Intro to schemes and their basic properties

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1 Intro: RS, LRS, and Sch

To do geometry, it makes sense to consider spaces and functions on them. A *ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space, and $\mathcal{O}_X$ is a sheaf of rings on X. A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$, where f is a continuous map $X \to Y$, and $f^\#$ is a morphism of sheaves of rings $\mathcal{O}_Y \to f_*\mathcal{O}_X$. The *category of ringed spaces* is denoted RS.

We have adjunctions

$$\mathrm{Hom}_{\mathrm{RS}}((X, 0_X), (Y, \mathcal{O}_Y)) \cong \mathrm{Hom}_{\mathrm{Top}}(X, Y) \cong \mathrm{Hom}_{\mathrm{RS}}((X, \mathcal{O}_X), (Y, \mathbb{Z}_Y)).$$

Where 0_X and $\mathbb{Z}_Y$ are the constant sheafs attached to the zero ring and $\mathbb{Z}$ respectively. Hence the forgetful functor RS $\to$ Top preserves all limits and colimits.

To reflect geometry better, we want a notion of functions vanishing at a point. The functions that vanish at a point should form a unique maximal ideal, in other words: the stalks should be local rings. This is equivalent to wanting the value of a function at a point to be an element of a field. Morphisms should preserve the local ring structure. This brings us to the definition of locally ringed spaces.

A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$, such that for each $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring (i.e., has a unique maximal ideal). A *morphism of locally ringed spaces* is a morphism of ringed spaces $(f, f^{\#})\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, such that for each $x \in X$ the induced map on stalks

$$f_x^{\#}\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

is a morphism of local rings (i.e., it maps the maximal ideal on the left *into* the maximal ideal on the right). The *category of ringed spaces* is denoted LRS.

The global sections functor

$$\begin{aligned} \Gamma\colon \quad \mathrm{LRS} &\longrightarrow \mathrm{CRing}^{\circ} \\ (X, \mathcal{O}_X) &\longmapsto \mathcal{O}_X(X) \end{aligned}$$

has a right adjoint

$$\mathrm{Spec}\colon \mathrm{CRing}^{\circ} \longrightarrow \mathrm{LRS},$$

so that

$$\mathrm{Hom}_{\mathrm{LRS}}((X, \mathcal{O}_X), \mathrm{Spec}(A)) \cong \mathrm{Hom}_{\mathrm{CRing}^{\circ}}(\mathcal{O}_X(X), A) \cong \mathrm{Hom}_{\mathrm{CRing}}(A, \mathcal{O}_X(X)).$$

Consequently, Spec maps coproducts (tensor product) in CRing to products in LRS. The functor Spec is fully faithful, and the essential image of CRing° is the *category of affine schemes*.

(Aside: the global section functor on ringed spaces also has a right adjoint. However, it is rather dull: map rings to the one point space with the ring as sheaf.)

A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$, such that X can be covered by open subsets U_i, so that $\left(U_i, \mathcal{O}_X|_{U_i}\right)$ is an affine scheme. The *category of schemes*, denoted Sch, is the full subcategory of LRS whose objects are schemes.

If we denote a scheme with one letter, say S, then it is common to denote the underlying topological space with $|S|$.

Abstract nonsense tells us that the Yoneda embedding

$$\mathrm{Sch} \longrightarrow [\mathrm{Sch}^{\circ}, \mathrm{Set}]$$

is fully faithful. If we postcompose this with the restriction to presheaves on CRing°, the resulting functor

$$\mathrm{Sch} \longrightarrow [\mathrm{CRing}, \mathrm{Set}]$$

is still fully faithful.

We can describe the Spec-functor in a bit more detail. Let A be a commutative ring. Then $\mathrm{Spec}(A)$ is

set: the set of prime ideals in A;

top: endowed with the *Zariski topology*: the *closed* subsets are precisely the subsets $Z(I) = \{p \in \mathrm{Spec}(A) \mid p \supset I\}$, for some ideal $I \subset A$.

For any element $f \in A$, let $I_f = (f)$ be the ideal generated by f, and $D_f = \mathrm{Spec}(A) - Z(I_f)$ the associated open subset. The D_f form a basis for the topology.

sheaf: Since the D_f form a basis for the topology, it suffices to specify the sheaf on these opens. On D_f the sheaf takes as value $A_f = A[\frac{1}{f}]$.

At a point $\mathfrak{p} \in \mathrm{Spec}(A)$ (i.e., a prime ideal $\mathfrak{p} \subset A$), the stalk is precisely $A_\mathfrak{p}$. (Recall that $A_\mathfrak{p} = S^{-1}A$, where $S = A - \mathfrak{p}$.)

The locally ringed space $\left(D_f, \mathcal{O}_{\mathrm{Spec}(A)}|_{D_f}\right)$ is isomorphic to $\mathrm{Spec}(A_f)$.

Intuition: view $f \in A$ as a function. Then $Z(I_f)$ is the set where f vanishes, and D_f is the set where f is non-zero.

1.1 Generic points

A topological space is *irreducible* if it cannot be written as a union of proper closed subsets. In particular, irreducible spaces are connected.

The underlying topological space of a scheme is *sober*, i.e., every irreducible closed subspace has a *generic point*.

Let A be a ring. Closed points of $\mathrm{Spec}(A)$ are precisely maximal ideals of A, and prime ideals that are not maximal correspond to non-trivial generic points of $\mathrm{Spec}(A)$.

Let us take the affine plane as an example. Let k be an algebraically closed field. Write $\mathbb{A}^2_k = \mathrm{Spec}(k[x, y])$. By the Hilbert Nullstellensatz, the closed points (i.e., maximal ideals) are in bijection with k^2, via the injective map

$$\begin{aligned} k^2 &\longrightarrow \mathbb{A}^2_k \\ (a, b) &\longmapsto (x - a, y - b). \end{aligned}$$

The ideal $(x) \subset k[x, y]$ is a prime ideal, and the closure of the point (x) in $\mathbb{A}^2_k$ consists of all the points $(x, y - b)$, with b ranging through k. (That are precisely "the points on the y-axis" in the domain of the map above!)

The ideal $(y - x^2) \subset k[x, y]$ is also prime (mod it out, to obtain the integral domain $k[x]$), and points in its closure are precisely points $(x - a, y - b)$, such that $(y - x^2) \subset (x - a, y - b)$. If $y - x^2 \in (x - a, y - b)$, then $a^2 - b$ is also an element of $(x - a, y - b)$, since

$$(y - b) - (x - a)^2 - 2a(x - a) - (y - x^2) = a^2 - b.$$

Therefore $a^2 - b = 0$. Thus the closure of $(y - x^2)$ corresponds to the parabola $y = x^2$, just like one would expect.

In this way, for each irreducible curve, there is a generic point in $\mathbb{A}^2_k$. Finally, there is the generic point (0), whose closure is the entire plane.

As another example, let us consider $\mathrm{Spec}(\mathbb{Z})$. Its closed points correspond to the ideals (p), where p is a prime number. The stalk of the structure sheaf $\mathcal{O}_{\mathrm{Spec}(\mathbb{Z})}$ at (p) is $\mathbb{Z}_{(p)}$: the ring of fractions $\frac{a}{b} \in \mathbb{Q}$, where p does not divide b. These stalks are local rings, with maximal ideal generated by p. The *residue field* is $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \cong \mathbb{Z}/p\mathbb{Z}$.

There is 1 generic point, namely (0). The stalk at (0) is precisely $\mathbb{Q}$.

1.2 Fiber products

The category of schemes has fibre products. As indicated above, fibre products of affine schemes correspond to tensor products of commutative rings. This provides us with the local picture of fibre products. Fibre products of arbitrary schemes are obtained by gluing the local picture.

Note that $\mathrm{Spec}(\mathbb{Z})$ is the final object of Sch (and even of LRS). Since the underlying topological space of $\mathrm{Spec}(\mathbb{Z})$ is not a point, the forgetful functor $\mathrm{Sch} \to \mathrm{Top}$ does not preserve fibre products.

In particular, the underlying topological space of the product of two schemes need not (and in practice, does not) agree with the product topology. But this is not a bad thing. Let us consider the example of the affine plane again. The affine line $\mathbb{A}^1_k$ is by definition $\mathrm{Spec}(k[x])$. Since $k[x, y] = k[x] \otimes_k k[y]$, the identity $\mathbb{A}^2_k = \mathbb{A}^1_k \times_{\mathrm{Spec}(k)} \mathbb{A}^1_k$ holds, as expected.

But in $\mathbb{A}^2_k$ we have generic points for all closed curves on the plane (e.g., the parabola we saw above), which one would not see in the product topology.

2 Open and closed subschemes

Let S be a scheme. Let U be an open subset of S. We claim that the locally ringed space $(U, \mathcal{O}_S|_U)$ is a scheme. Let $W \subset S$ be an affine open subscheme. Recall that the open subsets D_f form a basis for the topology of W. Besides that, these D_f are affine open subschemes of W. Consequently $W \cap U$ is covered by affine open subschemes. The locally ringed space $(U, \mathcal{O}_S|_U)$ is called an *open subscheme* of S.

Let $f\colon X \to S$ be a morphism of schemes. We say that f is an *open immersion* if f factors as $g\colon X \to U$ followed by an inclusion $U \to S$, where g is isomorphism between $(X, \mathcal{O}_X)$ and $(U, \mathcal{O}_S|_U)$.

With closed subschemes, the situation is a little bit more complicated. In the affine case we can see what is going on. Let A be a commutative ring. Closed subsets of $\mathrm{Spec}(A)$ are given by ideals $I \subset A$. The underlying topological space of $\mathrm{Spec}(A/I)$ is canonically homeomorphic to $Z(I)$. The *closed subschemes* of $\mathrm{Spec}(A)$ are precisely the schemes $\mathrm{Spec}(A/I)$.

The global picture requires the notion of *quasi-coherent ideal sheaves* $\mathcal{I} \subset \mathcal{O}_S$; but we will not go into the definition of those now. Important to observe is that a closed subset $Z \subset S$ does *not* determine the structure of a closed subscheme.

As with open immersions, a map of schemes is a *closed immersion* if it factors as an isomorphism with a closed subscheme, followed by the inclusion map.

3 Irreducible, reduced, and integral schemes

3.1 Irreducible schemes

A scheme is *irreducible* (resp. *connected*) if the underlying topological space is irreducible (resp. connected). In light of our discussion of generic points, a scheme S is irreducible if and only if there is a point $\eta \in S$ such that $S = \overline{\{\eta\}}$.

3.2 Reduced schemes

A commutative ring R is *reduced* if it contains no nilpotent elements: if for some $x \in R$ and $n > 0$, one has $x^n = 0$, then $x = 0$. The subset of nilpotent elements is an ideal, called the *nilradical*, and equals the intersection of all prime ideals. Hence, a ring is reduced if and only if the nilradical is the zero ideal.

A scheme $(S, \mathcal{O}_S)$ is *reduced* if for all open subsets $U \subset S$, the ring $\mathcal{O}_S(U)$ is reduced. Equivalently, a scheme is reduced if and only if all stalks are reduced.

The intuition of nilpotent elements in the structure sheaf is that they carry infinitesimal information. For example, if k is a field, one pictures $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ as a point with an infinitesimal tangent vector. (The vector has length ε, and $\varepsilon^2 = 0$.) This affine scheme represents the tangent space functor of (pointed) k-varieties.

If S is a scheme, there is an associated reduced scheme S_{red}, which is the closed subscheme corresponding to the ideal sheaf of nilradicals. This construction is functorial, and right adjoint to the inclusion of reduced schemes into schemes.

$$\mathrm{Hom}_{\mathrm{Sch}}(X, Y) \cong \mathrm{Hom}_{\mathrm{redSch}}(X, Y_{\mathrm{red}})$$

3.3 Integral schemes

A scheme $(S, \mathcal{O}_S)$ is *integral* if for all open subsets $U \subset S$, the ring $\mathcal{O}_S(U)$ is an integral domain.

A scheme is integral if and only if it is irreducible and reduced. Indeed, as a sketch of a proof: in an integral domain the zero ideal is a prime ideal; therefore the nilradical is trivial, and the zero ideal is a generic point for the entire scheme.

4 Separated schemes and morphisms

As pointed out above, schemes are almost never Hausdorff. However, we also saw that the topology of the product of schemes was not the product topology, i.e., if S is a scheme, then $|S \times S| \ncong |S| \times |S|$. And this comes to our rescue: a topological space X is Hausdorff if and only if the diagonal $\Delta \subset X \times X$ is closed.

A scheme is *separated* if $\Delta\colon S \to S \times S$ is a closed immersion. This actually happens "quite often". Most schemes that most algebraic geometers encounter in daily life are separated.

Often in algebraic geometry, one works in a relative setting, i.e., in the slice category $\mathrm{Sch}_{/S}$, of schemes over a fixed base scheme S. It makes sense to say that $X \to S$ is *separated* if $\Delta\colon X \to X \times_S X$ is a closed immersion. In that case, we also say that X is a separated S-scheme.

Separated morphisms are closed under composition and base change. Being closed under base change means that if $X \to S$ is separated and $S \to S$ is some morphism, then the projection $X \times_S S' \to S'$ is also separated.

Moreover, if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms, and $g \circ f$ is separated, then f is separated.

All affine schemes are separated, and the classical example of a non-separated scheme is the line with double origin: glue two affine lines along the complements of their origins.

5 Noetherian schemes

A commutative ring is a *Noetherian* ring if every increasing chain of ideals

$$I_1 \subset I_2 \subset I_3 \subset \dots$$

stabilises: there is some n such that $I_i = I_n$ for all $i \geqslant n$.

Examples:

- $\mathbb{Z}$;
- fields;
- localisations of Noetherian rings;
- completions of Noetherian rings at some ideal;
- rings that are finitely generated over a Noetherian ring.

A *locally Noetherian scheme* is a scheme that admits an open cover by spectra of Noetherian rings. A *Noetherian scheme* is a quasi-compact locally Noetherian scheme, i.e., the cover can be chosen to be finite.

Every affine subscheme of a locally Noetherian scheme is the spectrum of a Noetherian ring.

5.1 What are Noetherian schemes good for?

As far as I can see, we want to consider Noetherian schemes since they will allow for a Mayer–Vietoris sequence in the Nisnevich topology. The key ingredient for that is the fact that if S is a Noetherian scheme, then $|S|$ is a *Noetherian topological space:* every decreasing chain of closed subsets

$$Z_1 \supset Z_2 \supset Z_3 \supset \dots$$

stabalises.

The converse is not true: there exist non-Noetherian schemes S, such that $|S|$ is a Noetherian topological space. You can even choose S affine.