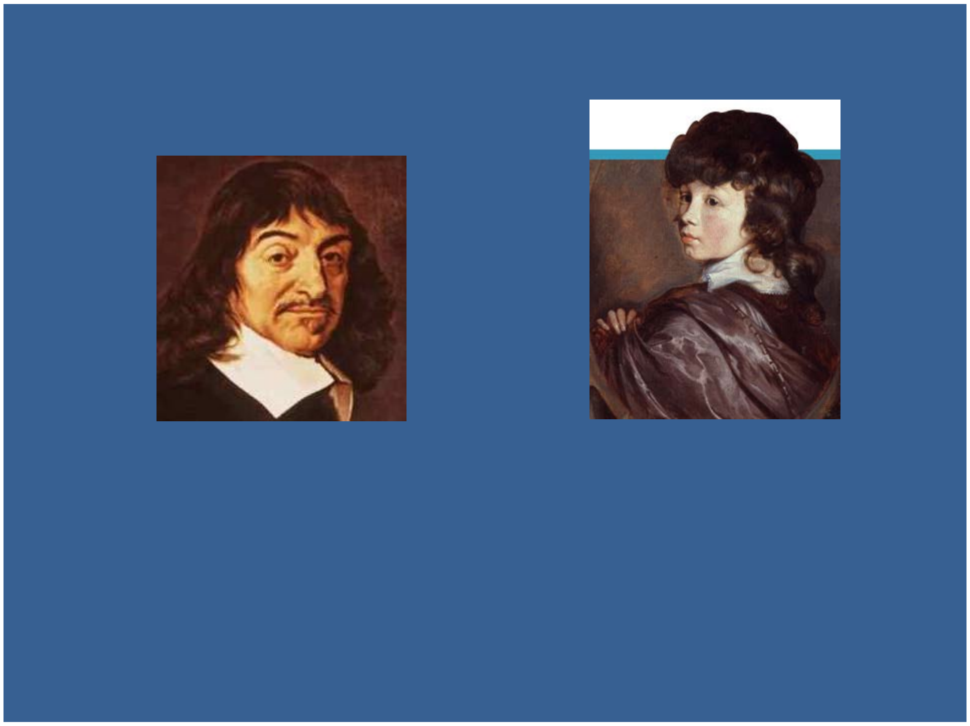

# Descartes and the Geometrization of Algebra

(Descartes-Huygens Lecture, 3 April 2012)

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Your Excellency the Ambassador of The Netherlands, etc, etc

Ladies and Gentlemen,

It is a great honour to be here, and receive the Descartes-Huygens prize for past collaboration, and as an encouragement for future interaction between Dutch and French mathematicians.

It is an equal honour to speak here at the *Académie des Sciences*. I regularly find myself at our own Dutch Royal Academy of Sciences in Amsterdam, which feels like the little brother (if not the great-great-grandson) of the Académie des Sciences. Indeed, the Dutch Academy was founded in 1808 at the instigation of Louis Napoléon, having been made King of The Netherlands by his brother Emperor Napoléon Bonaparte, more than one and a half century after this *Académie des Sciences* came into existence.

I would like to take this opportunity to congratulate my mirror prize winner, Francois Hammer, who by coincidence collaborates with some of my colleagues at the Radboud University in Nijmegen. I look forward to having the occasion to run into him on the corridors of our Science building - by another coincidence named the Huygens Building, and hear more from him about the formation of galaxies.

The two of us, professor Hammer and myself – are not the main characters of this ceremony, because our contributions to science vanish in comparison with those of Descartes (1596-1650) and Huygens. I believe that Huygens refers here to Christiaan Huygens (1629-1695), perhaps best known among mathematicians for his mathematical analysis of the pendulum, but of course he made so many more contributions to physics, optics and mathematics. However, it is his father Constantijn (1596-1687) who is the exact contemporary of Descartes[1]. Father Huygens and Descartes knew each other well, they met several times, and there is a record of the correspondence between them containing around 200 letters[2]. In Holland, it is Constantijn Huygens who is the better known among the public. He was Secretary to the Dutch Gouvernors (Stadhouders), but also a famous poet who is still being read (perhaps not quite voluntarily) by all Dutch pupils of the lycee.

Descartes grew up in France, went to university in Poitiers, but as you will know, he spent most of his life in Holland. Here he wrote the main parts of his famous treatise *Discours de la Méthode,* with Appendices on Optics, Meteorology and Geometry. He promoted a skeptical attitude, and clear reasoning and deduction, as a means for scientific discovery. In a sense, he anticipated the logical rules - the "Laws of Thought" - put forward much later by George Boole (1815-1864), the laws which lie at the origin of modern logic and computers. (All students of computer science are aware of this, because of the fact that the binary values 0 and 1, or "true" and "false", are usually referred to as "Booleans"). Descartes also introduced tools in mathematical reasoning which we still use today, and all learn in secondary school. The best known among these are his convention in expressions like $ax^2 + bx + c$ to indicate the unknown by x and the known variables by a, b, and c; and of course his use of two coordinates, the x-coordinate and the y-coordinate, to describe the position of a point on a plane or surface. Descartes's method made it possible to solve *geometric* problems by *algebra*. In other words, he made it possible to replace – or perhaps better, supplement - Euclid's axiomatic method by algebra and arithmetic, thus introducing a new level of rigour in geometry and becoming the founder of analytic geometry.

In fact, the interaction between algebra and geometry has been a recurrent theme in mathematics, and pervades the modern mathematics of today, as I will try to explain to you now.

Cartesian coordinates are not the only way in which algebra can help geometry. Another method is to try to describe geometric objects by numbers. One of the early examples of this method in *modern* geometry was invented by Henri Poincaré (1854-1912), cousin of your former President Raymond. He found a method for counting all the trajectories in a space which take you back to where you started – or in other words, all the ways to wind an elastic band around a geometric object – thereby ignoring small curvings or stretchings of the band. For example, for the 2-dimensional sphere, this number is zero, essentially because you cannot put an elastic band around a football in a way so that it does not easily come off. This property in fact uniquely characterizes the 2-dimensional sphere among similar 2-dimensional objects. The same assertion is true for the 3-dimensional sphere - an object which you probably have more difficulty imagining[3]. This, however, was only proved a few years ago by the Russian mathematician Grigori Perelman. His discovery stunned the mathematical world, and won Perelman some of the most prestigious prizes, which – by the way - he all refused. It was publicized a few years ago by the journal Science as "breakthrough of the year" (in all of science).

There are many other ways in which algebra is called to assistance by geometry. In fact, there is a purely algebraic way to describe geometric objects, a kind of lego game, a calculus of how to construct all of them from elementary building blocks. The Dutch mathematician Dan Kan (1927 - ) was one of the founders of this calculus, usually referred to as the *simplicial method*, a method which I and many of my French colleagues use every day.

One can carry the algebra of geometry even one step higher, going from Descartes's coordinates and the elastic bands of Poincaré and the building blocks of Kan, to viewing the totality of all geometric objects as a certain kind of algebra - an algebra which encodes how one can model one object in another, or parametrise one object by the points of another. This approach, now known as *categorical algebra,* originated with the American mathematicians S. Eilenberg(1913-1998) and S. Mac Lane (1909-2005), and was promoted and used with stunning effects in the French school of A. Grothendieck (1928 - ). [I was lucky enough to be able to have been involved in a long-term collaboration with Mac Lane, and we wrote a book together, explaining how categorical algebra helps to relate geometry to logic[4], a book which is read by surprisingly many current students of computer science as well.]

Now those of you who have looked at the program and remember the title of my address must start to think that something has gone wrong. Because, haven't we been speaking about how Descartes and modern mathematicians have achieved an *algebraization of geometry,* while the speech was announced as being about the *geometrization of algebra*? Was there an apparent miscommunication concerning my title?

No, -- in fact I claim that in modern mathematics, the construction of new and unusual geometries which are then called to assistance to help the algebra, are at least as frequent as the reverse process and incredibly productive and fruitful. An instance of this phenomenon which has dominated post-war mathematics has been the "algebraic geometry" of Grothendieck, who managed to construct the most incredible geometric objects with the purpose of studying properties of number systems and other algebras. A basic idea is that given an algebra, a system in which one can add and multiply

A + B, A x B,

one tries to construct some sort of imaginary space so that the algebra can be thought of as quantities varying over the space, you might like to think of quantities like the temperature in a point of the space.

When mathematicians had finally gotten used to this new idea of constructing such geometries from algebras, they realized that the method didn't work very well for systems in which the multiplication depends on the order, systems in which A x B need not be the same as B x A. To view such systems as measurements on a geometric object, on a space, a *new* kind of geometry was needed. It was indeed invented by another French mathematician and member of your Académie, Alain Connes (1947 - ), who thus became the founder of the field of *non-commutative geometry*, an area with many active researchers in France but also in Holland[5].

There is another law for the algebraic systems which we study, namely that when we multiply three terms A, B and C, we can do it in two ways and obtain the same result: we can first multiply A and B together, and then multiply the result with C; or we can first multiply B and C together, and then multiply A with the result. However, it turns out that when the algebraic system is subject to small change, to a *deformation* as mathematicians say, the simple equation

(A x B) x C = A x (B x C)

suddenly explodes into a system with infinitely many different multiplications and infinitely many rules for calculation between them[6]. Such a system seems at first impossible to comprehend, but again, geometry can be invoked to help us. Indeed, these infinitely many multiplications and equations fit into a neat sequence of geometric objects known as the *Stasheff polytopes*. These Stasheff polytopes exist in arbitrary high dimensions, and you see a picture of the one in dimension 3 here. I study these polytopes and related more complicated structures very intensively with my mathematical colleagues from Toulouse, Nice, Paris and Strassbourg[7].

As you see, while Descartes knew that algebra could help geometry, in modern mathematics geometry is often called to assistance by algebra[8].

In fact, to conclude my speech, I should take you back in time a bit further, to the "laws of thought" that George Boole wrote down in the 19th century. These "laws" also form some sort of algebra, the algebra underlying logical reasoning. Although an algebra of a quite different sort, it can also be viewed as the algebra of "measurements" on a space, where instead of the temperature in a point we "measure" whether a certain statement holds at that point or not. This space was discovered by the American mathematician Marshall Stone (1903 – 1989), and is now known as the Stone space. Interestingly enough, using the same method, variations of Stone's space give rise to "measurements" with a different kind of logic, the *intuitionistic logic* of the Dutch mathematician Brouwer (1881-1966). This logic underlies a more constructive approach to mathematical reasoning, and was viewed as esoteric for many decades (just like his inventor Brouwer was a bit esoteric). However, more recently this logic has turned out to be an extremely effective basis for computer programming, because a mathematical proof based on Brouwer's logic can quite directly be viewed as a computer programme[9]. Modern versions of this logic are studied in many places in the world, among others at the laboratoire here in Paris which has nominated me for the prize. The laboratoire is named PPS, which stands for "Proofs, Programs and Systems". Perhaps there is a better name, because PPS might equally well stand for *Proofs, Programs and Spaces*! I look very much forward to visiting the laboratoire, and discussing how new geometric *spaces* can help to understand proofs and programs and their underlying logic.

Thank you for your attention[10].

Notes

[1] The confusion between Huygens father and son is of all times, and illustrated by the following extract from a letter of Huygens the son to Pierre Bayle:

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perſonnes de marque de notre pays ce qui m'a fait penſer s'il ne ſembleroit pas qu'il y euſt de l'affectation et de la ſuperfluitè lors qu'on verroit a peu pres les meſmes choſes touchant cette vie paroiſtre en meſme temps dans ces deux dictionaires. J'avois ſongè apres cela ſi en reprenant les fautes de Mr. Baillet dans ſa vie de Mr. Des Cartes, ou il me confond perpetuellement avec mon pere, et me fait Curateur de l'Academie de Breda lors que j'y eſtudiay et que je n'avois que 17 ans, ſi disje a l'occaſion de cette critique, vous ne pourriez pas raporter quelques particularitez de ſa vie. Mais voila Mr. Baillet luy meſme qui par Mr. de Beauval m'a priè que je luy fiſſe un memoire des fautes qu'on luy avoit dit que j'avois trouvè dans ſon ouvrage [5]); dans l'intention, comme il ſemble de les redreſſer dans quelque nouvelle edition ou autrement, et d'echaper peut eſtre par la a voſtre cenſure. En quoy m'eſtant trouvè obligè de le ſatisfaire c'eſt a vous Mr. à juger, s'il ne vous oſte pas, par cette diligence tout ſujet de rien dire a ſon deſavantage, dans l'endroit ou vous parleriez de mon Pere et qu'ainſi cet article ne pourroit pas avoir ce pretexte [6]). S'il vous plaiſt de me mander voſtre ſentiment ſur ces raiſons de douter, j'y ſouſcriray volontiers et feray tout ce que vous ſouhaiterez eſtant entierement

MONSIEUR Voſtre

N° 2791.

CHRISTIAAN HUYGENS.

[1693].

*Appendice au No. 2790.*

*La pièce se trouve à Leiden, coll. Huygens* [1]).

*Elle a été imprimée par V. Cousin* [2]).

De la vie de M. DES CARTES par BAILLET.

2 vol. [3])

Page 485. C'eſt Wilkins qui a donnè des eſſais d'une langue univerſelle [4]) et

[2] To illustrate that Descartes knew the Huygens family rather well, I showed the following transcript during the lecture:

DESCARTES aan HUYGENS: Leiden, 30 oktober 1636

Monsieur,

Je vous suis extrêmement obligé de la souvenance que vous me faites l'honneur d'avoir de mes écrits. Nous en sommes à la fin de la Dioptrique, et il y a déjà plus de huit jours qu'elle aurait pu être achevée; mais à cause que les figures des Météores et de la Géométrie, qui doivent suivre, ne sont pas encore prêtes, l'imprimeur ne se hâte pas, et ne me promet le tout que vers Pâques. J'ai suivi entièrement les instructions que vous m'avez fait la faveur de me donner touchant les figures, car je les fais mettre vis-à-vis du texte en chaque page, et elles seront toutes en bois. Celui qui les taille me contente assez, et le libraire le tient en son logis de peur qu'il ne lui échappe. Il en est maintenant à ce que vous avez jugé le plus difficile, qui est de représenter comment les anguilles de l'eau se disposent dans les nues, et vous pourrez voir déjà ici comment il les a étendues en vapeurs pour causer le vent qui sort d'une aeolipile, et comment il les a entortillées autour des parties du sel dans l'eau de la mer, qui sont les 2 dernières figures qu'il ait faites. J'espère qu'avant qu'il vienne aux étoiles de la neige, l'hiver qui s'approche en pourra faire tomber du ciel quelques unes qui lui serviront de patron. Cependant je passe ici le temps sans rien faire sinon lire quelquefois une épreuve pleine de fautes, et sans rien apprendre sinon ce que c'est qu'un hypocolon; ce qui m'ennuierait fort si je ne savais que mon esprit est semblable à ces terres infertiles qu'il faut laisser reposer quelques années afin qu'elles rapportent après un peu de fruit. Au reste, Monsieur, je remarque bien que je vous ai entretenu ici trop privément de mes écrits; mais c'est vous-même qui m'avez mis sur ce discours, et ayant de très beaux enfants comme vous avez, je m'assure que vous ne trouverez pas étrange qu'un père à qui vous avez demandé des nouvelles des siens n'ait pas épargné les paroles pour vous répondre, et je suis….

Descartes, *Œuvres*, ed. Adam/Tannery (AT( I, 613–614

[3] During the lecture, I showed a model of the 3- sphere as constructed by gluing two solid tori together.

[4] S. Mac Lane, I. Moerdijk, Sheaves in Geometry and Logic: a first introduction to topos theory, Springer-Verlag, 1992, 1995.

[5] There is quite some research related to non-commutative geometry done, for example, in the group around my former PhD student Crainic at Utrecht, and around Solleveld and Van Suijlekom in Nijmegen.

[6] I am referring here to what mathematicians call A-infinity algebras.

[7] Cf. my joint work on the theory of operads with Berger (Nice) and Cisinski (Toulouse), and that of many other French mathematicians on the same subject, such as for example Loday (Strassbourg) and Livernet (Paris 13).

[8] I have often made the point that it would be an interesting challenge for philosophers of mathematics to describe and explain this phenomenon in more detail: unlike computers which can easily manipulate large algebraic data, the human mind is better suited for recognizing and manipulating geometric data like patterns and symmetry. The fact that geometry is called to help by mathematicians to assist in algebra should be directly related to the structure of the human brain.

[9] One of the theories which makes this idea more precise is the type theory of Martin-Löf. Together with several collaborators, I have done quite a bit of research on geometric models for this and similar type theories.

[10] I am grateful to my colleague Professor Th. Verbeek for helpful information about the life and correspondence of Rene Descartes.