Smooth and étale morphisms of schemes

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May 6, 2015

1 Introduction

Smooth morphisms are the analogue in algebraic geometry of what submersions are in diffential geometry. Étale morphisms are smooth morphisms with discrete fibres; they are the analogue of local isomorphisms (though they are *not* local isomorphisms for the Zariski topology).

We finish this talk with a discussion of Henselian local rings and the Nisnevich topology.

2 Formally (smooth | étale | unramified)

Define the following two classes of morphisms

$$\begin{aligned} \mathrm{Inf} &= \big\{ q\colon \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A/I) \,\big|\, A \in \mathrm{CRing}, I \subset A \text{ a nilpotent ideal} \big\} \\ \mathrm{Inf}_2 &= \big\{ q\colon \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A/I) \,\big|\, A \in \mathrm{CRing}, I \subset A \text{ a square-zero ideal} \big\}. \end{aligned}$$

A map of schemes $f\colon X \to Y$ is *formally smooth* if it satisfies the right lifting property with respect to Inf_2. It is a theorem that they satisfy the right lifing property with respect to Inf.

Suppose X, Y are k-schemes of finite type, and f is a k-morphism. If f is formally smooth, then it is indeed a "submersion":

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{\ p\ } & X \\ \big\downarrow q & \overset{?}{\nearrow} & \big\downarrow f \\ \mathrm{Spec}(k[\varepsilon]) & \xrightarrow{\ t\ } & Y; \end{array}$$

Recall that $\mathrm{Spec}(k[\varepsilon]) = \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ represents the tangent space functor.

The map p is a k-point of X, and via f it is mapped to Y. The map t gives a tangent vector at the point $f \circ p$. By definition of formally smooth, this tangent vector lifts to $p \in X(k)$. Thus on tangent spaces f is surjective.

Because not all schemes are finite type over a field, the definition refers to the general class Inf_2. Still, using the functor of points, the right lifting property is equivalent to the map of "generalized tangent spaces"

$$X(A/I)_p \longrightarrow Y(A/I)_{f(p)}$$

being surjective for all $p \in X(A/I^n)$, where $X(A/I)_p$ is the subset of $X(A/I)$ that maps to p via q (i.e., "generalized tangent vectors at p").

We can make this precise using sheaves on $\mathrm{Spec}(A/I)$: If $Z \to Y$ is some Y-scheme, then $(X/Y)_Z$ is the sheaf on Z mapping $U \subset Z$ to $\mathrm{Hom}_Y(U, X)$. Being formally smooth amounts to saying that

$$(X/Y)_{\mathrm{Spec}(A/I^n)} \longrightarrow (X/Y)_{\mathrm{Spec}(A/I)}$$

is surjective, where we view both sheaves as sheaves on $\mathrm{Spec}(A/I)$, using that $q\colon \mathrm{Spec}(A/I) \to \mathrm{Spec}(A/I^n)$ is a homeomorphism.

The map f is *formally étale* if it is formally smooth, and the lift in the right lifting property is *unique*. Again, it follows that formally étale maps satisfy unique right lifting with respect to Inf. So, in light of the above, morally speaking, "the maps on tangent spaces are isomorphisms".

The map f is *formally unramified* if there is at most one lift in the diagrams for the right lifting property. Again, it formally unramified maps satisfy this property with respect to Inf. So, in light of the above, morally speaking, "the maps on tangent spaces are injections".

It follows from the definitions that formally (smooth | étale | unramified) morphisms are stable under base change and composition.

3 Smooth, étale and unramified morphisms

A morphism $f\colon X \to Y$ of schemes is

$$\begin{cases} \textit{smooth} \\ \textit{étale} \\ \textit{unramified} \end{cases} \text{if it is formally} \begin{cases} \text{smooth} \\ \text{étale} \\ \text{unramified} \end{cases} \text{and locally of finite} \begin{cases} \text{presentation} \\ \text{presentation} \\ \text{type} \end{cases}.$$

Remark. It seems a bit strange that *unramified* diverges from the pattern that *smooth* and *étale* seemed to start. However, there is some justification for this definition. Recall from last week that a proper map was also required to be locally of finite type (as opposed to locally of finite presentation). In both cases one of the reasons for the weaker definition is that it implies that closed immersions are unramified (resp. proper). Note that the distinction is only visible in the non-Noetherian situation. So, for example take $A = k[X_1, X_2, \ldots]$, and $I = (X_2, X_4, \ldots)$. Then $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ is a closed immersion, but is not of finite presentation.

3.1 Intuition

Let's work over $\mathbb{C}$. Let $X \to \mathrm{Spec}(\mathbb{C})$ be a connected affine complex scheme of finite type. Thus $X = \mathrm{Spec}(\mathbb{C}[t_1, \ldots, t_n]/(P_1, \ldots, P_m))$. It is smooth if and only if $X(\mathbb{C})$ is a smooth complex manifold. What I mean is this: we can view $X(\mathbb{C})$ as the locus of points in $\mathbb{C}^n$ that are solutions to the equations $P_1 = 0$, $\ldots$, $P_m = 0$. If this solution set defines a smooth complex manifold, then X is a smooth scheme.

In particular, we can use the Jacobian criterion to verify smoothness: Check that the Jacobian matrix $J = (\partial P_i/\partial t_j)_{ij}$ has rank m everywhere on X. Equivalently, check that the locus where all $(m \times m)$-minors of J vanish does not intersect X.

At each point $x \in X(\mathbb{C})$, the Jacobian J defines a map $\mathbb{C}^n \to \mathbb{C}^m$, and its kernel is tangent space at x. If X is smooth at x, the dimension of the tangent space equals the dimension of X, that is $n - m$. If X is not smooth at x, then the dimension of the tangent space is bigger then the dimension of X.

This picture can be translated to other fields then $\mathbb{C}$. In general a morphism is smooth if

» all its fibres are smooth,
» it is locally of finite presentation, and
» it is flat.

We will not go into flatness here, but it is a condition that makes sure that dimensions of the fibres don't jump. (After all, we don't the following example to be smooth: let X be the union of the axes in the plane, $\mathrm{Spec}(k[x, y]/(xy))$; let $f\colon X \to \mathbb{A}^1_k$ be the projection onto the x-coordinate. Then all fibres are smooth, and the map is of finite presentation, but it is not a "submersion".)

The entire intuition caries over to étale morphisms by additionally requiring $n = m$. In other words, étale morphisms are smooth morphisms with 0-dimensional fibres. Moreover, the Jacobian criterion says that in the case of an étale morphism, the Jacobian J is invertible at each point.

However, the inverse function theorem does not apply verbatim in algebraic geometry. If $f\colon X \to Y$ is an étale morphism, there need not be a Zariski-local inverse. By switching to the étale topology, we do recover an inverse function theorem: For every $y \in f(X)$ there exists an étale map $U \to Y$ with y in its image, such that the base change $X \times_Y U \to U$ is a disjoint cover of copies of U.

The fact that étale morphisms behave a lot like local diffeomorphisms in differentiable geometry is made precise by the following theorem of M. Artin:

Let $X \to \mathrm{Spec}(\mathbb{C})$ be a smooth complex scheme. Let $X(\mathbb{C})$ denote the associated complex manifold. Let $X_{\text{ét}}$ be the (small étale) site whose objects are étale morphisms to X (morphisms: those that make the obvious triangles commute), and with covers the topological covers. Let A be a finite abelian group. Then $H^i(X(\mathbb{C}), A)$ is canonically isomorphic to $H^i(X_{\text{ét}}, A)$.

Using the site $X_{\text{ét}}$, one can define the étale fundamental group $\pi_1(X_{\text{ét}}, x)$, where x is some base point. As a consequence of M. Artin's theorem, the finite quotients of $\pi_1(X_{\text{ét}}, x)$ and $\pi_1(X(\mathbb{C}), x)$ coincide. One uses this to prove that $\pi_1(X_{\text{ét}}, x)$ is the profinite completion of $\pi_1(X(\mathbb{C}), x)$.

3.2 Henselian local rings

Let us first recall *Hensel's lemma*: Let $f \in \mathbb{Z}_p[X]$ be a monic polynomial. Suppose $a \in \mathbb{Z}_p$ satisfies $f(a) \cong 0 \mod p$ and $f'(a) \not\cong 0 \mod p$. Then there is a $b \in \mathbb{Z}_p$ such that $f(b) = 0$ and $a \cong b \mod p$.

Let $(A, \mathfrak{m}, \kappa)$ be a local ring. It is *Henselian* if it satisfies Hensel's lemma: if f is a monic polynomial in $A[X]$ then any factorization of $\bar{f} \in \kappa[X]$ into coprime monic polynomials can be lifted to a factorization of f in $A[X]$.

A Henselian local ring is *strict* if κ is separably closed.

Let $(A, \mathfrak{m}, \kappa)$ be a local ring. There exists a Henselian local ring A^h, such that every map from A to a Henselian local ring B factors uniquely via A^h. The ring A^h is called the *Henselisation* of A, and is unique up to unique isomorphism.

There also exists a strict Henselian local ring A^{sh}, such that every map from A to a strict Henselian local ring B factors via A^{sh}. The ring A^{sh} is called the *strict Henselisation* of A, and is *not* unique up to unique isomorphism: It depends on the choice of a separable closure of κ. The isomorphism class of A^{sh} is unique.

Examples of Henselian rings:

- complete (Hausdorff) local rings, like $\mathbb{Z}_p$;
- quotients of Henselian rings.

Examples of strict Henselian rings:

- the ring of germs of holomorphic functions at a point of a Riemann surface, e.g., $\mathcal{O}_{\mathbb{C},0}$.

To prove that the last example is indeed an example boils down precisely to the inverse function theorem.

In light of M. Artin's theorem and the last example above, it might come as no surprise that the stalks of $(X_{\text{ét}}, \mathcal{O}_X)$ are strict Henselian local rings. Moreover, they are the strict Henselisations of the local rings for the Zariski topology.

4 Nisnevich topology

Recall that an étale cover of a scheme X is a collection of étale morphisms $U_i \to X$ such that $\bigcup_i U_i \to X$ is surjective.

For a cover to be Nisnevich, we impose the extra condition that for every $x \in X$, there is an i and $u \in U_i$ such that the map of residue fields $\kappa(x) \to \kappa(u)$ is an isomorphism.

The *small Nisnevich site* of X has the same underlying category as $X_{\text{ét}}$ but takes Nisnevich covers as coverings.

The local rings for the Nisnevich topology are the Henselisations of the local rings for the Zariski topology.