Good reduction

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1 Introduction

In a course on elliptic curves the topic of *good reduction* will pass by sooner or later. If one takes a close look, it is usually a bit vague what is really meant by *good reduction*. In this talk we will make it more precise.

NB: We will use the term *elliptic curve* in our motivation quite a bit, but postpone the definition till we want to make precise statements about them.

The word *reduction* suggests that we want to look at quotient maps, or something similar. E.g., we have local ring R, with residue field κ, an R-scheme X, and then we define the reduction $\tilde{X}$ as $X_\kappa = X \times_R \operatorname{Spec} \kappa$.

On the other hand, given an elliptic curve $E/\mathbb{Q}$, we are 'used' to reducing the curve modulo a prime p, to a curve $\tilde{E}/\mathbb{F}_p$. This does not really fit into our first attempt, as we cannot base change from $\mathbb{Q}$ to $\mathbb{F}_p$.

We see that a solution to this problem would be passing through the base $\mathbb{Z}$. We have to do this in the right manner, though. Of course every $\mathbb{Q}$-scheme is also a $\mathbb{Z}$-scheme, so we could base change the composition $E/\mathbb{Z}$ to $\mathbb{F}_p$. However, this would give us the empty scheme, which is of course not very interesting. Indeed the only non-trivial fibre of $E \to \operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$ lies over the generic point of $\operatorname{Spec} \mathbb{Z}$. We would like to extend E to $\operatorname{Spec} \mathbb{Z}$ such that the other fibres are non-trivial.

To generalise this picture, we might say that we have an integral scheme S, with generic point η. Further we have a scheme X over the function field, $\kappa(\eta)$. To 'reduce' X at some point $s \in S$, we first need to extend $X/\kappa(\eta)$ to a scheme over S, and then we can look at the fibre above s. I.e., we are looking for a solution to the problem

$$\begin{array}{ccc} X & \longrightarrow & ? \\ \big\downarrow & & \big\downarrow \\ \kappa(\eta) & \longrightarrow & S \end{array}$$

such that the diagram is Cartesian. As we have seen, we could take $? = X$, but then we do not get any interesting reduction. To prevent this kind of solutions we want nice properties of the extension of X. For this we will introduce the notions of *flat morphism* and *smooth morphism* below.

However, we can only hope for nice properties of the extension if $X \to \kappa(\eta)$ has those nice properties, since we will see that the nice properties are stable under base change.

2 Flat and smooth morphisms

For a good and readable introduction I refer to the notes [1] on smooth morphisms by Peter Bruin.

Let $f\colon X \to S$ be a morphism of schemes.

Recall from commutative algebra that a ring map $A \to B$ is of finite presentation if there is an isomorphism of A-algebras $B \cong A[X_1, \dots, X_n]/(P_1, \dots, P_m)$. I.e., B can be generated as A-algebra by finitely many elements, with finitely many relations on the generators.

2.1 **Definition.** We say that f is *locally of finite presentation* if there exists an open affine cover $(\operatorname{Spec} B_i)_i$ of S, such that for every i the inverse image $f^{-1}(\operatorname{Spec} B_i)$ has a cover by spectra of finitely presented B_i-algebras. «

Recall that for a commutative ring A an A-module M is called *flat* if the functor $M \otimes \_$ is exact (or equivalently, left-exact). A ring map $A \to B$ is called flat if B is flat as A-module.

2.2 **Definition.** The morphism f is called *flat* if for each $x \in X$ the induced map on stalks $\mathcal{O}_{S,f(x)} \to \mathcal{O}_{X,x}$ is flat. «

2.3 **Example.** Since flatness (of modules) is a stalk local property, a map of rings $A \to B$ is flat if and only if $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat.

Localisation maps are flat. In particular $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$ is flat. Thus, if we have a flat $\mathbb{Q}$-scheme X, then $X \to \operatorname{Spec} \mathbb{Z}$ is a flat extension to $\mathbb{Z}$.

Any scheme over a field is flat. «

2.4 **Lemma.** *The composition of two flat morphisms is flat. Flat morphisms are stable under base change.* «

The following proposition will be handy in the near future.

2.5 **Proposition.** *Assume that*

- *S is a Dedekind scheme, and*
- *X is integral, and*
- *f is non-constant,*

then f is flat.

Proof. See [2, Corollary 4.3.10]. □

2.6 **Definition.** The morphism f is called *smooth* if

- it is locally of finite presentation, and
- it is flat, and

- for each geometric point $\bar{s} \to S$ the fibre $X_{\bar{s}} = X \times_S \bar{s}$ is regular (i.e., a nonsingular variety). «

This definition shows one way of thinking about smooth morphisms: they are 'continuous' families (flat) of nonsingular varieties (locally of finite presentation, geometrically regular fibres) parametrized by the base.

2.7 **Remark.** Recall that we have the Jacobian criterion to determine if a geometric fibre is regular. «

2.8 **Lemma.** *The composition of two smooth morphisms is smooth. Smooth morphisms are stable under base change.* «

2.9 **Example.** For any scheme S, and any positive integer n, the structure morphisms $\mathbb{A}^n_S \to S$ and $\mathbb{P}^n_S \to S$ are smooth. «

3 Good reduction

We give the following definition in a local setting. Let k be a local field, with valuation v, and valuation ring $(A, \mathfrak{m}, \kappa)$. Let X/k be a proper smooth scheme.

3.1 **Definition.** We say that X has *good reduction at v* if there exists a proper smooth A-scheme $\mathcal{X}$ such that the generic fibre $\mathcal{X}_k$ is isomorphic to X. «

It is now immediate that the special fibre $\mathcal{X}_\kappa$ is a proper smooth scheme over $\operatorname{Spec} \kappa$.

3.2 **Example.** Take $k = \mathbb{Q}_5$, and let X be the scheme cut out of $\mathbb{P}^2_k$ by the homogenous equation $y^2 z = x^3 + 2xz^2 + 4z^3$. Observe that X/k is projective, hence proper. Further (by theory of elliptic curves) we know that X is smooth, since $\Delta = -16(4 \cdot 2^3 + 27 \cdot 4^2) = -7424 \neq 0$. Also Δ is not divisible by 5, and therefore we know that $y^2 z = x^3 + 2xz^2 + 4z^3$ also defines a proper smooth scheme $\mathcal{X}$ over $\mathbb{Z}_5$. Indeed $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}_5$ is flat (by proposition 2.5), and locally of finite presentation. Further both fibres are geometrically regular.

The same equation also defines a smooth scheme over $\mathbb{Q}_2$, but not over $\mathbb{Z}_2$ (since then the fibre over $\mathbb{F}_2$ is singular). «

One can generalise this definition of good reduction to the setting of global fields. However, via completion at primes, it is clear that it is in essence a local problem.

4 The criterion of Néron-Ogg-Shafarevich

The criterion of Néron-Ogg-Shafarevich provides us with an alternative for testing whether an elliptic curve (or more general an abelian variety) has good reduction at a certain prime. We give the statement in a local setting. But before we proceed we give a proper (pun unintended) definition of an elliptic curve.

4.1 Definition. Let S be a scheme. An *elliptic curve over* S is a morphism $E \to S$ that is proper, smooth and with geometrically connected fibres all curves of genus 1, together with a fixed section $0 \in E(S)$. «

4.2 Remark. A couple of non-trivial facts, that 'should' be true, and indeed are true.

- One can show that $E \to S$ is projective.
- There is still a Weierstrass equation for E.
- For every S-scheme T the set $E(T)$ is a group in a functorial way. «

We now restrict our attention to the case of local fields. Let k be a local field, with valuation v, and valuation ring $(A, \mathfrak{m}, \kappa)$. Further l denotes a separable closure of k, and G the Galois group $\mathrm{Gal}(l/k)$. Let $I \subset G$ be the inertia group. For any integer n we denote with E_n the subgroup of n-torsion points in $E(l)$. Let ℓ be a prime number different from $\mathrm{char}\,\kappa$.

4.3 Definition. An elliptic curve E/k is said to have *good reduction* at v if there exists an elliptic curve $\mathcal{E}/A$ such that the generic fibre $\mathcal{E}_k$ is isomorphic to E/k.«

4.4 Remark. There is a subtle difference with the general definition of good reduction, since we also need an extension of the zero section. «

4.5 Definition. Let X be a set equipped with a G-action. We call X *unramified at* v if I acts trivially on X, i.e., I is contained in the kernel of $G \to \mathrm{Aut}(X)$. «

We can now formulate the celebrated criterion.

4.6 Theorem. *The following are equivalent:*

1. *E has good reduction at v;*
2. *E_n is unramified at v for all n coprime to* $\mathrm{char}\,\kappa$*;*
3. *E_n is unramified at v for infinitely many n coprime to* $\mathrm{char}\,\kappa$*;*

Proof. For a proof of the elliptic curve case, see [4]. The general case of abelian varieties is proven in [3]. □

References

[1] Peter Bruin. *Smooth morphisms*. URL: user.math.uzh.ch/bruin/smooth.pdf.

[2] Qing Liu. *Algebraic Geometry and Arithmetic Curves*. Trans. by R.Q. Erne. Oxford Graduate Texts in Mathematics. OUP Oxford, 2006.

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[4] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*. Graduate Texts in Mathematics. Springer, 2010. ISBN: 9781441918581.