EXERCISES, DIRAC GEOMETRY - GQT SCHOOL

Exercise 1. *Let $\omega \in \Omega^2(M)$ be a non-degenerate 2-form. For $f \in C^\infty(M)$, let $X_f \in \mathfrak{X}(M)$ be the vector field defined by the equation:*

$$\mathrm{d}f = \iota_{X_f}\omega.$$

Consider also the operation:

$$\{f, g\} := X_f(g), \quad f, g \in C^\infty(M).$$

Prove that ω is closed: $\mathrm{d}\omega = 0$, iff $\{\cdot,\cdot\}$ satisfies the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Exercise 2. *Let $\eta \in \Omega^2(M)$ be a closed 2-form of constant rank. Prove that $\ker(\eta) \subset TM$ is an involutive distribution.*

Exercise 3. *Let $\pi \in \mathfrak{X}(M)$ be a Poisson structure of constant rank. Prove that $\pi(T^*M) \subset TM$ is an involutive distribution.*

Exercise 4. *Recall that the Dorfman bracket is defined by:*

$$[v + \alpha, w + \beta] = [v, w] + \mathscr{L}_v\beta - \iota_w \mathrm{d}\alpha.$$

Prove that it satisfies the following relations:

(a) The Leibniz type rule:

$$[a_1, f a_2] = \mathscr{L}_{p_T(a_1)}(f)a_2 + f[a_1, a_2].$$

(b) It is skew-symmetric up to an exact form:

$$[a_1, a_2] + [a_2, a_1] = \mathrm{d}(a_1, a_2).$$

(c) It satisfies the Jacobi identity written in the following form:

$$[a_1, [a_2, a_3]] = [[a_1, a_2], a_3] + [a_2, [a_1, a_3]].$$

(d) It preserves the metric $(\cdot,\cdot)$ in the following sense:

$$\mathscr{L}_{p_T(a_1)}(a_2, a_3) = ([a_1, a_2], a_3) + (a_2, [a_1, a_3]),$$

for all $a_1, a_2, a_3 \in \Gamma(\mathbb{T}M)$ and all $f \in C^\infty(M)$, where we have denoted

$$p_T : \mathbb{T}M \longrightarrow TM, \quad p_T(v + \alpha) = v.$$

Exercise 5. *Let $L \subset \mathbb{T}M$ be a Lagrangian subbundle.*

(a) Show that the following defines an element $N_L \in \Gamma(\wedge^3 L^*)$:

$$N_L(\alpha, \beta, \gamma) := ([a, b], c)(p),$$

where $\alpha, \beta, \gamma \in L_p$ and $a, b, c \in \Gamma(L)$ are any sections extending α, β and γ, respectively.

(b) Prove that $L \in \mathrm{Dir}(M)$ if and only if $N_L = 0$.

Exercise 6. *(a) If $\omega \in \Omega^2(M)$ is closed, prove that $TM^\omega \in \mathrm{Dir}(M)$, where*

$$TM^\omega := \{v + \iota_v\omega \ : \ v \in TM\}.$$

Conversely, if $L \in \mathrm{Dir}(M)$ satisfies $L \cap T^*M = 0$, prove that $L = TM^\omega$ for some closed 2-form ω.

(b) If $\pi \in \mathfrak{X}^2(M)$ is a Poisson structure, prove that $T^*M_\pi \in \mathrm{Dir}(M)$, where

$$T^*M_\pi := \{\pi(\alpha) + \alpha \ : \ \alpha \in T^*M\}.$$

Conversely, if $L \in \mathrm{Dir}(M)$ satisfies $L \cap TM = 0$, prove that $L = T^*M_\pi$ for some Poisson structure π. (Hint: use the tensor N_L).

Exercise 7. *Let $Lagr(\mathbb{V})$ denote the set of Lagrangian subspace of $\mathbb{V} = V \oplus V^*$. For a pair (W, ω), where $W \subset V$ is a linear space and $\omega \in \wedge^2 W^*$ define*

$$L(W,\omega) := \{v + \alpha \ : \ v \in W, \ \alpha|_W = \iota_v\omega\} \subset \mathbb{V}.$$

Prove that $L(W,\omega) \in Lagr(\mathbb{V})$, and that this map

$$(W,\omega) \mapsto L(W,\omega)$$

is a 1-to-1 correspondence between such pairs (W,ω) and $Lagr(\mathbb{V})$.

Exercise 8. *Prove that $Lagr(\mathbb{V})$ has two connected components:*

$$Lagr(\mathbb{V})^i := \{L(W,\omega) \ : \ \dim(W) \equiv i \ mod \ 2\}, \quad i = 0, 1.$$

Conclude that, if $L \in \mathrm{Dir}(M)$, where M is a connected manifold, then either all leaves of L are odd-dimensional, or all leaves of L are even-dimensional.

We denote $\mathrm{Dir}(M)^0$ the Dirac structure with even-dimensional leaves and by $\mathrm{Dir}(M)^1$ the Dirac structures with odd-dimensional leaves.

Exercise 9. *(a) If $\dim(M) = 1$ and M is connected, show that $\mathrm{Dir}(M) = \{TM, T^*M\}$.*
(b) If $\dim(M) = 2$, think about $\mathrm{Dir}(M)^0$ and $\mathrm{Dir}(M)^1$.
(c) If $\dim(M) = 3$ and $L \in \mathrm{Dir}(M)^0$, and $f \in C^\infty(M)$, prove that

$$e^f L := \{v + e^f(x)\alpha \ : \ v + \alpha \in L_x, \ x \in M\} \in \mathrm{Dir}(M).$$

Exercise 10. *Let $\Psi : \mathbb{T}M \to \mathbb{T}M$ be a vector bundle isomorphism covering the identity. If Ψ preserves the metric and the Dorfman bracket, prove that there exists a unique closed 2-form ω so that $\Psi = \exp(\omega)$, i.e.*

$$\Psi(v + \alpha) = v + \alpha + \iota_v\alpha.$$

Exercise 11. *For $L_1, L_2 \in Lagr(\mathbb{V})$ define:*

$$L_1 \star L_2 := \{v + \alpha + \beta \ : \ v + \alpha \in L_1, \ v + \beta \in L_2\}.$$

(a) Prove that $L_1 \star L_2 \in Lagr(\mathbb{V})$.
(b) Prove that

$$L(W_1,\omega_1) \star L(W_2,\omega_2) = L(W_1 \cap W_2, \omega_1|_{W_1\cap W_2} + \omega_2|_{W_1\cap W_2}).$$

(c) Prove that $\star$ restricts to a smooth map on

$$\{(L_1, L_2) \ : \ \mathrm{pr}_V(L_1) + \mathrm{pr}_V(L_2) = V\}.$$

Exercise 12. *(a) Let $L_1, L_2 \in \mathrm{Dir}(M)$. If L_1 and $-L_2$ are transverse:*

$$L_1 \oplus (-L_2) = \mathbb{T}M,$$

prove that $L_1 \star L_2$ is a Poisson structure.
(b) Let $\pi_1, \pi_2 \in \mathfrak{X}^2(M)$ be two Poisson structures so that $\pi_1 + \pi_2$ is an invertible bivector field. Prove that

$$\pi_1 \star \pi_2 := \pi_1 \circ (\pi_1 + \pi_2)^{-1} \circ \pi_2$$

is a Poisson bivector field.

Exercise 13. *For $L \in Lagr(\mathbb{V}_2)$ and a linear map $A : V_1 \to V_2$, define:*

$$A^*L := \{v_1 + A^*(\alpha_2) \ : \ A(v_1) + \alpha_2 \in L\}.$$

(a) *Show that* $A^*L \in Lagr(\mathbb{V}_1)$*, and that*

$$A^*(L(W,\omega)) = L(A^{-1}(W), A^*\omega).$$

(b) *Show that* $(A, L) \mapsto A^*L$ *is not continuous, but its restriction to the following set is in fact smooth:*

$$\{(A, L)\ :\ A(V_1) + \mathrm{pr}_{V_2}(L_2) = V_2\}.$$

Exercise 14. *Show that:*

$$A^*(L_1 \star L_2) = A^*(L_1) \star A^*(L_2),$$

but, that in general

$$A_*(L_1 \star L_2) \neq A_*(L_1) \star A_*(L_2).$$

Exercise 15. *Let* $f : M \to N$ *be a smooth map. Prove that* $f^*(T^*N) \in \mathrm{Dir}(M)$ *if and only if* f *has constant rank.*

Exercise 16. *Let* $f : M \to N$ *be a smooth map, and let* $L \in \mathrm{Dir}(N)$ *so that* $f^*(L)$ *is a Poisson structure. Prove that* f *is an immersion.*

Exercise 17. *Let* (M, ω) *be a symplectic manifold, and let* $f : M \to N$ *be a surjective submersion with connected fibers. Denote by* D *the symplectic orthogonal to the fibers of* f*:*

$$D_x := \{v \in T_xM\ :\ \omega(v, w) = 0\ \forall\ w \in \ker(\mathrm{d}_x f)\}.$$

Prove Libermann's Theorem: There exists a Poisson structure π *on* N *so that* $f : (M, \omega) \to (N, \pi)$ *is a Poisson map, if and only if* D *is an involutive distribution.*