frac{1}{2}} x^{\ell - j} y^{\ell + j}, \quad j = -\ell, -\ell + 1, \ldots, \ell - 1, \ell \tag{4.2}$$

We stipulate that this is an orthonormal basis with respect to a Hermitian inner product that is linear in the first variable. The representation $T^\ell : K \to \mathrm{GL}(H^\ell)$ is irreducible and unitary.

The equivalence classes of the unitary irreducible representations of U are paramatrized by $\widehat{U} = \widehat{K} \times \widehat{K} = \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$. An element $(\ell_1, \ell_2) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$ gives rise to the Hilbert space $H^{\ell_1,\ell_2} := H^{\ell_1} \otimes H^{\ell_2}$ and in turn to the irreducible unitary representation on this space, given by the outer tensor product

$$T^{\ell_1,\ell_2}(k_1, k_2)(\psi_{j_1}^{\ell_1} \otimes \psi_{j_2}^{\ell_2}) = T^{\ell_1}(k_1)(\psi_{j_1}^{\ell_1}) \otimes T^{\ell_2}(k_2)(\psi_{j_2}^{\ell_2}).$$

The restriction of $(T^{\ell_1,\ell_2}, H^{\ell_1,\ell_2})$ to K_* decomposes multiplicity free in summands of type $\ell \in \frac{1}{2}\mathbb{N}$ with

$$|\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2 \quad \text{and} \quad \ell_1 + \ell_2 - \ell \in \mathbb{Z}. \tag{4.3}$$

Conversely, the representations of U that contain a given $\ell \in \frac{1}{2}\mathbb{N}$ are the pairs $(\ell_1, \ell_2) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$ that satisfy (4.3). We have pictured this parametrization in Figure 4.1 for $\ell = 3/2$.

The following theorem is standard, see [Koo81].

Theorem 4.2.1. *The space H^{ℓ_1,ℓ_2} has a basis*

$$\{\phi_{\ell,j}^{\ell_1,\ell_2} : \ell \textit{ satisfies (4.3) and } |j| \leq \ell\}$$

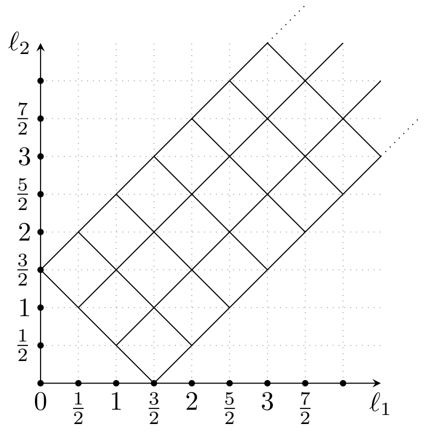

Figure 4.1: Plot of the parametrization of the pairs (ℓ_1, ℓ_2) that contain ℓ upon restriction.

such that for every ℓ the map $\beta_\ell^{\ell_1,\ell_2} : H^\ell \to H^{\ell_1,\ell_2}$ defined by $\psi_j^\ell \mapsto \phi_{\ell,j}^{\ell_1,\ell_2}$ is a K-intertwiner. The base change with respect to the standard basis $\{\psi_{j_1}^{\ell_1} \otimes \psi_{j_2}^{\ell_2}\}$ of H^{ℓ_1,ℓ_2} is given by

$$\phi_{\ell,j}^{\ell_1,\ell_2} = \sum_{j_1=-\ell_1}^{\ell_1} \sum_{j_2=-\ell_2}^{\ell_2} C_{j_1,j_2,j}^{\ell_1,\ell_2,\ell} \, \psi_{j_1}^{\ell_1} \otimes \psi_{j_2}^{\ell_2},$$

where the $C_{j_1,j_2,j}^{\ell_1,\ell_2,\ell}$ are the Clebsch-Gordan coefficients, normalized in the standard way. The Clebsch-Gordan coefficient satisfies $C_{j_1,j_2,j}^{\ell_1,\ell_2,\ell} = 0$ if $j_1 + j_2 \neq j$.

Definition 4.2.2 (Spherical Function)**.** *Fix a K-type $\ell \in \frac{1}{2}\mathbb{N}$ and let $(\ell_1, \ell_2) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$ be a representation that contains ℓ upon restriction to K_*. The spherical function of type $\ell \in \frac{1}{2}\mathbb{N}$ associated to $(\ell_1, \ell_2) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$ is defined by*

$$\Phi_{\ell_1,\ell_2}^\ell : U \to \mathrm{End}(H^\ell) : x \mapsto \left(\beta_\ell^{\ell_1,\ell_2}\right)^* \circ T^{\ell_1,\ell_2}(x) \circ \beta_\ell^{\ell_1,\ell_2}. \tag{4.4}$$

If $\Phi_{\ell_1,\ell_2}^\ell$ is a spherical function of type ℓ then it satisfies the following properties:

(i) $\Phi_{\ell_1,\ell_2}^\ell(e) = I$, where e is the identity element in the group U and I is the identity transformation in H^ℓ,

(ii) $\Phi_{\ell_1,\ell_2}^\ell(k_1 x k_2) = T^\ell(k_1)\Phi_\ell(x)T^\ell(k_2)$ for all $k_1, k_2 \in K_*$ and $x \in U$,

(iii) $\Phi_{\ell_1,\ell_2}^\ell(x)\Phi_{\ell_1,\ell_2}^\ell(y) = \int_{K^*} \chi_\ell(k^{-1})\Phi_{\ell_1,\ell_2}^\ell(xky)dk$, for all $x, y \in U$. Here ξ_ℓ denotes the character of T^ℓ and $\chi_\ell = (2\ell + 1)\xi_\ell$.

Remark 4.2.3. Definition 4.2.2 is not the definition of a spherical function given by Godement [God52], Gangolli and Varadarajan [GV88] or Tirao [Tir77] but it follows from property (iii) that it is equivalent in this situation. The point where our definition differs is essentially that we choose one space, namely $\mathrm{End}(H^\ell)$, in which all the spherical functions take their values, instead of different endomorphism rings for every U-representation. We can do this because of the multiplicity free splitting of the irreducible representations.

Proposition 4.2.4. *Let $\mathrm{End}_M(H^\ell)$ be the the algebra of elements $Y \in \mathrm{End}(H^\ell)$ such that $T^\ell(m)Y = YT^\ell(m)$ for all $m \in M$. Then $\Phi^\ell_{\ell_1,\ell_2}(A_*) \subset \mathrm{End}_M(H^\ell)$. The restriction of $\Phi^\ell_{\ell_1,\ell_2}$ to A_* is diagonalizable.*

Proof. This is an observation [Koo85, (2.6)]. Another proof, similar to [GPT02, Prop. 5.11], uses $ma = am$ for all $a \in A_*$ and $m \in M$ so that by (2)

$$\Phi^\ell_{\ell_1,\ell_2}(a) = T^\ell(m)\Phi^\ell_{\ell_1,\ell_2}(a)T^\ell(m)^{-1}.$$

The second statement follows from the fact that the restriction of any irreducible representation of $K_* \cong \mathrm{SU}(2)$ to $M \cong \mathrm{U}(1)$ decomposes multiplicity free. □ The standard weight basis (4.2) is a weight basis in which $\Phi^\ell_{\ell_1\ell_2}|_{A_*}$ is diagonal. The restricted spherical functions are given by

$$\left(\Phi^\ell_{\ell_1,\ell_2}(a_t)\right)_{j,j} = \sum_{j_1=-\ell_1}^{\ell_1} \sum_{j_2=-\ell_2}^{\ell_2} e^{i(j_2-j_1)t} \left(C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j}\right)^2, \tag{4.5}$$

which follows from Definition 4.2.2 and Theorem 4.2.1.

4.3 Recurrence Relation for the Spherical Functions

A zonal spherical function is a spherical function $\Phi^\ell_{\ell_1,\ell_2}$ for the trivial K-type $\ell = 0$. We have a diffeomorphism $U/K_* \to K : (k_1,k_2)K_* \mapsto k_1k_2^{-1}$ and the left K_*-action on U/K_* corresponds to the action of K on itself by conjugation. The zonal spherical functions are the characters on K [Vil68] which are parametrized by pairs (ℓ_1,ℓ_2) with $\ell_1 = \ell_2$ and we write $\varphi_\ell = \Phi^0_{\ell,\ell}$. Note that $\varphi_\ell = (-1)^{-j+l}(2\ell+1)^{-1/2}U_{2\ell}(\cos t)$ by (4.5) and $C^{\ell,\ell,0}_{j,-j,0} = (-1)^{-j+l}(2\ell+1)^{-1/2}$, where U_n is the Chebyshev polynomial of the second kind of degree n. The zonal spherical function $\varphi_{\frac{1}{2}}$ plays an important role and we denote it by $\varphi = \varphi_{\frac{1}{2}}$. Any other zonal spherical function φ_n can be expressed as a polynomial in φ, see e.g. [Vil68], [Vre76]. For the spherical functions we obtain a similar result. Namely, the product of φ and a spherical function of type ℓ can be written as a linear combination of at most four spherical functions of type ℓ.

Proposition 4.3.1. *We have as functions on U*

$$\varphi \cdot \Phi^{\ell_1,\ell_2}_\ell = \sum_{m_1=|\ell_1-\frac{1}{2}|}^{\ell_1+\frac{1}{2}} \sum_{m_2=|\ell_2-\frac{1}{2}|}^{\ell_2+\frac{1}{2}} \left|a^{(\ell_1,\ell_2)}_{(m_1,m_2),\ell}\right|^2 \Phi^{m_1,m_2}_\ell \tag{4.6}$$

where the coefficients $a^{(\ell_1,\ell_2)}_{(m_1,m_2),\ell}$ are given by

$$a^{(\ell_1,\ell_2)}_{(m_1,m_2),\ell} = \sum_{j_1,j_2,i_1,i_2,n_1,n_2} C^{\ell_1,\ell_2,\ell}_{j_1,j_2,\ell} C^{\frac{1}{2},\frac{1}{2},0}_{i_1,i_2,0} C^{\ell_1,\frac{1}{2},m_1}_{j_1,i_1,n_1} C^{\ell_2,\frac{1}{2},m_2}_{j_2,i_2,n_2} C^{m_1,m_2,\ell}_{n_1,n_2,\ell}. \tag{4.7}$$

where the sum is taken over

$$|j_1| \leq \ell_1, \quad |j_2| \leq \ell_2, \quad |i_1| \leq \frac{1}{2}, \quad |i_2| \leq \frac{1}{2}, \quad |n_1| \leq m_1 \quad \textit{and} \quad |n_2| \leq m_2. \tag{4.8}$$

Moreover, $a^{(\ell_1,\ell_2)}_{(\ell_1+1/2,\ell_2+1/2),\ell} \neq 0$*. Note that the sum in (4.7) is a double sum because of Theorem 4.2.1.*

Proposition 4.3.1 should be compared to Theorem 5.2 of [PT04], where a similar calculation is given for the case $(\mathrm{SU}(3), \mathrm{U}(2))$.

Proof. On the one hand the representation $T^{\ell_1,\ell_2} \otimes T^{\frac{1}{2},\frac{1}{2}}$ can be written as a sum of at most 4 irreducible U-representations that contain the representation T^{ℓ} upon restriction to K_*. On the other hand we can find a 'natural' copy $\mathcal{H}^{\ell}$ of H^{ℓ} in the space $H^{\ell_1,\ell_2} \otimes H^{\frac{1}{2},\frac{1}{2}}$ that is invariant under the K_*-action. Projection onto this space transfers via α, defined below, to a linear combination of projections on the spaces H^{ℓ} in the irreducible summands. The coefficients can be calculated in terms of Clebsch-Gordan coefficients and these in turn give rise to the recurrence relation. The details are as follows.

Consider the U-representation $T^{\ell_1,\ell_2} \otimes T^{\frac{1}{2},\frac{1}{2}}$ in the space $H^{\ell_1,\ell_2} \otimes H^{\frac{1}{2},\frac{1}{2}}$. By Theorem 4.2.1 we have

$$\alpha : H^{\ell_1,\ell_2} \otimes H^{\frac{1}{2},\frac{1}{2}} \to \bigoplus_{m_1=|\ell_1-\frac{1}{2}|}^{\ell_1+\frac{1}{2}} \bigoplus_{m_2=|\ell_2-\frac{1}{2}|}^{\ell_2+\frac{1}{2}} H^{m_1,m_2}$$

which is a U-intertwiner given by

$$\alpha : \left(\psi^{\ell_1}_{j_1} \otimes \psi^{\ell_2}_{j_2}\right) \otimes \left(\psi^{\frac{1}{2}}_{i_1} \otimes \psi^{\frac{1}{2}}_{i_2}\right) \mapsto \sum_{m_1=|\ell_1-\frac{1}{2}|}^{\ell_1+\frac{1}{2}} \sum_{n_1=-m_1}^{m_1} \sum_{m_2=|\ell_2-\frac{1}{2}|}^{\ell_2+\frac{1}{2}} \sum_{n_2=-m_2}^{m_2} C^{\ell_1,\frac{1}{2},m_1}_{j_1,i_1,n_1} C^{\ell_2,\frac{1}{2},m_2}_{j_2,i_2,n_2} \psi^{m_1}_{n_1} \otimes \psi^{m_2}_{n_2}.$$

Let $\mathcal{H}^{\ell} \subset H^{\ell_1,\ell_2} \otimes H^{\frac{1}{2},\frac{1}{2}}$ be the space that is spanned by the vectors

$$\left\{\phi^{\ell_1,\ell_2}_{\ell,j} \otimes \phi^{\frac{1}{2},\frac{1}{2}}_{0,0} : -\ell \leq j \leq \ell\right\}.$$

The element $\phi^{\ell_1,\ell_2}_{\ell,j} \otimes \phi^{\frac{1}{2},\frac{1}{2}}_{0,0}$ maps to

$$\sum_{j_1=-\ell_1}^{\ell_1} \sum_{j_2=-\ell_2}^{\ell_2} \sum_{i_1=-\frac{1}{2}}^{\frac{1}{2}} \sum_{i_2=-\frac{1}{2}}^{\frac{1}{2}} \sum_{m_1=|\ell_1-\frac{1}{2}|}^{\ell_1+\frac{1}{2}} \sum_{n_1=-m_1}^{m_1} \sum_{m_2=|\ell_2-\frac{1}{2}|}^{\ell_2+\frac{1}{2}} \sum_{n_2=-m_2}^{m_2} \sum_{p=|m_1-m_2|}^{m_1+m_2} \sum_{u=-p}^{p} C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} C^{\frac{1}{2},\frac{1}{2},0}_{i_1,i_2,0} C^{\ell_1,\frac{1}{2},m_1}_{j_1,i_1,n_1} C^{\ell_2,\frac{1}{2},m_2}_{j_2,i_2,n_2} C^{m_1,m_2,p}_{n_1,n_2,u} \phi^{m_1,m_2}_{p,u}.$$

Note that $u = n_1 + n_2 = j_1 + i_1 + j_2 + i_2 = j$, so the last sum can be omitted. Also, since α is a K_*-intertwiner, we must have $p = \ell$. For every pair (m_1, m_2) we have a projection

$$P^{(m_1,m_2)}_{\ell} : \bigoplus_{m_1=|\ell_1-\frac{1}{2}|}^{\ell_1+\frac{1}{2}} \bigoplus_{m_2=|\ell_2-\frac{1}{2}|}^{\ell_2+\frac{1}{2}} H^{m_1,m_2} \to \bigoplus_{m_1=|\ell_1-\frac{1}{2}|}^{\ell_1+\frac{1}{2}} \bigoplus_{m_2=|\ell_2-\frac{1}{2}|}^{\ell_2+\frac{1}{2}} H^{m_1,m_2}$$

onto the ℓ-isotypical summand in the summand H^{m_1,m_2}. Hence

$$P_\ell^{m_1,m_2}(\alpha(\phi_{\ell,j}^{\ell_1,\ell_2}\otimes\phi_{0,0}^{\frac12,\frac12}))=\sum_{j_1=-\ell_1}^{\ell_1}\sum_{j_2=-\ell_2}^{\ell_2}\sum_{i_1=-\frac12}^{\frac12}\sum_{i_2=-\frac12}^{\frac12}\sum_{n_1=-m_1}^{m_1}\sum_{n_2=-m_2}^{m_2} C_{j_1,j_2,j}^{\ell_1,\ell_2,\ell}C_{i_1,i_2,0}^{\frac12,\frac12,0}C_{j_1,i_1,n_1}^{\ell_1,\frac12,m_1}C_{j_2,i_2,n_2}^{\ell_2,\frac12,m_2}C_{n_1,n_2,j}^{m_1,m_2,\ell}\phi_{\ell,j}^{m_1,m_2}.$$

The map $P_\ell^{m_1,m_2}\circ\alpha$ is a K_*-intertwiner so Schur's lemma implies that

$$\sum_{j_1=-\ell_1}^{\ell_1}\sum_{j_2=-\ell_2}^{\ell_2}\sum_{i_1=-\frac12}^{\frac12}\sum_{i_2=-\frac12}^{\frac12}\sum_{n_1=-m_1}^{m_1}\sum_{n_2=-m_2}^{m_2} C_{j_1,j_2,j}^{\ell_1,\ell_2,\ell}C_{i_1,i_2,0}^{\frac12,\frac12,0}C_{j_1,i_1,n_1}^{\ell_1,\frac12,m_1}C_{j_2,i_2,n_2}^{\ell_2,\frac12,m_2}C_{n_1,n_2,j}^{m_1,m_2,\ell}$$

is independent of j. Hence it is equal to $a_{(m_1,m_2),\ell}^{(\ell_1,\ell_2)}$, taking $j=\ell$. We have

$$\alpha(\phi_{\ell,j}^{\ell_1,\ell_2}\otimes\phi_{0,0}^{\frac12,\frac12})=\sum_{m_1,m_2}a_{(m_1,m_2),j}^{(\ell_1,\ell_2)}\phi_{\ell,j}^{m_1,m_2}.$$

Moreover, the map

$$P=\sum_{m_1=|\ell_1-1/2|}^{\ell_1+1/2}\sum_{m_2=|\ell_2-1/2|}^{\ell_2+1/2}P_\ell^{m_1,m_2}\circ\alpha : H^{\ell_1,\ell_2}\otimes H^{\frac12,\frac12}\to\bigoplus_{m_1=|\ell_1-\frac12|}^{\ell_1+\frac12}\bigoplus_{m_2=|\ell_2-\frac12|}^{\ell_2+\frac12}H^{m_1,m_2}$$

is a K_*-intertwiner. To show that it is not the trivial map we note that the coefficient $a_{(\ell_1+\frac12,\ell_2+\frac12),\ell}^{(\ell_1,\ell_2)}$ is non-zero. Indeed, the equalities

$$C_{j_1,j_2,\ell}^{\ell_1,\ell_2,\ell}=\frac{(-1)^{\ell_1-j_1}(\ell+\ell_2-\ell_1)!}{(\ell_1+\ell_2+\ell+1)!\Delta(\ell_1,\ell_2,\ell)}\left[\frac{(2\ell+1)(\ell_1+j_1)!(\ell_2+\ell-j_1)!}{(\ell_1-j_1)!(\ell_2-\ell+j_1)!}\right]^{1/2},$$

$$C_{j_1,\frac12,j_1+\frac12}^{\ell_1,\frac12,\ell_1+\frac12}=\left[\frac{\ell_1+j_1+1}{2\ell_1+1}\right]^{1/2}\quad\text{and}\quad C_{j_1,-\frac12,j_1-\frac12}^{\ell_1,\frac12,\ell_1+\frac12}=\left[\frac{\ell_1-j_1+1}{2\ell_1+1}\right]^{1/2},$$

where $\Delta(\ell_1,\ell_2,\ell)$ is a positive function, can be found in [Vil68, Ch. 8] and plugging these into the formula for $a_{(\ell_1+\frac12,\ell_2+\frac12),\ell}^{(\ell_1,\ell_2)}$ shows that it is the sum of positive numbers, hence it is non-zero.

We conclude that P is non-trivial, so its restriction to $\mathcal{H}^\ell$ is an isomorphism and it intertwines the K_*-action. It maps K_*-isotypical summands to K_*-isotypical summands. Hence $\alpha|\mathcal{H}^\ell=P|\mathcal{H}^\ell$. Define

$$\gamma_\ell^{\ell_1,\ell_2}:H^\ell\to H^{\ell_1,\ell_2}\otimes H^{\frac12,\frac12}:\psi_j^\ell\mapsto\phi_{\ell,j}^{\ell_1,\ell_2}\otimes\phi_{0,0}^{\frac12,\frac12}.$$

This is a K-intertwiner. It follows that

$$\alpha\circ\gamma_\ell^{\ell_1,\ell_2}=\sum_{m_1,m_2}a_{(m_1,m_2),\ell}^{(\ell_1,\ell_2)}\beta_\ell^{m_1,m_2}. \tag{4.9}$$

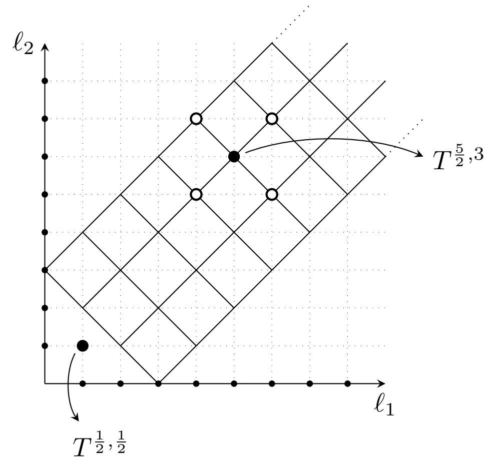

Figure 4.2: Plot of how the tensor product $T^{\frac{5}{2},3} \otimes T^{\frac{1}{2},\frac{1}{2}}$ splits into irreducible summands.

Define the $\mathrm{End}(H^\ell)$-valued function

$$\Psi^\ell_{\ell_1,\ell_2} : U \to \mathrm{End}(H^\ell) : x \mapsto (\gamma_\ell^{\ell_1,\ell_2})^* \circ (T^{\ell_1,\ell_1} \otimes T^{\frac{1}{2},\frac{1}{2}})(x) \circ \gamma_\ell^{\ell_1,\ell_2}.$$

Note that $\Psi^\ell_{\ell_1,\ell_2}(x) = \varphi(x)\Phi^\ell_{\ell_1,\ell_2}(x)$. On the other hand we have $T^{\ell_1,\ell_2} \otimes T^{\frac{1}{2},\frac{1}{2}} = \alpha \circ \left(\bigoplus_{m_1,m_2} T^{m_1,m_2}\right) \circ \alpha$. Together with (4.9) this yields

$$\Psi^\ell_{\ell_1,\ell_2} = \sum_{m_1,m_2} |a^{(\ell_1,\ell_2)}_{(m_1,m_2),\ell}|^2 (\beta_\ell^{\ell_1,\ell_2})^* \circ T^{m_1,m_2} \circ \beta_\ell^{m_1,m_2}.$$

Hence the result. □

In Figure 4.2 we have depicted the representations $(\frac{1}{2},\frac{1}{2})$ and $(\frac{5}{2},3)$ with black nodes. The tensor product decomposes into the four types $(\frac{5}{2} \pm \frac{1}{2}, 3 \pm \frac{1}{2})$ which are indicated with the white nodes.

Corollary 4.3.2. *Given a spherical function $\Phi^\ell_{\ell_1,\ell_2}$ there exist $2\ell+1$ elements $q^{\ell,j}_{\ell_1,\ell_2}$, $j \in \{-\ell, -\ell+1, \ldots, \ell\}$ in $\mathbb{C}[\varphi]$ such that*

$$\Phi^\ell_{\ell_1,\ell_2} = \sum_{j=-\ell}^{\ell} q^{\ell,j}_{\ell_1,\ell_2} \Phi^\ell_{(\ell+j)/2,(\ell-j)/2}. \tag{4.10}$$

The degree of $q^{\ell,j}_{\ell_1,\ell_2}$ is $\ell_1 + \ell_2 - \ell$.

Proof. We prove this by induction on $\ell_1 + \ell_2$. If $\ell_1 + \ell_2 = \ell$ then the statement is true with the polynomials $q^{\ell,j}_{(\ell-k)/2,(\ell+k)/2} = \delta_{j,k}$. Suppose $\ell_1 + \ell_2 > \ell$ and that the statement holds for (ℓ_1', ℓ_2') with $\ell \leq \ell_1' + \ell_2' < \ell_1 + \ell_2$. We can write $|a^{(\ell_1-1/2,\ell_2-1/2)}_{(\ell_1,\ell_2),\ell}|^2\Phi_{\ell_1,\ell_2}$ as

$$\varphi \cdot \Phi_{\ell_1-\frac{1}{2},\ell_2-\frac{1}{2}} - |a^{\ell_1-\frac{1}{2},\ell_2-\frac{1}{2}}_{(\ell_1-1,\ell_2),\ell}|^2\Phi_{\ell_1-1,\ell_2} - |a^{\ell_1-\frac{1}{2}\ell_2-\frac{1}{2}}_{(\ell_1,\ell_2-1),\ell}|^2\Phi_{\ell_1,\ell_2-1} - |a^{\ell_1-\frac{1}{2}\ell_2-\frac{1}{2}}_{(\ell_1-1,\ell_2-1),\ell}|^2\Phi_{\ell_1-1,\ell_2-1}$$

by means of Proposition 4.3.1. The result follows from the induction hypothesis and $a^{(\ell_1-1/2,\ell_2-1/2)}_{(\ell_1,\ell_2),\ell} \neq 0$. □

Remark 4.3.3. The fact that these functions $q^{\ell,j}_{\ell_1,\ell_2}$ are polynomials in $\cos(t)$ has also been shown by Koornwinder in Theorem 3.4 of [Koo85] using different methods.

4.4 Restricted Spherical Functions

For the restricted spherical functions $\Phi^\ell_{\ell_1,\ell_2} : A_* \to \mathrm{End}(H^\ell)$ we define a pairing.

$$\langle \Phi, \Psi \rangle_{A_*} = \frac{2}{\pi}\mathrm{tr}\left(\int_{A_*} \Phi(a)\,(\Psi(a))^*\,|D_*(a)|da\right) \tag{4.11}$$

where $D_*(a_t) = \sin^2(t)$, see [Hel62]. In [Koo85, Prop. 2.2] it is shown that on A_* the following orthogonality relations hold for the restricted spherical functions.

Proposition 4.4.1. *The spherical functions on U of type ℓ, when restricted to A_*, are orthogonal with respect to (4.11). In fact, we have*

$$\langle \Phi^\ell_{\ell_1,\ell_2}, \Phi^\ell_{\ell_1',\ell_2'} \rangle_{A_*} = \frac{(2\ell+1)^2}{(2\ell_1+1)(2\ell_2+1)}\delta_{\ell_1,\ell_1'}\delta_{\ell_2,\ell_2'} \tag{4.12}$$

This is a direct consequence of the Schur orthogonality relations and the integral formula corresponding to the $U = K_*A_*K_*$ decomposition.

The parametrization of the U-types that contain a fixed K-type ℓ is given by (4.3). For later purposes we reparametrize (4.3) by the function $\zeta : \mathbb{N} \times \{-\ell, \ldots, \ell\} \to \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$ given by

$$\zeta(d,k) = \left(\frac{d+\ell+k}{2}, \frac{d+\ell-k}{2}\right).$$

This new parametrization is pictured in Figure 4.3. For each degree d we have $2\ell+1$ spherical functions. By Proposition 4.2.4 the restricted spherical functions take their values in the vector space $\mathrm{End}_M(H^\ell)$ which is $2\ell+1$-dimensional. The appearance of the spherical functions in $2\ell+1$-tuples gives rise to the following definition.

Definition 4.4.2. *Fix a K-type $\ell \in \frac{1}{2}\mathbb{N}$ and a degree $d \in \mathbb{N}$. The function $\Phi^\ell_d : A_* \to \mathrm{End}(H^\ell)$ is defined by associating to each point $a \in A_*$ a matrix $\Phi^\ell_d(a)$ whose j-th row is the vector $\Phi^\ell_{\zeta(d,j)}(a)$. More precisely, we have*

$$\left(\Phi^\ell_d(a)\right)_{p,q} = \left(\Phi^\ell_{\zeta(d,p)}(a)\right)_{q,q} \quad \text{for all } a \in A_*. \tag{4.13}$$

The function Φ^ℓ_d is called the full spherical function of type ℓ and degree d.

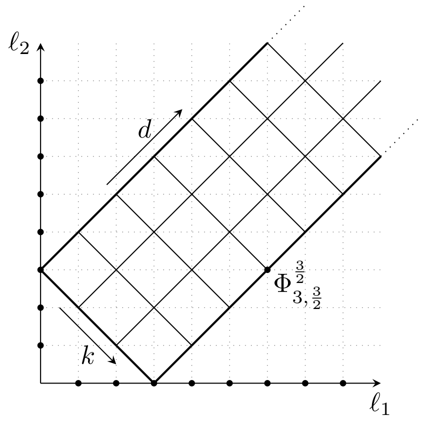

Figure 4.3: Another parametrization of the pairs (ℓ_1, ℓ_2) containing ℓ; the steps of the ladder are paramatrized by d, the position on a given step by k.

Let A'_* be the open subset $\{a_t \in A_* : t \notin \pi\mathbb{Z}\}$. This is the regular part of A_*. The following Proposition is shown in [Koo85, Prop. 3.2]. In Proposition 4.5.7 we prove this result independently for general points in A_* in a different way.

Proposition 4.4.3. *The full spherical function Φ^ℓ_0 of type ℓ and degree 0 has the property that its restriction to A'_* is invertible.*

Definition 4.4.4. *Fix a K-type $\ell \in \frac{1}{2}\mathbb{N}$ and a degree $d \in \mathbb{N}$. Define the function*

$$Q^\ell_d : A'_* \to \mathrm{End}(H^\ell) : a \mapsto \Phi^\ell_d(a)(\Phi^\ell_0(a))^{-1}. \tag{4.14}$$

The j-th row is denoted by $Q^\ell_{\zeta(d,j)}(a)$. Q^ℓ_d is called the full spherical polynomial of type ℓ and degree d.

The functions Q^ℓ_d and $Q^\ell_{\zeta(d,k)}$ are polynomials because $\left(Q^\ell_d\right)_{p,q} = q^{\ell,q}_{\zeta(d,p)}(\varphi)$. The degree of each row of Q^ℓ_d is d which justifies the name we have given these functions in Definition 4.4.4.

We shall show that the functions Φ^ℓ_d and Q^ℓ_d satisfy orthogonality relations that come from (4.11). We start with the Φ^ℓ_d. This function encodes $2\ell+1$ restricted spherical functions and to capture the orthogonality relations of (4.11) we need a matrix valued inner product.

Definition 4.4.5. *Let Φ, Ψ be $\mathrm{End}(H^\ell)$-valued functions on A_*. Define*

$$\langle \Phi, \Psi \rangle := \frac{2}{\pi} \int_{A_*} \Phi(a)\,(\Psi(a))^*\,|D_*(a)|da. \tag{4.15}$$

Proposition 4.4.6. *The pairing defined by (4.15) is a matrix valued inner product. The functions Φ^ℓ_d with $d \in \mathbb{N}$ form an orthogonal family with respect to this inner product.*

PROOF.The pairing satisfies all the linearity conditions of a matrix valued inner product. Moreover we have $\Phi(a)\,(\Phi(a))^*\,|D_*(a)| \geq 0$ for all $a \in A_*$. If $\langle \Phi, \Phi \rangle = 0$ then $\Phi\Phi^* = 0$ from which it follows that $\Phi = 0$. Hence the pairing is an inner product. The orthogonality follows from the formula

$$(\langle \Phi_d, \Psi_{d'} \rangle)_{p,q} = \langle \Phi^\ell_{\zeta(d,p)}, \Phi^\ell_{\zeta(d',q)} \rangle_{A_*} = \delta_{d,d'}\delta_{p,q}\frac{(2\ell+1)^2}{(d+\ell+p+1)(d+\ell-p+1)}$$

and Proposition 4.4.1. □

Define

$$V^\ell(a) = \Phi^\ell_0(a)\left(\Phi^\ell_0(a)\right)^*|D_*(a)|, \tag{4.16}$$

with $D_*(a_t) = \sin^2 t$. This is a weight matrix and we have the following corollary.

Corollary 4.4.7. *Let Q and R be $\mathrm{End}(H^\ell)$-valued functions on A_* and define the matrix valued paring with respect to the weight V^ℓ by*

$$\langle Q, R \rangle_{V^\ell} = \int_{A_*} Q(a)V^\ell(a)\,(R(a))^*\,da. \tag{4.17}$$

This pairing is a matrix valued inner product and the functions Q^ℓ_d form an orthogonal family for this inner product.

The functions Φ^ℓ_d and Q^ℓ_d being defined, we can now transfer the recurrence relations of Proposition 4.3.1 to these functions. Let $E_{i,j}$ be the elementary matrix with zeros everywhere except for the (i,j)-th spot, where it has a one. If we write $E_{i,j}$ with $|i| > \ell$ or $|j| > \ell$ then we mean the zero matrix.

Theorem 4.4.8. *Fix $\ell \in \frac{1}{2}\mathbb{N}$ and define the matrices A_d, B_d and C_d by*

$$\begin{aligned} A_d &= \sum_{k=-\ell}^{\ell} |a^{\zeta(d,k)}_{\zeta(d+1,k),\ell}|^2 E_{k,k}, \\ B_d &= \sum_{k=-\ell}^{\ell} \left(\left| a^{\zeta(d,k),}_{\zeta(d,k+1),\ell} \right|^2 E_{k,k+1} + \left| a^{\zeta(d,k)}_{\zeta(d,k-1),\ell} \right|^2 E_{k,k-1} \right), \\ C_d &= \sum_{k=-\ell}^{\ell} |a^{\zeta(d,k)}_{\zeta(d-1,k),\ell}|^2 E_{k,k}. \end{aligned} \tag{4.18}$$

For $a \in A_*$ we have

$$\varphi(a)\cdot\Phi^\ell_d(a) = A_d\Phi^\ell_{d+1}(a) + B_d\Phi^\ell_d(a) + C_d\Phi^\ell_{d-1}(a) \tag{4.19}$$

and similarly

$$\varphi(a)\cdot Q^\ell_d(a) = A_dQ^\ell_{d+1}(a) + B_dQ^\ell_d(a) + C_dQ^\ell_{d-1}(a). \tag{4.20}$$

Note $A_d \in GL_{2\ell+1}(\mathbb{R})$.

Proof. It is clear that (4.20) follows from (4.19) by multiplying on the right with the inverse of Φ_0^ℓ. To prove (4.19) we look at the rows. Let $p \in \{-\ell, -\ell+1, \ldots, \ell\}$ and multiply (4.19) on the left by $E_{p,p}$ to pick out the p-th row. The left hand side gives $\varphi(a)E_{p,p}\Phi_d^\ell(a)$ while the right hand side gives

$$|a^{\zeta(d,p)}_{\zeta(d+1,p),\ell}|^2 E_{p,p}\Phi^\ell_{d+1}(a) + |a^{\zeta(d,p)}_{\zeta(d,p+1),\ell}|^2 E_{p,p+1}\Phi^\ell_d(a) + \\ |a^{\zeta(d,p)}_{\zeta(d,p-1),\ell}|^2 E_{p,p-1}\Phi^\ell_d(a) + |a^{\zeta(d,p)}_{\zeta(d-1,p),\ell}|^2 E_{p,p}\Phi^\ell_{d-1}(a).$$

Now observe that these are equal by Proposition 4.3.1 and (4.13). This proves the result since p is arbitrary. □

Finally we discuss some symmetries of the full spherical functions. The Cartan involution corresponding to the pair (U, K_*) is the map $\theta(k_1, k_2) = (k_2, k_1)$. The representation T^{ℓ_1,ℓ_2} and $T^{\ell_2,\ell_1} \circ \theta$ are equivalent via the map $\psi^{\ell_1}_{j_1} \otimes \psi^{\ell_2}_{j_2} \mapsto \psi^{\ell_2}_{j_2} \otimes \psi^{\ell_1}_{j_1}$. It follows that $\theta^*\Phi^\ell_{\ell_1,\ell_2} = \Phi^\ell_{\ell_2,\ell_1}$. This has the following effect on the full spherical functions Φ^ℓ_d from Definition 4.4.2:

$$\theta^*\Phi^\ell_d = J\Phi^\ell_d, \tag{4.21}$$

where $J \in \mathrm{End}(H^\ell)$ is given by $\psi^\ell_j \mapsto \psi^\ell_{-j}$. The Weyl group $\mathcal{W}(U, K_*) = \{1, s\}$ consists of the identity and the reflection s in $0 \in \mathfrak{a}_*$. The group $\mathcal{W}(U, K_*)$ acts on A_* and on the functions on A_* by pull-back.

Lemma 4.4.9. *We have $s^*\Phi^\ell_{\ell_1,\ell_2}(a) = J\Phi^\ell_{\ell_1,\ell_2}(a)$ for all $a \in A_*$. The effect on the full spherical functions of type ℓ is*

$$s^*\Phi^\ell_d = \Phi^\ell_d J. \tag{4.22}$$

Proof. This follows from (4.5) and the fact that

$$C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} = (-1)^{\ell_1+\ell_2-\ell} C^{\ell_1,\ell_2,\ell}_{-j_1,-j_2,-j}.$$

□

Proposition 4.4.10. *The functions Φ^ℓ_d commute with J.*

Proof. The action of θ and s_α on A_* is just taking the inverse. Formulas (4.21) and (4.22) now yield the result. □

4.5 The Weight Matrix

We study the weight function $V^\ell : A_* \to \mathrm{End}(H^\ell)$ defined in (4.16), in particular its symmetries and explicit expressions for its matrix elements. First note that V^ℓ is real valued. Indeed, V^ℓ commutes with J,

$$JV^\ell(a)J = J\Phi^\ell_0(a)J(J\Phi^\ell_0(a)J)^*|D_*(a)| = \Phi^\ell_0(a)\Phi^\ell_0(a)^*|D_*(a)| = V^\ell(a), \tag{4.23}$$

since $J^* = J$ and $J^2 = 1$. This also shows that V is real valued,

$$\overline{V^\ell(a_t)} = V^\ell(a_{-t}) = JV^\ell(a_t)J = V^\ell(a_t). \tag{4.24}$$

Lemma 4.5.1. *The weight has the symmetries* $V^\ell_{p,q} = V^\ell_{q,p} = V^\ell_{-p,-q} = V^\ell_{-q,-p}$ *for* $p, q \in \{-\ell, \dots, \ell\}$.

Proof. The first equallity follows since $V^\ell(a)$ is self adjoint by (4.16) and real valued by (4.24). Since $V^\ell(a)$ commutes with J we see that $V^\ell_{p,q} = V^\ell_{-p,-q}$. □ Set

$$v^\ell(a_t) = \Phi^\ell_0(a)\Phi^\ell_0(a)^* \tag{4.25}$$

so that $V^\ell(a_t) = v^\ell(a_t)|D_*(a_t)| = v^\ell(a_t)\sin^2 t$. Note that for $-\ell \le p, q \le \ell$ the matrix coefficient

$$v^\ell(a_t)_{p,q} = \mathrm{tr}\left(\Phi^\ell_{\frac{\ell+p}{2},\frac{\ell-p}{2}}(a_t)\left(\Phi^\ell_{\frac{\ell+q}{2},\frac{\ell-q}{2}}(a_t)\right)^*\right) \tag{4.26}$$

is a linear combination of zonal spherical functions by the following lemma.

Lemma 4.5.2. *The function* $U \to \mathbb{C} : x \mapsto \mathrm{tr}\left(\Phi^\ell_{\ell_1,\ell_2}(x)\left(\Phi^\ell_{m_1,m_2}(x)\right)^*\right)$ *is a bi-K-invariant function and*

$$\mathrm{tr}\left(\Phi^\ell_{\ell_1,\ell_2}(a_t)\left(\Phi^\ell_{m_1,m_2}(a_t)\right)^*\right) = \sum_{n=\max(|\ell_1-m_1|,|\ell_2-m_2|)}^{\min(\ell_1+m_1,\ell_2+m_2)} c_n U_{2n}(\cos t) \tag{4.27}$$

if $\ell_1 + m_1 - (\ell_2 + m_2) \in \mathbb{Z}$ *and* $\mathrm{tr}\left(\Phi^\ell_{\ell_1,\ell_2}(a_t)\left(\Phi^\ell_{m_1,m_2}(a_t)\right)^*\right) = 0$ *otherwise.*

Proof. It follows from Property (2) that the function is bi-K-invariant, so it is natural to expand the function in terms of the zonal spherical functions U_{2n} corresponding to the spherical representations $T^{n,n}$, $n \in \frac{1}{2}\mathbb{N}$. Since T^{ℓ_1,ℓ_2} is equivalent to its contragredient representation, we see that the only spherical functions occurring in the expansion of $\mathrm{tr}\left(\Phi^\ell_{\ell_1,\ell_2}(x)\left(\Phi^\ell_{m_1,m_2}(x)\right)^*\right)$ are the ones for which $(n,n) \in A = \{(n_1,n_2) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N} : \ell_i + m_i - n_i \in \mathbb{Z}, |\ell_1 - m_1| \le n_1 \le \ell_1 + m_1, |\ell_2 - m_2| \le n_2 \le \ell_2 + m_2\}$ since the right hand side corresponds to the tensorproduct decomposition $T^{\ell_1,\ell_2} \otimes T^{m_1,m_2} = \bigoplus_{(n_1,n_2)\in A} T^{n_1,n_2}$, see Proposition 4.3.1 and Figure 4.4. □

Given $d, e \in \mathbb{N}$ and $-\ell \le p, q \le \ell$ we write $\zeta(d,p) = (\ell_1,\ell_2)$, $\zeta(e,q) = (m_1,m_2)$. Then we have

$$\begin{aligned}\left(\Phi^\ell_d(a_t)\left(\Phi^\ell_e\right)(a_t)^*\right)_{p,q} &= \mathrm{tr}\left(\Phi^\ell_{\zeta(d,p)}(a_t)\left(\Phi^\ell_{\zeta(e,q)}(a_t)\right)^*\right)\\ &= \sum_{j,j_1,j_2,i_1,i_2} \left(C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} C^{m_1,m_2,\ell}_{i_1,i_2,j}\right)^2 e^{i(j_2-j_1+i_1-i_2)t},\end{aligned} \tag{4.28}$$

where the sum is taken over

$$|j| \le \ell, \quad |j_1| \le \ell_1, \quad |j_2| \le \ell_2, \quad |i_1| \le m_1, \quad |i_2| \le m_2,$$

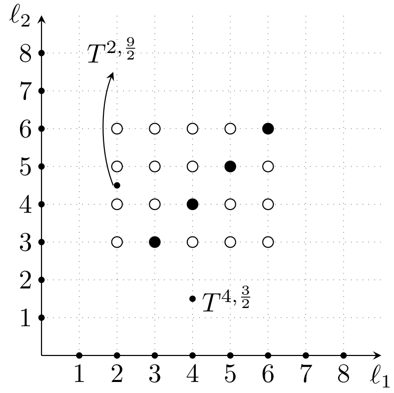

Figure 4.4: Plot of the decomposition of the tensor product $T^{4,\frac{3}{4}} \otimes T^{2,\frac{9}{2}}$ into irreducible representations. The big nodes indicate the irreducible summands, the big black nodes the ones that contain the trivial K_*-type upon restricting.

satisfying $j_1 + j_2 = i_1 + i_2 = j$. This equals

$$\left(\Phi_d^\ell(a_t)\left(\Phi_e^\ell(a_t)\right)^*\right)_{p,q} = \sum_{|s|\leq\min(\ell_1+m_1,\ell_2+m_2)} d^\ell_{\ell_1,\ell_2,m_1,m_2,s} e^{ist} \tag{4.29}$$

where

$$d^\ell_{\ell_1,\ell_2,m_1,m_2,s} = \sum_{j,j_1,j_2,i_1,i_2} \left(C^{\ell_1,\ell_2,\ell}_{j_1,j_2,j} C^{m_1,m_2,\ell}_{i_1,i_2,j}\right)^2, \tag{4.30}$$

where the sum is taken over

$$|j| \leq \ell, \quad |j_1| \leq \ell_1, \quad |j_2| \leq \ell_2, \quad |i_1| \leq m_1, \quad |i_2| \leq m_2,$$

satisfying $j_1 + j_2 = i_1 + i_2 = j$ and $j_2 - j_1 + i_1 - i_2 = s$. Since $U_n(\cos t) = e^{-int} + e^{-i(n-2)t} + \cdots + e^{int}$, it follows from (4.29) and Lemma 4.5.2 that we have the following summation result.

Corollary 4.5.3. *Let $|s| \leq \max(|\ell_1 - m_1|, |\ell_2 - m_2|)$. Then $d^\ell_{\ell_1,\ell_2,m_1,m_2,s}$ is independent of s.*

Note that the sum in (4.30) is a double sum of four Clebsch-Gordan coefficients, which in general are ${}_3F_2$-series [VK93].

We now turn to the case $\ell_1 = \frac{\ell+p}{2}, \ell_2 = \frac{\ell-p}{2}, m_1 = \frac{\ell+q}{2}, m_2 = \frac{\ell-q}{2}$. Because of Lemma 4.5.1 the next theorem gives an explicit expression for the weight matrix.

Theorem 4.5.4. *Let $q - p \leq 0$ and $q + p \leq 0$. For $n = 0, \ldots, \ell + q$ there are coefficients $c^\ell_n(p,q) \in \mathbb{Q}_{>0}$ such that*

$$\left(V^\ell(a_t)\right)_{p,q} = \sin^2(t) \sum_{n=0}^{\ell+q} c^\ell_n(p,q) U_{2\ell+p+q-2n}(\cos(t)) \tag{4.31}$$

where the coefficients are given by

$$c_n^\ell(p,q) = \frac{2\ell+1}{\ell+p+1}\frac{(\ell-q)!(\ell+q)!}{(2\ell)!}\frac{(p-\ell)_{\ell+q-n}}{(\ell+p+2)_{\ell+q-n}}(-1)^{\ell+q-n}\frac{(2\ell+2-n)_n}{n!}. \quad (4.32)$$

By $c_{n+p+q}(p,q) = c_n(-q,-p)$ the expansion (4.31) with (4.32) remains valid for $q \le p$.

PROOF. Since $\min(\ell_1+m_1, \ell_2+m_2) = \ell + \frac{p+q}{2}$ and $\max(|\ell_1-m_1|, |\ell_2-m_2|) = \frac{p-q}{2}$ in this case we find the expansion of the form as stated in (4.31). It remains to calculate the coefficients. Specializing (4.29) and writing

$$\left(C^{(\ell+m)/2,(\ell-m)/2,\ell}_{j_1,j_2,j}\right)^2 = \delta_{j,j_1+j_2}\frac{\binom{\ell+m}{j_1+(\ell+m)/2}\binom{\ell-m}{j_2+(\ell-m)/2}}{\binom{2\ell}{\ell-j}}. \quad (4.33)$$

we find

$$v^\ell_{pq}(\cos t) = \sum_{j=-\frac{\ell+p}{2}}^{\frac{\ell+p}{2}}\sum_{i=-\frac{\ell+q}{2}}^{\frac{\ell+q}{2}} F^\ell_{ij}(p,q)\exp(i(-2(i+j)t)) =$$

$$\sum_{r=-(\ell+\frac{p+q}{2})}^{\ell+\frac{p+q}{2}}\left(\sum_{i=\max(-\frac{\ell+q}{2},r-\frac{\ell+p}{2})}^{\min(\frac{\ell+q}{2},r+\frac{\ell+p}{2})} F^\ell_{i,r-i}(p,q)\right)e^{-2irt}, \quad (4.34)$$

with

$$F^\ell_{ij}(p,q) = \binom{\ell+p}{j+(\ell+p)/2}\binom{\ell+q}{i+(\ell+q)/2}\times$$
$$\sum_{k=\max(-j-\frac{\ell-p}{2},i-\frac{\ell-q}{2})}^{\min(-j+\frac{\ell-p}{2},i+\frac{\ell-q}{2})}\frac{\binom{\ell-p}{-k-j+(\ell-p)/2}\binom{\ell-q}{k-i+(\ell-q)/2}}{\binom{2\ell}{\ell-k}^2}. \quad (4.35)$$

From this we can obtain the explicit expression of $v^\ell(a_t)_{p,q}$ in Chebyshev polynomials. The details are presented in Appendix A. □

Proposition 4.5.5. *The commutant*

$$\{V^\ell(a) : a \in A_*\}' = \{Y \in \mathrm{End}(H^\ell) : V^\ell(a)Y = YV^\ell(a) \quad \text{for all } a \in A_*\}$$
$$\{v^\ell(a) : a \in A_*\}',$$

is spanned by the matrices I and J.

PROOF. By Proposition 4.4.10 we have $J \in \{V^\ell(a) : a \in A_*\}'$. It suffices to show that the commutant contains no other elements than those spanned by I and J.

Let $v^\ell(a_t) = \sum_{n=0}^{2\ell} U_n(\cos t)A_n$, with $A_n \in \mathrm{Mat}_{2\ell+1}(\mathbb{C})$ by Theorem 4.5.4. Then for B in the commutant it is necessary and sufficient that $A_nB = BA_n$ for all n. First, put $C = A_{2\ell}$. Then by Theorem 4.5.4 $C_{p,q} = \binom{2\ell}{\ell+p}^{-1}\delta_{p,-q}$. The equation $BC = CB$ leads to

$$B_{p,q} = (C^{-1}BC)_{p,q} = \frac{C_{q,-q}}{C_{p,-p}}B_{-p,-q} = \frac{C_{q,-q}C_{-q,q}}{C_{p,-p}C_{-p,p}}B_{p,q}$$

by iteration. Since $C_{q,-q} = C_{p,-p}$ if and only if $p = q$ or $p = -q$ we find $B_{p,q} = 0$ for $p \neq q$ or $p \neq -q$. Moreover, $B_{p,p} = B_{-p,-p}$ and $B_{p,-p} = B_{-p,p}$.

Secondly, put $C' = A_{2\ell-1}$. Then, by Theorem 4.5.4, we have

$$C'_{p,q} = \delta_{|p+q|,1}(2\ell+1)\binom{2\ell}{\ell-p}^{-1}\binom{2\ell}{\ell+q}^{-1},$$

so the non-zero entries are different up to the symmetries $C'_{p,q} = C'_{-p,-q} = C'_{q,p} = C'_{-q,-p}$. Now $BC' = C'B$ implies by the previous result $B_{p,p}C'_{p,q} + B_{p,-p}C'_{-p,q} = C'_{p,q}B_{q,q} + C'_{p,-q}B_{q,q}$. Take $q = 1-p$ to find $B_{p,p} = B_{1-p,1-p} = B_{p-1,p-1}$ unless $p = 0$ or $p = 1$, and take $q = p-1$ to find $B_{p,-p} = B_{1-p,p-1} = B_{p-1,1-p}$ unless $p = 0$ or $p = 1$. In particular, for $\ell \in \frac{1}{2} + \mathbb{N}$ this proves the result. In case $\ell \in \mathbb{N}$ we obtain one more equation: $B_{0,0} = B_{1,1} + B_{1,-1}$. This shows that B is in the span of I and J. □

The matrix J has eigenvalues ± 1 and two eigenspaces H^ℓ_- and H^ℓ_+. The dimensions are $\lfloor \ell + 1/2 \rfloor$ and $\lceil \ell + 1/2 \rceil$. A choice of (ordered) bases of the eigenspaces is given by

$$\{\psi^\ell_j - \psi^\ell_{-j} : -\ell \leq j < 0, \ell - j \in \mathbb{Z}\} \quad \text{and} \quad \{\psi^\ell_j + \psi^\ell_{-j} : 0 \leq j \leq \ell, \ell - j \in \mathbb{Z}\}. \tag{4.36}$$

Let Y_ℓ be the matrix whose columns are the normalized basis vectors of (4.36). Conjugating V^ℓ with Y_ℓ yields a matrix with two blocks, one block of size $\lceil \ell + 1/2 \rceil \times \lceil \ell + 1/2 \rceil$ and one of size $\lfloor \ell + 1/2 \rfloor \times \lfloor \ell + 1/2 \rfloor$.

Corollary 4.5.6. *The family $(Q^\ell_d)_{d\geq 0}$ and the weight V^ℓ are conjugate to a family and a weight in block form. More precisely*

$$Y_\ell^{-1} Q^\ell_d(a_t) Y_\ell = \begin{pmatrix} Q^\ell_{d,-}(a_t) & 0 \\ 0 & Q^\ell_{d,+}(a_t) \end{pmatrix}, \quad Y_\ell^{-1} V^\ell(a_t) Y_\ell = \begin{pmatrix} V^\ell_-(a_t) & 0 \\ 0 & V^\ell_+(a_t) \end{pmatrix}.$$

The families $(Q^\ell_{d,\pm})_{d\geq 0}$ are orthogonal with respect to the weight $V^\ell_\pm$. Moreover, there is no further reduction possible.

Proof. The functions Q^ℓ_d can be conjugated by Y_ℓ. Since the Q^ℓ_d commute with J we see that the $Y_\ell^{-1} Q^\ell_d Y_\ell$ has the same block structure as $Y_\ell^{-1} V^\ell Y_\ell$. The blocks of $Y_\ell^{-1} Q^\ell_d Y_\ell$ are orthogonal with respect to the corresponding block of $Y_\ell^{-1} V^\ell Y_\ell$. The polynomials $Q^\ell_{d,-}$ take their values in the (-1)-eigenspace H^ℓ_- of J, the polynomials $Q^\ell_{d,+}$ in the $(+1)$-eigenspace H^ℓ_+ of J. The dimensions are $\lfloor \ell + 1/2 \rfloor$ and $\lceil \ell + 1/2 \rceil$ respectively.

A further reduction would require an element in the commutant $\{V^\ell(a) : a \in A_*\}'$ not in the span of I and J. This is not possible by Proposition 4.5.5. □

The entries of the weight v^ℓ with the Chebyshev polynomials of the highest degree 2ℓ occur only on the antidiagonal by Theorem 4.5.4. This shows that the determinant of $v^\ell(a_t)$ is a polynomial in $\cos t$ of degree $2\ell(2\ell+1)$ with leading coefficient $(-1)^{\ell(2\ell+1)} \prod_{p=-\ell}^{\ell} c_0(p,-p) 2^{2\ell} \neq 0$. Hence v^ℓ is invertible on A_* away from the zeros of its determinant, of which there are only finitely many. We have proved the following proposition which should be compared to Proposition 4.4.3.

Proposition 4.5.7. *The full spherical function Φ_0^ℓ is invertible on A_* except for a finite set.*

In particular, Q_d^ℓ is well-defined in Definition 4.4.2, except for a finite set. Since Q_d^ℓ is polynomial, it is well-defined on A.

Mathematica calculations lead to the following conjecture.

Conjecture 4.5.8. $\det(v^\ell(a_t)) = (1-\cos^2 t)^{\ell(2\ell+1)} \prod_{p=-\ell}^{\ell}(2^{2\ell}c_0^\ell(p,-p))$.

Conjecture 4.5.8 is supported by Koornwinder [Koo85, Prop. 3.2], see Proposition 4.4.3. Conjecture 4.5.8 has been verified for $\ell \leq 16$.

4.6 The matrix valued orthogonal polynomials associated to $(\mathrm{SU}(2)\times\mathrm{SU}(2), \mathrm{diag})$

The main goal of Sections 4.3, 4.4 and 4.5 was to study the properties of the matrix valued spherical functions of any K-type associated to the pair $(\mathrm{SU}(2)\times\mathrm{SU}(2), \mathrm{diag})$. These functions, introduced in Definition 4.2.2, are the building blocks of the full spherical functions described in Definition 4.4.2. We have exploited the fact that the spherical functions diagonalize when restricted to the subgroup A. This allows us to identify each spherical function with a row vector and arrange them in a square matrix.

The goal of this section is to translate the properties of the full spherical functions obtained in the previous sections at the group level to the corresponding family of matrix valued orthogonal polynomials.

4.6.1 Matrix valued orthogonal polynomials

Let W be a complex $N\times N$ matrix valued integrable function on the interval (a,b) such that W is positive definite almost everywhere and with finite moments of all orders. Let $\mathrm{Mat}_N(\mathbb{C})$ be the algebra of all $N\times N$ complex matrices. The algebra over $\mathbb{C}$ of all polynomials in the indeterminate x with coefficients in $\mathrm{Mat}_N(\mathbb{C})$ is denoted by $\mathrm{Mat}_N(\mathbb{C})[x]$. Let $\langle\cdot,\cdot\rangle$ be the following Hermitian sesquilinear form in the linear space $\mathrm{Mat}_N(\mathbb{C})[x]$:

$$\langle P,Q\rangle = \int_a^b P(x)W(x)Q(x)^*dx. \tag{4.37}$$

The following properties are satisfied:

- $\langle aP+bQ,R\rangle = a\langle P,R\rangle + b\langle Q,R\rangle$, for all $P,Q,R\in \mathrm{Mat}_N(\mathbb{C})[x]$, $a,b\in\mathbb{C}$,
- $\langle TP,Q\rangle = T\langle P,Q\rangle$, for all $P,Q\in\mathrm{Mat}_N(\mathbb{C})[x]$, $T\in\mathrm{Mat}_N(\mathbb{C})$,
- $\langle P,Q\rangle^* = \langle Q,P\rangle$, for all $P,Q\in\mathrm{Mat}_N(\mathbb{C})[x]$,
- $\langle P,P\rangle\geq 0$ for all $P\in\mathrm{Mat}_N(\mathbb{C})[x]$. Moreover if $\langle P,P\rangle = 0$ then $P=0$.

Given a weight matrix W one constructs a sequence of matrix valued orthogonal polynomials, that is a sequence $\{R_n\}_{n\geq 0}$, where R_n is a polynomial of degree n with nonsingular leading coefficient and $\langle R_n, R_m\rangle = 0$ if $n\neq m$.

It is worth noting that there exists a unique sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ in $\mathrm{Mat}_N(\mathbb{C})[x]$. Any other sequence of $\{R_n\}_{n\geq 0}$ of orthogonal polynomials in $\mathrm{Mat}_N(\mathbb{C})[x]$ is of the form $R_n(x) = A_nP_n(x)$ for some $A_n\in \mathrm{GL}_N(\mathbb{C})$.

By following a well-known argument, see for instance [Kre71], [Kre49], one shows that the monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ satisfy a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + B_n(x)P_n(x) + C_nP_{n-1}(x),\quad n\geq 0,$$

where $Q_{-1}=0$ and B_n, C_n are matrices depending on n and not on x.

There is a notion of similarity between two weight matrices that was pointed out in [DG05a]. The weights W and $\widetilde{W}$ are said to be similar if there exists a nonsingular matrix M, which does not depend on x, such that $\widetilde{W}(x) = MW(x)M^*$ for all $x\in(a,b)$.

Proposition 4.6.1. *Let $\{R_{n,1}\}_{n\geq 0}$ be a sequence of orthogonal polynomials with respect to W and $M\in GL_N(\mathbb{C})$. The sequence $\{R_{n,2}(x) = R_{n,1}(x)M^{-1}\}_{n\geq 0}$ is orthogonal with respect to $\widetilde{W} = MWM^*$. Moreover, if $\{P_{n,1}\}$ is the sequence of monic orthogonal polynomials orthogonal with respect to W then $\{P_{n,2}(x) = MP_{n,1}(x)M^{-1}\}$ is the sequence of monic orthogonal polynomials with respect to $\widetilde{W}$.*

Proof. It follows directly by observing that

$$\begin{aligned}\int R_{n,2}(x)\widetilde{W}(x)R_{m,2}(x)^*dx &= \int R_{n,1}(x)M^{-1}\widetilde{W}(x)(M^{-1})^*R_{m,1}(x)^*dx\\ &= \int R_{n,1}(x)W(x)R_{m,1}(x)^*dx = 0,\quad \text{if } n\neq m.\end{aligned}$$

The second statement follows by looking at the leading coefficient of $P_{n,2}$ and the unicity of the sequence of monic orthogonal polynomials with respect to $\widetilde{W}$. ☐ A weight matrix W reduces to a smaller size if there exists a matrix M such that

$$W(x) = M\begin{pmatrix} W_1(x) & 0\\ 0 & W_2(x)\end{pmatrix}M^*,\quad \text{for all } x\in(a,b),$$

where W_1 and W_2 are matrix weights of smaller size. In this case the monic polynomials $\{P_n\}_{n\geq 0}$ with respect to the weight W are given by

$$P_n(x) = M\begin{pmatrix} P_{n,1}(x) & 0\\ 0 & P_{n,2}(x)\end{pmatrix}M^{-1},\quad n\geq 0,$$

where $\{P_{n,1}\}_{n\geq 0}$ and $\{P_{n,2}\}_{n\geq 0}$ are the monic orthogonal polynomials with respect to W_1 and W_2 respectively.

4.6.2 Polynomials associated to $\mathrm{SU}(2) \times \mathrm{SU}(2)$

In the rest of the paper we will be concerned with the properties of the matrix orthogonal polynomials Q_d. For this purpose we find convenient to introduce a new labeling in the rows and columns of the weight V. More precisely for any $\ell \in \frac{1}{2}\mathbb{Z}$ let W be the $(2\ell+1)\times(2\ell+1)$ matrix given by

$$\sqrt{1-x^2}\, W(x)_{n,m} = V(a_{\arccos x})_{-\ell+n,-\ell+m}, \quad n,m \in \{0,1,\ldots,2\ell\}. \tag{4.38}$$

It then follows from Theorem 4.5.4 that

$$W(x)_{n,m} = (1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}\frac{(2\ell+1)}{n+1}\frac{(2\ell-m)!m!}{(2\ell)!}\times \sum_{t=0}^{m}(-1)^{m-t}\frac{(n-2\ell)_{m-t}}{(n+2)_{m-t}}\frac{(2\ell+2-t)_t}{t!}U_{n+m-2t}(x), \tag{4.39}$$

if $n \leq m$ and $W(x)_{n,m} = W(x)_{m,n}$ otherwise.

We also consider the sequence of monic polynomials $\{P_d\}_{d\geq 0}$ given by

$$P_d(x)_{n,m} = \Upsilon_d^{-1} Q_d(a_{\arccos x})_{-\ell+n,-\ell+m}, \quad n,m \in \{0,1,\ldots,2\ell\}, \tag{4.40}$$

where Υ_d is the leading coefficient of the polynomial $Q_d(a_{\arccos x})$, which is non-singular by Theorem 4.4.8. Now we can rewrite the results on Section 4.5 in terms of the weight W and the polynomials P_d.

Corollary 4.6.2. *The sequence of matrix polynomials $\{P_d(x)\}_{d>0}$ is orthogonal with respect to the matrix valued inner product*

$$\langle P, Q\rangle = \int_{-1}^{1} P(x)W(x)Q(x)^* dx.$$

Theorem 4.4.8 states that there is a three term recurrence relation defining the matrix polynomials Q_d. These polynomials are functions on the group A. We can use (4.40) to derive a three term recurrence relation for the polynomials P_d.

Corollary 4.6.3. *For any $\ell \in \frac{1}{2}\mathbb{N}$ the matrix valued orthogonal polynomials P_d, are defined by the following three term recurrence relation*

$$xP_d(x) = P_{d+1}(x) + \Upsilon_d^{-1}B_d\Upsilon_d P_d(x) + \Upsilon_d^{-1}C_d\Upsilon_{d-1}P_{d-1}(x), \tag{4.41}$$

where the matrices A_d, B_d and C_d are given in Theorem 4.4.8 and taking into account the relabeling as in the beginning of this subsection.

4.6.3 Symmetries of the weight and the matrix polynomials

In this section we shall use the symmetries satisfied by the full spherical functions to derive symmetry properties for the matrix weight W and the polynomials P_d.

For any $n\in\mathbb{N}$, let I_n be the $n\times n$ identity matrix and let J_n and F_n be the following $n\times n$ matrices

$$J_n=\sum_{i=0}^{n-1}E_{i,n-1-i},\quad F_n=\sum_{i=0}^{n-1}(-1)^iE_{i,i}. \tag{4.42}$$

For any $n\times n$ matrix X the transpose X^t is defined by $(X^t)_{ij}=X_{ji}$ (reflection in the diagonal) and we define the reflection in the antidiagonal by $(X^d)_{ij}=X_{n-j,n-i}$. Note that taking transpose and taking antidiagonal transpose commute, and that

$$(X^t)^d = (X^d)^t = X^{dt} = J_nXJ_n.$$

Moreover, $(XZ)^d=Z^dX^d$ for arbitrary matrices X and Z. We also need to consider the $(2\ell+1)\times(2\ell+1)$ matrix Y defined by

$$\begin{aligned} Y&=\frac{1}{\sqrt{2}}\begin{pmatrix} I_{\ell+\frac12} & J_{\ell+\frac12}\\ -J_{\ell+\frac12} & I_{\ell+\frac12}\end{pmatrix},\ \text{if } \ell=\frac{2n+1}{2},\quad n\in\mathbb{N},\\ Y&=\frac{1}{\sqrt{2}}\begin{pmatrix} I_\ell & 0 & J_\ell\\ 0 & \sqrt{2} & 0\\ -J_\ell & 0 & I_\ell\end{pmatrix},\ \text{if } \ell\in\mathbb{N}.\end{aligned} \tag{4.43}$$

Proposition 4.6.4. *The weight matrix $W(x)$ satisfies the following symmetries*

(i) $W(x)^t=W(x)$ *and* $W(x)^d=W(x)$ *for all* $x\in[-1,1]$. *Thus*

$$J_{2\ell+1}W(x)J_{2\ell+1}=W(x),$$

for all $x\in[-1,1]$.

(ii) $W(-x)=F_{2\ell+1}W(x)F_{2\ell+1}$ *for all* $x\in[-1,1]$.

Here $F_{2\ell+1}$ *is the* $(2\ell+1)\times(2\ell+1)$ *matrix given in* (4.42).

Proof. The symmetry properties of W in (i) follow directly from Lemma 4.5.1. The proof of (ii) follows from (4.39) by using the fact that $U_n(-x)=(-1)^nU_n(x)$ for any Chebyshev polynomial of the second kind $U_n(x)$, so that $W(-x)_{n,m}=(-1)^{n+m}W_{n,m}(x)$. □

The weight matrix W can be conjugated into a 2×2 block diagonal matrix. In Corollary 4.5.6 we have pointed out this phenomenon for the weight V. The following theorem translates Corollary 4.5.6 to the weight matrix W.

Theorem 4.6.5. *For any* $\ell\in\frac12\mathbb{N}$, *the matrix W satisfies*

$$\widetilde{W}(x)=YW(x)Y^t=\begin{pmatrix} W_1(x) & 0\\ 0 & W_2(x)\end{pmatrix},$$

where Y is the matrix given by (4.43). *Moreover if* $\{P_{d,1}\}_{d\geq 0}$ *(resp.* $\{P_{d,2}\}_{d\geq0}$*) is a sequence of monic matrix orthogonal polynomials with respect to the weight* $W_1(x)$ *(resp.* $W_2(x)$*), then*

$$\widetilde{P}_d(x)=\begin{pmatrix} P_{d,1}(x) & 0\\ 0 & P_{d,2}\end{pmatrix},\quad d\geq 0, \tag{4.44}$$

is a sequence of matrix orthogonal polynomials with respect to $\widetilde{W}$. *There is no further reduction.*

The case $\ell = (2n+1)/2$, $n \in \mathbb{N}$, leads to weights matrices W of even size. In this case W splits into two blocks of size $\ell + \frac{1}{2}$. In Corollary 4.6.6 we prove that these two blocks are equivalent, hence the corresponding matrix orthogonal polynomials are equivalent.

It follows from Proposition 4.6.4 (1) that there exist $(n+1)\times(n+1)$ matrices $A(x)$ and $B(x)$ such that $A(x)^t = A(x)$ and

$$W(x) = \begin{pmatrix} A(x) & B(x) \\ B(x)^{dt} & A(x)^{dt} \end{pmatrix}, \tag{4.45}$$

for all $x \in [-1, 1]$.

Corollary 4.6.6. *Let* $\ell = (2n+1)/2$, $n \in \mathbb{Z}$. *Then*

$$YW(x)Y^t = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix},$$

where

$$W_1(x) = A(x) + B(x)J_{n+1}, \quad W_2(x) = J_{n+1}F_{n+1}W_1(-x)F_{n+1}J_{n+1}.$$

Here $A(x)$ *and* $B(x)$ *are the matrices described in* (4.39) *and* (4.45). *Moreover, if* $\{P_{d,1}\}_{d\geq 0}$ *is the sequence of monic orthogonal polynomials with respect to* $W_1(x)$ *then*

$$P_{d,2}(x) = (-1)^d J_{n+1}F_{n+1}P_{d,1}(-x)F_{n+1}J_{n+1}, \tag{4.46}$$

is the sequence of monic orthogonal polynomials with respect to $W_2(x)$.

Proof. In this proof we will drop the subindex in the matrices J_{n+1}, F_{n+1} and we will use J and F instead. It is a straightforward calculation that for $(n+1)\times(n+1)$-matrices A, B C and D the following holds

$$Y\begin{pmatrix} A & B \\ C & D \end{pmatrix}Y^t = \frac{1}{2}\begin{pmatrix} A + D^{dt} + J(C + B^{dt}) & B - C^{dt} + (D^{dt} - A)J \\ J(D^{dt} - A) + C - B^{dt} & D + A^{dt} - (C + B^{dt})J \end{pmatrix}$$

In particular for the weight function W we get

$$YW(x)Y^t = \\ Y\begin{pmatrix} A(x) & B(x) \\ B(x)^{dt} & A(x)^{dt} \end{pmatrix}Y^t = \begin{pmatrix} A(x) + B(x)J & 0 \\ 0 & J(A(x) - B(x)J)J \end{pmatrix}$$

This proves that

$$W_1(x) = A(x) + B(x)J, \quad W_2(x) = J(A(x) - B(x)J)J.$$

It follows from Proposition 4.6.4 (2) that $A(-x) = FA(x)F$ and $B(-x) = FB(x)F$. Therefore we have

$$JFW_1(-x)FJ = JFA(-x)FJ + FJB(-x)FJ = JA(x)J - JB(x) = W_2(x).$$

This proves the first assertion of the theorem.

The last statement follows from Proposition 4.6.1 □

4.7 Matrix Valued Differential Operators

In the study of matrix valued orthogonal polynomials an important ingredient is the study of differential operators which have these matrix valued orthogonal polynomials as eigenfunctions. In this section we discuss some of the differential operators that have the matrix valued orthogonal polynomials of the previous section as eigenfunctions. The calculations rest on the explicit form of the weight function (4.39).

4.7.1 Symmetric differential operators

We consider right hand side differential operators

$$D = \sum_{i=0}^{s} \partial^i F_i(x), \quad \partial = \frac{d}{dx}, \tag{4.47}$$

in such a way that the action of D on the polynomial $P(x)$ is

$$PD = \sum_{i=0}^{s} \partial^i (P)(x) F_i(x).$$

In [GT07, Propositions 2.6 and 2.7] one can find a proof of the following proposition.

Proposition 4.7.1. *Let $W = W(x)$ be a weight matrix of size N and let $\{P_n\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials in $\mathrm{Mat}_N(\mathbb{C})[x]$. If D is a right hand side ordinary differential operator as in (4.47) of order s such that*

$$P_n D = \Lambda_n P_n, \quad \textit{for all } n \geq 0,$$

with $\Lambda_n \in A$, then

$$F_i = F_i(x) = \sum_{j=0}^{i} x^j F_j^i, \quad F_j^i \in \mathrm{Mat}_N(\mathbb{C}),$$

is a polynomial of degree less than or equal to i. Moreover D is determined by the sequence $\{\Lambda_n\}_{n\geq 0}$ and

$$\Lambda_n = \sum_{i=0}^{s} [n]_i F_i^i(D), \quad \textit{for all } n \geq 0,$$

where $[n]_i = n(n-1)\cdots(n-i+1)$, $[n]_0 = 1$.

We consider the following algebra of right hand side differential operators with coefficients in $\mathrm{Mat}_N(\mathbb{C})[x]$.

$$\mathcal{D} = \{D = \sum_i \partial^i F_i : F_i \in \mathrm{Mat}_N(\mathbb{C})[x], \deg F_i \leq i\}.$$

Given any sequence of matrix valued orthogonal polynomials $\{R_n\}_{n\geq 0}$ with respect to W, we define

$$\mathcal{D}(W) = \{D \in \mathcal{D} \,:\, R_n D = \Gamma_n(D) R_n,\, \Gamma_n(D) \in \mathrm{Mat}_N(\mathbb{C}), \text{for all } n \geq 0\}.$$

We observe that the definition of $\mathcal{D}(W)$ does not depend on the sequence of orthogonal polynomials $\{R_n\}_{n\geq 0}$.

Remark 4.7.2. The mapping $D \mapsto \Gamma_n(D)$ is a representation of $\mathcal{D}(W)$ in $\mathbb{C}^N$ for each $n \geq 0$. Moreover the family of representations $\{\Gamma_n\}_{n\geq 0}$ separates the points of $\mathcal{D}(W)$. Note that $\mathcal{D}(W)$ is an algebra.

Definition 4.7.3. *A differential operator $D \in \mathcal{D}$ is said to be symmetric if $\langle PD, Q\rangle = \langle P, QD\rangle$ for all $P, Q \in \mathrm{Mat}_N(\mathbb{C})[x]$.*

Proposition 4.7.4 ([GT07])**.** *If $D \in \mathcal{D}$ is symmetric then $D \in \mathcal{D}(W)$.*

The main theorem in [GT07] says that for any $D \in \mathcal{D}$ there exists a unique differential operator $D^* \in \mathcal{D}(W)$, the adjoint of D, such that $\langle PD, Q\rangle = \langle P, QD^*\rangle$ for all $P, Q \in \mathrm{Mat}_N(\mathbb{C})[x]$. The map $D \mapsto D^*$ is a $*$-operation in the algebra $\mathcal{D}(W)$. Moreover we have $\mathcal{D}(W) = \mathcal{S}(W) \oplus i\mathcal{S}(W)$, where $\mathcal{S}(W)$ denotes the set of all symmetric operators. Therefore it suffices, in order to determine all the algebra $\mathcal{D}(W)$, to determine the symmetric operators $\mathcal{S}(W)$.

The condition of symmetry in Definition 4.7.3 can be translated into a set of differential equations involving the weight W and the coefficients of the differential operator D. For differential operators of order 2 this was proven in [DG04, Theorem 3.1].

Theorem 4.7.5. *Let $W(x)$ be a weight matrix supported on (a, b). Let $D \in \mathcal{D}$ be the differential operator*

$$D = \partial^2 F_2(x) + \partial F_1(x) + F_0^0,$$

Then D is symmetric with respect to W if and only if

$$F_2 W = W F_2, \tag{4.48}$$

$$2(F_2 W)' = W F_1^* + F_1 W, \tag{4.49}$$

$$(F_2 W)'' - (F_1 W)' + F_0 W = W F_0^*, \tag{4.50}$$

with the boundary conditions

$$\lim_{x\to a,b} F_2(x)W(x) = 0, \quad \lim_{x\to a,b} (F_2(x)W(x))' - F_1(x)W(x) = 0. \tag{4.51}$$

4.7.2 Matrix valued differential operators for the polynomials P_n

As in the previous section, we will denote by $\{P_n\}_{n\geq 0}$ the sequence of monic orthogonal polynomials with respect to the weight matrix W. We can write the weight as $W(x) = \rho(x)Z(x)$ where $\rho(x) = (1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}$ and $Z(x)$ is the $(2\ell+1)\times(2\ell+1)$ matrix whose (n, m)-entry is given by

$$\begin{aligned} Z(x)_{n,m} &= \frac{(2\ell+1)}{n+1}\frac{(2\ell-m)!m!}{(2\ell)!}\sum_{t=0}^{m}(-1)^{m-t}\frac{(n-2\ell)_{m-t}}{(n+2)_{m-t}}\frac{(2\ell+2-t)_t}{t!}U_{n+m-2t}. \quad (4.52)\\ &= \sum_{t=0}^{m} c(n,m,t)U_{n+m-2t}(x), \end{aligned}$$

if $n \leq m$ and $Z(x)_{n,m} = Z(x)_{m,n}$ otherwise.

Once we have an explicit expression for the weight matrix W we can use the symmetry equations in Theorem 4.7.5 to find symmetric differential operators. If we start with a generic second order differential operator

$$D = \sum_{i=0}^{2} \partial^i F_i(x), \quad F_i(x) = \sum_{j=0}^{i} x^j F_j^i, \quad F_j^i \in \mathrm{Mat}_N(\mathbb{C}),$$

then the equations (4.48), (4.49) and (4.50) lead to linear equations in the coefficients F_j^i. It is easy to solve these equations for small values of N using any software tool such as Maple. We have used the general expressions for small values of N to make an ansatz for the expressions of a first order and a second order differential operator. Then we prove that these operators are symmetric for all N by showing that they satisfy the conditions in Theorem 4.7.5.

In the following theorem we show the matrix polynomials P_n satisfy a matrix valued first order differential equation. This phenomenon, which does not appear in the scalar, case has been recently studied in the literature (see for instance [CG05], [Cas10]).

Theorem 4.7.6. *Let E be the first order matrix valued differential operator*

$$E = \left(\frac{d}{dx}\right) A_1(x) + A_0,$$

where the matrices $A_1(x)$ and A_0 are given by

$$A_1(x) = \sum_{i=0}^{2\ell} \left(\frac{2\ell - i}{2\ell}\right) E_{i,i+1} - \sum_{i=0}^{2\ell} x \left(\frac{\ell - i}{\ell}\right) E_{ii} - \sum_{i=0}^{2\ell} \left(\frac{i}{2\ell}\right) E_{i,i-1},$$
$$A_0 = \sum_{i=0}^{2\ell} \frac{(2\ell + 2)(i - 2\ell)}{2\ell} E_{ii}.$$

Then E is symmetric with respect to the weight W; hence $E \in D(W)$. Moreover for every integer $n \geq 0$,

$$P_n(x)E = \Lambda_n(E)P_n(x),$$

where

$$\Lambda_n(E) = \sum_{i=0}^{2\ell} \left(-\frac{n(\ell - i)}{\ell} + \frac{(2\ell + 2)(i - 2\ell)}{2\ell}\right) E_{ii}.$$

Proof. The proof of the theorem is performed by showing that the differential operator E is symmetric with respect to the weight W. It follows from Theorem 4.7.5, with $F_2 = 0$, that E is symmetric if and only if

$$W(x)A_1(x)^* + A_1(x)W(x) = 0, \tag{4.53}$$
$$-(A_1(x)W(x))' + A_0 W(x) = W(x)A_0^*, \tag{4.54}$$

with the boundary condition

$$\lim_{x\to\pm 1} A_1(x)W(x) = 0. \tag{4.55}$$

The second statement will then follow from Propositions 4.7.1 and 4.7.4.

The verification of (4.53) and (4.54) involves elaborate computations, see Appendix 4.B. □

Theorem 4.7.7. *Let D be the second order matrix valued differential operator*

$$D = (1-x^2)\frac{d^2}{dx^2} + \left(\frac{d}{dx}\right) B_1(x) + B_0,$$

where the matrices $B_1(x)$ and B_0 are given by

$$B_1(x) = \sum_{i=0}^{2\ell} \left(\frac{(4\ell+3)(i-2\ell)}{2\ell} E_{i,i+1} - x\frac{(2\ell+3)(i-2\ell)}{\ell} E_{ii} + \left(\frac{3i}{2\ell}\right) E_{i,i-1} \right),$$

$$B_0 = \sum_{i=0}^{2\ell} \frac{(i-2\ell)(i\ell-2\ell^2-5\ell-3)}{2\ell} E_{ii}.$$

Then D is symmetric with respect to the weight $W(x)$; hence $D \in D(W)$. Moreover for every integer $n \geq 0$,

$$P_n(x)D = \Lambda_n(D)P_n(x),$$

where

$$\Lambda_n(D) = \sum_{i=0}^{2\ell} \left(n(n-1) - \frac{(2\ell+3)(i-2\ell)}{\ell} + \frac{(i-2\ell)(i\ell-2\ell^2-5\ell-3)}{2\ell} \right) E_{ii}.$$

Proof. The proof of the theorem is similar to that of Theorem 4.7.6, see Appendix 4.B. □

Corollary 4.7.8. *The differential operators D and E commute.*

Proof. To see that D and E commute it is enough to verify that the corresponding eigenvalues commute. The eigenvalues commute because they are diagonal matrices. □

As we pointed out in Theorem 4.6.5, for any $\ell \in \frac{1}{2}\mathbb{N}$ the matrix weight W and the polynomials P_n are $(2\ell+1)\times(2\ell+1)$ matrices that can be conjugated into 2×2 block matrices. More precisely

$$YW(x)Y^t = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix}, \quad YP_n(x)Y^{-1} = \begin{pmatrix} P_{n,1}(x) & 0 \\ 0 & P_{n,2}(x) \end{pmatrix},$$

where Y is the orthogonal matrix introduced in (4.43) and W_1, W_2 are the square matrices described in Corollary 4.6.6. Here $\{P_{n,1}\}_{n\geq 0}$ and $\{P_{n,2}\}_{n\geq 0}$ are the sequences of monic orthogonal polynomials with respect to the weights W_1 and W_2 respectively.

Proposition 4.7.9. *Suppose $\ell = (2n+1)/2$ for some integer n, then E splits in $(n+1)\times(n+1)$ blocks in the following way*

$$Y\,E\,Y^t = \widetilde{E} = \begin{pmatrix} -(\ell+1)I_{n+1} & E_1 \\ E_2 & -(\ell+1)I_{n+1} \end{pmatrix},$$

where

$$E_1 = \left(\frac{d}{dx}\right)\widetilde{A}_1(x) + \widetilde{A}_0,$$
$$E_2 = \left(\frac{d}{dx}\right)F_{n+1}J_{n+1}\widetilde{A}_1(-x)J_{n+1}F_{n+1} + F_{n+1}J_{n+1}\widetilde{A}_0J_{n+1}F_{n+1}.$$

Here F_{n+1}, J_{n+1} are the matrices introduced in (4.42). The matrices A_1 and A_0 are given by

$$\widetilde{A}_1(x) = -\sum_{i=0}^{n-1}\frac{(2\ell-i)}{2\ell}E_{i,n-i-1} + x\sum_{i=0}^{n}\frac{(\ell-i)}{\ell}E_{i,n-i} + \sum_{i=1}^{n}\frac{i}{2\ell}E_{i,n-i+1} + \frac{(2\ell+1)}{4\ell}E_{n,n+1},$$

$$\widetilde{A}_0 = \sum_{i=0}^{n}\frac{(\ell+1)(\ell-i)}{\ell}E_{i,n-i}.$$

Proof. The proposition follows by a straightforward computation. □

Proposition 4.7.10. *Suppose $\ell \in \mathbb{N}$, then we have $Y\,E\,Y^t = \widetilde{E}$,*

$$\widetilde{E} = \left(\frac{d}{dx}\right)\begin{pmatrix} O_{(\ell+1)} & & \widetilde{A}_1(x) \\ & & v_1^t \\ F_\ell\widetilde{A}_1(x)F_\ell & v_2 & O_{\ell\times\ell} \end{pmatrix} + \begin{pmatrix} -(\ell+1)I_{(\ell+1)} & & \widetilde{A}_0(x) \\ & & v_0^t \\ F_\ell\widetilde{A}_0(x)F_\ell & v_0 & -(\ell+1)I_\ell \end{pmatrix},$$

where $\widetilde{A}_1$ and $\widetilde{A}_0$ are $n\times n$ matrices given by

$$\widetilde{A}_1(x) = -\sum_{i=0}^{\ell-2}\frac{(2\ell-i)}{2\ell}E_{i,\ell-i-1} + x\sum_{i=0}^{\ell-1}\frac{(\ell-i)}{\ell}E_{i,\ell-i} + \sum_{i=1}^{\ell-1}\frac{i}{2\ell}E_{i,\ell-i+1},$$
$$\widetilde{A}_0 = \sum_{i=0}^{\ell-1}\frac{(\ell+1)(\ell-i)}{\ell}E_{i,\ell-i},$$

and the vectors $v_0, v_1, v_2 \in \mathbb{C}^\ell$ are $v_0 = (0,0,\cdots,0)$,

$$v_1 = \left(\frac{(2\ell+1)\sqrt{2}}{4\ell}, 0, \cdots, 0\right), \quad v_2 = \left(-\frac{(2\ell+1)\sqrt{2}}{4(\ell+1)}, 0, \cdots, 0\right).$$

Proof. The proposition follows by a straightforward computation. □

Let us assume that $\ell = (2n+1)/2$ for some $n \in \mathbb{N}$ so that the weight W and the polynomials P_n are matrices of even dimension. Proposition 4.7.9 says that

$$\widetilde{P}_n(x)\widetilde{E} = Y\Lambda_n Y^t \widetilde{P}_n(x), \quad n \geq 0. \tag{4.56}$$

A simple computation shows that

$$Y\Lambda_n Y^t = -(\ell+1)I_{2n+2} + \begin{pmatrix} 0 & \Lambda_{n,1} \\ \Lambda_{n,2} & 0 \end{pmatrix},$$

where $\Lambda_{n,1}$ is a $(n+1)\times(n+1)$ matrix (depending on n) and

$$\Lambda_{n,2} = F_{n+1}J_{n+1}\Lambda_{n,1}J_{n+1}F_{n+1}.$$

It follows from (4.56) that the following matrix equation is satisfied

$$\begin{pmatrix} -(\ell+1)P_{n,1}(x) & P_{n,1}(x)E_1 \\ P_{n,2}(x)E_2 & -(\ell+1)P_{n,2}(x) \end{pmatrix} = \begin{pmatrix} -(\ell+1)P_{n,1}(x) & \Lambda_{n,1}P_{n,2}(x) \\ \Lambda_{n,2}P_{n,1}(x) & -(\ell+1)P_{n,2}(x) \end{pmatrix}.$$

Therefore the polynomials $P_{n,1}$ and $P_{n,2}$ satisfy the following differential equations

$$P_{n,1}E_1 - \Lambda_{n,1}P_{n,2} = 0, \tag{4.57}$$

$$P_{n,2}E_2 - \Lambda_{n,2}P_{n,1} = 0. \tag{4.58}$$

Finally it follows from (4.57), (4.58) and (4.46) that for every $n \geq 0$, the polynomial $P_{n,1}$ is a solution of the following second-order matrix valued differential equation

$$P_{n,1}E_1E_2 - \Lambda_{n,1}\Lambda_{n,2}P_{n,1} = 0.$$

We can also obtain a second order differential equation for $P_{n,2}$.

4.8 Examples

The purpose of this section is to study the properties of the monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ presented in Section 4.6 for small dimension. For $\ell = 0, \frac{1}{2}, 1, \frac{2}{3}, 2$, we show that these polynomials are solutions of certain matrix valued differential equations. We will show that the polynomials can be defined by means of Rodrigues' formulas and we will give explicit expressions for the three term recurrence relations.

4.8.1 The case $\ell = 0$; the scalar weight

In this case the polynomials $\{P_n\}_{n\geq 0}$ are scalar valued. The weight W reduces to the real function

$$W(x) = (1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}, \quad x \in [-1,1].$$

Therefore the polynomials P_n are a multiple of the Chebyshev polynomials of the second kind: $P_n(x) = 2^{-n}U_n(x)$, $n \in \mathbb{N}$.

4.8.2 The case $\ell = \frac{1}{2}$; weight of dimension 2

In this case the polynomials $\{P_n\}_{n\geq 0}$ are 2×2 matrices. The weight W is given by

$$W(x) = (1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}} \begin{pmatrix} 2 & 2x \\ 2x & 2 \end{pmatrix}, \quad x \in [-1,1].$$

It is a straightforward computation that

$$YW(x)Y^t = 2\begin{pmatrix} (1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}} & 0 \\ 0 & (1-x)^{\frac{3}{2}}(1+x)^{\frac{1}{2}} \end{pmatrix}, \quad Y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Observe that $W_1(x) = (1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{2}}$ and $W_2(x) = (1-x)^{\frac{3}{2}}(1+x)^{\frac{1}{2}}$ are Jacobi weights and therefore we have

$$P_{n,1} = \frac{2^n n!(n+2)!}{(2n+2)!} P_n^{(\frac{1}{2},\frac{3}{2})}(x), \quad P_{n,2}(x) = \frac{2^n n!(n+2)!}{(2n+2)!} P_n^{(\frac{3}{2},\frac{1}{2})}, \quad n \in \mathbb{N}_0,$$

where $\{P_n^{(\alpha,\beta)}\}_{n\geq 0}$ are the classical Jacobi polynomials

Differential equations

By Theorem 4.7.6 we have

$$\frac{d}{dx}P_n(x)\begin{pmatrix} -x & 1 \\ -1 & x \end{pmatrix} + P_n(x)\begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -n-3 & 0 \\ 0 & n \end{pmatrix} P_n(x).$$

We can conjugate the differential operator E by the matrix Y to obtain

$$\widetilde{E} = Y\,E\,Y^t = \left(\frac{d}{dx}\right)\begin{pmatrix} 0 & 1+x \\ x-1 & 0 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

The monic polynomials

$$\widetilde{P}_n(x) = YP_n(x)Y^t = \begin{pmatrix} P_{n,1}(x) & 0 \\ 0 & P_{n,2}(x) \end{pmatrix}, \quad n \in \mathbb{N}_0,$$

satisfy

$$\widetilde{P}_n(x)\widetilde{E} = \widetilde{\Lambda}_n\widetilde{P}_n(x), \text{ where } \widetilde{\Lambda}_n(x) = \begin{pmatrix} -3/2 & n+3/2 \\ n+3/2 & -3/2 \end{pmatrix}.$$

Now the fact that $\widetilde{P}_n(x)$ is an eigenfunction of $\widetilde{E}$ is equivalent to the following relations between Jacobi polynomials

$$(1+x)\frac{d}{dx}P^{(\frac{1}{2},\frac{3}{2})}(x) + \frac{3}{2}P_n^{(\frac{1}{2},\frac{3}{2})}(x) - (n+\frac{3}{2})P_n^{(\frac{3}{2},\frac{1}{2})}(x) = 0,$$
$$(1-x)\frac{d}{dx}P_n^{(\frac{3}{2},\frac{1}{2})}(x) + \frac{3}{2}P_n^{(\frac{3}{2},\frac{1}{2})}(x) - (n+\frac{3}{2})P_n^{(\frac{1}{2},\frac{3}{2})}(x) = 0.$$

4.8.3 Case $\ell = 1$; weight of dimension 3

Here we consider the simplest example of nontrivial matrix orthogonal polynomials for the weight W. The weight matrix W of size 3×3 is obtained by setting $\ell = 1$. We have

$$W(x) = (1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}} \begin{pmatrix} 3 & 3x & 4x^2-1 \\ 3x & x^2+2 & 3x \\ 4x^2-1 & 3x & 3 \end{pmatrix} \tag{4.59}$$

We know from Theorem 4.6.5 that the weight $W(x)$ splits into a block of size 2×2 and a block of size 1×1, namely

$$YW(x)Y^t = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix} = (1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}} \begin{pmatrix} 4x^2+2 & 3\sqrt{2}x & 0 \\ 3\sqrt{2}x & x^2+2 & 0 \\ 0 & 0 & 4(1-x^2) \end{pmatrix}.$$

From Theorem 4.6.5 the monic orthogonal polynomials $\tilde{P}_n(x)$ with respect to $\tilde{W}(x)$ reduce to

$$\tilde{P}_n = \begin{pmatrix} P_{n,1}(x) & 0 \\ 0 & P_{n,2}(x) \end{pmatrix}$$

where $\{P_{n,2}\}_{n\geq 0}$ are the monic polynomials with respect to $W_1(x)$ and $\{P_{n,2}\}_{n\geq 0}$ are the monic polynomials with respect to the weight $W_2(x)$.

Remark 4.8.1. The weight W_2 is a multiple of the Jacobi weight $(1-x)^\alpha(1+x)^\beta$ corresponding to $\alpha = 3/2$ and $\beta = 3/2$. The monic polynomials $\{P_{n,2}\}_{n\geq 0}$ are then a multiple of the Gegenbauer polynomials

$$P_{n,2}(x) = \frac{2^n n!(n+3)!}{(2n+3)!} P_n^{(\frac{3}{2},\frac{3}{2})}(x).$$

The first order differential operator

By Theorem 4.7.6 we have that the monic polynomials P_n are eigenfunctions of the differential operator E. More precisely the following equation holds

$$\frac{d}{dx}P_n(x) \begin{pmatrix} -x & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & x \end{pmatrix} + P_n(x) \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -n-4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & n \end{pmatrix} P_n(x).$$

Now we can conjugate the differential operator E by the matrix Y to obtain a differential operator $\widetilde{E} = Y\,E\,Y^t$. The fact that the polynomials P_n are eigenfunctions of E says that the polynomials $\widetilde{P}_n$ are eigenfunctions of $\widetilde{E}$. In other words

$$\frac{d}{dx}\widetilde{P}_n(x)\begin{pmatrix}0 & 0 & x\\ 0 & 0 & \frac{\sqrt{2}}{2}\\ x & -\sqrt{2} & 0\end{pmatrix}+\widetilde{P}_n(x)\begin{pmatrix}-2 & 0 & 2\\ 0 & -2 & 0\\ 2 & 0 & -2\end{pmatrix}=$$

$$\begin{pmatrix}-2 & 0 & n+2\\ 0 & -2 & 0\\ n+2 & 0 & -2\end{pmatrix}\widetilde{P}_n(x).$$

We can now rewrite the equation above in terms of the polynomials $P_{n,1}$ and $P_{n,2}$.

$$\frac{d}{dx}P_{n,1}(x)\begin{pmatrix}x\\ \frac{\sqrt{2}}{2}\end{pmatrix}+P_{n,1}(x)\begin{pmatrix}2\\ 0\end{pmatrix}=P_{n,2}(x)\begin{pmatrix}n+2\\ 0\end{pmatrix},$$

$$\frac{d}{dx}P_{n,2}(x)\begin{pmatrix}x & -\sqrt{2}\end{pmatrix}+P_{n,2}(x)\begin{pmatrix}2 & 0\end{pmatrix}=\begin{pmatrix}n+2 & 0\end{pmatrix}P_{n,1}(x).$$

Since $P_{n,2}$ is a Gegenbauer polynomial, we see that the elements of the first row of $P_{n,2}$ can be written explicitly in terms of Gegenbauer polynomials.

Second order differential operators

In this subsection we describe a set of linearly independent differential operators that have the polynomials $P_{n,1}$ as eigenfunctions.

Proposition 4.8.2. *The matrix orthogonal polynomials* $\{P_{n,1}\}_{n\geq 0}$ *satisfy*

$$P_{n,1}D_j=\Lambda_n(D_j)P_{n,1},\quad j=1,2,3,n\geq 0,$$

where the differential operators D_j *are*

$$D_1=(x^2-1)\left(\frac{d^2}{dx^2}\right)+\left(\frac{d}{dx}\right)\begin{pmatrix}5x & -4\\ -1 & 5x\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},$$

$$D_2=\left(\frac{d^2}{dx^2}\right)\begin{pmatrix}x^2 & -2x\\ \frac{x}{2} & -1\end{pmatrix}+\left(\frac{d}{dx}\right)\begin{pmatrix}5x & -6\\ 1 & 0\end{pmatrix}+\begin{pmatrix}4 & 0\\ 0 & 0\end{pmatrix},$$

$$D_3=\left(\frac{d^2}{dx^2}\right)\begin{pmatrix}-2x & 8x^2-4\\ x^2-2 & 2x\end{pmatrix}+\left(\frac{d}{dx}\right)\begin{pmatrix}-8 & 32x\\ 6x & -4\end{pmatrix}+\begin{pmatrix}0 & 16\\ 6 & 0\end{pmatrix}.$$

and the eigenvalues Λ_j *are given by*

$$\Lambda_n(D_1)=\begin{pmatrix}n(n+4)+1 & 0\\ 0 & n(n+4)\end{pmatrix},\quad \Lambda_n(D_2)=\begin{pmatrix}(n+2)^2 & 0\\ 0 & 0\end{pmatrix},$$

$$\Lambda_n(D_3)=\begin{pmatrix}0 & 8(n+2)(n+1)\\ (n+3)(n+2) & 0\end{pmatrix}.$$

Moreover, the differential operators D_1, D_2 *and* D_3 *satisfy*

$$D_1D_2=D_2D_1,\quad D_1D_3\neq D_3D_1,\quad D_2D_3\neq D_3D_2.$$

PROOF. The proposition follows by proving that the differential operators D_j, $j = 1, 2, 3$ are symmetric with respect to the weight matrix W_1. This is accomplished by a straightforward computation, showing that the differential equations (4.48), (4.49), (4.50) and the boundary conditions (4.51) are satisfied. As a consequence of Remark 4.7.2, the commutativity properties of the differential operators follow by observing the commutativity of the corresponding eigenvalues. □

Rodrigues' Formula

Proposition 4.8.3. *The matrix orthogonal polynomials* $\{P_{n,1}(x)\}_{n\geq 0}$ *satisfy the Rodrigues' formula*

$$P_{n,1}(x) = c\left[(1-x^2)^{\frac{1}{2}+n}\left(\begin{pmatrix} 4x^2+2 & 3\sqrt{2}x \\ 3\sqrt{2}x & x^2+2 \end{pmatrix} + \begin{pmatrix} \frac{2n}{n+2} & \frac{\sqrt{2}nx}{n+2} \\ -\frac{\sqrt{2}nx}{n+1} & -\frac{n}{n+1} \end{pmatrix}\right)\right]^{(n)} W_1^{-1}(x), \quad (4.60)$$

where

$$c = \frac{(-1)^n 2^{-2n-2}(n+2)(n+3)\sqrt{\pi}}{(2n+3)\Gamma(n+\frac{3}{2})}.$$

PROOF. The proposition can be proven in a similar way to Theorem 3.1 of [DG05b]. We include a sketch of the proof for the sake of completeness. First of all, we recall that the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy the Rodrigues' formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}[(1-x)^{\alpha+n}(1+x)^{\beta+n}]^{(n)}.$$

Let $R(x)$ and Y_n be the matrix polynomials of degree 2 and 1

$$R(x) = \begin{pmatrix} 4x^2+2 & 3\sqrt{2}x \\ 3\sqrt{2}x & x^2+2 \end{pmatrix}, \qquad Y_n(x) = \begin{pmatrix} \frac{2n}{n+2} & \frac{n\sqrt{2}x}{n+2} \\ -\frac{\sqrt{2}nx}{n+1} & -\frac{n}{n+1} \end{pmatrix},$$

so that the (4.60) can be rewritten as

$$P_{n,1}(x) = c\left[(1-x^2)^{\frac{1}{2}+n}\left(R(x)+Y_n(x)\right)\right]^{(n)} W_1^{-1}(x).$$

Then by applying the Leibniz rule on the right hand side of (4.60), it is not difficult to prove that

$$\begin{aligned} P_{n,1}(x) &= c\tfrac{1}{2}n(n-1)[(1-x)^{\frac{1}{2}+n}(1+x)^{\frac{1}{2}+n}]^{(n-2)}(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}}R''(x)R(x)^{-1} \\ &+ cn[(1-x)^{\frac{1}{2}+n}(1+x)^{\frac{1}{2}+n}]^{(n-1)}(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}}(R'(x)+Y'(x))R(x)^{-1} \\ &+ c[(1-x)^{\frac{1}{2}+n}(1+x)^{\frac{1}{2}+n}]^{(n)}(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}}(I+Y(x)R(x)^{-1}). \end{aligned}$$

Now by applying the Rodrigues' formula for the Jacobi polynomials we obtain

$$\begin{aligned} P_{n,1}(x) = 2^n n!(-1)^n c[P_n^{(\frac{1}{2},\frac{1}{2})}(x)(I+Y(x)R(x)^{-1}) \\ - \tfrac{1}{2}(1-x)(1+x)P_{n-1}^{(\frac{3}{2},\frac{3}{2})}(x)(Y'(x)+R'(x))R(x)^{-1} \\ + \tfrac{1}{8}(1-x)^2(1+x)^2 P_{n-2}^{(\frac{5}{2},\frac{5}{2})}(x)R''(x)R(x)^{-1}]. \quad (4.61) \end{aligned}$$

Now with a careful computation we can show that the expression above is a matrix polynomial of degree n with nonsingular leading term. Using integration by parts it is easy to show the orthogonality of $P_{n,1}$ and x^m, $m = 0, 1, \ldots, n-1$, with respect to the weight W_1. $\square$

Three term recurrence relations

In Corollary 4.6.3 we show that the matrix polynomials $P_n(x)$ of any size satisfy a three term recurrence relation. The recurrence relation for the polynomials $P_{n,1}$ can then be obtained by conjugating the recurrence relation for $P_n(x)$ by the matrix Y. The recurrence coefficients (4.18) are given in terms of Clebsch-Gordan coefficients and are difficult to manipulate. For $\ell = 1$ we can use the Rodrigues' formula (4.60) to derive explicit formulas for the three term recurrence relation for the polynomials $P_{n,1}$.

First we need to compute the norm of $P_{n,1}(x)$. The Rodrigues' formula (4.60) and integration by parts lead to

$$\|P_{n,1}\|^2 = 2^{-2n-1}\pi \begin{pmatrix} \frac{(n+3)}{(n+1)} & 0 \\ 0 & \frac{(n+3)^2}{8(n+1)^2} \end{pmatrix}.$$

If $\{\mathcal{P}_{n,1}\}_{n\geq 0}$ is a sequence of orthonormal polynomials with respect to W_1 with leading coefficients Ω_n, then it follows directly from the orthogonality relations for the monic polynomials that $\|P_{n,1}\|^2 = \Omega_n^{-1}(\Omega_n^*)^{-1}$. The orthonormal polynomials $\mathcal{P}_{n,1}$ with leading coefficient

$$\Omega_n = \begin{pmatrix} \sqrt{\frac{2^{2n+1}(n+1)}{\pi(n+3)}} & 0 \\ 0 & \frac{2^{n+2}(n+1)}{\sqrt{\pi}(n+3)} \end{pmatrix},$$

satisfy the three term recurrence relation

$$x\mathcal{P}_n(x) = A_{n+1}\mathcal{P}_{n+1}(x) + B_n\mathcal{P}_n(x) + A_n^*\mathcal{P}_{n-1}(x),$$

where $A_n = \Omega_{n-1}\Omega_n^{-1}$ and

$$B_n = \Omega_n[\text{coef. of } x^{n-1} \text{ in } P_{n,1} - \text{coef. of } x^n \text{ in } P_{n+1,1}]\Omega_n^{-1}.$$

The coefficient of x^{n-1} in $P_{n,1}$ can by obtained from (4.61). Now a careful computation shows that

$$A_n = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{n(n+3)}{(n+1)(n+2)}} & 0 \\ 0 & \frac{n(n+3)}{2(n+1)(n+2)} \end{pmatrix}, \quad B_n = \begin{pmatrix} 0 & \frac{4}{(n+2)(n+3)} \\ \frac{1}{2(n+1)(n+4)} & 0 \end{pmatrix}.$$

Therefore the monic polynomials $P_{n,1}$ satisfy the three term recurrence relation

$$xP_{n,1}(x) = P_{n+1,1}(x) + \tilde{B}_n P_{n,1}(x) + \tilde{C}_n P_{n-1,1}(x),$$

where

$$\tilde{B}_n = \begin{pmatrix} 0 & \frac{4}{(n+2)(n+3)} \\ \frac{1}{2(n+1)(n+2)} & 0 \end{pmatrix}, \qquad \tilde{C}_n = \begin{pmatrix} \frac{n(n+3)}{4(n+2)(n+1)} & 0 \\ 0 & \frac{n^2(n+3)^2}{4(n+2)^2(n+1)^2} \end{pmatrix}.$$

4.8.4 Case $\ell = 3/2$; weight of dimension 4

The weight matrix W of size 4×4 is obtained by setting $\ell = 3/2$.

$$W(x) = (1-x^2)^{\frac{1}{2}} \begin{pmatrix} 4 & 4x & \frac{4}{3}(4x^2-1) & 4x(2x^2-1) \\ 4x & \frac{4}{9}(4x^2+5) & \frac{4}{9}x(2x^2+7) & \frac{4}{3}(4x^2-1) \\ \frac{4}{3}(4x^2-1) & \frac{4}{9}x(2x^2+7) & \frac{4}{9}(4x^2+5) & 4x \\ 4x(2x^2-1) & \frac{4}{3}(4x^2-1) & 4x & 4 \end{pmatrix}$$

We know from Corollary 4.6.5 that the weight $W(x)$ splits in two blocks of size 2×2, namely

$$\tilde{W}(x) = YW(x)Y^t = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix},$$

where

$$W_1(x) = 4(1-x)^{1/2}(1+x)^{3/2} \begin{pmatrix} 2x^2-2x+1 & \frac{1}{3}(4x-1) \\ \frac{1}{3}(4x-1) & \frac{1}{9}(2x^2+2x+5) \end{pmatrix},$$

and

$$W_2(x) = J_2F_2W_1(-x)F_2J_2, \quad \text{where } F_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It follows from Corollary 4.6.6 that the monic orthogonal polynomials $P_{n,2}$ with respect to the weight W_2 are completely determined by the the monic orthogonal polynomials $P_{n,1}$ with respect to $P_{n,2}(x) = J_2F_2P_{n,1}(-x)F_2J_2$. Therefore we only need to study the polynomials $P_{n,1}$.

Differential operators

In this subsection we describe a set of linearly independent differential operators that have the polynomials $P_{n,1}$ as eigenfunctions.

Proposition 4.8.4. *The matrix orthogonal polynomials $\{P_{n,1}\}_{n\geq 0}$ satisfy*

$$P_{n,1}D_j = \Lambda_n(D_j)P_{n,1}, \quad j = 1,2,3, n \geq 0,$$

where the differential operators D_j are

$$D_1 = (x^2-1)\left(\frac{d^2}{dx^2}\right) + \left(\frac{d}{dx}\right)\begin{pmatrix} 6x & -3 \\ -1 & 6x-2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_2 = \left(\frac{d^2}{dx^2}\right)\begin{pmatrix} x^2-\frac{1}{4} & -\frac{3}{2}x+\frac{3}{4} \\ \frac{x}{2}+\frac{1}{4} & -\frac{3}{4} \end{pmatrix} + \left(\frac{d}{dx}\right)\begin{pmatrix} 6x & \frac{9}{2} \\ \frac{3}{2} & 0 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_3 = \left(\frac{d^2}{dx^2}\right)\begin{pmatrix} -3x+3 & 9x^2-9x \\ x^2+x-2 & 3x-3 \end{pmatrix} + \left(\frac{d}{dx}\right)\begin{pmatrix} -9 & 36x-18 \\ 8x+4 & -3 \end{pmatrix} + \begin{pmatrix} 0 & 18 \\ 12 & 0 \end{pmatrix},$$

and the eigenvalues Λ_j are given by

$$\Lambda_n(D_1) = \begin{pmatrix} n(n+5)+2 & 0 \\ 0 & n(n+5) \end{pmatrix}, \quad \Lambda_n(D_2) = \begin{pmatrix} (n+3)(n+2) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Lambda_n(D_3) = \begin{pmatrix} 0 & 9(n+2)(n+1) \\ (n+4)(n+3) & 0 \end{pmatrix}.$$

Moreover, the differential operators D_1, D_2 and D_3 satisfy

$$D_1D_2 = D_2D_1, \quad D_1D_3 \neq D_3D_1, \quad D_2D_3 \neq D_3D_2.$$

Rodrigues' Formula

The monic orthogonal polynomials $\{P_{n,1}(x)\}_{n\geq 0}$ satisfy the Rodrigues' formula

$$P_{n,1}(x) = c\left[(1-x)^{\frac{1}{2}+n}(1+x)^{\frac{3}{2}+n}(R(x)+Y_n(x))\right]^{(n)} W_1^{-1}(x),$$

where

$$c = \frac{2^{-2n-2}(-1)^n(n+3)(n+4)\sqrt{\pi}}{\Gamma(n+\frac{5}{2})},$$

and

$$R(x) = \begin{pmatrix} 2x^2-2x+1 & \frac{1}{3}(4x-1) \\ \frac{1}{3}(4x-1) & \frac{1}{9}(2x^2+2x+5) \end{pmatrix},$$

$$Y_n(x) = \begin{pmatrix} \frac{n}{n+3} & \frac{n}{3(n+3)}(2x+1) \\ \frac{n}{3(n+1)}(1-2x) & -\frac{n}{3(n+1)} \end{pmatrix}.$$

Three term recurrence relations

The orthonormal polynomials $\mathcal{P}_{n,1}(x) = \|P_{n,1}\|^{-1}P_{n,1}$, with leading coefficient

$$\Omega_n = \begin{pmatrix} \sqrt{\frac{2^{2n+1}(n+1)}{\pi(n+4)}} & 0 \\ 0 & \frac{9(n+1)\sqrt{2^{2n+1}(n+2)}}{\sqrt{\pi(n+3)(n+4)}} \end{pmatrix},$$

satisfy the three term recurrence relation

$$x\mathcal{P}_n(x) = A_{n+1}\mathcal{P}_{n+1}(x) + B_n\mathcal{P}_n(x) + A_n^*\mathcal{P}_{n-1}(x),$$

where

$$A_n = \begin{pmatrix} \frac{1}{2}\sqrt{\frac{n(n+4)}{(n+1)(n+3)}} & 0 \\ 0 & \frac{n(n+4)}{2(n+2)\sqrt{(n+1)(n+3)}} \end{pmatrix},$$

$$B_n = \begin{pmatrix} 0 & \frac{3}{2\sqrt{(n+1)(n+2)(n+3)(n+4)}} \\ \frac{3}{2\sqrt{(n+1)(n+2)(n+3)(n+4)}} & \frac{2}{(n+2)(n+3)} \end{pmatrix}.$$

Therefore the monic polynomials $P_{n,1}(x)$ satisfy the three term recurrence relation

$$xP_{n,1} = P_{n+1,1} + \tilde{B}_nP_{n,1} + \tilde{C}_nP_{n-1,1},$$

where

$$\tilde{B}_n = \begin{pmatrix} 0 & \frac{9}{2(n+3)(n+4)} \\ \frac{1}{2(n+1)(n+2)} & \frac{2}{(n+2)(n+3)} \end{pmatrix}, \quad \tilde{C}_n = \begin{pmatrix} \frac{n(n+4)}{4(n+1)(n+3)} & 0 \\ 0 & \frac{n^2(n+4)^2}{4(n+1)(n+2)^2(n+3)} \end{pmatrix}.$$

4.8.5 Case $\ell = 2$; weight of dimension 5

In this subsection we consider the 2×2 irreducible block in the case $\ell = 2$, where the matrix weight W is of dimension 5. This case completes the list of all irreducible 2×2 blocks obtained by conjugating the weight W by the matrix Y.

The 2×2 block is given by

$$W_1(x) = (1-x)^{\frac{3}{2}}(1+x)^{\frac{3}{2}} \begin{pmatrix} x^2+4 & 10x \\ 10x & 16x^2+4 \end{pmatrix}.$$

As before, we denote by $\{P_{n,1}\}_n$ the sequence of monic orthogonal polynomials with respect to W_1.

Differential operators

In this subsection we describe a set of linearly independent differential operators that have the polynomials $P_{n,1}$ as eigenfunctions.

Proposition 4.8.5. *The matrix orthogonal polynomials $\{P_{n,1}\}_{n\geq 0}$ satisfy*

$$P_{n,1}D_j = \Lambda_n(D_j)P_{n,1}, \quad j = 1, 2, 3, n \geq 0,$$

where the differential operators D_j are

$$D_1 = (x^2-1)\left(\frac{d^2}{dx^2}\right) + \left(\frac{d}{dx}\right)\begin{pmatrix} 7x & -1 \\ -4 & 7x \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_2 = \left(\frac{d^2}{dx^2}\right)\begin{pmatrix} x^2 & -\frac{1}{2}x \\ 2x & -1 \end{pmatrix} + \left(\frac{d}{dx}\right)\begin{pmatrix} 7x & -3 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix},$$

$$D_3 = \left(\frac{d^2}{dx^2}\right)\begin{pmatrix} \frac{3}{8}x & \frac{x^2}{16}-\frac{1}{4} \\ x^2-\frac{1}{4} & -\frac{3}{8}x \end{pmatrix} + \left(\frac{d}{dx}\right)\begin{pmatrix} -\frac{1}{4} & \frac{5}{8}x \\ 4x & -1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{5}{4} \\ 2 & 0 \end{pmatrix},$$

and the eigenvalues Λ_j are given by

$$\Lambda_n(D_1) = \begin{pmatrix} n(n+6)-3 & 0 \\ 0 & n(n+6) \end{pmatrix}, \quad \Lambda_n(D_2) = \begin{pmatrix} (n+1)(n+5) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Lambda_n(D_3) = \begin{pmatrix} 0 & \frac{1}{16}(n+5)(n+4) \\ (n+2)(n+1) & 0 \end{pmatrix}.$$

Moreover, the differential operators D_1, D_2 and D_3 satisfy

$$D_1D_2 = D_2D_1, \quad D_1D_3 \neq D_3D_1, \quad D_2D_3 \neq D_3D_2.$$

Rodrigues' Formula

The monic orthogonal polynomials $\{P_{n,1}(x)\}_{n\geq 0}$ satisfy the Rodrigues' formula

$$P_{n,1}(x) = c\left[(1-x)^{\frac{3}{2}+n}(1+x)^{\frac{3}{2}+n}(R(x)+Y_n(x))\right]^{(n)} W_1^{-1}(x),$$

where

$$c = \frac{(-1)^n 2^{-2n-4}(n+3)(n+4)(n+5)\sqrt{\pi}}{(2n+5)\Gamma(n+\frac{5}{2})},$$

and

$$R(x) = \begin{pmatrix} x^2+4 & 10x \\ 10x & 16x^2+4 \end{pmatrix}, \quad Y_n(x) = \begin{pmatrix} -\frac{3n}{n+1} & -\frac{6nx}{n+1} \\ \frac{6nx}{n+4} & \frac{12n}{n+4} \end{pmatrix}.$$

Three term recurrence relations

The orthonormal polynomials $\mathcal{P}_{n,1}(x) = \|P_{n,1}\|^{-1}P_{n,1}$ with leading coefficient

$$\Omega_n = \begin{pmatrix} 2^{2n+2}\frac{\sqrt{(n+2)}(n+1)}{\sqrt{\pi(n+4)(n+5)}} & 0 \\ 0 & 2^n\frac{\sqrt{2(n+1)}}{\sqrt{\pi(n+5)}} \end{pmatrix},$$

satisfy the three term recurrence relation

$$x\mathcal{P}_n(x) = A_{n+1}\mathcal{P}_{n+1}(x) + B_n\mathcal{P}_n(x) + A_n^*\mathcal{P}_{n-1}(x),$$

where

$$A_n = \begin{pmatrix} \frac{n(n+5)}{2\sqrt{(n+1)(n+2)(n+3)(n+4)}} & 0 \\ 0 & \frac{\sqrt{n(n+5)}}{\sqrt{(n+1)(n+4)}} \end{pmatrix},$$

$$B_n = \begin{pmatrix} 0 & \frac{2}{\sqrt{(n+1)(n+2)(n+4)(n+5)}} \\ \frac{2}{\sqrt{(n+1)(n+2)(n+4)(n+5)}} & 0 \end{pmatrix}.$$

Therefore the monic polynomials $P_{n,1}(x)$ satisfy the three term recurrence relation

$$xP_{n,1} = P_{n+1,1} + \tilde{B}_n P_{n,1} + \tilde{C}_n P_{n-1,1},$$

where

$$\tilde{B}_n = \begin{pmatrix} 0 & \frac{1}{2(n+1)(n+2)} \\ \frac{8}{(n+4)(n+5)} & 0 \end{pmatrix}, \tilde{C}_n = \begin{pmatrix} \frac{n^2(n+5)^2}{4(n+1)(n+2)(n+3)(n+4)} & 0 \\ 0 & \frac{n(n+5)}{4(n+1)(n+4)} \end{pmatrix}.$$

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4.A Transformation formulas

The goal of this appendix is to prove Theorem 4.5.4. We use the standard notation for the Pochhammer symbols and the hypergeometric series from [AAR99]. In the manipulations we only need the Chu-Vandermonde summation formula [AAR99, Corollary 2.2.3] which reads

$${}_2F_1\left(\begin{matrix}-n, a\\ c\end{matrix}; 1\right) = \frac{(c-a)_n}{(c)_n}. \tag{4.62}$$

and Sheppard's transformation formula for ${}_3F_2$'s [AAR99, Cor. 3.3.4] written as

$$\sum_{k=0}^{n}(e+k)_{n-k}(d+k)_{n-k}\frac{(-n)_k(a)_n(b)_n}{k!} = \sum_{k=0}^{n}(d-a)_{n-k}(e-a)_{n-k}\frac{(-n)_k(a)_k(a+b-n-d-e+1)_k}{k!}. \tag{4.63}$$

Proposition 4.A.1. *Let $\ell \in \frac{1}{2}\mathbb{N}$ and $p, q \in \frac{1}{2}\mathbb{Z}$ such that $|p|, |q| \le \ell$, $\ell - p, \ell - q \in \mathbb{Z}$, $q - p \le 0$ and $q + p \le 0$. Let $s \in \{0, \dots, \ell + q\}$ and define*

$$e_s(p,q) = \sum_{n=0}^{\ell+q-s} \binom{\ell+p}{n}\binom{\ell+q}{n+s} \sum_{m=0}^{\ell-p-s} \frac{\binom{\ell-p}{m+s}\binom{\ell-q}{m}}{\binom{2\ell}{m+n+s}^2}. \tag{4.64}$$

Then we have

$$e_s^{\ell}(p,q) =$$
$$\frac{(2\ell+1)}{(\ell+p+1)} \frac{(\ell-q)!(\ell+q)!}{(2\ell)!} \sum_{T=0}^{\ell+q-s} (-1)^{\ell+q-T} \frac{(p-\ell)_{\ell+q-T}(2+2\ell-T)_T}{(\ell+p+2)_{\ell+q-T}T!}.$$

Proof. First we reverse the inner summation using $M = \ell - p - s - m$ to get

$$e_s(p,q) = \sum_{n=0}^{\ell+q-s} \binom{\ell+p}{n}\binom{\ell+q}{n+s} \sum_{M=0}^{\ell-p-s} \frac{\binom{\ell-p}{M}\binom{\ell-q}{\ell-p-M-s}}{\binom{2\ell}{\ell-p-M+n}^2}. \tag{4.65}$$

We rewrite the inner summation:

$$\sum_{M=0}^{\ell-p-s} \frac{\binom{\ell-p}{M}\binom{\ell-q}{\ell-p-M-s}}{\binom{2\ell}{\ell-p-M+n}^2} = \frac{(\ell-q)!}{(2\ell)!}(-1)^s(-\ell+p)_s \binom{2\ell}{\ell-p+n}^{-1} \times$$
$$\sum_{M=0}^{\ell-p-s} \frac{(-\ell+p+s)_M(\ell+p-n+1)_M}{M!(-\ell+p-n)_M} B(M) \tag{4.66}$$

where $B(M) = (\ell - p - M + 1)_n (p - q + M + s + 1)_{\ell+q-s-n}$ is a polynomial in M of degree $\ell + q - s$ that depends on ℓ, p, q, n and s. The polynomial $B(M)$ has an expansion in $(-1)^t(-M)_t$,

$$B(M) = \sum_{t=0}^{\ell+q} A_t \cdot (-1)^t (-M)_t. \tag{4.67}$$

The coefficients $A_t = A_t(\ell, p, q, n, s)$ can be found by repeated application of the difference operator $\Delta_M f = f(M+1) - f(M)$. Let Δ_M^i be its i-th power. We have

$$\left.\frac{\Delta_M^t}{t!}\right|_{M=0} B = A_t. \tag{4.68}$$

In other words,

$$B(M) = \sum_{t=0}^{\ell+q} A_t \cdot (-1)^t (-M)_t \quad \text{with } A_t = \left.\frac{\Delta_M^t}{t!}\right|_{M=0} B. \tag{4.69}$$

We calculate (4.66) by substituting (4.67) in it. Interchanging summations, the inner sum can be evaluated using (4.62) (after shifting the summation parameter). We get

$$\sum_{M=0}^{\ell-p-s} \frac{\binom{\ell-p}{M}\binom{\ell-q}{\ell-p-M-s}}{\binom{2\ell}{\ell-p-M+n}^2} = \frac{(\ell-q)!}{(2\ell)!}(-1)^s(-\ell+p)_s \binom{2\ell}{\ell-p+n}^{-1} \times \sum_{t=0}^{\ell+q-s} A_t \frac{(-2\ell-1)_{\ell-p-s-t}(-\ell+p+s)_t(\ell+p-n+1)_t}{(-\ell+p-n)_{\ell-p-s}}. \quad (4.70)$$

Substituting (4.70) and (4.68) in (4.65) and simplifying gives

$$e_s(p,q) = \frac{(\ell-q)!(\ell+p)!}{(2\ell)!(2\ell)!}(-1)^{\ell-p-s}(-\ell+p)_s(-\ell-q)_s \times \sum_{t=0}^{\ell+q-s} (\ell+p+1)_t(-2\ell-1)_{\ell-p-s-t}(-\ell+p+s)_t \left.\frac{\Delta_M^t}{t!}\right|_{M=0} \times (p-q+s+M+1)_{\ell+q-s} \sum_{n=0}^{\ell+q-s} \frac{(-\ell-q-s)_n(-\ell-p)_n(\ell-p-M+1)_n}{n!(-\ell-p-t)_n(-\ell-p-M)_n}. \quad (4.71)$$

The inner sum over n is a ${}_3F_2$-series, which can be transformed using (4.63). Note that the t-order difference operator can now be evaluated yielding only one non-zero term in the sum over n. This gives

$$e_s(p,q) = \frac{2\ell+1}{\ell+p+1}\frac{(\ell-q)!(\ell+q)!}{(2\ell)!} \times \sum_{t=0}^{\ell+q-s} (-1)^{-s-t}\frac{(-\ell+p)_{s+t}(2+\ell-q+s+t)_{\ell+q-s-t}}{(\ell+p+2)_{s+t}(\ell+q-s-t)!}.$$

Reversing the order of summation using $T=\ell+q-s-t$ yields

$$e_s^\ell(p,q) = \frac{(2\ell+1)}{(\ell+p+1)}\frac{(\ell-q)!(\ell+q)!}{(2\ell)!} \times \sum_{T=0}^{\ell+q-s} (-1)^{\ell+q-T}\frac{(p-\ell)_{\ell+q-T}(2+2\ell-T)_T}{(\ell+p+2)_{\ell+q-T}T!}$$

as was to be shown. □

PROOF.[Proof of Theorem 4.5.4] We already argued that there is an expansion in Chebyshev polynomials (4.31). From (4.34) it follows that there are coefficients $d_r^\ell(p,q)$ such that

$$v_{p,q}^\ell(\cos t) = \sum_{r=-(\ell+\frac{p+q}{2})}^{\ell+\frac{p+q}{2}} d_r^\ell(p,q)e^{-2irt}. \quad (4.72)$$

The coefficients $d^\ell_r(p,q)$ and $c^\ell_n(p,q)$ are related by

$$d^\ell_r(p,q) = \sum_{n=0}^{\ell+q-(r+\frac{q-p}{2})} c^\ell_n(p,q). \tag{4.73}$$

Let $q-p \le 0$, $q+p \le 0$ and $r \ge \frac{p-q}{2}$ and substitute $r(s) = s + \frac{p-q}{2}$ in (4.34). Comparing this to (4.72) shows

$$d^\ell_{r(s)}(p,q) = \sum_{n=0}^{\ell+q-s} \binom{\ell+p}{n}\binom{\ell+q}{n+s} \sum_{m=0}^{\ell-p-s} \frac{\binom{\ell-p}{m+s}\binom{\ell-q}{m}}{\binom{2\ell}{m+n+s}^2} \tag{4.74}$$

for $s = 0, \ldots, \ell+q$. Now we use Proposition 4.A.1 to show that $d^\ell_r(p,q)$ equals

$$\frac{(2\ell+1)}{(\ell+p+1)} \frac{(\ell-q)!(\ell+q)!}{(2\ell)!} \sum_{n=0}^{\ell+q-(r+\frac{q-p}{2})} (-1)^{\ell+q-n} \frac{(p-\ell)_{\ell+q-n}(2+2\ell-n)_n}{(\ell+p+2)_{\ell+q-n} n!} \tag{4.75}$$

for $r = \frac{p-q}{2}, \ldots, \ell + \frac{p+q}{2}$. It follows that

$$c^\ell_n(p,q) = \frac{2\ell+1}{\ell+p+1} \frac{(\ell-q)!(\ell+q)!}{(2\ell)!} \frac{(p-\ell)_{\ell+q-n}}{(\ell+p+2)_{\ell+q-n}} (-1)^{\ell+q-n} \frac{(2\ell+2-n)_n}{n!}. \tag{4.76}$$

This proves the theorem. □

We can reformulate Proposition 4.A.1 in terms of hypergeometric series.

Corollary 4.A.2. *For $N \in \mathbb{N}$, $a, b, c \in \mathbb{N}$ so that $0 \le a \le N$, $0 \le b \le N$, and additionally $a \le b$, $N \le a+b$ and $0 \le c \le N-b$ we have*

$$\sum_{m=0}^{c} \frac{(-c)_m\,(b+1)_m\,(b+1)_m}{(N-a-c+1)_m\, m!\,(b-N)_m} \times \\ {}_4F_3\left(\begin{matrix} -b, N-a-b-c, N-b-m+1, N-b-m+1 \\ N-b-c+1, -b-m, -b-m \end{matrix} ; 1 \right) \\ = \frac{\binom{N+1}{a}}{\binom{b}{N-a-c}\binom{N-b}{N-b-c}} \sum_{n=0}^{c} \frac{(-a)_{N-b-n}\,(-1)^{N-b-n}\,(N+2-n)_n}{(N-a+2)_{N-b-n}\, n!}$$

The ${}_4F_3$-series in the summand is not balanced. Note that the case $s=0$ leads to single sums, and the ${}_4F_3$ boils down to a terminating ${}_2F_1$ which can be summed by the Chu-Vandermonde sum, so Corollary 4.A.2 can be viewed as an extension of Chu-Vandermonde sum (4.62).

The coefficients $d^\ell_r(p,q)$ of (4.34) with $|r| \le \frac{p-q}{2}$ are independent of r. Corollary 4.5.3 in case $\ell = \ell_1 + \ell_2 = m_1 + m_2$ can be stated as follows.

Corollary 4.A.3. *For $N \in \mathbb{N}$, $a, b, c \in \mathbb{N}$ so that $0 \le a \le N$, $0 \le b \le N$, $b \le a$, $a + b \le N$, and $0 \le c \le N - a - b$ we have*

$$\frac{\binom{N-b}{a+c}\binom{N-a}{c}}{\binom{N}{a+c}^2} \sum_{n=0}^{b} \frac{(-b)_n\,(c+a-N)_n\,(a+c+1)_n\,(a+c+1)_n}{n!\,(c+1)_n\,(a+c-N)_n\,(a+c-N)_n} \times {}_4F_3\left(\begin{matrix} -a, -a-c, N-a-c-n+1, N-a-c-n+1 \\ N-a-b-c+1, -a-c-n, -a-c-n \end{matrix};1\right)$$
$$= \frac{N+1}{N-a+1}\binom{N}{b}^{-1} \sum_{m=0}^{b} \frac{(-a)_m\,(N-b+m+2)_{b-m}\,(-1)^m}{(N-a+2)_m\,(b-m)!}.$$

In particular, the left hand side is independent of c in the range stated.

Different proofs of Corollaries 4.A.2, 4.A.3 using transformation and summation formulas for hypergeometric series have been communicated to us by Mizan Rahman.

4.B Proof of the symmetry for differential operators

Proof.[Proof of Theorem 4.7.6] In terms of $\rho(x)$ and $Z(x)$, the equations (4.53) and (4.54) are given by

$$0 = Z(x)A_1(x)^* + A_1(x)Z(x), \tag{4.77}$$
$$0 = -A_1'(x)Z(x) - \frac{\rho(x')}{\rho(x)}A_1(x)Z(x) - A_1(x)Z'(x) + A_0Z(x) - Z(x)A_0. \tag{4.78}$$

As a consequence of the properties of symmetry of the weight W, it suffices to verify the conditions above for all the (n, m)-entries with $n \le m$. Here we assume that $n < m$. The case $n = m$ can be done similarly. The first equation (4.77) holds true if and only if

$$Z_{n,m-1}A_1(x)_{m,m-1} + Z_{n,m}A_1(x)_{m,m} + Z_{n,m+1}A_1(x)_{m,m+1} + Z_{n-1,m}A_1(x)_{n,n-1} + Z_{n,m}A_1(x)_{n,n} + Z_{n+1,m}A_1(x)_{n,n+1} = 0,$$

for all $n \le m$. In order to prove the expression above we replace the coefficients of A_1 and Z in the left hand side and we obtain

$$-\frac{m}{2\ell}\sum_{t=0}^{m-1} c(n, m-1, t)U_{n+m-2t-1}(x) - \frac{\ell-m}{\ell}\sum_{t=0}^{m} c(n, m, t)\,x\,U_{n+m-2t}(x)$$
$$+\frac{2\ell-m}{2\ell}\sum_{t=0}^{m+1} c(n, m+1, t)U_{n+m-2t+1}(x) - \frac{n}{2\ell}\sum_{t=0}^{m} c(n-1, m, t)U_{n+m-2t-1}(x)$$
$$-\frac{\ell-n}{\ell}\sum_{t=0}^{m} c(n, m, t)\,x\,U_{n+m-2t}(x) + \frac{2\ell-n}{2\ell}\sum_{t=0}^{m} c(n+1, m, t)U_{n+m-2t+1}(x).$$

By using the recurrence relation $xU_r(x) = \frac{1}{2}U_{r-1}(x) + \frac{1}{2}U_{r+1}(x)$ we obtain

$$\sum_{t=0}^{m-1}\Big[-\frac{m}{2\ell}\,c(n,m-1,t) - \frac{n}{2\ell}\,c(n-1,m,t) - \frac{2\ell-n-m}{2\ell}\,c(n,m,t)\Big]\,U_{n+m-2t-1}(x)$$

$$\sum_{t=0}^{m}\Big[\frac{2\ell-m}{2\ell}\,c(n,m+1,t) + \frac{2\ell-n}{2\ell}\,c(n+1,m,t) - \frac{2\ell-n-m}{2\ell}\,c(n,m,t)\Big]\,U_{n+m-2t+1}(x)$$

$$+\Big[-\frac{n}{2\ell}c(n,m+1,m) - \frac{2\ell-m}{2\ell}c(n,m+1,m+1) - \frac{2\ell-m-n}{2\ell}c(n,m,m)\Big]\,U_{n-m-1}(x)$$

A simple computation shows that the coefficient of U_{n-m-1} in the expression above is zero. Now by changing the index of summation t we obtain

$$\Big[\frac{2\ell-m}{2\ell}\,c(n,m+1,0) + \frac{2\ell-n}{2\ell}\,c(n+1,m,0) - \frac{2\ell-n-m}{2\ell}\,c(n,m,0)\Big]\,U_{n+m+1}(x)$$
$$+\sum_{t=0}^{m-1}\Big[-\frac{m}{2\ell}\,c(n,m-1,t) - \frac{n}{2\ell}\,c(n-1,m,t) - \frac{2\ell-n-m}{2\ell}\,c(n,m,t)$$
$$\frac{2\ell-m}{2\ell}\,c(n,m+1,t+1) + \frac{2\ell-n}{2\ell}\,c(n+1,m,t+1) - \frac{2\ell-n-m}{2\ell}\,c(n,m,t+1)\Big]\,U_{n+m-2t-1}(x). \quad (4.79)$$

Using the explicit expression of $c(n,m,t)$ in (4.52) we obtain that (4.79) is given by

$$\sum_{t=0}^{m} c(n,m,t)\Big[-\frac{(m+1-t+n)(2\ell-m+1)}{2\ell(-m+1+t-n+2\ell)} - \frac{2\ell-n-m}{2\ell} - \frac{(-n+1+2\ell)(m+1-t+n)}{2\ell(-m+1+t-n+2\ell)} + \frac{(2\ell+1-t)(m+1)}{2\ell(t+1)} + \frac{(2\ell+1-t)(n+1)}{2\ell(t+1)} + \frac{(2\ell-n-m)(m+1-t+n)(2\ell+1-t)}{\ell(-m+1+t-n+2\ell)(t+1)}\Big]\,U_{n+m-2t-1}(x) = 0,$$

since the sum of the terms in the square brackets is zero. This completes the proof of (4.77).

Now we prove (4.78). The (n,m)-entry of the right hand side of (4.78) is given by

$$x(A_1(x)Z(x))_{n,m} - (1-x^2)(A_1(x)Z'(x))_{n,m} \\ + (1-x^2)[(A_0)_{n,n} - A_1'(x)_{n,n} - (A_0)_{m,m}]Z_{n,m},$$

Using (4.52) we obtain

$$(1-x^2)[(A_0)_{n,n} - A_1'(x)_{n,n} - (A_0)_{m,m}]Z_{n,m} \\ = \sum_{t=0}^{m} \frac{\ell(n-m+1)-m}{\ell} c(n,m,t)(1-x^2)U_{n+m-2t}(x). \quad (4.80)$$

$$x(A_1(x)Z(x))_{n,m} = \sum_{t=0}^{m} \Big[-\frac{n}{2\ell} c(n-1,m,t)\, x\, U_{n+m-2t-1}(x) \\ -\frac{\ell-n}{\ell} c(n,m,t)\, x^2\, U_{n+m-2t}(x) + \frac{2\ell-n}{2\ell} c(n+1,m,t)\, x\, U_{n+m-2t+1}(x) \Big] \quad (4.81)$$

$$(1-x^2)(A_1(x)Z'(x))_{n,m} = \sum_{t=0}^{m} \Big[-\frac{n}{2\ell} c(n-1,m,t)\, (1-x^2)\, U'_{n+m-2t-1}(x) \\ -\frac{\ell-n}{\ell} c(n,m,t)\, x(1-x^2)\, U'_{n+m-2t}(x) + \\ \frac{2\ell-n}{2\ell} c(n+1,m,t)\, (1-x^2)\, U'_{n+m-2t+1}(x) \Big] \quad (4.82)$$

Now we proceed as in the proof of the condition (4.77). In (4.80) and (4.81) we use the three term recurrence relation for the Chebychev's polynomials to get rid of the factors x and x^2. Equation (4.82) involves the derivative of the polynomials U. For this we use the following identity

$$U_n'(x) = \frac{(n+2)U_{n-1}(x) - nU_{n+1}(x)}{2(1-x^2)}, \quad n \geq 0, \quad (U_{-1} \equiv 0).$$

Finally we change the index of summation t and we use the explicit expression of the coefficients $c(n,m,t)$ to complete the proof.

The boundary condition (4.55) can be easily checked. □

PROOF.[Proof of Theorem 4.7.7] We will show that the conditions of symmetry in Theorem 4.7.5 hold true. The first equation (4.48) is satisfied because $A_2(x)$ is a scalar matrix. Equation (4.49) can be written in terms of $\rho(x)$ and $Z(x)$ in the following way

$$(6x - B_1(x))Z(x) + 2(x^2-1)Z'(x) - Z(x)B_1(x)^* = 0.$$

This can be checked by a similar computation to that of the proof of Theorem 4.7.6.

Now we give the proof of the third condition for symmetry. If we take the derivative of (4.49), we multiply it by 2 and we add it to (4.50) we obtain the following equivalent condition

$$(W(x)B_1(x)^* - B_1(x)W(x))' - 2(W(x)B_0 - B_0W(x)) = 0. \tag{4.83}$$

We shall prove instead that

$$W(x)B_1(x)^* - B_1(x)W(x) - 2\left(\int W(x)dx\right)B_0 - 2B_0\left(\int W(x)dx\right) = 0,$$

which is obtained by integrating (4.83) with respect to x. Then (4.83) will follow by taking the derivative with respect to x.

We assume $n < m$. The other cases can be proved similarly. We proceed as in the proof of (4.77) in Theorem 4.7.6 to show that

$$\begin{aligned}&(W(x)B_1(x)^* - B_1(x)W(x))_{n,m}\\ &= -\rho(x)\sum_{t=0}^{m} c(n,m,t)\frac{(m-n)(\ell+1)\left(4\,\ell^2 - \ell m - \ell n + 5\ell + 3\right)}{\ell\,(-m+1+t-n+2\ell)\,(t+1)}U_{n+m-2t-1}(x)\end{aligned} \tag{4.84}$$

On the other hand we have

$$\begin{aligned}&-\left(2\left(\int W(x)dx\right)B_0 + 2B_0\left(\int W(x)dx\right)\right)_{n,m}\\ &\qquad = \sum_{t=0}^{m}\left[2c(n,m,t)((B_0)_{m,m} - (B_0)_{n,n})\int \rho(x)U_{n+m-2t}(x)dx\right].\end{aligned} \tag{4.85}$$

It is easy to show that the following formula for the Chebyshev's polynomials holds

$$\int \rho(x)U_i(x) = \rho(x)\left(\frac{U_{i+1}(x)}{2(i+2)} - \frac{U_{i-1}(x)}{2i}\right), \qquad (U_{-1} \equiv 0).$$

Therefore we have that (4.85) is given by

$$\begin{aligned}&\rho(x)\sum_{t=0}^{m} c(n,m,t)\frac{(B_0)_{m,m} - (B_0)_{n,n}}{(n+m-2t)}\left(\frac{c(n,m,t+1)}{c(n,m,t)} - 1\right)U_{n+m-2t-1}(x)\\ &= \rho(x)\sum_{t=0}^{m} c(n,m,t)\frac{(m-n)(\ell+1)\left(4\,\ell^2 - \ell m - \ell n + 5\ell + 3\right)}{\ell\,(-m+1+t-n+2\ell)\,(t+1)}U_{n+m-2t-1}(x).\end{aligned} \tag{4.86}$$

Now (4.86) is exactly the negative of (4.84). This completes the proof of the theorem. □

Chapter 5

Matrix valued orthogonal polynomials related to $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$, II

Abstract

In a previous paper we have introduced matrix valued analogues of the Chebyshev polynomials by studying matrix valued spherical functions on $\mathrm{SU}(2) \times \mathrm{SU}(2)$. In particular the matrix-size of the polynomials is arbitrarily large. The matrix valued orthogonal polynomials and the corresponding weight function are studied. In particular, we calculate the LDU-decomposition of the weight where the matrix entries of L are given in terms of Gegenbauer polynomials. The monic matrix valued orthogonal polynomials P_n are expressed in terms of Tirao's matrix valued hypergeometric function using the matrix valued differential operator of first and second order to which the P_n's are eigenfunctions. From this result we obtain an explicit formula for coefficients in the three-term recurrence relation satisfied by the polynomials P_n. These differential operators are also crucial in expressing the matrix entries of P_nL as a product of a Racah and a Gegenbauer polynomial. We also present a group theoretic derivation of the matrix valued differential operators by considering the Casimir operators corresponding to $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

5.1 Introduction

Matrix valued orthogonal polynomials have been studied from different perspectives in recent years. Originally they have been introduced by Krein [Kre71], [Kre49]. Matrix valued orthogonal polynomials have been related to various different subjects, such as higher-order recurrence equations, spectral decompositions, and representation theory. The matrix valued orthogonal polynomials studied in this paper arise from the representation theory of the group SU(2) × SU(2) with the compact subgroup SU(2) embedded diagonally, see Chapter 4 for this particular case and Gangolli and Varadarajan [GV88], Tirao [Tir77], Warner [War72b] for general group theoretic interpretations of matrix valued spherical functions. An important example is the study of the matrix valued orthogonal polynomials for the case (SU(3), U(2)), which has been studied by Grünbaum, Pacharoni and Tirao [GPT02] mainly exploiting the invariant differential operators. In Chapter 4 we have studied the matrix valued orthogonal operators related to the case (SU(2) × SU(2), SU(2)), which lead to the matrix valued orthogonal polynomial analogues of Chebyshev polynomials of the second kind U_n, in a different fashion. In the current paper we study these matrix valued orthogonal polynomials in more detail.

In order to state the most important results for these polynomials we recall the weight function (4.39)

$$\begin{aligned} W(x)_{n,m} &= \sqrt{1-x^2}\sum_{t=0}^{m}\alpha_t(m,n)\,U_{n+m-2t}(x), \\ \alpha_t(m,n) &= \frac{(2\ell+1)}{n+1}\frac{(2\ell-m)!m!}{(2\ell)!}(-1)^{m-t}\frac{(n-2\ell)_{m-t}}{(n+2)_{m-t}}\frac{(2\ell+2-t)_t}{t!} \end{aligned} \tag{5.1}$$

if $n \geq m$ and $W(x)_{n,m} = W(x)_{m,n}$ otherwise. Here and elsewhere in this paper $\ell \in \frac{1}{2}\mathbb{N}$, $n, m \in \{0, 1, \cdots, 2\ell\}$, and U_n is the Chebyshev polynomial of the second kind. Note that the sum in (5.1) actually starts at $\min(0, n + m - 2\ell)$. It follows that $W\colon [-1, 1] \to M_{2\ell+1}(\mathbb{C})$, $W(x) = \big(W(x)_{n,m}\big)_{n,m=0}^{2\ell}$, is a $(2\ell + 1) \times (2\ell + 1)$-matrix valued integrable function such that all moments $\int_{-1}^{1} x^n W(x)\,dx$, $n \in \mathbb{N}$, exist. From the construction given in Section 4.5, it follows $W(x)$ is positive definite almost everywhere. By general considerations, e.g. [GT07], we can construct the corresponding monic matrix valued orthogonal polynomials $\{P_n\}_{n=0}^{\infty}$, so

$$\langle P_n, P_m\rangle_W = \int_{-1}^{1} P_n(x)\,W(x)\,\big(P_m(x)\big)^*\,dx = \delta_{nm}H_n, \quad 0 < H_n \in M_{2\ell+1}(\mathbb{C}) \tag{5.2}$$

where $H_n > 0$ means that H_n is a positive definite matrix, and $P_n(x) = \sum_{k=0}^{n} x^k P_k^n$ with $P_k^n \in M_{2\ell+1}(\mathbb{C})$ and $P_n^n = I$, the identity matrix. The polynomials P_n are the monic variants of the matrix valued orthogonal polynomials constructed in Chapter 4 from representation theoretic considerations. Note that (5.2) defines a matrix valued inner product $\langle\cdot,\cdot\rangle_W$ on the matrix valued polynomials. Using the orthogonality relations for the Chebyshev polynomials U_n it follows that

$$(H_0)_{nm} = \delta_{nm}\frac{\pi}{2}\frac{(2\ell+1)^2}{(n+1)(2\ell-n+1)} \tag{5.3}$$

which is in accordance with Proposition 4.4.6. From Chapter 4 we can also obtain an expression for H_n by translating the result of Proposition 4.4.6 to the monic case in (4.40), but since the matrix Υ_d in (4.40) is relatively complicated this leads to a complicated expression for the squared norm matrix H_n in (5.2). In Corollary 5.5.4 we give a simpler expression for H_n from the three-term recurrence relation.

These polynomials have a group theoretic interpretation as matrix valued spherical functions associated to $(SU(2) \times SU(2), SU(2))$, see Chapter 4 and Section 5.7. In particular, in Section 4.5 we have shown that the corresponding orthogonal polynomials are not irreducible, but can be written as a 2-block-diagonal matrix of irreducible matrix valued orthogonal polynomials. Indeed, if we put $J \in M_{2\ell}(\mathbb{C})$, $J_{nm} = \delta_{n+m,2\ell}$ we have $JW(x) = W(x)J$ for all $x \in [-1,1]$, and by Proposition 4.5.5 J and I span the commutant $\{Y \in M_{2\ell}(\mathbb{C}) \mid [Y, W(x)] = 0 \,\forall\, x \in [-1,1]\}$. Note that J is a self-adjoint involution, $J^2 = I$, $J^* = J$. It is easier to study the polynomials P_n, and we discuss the relation to the irreducible cases when appropriate.

In this paper we continue the study of the matrix valued orthogonal polynomials and the related weight function. Let us discuss in some more detail the results we obtain in this paper. Some of these results are obtained employing the group theoretic interpretation and some are obtained using special functions. Essentially, we obtain the following results for the weight function:

(a) explicit expression for $\det(W(x))$, hence proving Conjecture 4.5.8, see Corollary 5.2.3;

(b) an LDU-decomposition for W in terms of Gegenbauer polynomials, see Theorem 5.2.1.

Part (a) can be proved by a group theoretic consideration, and gives an alternative proof for a related statement by Koornwinder [Koo85], but we actually calculate it directly from (b). The LDU-decomposition hinges on expressing the integral of the product of two Gegenbauer polynomials and a Chebyshev polynomial as a Racah polynomial, see Lemma 5.2.7.

For the matrix valued orthogonal polynomials we obtain the following results:

(i) P_n as eigenfunctions to a second-order matrix valued differential operator $\tilde{D}$ and a first-order matrix valued differential operator $\tilde{E}$, compare Section 4.7, see Theorem 5.3.1 and Section 5.3;

(ii) the group-theoretic interpretation of $\tilde{D}$ and $\tilde{E}$ using the Casimir operators for $\mathrm{SU}(2)\times\mathrm{SU}(2)$, see Section 5.7, for which the paper by Casselman and Miličić [CM82] is essential;

(iii) explicit expressions for the matrix entries of the polynomials P_n in terms of matrix valued hypergeometric series using the matrix valued differential operators, see Theorem 5.4.5;

(iv) explicit expressions for the matrix entries of the polynomials P_nL in terms of (scalar-valued) Gegenbauer polynomials and Racah polynomials using the LDU-decomposition of the weight W and differential operators, see Theorem 5.6.2;

(v) explicit expression for the three-term recurrence satisfied by P_n, see Theorem 5.5.3.

In particular, (i) and (ii) follow from group theoretic considerations, see Section 5.3 and 5.7. This then gives the opportunity to link the polynomials to the matrix valued hypergeometric differential operator, leading to (iii). The explicit expression in (iv) involving Gegenbauer polynomials is obtained by using the LDU-decomposition of the weight matrix and the differential operator $\tilde{D}$. The expression of the coefficients as Racah polynomials involves the first order differential operator as well. Finally, in Theorem 4.4.8 we have obtained an expression for the coefficients of the three-term recurrence relation where the matrix entries of the coefficient matrices are given as sums of products of Clebsch-Gordan coefficients, and the purpose of (v) is to give a closed expression for these matrices. The case $\ell = 0$, or the spherical case, corresponds to the Chebyshev polynomials $U_n(x)$, which occur as spherical functions for $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$ or equivalently as characters on $\mathrm{SU}(2)$. For these cases almost all of the statements above reduce to well-known statements for Chebyshev polynomials, except that the first order differential operator has no meaning for this special case.

The structure of the paper is as follows. In Section 5.2 we discuss the LDU decomposition of the weight, but the main core of the proof is refererred to Appendix 5.A. In Section 5.3 we discuss the matrix valued differential operators to which the matrix valued orthogonal polynomials are eigenfunctions. We give a group theoretic proof of this result in Section 5.7. In Section 4.7 we have derived the same operators by a judicious guess and next proving the result. In order to connect to Tirao's matrix valued hypergeometric series, we switch to another variable. The connection is made precise in Section 5.4. This result is next used in Section 5.5 to derive a simple expression for the coefficients in the three-term recurrence of the monic orthogonal polynomials, improving a lot on the corresponding result in Theorem 4.4.8. In Section 5.6 we explicitly establish that the entries of the matrix valued orthogonal polynomials times the L-part of the LDU-decomposition of the weight W can be given explicitly as a product of a Racah polynomial and a Gegenbauer polynomial, see Theorem 5.6.2. Some of the above statements require somewhat lengthy and/or tedious manipulations, and in order to deal with these computations and also for various other checks we have used computer algebra.

As mentioned before, we consider the matrix valued orthogonal polynomials studied in this paper as matrix valued analogues of the Chebyshev polynomials of the second kind. As is well known, the group theoretic interpretation of the Chebyshev polynomials, or more generally of spherical functions, leads to more information on these special functions, and it remains to study which of these properties can be extended in this way to the explicit set of matrix valued orthogonal polynomials studied in this paper. This paper is mainly analytic in nature, and we only use the group theoretic interpretation to give a new way on how to obtain the first and second order matrix valued differential operator which have the matrix valued orthogonal polynomials as eigenfunctions. We note that all differential operators act on the right. The fact that we have both a first and a second order differential operator makes it possible to consider linear combinations, and this is useful in Section 5.4 to link to Tirao's matrix valued differential hypergeometric function and Section 5.6 in order to diagonalise (or decouple) a suitable matrix valued differential operator. The techniques in Section 5.4 and Section 5.6 are based on the techniques developed in [PR08], [PT07], [Rom07], [RT06].

We finally remark that J.A. Tirao has informed us that I. Zurrián, I. Pacharoni and J. Tirao have obtained results of a similar nature by considering matrix valued orthogonal polynomials for the closely related pair (SO(4), SO(3)), see [Zur08], along the lines of [GPT02]. We stress that our results and the results by Zurrián, Tirao and Pacharoni have been obtained independently.

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5.2 LDU-decomposition of the weight

In this section we state the LDU-decomposition of the weight matrix W in (5.1). The details of the proof, involving summation and transformation formulas for hypergeometric series (up to ${}_7F_6$-level), is presented in Appendix 5.A. Some direct consequences of the LDU-decomposition are discussed. The explicit decomposition is a crucial ingredient in Section 5.6, where the matrix valued orthogonal polynomials are related to the classical Gegenbauer and Racah polynomials.

In order to formulate the result we need the Gegenbauer, or ultraspherical, polynomials, see e.g. [AAR99], [Ism09], [KS98], defined by

$$C_n^{(\alpha)}(x) = \frac{(2\alpha)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+2\alpha \\ \alpha+\frac{1}{2}\end{matrix}; \frac{1-x}{2}\right). \tag{5.4}$$

The Gegenbauer polynomials are orthogonal polynomials;

$$\int_{-1}^{1} (1-x^2)^{\alpha-\frac{1}{2}} C_n^{(\alpha)}(x) C_m^{(\alpha)}(x)dx = \\ \delta_{nm}\frac{(2\alpha)_n\sqrt{\pi}\,\Gamma(\alpha+\frac{1}{2})}{n!\,(n+\alpha)\,\Gamma(\alpha)} = \delta_{nm}\frac{\pi\,\Gamma(n+2\alpha)\,2^{1-2\alpha}}{\Gamma(\alpha)^2\,(n+\alpha)\,n!} \tag{5.5}$$

Theorem 5.2.1. *The weight matrix W has the following LDU-decomposition;*

$$W(x) = \sqrt{1-x^2}\, L(x)\, T(x)\, L(x)^t, \qquad x \in [-1,1],$$

where $L\colon [-1,1] \to M_{2\ell+1}(\mathbb{C})$ is the unipotent lower triangular matrix

$$L(x)_{mk} = \begin{cases} 0, & k > m \\ \dfrac{m!\,(2k+1)!}{(m+k+1)!\,k!} C_{m-k}^{(k+1)}(x), & k \le m \end{cases}$$

and $T\colon [-1,1] \to M_{2\ell+1}(\mathbb{C})$ *is the diagonal matrix*

$$T(x)_{kk} = c_k(\ell)(1-x^2)^k, \quad c_k(\ell) = \frac{4^k (k!)^4 (2k+1)}{((2k+1)!)^2} \frac{(2\ell+k+1)!\,(2\ell-k)!}{((2\ell)!)^2}.$$

Note that the matrix-entries of L are independent of ℓ, hence of the size of the matrix valued weight W. Using

$$\frac{d^k}{dx^k} C_n^{(\alpha)}(x) = 2^k (\alpha)_k\, C_{n-k}^{(\alpha+k)}(x)$$

we can write uniformly $L(x)_{mk} = \frac{m!\,2^{-k}}{(k!)^2} \frac{(2k+1)!}{(m+k+1)!} \frac{d^k U_m}{dx^k}(x)$. In Theorem 5.6.2 we extend Theorem 5.2.1, but Theorem 5.2.1 is an essential ingredient in Theorem 5.6.2.

Since $W(x)$ is symmetric, it suffices to consider the (n,m)-matrix-entry for $m \leq n$ of Theorem 5.2.1. Hence Theorem 5.2.1 follows directly from Proposition 5.2.2 using the explicit expression (5.1) for the weight W.

Proposition 5.2.2. *The following relation*

$$\sum_{t=0}^{m} \alpha_t(m,n) U_{n+m-2t}(x) = \sum_{k=0}^{m} \beta_k(m,n)(1-x^2)^k\, C_{n-k}^{(k+1)}(x) C_{m-k}^{(k+1)}(x)$$

with the coefficients $\alpha_t(m,n)$ *given by* (5.1) *and*

$$\beta_k(m,n) = \frac{m!}{(m+k+1)!} \frac{n!}{(n+k+1)!} k!\, k!\, 2^{2k}(2k+1) \frac{(2\ell+k+1)!\ (2\ell-k)!}{(2\ell)!\,(2\ell)!}$$

holds for all integers $0 \leq m \leq n \leq 2\ell$, *and all* $\ell \in \frac{1}{2}\mathbb{N}$.

Before discussing the proof we list some corollaries of Theorem 5.2.1. First of all, we can use Theorem 5.2.1 to prove Conjecture 4.5.8, see (a) of Section 5.1.

Corollary 5.2.3. $\det\bigl(W(x)\bigr) = (1-x^2)^{2(\ell+\frac{1}{2})^2} \prod_{k=0}^{2\ell} c_k(\ell)$.

Remark 5.2.4. We also have another proof of this fact using a group theoretic approach to calculate $\det(\Phi_0(x))$, see Chapter 4 and Section 5.7 for the definition of Φ_0, and W is up to trivial factors equal to $(\Phi_0)(\Phi_0)^*$. This proof is along the lines of Koornwinder [Koo85].

Secondly, using the matrix $J \in M_{2\ell+1}(\mathbb{C})$, $J_{nm} = \delta_{n+m,2\ell}$ and $W(x) = JW(x)J$, see Proposition 4.5.5 and Subsection 4.6.2, we obtain from Theorem 5.2.1 the UDL-decomposition for W. For later reference we also recall $JP_n(x)J = P_n(x)$, since both are the monic matrix valued orthogonal polynomials with respect to $W(x) = JW(x)J$.

Corollary 5.2.5. $W(x) = \sqrt{1-x^2}\,\bigl(JL(x)J\bigr)\bigl(JT(x)J\bigr)\bigl(JL(x)J\bigr)^t$, $x \in [-1,1]$ *gives the UDL-decomposition of the weight* W.

Thirdly, considering the Fourier expansion of the weight function $W(\cos\theta)$, and using the expression of the weight in terms of Clebsch-Gordan coefficients, see (4.26), (4.29) and (4.30), we obtain a Fourier expansion, which is actually equivalent to Theorem 5.2.1.

Corollary 5.2.6. *We have the following Fourier expansion*

$$\sum_{k=0}^{m\wedge n}(-4)^k(2k+1)\frac{(m-k+1)_k\,(n-k+1)_k}{(m+1)_{k+1}\,(n+1)_{k+1}}\frac{(2\ell+k+1)!\,(2\ell-k)!}{(2\ell)!\,(2\ell)!}\times$$
$$e^{-i(n+m)t}(1-e^{2it})^{2k}\,{}_2F_1\left(\begin{matrix}k-n,k+1\\-n\end{matrix};e^{2it}\right)\,{}_2F_1\left(\begin{matrix}k-m,k+1\\-m\end{matrix};e^{2it}\right)=$$
$$\sum_{j=0}^{2\ell}\sum_{j_1=0}^{n}\sum_{\substack{j_2=0\\j_1+j_2=j}}^{2\ell-n}\sum_{i_1=0}^{m}\sum_{\substack{i_2=0\\i_1+i_2=j}}^{2\ell-m}\frac{\binom{n}{j_1}\binom{2\ell-n}{j_2}}{\binom{2\ell}{j}}\frac{\binom{m}{i_1}\binom{2\ell-m}{i_2}}{\binom{2\ell}{j}}e^{i((n-j_1+j_2)-(m-i_1+i_2))t}$$

Proof. In Subsections 4.5 and 4.6 the weight function $W(\cos t)$ was initially defined as a Fourier polynomial with the coefficients given in terms of Clebsch-Gordan coefficients. After relabeling this gives

$$\sum_{j=0}^{2\ell}\sum_{j_1=0}^{n}\sum_{\substack{j_2=0\\j_1+j_2=j}}^{2\ell-n}\sum_{i_1=0}^{m}\sum_{\substack{i_2=0\\i_1+i_2=j}}^{2\ell-m}\frac{\binom{n}{j_1}\binom{2\ell-n}{j_2}}{\binom{2\ell}{j}}\frac{\binom{m}{i_1}\binom{2\ell-m}{i_2}}{\binom{2\ell}{j}}e^{i((n-j_1+j_2)-(m-i_1+i_2))t}$$
$$=\left(L(\cos t)T(\cos t)L(\cos t)^t\right)_{nm}$$
$$=\sum_{k=0}^{\min(m,n)}\beta_k(m,n)\sin^{2k}t\,C_{n-k}^{(k+1)}(\cos t)C_{m-k}^{(k+1)}(\cos t)$$

where we have used (4.33) to express the Clebsch-Gordan coefficients in terms of binomial coefficients.

Using the result [BK92, Cor. 6.3] by Koornwinder and Badertscher together with the Fourier expansion of the Gegenbauer polynomial, see [BK92, (2.8)], [AAR99, (6.4.11)], [Ism09, (4.5.13)], we find the Fourier expansion of $\sin^k t\,C_{n-k}^{(k+\lambda)}(\cos t)$ in terms of Hahn polynomials defined by

$$Q_k(j;\alpha,\beta,N)={}_3F_2\left(\begin{matrix}-k,k+\alpha+\beta+1,-j\\\alpha+1,-N\end{matrix};1\right),\qquad k\in\{0,1,\cdots,N\},\tag{5.6}$$

see [AAR99, p. 345], [Ism09, §6.2], [KS98, §1.5]. For $\lambda=1$ the explicit formula is

$$\begin{aligned}\frac{i^k(n+1)_{k+1}\,(n-k)!}{2^k(\frac{3}{2})_k\,(2k+2)_{n-k}}\sin^k t\,C_{n-k}^{(k+1)}(\cos t)&=\sum_{j=0}^{n}Q_k(j;0,0,n)e^{i(2j-n)t}\\&=e^{-int}(1-e^{2it})^k\,{}_2F_1\left(\begin{matrix}k-n,k+1\\-n\end{matrix};e^{2it}\right)\end{aligned}\tag{5.7}$$

using the generating function [KS98, (1.6.12)] for the Hahn polynomials in the last equality. Plugging this in the identity gives the required result. □

In the proof of Proposition 5.2.2 and Theorem 5.2.1 given in Appendix 5.A we use a somewhat unusual integral representation of a Racah polynomial. Recall the Racah polynomials, [AAR99, p. 344], [KS98, §1.2], defined by

$$R_k(\lambda(t);\alpha,\beta,\gamma,\delta) \;=\; {}_4F_3\left(\begin{matrix}-k, k+\alpha+\beta+1, -t, t+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix};1\right) \tag{5.8}$$

where $\lambda(t) = t(t+\gamma+\delta+1)$, and one out of $\alpha+1$, $\beta+\delta+1$, $\gamma+1$ equals $-N$ with a non-negative integer N. The Racah polynomials with $0 \le k \le N$ form a set of orthogonal polynomials for $t \in \{0,1,\cdots,N\}$ for suitable conditions on the parameters. For the special case of the Racah polynomials in Lemma 5.2.7 the orthogonality relations are given in Appendix 5.A.

Lemma 5.2.7. *For integers $0 \le t, k \le m \le n$ we have*

$$\int_{-1}^{1} (1-x^2)^{k+\frac{1}{2}} C_{n-k}^{(k+1)}(x) C_{m-k}^{(k+1)}(x) U_{n+m-2t}(x)\, dx =$$
$$\frac{\sqrt{\pi}\,\Gamma(k+\frac{3}{2})}{(k+1)} \frac{(k+1)_{m-k}}{(m-k)!} \frac{(k+1)_{n-k}}{(n-k)!} \frac{(-1)^k\,(2k+2)_{m+n-2k}\,(k+1)!}{(n+m+1)!} \times$$
$$R_k(\lambda(t);0,0,-n-1,-m-1)$$

Remark 5.2.8. Lemma 5.2.7 can be extended using the same method of proof to

$$\int_{-1}^{1} (1-x^2)^{\alpha+k+\frac{1}{2}} C_{n-k}^{(\alpha+k+1)}(x) C_{m-k}^{(\alpha+k+1)}(x) C_{n+m-2t}^{(\beta)}(x)\, dx =$$
$$\frac{(\alpha+k+1)_{m-k}\,(2k+2\alpha+2)_{n-k}}{(m-k)!\,(n-m)!} \frac{(-m+\beta-\alpha-1)_{m-t}}{(m-t)!} \times$$
$$\frac{(\beta)_{n-t}\sqrt{\pi}\,\Gamma(\alpha+k+\frac{3}{2})}{\Gamma(\alpha+n+m-t+2)}\, {}_4F_3\left(\begin{matrix}k-m, -m-2\alpha-k-1, t-m, \beta+n-t\\ \beta-\alpha-1-m, -m-\alpha, n-m+1\end{matrix};1\right) \tag{5.9}$$

assuming $n \ge m$. Lemma 5.2.7 corresponds to the case $\alpha = 0$, $\beta = 1$ after using a transformation for a balanced ${}_4F_3$-series. Note that ${}_4F_3$-series can be expressed as a Racah polynomial orthogonal on $\{0,1,\cdots,m\}$ in case $\alpha = 0$ or $\beta = \alpha+1$, which corresponds to Lemma 5.2.7. We do not use (5.9) in the paper, and a proof follows the lines of the proof of Lemma 5.2.7 as given in Appendix 5.A.

In Theorem 4.6.5, see Section 5.1, we have proved that the weight function W is not irreducible, meaning that there exists $Y \in M_{2\ell+1}(\mathbb{C})$ so that

$$YW(x)Y^t \;=\; \begin{pmatrix} W_1(x) & 0\\ 0 & W_2(x)\end{pmatrix}, \qquad YY^t \;=\; I \;=\; Y^tY \tag{5.10}$$

and that there is no further reduction.

We can then obtain results by combining the reducibility and the LDU-decomposition. E.g. assuming $2\ell+1$ even and writing

$$Y \;=\; \begin{pmatrix} A & B\\ C & D\end{pmatrix}, \quad L(x) \;=\; \begin{pmatrix} l_1(x) & 0\\ r(x) & l_2(x)\end{pmatrix}, \quad T(x) \;=\; \begin{pmatrix} t_1(x) & 0\\ 0 & t_2(x)\end{pmatrix},$$

with A, D diagonal and B, C antidiagonal, see Corollary 4.5.6, $l_1(x)$, $l_2(x)$ lower-diagonal matrices and $r(x)$ a full matrix, we can work out the block-diagonal structure of $YL(x)T(x)L(x)^tY^t$. It follows that the off-diagonal blocks being zero is equivalent to

$$(Al_1(x)t_1(x) + Br(x)t_1(x))\left(l_1(x)^tC^t + r(x)^tD^t\right) + Bl_2(x)t_2(x)l_2(x)^tD^t = 0. \qquad (5.11)$$

This can be rewritten as an identity for four sums of products of two Gegenbauer polynomials involving the weight function and the constants in Theorem 5.2.1 being zero. We do not write the explicit results, since we do not need them.

5.3 Matrix valued orthogonal polynomials as eigenfunctions of matrix valued differential operators

In Section 4.7 we have derived that the matrix valued orthogonal polynomials are eigenfunctions for a second and a first order matrix valued differential operator by looking for suitable matrix valued differential operators that are self-adjoint with respect to the matrix valued inner product $\langle\cdot,\cdot\rangle_W$. The method was to establish relations between the coefficients of the differential operators and the weight W, next judiciously guessing the general result and next proving it by a verification. In this paper we show that essentially these operators can be obtained from the group theoretic interpretation by establishing that the matrix valued differential operators are obtainable from the Casimir operators for $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Since the paper is split into a first part of analytic nature and a second part of group theoretic nature, we state the result in this section whereas the proofs are given in Section 5.7. The Sections 5.4 and 5.6 depend strongly on the matrix valued differential operators in Theorem 5.3.1.

Recall that all differential operators act on the right, so for a matrix valued polynomial $P\colon \mathbb{R}\to M_N(\mathbb{C})$ depending on the variable x, the s-th order differential operator $D = \sum_{i=0}^{s}\frac{d^i}{dx^i}F_i(x)$, $F_i\colon\mathbb{R}\to M_N(\mathbb{C})$, acts by

$$\big(PD\big)(x) = \sum_{i=0}^{s}\frac{d^iP}{dx^i}(x)F_i(x), \qquad PD\colon\mathbb{R}\to M_N(\mathbb{C})$$

where $\left(\frac{d^iP}{dx^i}(x)\right)_{nm} = \frac{d^iP_{nm}}{dx^i}(x)$ is a matrix which is multiplied from the right by the matrix $F_i(x)$. The matrix valued orthogonal polynomial is an eigenfunction of a matrix valued differential operator if there exists a matrix $\Lambda\in M_N(\mathbb{C})$, the eigenvalue matrix, so that $PD=\Lambda P$ as matrix valued functions. Note that the eigenvalue matrix is multiplied from the left. For more information on differential operators for matrix valued functions, see e.g. [GT07], [Tir03].

We denote by E_{ij} the standard matrix units, i.e. E_{ij} is the matrix with all matrix entries equal to zero, except for the (i,j)-th entry which is 1. By convention, if either i or j is not in the appropriate range, the matrix E_{ij} is zero.

Theorem 5.3.1. *Define the second order matrix valued differential operator*

$$\tilde{D} = (1-x^2)\frac{d^2}{dx^2} + \left(\frac{d}{dx}\right)(\tilde{C} - x\tilde{U}) - \tilde{V}$$

$$\tilde{C} = \sum_{i=0}^{2\ell}(2\ell-i)E_{i,i+1} + \sum_{i=0}^{2\ell} iE_{i,i-1}, \quad \tilde{U} = (2\ell+3)I, \quad \tilde{V} = -\sum_{i=0}^{2\ell} i(2\ell-i)E_{ii}$$

and the first order matrix valued differential operator

$$\tilde{E} = \left(\frac{d}{dx}\right)(\tilde{B}_0 + x\tilde{B}_1) + \tilde{A}$$

$$\tilde{B}_0 = -\sum_{i=0}^{2\ell}\frac{(2\ell-i)}{4\ell}E_{i,i+1} + \sum_{i=0}^{2\ell}\frac{(\ell-i)}{2\ell}E_{ii} + \sum_{i=0}^{2\ell}\frac{i}{4\ell}E_{i,i-1},$$

$$\tilde{B}_1 = -\sum_{i=0}^{2\ell}\frac{(\ell-i)}{\ell}E_{i,i}, \qquad \tilde{A} = \sum_{i=0}^{2\ell}\frac{(2\ell+2)(i-2\ell)}{-4\ell}E_{i,i},$$

then the monic matrix valued orthogonal polynomials P_n satisfy

$$P_n\tilde{D} = \Lambda_n(\tilde{D})P_n, \quad \Lambda_n(\tilde{D}) = \sum_{i=0}^{2\ell}\left(-n(n-1) - n(2\ell+3) + i(2\ell-i)\right)E_{ii},$$

$$P_n\tilde{E} = \Lambda_n(\tilde{E})P_n, \quad \Lambda_n(\tilde{E}) = \sum_{i=0}^{2\ell}\left(\frac{n(\ell-i)}{2\ell} - \frac{(2\ell+2)(i-2\ell)}{4\ell}\right)E_{ii}.$$

and the operators $\tilde{D}$ and $\tilde{E}$ commute. The operators are symmetric with respect to W.

The group theoretic proof of Theorem 5.3.1 is given in Section 5.7. Theorem 5.3.1 has been proved in Theorems 4.7.6 and 4.7.7 analytically. The symmetry of the operators with respect to W means that $\langle PD, Q\rangle_W = \langle P, QD\rangle_W$ and $\langle PE, Q\rangle_W = \langle P, QE\rangle_W$ for all matrix valued polynomials with respect to the matrix valued inner product $\langle\cdot,\cdot\rangle_W$ defined in (5.2). The last statement follows immediately from the first by the results of Grünbaum and Tirao [GT07]. Also, $[D, E] = 0$ follows from the fact that the eigenvalue matrices commute. In the notation of [GT07] we have $\tilde{D}, \tilde{E} \in \mathcal{D}(W)$, where $\mathcal{D}(W)$ is the $*$-algebra of matrix valued differential operators having the matrix valued orthogonal polynomials as eigenfunctions.

Note that $\tilde{E}$ has no analogue in case $\ell = 0$, whereas $\tilde{D}$ reduces to the hypergeometric differential operator for the Chebyshev polynomials U_n.

The matrix-differential operator $\tilde{D}$ is J-invariant, i.e. $J\tilde{D}J = \tilde{D}$. The operator $\tilde{E}$ is almost J-anti-invariant, up to a multiple of the identity. This is explained in Theorem 5.7.15 and the discussion following this theorem. In particular, $\tilde{D}$ descends to the corresponding irreducible matrix valued orthogonal polynomials, but $\tilde{E}$ does not, see also Section 4.7.

5.4 Matrix valued orthogonal polynomials as matrix valued hypergeometric functions

The polynomial solutions to the hypergeometric differential equation, see (5.29), are uniquely determined. Many classical orthogonal polynomials, such as the Jacobi, Hermite, Laguerre and Chebyshev, can be written in terms of hypergeometric series. For matrix valued valued functions Tirao [Tir03] has introduced a matrix valued hypergeometric differential operator and its solutions. The purpose of this section is to link the monic matrix valued orthogonal polynomials to Tirao's matrix valued hypergeometric functions.

We want to use Theorem 5.3.1 in order to express the matrix valued orthogonal polynomials as matrix valued hypergeometric functions using Tirao's approach [Tir03]. In order to do so we have to switch from the interval $[-1,1]$ to $[0,1]$ using $x = 1-2u$. We define

$$R_n(u) \,=\, (-1)^n 2^{-n} P_n(1-2u), \qquad Z(u) \,=\, W(1-2u) \tag{5.12}$$

so that the rescaled monic matrix valued orthogonal polynomials R_n satisfy

$$Z(u) = W(1-2u), \qquad \int_0^1 R_n(u) Z(u) R_m(u)^* \, du \,=\, 2^{-1-2n} H_n. \tag{5.13}$$

In the remainder of Section 5.4 we work with the polynomials R_n on the interval $[0,1]$. It is a straightforward check to rewrite Theorem 5.3.1.

Corollary 5.4.1. *Let D and E be the matrix valued differential operators*

$$D \,=\, u(1-u)\frac{d^2}{du^2} + \left(\frac{d}{du}\right)(C - uU) - V, \qquad E \,=\, \left(\frac{d}{du}\right)(uB_1 + B_0) + A_0,$$

where the matrices C, U, V, B_0, B_1 and A_0 are given by

$$C = -\sum_{i=0}^{2\ell} \frac{(2\ell-i)}{2} E_{i,i+1} + \sum_{i=0}^{2\ell} \frac{(2\ell+3)}{2} E_{ii} - \sum_{i=0}^{2\ell} \frac{i}{2} E_{i,i-1}, \quad U = (2\ell+3)I,$$

$$V = -\sum_{i=0}^{2\ell} i(2\ell-i) E_{i,i} \quad A_0 = \sum_{i=0}^{2\ell} \frac{(2\ell+2)(i-2\ell)}{2\ell} E_{i,i},$$

$$B_0 = -\sum_{i=0}^{2\ell} \frac{(2\ell-i)}{4\ell} E_{i,i+1} + \sum_{i=0}^{2\ell} \frac{(\ell-i)}{2\ell} E_{ii} + \sum_{i=0}^{2\ell} \frac{i}{4\ell} E_{i,i-1}$$

$$B_1 = -\sum_{i=0}^{2\ell} \frac{(\ell-i)}{\ell} E_{i,i}.$$

Then D and E are symmetric with respect to the weight W, and D and E commute.

Moreover for every integer $n \geq 0$,

$$R_n D = \Lambda_n(D) R_n, \qquad \Lambda_n(D) = \sum_{i=0}^{2\ell} \left(-n(n-1) - n(2\ell+3) + i(2\ell-i)\right) E_{ii},$$

$$R_n E = \Lambda_n(E) R_n, \qquad \Lambda_n(E) = \sum_{i=0}^{2\ell} \left(-\frac{n(\ell-i)}{\ell} + \frac{(2\ell+2)(i-2\ell)}{2\ell}\right) E_{ii}.$$

It turns out that it is more convenient to work with $D_\alpha = D + \alpha E$ for $\alpha \in \mathbb{R}$, so that $R_n D_\alpha = \Lambda_n(D_\alpha) R_n$ with diagonal eigenvalue matrix $\Lambda_n(D_\alpha) = \Lambda_n(D) + \alpha \Lambda_n(E)$. By [GT07, Prop. 2.6] we have $\Lambda_n(D_\alpha) = -n^2 - n(U_\alpha - 1) - V_\alpha$. Since the eigenvalue matrix $\Lambda_n(D_\alpha)$ is diagonal, the matrix valued differential equation $R_n D_\alpha = \Lambda_n(D_\alpha) R_n$ can be read as $2\ell + 1$ differential equations for the rows of R_n. The i-th row of R_n is a solution to

$$u(1-u)p''(u) + p'(u)(C_\alpha - uU_\alpha) - p(u)(V_\alpha + \lambda) = 0, \qquad \lambda = \big(\Lambda_n(D_\alpha)\big)_{ii} \tag{5.14}$$

for $p\colon \mathbb{C} \to \mathbb{C}^{2\ell+1}$ a (row-)vector-valued polynomial function. Here $C_\alpha = C + \alpha B_0$, $U_\alpha = U - \alpha B_1$, $V_\alpha = V - \alpha A_0$ using the notation of Corollary 5.4.1. Now (5.14) allows us to connect to Tirao's matrix valued hypergeometric function [Tir03], which we briefly recall in Remark 5.4.2.

Remark 5.4.2. Given $d \times d$ matrices C, U and V we can consider the differential equation

$$z(1-z)F''(z) + (C - zU)F'(z) - VF(z) = 0, \quad z \in \mathbb{C}, \tag{5.15}$$

where $F\colon \mathbb{C} \to \mathbb{C}^d$ is a (column-)vector-valued function which is twice differentiable. It is shown by Tirao [Tir03] that if the eigenvalues of C are not in $-\mathbb{N}$, then the matrix valued hypergeometric function ${}_2H_1$ defined as the power series

$$\begin{gathered} {}_2H_1\left(\begin{matrix} U, V \\ C \end{matrix}\ ; z\right) = \sum_{i=0}^{\infty} \frac{z^i}{i!} [C, U, V]_i, \\ [C,U,V]_0 = 1, \quad [C,U,V]_{i+1} = (C+i)^{-1}\big(i^2 + i(U-1) + V\big)[C,U,V]_i \end{gathered} \tag{5.16}$$

converges for $|z| < 1$ in $M_d(\mathbb{C})$. Moreover, for $F_0 \in \mathbb{C}^d$ the (column-)vector-valued function

$$F(z) = {}_2H_1\left(\begin{matrix} U, V \\ C \end{matrix}\ ; z\right) F_0$$

is a solution to (5.15) which is analytic for $|z| < 1$, and any analytic (on $|z| < 1$) solution to (5.15) is of this form.

Comparing Tirao's matrix valued hypergeometric differential equation (5.15) with (5.14) and using Remark 5.4.2, we see that

$$p(u) = \left({}_2H_1\left(\begin{matrix} U_\alpha^t, V_\alpha^t + \lambda \\ C_\alpha^t \end{matrix}\ ; u\right) P_0\right)^t = P_0^t \left({}_2H_1\left(\begin{matrix} U_\alpha^t, V_\alpha^t + \lambda \\ C_\alpha \end{matrix}\ ; u\right)\right)^t, \tag{5.17}$$

$P_0 \in \mathbb{C}^{2\ell+1}$, are the solutions to (5.14) which are analytic in $|u| < 1$ assuming that eigenvalues of C^t_α are not in $-\mathbb{N}$. We first verify this assumption. Even though V_α and U_α are symmetric, we keep the notation for transposed matrices for notational esthetics.

Lemma 5.4.3. *For every $\ell \in \frac{1}{2}\mathbb{N}$, the matrix C_α is a diagonalisable matrix with eigenvalues $(2j+3)/2$, $j \in \{0, \ldots, 2\ell\}$.*

Proof. Note that C_α is tridiagonal, so that $v_\lambda = \sum_{n=0}^{2\ell} p_n(\lambda) e_n$ is an eigenvector for C_α for the eigenvalue λ if and only if

$$
\begin{aligned}
-(\lambda - \frac{3}{2}) p_n(\lambda) \;=\; & \frac{(2\ell-n)(2\ell+\alpha)}{4\ell}\, p_{n+1}(\lambda) \\
& - \left(\frac{(2\ell+\alpha)(2\ell-n) + n(2\ell-\alpha)}{4\ell} \right) p_n(\lambda) + \frac{(2\ell-\alpha)n}{4\ell}\, p_{n-1}(\lambda).
\end{aligned}
$$

The three-term recurrence relation corresponds precisely to the three-term recurrence relation for the Krawtchouk polynomials for $N \in \mathbb{N}$,

$$
K_n(x;p,N) \;=\; {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix} ; \frac{1}{p} \right), \qquad n, x \in \{0, 1, \cdots, N\},
$$

see e.g. [AAR99, p. 347], [Ism09, §6.2], [KS98, §1.10], with $N = 2\ell$, $p = \frac{2\ell+\alpha}{4\ell}$. The Krawtchouk polynomials are orthogonal with respect to the binomial distribution for $0 < p < 1$, or $\alpha \in (-2\ell, 2\ell)$, and we find

$$
p_n(\lambda) \;=\; K_n(\lambda; \frac{2\ell+\alpha}{4\ell}, \ell)
$$

and the eigenvalues of C_α are $\frac{3}{2} + x$, $x \in \{0, 1, \cdots, 2\ell\}$. This proves the statement for $\alpha \in (-2\ell, 2\ell)$.

Note that for $\alpha \neq \pm 2\ell$, the matrix C_α is tridiagonal, and the eigenvalue equation is solved by the same contiguous relation for the ${}_2F_1$-series leading to the same statement for $|\alpha| > 2\ell$. In case $\alpha = \pm 2\ell$ the matrix $C_{\pm 2\ell}$ is upper or lower triangular, and the eigenvalues can be read off from the diagonal. □

In particular, we can give the eigenvectors of C_α explicitly in terms of terminating ${}_2F_1$-hypergeometric series, but we do not use the result in the paper.

So (5.17) is valid and this gives a series representation for the rows of the monic polynomial R_n. Since each row is polynomial, the series has to terminate. This implies that there exists $n \in \mathbb{N}$ so that $[C^t_\alpha, U^t_\alpha, V^t_\alpha + \lambda]_{n+1}$ is singular and $0 \neq P_0 \in \operatorname{Ker}([C^t_\alpha, U^t_\alpha, V^t_\alpha + \lambda]_{n+1})$.

Suppose that n is the least integer for which $[C^t_\alpha, U^t_\alpha, V^t_\alpha + \lambda]_{n+1}$ is singular, i.e. $[C^t_\alpha, U^t_\alpha, V^t_\alpha + \lambda]_i$ is regular for all $i \leq n$. Since

$$
[C^t_\alpha, U^t_\alpha, V^t_\alpha + \lambda]_{n+1} = (C^t_\alpha + n)^{-1} \left(n^2 + n(U^t_\alpha - 1) + V^t_\alpha + \lambda \right) [C^t_\alpha, U^t_\alpha, V^t_\alpha + \lambda]_n \quad (5.18)
$$

and since the matrix $(C_\alpha + n)$ is invertible by Lemma 5.4.3, $[C_\alpha, U_\alpha, V_\alpha + \lambda]_{n+1}$ is a singular matrix if and only if the diagonal matrix

$$\begin{aligned} M_n^\alpha(\lambda) \;&=\; \big(n^2 + n(U_\alpha^t - 1) + V_\alpha^t + \lambda\big) \\ &=\; \big(n^2 + n(U_\alpha - 1) + V_\alpha + \lambda\big) \;=\; \lambda - \Lambda_n(D_\alpha) \end{aligned} \tag{5.19}$$

is singular. Note that the diagonal entries of $M_n^\alpha(\lambda)$ are of the form $\lambda - \lambda_j^\alpha(n)$, so that $M_n(\lambda)$ is singular if and only if $\lambda = \lambda_j^\alpha(n)$ for some $j \in \{0, 1, \cdots, 2\ell\}$. We need that the eigenvalues are sufficiently generic.

Lemma 5.4.4. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $(j, n) = (i, m) \in \{0, 1 \cdots, 2\ell\} \times \mathbb{N}$ if and only if $\lambda_j^\alpha(n) = \lambda_i^\alpha(m)$.*

Proof. Assume $\lambda_j^\alpha(n) = \lambda_i^\alpha(m)$ and let $(j, n), (i, m) \in \{0, 1 \cdots, 2\ell\} \times \mathbb{N}$, then

$$0 \;=\; \lambda_j^\alpha(n) - \lambda_i^\alpha(m) \;=\; (m-n)(n+m+2+2\ell) + (j-i)\left(\frac{\alpha(2\ell+4)}{2\ell} - j - i + 2\ell\right).$$

If $j \neq i$, then we solve for $\alpha = \frac{2\ell}{(2\ell+4)}\left(-\frac{(m-n)(m+n+2\ell+2)}{j-i} + j + i - 2\ell\right)$ which is rational.

Assume next that $j = i$, then $(m-n)(n+m+2+2\ell) = 0$. Since $n, m, \ell \geq 0$, it follows that $n = m$ and hence $(j, n) = (i, m)$. □

Assume α irrational, so that Lemma 5.4.4 shows that $M_n\big(\lambda_i^\alpha(m)\big)$ is singular if and only if $n = m$. So in the series (5.17) the matrix $[C_\alpha^t, U_\alpha^t, V_\alpha^t + \lambda]_{n+1}$ is singular and $[C_\alpha^t, U_\alpha^t, V_\alpha^t + \lambda]_i$ is non-singular for $0 \leq i \leq n$. Furthermore, by Lemma 5.4.4 we see that the kernel of $[C_\alpha^t, U_\alpha^t, V_\alpha^t + \lambda]_{n+1}$ is one-dimensional if and only if $\lambda = \lambda_n^\alpha(i)$, $i \in \{0, 1, \cdots, 2\ell\}$. In case $\lambda = \lambda_n^\alpha(i)$ we see that (5.17) is polynomial for

$$P_0 \;=\; [C_\alpha^t, U_\alpha^t, V_\alpha^t + \lambda_n^\alpha(i)]_n^{-1} e_i$$

determined uniquely up to a scalar, where e_i is the standard basis vector.

We can now state the main result of this section, expressing the monic polynomials R_n as a matrix valued hypergeometric function.

Theorem 5.4.5. *With the notation of Remark 5.4.2 the monic matrix valued orthogonal polynomials are given by*

$$\big(R_n(u)\big)_{ij} \;=\; \left({}_2H_1\left(\begin{matrix} U_\alpha^t, V_\alpha^t + \lambda_n^\alpha(i) \\ C_\alpha^t \end{matrix}\; ; u\right) n!\, [C_\alpha^t, U_\alpha^t, V_\alpha^t + \lambda_n^\alpha(i)]_n^{-1} e_i\right)_j^t$$

for all $\alpha \in \mathbb{R}$.

Note that the left hand side is independent of α, which is not obvious for the right hand side.

Proof. Let us first assume that α is irrational, so that the result follows from the considerations in this section using that the i-th row of $R_n(u)$ is a polynomial of (precise) degree n. The constant follows from monocity of R_n, so that $(R_n(u))_{ii} = u^n$.

Note that the left hand side is independent of α, and the right hand side is continuous in α. Hence the result follows for $\alpha \in \mathbb{R}$. □

In the scalar case $\ell = 0$ Theorem 5.4.5 reduces to

$$R_n(u) \;=\; (-4)^{-n}(n+1)\,{}_2F_1\left(\begin{matrix}-n,\, n+2\\ \frac{3}{2}\end{matrix};u\right), \tag{5.20}$$

which is the well-known hypergeometric expression for the monic Chebyshev polynomials, see [AAR99, §2.5], [Ism09, (4.5.21)], [KS98, (1.8.31)].

5.5 Three-term recurrence relation

Matrix valued orthogonal polynomials satisfy a three-term recurrence relation, see e.g. [DPS08], [GT07]. In Theorem 4.4.8 we have determined the three-term recurrence relation for the closely related matrix valued orthogonal polynomials explicitly in terms of Clebsch-Gordan coefficients. The matrix entries of the matrices occurring in the three-term recurrence relation have been given explicitly as sums of products of Clebsch-Gordan coefficients. The purpose of this section is to give simpler expressions for the monic matrix valued orthogonal polynomials using the explicit expression in terms of Tirao's matrix valued hypergeometric functions as established in Theorem 5.4.5.

From general theory the monic orthogonal polynomials $R_n\colon \mathbb{R} \to M_N(\mathbb{C})$ satisfy a three-term recurrence relation

$$uR_n(u) = R_{n+1}(u) + X_nR_n(u) + Y_nR_{n-1}(u),$$

$n \geq 0$, where $R_{-1} = 0$ and $X_n, Y_n \in M_{2\ell+1}(\mathbb{C})$ are matrices depending on n and not on x. Lemma 5.5.1 should be compared to [DPS08, Lemma 2.6].

Lemma 5.5.1. *Let $\{R_n\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials and write $R_n(u) = \sum_{k=0}^{n} R_k^n\, u^k$, $R_k^n \in M_N(\mathbb{C})$, and $R_n^n = I$. Then the coefficients X_n, Z_n of the three-term recurrence relation are given by*

$$X_n \;=\; R_{n-1}^n - R_n^{n+1}, \qquad Y_n \;=\; R_{n-2}^n - R_{n-1}^{n+1} - X_nR_{n-1}^n.$$

Proof. Let $\langle\cdot,\cdot\rangle$ denote the matrix valued inner product for which the monic polynomials are orthogonal. Using the three-term recursion, orthogonality relations and expanding the monic polynomial of degree $n+1$ gives

$$\langle uR_n - X_nR_n - Y_nR_{n-1}, R_n\rangle \;=\; \langle R_{n+1}, R_n\rangle \;=\; \langle u^{n+1}, R_n\rangle \;+\; R_n^{n+1}\langle u^n, R_n\rangle.$$

By the orthogonality relations the left hand side can be evaluated as

$$\langle uR_n - X_nR_n - Y_nR_{n-1}, R_n\rangle \;=\; \langle u^{n+1}, R_n\rangle \;+\; R_{n-1}^n\langle u^n, R_n\rangle \;-\; X_n\langle R_n, R_n\rangle$$

and comparing the two right hand sides gives the required expression for X_n, since $\langle R_n, R_n\rangle = \langle u^n, R_n\rangle$ is invertible.

The expression for Y_n follows by considering on the one hand

$$\begin{aligned}\langle uR_n - X_nR_n - Y_nR_{n-1}, R_{n-1}\rangle &= \langle R_{n+1}, R_{n-1}\rangle \\ &= \langle u^{n+1}, R_{n-1}\rangle + R_n^{n+1}\langle u^n, R_{n-1}\rangle + R_{n-1}^{n+1}\langle u^{n-1}, R_{n-1}\rangle.\end{aligned}$$

while on the other hand the left hand side also equals

$$\begin{aligned}&\langle u^{n+1}, R_{n-1}\rangle + R_{n-1}^{n}\langle u^n, R_{n-1}\rangle + R_{n-2}^{n}\langle u^{n-1}, R_{n-1}\rangle \\ &\quad - X_n\langle u^n, R_{n-1}\rangle - X_nR_{n-1}^n\langle u^{n-1}, R_{n-1}\rangle - Y_n\langle R_{n-1}, R_{n-1}\rangle\end{aligned}$$

and using the expression for X_n and cancelling common terms gives the required expression, since $\langle R_{n-1}, R_{n-1}\rangle = \langle u^{n-1}, R_{n-1}\rangle$ is invertible. □

In order to apply Lemma 5.5.1 for the explicit monic polynomials in this paper we need to calculate the coefficients, which is an application of Theorem 5.4.5.

Lemma 5.5.2. *Let $\{R_n\}_{n\geq 0}$ be the monic polynomials with respect to Z on $[0,1]$. Then*

$$R_{n-1}^n = \sum_{j=0}^{n} \frac{jn}{4(n+j)} E_{j,j-1} - \sum_{j=0}^{n} \frac{n}{2} E_{j,j} + \sum_{i=0}^{n} \frac{n(2\ell - j)}{4(2\ell - j + n)} E_{j,j+1}$$

$$\begin{aligned}R_{n-2}^n &= \sum_{j=0}^{n} \frac{n(n-1)j(j-1)}{32(n+j)(n+j-1)} E_{j,j-2} - \sum_{j=0}^{n} \frac{n(n-1)j}{8(n+j)} E_{j,j-1} \\ &\quad + \sum_{j=0}^{n} \frac{n(n-1)(3j^2 - 6\ell j - 2n^2 + n - 4n\ell)}{16(n+j)(i - 2\ell - n)} E_{j,j} \\ &\quad - \sum_{j=0}^{n} \frac{n(n-1)(2\ell - j)}{8(2\ell + n - j)} E_{j,j+1} + \sum_{j=0}^{n} \frac{n(n-1)(2\ell - j)(2\ell - j - 1)}{32(2\ell - j + n - 1)(2\ell + n - j)} E_{j,j+2}\end{aligned}$$

Proof. We can calculate R_{n-1}^n by considering the coefficients of u^{n-1} using the expression in Theorem 5.4.5. This gives

$$\begin{aligned}(R_{n-1}^n)_{ij} &= \frac{n!}{(n-1)!}\left([C_\alpha^t, U_\alpha^t, V_\alpha^t + \lambda_n^\alpha(i)]_{n-1}[C_\alpha^t, U_\alpha^t, V_\alpha^t + \lambda_n^\alpha(i)]_n^{-1} e_i\right)_j^t \\ &= n\left(M_{n-1}^\alpha\big(\lambda_n^\alpha(i)\big)^{-1}(C_\alpha^t + n - 1)e_i\right)_j^t\end{aligned}$$

using the recursive definition (5.18) of $[C_\alpha^t, U_\alpha^t, V_\alpha^t + \lambda_n^\alpha(i)]_n$.

Note that $M_{n-1}^\alpha\big(\lambda_n^\alpha(i)\big)$ is indeed invertible by Lemma 5.4.4 for irrational α. The explicit expression of the right hand side gives the result after a straightforward computation, since the resulting matrix is tridiagonal.

We can calculate R_{n-2}^n analogously,

$$(R_{n-2}^n)_{ij} = n(n-1)\left(M_{n-2}^\alpha\big(\lambda_n^\alpha(i)\big)^{-1}(C_\alpha^t + n - 2)M_{n-1}^\alpha\big(\lambda_n^\alpha(i)\big)^{-1}(C_\alpha^t + n - 1)e_i\right)_j^t$$

and a straightforward but tedious calculation gives the result. Note that R^n_{n-2} is a five-diagonal matrix, since it is the product of two tridiagonal matrices. □

Note that even though we have used the additional degree of freedom α in the proof of Lemma 5.5.2, the resulting expressions are indeed independent of α.

Now we are ready to obtain the coefficients in the recurrence relation satisfied by the polynomials R_n.

Theorem 5.5.3. *For any $\ell \in \frac{1}{2}\mathbb{N}$ the monic orthogonal polynomials R_n satisfy the three-term recurrence relation*

$$u\, R_n(u) \;=\; R_{n+1}(u) \;+\; X_n\, R_n(u) \;+\; Y_n\, R_{n-1}(u),$$

where the matrices X_n, Y_n are given by

$$X_n \;=\; -\sum_{i=0}^{2\ell}\left[\frac{i^2E_{i,i-1}}{4(n+i)(n+i+1)} - \frac{E_{i,i}}{2} + \frac{(2\ell-i)^2E_{i,i+1}}{4(2\ell+n-i)(2\ell+n-i+1)}\right],$$

$$Y_n \;=\; \sum_{i=0}^{2\ell}\frac{n^2(2\ell+n+1)^2}{16(n+i)(n+i+1)(2\ell+n-i)(2\ell+n-i+1)}E_{i,i}.$$

Proof. This is a straightforward computation using Lemma 5.5.1 and Lemma 5.5.2. The calculation of X_n is straightforward from Lemma 5.5.1 and Lemma 5.5.2. In order to calculate Y_n we need X_nR_{n-1}. A calculation shows

$$\begin{aligned} X_nR^n_{n-1} = &-\sum_{j=0}^{2\ell}\frac{nj^2(j-1)}{16(n+j-1)(n+j)(n+j+1)}E_{j,j-2} \\ &+\sum_{j=0}^{2\ell}\frac{nj(n+2j+1)}{8(n+j)(n+j+1)}E_{j,j-1}+ \\ &\sum_{j=0}^{2\ell}\Bigg(\frac{-i^2n(2\ell-i+1)}{16(n+i)(n+i+1)(2\ell-i+1+n)} - \frac{n}{4} - \\ &\frac{(2\ell-i)^2(i+1)n}{16(2\ell-i+1+n)(2\ell-i+n)(n+i+1)}\Bigg)E_{jj} \\ &-\sum_{j=0}^{2\ell}\frac{n(2\ell-i)(4\ell-2j+n+1)}{8(2\ell+n-j)(2\ell+n-j+1)}E_{j,j+1} \\ &+\sum_{j=0}^{2\ell}\frac{n(2\ell-j)^2(2\ell-j+1)}{16(2\ell-j+n-1)(2\ell-j+n)(2\ell-j+n+1)}E_{j,j+2} \end{aligned}$$

Now Lemma 5.5.1 and a computation show that Y_n reduces to a tridiagonal matrix. □

Now (5.12) and Theorem 5.5.3 give the three-term recurrence

$$x\, P_n(x) \;=\; P_{n+1}(x) \;+\; (1-2X_n)\, P_n(x) \;+\; 4Y_nP_{n-1}(x) \tag{5.21}$$

for the monic orthogonal polynomials with respect to the matrix valued weight W on $[-1,1]$. The case $\ell = 0$ corresponds to the three-term recurrence for the monic Chebyshev polynomials U_n. Note moreover, that $\lim_{n\to\infty} X_n = \frac{1}{2}$ and $\lim_{n\to\infty} Y_n = \frac{1}{16}$, so that the monic matrix valued orthogonal polynomials fit in the Nevai class, see [Dur99]. Note the matrix valued orthogonal polynomials P_n in this paper are considered as matrix valued analogues of the Chebyshev polynomials of the second kind, because of the group theoretic interpretation Chapter 4 and Section 5.7, but that these polynomials are not matrix valued Chebyshev polynomials in the sense of [Dur99, §3].

Using the three-term recurrence relation (5.21) and (5.2) we get

$$4Y_n H_{n-1} = \int_{-1}^{1} xP_n(x)W(x)P_{n-1}(x)^* dx = \int_{-1}^{1} P_n(x)W(x)(xP_{n-1}(x))^* dx = H_n \quad (5.22)$$

analogous to the scalar-valued case. Since H_0 is determined in (5.3) we obtain H_n.

Corollary 5.5.4. *The squared norm matrix H_n is*

$$(H_n)_{ij} = \delta_{ij} \frac{\pi}{2} \frac{(n!)^2\,(2\ell+1)_{n+1}^2}{(i+1)_n^2\,(2\ell-i+1)_n^2} \frac{2^{-2n}}{(n+i+1)(2\ell-i+n+1)}$$

and $JH_nJ = H_n$.

In Theorem 4.4.8 we have stated the three-term recurrence relation for the polynomials $Q_n^\ell(a)$, $a \in A_*$, see also Section 5.7 of this paper. Apart from a relabeling of the orthonormal basis the monic polynomials corresponding to Q_n^ℓ are precisely the polynomials P_n, see (4.40) for the precise identification

$$P_d(x)_{n,m} = \Upsilon_d^{-1} Q_d(a_{\arccos x})_{-\ell+n,-\ell+m}, \quad n,m \in \{0,1,\dots,2\ell\}, \qquad (5.23)$$

see also Section 5.7, where Υ_d in (5.23) is the leading coefficient of Q_d^ℓ.

Corollary 5.5.5. *The polynomials Q_n^ℓ as in Section 4.4 satisfy the recurrence*

$$\phi(a)\, Q_n^\ell(a) = A_n\, Q_{n+1}^\ell(a) + B_n\, Q_n^\ell(a) + C_n\, Q_{n-1}^\ell(a)$$

where

$$\begin{aligned}
A_n &= \sum_{q=-\ell}^{\ell} \frac{(n+1)(2\ell+n+2)}{2(\ell-q+n+1)(\ell+q+n+1)} E_{q,q}, \\
B_n &= \sum_{q=-\ell}^{\ell} \frac{(\ell-q+1)(\ell+q)}{2(\ell-q+n+1)(\ell+q+n+1)} E_{q,q-1} \\
&\quad + \sum_{q=-\ell}^{\ell} \frac{(\ell+q+1)(\ell-q)}{2(\ell-q+n+1)(\ell+q+n+1)} E_{q,q+1} \\
C_n &= \sum_{q=-\ell}^{\ell} \frac{n(2\ell+n+1)}{2(n+q+\ell+1)(n-q+\ell+1)} E_{q,q}.
\end{aligned}$$

Proof. We use $A_n = \Upsilon_n \Upsilon_{n+1}^{-1}$, $B_n = \Upsilon_n(1-2X_n)\Upsilon_n^{-1}$, $C_n = \Upsilon_n(4Y_n)\Upsilon_{n-1}^{-1}$ and Theorem 5.5.3 to obtain the result from a straighforward computation. The matrices Υ_n are given by

$$(\Upsilon_n)_{p,q} = \delta_{pq}\, 2^n \frac{(\ell-q+1)_n\, (\ell+q+1)_n}{n!\,(2\ell+2)_n} \tag{5.24}$$

which follows from [Koo85, (3.10), (3.16)] where we have to bear in mind that the polynomials in [Koo85] differ from ours by an application of J. Note that Chapter 4 does not give this value for Υ_n. □

Recall from Theorem 4.4.8 and (4.7) that the matrix entries of the matrices A_n, B_n and C_n are explicitly known as a square of a double sum with summand the product of four Clebsch-Gordan coefficients, hence Corollary 5.5.5 leads to an explicit expression for this square.

5.6 The matrix valued orthogonal polynomials related to Gegenbauer and Racah polynomials

The LDU-decomposition of the weight W of Theorem 5.2.1 has the weight functions of the Gegenbauer polynomials in the diagonal T, so we can expect a link between the matrix valued polynomials $P_n(x)L(x)$ and the Gegenbauer polynomials. We cannot do this via the orthogonality relations and the weight function, since the matrix L also depends on x. Instead we use an approach based on the differential operators $\tilde{D}$ and $\tilde{E}$ of Section 5.3, and because of the link to the matrix valued hypergeometric differential operator as in Theorem 5.4.5 we switch to the matrix valued orthogonal polynomials R_n and $x = 1-2u$. It turns out that the matrix entries of $P_n(x)L(x)$ can be given as a product of a Racah polynomial times a Gegenbauer polynomial, see Theorem 5.6.2.

We use the differential operators D and E of Corollary 5.4.1, and as in Section 5.4 it is handier to work with the second-order differential operator $D_{-2\ell} = D - 2\ell E$. By Theorem

5.2.1 we have $W(x) = L(x)T(x)L(x)^t$, hence $Z(u) = L(1-2u)T(1-2u)L(1-2u)^t$. For this reason we look at the differential operator conjugated by $M(u) = L(1-2u)$.

In general, for $D = \frac{d^2}{du^2}F_2(u) + \frac{d}{du}F_1(u) + F_0(u)$ a second order matrix valued differential operator, conjugation with the matrix valued function M, which we assume invertible for all u, gives

$$M^{-1}DM = \frac{d^2}{du^2}M^{-1}F_2M + \frac{d}{du}\left(M^{-1}F_1M + 2\frac{dM^{-1}}{du}F_0M\right) + \left(M^{-1}F_0M + \frac{dM^{-1}}{du}F_1M + \frac{d^2M^{-1}}{du^2}F_0M\right).$$

Note that differentiating $M^{-1}M = I$ gives $\frac{dM^{-1}}{du} = -M^{-1}\frac{dM}{du}M^{-1}$, and similarly we find $\frac{d^2M^{-1}}{du^2} = -M^{-1}\frac{d^2M}{du^2}M^{-1} + 2M^{-1}\frac{dM}{du}M^{-1}\frac{dM}{du}M^{-1}$. We are investigating the possibility of $M^{-1}DM$ being a diagonal matrix valued differential operator. We now assume that $F_2(u) = u(1-u)$, so that $M^{-1}F_2M = u(1-u)$. A straightforward calculation using this assumption and the calculation of the derivatives of M^{-1} shows that $M^{-1}DM = u(1-u)\frac{d^2}{du^2} + \frac{d}{du}T_1 + T_0$ with T_0 and T_1 matrix valued functions if and only if the following equations (5.25), (5.26) hold:

$$F_0M - \frac{dM}{du}T_1 - u(1-u)\frac{d^2M}{du^2} = MT_0 \tag{5.25}$$

$$F_1M - 2u(1-u)\frac{dM}{du} = MT_1. \tag{5.26}$$

Of course, T_0 and T_1 need not be diagonal in general, but this is the case of interest.

Proposition 5.6.1. *The differential operator $\mathcal{D} = M^{-1}D_{-2\ell}M$ is the diagonal differential operator*

$$\mathcal{D} = u(1-u)\frac{d^2}{du^2} + \left(\frac{d}{du}\right)T_1(u) + T_0,$$

where

$$T_1(u) = \frac{1}{2}T_1^1 - uT_1^1, \quad T_1^1 = \sum_{i=0}^{2\ell}(2i+3)E_{i,i}, \qquad T_0 = \sum_{i=0}^{2\ell}(2\ell-i)(2\ell+i+2)E_{i,i}$$

Moreover, $\mathcal{R}_n(u) = R_n(u)M(u)$ satisfies

$$\mathcal{R}_n\,\mathcal{D} = \Lambda_n(\mathcal{D})\,\mathcal{R}_n, \qquad \Lambda_n(\mathcal{D}) = \Lambda_n(D) - 2\ell\,\Lambda_n(E).$$

The proof shows that $M^{-1}D_\alpha M$ can only be a diagonal differential operator for $\alpha = -2\ell$. Note that $\mathcal{D}$ is a matrix valued differential operator as considered by Tirao, see Remark 5.4.2 and [Tir03], and diagonality of $\mathcal{D}$ implies that the matrix valued hypergeometric ${}_2H_1$-series can be given explicitly in terms of (usual) hypergeometric series. In particular, we find as in the proof of Theorem 5.4.5 that

$$\big(\mathcal{R}_n(u)\big)_{kj} = \left({}_2H_1 \left(\begin{matrix} T_1^1,\ \lambda_n(k) - T_0 \\ \frac{1}{2}T_1^1 \end{matrix} ; u \right) v \right)_j^t ,$$

$$v_k = \big(\mathcal{R}_n(0)\big)_{kj}, \quad \lambda_n(k) = \Lambda_n(\mathcal{D})_{kk}, \quad (5.27)$$

since the condition $\sigma(\frac{1}{2}T_1^1) \not\subset -\mathbb{N}$ is satisfied.

Proof. Consider $D_\alpha = D + \alpha E$, so that $F_2(u) = u(1-u)$ and the above considerations apply and $F_1(u) = C_\alpha - uU_\alpha$, and $F_0 = -V_\alpha$. We want to find out if we can obtain matrix valued functions T_1 and T_0 satisfying (5.25), (5.26) for this particular F_1, F_2 and $M(u) = L(1-2u)$. Since F_0 is diagonal, and assuming that T_0, T_1 can be taken diagonal it is clear that taking the (k,l)-th entry of (5.25) leads to

$$(F_0)_{kk}M_{kl} - \frac{dM_{kl}}{du}(T_1)_{ll} - u(1-u)\frac{d^2M_{kl}}{du^2} = M_{kl}(T_0)_{ll}. \qquad (5.28)$$

By Theorem 5.2.1 we have $M_{kl} = 0$ for $l > k$ and for $l \leq k$

$$M_{kl}(u) = \binom{k}{l}\, {}_2F_1 \left(\begin{matrix} l-k, k+l+2 \\ l+\frac{3}{2} \end{matrix} ; u \right)$$

so that (5.28) has to correspond to the second order differential operator

$$u(1-u)f''(u) + \big(c - (a+b+1)u\big)f'(u) - abf(u) = 0, \qquad f(u) = {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; u \right) \quad (5.29)$$

for the hypergeometric function. This immediately gives

$$(T_1)_{ll} = l + \frac{3}{2} - (2l+3)u, \qquad (T_0)_{ll} - (F_0)_{kk} = k^2 + 2k - (l^2 + 2l).$$

Since $(F_0)_{kk} = (-V_\alpha)_{kk} = -k^2 + (2\ell + \alpha\frac{(2\ell+2)}{2\ell})k - \alpha(2\ell+2)$, this is only possible for $\alpha = -2\ell$, and in that case

$$(T_1)_{ll} = l + \frac{3}{2} - (2l+3)u, \qquad (T_0)_{ll} = -l^2 - 2l + 2\ell(2\ell+2). \qquad (5.30)$$

It remains to check that for $\alpha = -2\ell$ the condition (5.26) is valid with the explicit values (5.30). For $\alpha = -2\ell$ the matrix valued function F_1 is lower triangular instead of tridiagonal, so that (5.26) is an identity in the subalgebra of lower triangular matrices. With the explicit expression for M we have to check that

$$\big((\frac{3}{2} + k) - u(3+2k)\big)M_{kl} - kM_{k-1,l} - 2u(1-u)\frac{dM_{kl}}{du} = M_{kl}\big(\frac{3}{2} + l - u(3+2l)\big)$$

which can be identified with the identity

$$(1-x^2)\,\frac{dC_{k-l}^{(l+1)}}{dx}(x) = (k+l+1)C_{k-l-1}^{(l+1)}(x) - x(k-l)C_{k-l}^{(l+1)}(x).$$

In turn, this identity can be easily obtained from [AS92, (22.7.21)] or from [Ism09, (4.5.3), (4.5.7)]. □

Since R_n and M are polynomial, Proposition 5.6.1 and the explicit expression for the eigenvalue matrix in Corollary 5.4.1 imply that $(\mathcal{R}_n)_{kj}$ is a polynomial solution to

$$\begin{aligned} u(1-u)f''(u) &+ \left((j+\frac{3}{2}) - u(2j+3)\right) f'(u) + (2\ell-j)(2\ell+j+2)f(u) \\ &= \Big(-n(n-1) - n(2\ell+3) + k(2\ell-k) + 2n(\ell-k) - (2\ell+2)(k-2\ell)\Big) f(u) \end{aligned}$$

which can be rewritten as

$$u(1-u)f''(u) + \left((j+\frac{3}{2}) - u(2j+3)\right) f'(u) - (j-k-n)(n+k+j+2)f(u) = 0$$

which is the hypergeometric differential operator for which the polynomial solutions are uniquely determined up to a constant. This immediately gives

$$\mathcal{R}_n(u)_{kj} = c_{kj}(n)\ {}_2F_1\left(\begin{matrix} j-k-n,\ n+k+j+2 \\ j+\frac{3}{2} \end{matrix};u\right). \tag{5.31}$$

for $j-k-n \leq 0$ and $\mathcal{R}_n(u)_{kj} = 0$ otherwise. The case $n=0$ corresponds to Theorem 5.2.1 and we obtain $c_{kj}(0) = \binom{k}{j}$. It remains to determine the constants $c_{kj}(n)$ in (5.31).

First, switching to the variable x, we find

$$\big(\mathcal{P}_n(x)\big)_{kj} = \big(P_n(x)L(x)\big)_{kj} = (-2)^n\, c_{kj}(n) \frac{(n+k-j)!}{(2j+2)_{n+k-j}} C^{(j+1)}_{n+k-j}(x) \tag{5.32}$$

so that by (5.32) the orthogonality relations (5.2) and (5.5) give

$$\begin{aligned} \delta_{nm}(H_n)_{kl} = (-2)^{n+m} \sum_{j=0}^{2\ell\wedge(n+k)} c_{kj}(n)\overline{c_{lj}(m)}c_j(\ell)\frac{(n+k-j)!}{(2j+2)_{n+k-j}} \times \\ \delta_{n+k,m+l}\frac{\sqrt{\pi}\,\Gamma(j+\frac{3}{2})}{(n+k+1)\,j!}. \end{aligned}$$

Using the explicit value for $c_j(\ell)$ as in Theorem 5.2.1 and Corollary 5.5.4 we find orthogonality relations for the coefficients $c_{kj}(n)$:

$$\begin{aligned} (H_n)_{kk}\, 2^{-2n}\, \delta_{nm} = \sum_{j=0}^{2\ell\wedge(n+k)} c_{kj}(n)\overline{c_{k+m-n,j}(m)} \times \\ \frac{(j!)^2\,(2j+1)\,(2\ell+j+1)!\,(2\ell-j)!\,(n+k-j)!}{(n+k+j+1)!\,(n+k+1)\,(2\ell)!^2} \end{aligned} \tag{5.33}$$

Note that we can also obtain recurrence relations for the coefficients $c_{kj}(n)$ using the three-term recurrence relation of Theorem 5.5.3.

Theorem 5.6.2. *The polynomials* $\mathcal{R}_n(u) = R_n(u)M(u)$ *satisfy*

$$\begin{aligned}\mathcal{R}_n(u)_{kj} \;&=\; c_{k,0}(n)(-1)^j \frac{(-2\ell)_j\,(-k-n)_j}{j!\,(2\ell+2)_j}\times \\ &{}_4F_3\left(\begin{matrix}-j, j+1, -k, -2\ell-n-1\\ 1, -k-n, -2\ell\end{matrix};1\right)\,{}_2F_1\left(\begin{matrix}j-k-n,\, n+k+j+2\\ j+\frac{3}{2}\end{matrix};u\right)\end{aligned}$$

with $\mathcal{R}_n(u)_{kj} = 0$ *for* $j-k-n>0$ *and*

$$c_{k,0}(n) \;=\; (-1)^n 4^{-n}\frac{n!\,(2\ell+2)_n}{(k+1)_n\,(2\ell-k+1)_n}$$

We view Theorem 5.6.2 as an extension of Theorem 5.2.1, but Theorem 5.2.1 is instrumental in the proof of Theorem 5.6.2. Since the inverse of $M(u)$, or of $L(x)$, does not seem to have a nice explicit expression we do not obtain an interesting expression for the matrix elements of the matrix valued monic orthogonal polynomials $R_n(u)$ or of $P_n(x)$. Note also that the case $\ell = 0$ gives back the hypergeometric representation of the Chebyshev polynomials of the second kind U_n, see (5.20).

Comparing with (5.8) we see that we can view the ${}_4F_3$-series in Theorem 5.6.2 as a Racah polynomial $R_k(\lambda(j); -2\ell-1, -k-n-1, 0, 0)$, respectively $R_{k+n-m}(\lambda(j); -2\ell-1, -k-n-1, 0, 0)$, see (5.8), where the N of the Racah polynomials equals 2ℓ in case $2\ell \le k+n$ and N equals $k+n$ in case $2\ell \ge k+n$. Using the first part of Theorem 5.6.2 we see that the orthogonality relations (5.33) lead to

$$\begin{aligned}(H_n)_{kk}\,2^{-2n}\,\delta_{nm} \;&=\; \frac{\pi}{2}\frac{2\ell+1}{(n+k+1)^2}|c_{k0}(n)|^2\times \\ &\sum_{j=0}^{2\ell\wedge(n+k)} \frac{(2j+1)\,(-2\ell)_j\,(-n-k)_j}{(2\ell+2)_j\,(n+k+2)_j}\times \\ &R_k(\lambda(j); -2\ell-1, -k-n-1, 0, 0)\,R_{k+n-m}(\lambda(j); -2\ell-1, -k-n-1, 0, 0), \quad (5.34)\end{aligned}$$

which corresponds to the orthogonality relations for the corresponding Racah polynomials, see [AAR99, p. 344], [KS98, §1.2]. From this we find that the sum in (5.34) equals

$$\delta_{nm}\,\frac{(2\ell+1)(n+k+1)}{(2\ell+1+n-k)}.$$

Hence,

$$\begin{aligned}|c_{k0}(n)|^2 \;&=\; (H_n)_{kk}\,2^{-2n}\,\frac{2}{\pi}\frac{(n+k+1)(2\ell+1+n-k)}{(2\ell+1)^2} \;= \\ &4^{-2n}\frac{(n!)^2\,(2\ell+2)_n^2}{(k+1)_n^2\,(2\ell-k+1)_n^2} \qquad (5.35)\end{aligned}$$

using Corollary 5.5.4.

We end this section with the proof of Theorem 5.6.2. The idea of the proof is to obtain a three-term recurrence for the coefficients $c_{kj}(n)$ with explicit initial conditions, and to compare the resulting three-term recurrence with well-known recurrences for Racah polynomials, see [AAR99], [Ism09], [KS98]. The three-term recurrence relation is obtained using the first-order differential operator E and the fact that the $\mathcal{R}_n$, being analytic eigenfunctions to $\mathcal{D}$, are completely determined by the value at 0, see Remark 5.4.2.

Proof.[Proof of Theorem 5.6.2] Since the matrix valued differential operators D and E commute and have the matrix valued orthogonal polynomials R_n as eigenfunctions by Corollary 5.4.1, we see that $\mathcal{E} = M^{-1}EM$ satisfies

$$\mathcal{E}\,\mathcal{R}_n = \Lambda_n(\mathcal{E})\,\mathcal{R}_n, \quad \Lambda_n(\mathcal{E}) = \Lambda_n(E), \qquad \mathcal{E}\,\mathcal{D} = \mathcal{D}\,\mathcal{E} \tag{5.36}$$

Moreover, in the same spirit as the proof of Proposition 5.6.1 we obtain

$$\begin{aligned}
\mathcal{E} &= \left(\frac{d}{du}\right) S_1(u) + S_0(u), \\
S_1(u) &= u(1-u)\sum_{i=0}^{2\ell} \frac{i^2(2\ell+i+1)}{\ell(2i-1)(2i+1)}\, E_{i,i-1} - \sum_{i=0}^{2\ell} \frac{(2\ell-i)}{4\ell}\, E_{i,i+1}, \\
S_0(u) &= (1-2u)\sum_{i=0}^{2\ell} \frac{i^2(2\ell+i+1)}{2\ell(2i-1)}\, E_{i,i-1} + \sum_{i=0}^{2\ell} \frac{i(i+1)-4\ell(\ell+1)}{2\ell}\, E_{i,i}
\end{aligned} \tag{5.37}$$

by a straightforward calculation.

Define the vector space of (row-)vector valued functions

$$V(\lambda) = \{F \text{ analytic at } u=0 \mid F\mathcal{D} = \lambda F\},$$

and $\nu\colon V(\lambda) \to \mathbb{C}^{2\ell+1}$, $F \mapsto F(0)$, is an isomorphism, see Remark 5.4.2 and [Tir03]. Because of (5.36) we have the following commutative diagram

$$\begin{array}{ccc}
V(\lambda) & \xrightarrow{\mathcal{E}} & V(\lambda) \\
{\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\nu} \\
\mathbb{C}^{2\ell+1} & \xrightarrow{N(\lambda)} & \mathbb{C}^{2\ell+1}
\end{array}$$

with $N(\lambda)$ a linear map. In order to determine $N(\lambda)$ we note that $F \in V(\lambda)$ can be written as, cf (5.27),

$$F_j(u) = \left({}_2H_1\left(\begin{matrix} T_1^1,\ \lambda - T_0 \\ \frac{1}{2}T_1^1 \end{matrix} ; u\right) F(0)^t \right)_j^t,$$

so that $\frac{dF_j}{du}(0) = F(0)(\lambda - T_0)(\frac{1}{2}T_1^1)^{-1}$ by construction of the ${}_2H_1$-series, see Remark 5.4.2. Now (5.36) gives

$$N(\lambda) = (\lambda - T_0)(\frac{1}{2}T_1^1)^{-1} S_1(0) + S_0(0)$$

acting from the right on row-vectors from $\mathbb{C}^{2\ell+1}$.

By Proposition 5.6.1 we have that the k-th row $\big((\mathcal{R}_n)_{kj}(\cdot)\big)_{j=0}^{2\ell}$ of R_n is contained in $V(\lambda_n(k))$, see (5.27). On the other hand, the k-th row of $\mathcal{R}_n$ is an eigenfunction of $\mathcal{E}$ for the eigenvalue $\mu_n(k) = \Lambda_n(\mathcal{E})_{kk}$. Since $\nu\Big(\big((\mathcal{R}_n)_{kj}\big)_{j=0}^{2\ell}\Big) = (c_{kj}(n))_{j=0}^{2\ell}$ we see that the row-vector $c_k = (c_{kj}(n))_{j=0}^{2\ell}$ satisfies $c_k N(\lambda_n(k)) = \mu_n(k)\, c_k$, which gives the recurrence relation

$$
\begin{aligned}
&-\frac{(i+k+n+1)(i-k-n-1)(2\ell-i+1)}{(2i+1)} c_{k,i-1}(n)\\
&\qquad + (i(i+1) - 4\ell(\ell+1))c_{k,i}(n) + \frac{(i+1)^2(2\ell+i+2)}{(2i+1)} c_{k,i+1}(n)\\
&\qquad\qquad = (-2n(\ell-k) + (2\ell+2)(k-2\ell))c_{k,i}(n), \qquad (5.38)
\end{aligned}
$$

with the convention $c_{k,-1}(n) = 0$. Note that $c_{jk}(0) = \binom{k}{j}$ indeed satisfies (5.38). Comparing (5.38) with the three-term recurrence relation for the Racah polynomials or the corresponding contiguous relation for balanced ${}_4F_3$-series, see e.g. [AAR99, p. 344], [KS98, §1.2], gives

$$
c_{kj}(n) = c_{k,0}(n)(-1)^j \frac{(-2\ell)_j\,(-k-n)_j}{j!\,(2\ell+2)_j}\; {}_4F_3\left(\begin{matrix} -j, j+1, -k, -2\ell-n-1 \\ 1, -k-n, -2\ell \end{matrix}; 1\right)
$$

and $c_{kj}(n) = 0$ for $j > k+n$.

It remains to determine the constants $c_{k0}(n)$, and we have already determined their absolute values in (5.35) by matching it to the orthogonality relations for Racah polynomials. From the three-term recurrence relation Theorem 5.5.3 we see that the constants $c_{kj}(n)$ are all real, so it remains to determine the sign of $c_{k0}(n)$. Theorem 5.5.3 gives a three-term recurrence for $\mathcal{R}_n(u)$, and taking the $(k,0)$-th matrix entry gives a polynomial identity in u using (5.31). Next taking the leading coefficient gives the recursion

$$
c_{k0}(n+1) = -\frac{(n+k+2)}{4(n+k+1)}\, c_{k0}(n) + \frac{(2\ell-k)^2}{4(2\ell+n-k)(2\ell+n-k+1)}\, c_{k+1,0}(n)
$$

and plugging in $c_{k0}(n) = \mathrm{sgn}(c_{k0}(n))|c_{k0}(n)|$ and using the explicit value for $|c_{k0}(n)|$ gives

$$
\begin{aligned}
&\mathrm{sgn}(c_{k0}(n+1))\,(n+1)(2\ell+n+2) = \\
&\qquad -\mathrm{sgn}(c_{k0}(n))(n+k+2)(2l-k+n+1) + \mathrm{sgn}(c_{k+1,0}(n))(2\ell-k)(k+1).
\end{aligned}
$$

This gives $\mathrm{sgn}(c_{k0}(n)) = \mathrm{sgn}(c_{k+1,0}(n))$ for the right hand side to factorise as in the left hand side, and then $\mathrm{sgn}(c_{k0}(n+1)) = -\mathrm{sgn}(c_{k0}(n))$. Since $c_{k0}(0) = 1$, we find $\mathrm{sgn}(c_{k0}(n)) = (-1)^n$. □

Remark 5.6.3. Theorem 5.6.2 can now be plugged into the three-term recurrence relation for $\mathcal{R}_n$ of Theorem 5.5.3, and this then gives a intricate three-term recurrence relation for Gegenbauer polynomials involving coefficients which consist of sums of two Racah polynomials. We leave this to the interested reader.

Remark 5.6.4. We sketch another approach to the proof of the value of $c_{k0}(n)$ by calculating the value $c_{k,2\ell}(n)$ in case $k+n \geq 2\ell$ or $c_{k,n+k}(n)$ in case $k+n < 2\ell$. For instance, in case $k+n \geq 2\ell$ we have

$$(\mathcal{R}_n(u))_{k,2\ell} = (R_n(u)M(u))_{k,2\ell} = (R_n(u))_{k,2\ell} = (R_n(u))_{2\ell-k,0}$$

using that M is a unipotent lower-triangular matrix valued polynomial and the symmetry $JR_n(u)J = R_n(u)$, see Section 4.5. Now the leading coefficient of the right hand side can be calculated using Theorem 5.4.5, and combining with (5.31), the value $c_{k,2\ell}(n)$ follows. Then the recurrence (5.38) can be used to find $c_{k0}(n)$.

5.7 Group theoretic interpretation

The purpose of this section is to give a group theoretic derivation of Theorem 5.3.1 complementing the analytic derivation of Section 4.7. For this we need to recall some of the results of Chapter 4.

5.7.1 Group theoretic setting of the matrix valued orthogonal polynomials

In this subsection we recall the construction of the matrix valued orthogonal polynomials and the corresponding weight starting from the pair $(\mathrm{SU}(2)\times\mathrm{SU}(2), \mathrm{SU}(2))$ and an SU(2)-representation T^ℓ. Then we discuss how the differential operators come into play and what their relation is with the matrix valued orthogonal polynomials. The goal of this section is to provide a map of the relevant differential operators in the group setting to the relevant differential operators for the matrix valued orthogonal polynomials in Theorem 5.7.8.

Let $U = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $K = \mathrm{SU}(2)$ diagonally embedded in U. Note that K is the set of fixed points of the involution $\theta : U \to U : (x, y) \mapsto (y, x)$. The irreducible representations of U and K are denoted by T^{ℓ_1,ℓ_2} and T^ℓ as is explained in Section 4.2. The representation space of T^ℓ is denoted by H^ℓ which is a $2\ell+1$-dimensional vector space. If T^ℓ occurs in T^{ℓ_1,ℓ_2} upon restriction to K we defined the spherical function $\Phi^\ell_{\ell_1,\ell_2}$ in Definition 4.2.2 as the T^ℓ-isotypical part of the matrix T^{ℓ_1,ℓ_2}. Let $A \subset U$ be the subgroup

$$A = \left\{ a_t = \left(\begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}, \begin{pmatrix} e^{-it/2} & 0 \\ 0 & e^{it/2} \end{pmatrix} \right), 0 \leq t < 4\pi \right\}$$

and let $M = Z_K(A)$. Recall the decomposition $U = KAK$, [Kna02, Thm. 7.38]. The restricted spherical functions $\Phi^\ell_{\ell_1,\ell_2}|_A$ take their values in $\mathrm{End}_M(H^\ell)$, see Proposition 4.2.4. Since $\mathrm{End}_M(H^\ell) \cong \mathbb{C}^{2\ell+1}$ this allows allows us to view the restricted spherical functions as being $\mathbb{C}^{2\ell+1}$-valued. The parametrization of the U-representations that contain T^ℓ indicates how to gather the restricted spherical functions. Following Figure 4.3 we write $(\ell_1, \ell_2) = \zeta(d, h)$ with $\zeta(d, h) = (\frac{1}{2}(d+\ell+h), \frac{1}{2}(d+\ell-h))$. Here $d \in \mathbb{N}$ and

$h \in \{-\ell, -\ell+1, \dots, \ell\}$. We recall the definition of the full spherical functions of type ℓ, Definition 4.4.2.

Definition 5.7.1. *The full spherical function of type ℓ and degree d is the matrix valued function $\Phi^\ell_d : A \to \mathrm{End}(\mathbb{C}^{2\ell+1})$ whose j-th row is the restricted spherical function $\Phi^\ell_{\ell_1,\ell_2}$ with $(\ell_1,\ell_2) = \zeta(d,j)$.*

The full spherical function of degree zero has the remarkable property of being invertible on the subset $A_{reg} := \{a_t | t \in (0,\pi) \cup (\pi, 2\pi) \cup (2\pi, 3\pi) \cup (3\pi, 4\pi)\}$, which was first proved by Koornwinder [Koo85, Prop. 3.2]. The invertibility follows also from Corollary 5.2.3. Let $\phi = \Phi^0_{1/2,1/2}$ be the minimal nontrivial zonal spherical function of Section 4.3. Together with the recurrence relations for the full spherical functions with ϕ, Proposition 4.3.1 this gives rise to the full spherical polynomials from Definition 4.4.4.

Definition 5.7.2. *The full spherical polynomial $Q^\ell_d : A \to \mathrm{End}(\mathbb{C}^{2\ell+1})$ is defined by $Q^\ell_d(a) = (\Phi^\ell_0(a))^{-1}\Phi^\ell_d(a)$.*

The name full spherical polynomial comes from the fact that the Q^ℓ_d are polynomials in ϕ. The full spherical polynomials Q^ℓ_d are orthogonal with respect to

$$\langle Q, P\rangle_{V^\ell} = \int_A Q(a)V^\ell(a)P(a)da, \quad V^\ell(a_t) = (\Phi^\ell_0(a_t))^*\Phi^\ell_0(a_t)\sin^2 t,$$

see Corollary 4.4.7.

In Section 4.5 we studied the weight functions V^ℓ extensively. It turns out that the matrix entries are polynomials in the function ϕ, apart from the common factor $\sin t$. Upon changing the variable $x = \phi(a)$ we obtain the following system of matrix valued orthogonal polynomials.

Definition 5.7.3. *Let $R^\ell_d : [0,1] \to \mathrm{End}(\mathbb{C}^{2\ell+1})$ be the polynomial defined by $R^\ell_d(\phi(a)) = Q^\ell_d(a)$. The degree of R^ℓ_d is d. The polynomials are orthogonal with respect to*

$$\langle R, P\rangle_{W^\ell} = \int_{-1}^1 R(x)W^\ell(x)P(x)\, dx,$$

where W^ℓ is defined by $W^\ell(\phi(a))d\phi = V^\ell(a)da$.

The weight $W^\ell(x)$ from Definition 5.7.3 is the same as the weight defined in (5.1) where we have to bear in mind that the basis is parametrised differently. The matrix valued polynomials R^ℓ_d correspond to the family $\{P_d\}_{d\geq 0}$ from Theorem 5.3.1 by means of making the R^ℓ_d monic. Given a system of matrix valued orthogonal polynomials as in Definition 5.7.3 it is of great interest to see whether there are interesting differential operators. More precisely we define the algebra $\mathcal{D}(W^\ell)$ as the algebra of differential operators that are self adjoint with respect to the weight W^ℓ and that have the R^ℓ_d as eigenfunctions. We define a map that associates to a certain left invariant differential operator on the group U an element in $\mathcal{D}(W)$.

Before we go into the construction we observe that the spherical functions may also be defined on the complexification $A^{\mathbb{C}}$, using Weyl's unitary trick. Indeed, all the representations that we consider are finite dimensional and unitary, so they give holomorphic representations of the complexifications $U^{\mathbb{C}}$ and $K^{\mathbb{C}}$.

A great part of the constructions that we are about to consider follows from Casselman and Miličić [CM82], where the differential operators act from the left. In this section we follow this convention, except that we transpose the results at the end in order to obtain the proof of Theorem 5.3.1 where the differential operators act from the right.

Let $U(\mathfrak{u}^{\mathbb{C}})$ be the universal enveloping algebra for the complexification $\mathfrak{u}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{u}$ of the group $U = \mathrm{SU}(2) \times \mathrm{SU}(2)$. Let $\theta\colon U(\mathfrak{u}^{\mathbb{C}}) \to U(\mathfrak{u}^{\mathbb{C}})$ be the flip on simple tensors extending the Cartan involution $\theta\colon \mathfrak{su}(2)\times\mathfrak{su}(2) \to \mathfrak{su}(2)\times\mathfrak{su}(2)$, $(X, Y) \mapsto (Y, X)$. Recall $\mathfrak{k} \cong \mathfrak{su}(2)$ is the fixed-point set of θ. Let $U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$ denote the subalgebra of elements that commute with $\mathfrak{k}^{\mathbb{C}}$. Let $\mathfrak{Z}$ denote the center of $U(\mathfrak{k}^{\mathbb{C}})$.

Lemma 5.7.4. $U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}} \cong \mathfrak{Z} \otimes \mathfrak{Z} \otimes \mathfrak{Z}$.

Proof. From [Kno90, Satz 2.1 and Satz 2.3] it follows that $U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}} \cong \mathfrak{Z} \otimes_{Z(\mathfrak{j})} (\mathfrak{Z} \otimes \mathfrak{Z})$ where $\mathfrak{j} \subset \mathfrak{u}^{\mathbb{C}}$ is the largest ideal of $\mathfrak{u}^{\mathbb{C}}$ contained in $\mathfrak{k}^{\mathbb{C}}$. Since $\mathfrak{j} = 0$ the result follows. □

Proposition 5.7.5. *The elements of the algebra $U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$ have the spherical functions $\Phi^{\ell}_{\ell_1,\ell_2}$ as eigenfunctions. This remains true when we extend $\Phi^{\ell}_{\ell_1,\ell_2}$ to $U^{\mathbb{C}}$.*

Proof. See [War72b, Thm. 6.1.2.3]. The second statement follows from Weyl's unitary trick. □

The spherical functions $\Phi^{\ell}_{\ell_1,\ell_2}$ have T^{ℓ}-transformation behaviour:

$$\Phi^{\ell}_{\ell_1,\ell_2}(k_1 u k_2) = T^{\ell}(k_1)\Phi^{\ell}_{\ell_1,\ell_2}(u)T^{\ell}(k_2) \tag{5.39}$$

for all $k_1, k_2 \in K$ and $u \in U$, see Definition 4.2.2. Let $C(A)$ denote the set of continuous ($\mathbb{C}$-valued) functions on A. Casselman and Miličić [CM82] define the map

$$\Pi_\ell : U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}} \to C(A) \otimes U(\mathfrak{a}^{\mathbb{C}}) \otimes \mathrm{End}(\mathrm{End}_M(H^{\ell}))$$

and prove the following properties [CM82, Thm. 3.1,Thm. 3.3].

Theorem 5.7.6. *Let $F : U \to \mathrm{End}(H^{\ell})$ be a smooth function that satisfies (5.39). Then $(DF)|_A = \Pi_\ell(D)(F|_A)$ for all $D \in U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$. Moreover, Π_ℓ is an algebra homomorphism.*

We call $\Pi_\ell(D)$ the T^{ℓ}-radial part of $D \in U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$. In particular we have

$$\Pi_\ell(D)(\Phi^{\ell}_{\ell_1,\ell_2}|_A) = \lambda^{\ell}_{D,\ell_1,\ell_2}\Phi^{\ell}_{\ell_1,\ell_2}|_A, \quad \lambda^{\ell}_{D,\ell_1,\ell_2} \in \mathbb{C}.$$

Upon identifying $\mathrm{End}_M(H^{\ell}) \cong \mathbb{C}^{2\ell+1}$ we observe that we may view $\Pi_\ell(D)$ as a differential operator of the $\mathrm{End}(\mathbb{C}^{2\ell+1})$-valued functions that act on from the left. In particular, let $C(A, \mathrm{End}(\mathbb{C}^{2\ell+1}), T^{\ell})$ denote the vector space generated by the $\Phi^{\ell}_d, d \geq 0$. The following lemma follows immediately from the construction.

Lemma 5.7.7. *Let $D \in U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$ be self-adjoint and consider $\Pi_\ell(D)$ as a differential operator acting on $C(A, \mathrm{End}(\mathbb{C}^{2\ell+1}), T^\ell)$ from the left. Then $\Pi_\ell(D)$ is self-adjoint for $\langle\cdot,\cdot\rangle_{V^\ell}$.*

Definition 5.7.8. *Let $f : U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}} \to \mathcal{D}(W^\ell)$ be defined by sending D to the conjugation of the differential operator $\Pi_\ell(D)$ by Φ_0^ℓ followed by changing the variable $x = \phi(a)$.*

The map $f : U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}} \to \mathcal{D}(W^\ell)$ is an algebra homomorphism. It gives an abstract description of a part of $\mathcal{D}(W^\ell)$. Note that f is not surjective because in Proposition 4.8.2 we have found a differential operator that does not commute with some of the other. However, the algebra $U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$ is commutative by Lemma 5.7.4.

By means of Lemma 5.7.4 we know that $U(\mathfrak{u}^{\mathbb{C}})^{\mathfrak{k}^{\mathbb{C}}}$ has $\Omega_1 = \Omega_{\mathfrak{k}} \otimes 1$ and $\Omega_2 = 1 \otimes \Omega_{\mathfrak{k}}$ among its generators, where $\Omega_{\mathfrak{k}} \in \mathfrak{Z}$ is the Casimir operator. In the following subsection we calculate $f(\Omega_1 + \Omega_2)$ and $f(\Omega_1 - \Omega_2)$ explicitly. Upon transposing and taking suitable linear combinations we find the differential operators $\tilde{D}$ and $\tilde{E}$ from Theorem 5.3.1.

5.7.2 Calculation of the Casimir operators

The goal of this subsection is to calculate $f(\Omega)$ and $f(\Omega')$ where f is the map described in Definition 5.7.8 and where $\Omega = \Omega_1 + \Omega_2$ and $\Omega' = \Omega_1 - \Omega_2$. We proceed in a series of six steps. (1) First we provide expressions for the Casimir operators Ω and Ω' which (2) we rewrite according to the infinitesimal Cartan decomposition defined by Casselman and Miličić [CM82, §2]. These calculations are similar to those in [War72b, Prop. 9.1.2.11]. (3) From this expression we can easily calculate the T^ℓ-radial parts, see Theorem 5.7.6. The radial parts are differential operators for $\mathrm{End}_M(H^\ell)$-valued functions on A. At this point we see that we can extend matters to the complexification $A^{\mathbb{C}}$ of A as in [CM82, Ex. 3.7]. (4) We identify $\mathrm{End}_M(H^\ell) \cong \mathbb{C}^{2\ell+1}$ and rewrite the radial parts of step 3 accordingly. (5) We conjugate these differential operators with Φ_0 and (6) we make a change of variables to obtain two matrix valued differential operators $f(\Omega)$ and $f(\Omega')$. Along the way we keep track of the differential equations for the spherical functions. Finally we give expressions for the eigenvalues Λ_d and Γ_d of $f(\Omega)$ and $f(\Omega')$ such that the full spherical polynomials Q_d are the corresponding eigenfunctions. Following Casselman and Miličić [CM82, §2] the roots are considered as characters, hence written multiplicatively.

(1). First we concentrate on one factor $K \cong \mathrm{SU}(2)$, with Lie algebra $\mathfrak{k}$ and standard Cartan subalgebra $\mathfrak{t}$. The complexifications are denoted by $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{t}^{\mathbb{C}}$ and we use the standard basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\alpha = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{\alpha^{-1}} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for $\mathfrak{k}^{\mathbb{C}}$. The Casimir of K is given by $\Omega_{\mathfrak{k}} = \frac{1}{2}H^2 + 4\{E_\alpha E_{\alpha^{-1}} + E_{\alpha^{-1}} E_\alpha\}$. It is well-known that the matrix-elements of the irreducible unitary representation T^ℓ of $\mathrm{SU}(2)$ are eigenfunctions of the Casimir operator $\Omega_{\mathfrak{k}}$ for the eigenvalue $\frac{1}{2}(\ell^2 + \ell)$, see e.g. [Kna02, Thm. 5.28]. The roots of the pair $(\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{t}^{\mathbb{C}})$ are given by

$$R = \{(\alpha, 1), (\alpha^{-1}, 1), (1, \alpha), (1, \alpha^{-1})\}.$$

The positive roots are choosen as $R^+ = \{(\alpha,1),(1,\alpha^{-1})\}$, so that the two positive roots restrict to the same root $R(\mathfrak{u}^{\mathbb{C}},\mathfrak{a}^{\mathbb{C}})$ which we declare positive. The corresponding root vectors are $E_{(\alpha,1)} = (E_\alpha,0)$, etc. Define

$$E = (E_\alpha,0)(E_{\alpha^{-1}},0) + (E_{\alpha^{-1}},0)(E_\alpha,0).$$

Then we have $\Omega_1 = \frac{1}{2}(H,0)^2 + 4E$ and $\Omega_2 = \theta(\Omega_1)$. In particular, the spherical function $\Phi^\ell_{\ell_1,\ell_2}$ is an eigenfunction of Ω_i for the eigenvalue $\frac{1}{2}(\ell_i^2+\ell_i)$ for $i=1,2$.

We have $(H,0) = \frac{1}{2}((H,-H)+(H,H))$ and $(0,H) = \frac{1}{2}((H,H)-(H,-H))$ and from this we find in $U(\mathfrak{u}^{\mathbb{C}})$

$$\begin{aligned} \Omega &= \Omega_1+\Omega_2 = \frac{1}{4}(H,H)^2 + \frac{1}{4}(H,-H)^2 + 4(E+\theta(E)), \\ \Omega' &= \Omega_1-\Omega_2 = \frac{1}{2}(H,-H)(H,H) + 4(E-\theta(E)). \end{aligned} \tag{5.40}$$

(2). Following Casselman and Miličić [CM82, §2] we can express Ω and Ω' according to the infinitesimal Cartan decomposition of $U(\mathfrak{u}^{\mathbb{C}})$. Let $\beta \in R$ and denote $X_\beta = E_\beta + \theta(E_\beta) \in \mathfrak{k}^{\mathbb{C}}$. Denote $Y^a = \mathrm{Ad}(a^{-1})Y$ for $a \in A$. In [CM82, Lemma 2.2] it is proved that the equality

$$(1-\beta(a)^2)X_\beta = \beta(a)(E^a_\beta - \beta(a)E_\beta)$$

holds for all $a \in A_{reg}$. This is the key identity in a straightforward but tedious calculation to prove the following proposition which we leave to the reader.

Proposition 5.7.9. *Let $a \in A_{reg}$ and $\beta \in R^+$. Then*

$$\begin{aligned} \Omega = \frac{1}{16}\left((H,-H)^2+(H,H)^2\right) - \frac{2}{(\beta(a)^{-1}-\beta(a))^2}\{X^a_\beta X^a_{\beta^{-1}} + \\ X^a_{\beta^{-1}}X^a_\beta + X_\beta X_{\beta^{-1}} + X_{\beta^{-1}}X_\beta - (\beta(a)+\beta(a)^{-1})(X^a_\beta X_{\beta^{-1}} + X^a_{\beta^{-1}}X_\beta)\} + \\ \frac{1}{4}\frac{\beta(a)+\beta(a)^{-1}}{\beta(a)-\beta(a)^{-1}}(H,-H). \end{aligned} \tag{5.41}$$

and

$$\begin{aligned} \Omega' = \frac{1}{8}(H,H)(H,-H) + \frac{1}{4}\frac{\beta(a)+\beta(a)^{-1}}{\beta(a)-\beta(a)^{-1}}(H,H) + \\ \frac{2}{\beta(a)-\beta(a)^{-1}}(X^a_{\beta^{-1}}X_\beta - X^a_\beta X_{\beta^{-1}}), \end{aligned} \tag{5.42}$$

The calculation of (5.41) is completely analogous to [War72b, Prop. 9.1.2.11] and it is clear that (5.41) is invariant for interchanging β and β^{-1}. The expression in (5.42) is also invariant for interchanging β and β^{-1} albeit that it is less clear in this case. In either case the expressions (5.41) and (5.42) do not depend on the choice of $\beta \in R^+$.

(3). Following Casselman and Miličić [CM82, §3] we calculate the T^ℓ-radial parts of Ω and Ω'. This is a matter of applying the map Π_ℓ from Theorem 5.7.6 to the expressions

(5.41) and (5.42). At the same time we note that the coefficients in (5.41) and (5.42) are analytic functions on A_{reg}. They extend to meromorphic functions on the complexification $A^{\mathbb{C}}$ of A which we identify with $\mathbb{C}^{\times}$ using the map

$$a : \mathbb{C}^{\times} \to A^{\mathbb{C}} : w \mapsto a(w) = \left(\begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}, \begin{pmatrix} w^{-1} & 0 \\ 0 & w \end{pmatrix} \right).$$

Under this isomorphism the differential operator $(H, -H)$ translates to $w\frac{d}{dw}$. To see this let $g : A^{\mathbb{C}} \to \mathbb{C}$ be holomorphic and consider $(H, -H)g(a(w))$ which is equal to

$$(H, -H)g(a(w)) = \left\{ \frac{d}{dt} g(a(e^t w)) \right\}_{t=0} = w\frac{d}{dw}(g \circ a)(w).$$

Following [CM82], [War72b] we find the following expressions for the T^{ℓ}-radial parts of Ω and Ω';

$$\begin{aligned}\Pi_{\ell}(\Omega) = \frac{1}{16}\left(w\frac{d}{dw}\right)^2 + \frac{1}{4}\frac{w^2 + w^{-2}}{w^2 - w^{-2}} w\frac{d}{dw} + \frac{1}{16}T^{\ell}(H)^2 + \\ - \frac{2}{(w^2 - w^{-2})^2}\left\{T^{\ell}(E_{\alpha})T^{\ell}(E_{\alpha^{-1}}) \bullet + T^{\ell}(E_{\alpha^{-1}})T^{\ell}(E_{\alpha}) \bullet + \right. \\ \left. \bullet T^{\ell}(E_{\alpha})T^{\ell}(E_{\alpha^{-1}}) + \bullet T^{\ell}(E_{\alpha^{-1}})T^{\ell}(E_{\alpha})\right\} + \\ 2\frac{w^2 + w^{-2}}{(w^2 - w^{-2})^2}\left\{T^{\ell}(E_{\alpha}) \bullet T^{\ell}(E_{\alpha^{-1}}) + T^{\ell}(E_{\alpha^{-1}}) \bullet T^{\ell}(E_{\alpha})\right\},\end{aligned} \tag{5.43}$$

and

$$\begin{aligned}\Pi_{\ell}(\Omega') = \frac{1}{8}T^{\ell}(H)w\frac{d}{dw} + \frac{1}{4}\frac{w^2 + w^{-2}}{w^2 - w^{-2}}T^{\ell}(H) + \\ \frac{2}{w^2 - w^{-2}}\left\{T^{\ell}(E_{\alpha^{-1}}) \bullet T^{\ell}(E_{\alpha}) + T^{\ell}(E_{\alpha}) \bullet T^{\ell}(E_{\alpha^{-1}})\right\}\end{aligned} \tag{5.44}$$

where the bullet ($\bullet$) indicates where to put the restricted spherical function. The matrices $T^{\ell}(E_{\alpha})$ and $T^{\ell}(H)$ are easily calculated in the basis of weight vectors. Note that $T^{\ell}(E_{\alpha^{-1}}) = JT^{\ell}(E_{\alpha})J$. We give the entries of $T^{\ell}(E_{\alpha})$ in the proof of Lemma 5.7.11.

The following proposition is a direct consequence of Theorem 5.7.6 and Proposition 5.7.5.

Proposition 5.7.10. *The restricted spherical functions are eigenfunctions of the radial parts of Ω and Ω',*

$$\begin{aligned}\Pi_{\ell}(\Omega)(\Phi^{\ell_1,\ell_2}_{\ell}|_{A^{\mathbb{C}}}) &= \frac{1}{2}(\ell_1^2 + \ell_1 + \ell_2^2 + \ell_2)\Phi^{\ell_1,\ell_2}_{\ell}|_{A^{\mathbb{C}}}, \\ \Pi_{\ell}(\Omega')(\Phi^{\ell_1,\ell_2}_{\ell}|_{A^{\mathbb{C}}}) &= \frac{1}{2}(\ell_1^2 + \ell_1 - \ell_2^2 - \ell_2)\Phi^{\ell_1,\ell_2}_{\ell}|_{A^{\mathbb{C}}}.\end{aligned}$$

(4). The spherical functions $\Phi^{\ell}_{\ell_1,\ell_2}$ restricted to the torus $A^{\mathbb{C}}$ take their values in $\mathrm{End}_M(H^\ell)$ and this is a $2\ell+1$-dimensional vector space. We identify

$$\mathrm{End}_M(H^\ell) \to \mathbb{C}^{2\ell+1} : Y \mapsto Y^{\mathrm{up}}$$

to obtain functions $(\Phi^{\ell}_{\ell_1,\ell_2}|_{A^{\mathbb{C}}})^{\mathrm{up}}$. The reason for putting the diagonals up is that we want to write the differential operators as differential operators with coefficients in the function algebra on A with values in $\mathrm{End}(\mathbb{C}^{2\ell+1})$ instead of the way $\Pi_\ell(\Omega)$ and $\Pi_\ell(\Omega')$ are defined. The differential operators that are conjugated to act on $\mathbb{C}^{2\ell+1}$-valued functions are also denoted by $(\cdot)^{\mathrm{up}}$. The differential operators (5.43) and (5.44) that are defined for $\mathrm{End}_M(H^\ell)$-valued functions conjugate to differential operators $\Pi_\ell(\Omega)^{\mathrm{up}}$ and $\Pi_\ell(\Omega')^{\mathrm{up}}$ for $\mathbb{C}^{2\ell+1}$-valued functions. All the terms except for the last ones in (5.43) and (5.44) transform straightforwardly.

Lemma 5.7.11. *The linear isomorphism* $\mathrm{End}_M(H^\ell) \to \mathbb{C}^{2\ell+1} : D \mapsto D^{\mathrm{up}}$ *conjugates the linear map* $\mathrm{End}_M(H^\ell) \to \mathrm{End}_M(H^\ell) : D \mapsto T^\ell(E_\alpha) D T^\ell(E_{\alpha^{-1}})$ *to* $\mathbb{C}^{2\ell+1} \to \mathbb{C}^{2\ell+1} : D^{\mathrm{up}} \mapsto C^\ell D^{\mathrm{up}}$*, where* $C^\ell \in \mathrm{End}(\mathbb{C}^{2\ell+1})$ *is the matrix given by*

$$C^\ell_{p,j} = \frac{1}{4}(\ell+j)(\ell-j+1)\delta_{j-p,1}, \quad \ell \le p, j \le \ell.$$

Likewise, $D \mapsto T^\ell(E_{\alpha^{-1}}) D T^\ell(E_\alpha)$ *transforms to* $D^{\mathrm{up}} \mapsto J C^\ell J D^{\mathrm{up}}$*, where* J *is the anti-diagonal defined by* $J_{ij} = \delta_{i,-j}$ *with* $-\ell \le i, j \le \ell$*.*

Proof. Working with the normalized weight-basis as in [Koo85, §1] we see that $T^\ell(E_\alpha)$ is the matrix given by

$$T^\ell(E_\alpha)_{ij} = \delta_{i,i+1}\frac{\ell+i+1}{2}\sqrt{\frac{(\ell-i-2)!(\ell+i+2)!}{(\ell-i-1)!(\ell+i+1)!}}$$

and $T^\ell(E_{\alpha^{-1}}) = J T^\ell(E_\alpha) J$. The lemma follows from elementary manipulations. □

We collect the expressions for the conjugation of the differential operators (5.43) and (5.44) by the linear map $Y \mapsto Y^{\mathrm{up}}$ where we have used Lemma 5.7.11.

$$\begin{aligned}\Pi_\ell(\Omega_1+\Omega_2)^{\mathrm{up}} = {} & \frac{1}{16}\left(w\frac{d}{dw}\right)^2 + \frac{1}{4}\frac{w^2+w^{-2}}{w^2-w^{-2}}w\frac{d}{dw} + \frac{1}{16}T^\ell(H)^2 + \\ & -\frac{4}{(w^2-w^{-2})^2}\left\{T^\ell(E_\alpha)T^\ell(E_{\alpha^{-1}}) + T^\ell(E_{\alpha^{-1}})T^\ell(E_\alpha)\right\} + \\ & 2\frac{w^2+w^{-2}}{(w^2-w^{-2})^2}\left\{JC^\ell J + C^\ell\right\},\end{aligned} \tag{5.45}$$

$$\Pi_\ell(\Omega_1-\Omega_2)^{\mathrm{up}} = \frac{1}{8}T^\ell(H)w\frac{d}{dw} + \frac{1}{4}\frac{w^2+w^{-2}}{w^2-w^{-2}}T^\ell(H) + \frac{2}{w^2-w^{-2}}\left\{JC^\ell J - C^\ell\right\}. \tag{5.46}$$

The differential operators (5.45) and (5.46) also act on the full spherical functions $\Phi_d^{\ell,t}$. Collecting the eigenvalues of the columns in $\Phi_d^{\ell,t}$ in diagonal matrices we obtain the following differential equations:

$$\Pi_\ell(\Omega_1+\Omega_2)^{\mathrm{up}}\Phi_d = \Phi_d\Lambda_d, \tag{5.47}$$
$$\Pi_\ell(\Omega_1-\Omega_2)^{\mathrm{up}}\Phi_d = \Phi_d\Gamma_d, \tag{5.48}$$

where $(\Lambda_d)_{pj} = \frac{1}{4}\delta_{p,j}(d^2+j^2+2d(\ell+1)+\ell(\ell+2))$ and $(\Gamma_d)_{pj} = \frac{1}{2}\delta_{p,j}j(\ell+d+1)$. For further reference we write

$$\Pi_\ell(\Omega_1+\Omega_2)^{\mathrm{up}} = a_2(w)\frac{d^2}{dw^2}+a_1(w)\frac{d}{dw}+a_0(w), \tag{5.49}$$
$$\Pi_\ell(\Omega_1-\Omega_2)^{\mathrm{up}} = b_1(w)\frac{d}{dw}+b_0(w). \tag{5.50}$$

(5). Recall from Definition 5.7.1 that the full spherical polynomials $Q_d^{\ell,t}$ are obtained from the full spherical functions $\Phi_d^{\ell,t}$ by the description $Q_d^{\ell,t} = (\Phi_0^{\ell,t})^{-1}\Phi_d^{\ell,t}$. We conjugate the differential operators (5.45) and (5.46) with Φ_0 to obtain differential operators to which the polynomials Q_d are eigenfunctions. We need a technical lemma.

Lemma 5.7.12.
- *Let $\sigma^\ell : \mathbb{C}^\times \to \mathrm{End}(\mathbb{C}^{2\ell+1})$ be the map given by $\sigma^\ell(w) = \ell(w^2+w^{-2})I+S^\ell$ where S^ℓ is defined by $\left(S^\ell\right)_{p,j} = -(\ell-j)\delta_{p-j,1}-(\ell+j)\delta_{j-p,1}$. Then*

$$\frac{1}{2}w(w^2-w^{-2})\frac{d}{dw}\Phi_0^{\ell,t}(w) = \Phi_0^{\ell,t}(w)\sigma^\ell(w). \tag{5.51}$$

- *Let $\upsilon^\ell : \mathbb{C}^\times \to \mathrm{End}(\mathbb{C}^{2\ell+1})$ be the map given by*

$$\upsilon^\ell(w) = \frac{1}{8}\frac{w^3}{w^4-1}\left(\frac{1+w^4}{w^2}U^\ell_{\mathrm{diag}}+U^\ell_{lu}\right),$$

where $\left(U^\ell_{lu}\right)_{i,j} = (-2\ell+2j)\delta_{i,j+1}+(2\ell+2j)\delta_{i+1,j}$ and $\left(U^\ell_{\mathrm{diag}}\right)_{i,j} = -2i\delta_{ij}$. Then

$$b_1(w)\Phi_0^{\ell,t}(a(w)) = \Phi_0^{\ell,t}(a(w))\upsilon^\ell(w). \tag{5.52}$$

Proof. The matrix coefficients of $\Phi_0^{\ell,t}(a(w))$ are given by

$$(\Phi_0^{\ell,t}(a(w)))_{p,j} = \frac{(\ell-j)!(\ell+j)!(\ell-p)!(\ell+p)!}{(2\ell)!}\times \sum_{r=\max(0,-p-j)}^{\min(\ell-p,\ell-j)} \frac{w^{4r-2\ell+2p+2j}}{r!(\ell-p-r)!(\ell-j-r)!(p+j+r)!}, \tag{5.53}$$

see [Koo85, Prop. 3.2]. The matrix valued function $b_1(w)$ is equal to the constant matrix $\frac{1}{8}T^\ell$ where $T^\ell(H)_{ij} = 2\delta_{ij}j$. We can now express the matrix coefficients of the matrices in equations (5.51) and (5.52) in Laurent polynomials in the variable w and comparing coefficients of these polynomials shows that the equalities hold. □

Definition 5.7.13. *Define* $\Omega_\ell = (\Phi_0^{\ell,t})^{-1} \circ \Pi_\ell(\Omega)^{\mathrm{up}} \circ \Phi_0^{\ell,t}$ *and* $\Delta_\ell = (\Phi_0^{\ell,t})^{-1} \circ \Pi_\ell(\Omega')^{\mathrm{up}} \circ \Phi_0^{\ell,t}$.

Theorem 5.7.14. *The differential operators* Ω_ℓ *and* Δ_ℓ *are given by*

$$\begin{aligned}\Omega_\ell &= \frac{1}{16}\left(w\frac{d}{dw}\right)^2 + \frac{1}{4}\left\{(\ell+1)(w^2+w^{-2}) + S^\ell\right\}\frac{w}{w^2-w^{-2}}\frac{d}{dw} + \Lambda_0,\\ \Delta_\ell &= v^\ell(w)\frac{d}{dw} + \Gamma_0.\end{aligned}$$

Proof. This is a straightforward calculation using the expressions (5.49) and (5.50), bearing in mind that the coefficients are matrix valued. In both calculations the difficult parts are taken care of by Lemma 5.7.12. □

(6). The elementary zonal spherical function $\Phi_0^{\frac{1}{2},\frac{1}{2}}$ is denoted by ϕ and we have $\phi(a(w)) = \frac{1}{2}(w^2+w^{-2})$. In this final step we note that the differential operators Ω_ℓ and Δ_ℓ are invariant under the maps $w \mapsto -w$ and $w \mapsto w^{-1}$. This shows that the differential operators can be pushed forward by $\phi \circ a$ to obtain differential operators on $\mathbb{C}$ in a coordinate $z = \phi(a(w))$. Using the identities $w\frac{d}{dw}(h\circ\phi)(w) = (w^2-w^{-2})h'(\phi(w))$, $(w\frac{d}{dw})^2(h\circ\phi)(w) = (w^2-w^{-2})^2h''(\phi(a(w)))+2(w^2+w^{-2})h'(\phi(a(w)))$ and $(w^2-w^{-2})^2 = 4(\phi(a(w))^2-1)$ we transform (5.54) and (5.54) into

$$\widetilde{\Omega_\ell} = \frac{1}{4}(z^2-1)\left(\frac{d}{dz}\right)^2 + \frac{1}{4}\left\{(2\ell+3)z + S^\ell\right\}\frac{d}{dz} + \Lambda_0, \tag{5.54}$$

$$\widetilde{\Delta_\ell} = \frac{1}{8}\left(2zU^\ell_{\mathrm{diag}} + U^\ell_{ul}\right)\frac{d}{dz} + \Gamma_0. \tag{5.55}$$

Recall that the $\mathrm{End}(\mathbb{C}^{2\ell+1})$-valued polynomials $R_d^{\ell,t}$ are defined by pushing forward the $\mathrm{End}(\mathbb{C}^{2\ell+1})$-valued functions $Q_d^{\ell,t}$ over $\phi\circ a$, see Definition 5.7.3.

Theorem 5.7.15. *The members of the family* $\{R_d^{\ell,t}\}_{d\geq 0}$ *of* $\mathrm{End}(\mathbb{C}^{2\ell+1})$*-valued polynomials of degree d are eigenfunctions of the differential operators* $\widetilde{\Omega_\ell}$ *and* $\widetilde{\Delta_\ell}$ *with eigenvalues* Λ_d *and* Γ_d *respecively. The transposed differential operators* $(\widetilde{\Omega_\ell})^t$ *and* $(\widetilde{\Delta_\ell})^t$ *satisfy*

$$-4(\widetilde{\Omega_\ell})^t + 2(\ell^2+\ell) = \tilde{D}, \tag{5.56}$$

$$-\frac{2}{\ell}(\widetilde{\Delta_\ell})^t - (\ell+1) = \tilde{E}, \tag{5.57}$$

where $\tilde{D}$ *and* $\tilde{E}$ *are defined in Theorem 5.3.1.*

Proof. The only things that need proofs are the equalities of the differential operators. These follow easily upon comparing coefficients where one has to bear in mind the different labeling of the matrices involved in the two cases. □

Note that the differential operators $\tilde{D}$ and $\widetilde{\Omega_\ell}$ are invariant under conjugation by the matrix J, where $J_{i,j} = \delta_{i,-j}$. The differential operator $\widetilde{\Delta_\ell}$ is anti-invariant for this conjugation. The differential operator $\tilde{E}$ does not have this nice property.

5.A Proof of Theorem 5.2.1

The purpose of this appendix is to prove the LDU-decomposition of Theorem 5.2.1. We prove instead the equivalent Proposition 5.2.2, and we start with proving Lemma 5.2.7.

Note that the integral in Lemma 5.2.7 is zero by (5.5) in case $t > m$, since

$$C_{m-k}^{(k+1)}(x)U_{n+m-2t}(x)$$

is a polynomial of degree $n + 2m - k - 2t < n - k$.

We start by proving Lemma 5.2.7 in the remaining case for which we use the following well-known formulas for connection and linearisation formulas of Gegenbauer polynomials, see e.g. [AAR99, Thm. 6.8.2], [Ism09, Thm. 9.2.1];

$$\begin{aligned} C_n^{(\gamma)}(x) &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(\gamma-\beta)_k(\gamma)_{n-k}}{k!\,(\beta+1)_{n-k}} \left(\frac{\beta+n-2k}{\beta}\right) C_{n-2k}^{(\beta)}(x), \\ C_n^{(\alpha)}(x)C_m^{(\alpha)}(x) &= \sum_{k=0}^{m\wedge n} \frac{(n+m-2k+\alpha)(n+m-2k)!(\alpha)_k}{(n+m-k+\alpha)k!} \\ &\quad \times \frac{(\alpha)_{n-k}(\alpha)_{m-k}(2\alpha)_{n+m-k}}{(n-k)!(m-k)!(\alpha)_{n+m-k}(2\alpha)_{n+m-2k}} C_{n+m-2k}^{(\alpha)}(x). \end{aligned} \tag{5.58}$$

Proof.[Proof of Lemma 5.2.7.] We indicate the proof of Lemma 5.2.7, so that the reader can easily fill in the details. Calculating the product of two Gegenbauer polynomials as a sum using the linearisation formula of (5.58) and expanding the Chebyshev polynomial $U_{n+m-2t}(x) = C_{n+m-2t}^{(1)}(x)$ in terms of Chebyshev polynomials with parameter $k+1$ using the linearisation formula of (5.58), we can rewrite the integral as a double sum with an integral of Chebyshev polynomials that can be evaluated using the orthogonality relations (5.5) reducing the integral of Lemma 5.2.7 to the single sum

$$\sum_{r=\max(0,t-k)}^{\min(t,m-k)} \frac{(k+1)_r(k+1)_{n-k-r}(k+1)_{m-k-r}(2k+2)_{m+n-2k-r}}{r!\,(m-k-r)!\,(n-k-r)!\,(k+1)_{m+n-2k-r}} \times \frac{(m+n-k+1-2r)}{(m+n-k+1-r)} \frac{(-k)_{k+r-t}(n+m-t-k-r)!}{(k-t+r)!\,(k+2)_{n+m-t-k-r}} \frac{\sqrt{\pi}\,\Gamma(k+\frac{3}{2})}{(k+1)\,\Gamma(k+1)}.$$

Assuming for the moment that $k \geq t$, so the sum is $\sum_{r=0}^{\min(t,m-k)}$. Then this sum can be written as a very-well-poised ${}_7F_6$-series

$$\frac{\sqrt{\pi}\,\Gamma(k+\frac{3}{2})}{(k+1)\,\Gamma(k+1)} \frac{(k+1)_{m-k}}{(m-k)!} \frac{(k+1)_{n-k}}{(n-k)!} \frac{(2k+2)_{m+n-2k}}{(k+1)_{m+n-2k}} \times \frac{(-k)_{k-t}}{(k-t)!} \frac{(n+m-t-k)!}{(k+2)_{n+m-t-k}} \, {}_7F_6\left(\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7 \\ s_1, s_2, s_3, s_4, s_5, s_6 \end{matrix} ; 1\right).$$

with $r_1 = \frac{1}{2}(k-m-n+1), r_2 = k+1, r_3 = k-m, r_4 = k-n, r_5 = k-m-n-1, r_6 = -t, r_7 = t-m-n-1$ and $s_1 = \frac{1}{2}(k-m-n-1), s_2 = -m, s_3 = -n, s_4 = -m-n-1, s_5 =$

$k-t+1, s_6 = -n-m+k+t$. Using Whipple's transformation [AAR99, Thm. 3.4.4], [Bai64, §4.3] of a very-well-poised ${}_7F_6$-series to a balanced ${}_4F_3$-series, we find that the ${}_7F_6$-series can be written as

$$\frac{(k-m-n)_t\,(-t)_t}{(k-t+1)_t\,(-m-n-1)_t}\,{}_4F_3\left(\begin{matrix}-k, k+1, -t, t-m-n-1 \\ -n,\, -m,\, 1\end{matrix}\,;1\right).$$

Simplifying the shifted factorials and recalling the definition of the Racah polynomials (5.8) in terms of a balanced ${}_4F_3$-series gives the result in case $k \geq t$.

In case $k \leq t$ we have to relabel the sum, which turns out again to be a very-well-poised ${}_7F_6$-series which can be transformed to a balanced ${}_4F_3$-series. The resulting balanced ${}_4F_3$-series is not a Racah polynomial as in the statement of Lemma 5.2.7, but it can be transformed to a Racah polynomial using Whipple's transformation for balanced ${}_4F_3$-series [AAR99, Thm. 3.3.3]. Keeping track of the constants proves Lemma 5.2.7 in this case. □

As remarked in Section 5.2, Theorem 5.2.1 follows from Proposition 5.2.2. In order to prove Proposition 5.2.2 we assume $\alpha_t(m,n)$ to be given by (5.1) and we want to find $\beta_k(m,n)$. Given the explicit expression for $\alpha_t(m,n)$, we see that multiplying by $\sqrt{1-x^2}\,U_{n+m-2t}(x)$, integrating over $[-1,1]$ and using Lemma 5.2.7 we find

$$\alpha_t(m,n)\frac{\pi}{2} = \sum_{k=0}^{m} \beta_k(m,n)C_k(m,n)R_k(\lambda(t);0,0,-m-1,-n-1) \tag{5.59}$$

where

$$C_k(m,n) = \frac{\sqrt{\pi}\,\Gamma(k+\frac{3}{2})}{(k+1)}\,\frac{(k+1)_{m-k}}{(m-k)!}\,\frac{(k+1)_{n-k}}{(n-k)!}\,\frac{(-1)^k\,(2k+2)_{m+n-2k}\,(k+1)!}{(n+m+1)!}.$$

Using the orthogonality relations for the Racah polynomials, see [AAR99, p. 344], [KS98, §1.2],

$$\begin{aligned}\sum_{t=0}^{m}(m+n+1-2t)\,R_k(\lambda(t);0,0,-m-1,-n-1)\,\times \\ R_l(\lambda(t);0,0,-m-1,-n-1) = \delta_{k,l}\frac{(n+1)(m+1)}{(2k+1)}\,\frac{(m+2)_k(n+2)_k}{(-m)_k(-n)_k}\end{aligned}$$

we find the following explicit expression for $\beta_k(m,n)$

$$\begin{aligned}\beta_k(m,n) =& \frac{1}{C_k(m,n)}\,\frac{(2k+1)}{(n+1)(m+1)}\,\frac{(-m)_k(-n)_k}{(m+2)_k(n+2)_k} \\ &\times\sum_{t=0}^{m}(m+n+1-2t)R_k(\lambda(t);0,0,-m-1,-n-1)\alpha_t(m,n)\frac{\pi}{2}\end{aligned} \tag{5.60}$$

Now Proposition 5.2.2, and hence Theorem 5.2.1, follows from the following summation and simplifying the result.

Lemma 5.A.1. *For $\ell \in \frac{1}{2}\mathbb{N}$, $n, m, k \in \mathbb{N}$ with $0 \le k \le m \le n$ we have*

$$\sum_{t=0}^{m} (-1)^t \frac{(n-2\ell)_{m-t}}{(n+2)_{m-t}} \frac{(2\ell+2-t)_t}{t!} (m+n+1-2t) \times \\ R_k(\lambda(t); 0, 0, -m-1, -n-1) = (-1)^{m+k} \frac{(2\ell+k+1)!\,(2\ell-k)!}{(2\ell+1)!} \frac{(n+1)}{m!\,(2\ell-m)!}$$

Proof. Start with the left hand side and insert the ${}_4F_3$-series for the Racah polynomial and interchange summations to find

$$\sum_{j=0}^{k} \frac{(-k)_j\,(k+1)_j}{j!\,j!\,(-m)_j\,(-n)_j} \frac{(n-2\ell)_m}{(n+2)_m} \\ \times \sum_{t=j}^{m} \frac{(-1-n-m)_t}{(2\ell-n-m+1)_t} \frac{(-2\ell-1)_t}{t!} (m+n+1-2t)(-t)_j (t-m-n-1)_j$$

Relabeling the inner sum $t = j + p$ shows that the inner sum equals

$$(-1)^j \frac{(-1-n-m)_{2j}}{(2\ell-n-m+1)_j} (-2\ell-1)_j (1+m+n-2j) \\ \times \sum_{p=0}^{m-j} \frac{(-1-n-m+j)_p}{(2\ell-n-m+1+j)_p} \frac{(-2\ell-1+j)_p}{p!} \times \\ \frac{(1+\frac{1}{2}(-1-m-n+2j))_p}{(\frac{1}{2}(-1-m-n+2j))_p} \frac{(-1-m-n+2j)_p}{(-1-m-n+j)_p}$$

and the sum over p is a hypergeometric sum. Multiplying by $\frac{(j-m)_p (j-n)_p}{(j-m)_p (j-n)_p}$ the sum can be written as a very-well-poised ${}_5F_4$-series

$${}_5F_4 \left(\begin{matrix} \frac{1}{2} - m - n + 2j,\ -1-m-n+2j,\ -1-2\ell+j,\ j-m,\ j-n \\ \frac{1}{2}(-1-m-n+2j), 2\ell-n-m+j+1,\ j-n,\ j-m \end{matrix} ; 1 \right) \\ = \frac{(-m-n+2j)_{m-j}}{(-m-n+j+1+2\ell)_{m-j}} \frac{(-m+1+2\ell)_{m-j}}{(-m+j)_{m-j}}$$

by the terminating Rogers-Dougall summation formula [Bai64, §4.4].

Simplifying shows that the left hand side of the lemma is equal to the single sum

$$\frac{(n-2\ell)_m}{(n+2)_m} (-1)^m (n+m+1) \frac{(-n-m)_m}{(2\ell-n-m+1)_m} \times \\ \frac{(2\ell+1-m)_m}{m!} \sum_{j=0}^{k} \frac{(-k)_j\,(k+1)_j}{j!\,j!} \frac{(-2\ell-1)_j}{(-2\ell)_j}$$

which can be summed by the Pfaff-Saalschütz summation [AAR99, Thm. 2.2.6], [Ism09, (1.4.5)]. This proves the lemma after some simplifications. □

5.B Moments

In this appendix we give an explicit sum for the generalised moments for W. By the explicit expression

$$U_r(x) \,=\, (r+1)\; {}_2F_1\left(\begin{matrix} -r,\; r+2 \\ \frac{3}{2} \end{matrix}\; ; \frac{1-x}{2}\right)$$

we find

$$\begin{aligned} &\int_{-1}^{1} (1-x)^n U_r(x) \sqrt{1-x^2}\, dx \\ &\qquad = \,(r+1) \sum_{k=0}^{r} \frac{(-r)_k (r+2)_k}{k!\,(\frac{3}{2})_k} 2^{-k} 2^{n+k+2} \frac{\Gamma(n+k+\frac{3}{2})\,\Gamma(\frac{3}{2})}{\Gamma(n+k+3)} \\ &\qquad = \,(r+1) 2^{n+2} \frac{\Gamma(n+\frac{3}{2})\,\Gamma(\frac{3}{2})}{\Gamma(n+3)}\; {}_3F_2\left(\begin{matrix} -r, r+2, n+\frac{3}{2} \\ \frac{3}{2},\, n+3 \end{matrix}\; ; 1\right) \\ &\qquad\qquad = (r+1) 2^{n+2} \frac{\Gamma(n+\frac{3}{2})\,\Gamma(\frac{3}{2})}{\Gamma(n+3)} \frac{(-n)_r}{(n+3)_r} \end{aligned}$$

using the beta-integral in the first equality and the Pfaff-Saalschütz summation [AAR99, Thm. 2.2.6], [Ism09, (1.4.5)] in the last equality. For $m \leq n$, the explicit expression (5.1) gives the following generalised moments

$$\begin{aligned} \int_{-1}^{1} (1-x)^p W(x)_{nm}\, dx = 2^{p+2} \frac{\Gamma(p+\frac{3}{2})\,\Gamma(\frac{3}{2})}{\Gamma(p+3)} \frac{(2\ell+1)}{n+1} \frac{(2\ell-m)!m!}{(2\ell)!} \times \\ \sum_{t=0}^{m} (-1)^{m-t} \frac{(n-2\ell)_{m-t}}{(n+2)_{m-t}} \frac{(2\ell+2-t)_t}{t!} (n+m-2t+1) \frac{(-p)_{n+m-2t}}{(p+3)_{n+m-2t}}. \end{aligned} \tag{5.61}$$

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Samenvatting

Speciale functies zijn innig verbonden met groepentheorie. Een exponent van deze verbinding vinden we in de Jacobipolynomen die voor sommige parameters gerealiseerd kunnen worden als matrixcoëfficiënten op compacte Liegroepen. De eigenschappen die Jacobipolynomen kenmerken, zoals orthogonaliteit, het voldoen aan een drie-terms-recurrente betrekking en het optreden als eigenfunctie van een zekere tweede orde differentiaaloperator, kunnen we bewijzen door de polynomen te interpreteren als functies op een compacte Liegroep.

De Liegroepen waarop we Jacobipolynomen kunnen realiseren als matrixcoëfficiënten komen met een compacte ondergroep, het zijn de compacte rang één Gel'fandparen (G, K) van samenhangende Liegroepen G en K. De constructie van de Jacobipolynomen berust in wezen op de kenmerkende eigenschap van een Gel'fandpaar (G, K), namelijk dat de triviale K-representatie ten hoogste één keer voorkomt in de ontbinding van de beperking van iedere irreducibele G-representatie tot K.

In deel één van deze dissertatie formuleren we een definitie die de definitie van een Gel'fandpaar veralgemeniseert. Een multipliciteitsvrij systeem (G, K, F) bestaat uit twee compacte samenhangende Liegroepen $K \subset G$ en een verzameling F van dominante gehele gewichten voor K waarvoor het volgende geldt: (1) F is de doorsnijding is van alle dominante gewichten van K met een relatief open facet van de gesloten positieve Weylkamer van K en (2) voor elke irreducibele K-representatie met hoogste gewicht in F geldt dat ze met multipliciteit ten hoogste één voorkomt in de ontbinding van de beperking van iedere irreducibele G-representatie tot K.

We bewijzen dat voor een multipliciteitsvrij systeem (G, K, F) het paar (G, K) een Gel'fandpaar is. We classificeren vervolgens alle multipliciteitsvrije systemen op equivalentie na, waarvoor (G, K) een rang één Gel'fandpaar is. Dit levert een lijstje op van zeven wezenlijk verschillende gevallen en onder deze zeven gevallen zijn vier families. Bij ieder element uit de lijst geven we een beschrijving van de toegestane verzamelingen F in termen van de dimensie.

Vervolgens beschrijven we een constructie van een familie van matrixwaardige polynomen voor ieder drietal (G, K, μ) waarbij $\mu \in F$ en (G, K, F) een multipliciteitsvrij systeem met (G, K) een Gel'fand paar van rang één en waarbij G niet van type F_4 is[1]. Deze con-

[1]Enkele dagen voor het ter perse gaan van dit proefschrift ontdekten we dat er voor dit geval nog meer

structie veralgemeniseert die van families van Jacobipolynomen voor zekere parameters. We leiden allerlei eigenschappen af voor deze families: de polynomen in de familie zijn oplossingen van een matrixwaardige hypergeometrische differentiaalvergelijking (vanwege de Casimir operator), de polynomen in een familie zijn orthogonaal ten aanzien van een matrixwaardig inproduct (vanwege Schur orthogonaliteit) en de polynomen in de familie voldoen aan een drie-terms-recurrente betrekking waarvoor we de coëfficiënten uit kunnen drukken in Clebsch-Gordancoëfficiënten (de ontbinding van tensorproductrepresentaties).

In deel twee wordt het voorbeeld (G, K, μ) uitgewerkt met $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$, K de diagonale inbedding van $\mathrm{SU}(2)$ in G en μ eender welk dominant geheel K-gewicht. In dit geval kunnen we allerlei specifieke verbanden leggen met ($\mathbb{C}$-waardige) speciale functies. Veel van deze verbanden zijn vooralsnog nog niet begrepen op het niveau van de groepen.

facetten zijn dan alleen $\{0\}$. Helaas was er niet genoeg tijd meer om de analyse op de nieuwe putten uit te voeren.

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Curriculum vitae

Maarten van Pruijssen was born on the 4th of June 1979 in Oosterhout, the Netherlands. He grew up in Oosterhout where he attended the Monseigneur Frencken College to obtain his VWO-diploma in 1999. Two years later he began the studies of mathematics at the University of Utrecht. In 2008 he concluded his studies with a master thesis on a subject in the field of complex geometry.

Between 2008 and 2012 Maarten did his Ph.D. research at the Radboud University under the supervision of Erik Koelink and Gert Heckman. His focus migrated from geometry to the areas of special functions and Lie theory, fields of interest of both of his supervisors. During his studies Maarten co-organized several student seminars. He also taught a course for freshmen on dynamical systems and he was teacher in several exercise classes. He also participated in three summer schools and he reported on his research on various occasions.

After finishing his Ph.D.-thesis Maarten spent two months as a guest researcher at the University of Córdoba as a guest of Pablo Róman.