Derivations and differentials

Johan Commelin

April 24, 2012

In the following text all rings are commutative with 1, unless otherwise specified.

1 Modules of derivations

Let A be a ring, $\alpha\colon A \to B$ an A algebra, and M a B-module.

1.1 **Definition.** An *A-derivation* is an A-linear map $d\colon B \to M$, satisfying the *Leibniz rule*, i.e., $d(fg) = fd(g) + gd(f)$ for all $f, g \in B$. «

Observe that the set of derivations $\mathrm{Der}_A(B, M)$ is an B-module in a natural way. We write $\mathrm{Der}_A(B)$ for $\mathrm{Der}_A(B, B)$.

We are used to the fact that the derivative of a constant is 0. An analogues statement is true in this setting.

1.2 **Lemma.** *For all $c \in \alpha(A)$ and $d \in \mathrm{Der}_A(B, M)$ we have $d(c) = 0$.*

Proof.

$$d(c) = cd(1) = cd(1) + 1 \cdot d(c) - d(c) = d(1 \cdot c) - d(c) = 0 \qquad \square$$

Other facts that we are used to are also true.

1.3 **Exercise.**

1. Prove that, for $d \in \mathrm{Der}_A(B, M)$ we have

$$d(a^n) = na^{n-1}d(a)$$

and therefore, in a ring of characteristic n, we have $d(a^n) = 0$.

2. Prove that, for $d \in \mathrm{Der}_A(B)$ we have a Leibniz formula for powers of d,

$$d^n(ab) = \sum_{i=0}^{n} \binom{n}{i} d^i(a) \cdot d^{n-i}(b);$$

and if A is of characteristic n, this reduces to $d^n(ab) = ad^n(b) + bd^n(a)$. «

Let A be a ring, and let a commutative diagram

$$\begin{array}{ccc} C/I & \xleftarrow[\beta]{} & B \\ \uparrow{\scriptstyle q} & & \\ C & & \end{array}$$

of A-algebras be given, where I is an ideal of C, and q is the reduction map.

A map $\gamma\colon B \to C$ is called a *lift* of β if the obvious triangle commutes.

1.4 Lemma. *Suppose $I^2 = 0$. Let S be the set of lifts $B \to C$. Let γ be a fixed element of S. Then $S - \gamma = \mathrm{Der}_A(B, I)$.*

Proof. Let $\gamma' \in S$ be given. Clearly $\gamma' - \gamma$ maps into I, since $q \circ (\gamma' - \gamma) = \beta - \beta = 0$.

Also $\gamma' - \gamma$ is clearly A-linear. It satisfies the Leibniz rule, since

$$0 = (\gamma' - \gamma)(f)(\gamma' - \gamma)(g) = \gamma'(fg) + \gamma(fg) - \gamma(f)\gamma'(g) - \gamma'(f)\gamma(g),$$

and therefore

$$(\gamma' - \gamma)(fg) = \gamma(f)\gamma'(g) + \gamma(g)\gamma'(f) - 2\gamma(fg) = \\ \gamma(f)(\gamma' - \gamma)(g) + \gamma(g)(\gamma' - \gamma)(f).$$

Finally observe that I inherits a B-module structure via γ, and that this does not depend on the choice of γ, since $I^2 = 0$.

This shows that $S - \gamma \subset \mathrm{Der}_A(B, I)$. It is left as an easy exercise to verify that for $d \in \mathrm{Der}_A(B, I)$ the map $d + \gamma$ is a lift of β. □

1.5 Lemma. *The functor $M \mapsto \mathrm{Der}_A(B, M)$ is representable.*

Proof. We want to show that there exists a B-module $\Omega_{B/A}$ together with an A-derivation $d_{B/A}\colon B \to \Omega_{B/A}$, such that $\mathrm{Der}_A(B, M) \cong \mathrm{Hom}_B(\Omega_{B/A}, M)$ in a functorial way.

Define $\mu\colon B \otimes_A B \to B$ by $f \otimes g \mapsto fg$. Put $I = \ker \mu$, $\Omega_{B/A} = I/I^2$ and $C = (B \otimes_A B)/I^2$. Then μ induces a map $\mu'\colon C \to B$, and

$$0 \to \Omega_{B/A} \to C \xrightarrow{\mu'} B \to 0$$

is an exact sequence of B-modules. For $i \in \{1, 2\}$ define $\lambda_i\colon B \to C$ by $\lambda_1(f) = f \otimes 1$ and $\lambda_2(f) = 1 \otimes f$. Observe that these are splittings of the exact sequence.

Observe that as $(\Omega_{B/A})^2 = 0$ as an ideal of C. Define $d = d_{B/A} = \lambda_1 - \lambda_2$. Indeed, d is a derivation, by lemma 1.4.

We claim that said isomorphism is given by

$$\begin{aligned} \mathrm{Der}_A(B, M) &\cong \mathrm{Hom}_B(\Omega_{B/A}, M) \\ \delta &\mapsto (f \otimes g \mapsto f\delta g) \\ \phi \circ d_{B/A} &\mapsfrom \phi. \end{aligned}$$

It is left as an exercise to prove that the maps are each others inverses. □

The module $\Omega_{B/A}$ is called the *module of Kähler differentials*.

We also give another construction of $\Omega_{B/A}$. For all $b \in B$, we denote with db an abstract symbol. Write Ω' for the free A-module generated by $\{db : b \in B\}$. We define $\Omega_{B/A}$ to be the quotient of Ω' by the submodule generated by

$$\begin{aligned} d(cf + c'g) - cdf + c'dg \qquad & \forall c, c' \in A, \quad f, g \in B \\ d(fg) - fdg - gdf \qquad & \forall f, g \in B. \end{aligned}$$

Finally, we define $d_{B/A} \colon B \to \Omega_{B/A}$ by $f \mapsto df$.

By definition this module represents the functor $\mathrm{Der}_A(B, \_)$, and therefore it is isomorphic to the previous construction in lemma 1.5.

1.6 Lemma. *If the map $A \to B$ is surjective, then $\Omega_{B/A} = 0$.*

Proof. Immediate from the construction of $\Omega_{B/A}$, and the result of lemma 1.2. □

1.7 Exercise. Prove that, if B is generated by $S \subset B$ as an A-algebra, then $\Omega_{B/A}$ is generated by $d(S)$, using the Leibniz rule repeatedly.

In particular, if B is finitely generated as A algebra, then $\Omega_{B/A}$ is finitely generated as B-module. «

1.8 Exercise. If $B = A[x_1, \ldots, x_n]$, then $\Omega_{B/A} = \oplus_{i=1}^{n} A dx_i$.

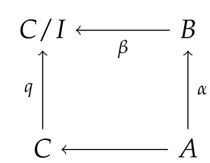
Use this to show that if $B = A[x_1, \ldots, x_n]/I$ with $I = (f_1, \ldots, f_m)$ then $\Omega_{B/A} = \mathrm{coker}(d \colon I/I^2 \to \oplus_{i=1}^{n} B dx_i$. If we precompose d with $\oplus_{i=1}^{m} S e_i \to I/I^2 \colon e_i \mapsto f_i$, then $\Omega_{B/A}$ is the cokernel of the jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}.$$

«

2 Formally smooth, unramified, etale

We return to the situation before, about lifts of ring maps. Let A be a ring, and let a commutative diagram



of A-algebras be given, where I is an ideal of C, and q is the reduction map. Let S be the set of lifts $B \to C$ of β.

2.1 Definition. If for all pairs (C, I), where $I \subset C$ is an ideal satisfying $I^2 = 0$, we have

$\#S \geq 1$ we say that α is *formally smooth*;

$\#S \leq 1$ we say that α is *formally unramified*.

If α is both formally smooth and formally unramified, then we say that α is *formally etale*. «

2.2 Lemma. *The map α is formally unramified if and only if $\Omega_{B/A} = 0$.*

Proof. The implication to the left is clear from the definition and lemma 1.4.

For the other implication take consider the construction of $\Omega_{B/A}$ via $C = (B \otimes_A B)/I^2$ and $\Omega_{B/A} = I/I^2$. Then $(\Omega_{B/A})^2 = 0$. Also we had two lifts λ_1 and λ_2. If α is formally unramified, we have $\lambda_1 = \lambda_2$, and hence $d = \lambda_1 - \lambda_2 = 0$. Since $d(B)$ generates $\Omega_{B/A}$ we conclude that $\Omega_{B/A} = 0$. □

2.3 Exercise. Let A be a ring, and $S \subset A$ a set. Prove that the localization $A \to S^{-1}A$ is formally etale. «

3 Two exact sequences

Let a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\phi} & B' \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\alpha'} \\ A & \xrightarrow{\psi} & A' \end{array}$$

of ring maps be given.

3.1 Exercise. Show that an A'-derivation $d\colon B \to M$ induces an A-derivation $d \circ \psi$. «

By the universal property of $\Omega_{B/A}$ we see that ϕ induces a B-linear map $\Omega_{B/A} \to \Omega_{B'/A'}$. Explicitly the map is given by $fdg \mapsto \phi(f)d\phi(g)$.

3.2 Lemma. *If ϕ is surjective, so is the induced map $\Omega_{B/A} \to \Omega_{B'/A'}$, and its kernel is generated by $\{df | \phi(f) \in A'\}$.*

Proof. Note that $\Omega_{B/A}$ is generated by $d(B)$, while $\Omega_{B'/A'}$ is generated by $d(B')$. Clearly $d(B') = d(\phi(B))$, and hence the image of the induced map generates $\Omega_{B'/A'}$. But then it is surjective, since it is B-linear.

If fdg is mapped to 0, then $\phi(f)d\phi(g) = 0$. This shows that the kernel is generated by elements of the form idf with $i \in \ker \phi$ and $f \in B$, together with elements of the form df satisfying $\phi(f) \in A'$. But since $dif = idf + fdi$, we see that $idf = dif - fdi$. Hence the elements of the second form suffice. □

All these preperations lead to two important exact sequences.

3.3 Proposition. *Let $A \to B \to C$ be ring maps. Then there is a canonical exact sequence of C-modules*

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

Proof. Observe that this sequence is exact if for every C-module M the sequence

$$\mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, M) \leftarrow \mathrm{Hom}_C(\Omega_{C/A}, M) \leftarrow \mathrm{Hom}_C(\Omega_{C/B}, M) \leftarrow 0$$

is exact. But that sequence actually is

$$\mathrm{Der}_A(B, M) \leftarrow \mathrm{Der}_A(C, M) \leftarrow \mathrm{Der}_B(C, M) \leftarrow 0.$$

Now the exactness is clear. □

3.4 **Exercise.** In the above situation, if $B \to C$ is formally smooth, then the sequence actually is short exact (i.e., the first map is injective) and split. «

We now consider the case where the second map is surjective. In the above sequence we would have $\Omega_{C/B} = 0$, by lemma 1.6. Therefore we are interested in the kernel of the first map of the sequence.

3.5 **Proposition.** *Let $A \to B \to C$ be ring maps, where the second map is surjective, with kernel $I \subset B$. Then there exists a canonical exact sequence of C-modules*

$$I/I^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0,$$

where the first map is given by $f \mapsto df \otimes 1$.

Proof. We leave it as an exercise to the reader to show that the first map is well-defined. By proposition 3.3 it is clear that we only need to show exactness at the second term.

Since I maps to 0 in C, it follows that for $f \in I$, df maps to 0 in $\Omega_{C/A}$. Therefore the composition of the first two maps is zero.

Again let M be an arbitrary C-module. Then

$$\mathrm{Hom}_C(I/I^2, M) \leftarrow \mathrm{Der}_A(B, M) \leftarrow \mathrm{Der}_A(C, M)$$

is exact, since if $\delta \in \mathrm{Der}_A(B, M)$ maps to 0, this means that $\delta(I) = 0$. But then δ comes from a derivation $C = B/I \to M$. This proves the exactness of the sequence. □

3.6 **Exercise.** In the above situation, if the compostion $A \to C$ is formally smooth, then the sequence actually is short exact (i.e., the first map is injective) and split. «

4 Colimits

Let A be ring.

4.1 **Lemma.** *Let B, A' be A-algebras. Put $B' = B \otimes_A A'$. Then $\Omega_{B'/A'} \cong \Omega_{B/A} \otimes_A A'$. I.e., formation of differentials commutes with arbitrary change of base.*

Proof. Observe that $d \otimes 1\colon B' \to \Omega_{B/A} \otimes_A A'$ is an A'-derivation, which gives a map $\phi\colon \Omega_{B'/A'} \to \Omega_{B/A} \otimes_A A'$, by the universal property.

On the other hand, the composite map

$$B = B \otimes_A A \to B \otimes_A A' \xrightarrow{d} \Omega B'/A'$$

is an A-derivation, which induces a map $\psi\colon \Omega_{B/A} \to \Omega_{B'/A'}$. Since the target is a B'-module, we get another induced map, which is the inverse of ϕ. □

The following fact we shall not prove. It is an analogue to the fact that for two manifolds X and Y we have the identity

$$T_{(x,y)}X \times Y = T_x X \oplus T_y Y.$$

Observe that in the following lemma, instead of taking a product of algebras, we take a coproduct, since moving between geometric categories (manifolds, varieties, schemes) and the category of rings is contravariant.

4.2 **Lemma.** *Let B_i be A-algebras, for $i \in I$. Let T denote the coproduct $\otimes_A B_i$. Then*

$$\Omega_{T/B} \cong \oplus_i (\Omega_{B_i/A} \otimes_{B_i} T).$$

Proof. See Eisenbud, 394. □

4.3 **Remark.** In Eisenbud, 397, is proven that formation of differentials commutes with arbitrary colimits. «

As a consequence we state the following lemma, that we also do not prove.

4.4 **Lemma.** *Let $A \to B$ be a map of rings. Let $S \subset B$ be a subset. Then*

$$\Omega_{S^{-1}B/A} \cong \Omega_{B/A} \otimes_B S^{-1}B.$$

Proof. See Eisenbud, 397. □

Finally, we prove that formation of differentials commutes with finite products.

4.5 **Lemma.** *Let B_1 and B_2 be A-algebras. Write $B = B_1 \times B_2$. Then*

$$\Omega_{B/A} = \Omega_{B_1/A} \times \Omega_{B_2/A}.$$

Proof. Let e_1 denote $(1,0) \in B$ and $e_2 = (0,1)$. Let M be an arbitrary B-module, and $\delta\colon B \to M$ an A-derivation. Since e_i is idempotent, $\delta e_i = 0$. Hence by the Leibniz rule $\delta(e_i f) = e_i \delta f$. But then δ maps $B_i = e_i B$ to $e_i M$. Thus δ corresponds to a map $\Omega_{B_i/A} \to e_i M$. It follows that $\Omega_{B_1/A} \times \Omega_{B_2/A}$ satisfies the universal property for Kähler differentials. □

4.6 **Exercise.** Let $A \to B$ be a map of rings. Let $\delta\colon B \to M$ be an A-derivation. Let $e \in B$ be an idempotent. Prove that $\delta e = 0$. «

5 Differential forms

Let B be a ring, and M an B-module.

5.1 **Definition.** Let k be an integer. Let $M^{\otimes k}$ denote the k-fold tensor product of M over B. Let N denote the submodule generated by

$$m_1 \otimes m_2 \otimes \ldots \otimes m_k, \qquad \exists i, j : i \neq j, m_i = m_j.$$

The *k-th exterior power of M* is defined as $M^{\otimes k}/N$, and is denoted $\Lambda^k M$. An element of Λ^k is written as a wedge product: $m_1 \wedge \ldots \wedge m_k$ with $m_i \in M$. «

Observe that $\Lambda^0 M = B$.

Let $\alpha\colon A \to B$ be an A-algebra. Recall that $\Omega_{B/A}$ is a B-module. We write $\Omega^k_{B/A}$ for $\Lambda^k \Omega_{B/A}$. We define maps

$$\begin{aligned} d^i : \Omega^i_{B/A} &\to \Omega^{i+1}_{B/A} \\ f \cdot \omega_1 \wedge \ldots \wedge \omega_i &\mapsto df \wedge \omega_1 \ldots \wedge \omega_i. \end{aligned}$$

5.2 **Exercise.** Prove that the maps d^i are well-defined. «

Observe that $d^{i+1} d^i = 0$. Therefore we have an (algebraic) de Rham complex associated to $\alpha\colon A \to B$

$$\Omega^\bullet_{B/A} : 0 \to \Omega^0_{B/A} \xrightarrow{d} \Omega^1_{B/A} \xrightarrow{d^1} \cdots \to \Omega^i_{B/A} \xrightarrow{d^i} \ldots \Omega^{i+1}_{B/A} \to \ldots$$

We can therefore define de Rham cohomology groups

$$H^i_{\mathrm{dR}}(B/A) = \ker d^i / \operatorname{im} d^{i-1}$$

associated to B/A.

5.3 **Remark.** It can be proven (see e.g., Hartshorne) that if $A = \mathbb{C}$ and B is the ring of global sections of a smooth affine variety X over $\mathbb{C}$, then H^i_{dR} coincides with the usual singular homology group $H^{\mathrm{sing}}_i(X, \mathbb{C})$. «

6 Tangent spaces

In view of the previous it makes sense to view $\mathrm{Der}_A(B)$ as the tangent space, since it is the dual module to $\Omega_{B/A}$.

6.1 **Exercise.** The map $\mathrm{Der}_A(B) \times \mathrm{Der}_A(B) \to \mathrm{Der}_A(B)$ defined by $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$ is a Lie bracket. In particular if A is a field, this turns $\mathrm{Der}_A(B)$ into a Lie algebra.

Proof. We verify that the map is well-defined. Clearly $[d_1, d_2]$ is an A-linear map $B \to B$. Also

$$\begin{aligned} [d_1, d_2](fg) &= d_1 d_2(fg) - d_2 d_1(fg) \\ &= d_1(f d_2(g) + g d_2(f)) - d_2(f d_1(g) + g d_1(f)) \\ &= f d_1 d_2(g) + d_1(f) d_2(g) + g d_1 d_2(f) + d_1(g) d_2(f) \\ &\quad - f d_2 d_1(g) - d_2(f) d_1(g) - g d_2 d_1(f) - d_2(g) d_1(f) \\ &= f[d_1, d_2](g) + g[d_1, d_2](f) \end{aligned}$$

shows that $[d_1, d_2]$ is a derivation.

Clearly $[\_,\_]$ is anti-symmetric. To show that $[\_,\_]$ is a Lie bracket, we must prove that it satisfies the Jacobi identity.

$$\begin{aligned} [d,[e,f]] + [f,[d,e]] + [e,[f,d]] &= def - dfe - efd + fed + \\ &\quad fde - edf - def + dfe \\ &\quad efd - fed - fde + edf \\ &= 0 \end{aligned}$$

Consequently, $\mathrm{Der}_A(B)$ is a Lie algebra. □

Let $(A, \mathfrak{m})$ be a local ring containing a field k isomorphic to its residue field $A/\mathfrak{m}$.

6.2 Lemma. *The map*

$$d\colon \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k} \otimes_A k$$

is an isomorphism.

Proof. We have a sequence $k \to A \to k$, where the second map is surjective. Therefore, by proposition 3.5 we see that

$$\mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A/k} \otimes_A k \to \Omega_{k/k} \to 0$$

is an exact sequence. The surjectivity follows immediately.

(Since $k \to k$ is formally smooth, we also get injectivity from this sequence. We will give another proof below.)

To prove the injectivity we pas to the dual modules, and prove surjectivity there.

$$d^*\colon \mathrm{Hom}_k(\Omega_{A/k} \otimes_A k, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$$

Observe that the left hand side is isomorphic to $\mathrm{Hom}_A(\Omega_{A/k}, k) \cong \mathrm{Der}_k(A, k)$. Note that for any derivation $\delta\colon A \to k$ we have $\delta(\mathfrak{m}^2) = 0$ by the Leibniz rule.

Let $h \in \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ be given. Note that $f \in A$ can be uniquely written as $c + f'$, with $c \in k$ and $f' \in \mathfrak{m}$. Define $\delta f = h(\overline{f'})$. We claim that δ is a k-derivation, and that $d^*(\delta) = h$. Clearly, δ is k-linear. Also, if we write $f = c + f'$ and $g = c' + g'$, then

$$\delta(fg) = h(\overline{c'f' + cg'}) = c'h(\overline{f'}) + ch(\overline{g'}) = g\delta f + f\delta g.$$

Thus δ is indeed a k-derivation. Also, since $\delta(\mathfrak{m}^2) = 0$, we see that the restriction of δ to $\mathfrak{m}$ yields h, i.e., $d^*(\delta) = h$. Hence d^* is surjective, and therefore d is injective. □

As a consequence of the provious lemma we see that the notions of tangent space in algebraic geometry and differential geometry coincide.