Motivic cohomology

Johan M. Commelin

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1 Motivic Eilenberg–MacLane spectrum

We can try to mimic the definition of Eilenberg–MacLane spaces to obtain a motivic Eilenberg–MacLane spectrum; but the naive approach does not work.

Let (X, x) be a pointed motivic space (i.e., an object in $\Delta^{\circ}\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}/k)$). Then we define the homotopy groups of (X, x) as

$$\pi_{i,U}(X, x) = \mathrm{Hom}_{H_{\bullet}^{\mathbb{A}^1}}(S^i_s \wedge U_+, (X, x)),$$

where $U \in \mathrm{Sm}/k$; so that we obtain a presheaf on Sm/k.

Temporarily define $K(\mathbb{Z}, n)$ to be the motivic space that has the property that

$$\pi_{i,U}(K(\mathbb{Z}, n)) = \begin{cases} \mathbb{Z} & \text{if } i = n, \\ 0 & \text{if } i \neq n. \end{cases}$$

To turn this into a $\mathbb{P}^1$-spectrum we would need structure morphisms

$$(\mathbb{P}^1, \infty) \wedge K(\mathbb{Z}, n) \to K(\mathbb{Z}, n+1),$$

however, all such morphisms are trivial in the $\mathbb{A}^1$-homotopy category (according to Voevodsky). This sounds like something we do not want.

Now there is a theorem in algebraic topology that I do not know—there are actually many! But the theorem I have in mind is the Dold–Thom theorem. Let $(T, \star)$ be a pointed CW-complex. Let $\mathrm{Sym}^n(T, \star)$ denote $(T, \star)^{\times n}/\mathfrak{S}_n$. Let $\mathrm{Sym}^n(T, \star) \to \mathrm{Sym}^{n+1}(T, \star)$ be the map $(x_1, \dots, x_n) \to (x_1, \dots, x_n, \star)$. Write $\mathrm{Sym}^{\infty}(T, \star)$ for $\mathrm{colim}_n \mathrm{Sym}^n(T, \star)$. The Dold–Thom theorem states that:

$$(\mathrm{Sym}^{\infty}(T, \star))^+ = \prod_{i \geqslant 0} K(H_i(T), i),$$

where $(\_)^+$ denotes the group associated with a monoid.

In particular, one can define $K(\mathbb{Z}, n)$ as $(\mathrm{Sym}^{\infty} S^n)^+$.

It is this construction that we mimic, and that gives a working definition of motivic Eilenberg–MacLane spectrum. If we copy the definitions verbatim, we get a space $\mathrm{Sym}^{\infty}(X_+)$, for $X \in \mathrm{Sm}/k$, and this should more or less be a space of formal linear combinations of points of X (with positive coefficients). There are some subtleties involved (think about generic points, etc); and the correct definition is the following related notion:

For $X, U \in \mathrm{Sm}/k$ with U connected, define $c(U, X)$ as the free abelian group generated by closed irreducible subsets of $U \times X$ that are finite and surjective over U. This is a contravariant functor in U, and we obtain a Nisnevich sheaf $c(\_, X)$ on Sm/k. It is a pointed space if we forget the group structure, but remember 0. Denote it with $L(X)$. The graph of a morphism $U \to X$ is a closed irreducible subset of $U \times X$ (finite and surjective over U), so we obtain canonical maps $\mathrm{Hom}(U, X) \to c(U, X)$, and therefore $X \to L(X)$.

Instead of $(\mathrm{Sym}^\infty(X_+)^+$ we use $L(X)$ for our definition of motivic Eilenberg–MacLane spaces. We want to use $L(\mathbb{P}^1, \infty)^{\wedge n}$ but this is not yet defined. Voevodsky has a way out: call a space *scheme-like* if it is of the form $X/(\cup_i Z_i)$ where the Z_i are smooth subschemes of a scheme X, such that all intersections of Z_i's are also smooth. The class of scheme-like spaces is closed under smash-products and contains S^1_s.

Define $K(\mathbb{Z}(n), 2n) = L(\mathbb{P}^1, \infty)^{\wedge n}$. The weird notation has everything to do with the associated cohomology theory (and its weird notation, which has everything to do with motives). (Aside: apparently the wrong naive definition outlined in the beginning is weakly equivalent to $L(S^n_s)$.)

We now have to assemble these spaces into a spectrum. The external product

$$\begin{aligned} c(U, X) \times c(U, X') &\longrightarrow c(U, X \times X') \\ (Z, Z') &\longmapsto Z \times_U Z' \end{aligned}$$

induces a bilinear map $L(X) \times L(X') \to L(X \times X')$, which in turn induces a map $L(X) \wedge L(X') \to L(X \wedge X')$. Thus we obtain morphisms

$$\tau_{m,n} \colon K(\mathbb{Z}(m), 2m) \wedge (\mathbb{Z}(n), 2n) \to (\mathbb{Z}(m+n), 2(m+n)).$$

Recall the canonical map $i \colon (\mathbb{P}^1, \infty) \to L(\mathbb{P}^1, \infty)$. The maps $\sigma_n = \tau_{1,n} \circ (i \wedge \mathrm{id})$ are the structure morphisms for the *motivic Eilenberg–MacLane spectrum*:

$$\mathbf{H}\mathbb{Z} \colon (K(\mathbb{Z}(n), 2n), \sigma_n)$$

For a $X \in \mathrm{Sm}/k$ recall that

$$\mathbf{H}\mathbb{Z}^{p,q}(X) = \mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty_{s,t}, S^{p-q}_s \wedge S^q_t \wedge \mathbf{H}\mathbb{Z}).$$

This cohomology theory is called *motivic cohomology* (with integral coefficients). One should think of p as indexing *degree*, and q indexing *weight*. Observe that when $p = 2q$, in the definition of cohomology, we are smashing with $(\mathbb{P}^1)^{\wedge q}$.

One can show that $\mathbf{H}\mathbb{Z}^{p,q}(X)$ is

— isomorphic to $\mathrm{CH}^q(X, 2q - p)$, the higher Chow group defined by Bloch;

— the Zariski hypercohomology of certain complexes of sheaves

$$\mathbf{H}^{p,q}(X, \mathbb{Z}) = H^p_{\mathrm{Zar}}(X, \mathbb{Z}(q));$$

conjectured to exist by Beilinson.

So far the only motivation for what we have done has been the Eilenberg–MacLane spaces in "ordinary" topology, and we have tried to find an analogue. Now suddenly these higher Chow groups pop up, and we have no idea what they are. Let me try to introduce them briefly.

2 Chow groups

First of all, some motivation. We are going to define so called *higher Chow groups* $CH^q(X,n)$, first defined by S. Bloch in 1986. To motivate the people who participated in the algebraic K-theory seminar, note the following theorem:

Let X be a quasi-projective scheme over a field k (i.e., open in closed in $\mathbb{P}^n_k$). Then there is an isomorphism

$$K_n(X)_{\mathbb{Q}} \to \oplus_q CH^q(X,n).$$

where $K_n(X)$ is the n-th higher K-group of coherent sheaves as defined by Quillen. So apparently, motivic cohomology can compute higher K-theory (if we switch to rational coefficients).

2.1 Definition

The definition of $CH^q(X,n)$ mimics the simplicial definition of homology. Let $z^q(Y)$ be the free abelian group with generators the irreducible closed subvarieties $Z \subset Y$ of codimension q (i.e., $\dim Y - \dim Z = q$). We assemble these in a graded group $z^*(Y)$.

For $n \geqslant 0$, define

$$\Delta^n = \mathrm{Spec}\left(k[t_0,\dots,t_n]/\left(\sum t_i - 1\right)\right) \cong \mathbb{A}^n_k.$$

Just as in topology, we have face maps and degeneracy maps: if $\rho\colon \{0,\dots,m\} \to \{0,\dots,n\}$ is an increasing map, define $\tilde{\rho}\colon \Delta^m \to \Delta^n$ by $\tilde{\rho}^*(t_i = \sum_{\rho(j)=i} t_j$. The map $\tilde{\rho}$ is a face (resp. degeneracy), if ρ is injective (resp. surjective).

Define $z^*(X,n) \subset z^*(X \times \Delta^n)$ to be the subgroup generated by irreducible subvarieties Z intersecting all faces $X \times \Delta^m \subset X \times \Delta^n$ transversally (this means that the intersection has the expected codimension: $\mathrm{codim}(Z \cap (X \times \Delta^m) = \mathrm{codim}(Z) + \mathrm{codim}(X \times \Delta^m))$. The face and degeneracy maps induce a simplicial complex $z^*(X,\_)$. The higher Chow groups are defined to be the homology of this complex:

$$CH^*(X,n) = \frac{\text{cycles} \subset z^*(X,n)}{\text{boundaries} \subset z^*(X,n)}$$

(Aside, for the algebraic geometers: the classical theory of algebraic cycles is recovered by putting $n = 0$. In other words $CH^q(X,0)$ is the group of algebraic cycles of codimension q, modulo rational equivalence. The rational equivalence is captured by modding out the boundaries above.)

One Chow group that is particularly well-known is $CH^1(X,0)$. It is isomorphic to the Picard group of X: the group of line bundles up to isomorphism.

References

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[2] Voevodsky, Vladimir. $\mathbf{A}^1$-homotopy theory. Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998). Doc. Math. 1998, Extra Vol. I, 579–604 (electronic). MR1648048