Gabber's lemma

by J.M. Commelin Tuesday, the 24th of November, 2015

1 Tempered dream

Let k be a number field. The following lemma would take us a long way towards our end goal: a proof of the Tate, Shafarevich and Mordell conjectures.

1.1 Lemma. — *Let h and g be integers. Let m be an integer divisible by two different primes ≥ 3. There exists only finitely many isomorphism classes of principally polarized g-dimenstional abelian varieties (A, Θ) over k, equipped with level-m structure $\varepsilon\colon (\mathbb{Z}/m\mathbb{Z})^{2g} \xrightarrow{\sim} A[m]$, and $h_{\mathrm{Fal}}(A) \leq h$.* «

Note that this lemma is still a long way from our Big Dream, because we would need to get rid of the level structure and polarisation. I will not touch on those issues here.

2 Problem

Let M be the moduli space of such abelian varieties. If M had a projective compactification $\bar{M}$, with a line bundle L, such that $h_{\mathrm{Fal}}(A) \sim h_{\bar{M},L}(A)$, we would win.

However, as we know by now, we are not so lucky, and even if $\bar{M}$ and L exist, we cannot compare the Faltings height to $h_{\bar{M},L}$.

The main problem is that we do not have a good modular interpretation of compactifications $\bar{M}$. In particular, there is not a universal semi-abelian scheme over any such $\bar{M}$. That would more or less amount to a generalisation of the semistable reduction theorem to higher-dimensional base schemes (dvr's and Dedekind schemes are 1-dimensional).

2.1 Theorem (Semistable reduction theorem). — *Let R be a discrete valuation ring. Let K be the field of fractions of R. Let A be an abelian variety over K. There exists a finite separable extension K'/K, such that for the R-finite integral closure R' of R in K' the base change $A_{K'}$ extends to a semi-abelian scheme over* $\mathrm{Spec}(R')$. «

3 Workaround

The goal of today is to prove Gabber's lemma, which is a technical tool that provides us with sufficient leeway to prove the desired properties of the Faltings height.

Gabber's lemma may really be thought of as an analogue of theorem 2.1 to higher-dimensional base schemes, though it is not a formal generalisation of theorem 2.1.

3.1 Theorem (Gabber's lemma). — *Let S be a noetherian scheme. Let $f\colon X \to S$ be a separated map of finite type. Let $u\colon A \to X$ be an abelian scheme.*

There exists a proper surjection $\pi\colon X' \to X$, and an open immersion $j\colon X' \to \bar{X}'$ into a proper S-scheme such that the pullback $A_{X'} \to X'$ extends to a semi-abelian scheme over $\bar{X}'$.

$$\begin{array}{ccccc}
A & \longleftarrow & A_{X'} & \hookrightarrow & A' \\
\downarrow u & & \downarrow u_{X'} & & \downarrow \bar{u} \\
X & \overset{\pi}{\twoheadleftarrow} & X' & \overset{j}{\hookrightarrow} & \bar{X}' \\
& \searrow f & & \swarrow \bar{f}' & \\
& & S & &
\end{array}$$

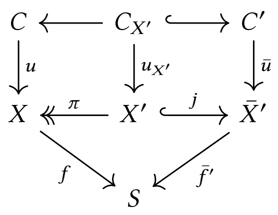
(In the diagram both squares are cartesian; π is a proper surjection; j is an open immersion; $\bar{f}'$ is proper; and $\bar{u}$ is a semi-abelian scheme.) «

4 A crucial tool: the analogue for curves

In one line, the main idea in the proof of Gabber's lemma is: reduce to the analogous statement for curves, using the fact that every abelian scheme is the quotient of the Picard scheme of a curve. So, let us now formulate and prove this analogous statement for curves.

4.1 Theorem. — *Let S be a noetherian scheme. Let $f\colon X \to S$ be a separated map of finite type. Let $u\colon C \to X$ be a smooth proper map whose geometric fibres are connected curves of genus ≥ 2.*

There exists a proper surjection $\pi\colon X' \to X$, and an open immersion $j\colon X' \to \bar{X}'$ into a proper S-scheme such that the pullback $C_{X'} \to X'$ extends to a semistable curve over $\bar{X}'$.



(In the diagram both squares are cartesian; π is a proper surjection; j is an open immersion; $\bar{f}'$ is proper; and $\bar{u}$ is a proper flat map whose geometric fibres are connected semistable curves.) «

(A semistable curve is a curve whose singularities are ordinary double points, and all whose rational components meet the other components in at least 2 points.)

We will need some ingredients to prove theorem 4.1.

4.2 Lemma (Chow's lemma). — *Let S be a noetherian scheme. Let $X \to S$ be a separated S-scheme of finite type. There exists a surjective proper map $X' \to X$ such that $X' \to S$ is quasi-projective.*

Proof. What follows is more or less a verbatim copy of [Stacks, tag 02O2].

The scheme X has only finitely many irreducible components Say $X = X_1 \cup \ldots \cup X_r$ is the decomposition of X into irreducible components. Let $\eta_i \in X_i$ be the generic point. For every point $x \in X$ there exists an affine open $U_x \subset X$ which contains x and each of the generic points η_i. Since X is quasi-compact, we can find a finite affine open covering $X = U_1 \cup \ldots \cup U_m$ such that each U_i contains $\eta_1, \ldots, \eta_r$. In particular we conclude that the open $U = U_1 \cap \ldots \cap U_m \subset X$ is a dense open. This and the fact that the U_i are affine opens covering X is all that we will use below.

Let $X^* \subset X$ be the scheme theoretic closure of $U \to X$. Let $U_i^* = X^* \cap U_i$. Note that U_i^* is a closed subscheme of U_i. Hence U_i^* is affine. Since U is dense in X the morphism $X^* \to X$ is a surjective closed immersion. It is an isomorphism over U. Hence we may replace X by X^* and U_i by U_i^* and assume that U is scheme theoretically dense in X.

We can find an immersion $j_i : U_i \to \mathbb{P}^{n_i}_S$ for each i. We can find closed subschemes $Z_i \subset \mathbb{P}^{n_i}_S$ such that $j_i : U_i \to Z_i$ is a scheme theoretically dense open immersion. Note that $Z_i \to S$ is proper. Consider the morphism

$$j = (j_1|_U, \ldots, j_n|_U) : U \longrightarrow \mathbb{P}^{n_1}_S \times_S \ldots \times_S \mathbb{P}^{n_n}_S.$$

We can find a closed subscheme Z of $\mathbb{P}^{n_1}_S \times_S \ldots \times_S \mathbb{P}^{n_n}_S$ such that $j : U \to Z$ is an open immersion and such that U is scheme theoretically dense in Z. The morphism $Z \to S$ is proper. Consider the ith projection

$$\mathrm{pr}_i|_Z : Z \longrightarrow \mathbb{P}^{n_i}_S.$$

This morphism factors through Z_i. Denote $p_i : Z \to Z_i$ the induced morphism. This is a proper morphism. At this point we have that $U \subset U_i \subset Z_i$ are scheme theoretically dense open immersions. Moreover, we can think of Z as the scheme theoretic image of the "diagonal" morphism

$$U \to Z_1 \times_S \ldots \times_S Z_n.$$

Set $V_i = p_i^{-1}(U_i)$. Note that $p_i|_{V_i} : V_i \to U_i$ is proper. Set $X' = V_1 \cup \ldots \cup V_n$. By construction X' has an immersion into the scheme $\mathbb{P}^{n_1}_S \times_S \ldots \times_S \mathbb{P}^{n_n}_S$. Thus by the Segre embedding we see that X' has an immersion into a projective space over S.

We claim that the morphisms $p_i|_{V_i} : V_i \to U_i$ glue to a morphism $X' \to X$. Namely, it is clear that $p_i|_U$ is the identity map from U to U. Since $U \subset X'$ is scheme theoretically dense by construction, it is also scheme theoretically dense in the open subscheme $V_i \cap V_j$. Thus we see that $p_i|_{V_i \cap V_j} = p_j|_{V_i \cap V_j}$ as morphisms into the separated S-scheme X. We denote the resulting morphism $\pi : X' \to X$.

We claim that $\pi^{-1}(U_i) = V_i$. Since $\pi|_{V_i} = p_i|_{V_i}$ it follows that $V_i \subset \pi^{-1}(U_i)$. Consider the diagram

$$\begin{array}{ccc} V_i & \longrightarrow & \pi^{-1}(U_i) \\ & \searrow^{p_i|_{V_i}} & \downarrow \\ & & U_i \end{array}$$

Since $V_i \to U_i$ is proper we see that the image of the horizontal arrow is closed. Since $V_i \subset \pi^{-1}(U_i)$ is scheme theoretically dense (as it contains U) we conclude that $V_i = \pi^{-1}(U_i)$ as claimed.

This shows that $\pi^{-1}(U_i) \to U_i$ is identified with the proper morphism $p_i|_{V_i} : V_i \to U_i$. Hence we see that X has a finite affine covering $X = \bigcup U_i$ such that the restriction of π is proper on each member of the covering. We conclude that π is proper. ☐

4.3 Lemma. — *Let S be a noetherian scheme. Let X be an S-scheme of finite type. Let $U \subset X$ be a dense open subscheme. Let $f : U \to Y$ be a morphism of S-schemes. Assume Y is proper.*

There exists a proper surjective morphism $\pi : X' \to X$, a morphism $\bar{f} : X' \to Y$ of S-schemes, such that $\bar{f}|_{\pi^{-1}(U)} = f \circ \pi|_{\pi^{-1}(U)}$.

Proof. Consider the composition

$$U \longrightarrow U \times_S Y \longrightarrow X \times_S Y,$$

and call the schematic closure of the image X'. Observe that $\bar{f} : X' \to Y$ extends f in the required sense. Moreover, $\pi : X' \to X$ is proper, since $X \times_S Y$ is proper. Since π is an isomorphism above U, the map π is dominant, hence surjective. ☐

Proof (of theorem 4.1). Superficially, the previous two lemmata immediately give theorem 4.1. The only problem is that suddenly algebraic stacks pop out their heads. But we will be courageous and ignorant, and wave our hands about important details.

By lemma 4.2, we may assume that X is quasi-projective over S. Let $j : X \to \bar{X}$ be a dense open immersion into a projective scheme over S. Write M_S for $\overline{\mathscr{M}_g} \times_{\mathrm{Spec}(\mathbb{Z})} S$. The curve $u : C \to X$ corresponds with a morphism of S-stacks

$$u : X \longrightarrow M_S$$

By a stacky version of lemma 4.3, we may find a proper surjective map $\bar{X}' \to \bar{X}$, and a morphism of S-stacks $\bar{X}' \to \overline{\mathcal{M}_g} \times_{\mathrm{Spec}(\mathbb{Z})} S$, extending u. In other words, we find a semistable $\bar{u} : C' \to \bar{X}'$, as requested. The only problem is that C' is a proper DM-stack over S, instead of a proper scheme over S. However, this is solved by the next lemma. ☐

4.4 Lemma (Chow's lemma for DM-stacks). — *Let S be a noetherian scheme. Let M be a separated DM-stack over S, of finite type. There exists a quasi-projective S-scheme X, and a proper surjective S-morphism $X \to M$. Moreover, M is proper over S if and only if X is projective over S.*

Proof. This follows from corollaire 16.6.1 of [2]. The last statement follows from an inspection of the proof of lemma 4.2. ☐

5 Proof of Gabber's lemma

With theorem 4.1 under our belt, we are in good shape to attack theorem 3.1. We will need a few more tools, which we state (and sometimes prove) along the way. We start of with the following lemma due to Faltings.

5.1 Lemma. — *Let S be a normal noetherian scheme. Let $U \subset S$ be a dense open subscheme. The restriction functor from semi-abelian schemes over S to semi-abelian schemes over U, given by $A \mapsto A_U$ is fully faithful.* «

Proof (of theorem 3.1). First of all, we may assume that X is quasi-projective, using lemma 4.2. By replacing X with the disjoint union of its irreducible components, we see that it suffices to prove the theorem for irreducible X. We may also assume that X is reduced, hence integral.

We may even reduce the entire problem to the case where S is of finite type over $\mathrm{Spec}(\mathbb{Z})$, since all input is of finite presentation over S, and S is a limit of schemes of finite type over $\mathrm{Spec}(\mathbb{Z})$. (This feels a bit shaky, but we only need the case where S is of finite type over $\mathrm{Spec}(\mathbb{Z})$ anyway.) This reduces us to the case where S is *excellent*, which implies that the normalisation map $\tilde{X} \to X$ is finite. (Sadly, I have no reference for this statement, other then Brian Conrad's claim in his notes on Gabber's lemma [1].) In the end, we may assume that X is normal; hence we can apply lemma 5.1.

To prove theorem 3.1, we proceed as follows. Let η be the generic point of X. Let B_η be an abelian scheme such that $A_\eta \times B_\eta$ is isogenous to the Jacobian of a smooth, proper, geometrically connected curve of genus $g \geq 2$ over η. (This can always be done!) Since η is the generic point, there is a dense open $U \subset X$, and an abelian scheme B_U with generic fibre B_η, and a curve (smooth, proper, geometrically connected fibres of genus g) C_U such that

$$\mathrm{Pic}^0_{C_U/U} \to A_U \times B_U$$

is an isogeny of abelian schemes over U.

By theorem 4.1, there is a proper surjection $X' \to X$ and an open immersion $X' \to \bar{X}'$ such that C' extends to a proper semistable curve $\bar{C}'$ over $\bar{X}'$, where C' is the pullback of C_U to $U' = U \times_X X'$. By another proper surjective base change, we may assume that $\bar{X}'$ is normal. In the end, the abelian scheme $(\mathrm{Pic}^0_{C_U/U})_{U'}$ extends to a semi-abelian scheme $\mathrm{Pic}^0_{\bar{C}'/\bar{X}'}$ over $\bar{X}'$. Except that there is one catch: why is $\mathrm{Pic}^0_{\bar{C}'/\bar{X}'}$ a scheme? Well, it turn out that it is (in the case of proper semistable curves) by results of Deligne and Raynaud.

Write J for $\mathrm{Pic}^0_{\bar{C}'/\bar{X}'}$. By construction $J_{U'} \to A_{U'} \times B_{U'}$ is an isogeny of abelian schemes. If we show that this implies that $A_{U'}$ extends to $\bar{X}'$, then we win. This is the content of the next lemmata. □

5.2 Lemma. — *Let X be a noetherian scheme. Let $U \subset X$ be an open subscheme. Let $J \to X$ be a semi-abelian scheme. Assume J_U is an abelian scheme. Let $f \colon J_U \to J'$ be an isogeny of abelian schemes over U.*

There exists a proper surjective morphism $X' \to X$, such that the pullback $J'_{U'}$ (where $U' = U \times_X X'$) extends to a semi-abelian scheme over X'.

Proof. By another one of those proper surjective base changes, we may assume that X is normal and integral. If U is empty, then f is an isomorphism, so the lemma is trivial. Thus assume that U is not empty, and therefore dense.

Let H denote $\ker(f)$. Note that H is a finite flat closed subgroup scheme of J_U over U (because f is an isogeny of abelian schemes f is flat above each point of U and J_U is flat over U, hence we can apply the fibrewise criterion for flatness). We try to extend H to a quasi-finite flat closed subgroup scheme of J over X (up to proper surjective base change).

By a special case of the hard and deep theorem 5.2.2 of [3], there exists a proper surjection $X' \to X$ that is an isomorphism above U, such that the flat closed subscheme $H \subset J_U$ extends to a flat closed subscheme $G \subset J_{X'}$. Replace X by X', and assume again that X' is normal and integral.

We have now arranged that G is flat over X, which is crucial. It follows immediately that G is quasi-finite over X. Moreover, since X is integral, and J is separated over X, the 0-section $X \to J$ factors via G, because it does so over the dense open U. Similarly, G is closed under inversion. Note that $G \times_X G$ is flat over X, and therefore the Zariski closure $H \times_U H$ (which is flat over U) in $J \times_X J$. Hence G is also closed under multiplication. We conclude that G is a quasi-finite flat closed subgroup scheme of J over X.

We will now prove that J/G is a semi-abelian scheme extending J' to X. Since $H \subset J_U[n]$ for some n, we may conclude that $G \subset J[n]$.

By work of Artin, the quotient J/G is a smooth separated algebraic space over X. Now consider the map $J/G \to J/J[n]$. This map is quasi-finite and separated (because everything in sight is separated). Since $[n]\colon J \to J$ is an fppf cover, we get a short exact sequence of abelian sheaves for the fppf-topology:

$$0 \longrightarrow J[n] \longrightarrow J \xrightarrow{[n]} J \longrightarrow 0$$

This shows that $J/J[n]$ is a scheme. We win because algebraic spaces that are separated and quasi-finite over schemes are themselves schemes. □

5.3 Lemma. — *Let X be a connected normal noetherian scheme. Let $U \subset X$ be a dense open subscheme. Let $J \to X$ be a semi-abelian scheme. Assume $J_U \cong A_1 \times A_2$ for abelian schemes A_1 and A_2 over U.*

The abelian schemes A_i extend as semi-abelian schemes to X.

Proof. Let $e \in \mathrm{End}_U(J_U)$ be the idempotent projecting onto A_1. By lemma 5.1, e extends to an idempotent endomorphism $\bar{e}$ of J. It follows at once from Yoneda that $\bar{A}_1 = \ker(\bar{e})$ and $\bar{A}_2 = \ker(1-\bar{e})$ are subgroup schemes of J such that $J = \bar{A}_1 \times \bar{A}_2$.

The geometric fibres of the $\bar{A}_i$ are semi-abelian varieties by lemma 5.4. Hence we are done if we prove that $\bar{A}_i$ is smooth over X. Note that $\bar{A}_i$ is a closed subgroup scheme of J, hence $\bar{A}_i$ is locally of finite presentation over X. Since J is smooth, the map $J(R) \to J(R/I)$ will always be surjective for each pair (R, I) where R is a ring, with a map $\mathrm{Spec}(R) \to X$, and $I \subset R$ is an ideal with $I^2 = 0$. But then the map $\bar{A}_i(R) \to \bar{A}_i(R/I)$ is also surjective, since $J(T) \cong \bar{A}_1(T) \times \bar{A}_2(T)$ is a product of non-empty sets (we always have a 0-section), for each test object $T \to X$. We conclude that the $\bar{A}_i$ are indeed smooth over X. □

5.4 Lemma. — *Let k be an algebraically closed field. Let $1 \to G' \to G \to G'' \to 1$ be a short exact sequence of smooth group schemes over k. The group schemes G' and G'' are semi-abelian if and only if G is semi-abelian.*

Proof. By a theorem of Chevalley, a smooth group scheme over k decomposes as

$$1 \to G^{\mathrm{aff}} \to G \to G/G^{\mathrm{aff}} \to 1,$$

where G^{aff} is an affine group scheme, and the quotient G/G^{aff} is an abelian variety. Hence G is semi-abelian if and only if G^{aff} is a torus. Moreover, smooth affine group schemes over $k = \bar{k}$ are products of tori and unipotent groups.

The lemma is now immediate. Indeed G^{aff} surjects onto $(G'')^{\mathrm{aff}}$, so the latter may only have a non-trivial unipotent factor if the former has as well. If G'' is semi-abelian, and G is not, then the unipotent part of G^{aff} is in the kernel of $G \to G''$. Hence $(G')^{\mathrm{aff}}$ contains a unipotent part. □

Bibliography

[1] Brian Conrad. *Gabber's lemma.*

[2] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*. Vol. 39. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2000, pp. xii+208.

[3] Michel Raynaud and Laurent Gruson. "Critères de platitude et de projectivité. Techniques de "platification" d'un module". In: *Invent. Math.* 13 (1971), pp. 1–89.

[Stacks] The Stacks Project Authors. *Stacks Project*. `http://stacks.math.columbia.edu`. 2015.