Exercises Complex Functions

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1 Holomorphic Functions

Exercise 1.1. *Show that for all* $z, w \in \mathbb{C}$

1. $|z + w| \leq |z| + |w|$
2. $|z - w| \geq ||z| - |w||$

Exercise 1.2. *Explain the geometric meaning of the following relations (with* $z = x + iy \in \mathbb{C}$ *and* $z_1 \neq z_2 \in \mathbb{C}$*)*

1. $|z - 1 + i| = 2$
2. $1/z = \overline{z}$
3. $|z - 2| + |z + 2| < 6$
4. $|z - 2| - |z + 2| > 3$
5. $|z - z_1| \leq |z - z_2|$
6. $\Re z > 1$
7. $\Im z < |\Re z| + 1$
8. $|\Im z| > |\Re z| + 1$
9. $|z| = \Re z + 2$
10. $\pi/4 < \arg(iz - i) < 3\pi/4$

Which of these are regions, and which regions are convex or starlike?

Exercise 1.3. *Check the Cauchy–Riemann equations for the complex function* $z = (x + iy) \mapsto e^z = e^x(\cos y + i \sin y)$ *on* $\mathbb{C}$*. What is the derivative of the function* $z \mapsto e^z$*?*

Exercise 1.4. *Using that* $(e^z)' = e^z$ *compute the derivative of the complex functions* $\sin z = (e^{iz} - e^{-iz})/2i$ *and* $\cos z = (e^{iz} + e^{-iz})/2$ *on* $\mathbb{C}$*.*

Exercise 1.5. *Suppose that f is a holomorphic function on a region Ω. Show that f is constant once its real part $\Re f$ (or likewise its imaginary part $\Im f$) is constant.*

2 Power series

Exercise 2.1. *In each of the following problems determine the radius of convergence of the power series $\sum_0^\infty a_n z^n$*

1. $a_n = 1/(n+1)$
2. $a_n = n^n$
3. $a_n = (2 + (-1)^n)^{-n}$
4. $a_n = n!/n^n$
5. $a_n = (n!)^5/(5n)!$
6. $a_n = 0$ *unless* $n = m!$, *and* $a_{m!} = 2^m$ *for* $m \in \mathbb{N}$

Hint: Use Stirling's fomula, which says that $n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$ *for* $n \to \infty$.

Exercise 2.2. *Suppose that the radius of convergence of the power series $\sum_0^\infty a_n z^n$ is equal to R with $0 < R < \infty$. Determine the radius of convergence of the power series $\sum_0^\infty b_n z^n$ in case*

1. $b_n = n^k a_n$
2. $b_n = n^{-n} a_n$
3. $b_n = a_n^k$
4. $b_n = a_n/n!$
5. $b_n = (z_0^n - 1)a_n$

with $k \in \mathbb{N}$ and $z_0 \in \mathbb{C}$ not on the unit circle.

Exercise 2.3. *Sum the following power series for $|z| < 1$*

1. $\sum_0^\infty z^n$
2. $\sum_1^\infty n z^n$
3. $\sum_1^\infty z^n/n$
4. $\sum_1^\infty n(n+1)z^n$

3 Contour integral

Exercise 3.1. *Suppose Ω is a region, and $f : \Omega \to \mathbb{C}$ a continuous function. Let $\gamma : [a, b] \in t \mapsto z(t)$ be a smooth curve in Ω, and let $\delta : [c, d] \in s \mapsto z(t(s))$*

be a smooth reparametrization via a diffeomorphism $[c,d] \ni s \mapsto t(s) \in [a,b]$ *with* $a = t(c) < b = t(d), c < d$. *Show that*

$$\int_\delta f(z)dz = \int_\gamma f(z)dz$$

and so the contour integral is independent of the parametrization.

Exercise 3.2. *For* γ *a smooth curve in a region* Ω *with parametrization* $\gamma : [a,b] \ni t \mapsto z(t)$ *denote by* $-\gamma : [-b,-a] \ni t \mapsto z(-t)$ *the curve traversed in opposite direction. Show that*

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$$

for any continuous function f *on* Ω.

Exercise 3.3. *Compute* $\oint z^n dz$ *over the unit circle* $|z| = 1$ *for any* $n \in \mathbb{Z}$.

Exercise 3.4. *Prove the following result, which is called Jordan's lemma. Suppose* $z \mapsto f(z)$ *is a continuous function on* $\{|z| \geq R_0, \Im z \geq 0\}$, *and suppose that*

$$\lim_{R\to\infty} \max\{|f(z)|; z \in \gamma_R\} = 0$$

with γ_R *the semicircle* $[0,\pi] \ni \theta \mapsto Re^{i\theta}$. *Then*

$$\lim_{R\to\infty} \int_{\gamma_R} e^{imz} f(z)dz = 0$$

for any positive number m. *Hint: Estimate the modulus of the integrand using the inequality* $\sin\theta \geq 2\theta/\pi$ *for* $0 \leq \theta \leq \pi/2$.

4 Cauchy theorem

Exercise 4.1. *Prove the Fresnel integrals*

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4} .$$

Hint: Consider the contour integral of the function $f(z) = e^{iz^2}$ *along the boundary of the sector* $\{0 \leq |z| \leq R,\ 0 \leq \arg z \leq \pi/4\}$ *for* $R > 1$, *use Jordan's lemma, and use the familiar integral* $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

Exercise 4.2. *Prove the Dirichlet integral*

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \ .$$

Hint: Integrate the function $f(z) = e^{iz}/z$ *along the boundary of the sector* $\{\epsilon \leq |z| \leq R,\ 0 \leq \arg z \leq \pi\}$ *for* $0 < \epsilon < 1 < R$*, and use Jordan's lemma.*

Exercise 4.3. *Prove the integral formulas*

$$\int_0^\infty x^{s-1} \cos x dx = \Gamma(s) \cos \frac{\pi s}{2} \ , \ \int_0^\infty x^{s-1} \sin x dx = \Gamma(s) \sin \frac{\pi s}{2}$$

for $0 < s < 1$*. Hint: Integrate the function* $f(z) = z^{s-1} e^{iz}$ *along the boundary of the sector* $\{\epsilon \leq |z| \leq R,\ 0 \leq \arg z \leq \pi/2\}$ *for* $0 < \epsilon < 1 < R$*, use Jordan's lemma, and the integral representation* $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$ *of the gamma function for* $s > 0$*.*

5 Cauchy's integral formula

Exercise 5.1. *Expand the given functions in a power series* $\sum_0^\infty a_n z^n$ *around the origin and find the radius of convergence.*

1. $\cosh z$
2. $\sin^2 z$
3. $z/(z^2 - 2z + 5)$
4. $\log\{(1+z)/(1-z)\}$
5. $\int_0^z \zeta^{-1} \sin \zeta d\zeta$

Exercise 5.2. *The Fundamental Theorem of Algebra states that a monic polynomial*

$$p(z) = z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$$

of degree $n \geq 1$ *with complex coefficients must vanish somewhere in* $\mathbb{C}$*. Here is a proof using Cauchy's integral formula.*

1. *Show that there exists* $\rho = \rho_p > 0$ *such that* $|p(z)| \geq |z|^n/2$ *for* $|z| \geq \rho$*.*
2. *If* $p(z)$ *is nowhere* 0 *on* $\mathbb{C}$ *then* $f(z) = 1/p(z)$ *is a nowhere vanishing holomorphic function on* $\mathbb{C}$ *and* $|f(z)| \leq 2|z|^{-n}$ *for* $|z| \geq \rho$*. Taking the limit for* $R \to \infty$ *in Cauchy's integral formula*

$$f(0) = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z} dz$$

show that $f(0) = 0$*, which gives a contradiction.*

Exercise 5.3. *The theorem of Liouville states that a bounded holomorphic function f on $\mathbb{C}$ is constant. Prove this theorem by evaluating the integral (for $|a| < R, |b| < R$ and $a \neq b$)*

$$\oint_{|z|=R} \frac{f(z)dz}{(z-a)(z-b)}$$

using partial fraction and taking the limit for $R \to \infty$.

Exercise 5.4. *Suppose f is a holomorphic function on a domain Ω containing the closed disc $|z| \leq R$ with radius $R > 0$.*

1. *Show that*
$$\Re\left(\frac{\zeta+z}{\zeta-z}\right) = \frac{\zeta}{\zeta-z} - \frac{\zeta}{\zeta - R^2/\overline{z}}$$
for $|\zeta| = R$ and $z(z-\zeta) \neq 0$. Hint: Put $w = z/\zeta$.
2. *Show that for $|z| < R$*
$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta|=R} f(\zeta) \Re\left(\frac{\zeta+z}{\zeta-z}\right) \frac{d\zeta}{\zeta}$$
by using the Cauchy integral formula and the Cauchy theorem.
3. *Deduce that the real part $u(z) = \Re f(z)$ is given by*
$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\varphi}) \Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right) d\varphi$$
for $|z| < R$. This real analogue of the Cauchy integral formula is called the Poisson integral formula. It shows that a harmonic function on a disc is completely determined by its values on the boundary of the disc. This is in accordance with the intuition from physics: a stationary temperature distribution on a domain is a harmonic function, as follows from the heat equation.
4. *Rewrite the so called Poisson kernel function in the form*
$$\Re\left(\frac{\zeta+z}{\zeta-z}\right) = \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - \varphi) + r^2}$$
if $\zeta = Re^{i\varphi}$ and $z = re^{i\theta}$.

6 Laurent series

Exercise 6.1. *Expand the given function in a Laurent series either in the given ring or in the neighbourhood of the given point(s). In the latter case determine the domain of convergence of the series expansion.*

1. $1/((z-a)(z-b))$ *for* $0 < |a| < |b|$, *in the neighbourhood of the points* $z = 0$, $z = a$, $z = \infty$ *and on the ring* $|a| < |z| < |b|$.
2. $(z^2 - 2z + 1)/((z-2)(z^2+1))$ *around the point* $z = 2$ *and on the ring* $1 < |z| < 2$.
3. $z^3 \log((z-a)/(z-b))$ *for* $a, b \in \mathbb{C}$, *in the neighbourhood of* $z = \infty$.

Exercise 6.2. *Find the singular points in the extended complex plane* $\mathbb{C} \sqcup \{\infty\}$ *and explain their nature (pole (of which order), essential singular point or a nonisolated singular point) for the given functions. After giving the function an appropriate value removable singular points will be considered regular points.*

1. $\cot z - 1/z$
2. $(\sin z)/z$
3. $z \sin(\pi(z+1)/(z-1))$

Exercise 6.3. *The gamma function* $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ *is a holomorphic function for* $\Re z > 0$ *and satisfies the functional equation* $\Gamma(z+1) = z\Gamma(z)$.

1. *Show that* $\Gamma(1) = 1$ *and* $\Gamma(n+1) = n!$ *for* $n \in \mathbb{N}$.
2. *Using* $\Gamma(z) = \Gamma(z+1)/z$ *the gamma function has a meromorphic continuation to* $\Re z > -1$. *Show that* $\Gamma(z)$ *has a simple pole at* $z = 0$ *with residue equal to* 1.
3. *Using* $\Gamma(z) = \Gamma(z+n+1)/(z(z+1)\cdots(z+n))$ *the gamma function has a meromorphic continuation to all of* $\mathbb{C}$. *Show that* $\Gamma(z)$ *has a simple pole with residue equal to* $(-1)^n/n!$ *at* $z = -n \in -\mathbb{N}$.

7 Residue formula

Exercise 7.1. *Find the residues of the given functions at the isolated singular points in the extended complex plane* $\mathbb{C} \sqcup \{\infty\}$.

1. $1/\sin z$
2. $\sin \pi z/(z-1)^3$
3. $z^3 \cos(1/z)$
4. $1/(1+z^2)^2$

Exercise 7.2. *Show that*

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \pi/\sqrt{2}.$$

Exercise 7.3. *Show that*

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\pi$$

for $n = 1, 2, 3, \cdots$.

Exercise 7.4. *Show that*

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+b^2} = \pi\frac{e^{-ab}}{b}$$

for $a, b > 0$. *Hint: Integrate* $f(z) = e^{iaz}/(z^2+b^2)$ *over the boundary of the semicircle* $\{z \in \mathbb{C}; |z| \leq R, \Im z \geq 0\}$.

Exercise 7.5. *Show that*

$$\int_{-\infty}^{\infty} \frac{x\sin ax}{x^2+b^2} = \pi e^{-ab}$$

for $a, b > 0$.

Exercise 7.6. *Show that*

$$\int_{0}^{2\pi} \frac{d\theta}{a+\cos\theta} = 2\pi/\sqrt{a^2-1}$$

for $a > 1$. *Hint: Put* $z = e^{i\theta}$ *and integrate over the unit circle* $|z| = 1$.

Exercise 7.7. *Show that*

$$\int_{0}^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = 2\pi a/(a^2-1)^{3/2}$$

for $a > 1$.

Exercise 7.8. *Show that*

$$\int_{0}^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta} = 2\pi/(a^2-1)$$

for $a > 1$.

Exercise 7.9. *Show that*

$$\int_0^\infty \frac{dx}{x^p(x+1)} = \pi/\sin(\pi p)$$

for $0 < p < 1$. *Hint: Integrate the function* $f(z) = z^{-p}/(z+1)$ *over the boundary of* $\{z \in \mathbb{C}; \epsilon \le |z| \le R, 0 < \arg z < 2\pi\}$ *and let* $\epsilon \downarrow 0$ *and* $R \to \infty$. *Pay attention to the multivalued character of* $\log z = \ln|z| + i \arg z$ *in* $z^{-p} = e^{-p \log z}$.

Exercise 7.10. *Show that*

$$\int_0^\infty \frac{\log x}{x^2+a^2} dx = \frac{\pi \log a}{2a}$$

for $a > 0$. *Hint: Integrate the function* $f(z) = \log z/(z^2+a^2)$ *over the boundary of* $\{z \in \mathbb{C}; \epsilon \le |z| \le R, 0 \le \arg z \le \pi\}$ *for* $0 < \epsilon < a < R$.

Exercise 7.11. *Show that*

$$\int_0^\infty \frac{x^p dx}{1+x^2} = \tfrac{1}{2}\pi/\cos(\pi p/2)$$

for $-1 < p < 1$.

Exercise 7.12. *Show that*

$$\int_0^\infty \frac{x^p dx}{(1+x^2)^2} = \tfrac{1}{4}\pi(1-p)/\cos(\pi p/2)$$

for $-1 < p < 3$.