Functional equation for GL_2 and cuspidal local constants

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Let π be an irreducible smooth representation of $G = \mathrm{GL}_2(F)$. We want to attach an L-function $L(\pi, s)$, and a so called *local constant* $\varepsilon(\pi, s, \phi)$ to π. We give the definition for general smooth representations of G. Further, we compute these invariants in the cuspidal case. The non-cuspidal case will be done by Julius.

The main results of this talk are:

(1) If π is cuspidal, the L-function is constant 1.

(2) Let $(\mathfrak{A}, J, \Lambda)$ be a cuspidal type. Set $\pi = \text{c-Ind}_J^G \Lambda$. The epsilon factor is determined by

$$\varepsilon(\pi, \tfrac{1}{2}, \psi) = q^a \sum_x \mathrm{tr}\Lambda^\vee(cx)\psi_A(cx),$$

for some $c \in J$ and q^a that will be described later.

To arrive at our destination, we will have to (i) define Fourier transforms of coefficients of G; (ii) define L-functions/functional equation/local constants (including a review of the GL_1 case); (iii) work our way through some technical results with strata and cuspidal types.

1 Fourier transforms

We simultaneously define the Fourier transform for the case GL_1 and GL_2.

Notation:

- n Either 1 or 2.
- F be a non-archimedean local field.
- A be the ring $\mathrm{M}_n(F)$.
- $\mathfrak{M}$ $\mathrm{M}_n(\mathfrak{o})$, the integral part of A.
- G be the group $\mathrm{GL}_n(F) = A^\star$.
- μ be a Haar measure on A.
- ψ $\in \hat{F}$, $\psi \neq 1$, a non-trivial character of F.
- ψ_A $= \psi \circ \mathrm{tr}_A$.

$C_c^\infty(A)$ be the space of locally constant functions $A \to \mathbb{C}$ that have compact support.

Given $\Phi \in C_c^\infty(A)$, the *Fourier transform* $\hat{\Phi}$ (relative to μ and ψ) is given by

$$\hat{\Phi} = \int_A \Phi(y)\psi_A(xy)\,\mathrm{d}\mu(y).$$

Lemma 1 *(a) The function $\hat{\Phi}$ is an element of $C_c^\infty(A)$.*

(b) There is a positive real number c (depending on μ and ψ) such that

$$\hat{\hat{\Phi}} = c\Phi(-x), \quad \Phi \in C_c^\infty(A), x \in A. \tag{1}$$

(c) Fixing ψ, there is a unique μ_ψ so that this c is equal to 1. This measure satisfies $\mu_\psi(\mathfrak{M}) = q^{ln^2/2}$, where l is the level of ψ.

(d) For $a \in F^\star$, we have $\mu_{a\psi} = \|a\|^{n^2/2}\mu_\psi$.

Proof First note that $C_c^\infty(A)$ is spanned by the characteristic functions (indicator functions) of the sets $a + \mathfrak{p}^j\mathfrak{M}$ (for $a \in A$ and $j \in \mathbb{Z}$).

One then proves the first two items for the characteristic functions of the sets $\mathfrak{p}^j\mathfrak{M}$. The next step is proving it for their translations. Consequently the first two items hold for all $\Phi \in C_c^\infty(A)$. For these characteristic functions one computes $c = \mu(\mathfrak{M})^2 q^{-ln^2}$.

It is immediate from (1) that scaling μ with $b > 0$ has the effect of scaling c with b^2. Hence, it is clear that there exists a measure with $c = 1$, and since μ is determined by $\mu(\mathfrak{M})$, it is unique. This proves the third item. The fourth item is an immediate consequence.

The measure μ_ψ is the *self-dual* Haar measure (with respect to ψ).

2 Introducing the main actors

2.1 The L-function

Let (π, V) be an irreducible smooth representation of G. Let $\mathcal{C}(\pi)$ be the space of coefficients of π. It is the space of smooth functions on G generated by functions of the form

$$\gamma_{\check{v}\otimes v} \colon \begin{array}{rcl} G & \longrightarrow & \mathbb{C} \\ g & \longmapsto & \langle \check{v}, \pi(g)v \rangle \end{array}$$

Given $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$, we define a function on the complex plane

$$\zeta(\Phi, f, s) = \int_G \Phi(x) f(x) \|\det x\|^s \,\mathrm{d}\mu^*(x)$$

where $\mathrm{d}\mu^*$ is a Haar measure on G.

There is one small little problem: the integral does not converge in general.

Lemma 2 (p148) *There exists an $s_0 \in \mathbb{R}$ such that the integral converges, absolutely and uniformly in vertical strips in the region $\Re s > s_0$, for all Φ and f. The integral represents a rational function in q^{-s}.*

Next we define

$$\mathcal{Z}(\pi) = \{\zeta(\Phi, f, s + \tfrac{1}{2}) \mid \Phi \in C_c^\infty(A), f \in \mathcal{C}(\pi)\} \subset \mathbb{C}(q^{\pm s}).$$

It turns out that $\mathcal{Z}(\pi)$ is a principal fractional ideal.

Lemma 3 (p148) *There is a unique polynomial $P_\pi(X) \in \mathbb{C}[X]$, with $P_\pi(0) = 1$, and*

$$\mathcal{Z}(\pi) = \underbrace{P_\pi(q^{-s})^{-1}}_{\textit{The L-function!}} \mathbb{C}[q^{\pm s}].$$

Moreover, this polynomial does not depend on μ^*.

Finally, we define the L-function

$$L(\pi, s) = P_\pi(q^{-s})^{-1}.$$

2.2 The local constant and the functional equation

So far we haven't applied the Fourier transform yet. It is about time to change that.

Observe that if $f \in \mathcal{C}(\pi)$, then

$$\begin{aligned} \check{f}\colon\ G &\longrightarrow \mathbb{C} \\ g &\longmapsto f(g^{-1}) \end{aligned}$$

lies in $\mathcal{C}(\check{\pi})$. Moreover,

$$\begin{aligned} \mathcal{C}(\pi) &\longrightarrow \mathcal{C}(\check{\pi}) \\ f &\longmapsto \check{f} \end{aligned}$$

is a linear isomorphism.

We use this duality to obtain a relation between the L-function of π and $\check{\pi}$. Fix a non-trivial $\psi \in \hat{F}$, and use the Haar measure that is self-dual with respect to ψ.

Theorem 1 (p148) *There is a unique rational function $\gamma(\pi, s, \psi) \in \mathbb{C}(q^{-s})$ such that*

$$\zeta(\hat{\Phi}, \check{f}, \tfrac{3}{2} - s) = \gamma(\pi, s, \psi)\zeta(\Phi, f, \tfrac{1}{2} + s),$$

for all Φ and f.

This rational function allows us to define the local constant. It is

$$\varepsilon(\pi, s, \phi) = \gamma(\pi, s, \psi)\frac{L(\pi, s)}{L(\check{\pi}, 1 - s)}.$$

Observe that $\varepsilon(\pi, s, \psi)\varepsilon(\check{\pi}, 1-s, \psi) = \gamma(\pi, s, \psi)\gamma(\check{\pi}, 1-s, \psi)$. Using the theorem, we compute

$$\begin{aligned} \zeta(\hat{\hat{\Phi}}, f, \tfrac{1}{2} + s) =& \gamma(\check{\pi}, 1 - s, \psi)\zeta(\hat{\Phi}, \check{f}, \tfrac{3}{2} - s) \\ =& \gamma(\check{\pi}, 1 - s, \psi)\gamma(\pi, s, \psi)\zeta(\Phi, f, \tfrac{1}{2} + s) \end{aligned}$$

and since we are dealing with the self-dual Haar measure, we derive

$$\varepsilon(\pi, s, \psi)\varepsilon(\check{\pi}, 1 - s, \psi) = \gamma(\check{\pi}, 1 - s, \psi)\gamma(\pi, s, \psi) = \omega_\pi(-1)$$

Lemma 4 *There exist $a \in \mathbb{C}^\star$ and $b \in \mathbb{Z}$ such that $\varepsilon(\pi, s, \psi) = aq^{bs}$.*

We now investigate how the γ- and ε-factors depend on ψ. We compute

$$\begin{aligned}
\zeta(\hat{\Phi}, \check{f}, \tfrac{3}{2} - s) &= \int_G \hat{\Phi}(x)\check{f}(x)\|\det x\|^{\frac{3}{2}-s}\, \mathrm{d}\mu^*(x) \\
&= \int_G \int_A \Phi(y) a\psi_A(xy)\, \mathrm{d}\mu_{a\psi}(y)\check{f}(x)\|\det x\|^{\frac{3}{2}-s}\, \mathrm{d}\mu^*(x) \\
&= \int_G \int_A \Phi(y)\psi_A(axy)\|a\|^2\, \mathrm{d}\mu_\psi(y)\check{f}(x)\|\det x\|^{\frac{3}{2}-s}\, \mathrm{d}\mu^*(x) \\
&= \int_G \int_A \Phi(y)\psi_A(ty)\|a\|^2\, \mathrm{d}\mu_\psi(y)\check{f}(a^{-1}t)\|\det ta^{-1}\|^{\frac{3}{2}-s}\, \mathrm{d}\mu^*(t) \\
&= \|a\|^2 \int_G \int_A \Phi(y)\psi_A(ty)\, \mathrm{d}\mu_\psi(y)\omega_\pi(a)\check{f}(t)\|a\|^{2s-3}\|\det t\|^{\frac{3}{2}-s}\, \mathrm{d}\mu^*(t) \\
&= \omega_\pi(a)\|a\|^{2s-1} \int_G \int_A \Phi(y)\psi_A(ty)\, \mathrm{d}\mu_\psi(y)\check{f}(t)\|\det t\|^{\frac{3}{2}-s}\, \mathrm{d}\mu^*(t)
\end{aligned}$$

from which we derive

$$\begin{aligned}
\gamma(\pi, s, a\psi) &= \omega_\pi(a)\|a\|^{2s-1}\gamma(\pi, s, \psi), \\
\varepsilon(\pi, s, a\psi) &= \omega_\pi(a)\|a\|^{2s-1}\varepsilon(\pi, s, \psi).
\end{aligned}$$

Using certain truncations of $\zeta(\Phi, f, s)$ we build approximations $Z(\Phi, f, X)$ which are formal Laurent series. With these the book tackles the following proposition.

Proposition 1 *Let (π, V) be an irreducible cuspidal representation of G. Then*

$$\mathcal{Z}(\pi) = \mathbb{C}[X, X^{-1}].$$

As an immediate consequence, the L-function of an irreducible cuspidal representation is the constant function 1.

3 Technical results concerning the local constants

So, the upshot of Maarten's talks was the following theorem.

Theorem 2 (p108) *The map*

$$(\mathfrak{A}, J, \Lambda) \mapsto \pi_\Lambda = \text{c-Ind}_J^G \Lambda$$

induces a bijection between the set of conjugacy classes of cuspidal types in G and the set of equivalence classes of irreducible cuspidal representations of G.

In a similar spirit is the following proposition.

Proposition 2 (p110) *The map $(\mathfrak{A}, \Xi) \to \pi_\Xi = \text{c-Ind}_{K_\mathfrak{A}}^G \Xi$ induces a bijection between the set of G-conjugacy classes of cuspidal inducing data in G and the set of equivalence classes of irreducible cuspidal representations of G.*

Recall that a cuspidal inducing datum in G is a pair $(\mathfrak{A}, \Xi)$, where $\mathfrak{A}$ is a chain order, and Ξ an irreducible smooth representation of $K_{\mathfrak{A}}$ of the form $\Xi = \mathrm{Ind}_J^{K_{\mathfrak{A}}} \Lambda$ for some cuspidal type $(\mathfrak{A}, J, \Lambda)$.

(Oh, and $K_{\mathfrak{A}}$ is the group $\{g \in G \mid g\mathfrak{A}g^{-1} = \mathfrak{A}\}$; J is a subgroup of $K_{\mathfrak{A}}$; and Λ a representation of J.)

Now that everyone is back on track (including me), we can introduce the main tool for the calculations of this section. It is the so-called *non-abelian Gauss sum.* Write $\mathfrak{P}$ for the Jacobson radical of $\mathfrak{A}$. Let W be the space underlying the representation Ξ. Define an element of $\mathrm{End}_{\mathbb{C}}(\check{W})$,

$$\mathcal{T}(\Xi, \psi) = \sum_{x \in U_{\mathfrak{A}}/U_{\mathfrak{A}}^{n+1}} \check{\Xi}(cx)\psi_A(cx),$$

where $c \in K_{\mathfrak{A}}$ satisfies $c\mathfrak{A} = \mathfrak{P}^{-n}$.

Lemma 5 *The sum $\mathcal{T}(\Xi, \psi)$ does not depend on the choices of coset representatives x, nor on the choice of c.*

Moreover, the endomorphism is scalar. Write $\tau(\Xi, \psi)$ for the complex number satisfying $\mathcal{T}(\Xi, \psi) = \tau(\Xi, \psi)1_{\check{W}}$.

The number $\tau(\Xi, \psi) \in \mathbb{C}$ is the *non-abelian Gauss sum* of Ξ. Observe that

$$\begin{aligned}\tau(\Xi, \psi) =& \frac{1}{\dim(\Xi)} \mathrm{tr}\,(\mathcal{T}(\Xi, \psi)) \\ =& \frac{1}{\dim(\Xi)} \sum_{x \in U_{\mathfrak{A}}/U_{\mathfrak{A}}^{n+1}} \mathrm{tr}(\check{\Xi}(cx))\psi_A(cx).\end{aligned}$$

We formulate the first computational results with $\tau(\Xi, \psi)$. Let (π, V) be an irreducible cuspidal representation of G. **From now on,** fix $\psi \in \hat{F}$, a character of level one. Recall that $\ell(\pi)$ denotes

$$\min \left\{ n/e_{\mathfrak{A}} \;\middle|\; \begin{array}{l} \text{there exists a chain order } \mathfrak{A} \text{ such that} \\ \pi \text{ contains the trivial character of } U_{\mathfrak{A}}^{n+1} \end{array} \right\}$$

Theorem 3 *We have*

$$\begin{aligned}\varepsilon(\pi, s, \psi) =& (\mathfrak{P}^{-n} : \mathfrak{A})^{(\frac{1}{2}-s)/2} \frac{\tau(\Xi, \psi)}{(\mathfrak{A} : \mathfrak{P}^{n+1})^{\frac{1}{2}}} \\ =& q^{2\ell(\pi)(\frac{1}{2}-s)} \frac{\tau(\Xi, \psi)}{(\mathfrak{A} : \mathfrak{P}^{n+1})^{\frac{1}{2}}}.\end{aligned}$$

The next thing we want to do is translate the formalism of Gauss sums from inducing data back to cuspidal types. Let $(\mathfrak{A}, J, \Lambda)$ be a cuspidal type that induces Ξ.

Lemma 6 *The Gauss sum $\tau(\Xi, \psi)$ is the unique eigenvalue of the scalar operator*

$$\sum_{x \in J \cap U_{\mathfrak{A}}/U_{\mathfrak{A}}^{n+1}} \check{\Lambda}(cx)\psi_A(cx),$$

for any $c \in J$ such that $c\mathfrak{A} = \mathfrak{P}^{-n}$.

For some reason (currently unknown to me) we can choose $c \in J$ such that $\Lambda|_{U_{\mathfrak{A}}^{[n/2]+1}}$ is a multiple of ψ_c.

Proposition 3 *The Gauss sum $\tau(\Xi, \psi)$ is the unique eigenvalue of the scalar operator*

$$(\mathfrak{A} : \mathfrak{P}^{(n+1)/2}) \sum_{y \in U_{\mathfrak{A}}^{(n+1)/2} / U_{\mathfrak{A}}^{n/2+1}} \check{\Lambda}(cy)\psi_A(cy).$$

The theorem, lemma and proposition together give the following corollary.

Corollary 1 *Let $(\mathfrak{A}, J, \Lambda)$ be a cuspidal type in G. Let n be the least integer ≥ 0 such that $U_{\mathfrak{A}}^{n+1} \subset \mathrm{Ker}\Lambda$. Choose $c \in J$ such that $\Lambda|_{U_{\mathfrak{A}}^{[n/2]+1}}$ is a multiple of ψ_c. If $\pi = \text{c-Ind}_J^G \Lambda$, then*

$$\varepsilon(\pi, \tfrac{1}{2}, \psi) = q^a \sum_x \mathrm{tr}\check{\Lambda}(cx)\psi_A(cx),$$

where x ranges over $U_{\mathfrak{A}}^{(n+1)/2} / U_{\mathfrak{A}}^{n/2+1}$ and

$$q^a \dim \Lambda = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ (\mathfrak{A} : \mathfrak{P})^{-1/2} & \text{if } n \text{ is even.} \end{cases}$$