On ℓ-adic compatibility for abelian motives

&

the Mumford–Tate conjecture for products of K3 surfaces

Proefschrift

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To Milan and Remy,
my fellow apprentices at the mathematical workbench

Contents

0 Introduction

Readme. — This section starts with a general introduction of the topics of this thesis and gradually transforms into a more detailed overview of the structure and contents. We state our main results in §0.3. See §0.13 for a short word on how to navigate through this text. In §0.14 we explain our conventions, and we set up some basic notation and terminology.

0.1 — The theory of motives has many facets. This thesis focuses on two of these: compatible systems of Galois representations and the Mumford–Tate conjecture. We will return to the theory of motives later on (§0.4 and section 2). In fact, motives lie at the heart of most statements and arguments in this thesis. But let us first take a moment to admire the beauty of the two facets that have our interest.

Let K be a number field, let $\bar{K}$ be an algebraic closure of K, and let X be a smooth projective variety over K. Fix an integer i. For each prime number ℓ we may form the ℓ-adic étale cohomology group $H^i_\ell(X) = H^i_{ét}(X_{\bar{K}}, \mathbb{Q}_\ell)$. The cohomology group $H^i_\ell(X)$ is a finite-dimensional $\mathbb{Q}_\ell$-vector space equipped with a continuous representation of $\mathrm{Gal}(\bar{K}/K)$. As the prime ℓ varies, these Galois representations have a lot of structure in common. This common ground is captured in the notion of a compatible system of Galois representations; a concept first introduced by Serre [Ser98]. We will now recall this notion. Let v be a finite place of K, let ℓ be a prime number different from the residue characteristic of v. Assume that the Galois representation $H^i_\ell(X)$ is unramified at v. Then one may attach to v an endomorphism $F_{v,\ell}$ of $H^i_\ell(X)$, well-defined up to conjugation; the so-called "Frobenius endomorphism". A priori, the characteristic polynomial of $F_{v,\ell}$ has coefficients in $\mathbb{Q}_\ell$. The compatibility condition that the representations $H^i_\ell(X)$ satisfy, is that the characteristic polynomial of $F_{v,\ell}$ has coefficients in $\mathbb{Q}$ and does not depend on ℓ. The fact that the Galois representations $H^i_\ell(X)$ indeed satisfy this compatibility condition is a consequence of the Weil conjectures proven by Deligne in [Del74a].

Actually, the Galois representations should also be compatible in another way. The Zariski closure of the image of the Galois group $\mathrm{Gal}(\bar{K}/K)$ in $\mathrm{GL}(H^i_\ell(X))$ should in a suitable sense be independent of ℓ. This is where the Mumford–Tate conjecture comes into play. Fix an embedding $\bar{\sigma} : \bar{K} \hookrightarrow \mathbb{C}$ and write σ for the composition $K \subset \bar{K} \hookrightarrow \mathbb{C}$. Write $H^i_\sigma(X)$ for the singular cohomology group $H^i_{\mathrm{sing}}(X_\sigma(\mathbb{C}), \mathbb{Q})$. Artin showed that $H^i_\sigma(X) \otimes \mathbb{Q}_\ell$ and $H^i_\ell(X)$ are canonically isomorphic as vector spaces (théorème 4.4.(iii) of exposé xi in [SGA4-3]). This comparison isomorphism is a remarkable fact in and of itself, and it is rather striking that the story does not end here.

Just as the vector space $H^i_\ell(X)$ is naturally endowed with a Galois representation, so also $H^i_\sigma(X)$ comes with extra structure, namely a Hodge structure. The Hodge structure on $H^i_\sigma(X)$ is completely described in terms of a representation $\mathbb{S} \to \mathrm{GL}(H^i_\sigma(X))_\mathbb{R}$, where $\mathbb{S}$ denotes the Deligne torus $\mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{G}_m)$. The Mumford–Tate group G_σ of the $\mathbb{Q}$-Hodge structure $H^i_\sigma(X)$ is the smallest algebraic subgroup of $GL(H^i_\sigma(X))$ such that $G_\sigma \otimes \mathbb{R}$ contains the image of $\mathbb{S}$. Analogously, let G_ℓ be the smallest algebraic subgroup of $GL(H^i_\ell(X))$ that contains the image of $\mathrm{Gal}(\bar{K}/K)$.

The Mumford–Tate conjecture asserts that Artin's comparison isomorphism identifies $G_\sigma \otimes \mathbb{Q}_\ell$ with the identity component G°_ℓ of G_ℓ. Let us step back for a moment to contemplate this. In a certain sense, Artin's comparison theorem is a topological statement, and his proof reflects that. But the Mumford–Tate conjecture extends the comparison considerably. In that light, it is all the more remarkable that the complex manifold $X_\sigma(\mathbb{C})$ discards all Galois-theoretic information, and considers X over the algebraic closure of one of the archimedean primes of K. The Hodge structure on $H^i_\sigma(X)$ captures analytical information about holomorphic differential forms. On the other hand, $H^i_\ell(X)$ is the 'mere' topological data contained in Artin's comparison theorem enhanced with Galois-theoretic information contained in the projection $X_{\bar{K}} \to X$. A priori, these pieces of data appear to me almost orthogonal, and it is truly astonishing that they should be so intimately related.

Further inquiry of 'why' the conjecture might be true leads us into the realm of motives. But it is not the purpose of this introduction to consider the philosophical underpinnings of this conjecture. Therefore we will conclude this strand of thought with the following remark. The invariants in the tensor algebra $H^{2i}_\sigma(X)(i)^\otimes$ under the action of G_σ are precisely the Hodge classes, and the invariants in the tensor algebra $H^{2i}_\ell(X)(i)^\otimes$ under G°_ℓ are precisely the Tate classes. Using this fact, one can show that if two out of the following three conjectures are true, then so is the third:

» the Hodge conjecture for all powers of X;
» the Tate conjecture for all powers of X;
» the Mumford–Tate conjecture for all powers of X.

0.2 — In §3.7 we give some details about known cases of the Mumford–Tate conjecture. For the purpose of this introduction, let us note that the Mumford–Tate conjecture is known for K3 surfaces, by Tankeev in [Tan90] and [Tan95], and independently by André [And96a]. Suffice it to say that in general, the conjecture is wide open.

0.3 Main results. — We now state and explain the main results of this thesis. Then we highlight the key techniques and outline the proofs. The first main result is theorem 10.1:

> Theorem. *Let* M *be an abelian motive over a finitely generated field of characteristic* 0. *Let* E *be a subfield of* End(M)*, and let* Λ *be the set of finite places of* E*. Then the system* $H_\Lambda(M)$ *is a quasi-compatible system of Galois representations.*

To understand the statement of the first main result we need to explain what we mean by: (*i*) the words 'abelian motive'; (*ii*) the notation $H_\Lambda(M)$; and (*iii*) a 'quasi-compatible' system of Galois representations. We will do that in the following paragraphs. In [Las14], related results are obtained that imply that for a large class of abelian motives the ℓ-adic realisations form a compatible system of Galois representations in the sense of Serre. The main contribution of our result is that it takes the field E into account and that M is an arbitrary abelian motive; although we need to weaken the concept of compatibility to quasi-compatibility to achieve this. See remark 10.9 for a more extensive comparison.

Shimura showed that if $M = H^1(A)$, with A an abelian variety, then the system $H_\Lambda(M)$ is an E-rational compatible system in the sense of Serre. In theorem 7.2 we recall this result of Shimura in the setting of quasi-compatible systems of Galois representations.

The second main result of this thesis is the following.

THEOREM. *The Mumford–Tate conjecture is true for products of K3 surfaces.*

Actually we prove a more general statement in theorem 17.4, but this is the key example to keep in mind. We remark that the second main result is not a formal consequence of the results of Tankeev and André mentioned above. Indeed, let X_1 and X_2 be two K3 surfaces over a finitely generated field K of characteristic 0, and let $\sigma \colon K \hookrightarrow \mathbb{C}$ be a complex embedding. Recall that a K3 surface has no cohomology in degree 1. By Künneth's theorem we have

$$H^2(X_1 \times X_2) \cong H^2(X_1) \oplus H^2(X_2).$$

It is true that $G_\sigma(H^2(X_1) \oplus H^2(X_2))$ is a subgroup of $G_\sigma(H^2(X_1)) \times G_\sigma(H^2(X_2))$, but it may range from the graph of an isogeny to the full product; and likewise on the ℓ-adic side. Thus we will need to use more input to derive the Mumford–Tate conjecture for products of K3 surfaces.

0.4 ABELIAN MOTIVES. — In this text we use motives in the sense of André [And96b]. Alternatively we could have used the notion of absolute Hodge cycles. Let K be a finitely generated field of characteristic 0. An *abelian motive* over K is a summand of (a Tate twist of) the motive of an abelian variety over K. In practice this means that an abelian motive M is a package consisting of a Hodge structure $H_\sigma(M)$ for each complex embedding $\sigma \colon K \hookrightarrow \mathbb{C}$, and an ℓ-adic Galois representation $H_\ell(M)$ for each prime ℓ, that arise in a compatible way as summands of Tate twists of the cohomology of an abelian variety.

Caution: we do not know in general that for a motive M over K the ℓ-adic Galois representations $H_\ell(M)$ form a compatible system of Galois representations in the sense of Serre.

0.5 THE λ-ADIC REALISATIONS. — Let M be an abelian motive over a finitely generated subfield $K \subset \mathbb{C}$. Let E be a subfield of End(M), and let Λ be the set of finite places of E. For each prime number ℓ, the field E acts on the Galois representation $H_\ell(M)$ via $E_\ell = E \otimes \mathbb{Q}_\ell = \prod_{\lambda|\ell} E_\lambda$ and accordingly we get a decomposition of Galois representations $H_\ell(M) = \bigoplus_{\lambda|\ell} H_\lambda(M)$. We denote with $H_\Lambda(M)$ the system of λ-adic Galois representations $H_\lambda(M)$ as λ runs through Λ.

0.6 Key ingredients. — We now list the main techniques that we use, and we will explain them in more detail below. For the first result we use (*i*) a slight variation in the definition of a compatible system of Galois representations, which gives us the flexibility to take extensions of the base field; and (*ii*) an argument to deform the problem to an abelian CM motive. The key ingredients in the proof of the second main result are (*i*) hyperadjoint motives; and (*ii*) Hodge–Tate maximality.

0.7 Quasi-compatible systems of Galois representations. — In section 6 we develop a variation on the concept of compatible systems of Galois representations that has its origins in the work of Serre [Ser98]. Besides the original work of Serre, we draw inspiration from Ribet [Rib76] and Larsen–Pink [LP92]. The main features of our variant are:

» Recall the Frobenius endomorphism $F_{v,\ell}$ that we mentioned above. Our variant replaces the compatibility condition on the characteristic polynomial of $F_{v,\ell}$ by the analogous condition for a power of $F_{v,\ell}$ that is allowed to depend on v. In other words, one may replace the local field K_v by a finite field extension before checking the compatibility. This gives robustness with respect to extension of the base field.

» We take endomorphisms into account. Instead of only considering systems of Galois representations that are indexed by finite places of $\mathbb{Q}$, we also consider systems that are indexed by finite places of a number field E. This was already suggested by Serre [Ser98], and Ribet pursued this further in [Rib76].

  The compatibility condition mentioned in the previous item must then be adapted as follows. Let $\rho_\lambda \colon \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_n(E_\lambda)$ and $\rho_{\lambda'} \colon \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_n(E_{\lambda'})$ be two Galois representations. We say that ρ_λ and $\rho_{\lambda'}$ are quasi-compatible at v if there is a positive integer n such that the characteristic polynomials of $\rho_\lambda(F^n_{\bar{v}/v})$ and $\rho_{\lambda'}(F^n_{\bar{v}/v})$ have coefficients in E and are equal to each other.

0.8 Deformations of abelian motives. — Roughly speaking, the proof of the first main result works by placing the abelian motive in a family of motives over a Shimura variety. The problem may then be deformed to a CM point on the Shimura variety, where we can prove the result by reducing to the case of abelian varieties mentioned in §0.3. To actually make this work we need a recent result of Kisin [Kis17]: Let $\mathscr{S}$ be an integral model of a Shimura variety of Hodge type over a the ring of integers $\mathcal{O}_K$ of a p-adic field K. Then every point in the special fibre of $\mathscr{S}$ is *isogenous* to a point that lifts to a CM point of the generic fibre. For details we refer to the main text (§10.6 and §10.7).

0.9 Motives of K3 type. — A Hodge structure V is said to be *of K3 type* if it is polarisable, pure of weight 0, and if $\dim(V^{-1,1}) = 1$, and $\dim(V^{-n,n}) = 0$ for $n > 1$. Analogously, a motive M over a finitely generated field K of characteristic 0 is called *of K3 type* if $H_\sigma(M)$ is a Hodge structure of K3 type, for one (and hence every) complex embedding $\sigma\colon K \hookrightarrow \mathbb{C}$. Note that if X denotes a K3 surface, then $H^2(X)(1)$ is a motive of K3 type. It is expected that every motive of K3 type is an abelian motive, but this is currently not known; see also remark 13.9.

0.10 Hyperadjoint motives. — Let M be an abelian motive. In theorem 5.6 we show that the Mumford–Tate conjecture for M is true on centres. In other words, $Z_\sigma(M) \otimes \mathbb{Q}_\ell \cong Z^\circ_\ell(M)$, where $Z_\sigma(M)$ denotes the centre of the Mumford–Tate group $G_\sigma(M)$ and similarly $Z^\circ_\ell(M)$ denotes the centre of $G^\circ_\ell(M)$. This means that in order to prove the full Mumford–Tate conjecture for M we only need to focus on the semisimple parts of $G_\sigma(M)$ and $G^\circ_\ell(M)$.

Hyperadjoint motives allow one to do just that. Roughly speaking, the hyperadjoint motive M^{ha} is the motive that corresponds to the adjoint representation of $G_\sigma(M)^{ad}$ via Tannaka duality. We warn the reader in advance that the notion is rather treacherous and ill-behaved; but the lemmas in section 4 should help in handling these objects. (The hyperadjoint motive is obtained by applying a construction that works in any neutral Tannakian category. Roughly speaking, one iteratively replaces an object with the adjoint representation of the associated group scheme. This iterative process stabilises at a *hyperadjoint* object.)

0.11 Hodge–Tate maximality. — Let M be an abelian motive of K3 type. We show that there is no abelian motive N such that M is contained in the Tannakian category generated by N and such that the natural quotient map $G^\circ_\ell(N) \to G^\circ_\ell(M)$ is a non-trivial isogeny. The Hodge-theoretic analogue of this result was proven by Cadoret and Moonen [CM15], and our proof mimicks their proof.

0.12 On the proof of the second main result. — Let K be a finitely generated field of characteristic 0. Let M be an abelian motive of K3 type over K. In theorem 14.1 we prove the Mumford–Tate conjecture for M. The final goal of this thesis is to prove the Mumford–Tate conjecture for a finite sum of abelian motives of K3 type, which we do in sections 16 and 17. The essential case is that of the sum of two abelian motives of K3 type. Let M_1 and M_2 be two abelian motives of K3 type. In §16.4 we give an outline of the proof of the Mumford–Tate conjecture for $M_1 \oplus M_2$.

In this paragraph we give a rough sketch of the ideas that go into the proof. For purposes of this introduction we will omit certain points that will be treated in detail in section 16. Three main steps in the proof are as follows. The first step amounts to replacing M_1 and M_2 with the hyperadjoint motives M_1^{ha} and M_2^{ha}. Without loss of generality we may assume that M_1^{ha} and M_2^{ha} are irreducible and we may assume that the algebraic group $G_\ell(M_1^{ha} \oplus M_2^{ha})$ is connected. There is a natural inclusion $G_\ell(M_1^{ha} \oplus M_2^{ha}) \hookrightarrow G_\ell(M_1^{ha}) \times G_\ell(M_2^{ha})$. There are two possibilities: the inclusion is an equality or a strict inclusion. The non-trivial case is when the inclusion is strict. In that case we show that $\mathrm{End}(M_1^{ha}) = \mathrm{End}(M_2^{ha})$ and $H_\Lambda(M_1^{ha}) \cong H_\Lambda(M_2^{ha})$. In the second step we use the results on Hodge–Tate maximality that we mentioned above, and we show that we can lift these results from the hyperadjoint motives to the motives of K3 type: we get $\mathrm{End}(M_1) = \mathrm{End}(M_2)$ and $H_\Lambda(M_1) \cong H_\Lambda(M_2)$. As a third step we apply the Kuga–Satake construction to M_1 and M_2 to obtain to abelian varieties A_1 and A_2. We deduce that $H^1_\ell(A_1) \cong H^1_\ell(A_2)$ for all prime numbers ℓ. Finally, Faltings's theorem implies that A_1 is isogenous to A_2 which in turn implies that $M_1 \cong M_2$. We may then deduce the Mumford–Tate conjecture for $M_1 \oplus M_2$.

0.13 HOW TO READ THIS TEXT. — Every section starts with a paragraph labeled "README". The purpose of these paragraphs is the following: (*i*) give a concise overview of the contents of the section; (*ii*) point to the important results; and (*iii*) describe the philosophy or intuition behind certain concepts. As such, these paragraphs may sacrifice mathematical precision and rather give rough sketches.

Roughly speaking, there are two kinds of sections in this text. Certain sections have one or two main results, and the rest of the section is devoted to proving them. In this case, we usually state these results at the beginning of the section, and the reader may treat them as black boxes, and skip the entire section on first reading.

The other kind of sections has a high density of definitions, terminology, and basic results that are used throughout the text. These sections will have to be read in their entirety, unless the reader is already familiar with their contents.

0.14 CONVENTIONS, TERMINOLOGY AND NOTATION. — Unless specified, reductive and semisimple algebraic groups are not assumed to be connected. If X is a scheme, then X^{cl} denotes the set of closed points of X. We say that a field is a *finitely generated field* if it is finitely generated over its prime field. Let p be a prime number; we call a field a p-adic field if it is a local field that is a finite extension of $\mathbb{Q}_p$. If K is a field, V a vector space over K, and g an endomorphism of V, then we denote with $\mathrm{c.p.}_K(g|V)$ the characteristic polynomial of g. If there is no confusion possible, then we may drop K or V from the notation, and write $\mathrm{c.p.}(g|V)$ or simply $\mathrm{c.p.}(g)$.

Let K be a field, and let $\bar{K}$ be an algebraic closure of K. If $\bar{\sigma} : \bar{K} \hookrightarrow \mathbb{C}$ is a complex embedding, then we denote with σ the composition of $\bar{\sigma}$ with the inclusion $K \subset \bar{K}$.

Let E be a number field. Recall that E is called *totally real* (TR) if for all complex embeddings $\sigma : E \hookrightarrow \mathbb{C}$ the image $\sigma(E)$ is contained in $\mathbb{R}$. The field E is called a *complex multiplication* field (CM) if it is a quadratic extension of a totally real field E^0, and if all complex embeddings $\sigma : E \hookrightarrow \mathbb{C}$ have an image that is not contained in $\mathbb{R}$.

Let C be a Tannakian category, and let V be an object of C. If a and b are non-negative integers, then $\mathbb{T}^{a,b}V$ denotes the object $V^{\otimes a} \otimes \check{V}^{\otimes b}$. With $\langle V \rangle^{\otimes}$ we denote the smallest strictly full Tannakian subcategory of C that contains V. That $\langle V \rangle^{\otimes}$ is a strict subcategory means that if $W \in \langle V \rangle^{\otimes}$, then every object in C that is isomorphic to W is also an object of $\langle V \rangle^{\otimes}$. That $\langle V \rangle^{\otimes}$ is a full Tannakian subcategory means that $\langle V \rangle^{\otimes}$ is closed under directs sums, tensor products, duals, and subquotients. The irreducible objects in $\langle V \rangle^{\otimes}$ are precisely the irreducible objects of C that are isomorphic to a subquotient of $\mathbb{T}^{a,b}$ for some $a, b \geq 0$.

Preliminaries

1 Representation theory

Readme. — Important: proposition 1.2.

1.1 — Let K be a field, and let G be an algebraic group over K. Let $\rho\colon G \to GL(V)$ be a finite-dimensional algebraic representation of G. If we say that ρ is invariant under all automorphisms of G, then we mean that for every automorphism f of G the representation $\rho \circ f$ is isomorphic to ρ (as representation of G). This terminology extends naturally to representations of Lie algebras.

1.2 Proposition. — *Let* K *be a field of characteristic* 0.

1. *Let* G *be a linear algebraic group over* K*. Then the adjoint representation* $\mathrm{ad}\colon G \to GL(\mathrm{Lie}(G))$ *is invariant under all automorphisms of* G.
2. *Let* V *be a finite-dimensional vector space over* K *endowed with a non-degenerate symmetric bilinear form* ϕ*. Then the representation* V *of the algebraic group* $SO(V,\phi)$ *over* K *is invariant under all automorphisms of* $SO(V,\phi)$.
3. *Let* V *be a finite-dimensional vector space over* K*. Then the representation* $V \oplus \check{V}$ *of the algebraic group* $GL(V)$ *over* K *is invariant under all automorphisms of* $GL(V)$.
4. *Let* L/K *be a quadratic extension of* K*. Let* V *be a finite-dimensional vector space over* L *endowed with a non-degenerate skew-Hermitian form* ϕ*. Then the representation* V *of the algebraic group* $U(V,\phi)$ *over* K *is invariant under all automorphisms of* $U(V,\phi)$.

1.3 — For the proof of this proposition we first state a lemma that will also be useful later on; and we recall some facts about representations of simple Lie algebras. We finish the proof in §1.6.

1.4 Lemma. — *Let* K *be an infinite field, let* L/K *be a field extension, and let* G *be a linear algebraic group over* K*. Let* V_1 *and* V_2 *be finite-dimensional algebraic representations of* G *over* K*. If* $V_{1,L} \cong V_{2,L}$

(as representations of G_L*), then* $V_1 \cong V_2$ *(as representations of* G*).*

Proof. Observe that $\mathrm{Hom}_G(V_1, V_2)$ is in a natural way an affine space, and $\mathrm{Hom}_{G_L}(V_{1,L}, V_{2,L}) = \mathrm{Hom}_G(V_1, V_2)_L$. Since $V_{1,L} \cong V_{2,L}$, we know that V_1 and V_2 have the same dimension.

The locus of isomorphisms $\mathrm{Isom}_G(V_1, V_2) \subset \mathrm{Hom}_G(V_1, V_2)$ is a Zariski open subset, which is non-empty, because $\mathrm{Isom}_{G_L}(V_{1,L}, V_{2,L}) = \mathrm{Isom}_G(V_1, V_2)_L$ has a rational point. We conclude that $V_1 \cong V_2$, because a non-empty Zariski open subset of an affine space over an infinite field always has a rational point. □

1.5 — Let K be an algebraically closed field of characteristic 0. Let $\mathfrak{g}$ be a semisimple Lie algebra over K. Recall that an irreducible representation of $\mathfrak{g}$ is called a *minuscule* (resp. *quasi-minuscule*) representation if the Weyl group acts transitively on its weights (resp. non-zero weights). The Lie algebras that are relevant to our discussion are those of type A_k, B_k, and D_k (*cf.* theorem 11.2.4). We list their (quasi-)minuscule representations and their dimensions. Let V be a finite-dimensional vector space over K. If $\dim(V) \geq 3$, let ϕ be a non-degenerate symmetric bilinear form on V.

Lie alg.	dim(V)	*D.t.*	*Minuscule*	*Quasi-minuscule*
$\mathfrak{sl}(V)$	$k+1$	A_k	$\bigwedge^i V$ $(\binom{k+1}{i})$	ad (k^2+2k)
$\mathfrak{so}(V,\phi)$	$2k+1$	B_k	spin (2^k)	V $(2k+1)$
$\mathfrak{so}(V,\phi)$	$2k$	D_k	V $(2k)$; two half-spins (2^{k-1})	ad $(2k^2-k)$

(Here 'D.t.' stands for 'Dynkin type', and ad denotes the adjoint representation.)

1.6 *Proof* (of proposition 1.2). —

1. Let ϕ be an automorphism of G. Then $\mathrm{Lie}(\phi)$ is an automorphism of Lie(G) that is compatible with the adjoint representation, since ϕ respects conjugation. Thus the adjoint representation is invariant under all automorphisms of G.
2. By lemma 1.4 it suffices to assume that K is algebraically closed. We distinguish two cases based on the parity of dim(V).
   a. Suppose that $\dim(V) = 2k+1$ is odd. Then V is, up to isomorphism, the unique quasi-minuscule representation of $\mathfrak{so}(V,\phi)$ of dimension $2k+1$. Hence V is invariant under all automorphisms of $SO(V,\phi)$.
   b. Suppose that $\dim(V) = 2k$ is even. Then V is, up to isomorphism, the unique $2k$-dimensional minuscule representation of $\mathfrak{so}(V,\phi)$ that integrates to a representation of $SO(V,\phi)$. Hence V is invariant under all automorphisms of $SO(V,\phi)$.

      (A remark about D_4, or equivalently $\dim(V) = 8$: The Lie algebra $\mathfrak{so}_8$ has 3 minuscule representations of dimension 8. This phenomenon goes by the name *triality*. However, only one of these representations integrates to a representation of SO_8.)

3. By lemma 1.4 it suffices to assume that K is algebraically closed. We distinguish several cases based on dim(V).
   a. Suppose that $\dim(V) = 1$. Then $GL(V) \cong \mathbb{G}_m$ has 2 automorphisms: the identity and the inverse. It is clear that $V \oplus \check{V}$ is invariant under these automorphisms.
   b. Suppose that $\dim(V) = 2$. Note that $V \cong \check{V}$ as representations of SL(V).

      Let f be an automorphism of GL(V). The Dynkin diagram of $\mathfrak{sl}(V)$ is A_1, which has only one automorphism: the identity. Therefore $f|_{SL(V)}$ is an inner automorphism, in other words, it is the conjugation by some element $\alpha \in SL(V)$. Let c_α be the automorphism of GL(V) that is conjugation by α. Then $g = f \circ c_\alpha^{-1}$ is an automorphism of GL(V) that is the identity on SL(V). On the centre of GL(V) we see that g is either the identity or the inverse automorphism. We conclude that $V \oplus \check{V}$ is invariant under g, and hence under f.
   c. Suppose that $\dim(V) = n = k + 1 \geq 3$. Note that $\mathfrak{sl}(V)$ has precisely two minuscule representations of dimension dim(V), namely V and $\check{V}$. Therefore $V \oplus \check{V}$ is invariant under all automorphisms of SL(V). Let f be an automorphism of GL(V). Then f restricts to an automorphism f^{der} of SL(V). The Dynkin diagram of $\mathfrak{sl}(V)$ is A_k, which has two automorphisms. Thus we may find an element $\alpha \in SL(V)$ like before, such that $g = f \circ c_\alpha^{-1}$ is either the identity or the inverse-transpose on SL(V) (with respect to some chosen basis of V). Write Z for the centre of GL(V). Recall that $Z \cap SL(V) = \mu_n$, with $n \geq 3$. If g is the identity on SL(V), then g is the identity on μ_n, and hence the identity on Z. On the other hand, if g is the inverse-transpose on SL(V), then g is the inverse on μ_n, and hence the inverse on Z. We conclude that $V \oplus \check{V}$ is invariant under g, and hence under f.
4. Let $\bar{K}$ be an algebraic closure of K that contains the quadratic extension L/K. By lemma 1.4 it suffices to prove the statement for the representation $V \otimes_K \bar{K}$ of the group $U(V, \phi) \otimes_K \bar{K}$. Note that $U(V, \phi) \otimes_K \bar{K} \cong GL(V) \otimes_L \bar{K}$. The representation $V \otimes_K \bar{K}$ is isomorphic to $(V \otimes_L \bar{K}) \oplus (\check{V} \otimes_L \bar{K})$, and thus the result follows from the previous point. □

1.7 Lemma. — *Let* K *be a field of characteristic* 0*. Let* G *be a reductive group over* K*. Let* ρ_1 *and* ρ_2 *be two finite-dimensional representations of* G*. If there is a Zariski-dense subset* $S \subset G(K)$ *such that* $\mathrm{tr}(\rho_1(g)) = \mathrm{tr}(\rho_2(g))$ *for all* $g \in S$*, then* $\rho_1 \cong \rho_2$ *as representations of* G.

Proof. Note that $\mathrm{tr} \circ \rho_i$ is a separated morphism of schemes $G \to \mathbb{A}^1_K$. Therefore we have $\mathrm{tr}(\rho_1(g)) = \mathrm{tr}(\rho_2(g))$ for all $g \in G(K)$. By linearity, we find that $\mathrm{tr}(\rho_1(\alpha)) = \mathrm{tr}(\rho_2(\alpha))$ for all α in the group algebra K[G(K)]. By proposition 3 in §12, №1 of [Bou12], we conclude that $\rho_1 \cong \rho_2$ as representations of G(K), hence as representations of G. □

Readme. — This section specifies which type of motives we use (namely, motives in the sense of André), and it introduces a lot of notation. It does not contain any new results, but it provides the context for the rest of the text.

2.1 — Let K be a field. Denote with SmPr_K the category of smooth projective varieties over K. Let X be a smooth projective variety over K. We will use the following notation:
- » $H^i_{dR}(X)$ for the filtered K-vector space $H^i_{dR}(X/K)$;
- » $H^i_\ell(X)$ (ℓ prime) for the Galois representation $H^i_{ét}(X_{\bar{K}}, \mathbb{Q}_\ell)$;
- » $H^i_B(X)$ (if $K = \mathbb{C}$) for the Hodge structure $H^i_{sing}(X^{an}, \mathbb{Q})$;
- » $H^i_\sigma(X)$ ($\sigma : K \hookrightarrow \mathbb{C}$ an embedding) for the Hodge structure $H^i_B(X \times_{K,\sigma} \mathbb{C})$.

All these constructions are functorial in X.

2.2 — Let K be a field of characteristic 0. In this text a *motive* over K shall mean a motive in the sense of André [And96b]. (To be precise, our category of base pieces is the category of smooth projective varieties over K, and our reference cohomology is de Rham cohomology, $H_{dR}(\_)$. The resulting notion of motive does not depend on the chosen reference cohomology, see proposition 2.3 of [And96b].) We denote the category of motives over K with Mot_K.

For the reader who is unfamiliar with motives in the sense of André, let us note that the most crucial property of Mot_K is that it is a Tannakian category that fits into the following diagram. (Also see théorème 0.4 of [And96b] and §2.3 below.)

$$\begin{array}{ccccc} & & & \nearrow^{H^i_{dR}} & \mathrm{Filt}_K \\ \mathrm{SmPr}_K & \xrightarrow{H^i} & \mathrm{Mot}_K & \xrightarrow{H^i_\ell} & \mathrm{Rep}_{\mathbb{Q}_\ell}(\mathrm{Gal}(\bar{K}/K)) \\ & & & \searrow_{H^i_\sigma} & \mathbb{Q}\mathrm{HS} \end{array}$$

We name three other strengths of motives in the sense of André: (*i*) Künneth projectors exist in Mot_K; (*ii*) the category Mot_K is a *semisimple* neutral Tannakian category and therefore the motivic Galois group of a motive is a reductive algebraic group; and (*iii*) if $K = \mathbb{C}$, then we know that the Betti realisation functor is fully faithful on the Tannakian subcategory generated by motives of abelian varieties, see theorem 5.2.1. All these results are due to André [And96b].

2.3 — Let K be a field of characteristic 0. If X is a smooth projective variety over K, then we write

$H^i(X)$ for the motive in degree i associated with X. The cohomology functors mentioned in §2.1 induce realisation functors on the category of motives over K, and we have $H^i_{dR} = H_{dR} \circ H^i$, etc. Let M be a motive over K.

- We write $H_{dR}(M)$ for the de Rham realisation; it is a finite-dimensional K-vector space with a decreasing filtration.
- For every prime ℓ, we write $H_\ell(M)$ for the ℓ-adic realisation; it is a finite-dimensional $\mathbb{Q}_\ell$-vector space equipped with a continuous representation of $\mathrm{Gal}(\bar{K}/K)$.
- If $K = \mathbb{C}$, then we write $H_B(M)$ for the Betti realisation; it is a polarisable $\mathbb{Q}$-Hodge structure.
- For every complex embedding $\sigma \colon K \hookrightarrow \mathbb{C}$, we write $H_\sigma(M)$ for the polarisable $\mathbb{Q}$-Hodge structure $H_B(M_\sigma)$.

2.4 — There are several theorems that compare the various realisations of motives. Let K be a field of characteristic 0; and let M be a motive over K. Then we have the following isomorphisms, that are functorial in M.

1. If $K = \mathbb{C}$, then there is an isomorphism of filtered complex vector spaces $H_{dR}(M) \cong H_B(M) \otimes_{\mathbb{Q}} \mathbb{C}$. Consequently, if $\sigma \colon K \hookrightarrow \mathbb{C}$ is a complex embedding, then there is an isomorphism of filtered complex vector spaces

$$H_{dR}(M) \otimes_{K,\sigma} \mathbb{C} \cong H_\sigma(M) \otimes_{\mathbb{Q}} \mathbb{C}.$$

   This isomorphism was proven for varieties by Grothendieck [Gro66]. The generalisation to motives follows from the fact that the isomorphism is compatible with cycle class maps.
2. If $K = \mathbb{C}$, and ℓ is a prime number, then there is an isomorphism of $\mathbb{Q}_\ell$-vector spaces $H_B(M) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong H_\ell(M)$. If $\sigma \colon \bar{K} \hookrightarrow \mathbb{C}$ is a complex embedding, and ℓ is a prime number, then there is an isomorphism of $\mathbb{Q}_\ell$-vector spaces $H_\ell(M) \cong H_\ell(M_\sigma)$; and therefore

$$H_\sigma(M) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong H_\ell(M).$$

   This isomorphism was proven for varieties by Artin in exposé XI in [SGA4-3]. The generalisation to motives again follows from the fact that the isomorphism is compatible with cycle class maps.
3. Suppose that K embeds into a p-adic field, and fix such an embedding $K \hookrightarrow K_v$. Let B_{dR,K_v} be the p-adic period ring associated with K_v, in the sense of Fontaine [Fon94]. Then p-adic Hodge theory gives an isomorphism of filtered modules with $\mathrm{Gal}(\bar{K}_v/K_v)$-action

$$B_{dR,K_v} \otimes_K H_{dR}(M) \cong B_{dR,K_v} \otimes_{\mathbb{Q}_p} H_p(M).$$

   This was proven for varieties by Faltings in theorem 8.1 of [Fal89]. Once again, the generalisation to motives follows from the fact that the isomorphism is compatible with cycle class maps.

2.5 — The categories of motives, Hodge structures, and Galois representations form neutral Tannakian categories. That means that we may attach algebraic groups to objects in those categories. In this paragraph we introduce notation for these groups.

- Let $\mathcal{G}_B$ be the affine group scheme associated with the forgetful functor $\mathbb{Q}HS \to \mathrm{Vect}_{\mathbb{Q}}$. Let V be a $\mathbb{Q}$-Hodge structure. The *Mumford–Tate group* $G_B(V)$ of V is the linear algebraic group over $\mathbb{Q}$ associated with the Tannakian category $\langle V\rangle^{\otimes}$ generated by V. Note that $G_B(V)$ is the image of $\mathcal{G}_B$ in $GL(V)$.

  For an alternative description, recall that the Hodge structure on V is determined by a homomorphism of algebraic groups $\mathbb{S} \to GL(V \otimes_{\mathbb{Q}} \mathbb{R})$, where $\mathbb{S}$ is the Deligne torus $\mathrm{Res}_{\mathbb{R}}^{\mathbb{C}}\mathbb{G}_m$. The Mumford–Tate group is the smallest algebraic subgroup G of GL(V) such that $G_{\mathbb{R}}$ contains the image of $\mathbb{S}$. Since $\mathbb{S}$ is connected, so is $G_B(V)$.

  If V is polarisable, then the Tannakian category $\langle V\rangle^{\otimes}$ is semisimple; which implies that $G_B(V)$ is reductive. We write $Z_B(V)$ for the centre of $G_B(V)$.
- Let K be a field; and let ℓ be prime number. Let $\mathcal{G}_{K,\ell}$ be the affine group scheme associated with the forgetful functor from the Tannakian category of ℓ-adic representations of $\mathrm{Gal}(\bar{K}/K)$ to $\mathrm{Vect}_{\mathbb{Q}_\ell}$. Let V_ℓ be a finite-dimensional $\mathbb{Q}_\ell$-vector space equipped with a representation of $\mathrm{Gal}(\bar{K}/K)$. We write $G_\ell(V_\ell)$ for the linear algebraic group over $\mathbb{Q}_\ell$ associated with the Tannakian category $\langle V_\ell\rangle^{\otimes}$. Note that $G_\ell(V_\ell)$ is the image of $\mathcal{G}_{K,\ell}$ in $GL(V_\ell)$.

  Alternatively, $G_\ell(V_\ell)$ is the Zariski closure of the image of $\mathrm{Gal}(\bar{K}/K)$ in $GL(V_\ell)$. In general, $G_\ell(V_\ell)$ is neither reductive nor connected. We write $G_\ell^\circ(V_\ell)$ for the identity component of $G_\ell(V_\ell)$. The centre of $G_\ell^\circ(V_\ell)$ is denoted with $Z_\ell^\circ(V_\ell)$.
- Let K be a field of characteristic 0. There are many fibre functors $\mathrm{Mot}_K \to \mathrm{Vect}_{\mathbb{Q}}$, but among those there is no natural choice presented to us.

  Let E be a field of characteristic 0. If ω is a fibre functor from Mot_K to finite-dimensional E-vector spaces, then we write $\mathcal{G}_{\mathrm{mot},K,\omega}$ for the associated affine group scheme $\underline{\mathrm{Aut}}(\omega)^{\otimes}$ over E.

  Let M be a motive over K. We write $G_{\mathrm{mot},\omega}(M)$ for the affine group scheme associated with $\langle M\rangle^{\otimes}$ (and fibre functor ω). Note that $G_{\mathrm{mot},\omega}(M)$ is the image of $\mathcal{G}_{\mathrm{mot},K,\omega}$ in $GL(\omega(M))$. We write $Z_{\mathrm{mot},\omega}(M)$ for the centre of $G_{\mathrm{mot},\omega}(M)$.

  For each embedding $\sigma\colon K \hookrightarrow \mathbb{C}$, we may take $\omega = H_\sigma$, and we write $\mathcal{G}_{\mathrm{mot},K,\sigma}$ for $\mathcal{G}_{\mathrm{mot},K,H_\sigma}$, and similarly we write $G_{\mathrm{mot},\sigma}(M)$ and $Z_{\mathrm{mot},\sigma}(M)$. These are affine group schemes over $\mathbb{Q}$. For each prime number ℓ, we may take $\omega = H_\ell$, and we write $\mathcal{G}_{\mathrm{mot},K,\ell}$ for $\mathcal{G}_{\mathrm{mot},K,H_\ell}$, and similarly we write $G_{\mathrm{mot},\ell}(M)$ and $Z_{\mathrm{mot},\ell}(M)$. These are affine group schemes over $\mathbb{Q}_\ell$. By the comparison theorem of Artin (see §2.4.2) we have a canonical isomorphism $\mathcal{G}_{\mathrm{mot},K,\sigma} \otimes \mathbb{Q}_\ell \cong \mathcal{G}_{\mathrm{mot},K,\ell}$, and in particular, for every motive M over K a canonical isomorphism $G_{\mathrm{mot},\sigma}(M) \otimes \mathbb{Q}_\ell \cong G_{\mathrm{mot},\ell}(M)$.

Let K be a field; and let M be a motive over K. If $K = \mathbb{C}$, then we write $G_B(M)$ for $G_B(H_B(M))$, and $Z_B(M)$ for $Z_B(H_B(M))$. For every complex embedding $\sigma\colon K \hookrightarrow \mathbb{C}$, we write $G_\sigma(M)$ for $G_B(H_\sigma(M))$, and $Z_\sigma(M)$ for $Z_B(H_\sigma(M))$. For every prime number ℓ, we write $G_\ell(M)$ for $G_\ell(H_\ell(M))$; and $G_\ell^\circ(M)$ for $G_\ell^\circ(H_\ell(M))$. Also, we write $Z_\ell^\circ(M)$ for $Z_\ell^\circ(H_\ell(M))$.

2.6 Remark. — Let K be a field of characteristic 0, and let M be a motive over K. By construction, we have inclusions $G_\sigma(M) \subset G_{mot,\sigma}(M)$ and $G_\ell(M) \subset G_{mot,\ell}(M)$. However, we stress that these inclusions do not need to be equalities: in general $G_\sigma(M)$ is connected, while $G_{mot,\sigma}(M)$ is not. The following conjectures (implied by respectively the Hodge conjecture and the Tate conjecture) show what the expectations are. For so-called abelian motives (see section 5) we do know conjecture 2.7.1. This was proven by André, and we quote his result in theorem 5.2.1.

2.7 Conjecture. — *Let* K *be a field of characteristic* 0*, and let* M *be a motive over* K*.*

1. *Assume* $K = \mathbb{C}$.
   a. *Every Hodge class in* $H_B(M)$ *is motivated. In other words, every element of* $H_B(M)$ *that is fixed by* $G_B(M)$ *is also fixed by* $G_{mot,B}(M)$*.*
   b. *The inclusion* $G_B(M) \subset G_{mot,B}(M)$ *is an equality.*
2. *Assume* K *is finitely generated. Let* ℓ *be a prime number.*
   a. *Every Tate class in* $H_\ell(M)$ *is motivated. In other words, every element of* $H_\ell(M)$ *that is fixed by* $G_\ell(M)$ *is also fixed by* $G_{mot,\ell}(M)$*.*
   b. *The inclusion* $G_\ell(M) \subset G_{mot,\ell}(M)$ *is an equality.*

2.8 Remark. — If conjecture 2.7.1.a is true for all $M' \in \langle M\rangle^{\otimes}$, then conjecture 2.7.1.b is true for M. Similarly, if conjecture 2.7.2.a is true for all $M' \in \langle M\rangle^{\otimes}$, then conjecture 2.7.2.b is true for M. Conversely, of course conjecture 2.7.1.b implies conjecture 2.7.1.a; and conjecture 2.7.2.b implies conjecture 2.7.2.a.

The following discussion (§2.9 through §2.11) shows that conjecture 2.7.1.b implies that for a motive M over a finitely generated field K of characteristic 0 the group $G_\sigma(M)$ is the connected component of the identity of $G_{mot,\sigma}(M)$ (also see theorem 5.2.2).

2.9 — Let L be a field extension of K. The base change functor $\mathrm{Mot}_K \to \mathrm{Mot}_L$, given by $M \mapsto M_L$, is a tensor functor and therefore induces a homomorphism $\iota\colon \mathcal{G}_{mot,L,\omega} \to \mathcal{G}_{mot,K,\omega}$ for every fibre functor ω on Mot_L. We highlight two cases.

1. If K and L are algebraically closed, then the base change functor is fully faithful. Hence the homomorphism ι is surjective, and for every motive M over K the homomorphism $\iota\colon G_{mot,\omega}(M_L) \to G_{mot,\omega}(M)$ is an isomorphism. (See théorème 0.6.1, and remarque (ii) after théorème 5.2 of [And96b].)
2. Suppose that $L = \bar{K}$ is an algebraic closure of K. The homomorphism ι is injective, for the

following reason. It suffices to show that every motive N in Mot_L is a subquotient of a motive M_L, with $M \in \mathrm{Mot}_K$. The motive N is defined over an intermediate field $K \subset K' \subset L$ such that K'/K is finite. Now take $M = \mathrm{Res}_K^{K'} N$. (This argument comes from proposition II.6.23(a) of [Del+82].)

There is a natural exact sequence

$$1 \to \mathscr{G}_{\mathrm{mot},\bar{K},\omega} \to \mathscr{G}_{\mathrm{mot},K,\omega} \to \mathrm{Gal}(\bar{K}/K) \to 1$$

(see the last sentences of §4 in [And96b]). For every motive M over K this gives a sequence

$$1 \to G_{\mathrm{mot},\omega}(M_{\bar{K}}) \to G_{\mathrm{mot},\omega}(M) \to \Gamma \to 1$$

where Γ is naturally a finite quotient of $\mathrm{Gal}(\bar{K}/K)$ by some normal subgroup $\mathrm{Gal}(\bar{K}/K')$. The group Γ (or more precisely, the field extension K'/K) does not depend on the choice of fibre functor ω. We have $G_{\mathrm{mot},\omega}(M_{\bar{K}}) = G_{\mathrm{mot},\omega}(M_{K'})$, and in particular $G_{\mathrm{mot},\omega}(M_{K'})$, does not change if we replace K' by an extension inside $\bar{K}$.

2.10 — Let L/K be a Galois extension. By proposition II.6.23(a,d) of [Del+82] the exact sequences of §2.9.2 can be placed in commutative diagrams (with exact rows) as follows.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathscr{G}_{L,\ell} & \longrightarrow & \mathscr{G}_{K,\ell} & \longrightarrow & \mathrm{Gal}(L/K) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
1 & \longrightarrow & \mathscr{G}_{\mathrm{mot},L,\ell} & \longrightarrow & \mathscr{G}_{\mathrm{mot},K,\ell} & \longrightarrow & \mathrm{Gal}(L/K) & \longrightarrow & 1
\end{array}$$

Let M be a motive over K. Then there exists a quotient Γ_ℓ of $\mathrm{Gal}(L/K)$ and a quotient Γ of Γ_ℓ such that the following commutative diagram has exact rows.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_\ell(M_L) & \longrightarrow & G_\ell(M) & \longrightarrow & \Gamma_\ell & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G_{\mathrm{mot},\ell}(M_L) & \longrightarrow & G_{\mathrm{mot},\ell}(M) & \longrightarrow & \Gamma & \longrightarrow & 1
\end{array} \tag{2.10.1}$$

(Note that the group Γ in this diagram is not the same as the group Γ in §2.9.2, unless $L = \bar{K}$.) The two diagrams may be placed in a bigger commutative diagram with quotient maps from every object in the first diagram to the corresponding object in the second diagram.

2.11 — Let K be a finitely generated field of characteristic 0; and let X be a smooth projective variety over K. Recall that the ℓ-adic Galois representations $H^i_\ell(X)$ form a compatible system of Galois representations in the sense of Serre [Ser98]. Also recall that we do not know in general that for a motive M over K the ℓ-adic Galois representations $H_\ell(M)$ form a compatible system of Galois representations in the sense of Serre.

Let M denote the motive $H^i(X)$. Serre [Ser13] proved, in a letter to Ribet (Jan. 29, 1981) that (*i*) the group $G_\ell(M)$ has finitely many connected components, (*ii*) its group of components does not depend on ℓ, and (*iii*) the kernel of $\mathrm{Gal}(\bar{K}/K) \to G_\ell(M)/G_\ell^\circ(M)$ does not depend on ℓ. This has two related implications. First of all, the group H_ℓ in diagram 2.10.1 does not depend on ℓ. Secondly, there is a finite field extension L/K such that $G_\ell(M)$ is connected for all prime numbers ℓ. By §2.9 and diagram 2.10.1 we know that $G_{\mathrm{mot},\ell}(M_L)$ does not change if we replace L by a field extension.

We stress that the above remarks are only known to be true for motives M for which the ℓ-adic Galois representations $H_\ell(M)$ form a compatible system of Galois representations in the sense of Serre. Let M be a motive over K. Then we do not know that the group of components of $G_\ell(M)$ is independent of ℓ, and in particular, we do not know that the group Γ_ℓ in diagram 2.10.1 is independent of ℓ. Still, there is something to be said. By definition, M is a subobject of $H(X)(n)$ for some smooth projective variety X over K. Therefore, we may take a finite field extension L/K such that $G_\ell(H(X_L)(n))$ is connected for all prime numbers ℓ. It follows that the algebraic group $G_\ell(M_L)$ is connected for all prime numbers ℓ. Once again, diagram 2.10.1 shows that if $G_\ell(M)$ is connected for some prime number ℓ, then $G_{\mathrm{mot},\ell}(M)$ does not change if we replace K by a field extension. In conclusion, this discussion leads to the following lemma.

2.12 Lemma. — *Let* K *be a field of characteristic* 0, *and let* M *be a motive over* K.

1. *There exists a finite field extension* L/K *such that* $G_\ell(M)$ *is connected for all prime numbers* ℓ.
2. *Let* ω *be a fibre functor on* Mot_K. *If* $G_\ell(M)$ *is connected for some prime number* ℓ *then* $G_{\mathrm{mot},\omega}(M)$ *does not change if we replace* K *by a field extension.*

2.13 Corollary. — *Let* K *be a field of characteristic* 0, *and let* M *be a motive over* K. *If* $G_\ell(M)$ *is connected for some prime number* ℓ, *then* $\mathrm{End}(M) = \mathrm{End}(M_{\bar{K}})$.

2.14 — Let K be a field of characteristic 0, and let M be a motive over K. The motive $M_{\bar{K}}$ decomposes into a sum of isotypical components. We denote with $M_{\bar{K}}^{\mathrm{alg}}$ the trivial isotypical component, and with $M_{\bar{K}}^{\mathrm{tra}}$ the sum of the non-trivial isotypical components. The decomposition $M_{\bar{K}} \cong M_{\bar{K}}^{\mathrm{alg}} \oplus M_{\bar{K}}^{\mathrm{tra}}$ is defined over K, so that we get a decomposition $M \cong M^{\mathrm{alg}} \oplus M^{\mathrm{tra}}$. We call M^{alg} the *algebraic* part of M, and M^{tra} the *transcendental* part of M. We call $\mathrm{End}(M_{\bar{K}})$ the *geometric* endomorphism algebra of M.

3 The Mumford–Tate conjecture

Readme. — In this section we state the Mumford–Tate conjecture, we show that it is invariant under finitely generated extensions of the base field (lemma 3.4), and we explain that the conjecture is not "additive" (§3.5). See §3.7 for some historical remarks about the Mumford–Tate conjecture.

3.1 Conjecture (Mumford–Tate). — *Let* K *be a finitely generated field of characteristic* 0 *and let* M *be a motive over* K. *Let* $\bar\sigma : \bar{K} \hookrightarrow \mathbb{C}$ *be an embedding, and let* ℓ *be a prime number. Then under Artin's comparison isomorphism (§2.4.2) we have*

$$G_\sigma(M) \otimes \mathbb{Q}_\ell \cong G_\ell^\circ(M).$$

3.2 Remark. — 1. Note that conjecture 3.1 depends on $\bar\sigma$ and ℓ. We denote the conjectural statement $G_\sigma(M) \otimes \mathbb{Q}_\ell \cong G_\ell^\circ(M)$ with $\mathrm{MTC}_{\bar\sigma,\ell}(M)$. If we mean the conjecture for all embeddings $\bar\sigma$ and prime numbers ℓ, then we write MTC(M).

2. Let K be a finitely generated field of characteristic 0. Let $\bar\sigma : \bar{K} \hookrightarrow \mathbb{C}$ be a complex embedding. Let ℓ be a prime number. Recall that Artin's comparison isomorphism (see §2.4.2) gives an isomorphism $G_{\mathrm{mot},\sigma}(M)^\circ \otimes \mathbb{Q}_\ell \cong G_{\mathrm{mot},\ell}(M)^\circ$. In light of the discussion in §2.9 and §2.10 we see that there is a "2-out-of-3" principle for conjecture 3.1, conjecture 2.7.1.b, and conjecture 2.7.2.b. If any two of the following three conjectures are true, then so is the third: (*i*) the conjecture $\mathrm{MTC}_{\bar\sigma,\ell}(M)$, (*ii*) conjecture 2.7.1.b for M_σ, and (*iii*) conjecture 2.7.2.b for M and the prime ℓ.

3.3 — Let K be a finitely generated field of characteristic 0; and let M be a motive over K. If the Mumford–Tate conjecture is true for M, then it is also true for all motives in $\langle M\rangle^\otimes$. Indeed, let M′ be a motive in $\langle M\rangle^\otimes$. The diagram

$$\begin{array}{ccc} G_\sigma(M) \otimes \mathbb{Q}_\ell & \longrightarrow & GL(H_\sigma(M')) \otimes \mathbb{Q}_\ell \\ \downarrow \simeq & & \downarrow \simeq \\ G_\ell^\circ(M) & \longrightarrow & GL(H_\ell(M')) \end{array}$$

commutes; and $G_\sigma(M')$ is the image of $G_\sigma(M)$ in $GL(H_\sigma(M'))$ and analogously $G_\ell^\circ(M')$ is the image of $G_\ell^\circ(M)$ in $GL(H_\ell(M'))$. This shows that $G_\sigma(M') \otimes \mathbb{Q}_\ell \cong G_\ell^\circ(M')$.

3.4 Lemma. — *Let* $K \subset L$ *be finitely generated fields of characteristic* 0. *Let* M *be a motive over* K. *Then* $\mathrm{MTC}(M) \iff \mathrm{MTC}(M_L)$.

Proof. See proposition 1.3 of [Moo16]. □

3.5 — Let K be a finitely generated field of characteristic 0; and let M_1 and M_2 be two motives over K. Suppose that $\mathrm{MTC}(M_1)$ and $\mathrm{MTC}(M_2)$ hold. We point out that $\mathrm{MTC}(M_1 \oplus M_2)$ is not a formal consequence of these assumptions.

The reason is best illuminated by recalling the following fact from representation theory. Let G be an algebraic group over a field F; and let V_1 and V_2 be two finite-dimensional representations of G. We expressly do *not* assume that V_1 and V_2 are faithful. For $i = 1, 2$, let G_i be the image of G in $\mathrm{GL}(V_i)$. Then the image G_{12} of G in $\mathrm{GL}(V_1 \oplus V_2)$ can not be determined from only the data $G_i \subset \mathrm{GL}(V_i)$. Even though in the "generic" situation one expects $G_{12} \cong G_1 \times G_2$, it is clear that if $V_1 \cong V_2$, then $G_1 \cong G_{12} \cong G_2$. Remark that it is always the case that G_{12} is a subgroup of $G_1 \times G_2$, and that the projections $G_{12} \to G_1$ and $G_{12} \to G_2$ are surjective homomorphisms.

We turn our attention back to the motives M_1 and M_2, and the Mumford–Tate conjecture for $M_1 \oplus M_2$. Recall that we supposed that $\mathrm{MTC}(M_1)$ and $\mathrm{MTC}(M_2)$ hold. The remark above shows that the algebraic groups $G_\sigma(M_1 \oplus M_2) \otimes \mathbb{Q}_\ell$ and $G_\ell^\circ(M_1 \oplus M_2)$ are both subgroups of $G_\ell^\circ(M_1) \times G_\ell^\circ(M_2)$, and they both project surjectively onto $G_\ell^\circ(M_i)$, $i = 1, 2$. However, this is not enough to conclude that $G_\sigma(M_1 \oplus M_2) \otimes \mathbb{Q}_\ell \cong G_\ell^\circ(M_1 \oplus M_2)$.

3.6 — A weaker version of conjecture 3.1 might ask whether the centres of $G_\sigma(M)$ and $G_\ell^\circ(M)$ coincide, *i.e.*, whether $Z_\sigma(M) \otimes \mathbb{Q}_\ell \cong Z_\ell^\circ(M)$. In theorem 5.6 we prove that this is the case for so-called abelian motives, which form the subject of section 5.

3.7 Historical remarks. — For an overview of the early history of the Mumford–Tate conjecture, we refer to Ribet's review [Rib90] of [Ser98]. We give a brief selection of further results since [Rib90].

- The most impressive result related to the Mumford–Tate conjecture is Deligne's "Hodge = absolute Hodge" theorem (proposition 2.9 and theorem 2.11 of [Del+82]). This theorem implies that if A is an abelian variety, then Artin's comparison isomorphism maps Hodge classes (invariants under $G_\sigma(A)$) to Tate classes (invariants under $G_\ell^\circ(A)$). An immediate corollary to this theorem is $G_\ell^\circ(A) \subset G_\sigma(A) \otimes \mathbb{Q}_\ell$, if A is an abelian variety. (Although [Del+82] is mentioned in [Rib90], Ribet does not explicitly mention this particular result.)
- Tankeev proved the Mumford–Tate conjecture for K3 surfaces, in [Tan90] and [Tan95].
- Independently, André [And96a] also proved the Mumford–Tate conjecture for K3 surfaces, using monodromy arguments and the theory of Kuga–Satake varieties.
- In [Pin98], Pink leveraged p-adic Hodge theory to prove the Mumford–Tate conjecture for a vast class of abelian varieties.
- Vasiu [Vas08] and Ullmo–Yafaev [UY13] showed that if A is an abelian variety, then the Mumford–Tate conjecture is true for centres (*cf.* §3.6): $Z_\sigma(A) \otimes \mathbb{Q}_\ell \cong Z_\ell^\circ(A)$.

» Moonen [Moo16] extended arguments of André to take generic endomorphisms into account. This proves the Mumford–Tate conjecture for surfaces with $p_g = 1$ that can be placed in a family with non-constant period map.

To put this account of the status of the Mumford–Tate conjecture in a sobering perspective: there are examples of abelian varieties of dimension 4 for which we currently do not know the Mumford–Tate conjecture.

4 Hyperadjoint objects in Tannakian categories

Readme. — Important: definition 4.4; lemma 4.8.

In this section we consider a construction that we will apply to motives. The philosophy is as follows: Let M be a motive over a field K of characteristic 0, and let ω be a fibre functor $\mathrm{Mot}_K \to \mathrm{Vect}_{\mathbb{Q}}$. The adjoint representation of the largest semisimple quotient of $G_{\mathrm{mot},\omega}(M)$ is an object in $\langle M\rangle^{\otimes}$ that, roughly speaking, captures the “semisimple data” of M. We prove that the isomorphism class of this object does not depend on the choice of the fibre functor ω.

Recall our convention that reductive groups and semisimple groups need not be connected. The main reason that we have to work with groups that are not connected is that we do not know that motivic Galois groups are connected.

4.1 — Let K be a field of characteristic 0; and let G be a linear algebraic group over K. Recall that the adjoint representation of G is the natural representation $\mathrm{ad}\colon G \to \mathrm{GL}(\mathrm{Lie}(G))$ via conjugation. With G^{ad} we mean the image of ad. Note that the kernel of the representation ad is the centraliser of the identity component G° in G. Therefore $G^{\mathrm{ad}} \cong G/Z_G(G^{\circ})$.

4.2 Lemma. — *Let* K *be a field of characteristic* 0*, and let* $f : G_1 \twoheadrightarrow G_2$ *be a surjective homomorphism of linear algebraic groups. Then* $\mathrm{Lie}(f)\colon \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$ *is a surjective homomorphism of representations of* G_1.

Proof. It is clear that $\mathrm{Lie}(f)$ is a homomorphism of representations of G_1. Note that $\mathrm{Lie}(f)$ is surjective because f is surjective and K has characteristic 0. □

4.3 Lemma. — *Let* K *be a field of characteristic* 0*, and let* C *be a* K*-linear neutral Tannakian category. Let* V *be an object of* C*. For* $i = 1, 2$*, let* ω_i *be a fibre functor from* $\langle V\rangle^{\otimes}$ *to the category of finite-dimensional* K*-vector spaces. and let* G_i *be the affine group scheme associated with* $\langle V\rangle^{\otimes}$ *via the fibre functor* ω_i*. In other words,* $G_i = \underline{\mathrm{Aut}}(\omega_i)^{\otimes}$*. Consider* $\mathrm{Lie}(G_i)$ *as an object of* $\langle V\rangle^{\otimes}$*, via the natural equivalences* $\langle V\rangle^{\otimes} \cong \mathrm{Rep}_K(G_i)$*. Then we have* $\mathrm{Lie}(G_1) \cong \mathrm{Lie}(G_2)$ *as objects in* C *(but not necessarily as*

Lie algebra objects).

Proof. There exists a field extension L/K such that $\underline{\mathrm{Isom}}(\omega_1, \omega_2)$ has an L-rational point. By lemma 1.4 it suffices to show $\mathrm{Lie}(G_1) \cong \mathrm{Lie}(G_2)$ after extending scalars from K to L. In other words, we may assume without loss of generality that $\underline{\mathrm{Isom}}(\omega_1, \omega_2)$ has a K-rational point. A rational point $f \in \underline{\mathrm{Isom}}(\omega_1, \omega_2)(K)$ induces an isomorphism $G_1 \to G_2$, given by $\alpha \mapsto f \circ \alpha \circ f^{-1}$. In turn, this isomorphism induces an equivalence $\mathrm{Rep}_K(G_2) \to \mathrm{Rep}_K(G_1)$. For $W \in \langle V \rangle^{\otimes}$, this equivalence maps $\omega_2(W)$ to $\omega_1(W)$. But since this equivalence is induced by an isomorphism $G_1 \to G_2$, we also find $\mathrm{Lie}(G_2) \mapsto \mathrm{Lie}(G_1)$. □

4.4 Definition. — Let K be a field of characteristic 0, and let C be a K-linear neutral Tannakian category. Let V be an object of C. Let ω be a fibre functor on $\langle V \rangle^{\otimes}$, and let G be the linear algebraic group $\underline{\mathrm{Aut}}(\omega)^{\otimes}$. With V^a we mean an object that corresponds with Lie(G) under the equivalence $\langle V \rangle^{\otimes} \cong \mathrm{Rep}_K(G)$. By lemma 4.3 we know that the isomorphism class of V^a does not depend on the choice of ω. (The superscript $(\_)^a$ indicates that V^a is the “adjoint representation” object in $\langle V \rangle^{\otimes}$.)

If W is an object of $\langle V \rangle^{\otimes}$, then W^a is a quotient of V^a, by lemma 4.2. Write $V^{(1)}$ for V^a, and inductively write $V^{(i+1)}$ for $(V^{(i)})^a$. Since $V^{(i+1)}$ is a quotient of $V^{(i)}$, and the dimension of $V^{(1)}$ is finite, we know that there is an integer $n \geq 1$ such that $V^{(n)} \cong V^{(i)}$ for all $i \geq n$. We denote $V^{(n)}$ with V^{ha}, and call V^{ha} the *hyperadjoint object* in $\langle V \rangle^{\otimes}$.

4.5 Caveat. — The constructions $V \rightsquigarrow V^a$ and $V \rightsquigarrow V^{\mathrm{ha}}$ are not functorial. They do not in general commute with tensor functors between neutral Tannakian categories. Also, the constructions are not in general compatible with direct sums; that is $(V_1 \oplus V_2)^{\mathrm{ha}} \ncong V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}}$.

4.6 — Let us take a step back to see what this construction means in the context of representations of a linear algebraic group. Let G be a linear algebraic group over a field K of characteristic 0. Let V be a representation of G. Write $G^{(i)}$ for the image of G in $\mathrm{GL}(V^{(i)})$. Then $V^{(1)} = V^a = \mathrm{Lie}(G)$, and $G^{(1)} = G/Z_G(G^\circ)$. Hence $G^{(1),\circ} = G^\circ/Z(G^\circ)$.

Since $G^{(1)}$ is a quotient of G, we get a representation of G on $\mathrm{Lie}(G^{(1)})$, and this is $V^{(2)}$. Although $V^{(1)}$ need not be a quotient of V, we do know that $V^{(2)}$ is a quotient of $V^{(1)}$ by lemma 4.2. The quotient map $V^{(1)} \twoheadrightarrow V^{(2)}$ is an isomorphism if and only if $Z_G(G^\circ)$ is finite, or equivalently, if and only if $Z(G^\circ)$ is finite.

Since the definition of $V^{(i)}$ for $i > 1$ is inductive we see that we get a sequence of quotients $V^{(1)} \twoheadrightarrow V^{(2)} \twoheadrightarrow V^{(3)} \twoheadrightarrow \ldots$ and if the quotient $V^{(i)} \twoheadrightarrow V^{(i+1)}$ is an isomorphism then so are all the following quotients.

In the following remark we translate these statements back in the context of Tannakian categories.

4.7 REMARK. — Let K be a field of characteristic 0, and let C be a neutral K-linear Tannakian category.

1. Let V be an object of C. If $W \in \langle V \rangle^{\otimes}$, then W^{a} is a quotient of V^{a}, by lemma 4.2.
2. By induction, the previous point shows that W^{ha} is a quotient of V^{ha}, for all objects $W \in \langle V \rangle^{\otimes}$.

Let G be a linear algebraic group over K, and let V be a faithful representation of G.

3. If G is semisimple, then $\dim(V^{a}) = \dim(G) = \dim(G^{\mathrm{ad}}) = \dim(V^{(2)})$. Hence $V^{\mathrm{ha}} = V^{(1)} = V^{a} = \mathrm{Lie}(G)$.
4. If G is reductive, then G^{ad} is semisimple. Hence $V^{\mathrm{ha}} = V^{(2)} = \mathrm{Lie}(G^{\mathrm{ad}})$.

4.8 LEMMA. — *Let* K *be a field of characteristic* 0*, and let* C *be a neutral* K*-linear Tannakian category. Let* V_1 *and* V_2 *be two objects of* C.

1. *Then* $(V_1 \oplus V_2)^{a}$ *is a subobject of* $V_1^{a} \oplus V_2^{a}$.
2. *We have* $(V_1 \oplus V_2)^{\mathrm{ha}} \cong (V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}})^{\mathrm{ha}}$.
3. *If* C *is a semisimple category, then* $(V_1 \oplus V_2)^{\mathrm{ha}}$ *is a direct summand of* $V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}}$.

Proof.

1. By lemma 4.3 this statement is independent of the choice of a fibre functor. Therefore, we choose a fibre functor ω on the Tannakian category $\langle V_1 \oplus V_2 \rangle^{\otimes}$; we write G for the linear algebraic group $\underline{\mathrm{Aut}}(\omega)^{\otimes}$ over K; and we view $V_1 \oplus V_2$ as a faithful representation of G. Let G_i be the image of G in $\mathrm{GL}(V_i)$. The natural map $G \to G_1 \times G_2$ is injective, and its composition with the projections $G_1 \times G_2 \twoheadrightarrow G_i$ is surjective.

   We identify $(V_1 \oplus V_2)^{a}$ with $\mathrm{Lie}(G)$, and V_i^{a} with $\mathrm{Lie}(G_i)$. By lemma 4.2 we know that $\mathrm{Lie}(G_i)$ is a quotient of $\mathrm{Lie}(G)$ as representations of G. Thus we obtain a homomorphism of representations $\mathrm{Lie}(G) \to \mathrm{Lie}(G_1) \oplus \mathrm{Lie}(G_2)$, which is injective since $G \to G_1 \times G_2$ is injective. In other words: $(V_1 \oplus V_2)^{a}$ is a subobject of $V_1^{a} \oplus V_2^{a}$.
2. Since $V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}} \in \langle V_1 \oplus V_2 \rangle^{\otimes}$, we get a quotient map $q\colon (V_1 \oplus V_2)^{\mathrm{ha}} \twoheadrightarrow (V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}})^{\mathrm{ha}}$. We claim that $(V_1 \oplus V_2)^{\mathrm{ha}} \in \langle V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}} \rangle^{\otimes}$. If we take the claim for granted, then we get another quotient map $q'\colon (V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}})^{\mathrm{ha}} \twoheadrightarrow ((V_1 \oplus V_2)^{\mathrm{ha}})^{\mathrm{ha}} = (V_1 \oplus V_2)^{\mathrm{ha}}$. Hence $\dim((V_1 \oplus V_2)^{\mathrm{ha}}) = \dim((V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}})^{\mathrm{ha}})$ and q and q' are isomorphisms. Thus we are done if we prove the claim.

   Assume that $(V_1 \oplus V_2)^{(i)}$ is contained in $\langle V_1^{(i)} \oplus V_2^{(i)} \rangle^{\otimes}$. By point 1, we know that this is true for $i = 1$. Since $(V_1 \oplus V_2)^{(i)}$ is contained in $\langle V_1^{(i)} \oplus V_2^{(i)} \rangle^{\otimes}$, we know that $(V_1 \oplus V_2)^{(i+1)}$ is a quotient of $(V_1^{(i)} \oplus V_2^{(i)})^{a}$, by remark 4.7.1. In turn, $(V_1^{(i)} \oplus V_2^{(i)})^{a}$ is a subobject of $V_1^{(i+1)} \oplus V_2^{(i+1)}$ by another application of point 1. Therefore $(V_1 \oplus V_2)^{(i+1)}$ is contained in $\langle V_1^{(i+1)} \oplus V_2^{(i+1)} \rangle^{\otimes}$. By induction we conclude that $(V_1 \oplus V_2)^{\mathrm{ha}}$ is contained in $\langle V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}} \rangle^{\otimes}$. This proves the claim.
3. By point 1, $(V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}})^{a}$ is a subobject of $(V_1^{\mathrm{ha}})^{a} \oplus (V_2^{\mathrm{ha}})^{a} \cong V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}}$. Recall that $(V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}})^{\mathrm{ha}}$ is a quotient of $(V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}})^{a}$ and thus a subquotient of $V_1^{\mathrm{ha}} \oplus V_2^{\mathrm{ha}}$. The result follows from point 2 and the semisimplicity of C. □

5 Abelian motives

Readme. — We define abelian motives, and then we list theorems that capture the main reason why abelian motives are easier to work with than general motives.

5.1 — Let K be a finitely generated field of characteristic 0. An *abelian motive* over K is an object of the Tannakian subcategory of motives over K generated by the motives of abelian varieties over K. Recall that $H(A) \cong \bigwedge^\star H^1(A)$ for every abelian variety A over K, and thus we have $\langle H(A)\rangle^{\otimes} = \langle H^1(A)\rangle^{\otimes}$. If A is a non-trivial abelian variety, then the class of any effective non-zero divisor realises $\mathbb{1}(-1)$ as a subobject of $H^2(A)$, and therefore $\mathbb{1}(-1) \in \langle H^1(A)\rangle^{\otimes}$. In particular $\mathbb{1}(-1)$ is an abelian motive. We claim that every abelian motive M is contained in $\langle H^1(A)\rangle^{\otimes}$ for some abelian variety A over K. By definition there are abelian varieties $(A_i)_{i=1}^k$ such that M is contained in the Tannakian subcategory generated by the $H(A_i)$. Put $A = \prod_{i=1}^k A_i$, so that $H^1(A) \cong \bigoplus_{i=1}^k H^1(A_i)$. It follows that M is contained in $\langle H^1(A)\rangle^{\otimes}$.

5.2 Theorem. — *1. The Betti realisation functor* $H_B(\_)$ *is fully faithful on the subcategory of abelian motives over* $\mathbb{C}$*. In other words, conjecture 2.7.1 is true for every abelian motive over* $\mathbb{C}$*.*

2. Let K *be a field, and let* $\sigma : \bar{K} \hookrightarrow \mathbb{C}$ *be an embedding. If* M *is an abelian motive over* K*, then the natural inclusion* $G_\sigma(M) \hookrightarrow G_{\mathrm{mot},\sigma}(M)^\circ$ *is an isomorphism, and* $G_\ell^\circ(M) \subset G_\sigma(M) \otimes \mathbb{Q}_\ell$*.*

Proof. 1. See théorème 0.6.2 of [And96b].

2. Note that by the previous point, $G_\sigma(M) \cong G_{\mathrm{mot},\sigma}(M_\sigma)$ is connected. By §2.9 we know that $G_{\mathrm{mot},\sigma}(M_\sigma) \hookrightarrow G_{\mathrm{mot},\sigma}(M)$ is an inclusion of algebraic groups with the same dimension. Therefore $G_\sigma(M)$ is the identity component of $G_{\mathrm{mot},\sigma}(M)$. Since $G_{\mathrm{mot},\ell}(M) \cong G_{\mathrm{mot},\sigma}(M) \otimes \mathbb{Q}_\ell$ (by Artin's comparison theorem, §2.4.2) we conclude that $G_\ell^\circ(M) \subset G_\sigma(M) \otimes \mathbb{Q}_\ell$. □

5.3 Remark. — Let K be a field of characteristic 0, and let M be an abelian motive over K. Theorem 5.2.1 tells us that for every complex embedding $\sigma : \bar{K} \hookrightarrow \mathbb{C}$ the subspace $H_\sigma(M^{\mathrm{alg}}) \subset H_\sigma(M)$ is exactly the subspace of Hodge classes in $H_\sigma(M)$, and $\mathrm{End}(M_{\bar{K}}) \cong \mathrm{End}(H_\sigma(M))$.

5.4 Theorem. — *Let* M *be an abelian motive over a finitely generated field of characteristic* 0*. Then the algebraic group* $G_\ell(M)$ *is reductive and* $H_\ell(M)$ *is a semisimple Galois representation.*

Proof. The result is true for abelian varieties by Satz 3 in §5 of [Fal83] (also see [Fal84]).

By §5.1, there is an abelian variety A such that M is contained in the Tannakian subcategory $\langle H^1(A)\rangle^{\otimes}$ generated by the motive $H^1(A)$. This yields a surjection $G_\ell(A) \twoheadrightarrow G_\ell(M)$ and therefore $G_\ell(M)$ is reductive. Consequently, $H_\ell(M)$ is a semisimple representation of $G_\ell(M)$, and thus a semisimple Galois representation. □

5.5 Theorem. — *Let* A *be an abelian variety over a finitely generated field* K *of characteristic* 0. *Let* $\bar{\sigma} : \bar{\mathrm{K}} \hookrightarrow \mathbb{C}$ *be a complex embedding, and let* ℓ *be a prime number. Under Artin's comparison isomorphism (§2.4.2) we have* $\mathrm{Z}^\circ_\ell(\mathrm{A}) \cong \mathrm{Z}_\sigma(\mathrm{A}) \otimes \mathbb{Q}_\ell$.

Proof. See theorem 1.3.1 of [Vas08] or corollary 2.11 of [UY13]. The following proof is due to Moonen (private communication). We first prove two claims.

1. Let S be a connected K-scheme of finite type, $f : \mathscr{A} \to \mathrm{S}$ an abelian scheme, $\eta \in \mathrm{S}$ a Hodge generic point, and $s \in \mathrm{S}$ any point. Identify $\mathrm{H}^1_\sigma(\mathscr{A}_\eta)$ with $\mathrm{H}^1_\sigma(\mathscr{A}_s)$ via a trivialisation of $\mathrm{R}^1 f_* \mathbb{Q}$ on the universal covering $\tilde{\mathrm{S}}$ of S_σ, so that we get $\mathrm{G}_\sigma(\mathscr{A}_s) \hookrightarrow \mathrm{G}_\sigma(\mathscr{A}_\eta)$. We claim that the induced homomorphism $\mathrm{G}_\sigma(\mathscr{A}_s)^{\mathrm{ab}} \hookrightarrow \mathrm{G}_\sigma(\mathscr{A}_\eta)^{\mathrm{ab}}$ is surjective.

   Indeed, we may assume that $\mathscr{A} \to \mathrm{S}$ is a universal family of abelian varieties over a Shimura variety defined by the group $\mathrm{G} = \mathrm{G}_\sigma(\mathscr{A}_\eta)$. This gives a family of homomorphisms $\{h_t : \mathbb{S} \to \mathrm{G}_\mathbb{R}\}_{t \in \tilde{\mathrm{S}}}$, and $h_t^{\mathrm{ab}} : \mathbb{S} \to \mathrm{G}^{\mathrm{ab}}_\mathbb{R}$ is independent of $t \in \tilde{\mathrm{S}}$.

   As G is the generic Mumford–Tate group of the family, G^{ab} does not contain a proper algebraic subgroup $\mathrm{H} \subsetneq \mathrm{G}^{\mathrm{ab}}$ such that h_t^{ab} factors through $\mathrm{H}_\mathbb{R}$. On the other hand, we have a diagram

$$\begin{array}{ccc} \mathrm{G}_\sigma(\mathscr{A}_s) & \longrightarrow & \mathrm{G}_\sigma(\mathscr{A}_\eta) \\ \downarrow & & \downarrow \\ \mathrm{G}_\sigma(\mathscr{A}_s)^{\mathrm{ab}} & \xrightarrow{j} & \mathrm{G}_\sigma(\mathscr{A}_\eta)^{\mathrm{ab}} \end{array}$$

   and $h_{\tilde{s}}^{\mathrm{ab}}$, for some $\tilde{\mathrm{S}} \ni \tilde{s} \mapsto s$, factors through $j_\mathbb{R}$. Hence j is surjective.
2. We claim that $\mathrm{Z}^\circ_\ell(\mathrm{A}) = \mathrm{G}^\circ_\ell(\mathrm{A}) \cap (\mathrm{Z}_\sigma(\mathrm{A}) \otimes \mathbb{Q}_\ell)$. Indeed, by theorem 5.2.2 we have $\mathrm{G}^\circ_\ell(\mathrm{A}) \subset \mathrm{G}_\sigma(\mathrm{A}) \otimes \mathbb{Q}_\ell$ and hence $\mathrm{G}^\circ_\ell(\mathrm{A}) \cap (\mathrm{Z}_\sigma(\mathrm{A}) \otimes \mathbb{Q}_\ell) \subset \mathrm{Z}^\circ_\ell(\mathrm{A})$. On the other hand, the results of Faltings (Satz 4 of [Fal83], see also [Fal84]) show that $\mathrm{Z}^\circ_\ell(\mathrm{A}) \subset \mathrm{Z}_\sigma(\mathrm{A}) \otimes \mathbb{Q}_\ell$.

We now return to the abelian variety A over K. We may arrange a situation as in point 1, in such a way that $\mathrm{A} \cong \mathscr{A}_\eta$, and $s \in \mathrm{S}$ is a special point. Recall that $\mathrm{G}^\circ_\ell(\mathscr{A}_s) = \mathrm{Z}^\circ_\ell(\mathscr{A}_s) = \mathrm{Z}_\sigma(\mathscr{A}_s) \otimes \mathbb{Q}_\ell = \mathrm{G}_\sigma(\mathscr{A}_s) \otimes \mathbb{Q}_\ell$ by [Poh68]. Consider the diagram

$$\begin{array}{ccccc} \mathrm{G}^\circ_\ell(\mathscr{A}_s) & \xrightarrow{\simeq} & \mathrm{G}_\sigma(\mathscr{A}_s) \otimes \mathbb{Q}_\ell & & \\ & \searrow & & \searrow & \\ & & \mathrm{G}^\circ_\ell(\mathscr{A}_\eta) & \hookrightarrow & \mathrm{G}_\sigma(\mathscr{A}_\eta) \otimes \mathbb{Q}_\ell \\ & & \downarrow & & \downarrow \\ & & \mathrm{G}^\circ_\ell(\mathscr{A}_\eta)^{\mathrm{ab}} & \longrightarrow & \mathrm{G}_\sigma(\mathscr{A}_\eta)^{\mathrm{ab}} \otimes \mathbb{Q}_\ell \end{array}$$

The fact proven in point 1 tells us that the composition $\mathrm{G}^\circ_\ell(\mathscr{A}_s) \to \mathrm{G}_\sigma(\mathscr{A}_\eta)^{\mathrm{ab}} \otimes \mathbb{Q}_\ell$ is surjective.

Hence $G^\circ_\ell(A)^{ab} \to G_\sigma(A)^{ab} \otimes \mathbb{Q}_\ell$ is also surjective. (Recall that $A = \mathscr{A}_{\bar\eta}$.) This implies that

$$\mathrm{rk}(Z^\circ_\ell(A)) = \mathrm{rk}(G^\circ_\ell(A)^{ab}) \geq \mathrm{rk}(G_\sigma(A)^{ab}) = \mathrm{rk}(Z_\sigma(A))$$

and point 2 then gives $Z^\circ_\ell(A) = Z_\sigma(A) \otimes \mathbb{Q}_\ell$. □

5.6 Theorem. — *Let* M *be an abelian motive over a finitely generated field* K *of characteristic* 0*. Let* $\sigma : \bar{K} \hookrightarrow \mathbb{C}$ *be a complex embedding, and let* ℓ *be a prime number. Under Artin's comparison isomorphism (§2.4.2) we have* $Z^\circ_\ell(M) \cong Z_\sigma(M) \otimes \mathbb{Q}_\ell$*.*

Proof. By §5.1, there is an abelian variety A such that M is contained in the Tannakian subcategory $\langle H^1(A)\rangle^\otimes$ generated by the motive $H^1(A)$. This yields a surjection of reductive groups $G_\sigma(A) \twoheadrightarrow G_\sigma(M)$, and therefore $Z_\sigma(M)$ is the image of $Z_\sigma(A)$ under this map. The same is true on the ℓ-adic side, since by theorem 5.4, the algebraic group $G^\circ_\ell(M)$ is reductive: $Z^\circ_\ell(M)$ is the image of $Z^\circ_\ell(A)$ under the surjection $G^\circ_\ell(A) \twoheadrightarrow G^\circ_\ell(M)$.

Hence we obtain a commutative diagram with solid arrows

$$\begin{array}{ccccc} Z^\circ_\ell(A) & \twoheadrightarrow & Z^\circ_\ell(M) & \hookrightarrow & G^\circ_\ell(M) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \\ Z_\sigma(A) \otimes \mathbb{Q}_\ell & \twoheadrightarrow & Z_\sigma(M) \otimes \mathbb{Q}_\ell & \hookrightarrow & G_\sigma(M) \otimes \mathbb{Q}_\ell \end{array}$$

where the vertical arrow on the left is an isomorphism by theorem 5.5, and the vertical arrow on the right is an inclusion by theorem 5.2.2. The diagram shows that $Z^\circ_\ell(M)$ and $Z_\sigma(M) \otimes \mathbb{Q}_\ell$ are both the image of the morphism $Z^\circ_\ell(A) \to G_\sigma(M) \otimes \mathbb{Q}_\ell$; and therefore the the dashed arrow exists and is an isomorphism. □

5.7 — We now apply the concept of hyperadjoint objects (developed in section 4) to abelian motives. If M is a motive, then we denote the hyperadjoint object in $\langle M\rangle^\otimes$ with M^{ha}. We call M^{ha} the *hyperadjoint motive* associated with M.

Nota bene: the caveats of §4.5 apply undiminished to the context of motives.

5.8 Lemma. — *Let* M *be an abelian motive over a finitely generated field* K *of characteristic* 0*.*

1. *Then* M^{ha} *is an abelian motive.*
2. *Let* ω *be a fibre functor on the category of motives over* K*. Then* $G_{mot,\omega}(M^{ha}) \cong G_{mot,\omega}(M)^{ad}$ *and* $\omega(M^{ha}) \cong \mathrm{Lie}(G_{mot,\omega}(M)^{ad})$*.*
3. *Then* $G_\sigma(M^{ha}) \cong G_\sigma(M)^{ad}$ *and* $H_\sigma(M^{ha}) \cong H_\sigma(M)^{ha} \cong \mathrm{Lie}(G_\sigma(M)^{ad})$*.*
4. *Assume that* $G_\ell(M)$ *is connected. Then* $G_\ell(M^{ha}) \cong G_\ell(M)^{ad}$*. If in addition* MTC(M) *holds, then* $H_\ell(M^{ha}) \cong H_\ell(M)^{ha} \cong \mathrm{Lie}(G_\ell(M)^{ad})$*.*

Proof. 1. Note that M^{ha} is an abelian motive, since $M^{ha} \in \langle M \rangle^{\otimes}$.

2. Recall that $G_{mot,\omega}(M^{ha})$ is a reductive group. Thus the claim follows from remark 4.7.4.
3. This follows from theorem 5.2 and the previous point.
4. Since M is an abelian motive, and since $G_\ell(M)$ is connected, the group $G_{mot,\ell}(M)$ is connected (*cf.* lemma 2.12). Note that $G_\ell(M^{ha})$ is the image of $G_\ell(M)$ in $G_{mot,\ell}(M^{ha})$. Since $G_{mot,\ell}(M)$ is reductive, the kernel of $G_{mot,\ell}(M) \to G_{mot,\ell}(M^{ha})$ is $Z_{mot,\ell}(M)$. The kernel of the map $G_\ell(M) \to G_{mot,\ell}(M^{ha})$ is $G_\ell(M) \cap Z_{mot,\ell}(M)$. By theorem 5.6, we know that $Z_\ell(M) = Z_{mot,\ell}(M)$, and therefore $G_\ell(M) \cap Z_{mot,\ell}(M) = Z_\ell(M)$. In other words, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_\ell(M) & \hookrightarrow & G_\ell(M) & \twoheadrightarrow & G_\ell(M^{ha}) & \longrightarrow & 0 \\
 & & \downarrow \simeq & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_{mot,\ell}(M) & \hookrightarrow & G_{mot,\ell}(M) & \twoheadrightarrow & G_{mot,\ell}(M^{ha}) & \longrightarrow & 0
\end{array}$$

We conclude that $G_\ell(M^{ha}) \cong G_\ell(M)^{ad}$. If in addition MTC(M) holds, then $G_\ell(M) \cong G_{mot,\ell}(M)$, and hence $H_\ell(M^{ha}) \cong H_\ell(M)^{ha} \cong \mathrm{Lie}(G_\ell(M)^{ad})$. □

5.9 Proposition. — *Let* M *be an abelian motive over a finitely generated field* K *of characteristic* 0. *Let* $\bar{\sigma} : K \hookrightarrow \mathbb{C}$ *be a complex embedding and let* ℓ *be a prime number. Then* $\mathrm{MTC}_{\bar{\sigma},\ell}(M)$ *is equivalent to* $\mathrm{MTC}_{\bar{\sigma},\ell}(M^{ha})$.

Proof. By lemma 5.8, the rows of the following commutative diagram are exact

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z^\circ_\ell(M) & \hookrightarrow & G^\circ_\ell(M) & \twoheadrightarrow & G^\circ_\ell(M^{ha}) & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
0 & \longrightarrow & Z_\sigma(M) \otimes \mathbb{Q}_\ell & \hookrightarrow & G_\sigma(M) \otimes \mathbb{Q}_\ell & \twoheadrightarrow & G_\sigma(M^{ha}) \otimes \mathbb{Q}_\ell & \longrightarrow & 0
\end{array}$$

where the inclusions b, and c exist by theorem 5.2.2; and a is an isomorphism by theorem 5.6. In conclusion, b is an isomorphism if and only if c is an isomorphism, which means to say $\mathrm{MTC}_{\bar{\sigma},\ell}(M) \iff \mathrm{MTC}_{\bar{\sigma},\ell}(M^{ha})$. □

Quasi-compatible systems of Galois representations

6 Quasi-compatible systems of Galois representations

Readme. — Important: definition 6.13; lemma 6.21.

In this section we develop a variant of Serre's notion of a compatible system of Galois representations [Ser98]. We follow Serre's suggestion of developing an E-rational version (where E is a number field); which has also been done by Ribet [Rib76] and Chi [Chi92]. The main benefit of the variant that we develop is that we relax the compatibility condition, thereby gaining a certain robustness with respect to extensions of the base field and residue fields. We will need this property in a crucial way in the proof of theorem 10.1.

6.1 — Let κ be a finite field with q elements, and let $\bar{\kappa}$ be an algebraic closure of κ. We denote with $F_{\bar{\kappa}/\kappa}$ the geometric Frobenius element (that is, the inverse of $x \mapsto x^q$) in $\mathrm{Gal}(\bar{\kappa}/\kappa)$.

6.2 — Let K be a number field. Let v be a finite place of K, and let K_v denote the completion of K at v. Let $\bar{K}_v$ be an algebraic closure of K_v. Let $\bar{\kappa}/\kappa$ be the extension of residue fields corresponding with $\bar{K}_v/K_v$. The inertia group, denoted I_v, is the kernel of the natural surjection $\mathrm{Gal}(\bar{K}_v/K_v) \twoheadrightarrow \mathrm{Gal}(\bar{\kappa}/\kappa)$. The inverse image of $F_{\bar{\kappa}/\kappa}$ in $\mathrm{Gal}(\bar{K}_v/K_v)$ is called the *Frobenius coset* of v. An element $\alpha \in \mathrm{Gal}(\bar{K}/K)$ is called a *Frobenius element with respect to* v if there exists an embedding $\bar{K} \hookrightarrow \bar{K}_v$ such that α is the restriction of an element of the Frobenius coset of v.

6.3 — Let K be a finitely generated field. A *model* of K is an integral scheme X of finite type over $\mathrm{Spec}(\mathbb{Z})$ together with an isomorphism between K and the function field of X. Remark that if K is a number field, and $R \subset K$ is an order, then Spec(R) is naturally a model of K. The only model of a number field K that is normal and proper over $\mathrm{Spec}(\mathbb{Z})$ is $\mathrm{Spec}(\mathcal{O}_K)$.

6.4 — Let K be a finitely generated field, and let X be a model of K. Recall that we denote the set of closed points of X with X^{cl}. Let $x \in X^{cl}$ be a closed point. Let K_x be the function field of the Henselisation of X at x; and let $\kappa(x)$ be the residue field at x. We denote with I_x the kernel of $\mathrm{Gal}(\bar{K}_x/K_x) \twoheadrightarrow \mathrm{Gal}(\bar{\kappa}(x)/\kappa(x))$. Every embedding $\bar{K} \hookrightarrow \bar{K}_x$ induces an inclusion $\mathrm{Gal}(\bar{K}_x/K_x) \hookrightarrow \mathrm{Gal}(\bar{K}/K)$.

Like in §6.2, the inverse image of $F_{\bar{\kappa}(x)/\kappa(x)}$ is called the Frobenius coset of x. An element $\alpha \in \mathrm{Gal}(\bar{K}/K)$ is called a *Frobenius element with respect to x* if there exists an embedding $\bar{K} \hookrightarrow \bar{K}_x$ such that α is the restriction of an element of the Frobenius coset of x.

6.5 — Let K be a field and let E be a number field. Let λ be a place of E. With a *λ-adic Galois representation* of K we mean a representation of $\mathrm{Gal}(\bar{K}/K)$ on a finite-dimensional E_λ-vector space that is continuous for the λ-adic topology.

Recall from §2.5 that we defined the group $G_\ell(\rho)$ for every ℓ-adic Galois representation ρ. We extend this definition to λ-adic Galois representations in the natural way: If $\rho\colon \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}(V)$ is a λ-adic Galois representation, then $G_\lambda(\rho)$ denotes the Zariski closure of the image of ρ in $\mathrm{GL}(V)$.

6.6 — Let K be a finitely generated field. Let X be a model of K, and let $x \in X^{cl}$ be a closed point. We use the notation introduced in §6.4. Let ρ be a λ-adic Galois representation of K. We say that ρ is *unramified at x* if there is an embedding $\bar{K} \hookrightarrow \bar{K}_x$ for which $\rho(I_x) = \{1\}$. If this is true for one embedding, then it is true for all embeddings.

Let F_x be a Frobenius element with respect to x. If ρ is unramified at x, then the element $F_{x,\rho} = \rho(F_x)$ is well-defined up to conjugation. We write $P_{x,\rho,n}(t)$ for the characteristic polynomial c.p.$(F^n_{x,\rho})$. Note that $P_{x,\rho,n}(t)$ is well-defined, since conjugate endomorphisms have the same characteristic polynomial.

6.7 — In the following definitions, one recovers the notions of Serre [Ser98] by demanding $n = 1$ everywhere. By not making this demand we gain a certain flexibility that will turn out to be crucial for our proof of theorem 10.1.

6.8 Definition. — Let K be a finitely generated field. Let E be a number field, and let λ be a finite place of E. Let ρ be a λ-adic Galois representation of K. Let X be a model of K, and let $x \in X^{cl}$ be a closed point. The representation ρ is said to be *E-rational at x* if ρ is unramified at x, and $P_{x,\rho,n}(t) \in E[t]$, for some $n \geq 1$.

6.9 Definition. — Let K be a finitely generated field. Let E be a number field, and let λ_1 and λ_2 be two finite places of E. Let ρ_1 (resp. ρ_2) be a λ_1-adic (resp. λ_2-adic) Galois representation of K.

1. Let X be a model of K, and let $x \in X^{cl}$ be a closed point. Then ρ_1 and ρ_2 are said to be *quasi-compatible at* x if ρ_1 and ρ_2 are both E-rational at x, and if there is an integer n such that $P_{x,\rho_1,n}(t) = P_{x,\rho_2,n}(t)$ as polynomials in $E[t]$.
2. Let X be a model of K. The representations ρ_1 and ρ_2 are *quasi-compatible with respect to* X if there is a non-empty open subset $U \subset X$, such that ρ_1 and ρ_2 are quasi-compatible at x for all $x \in U^{cl}$.
3. The representations ρ_1 and ρ_2 are *quasi-compatible* if they are quasi-compatible with respect to every model of K.
4. Let X be a model of K. The representations ρ_1 and ρ_2 are *strongly quasi-compatible with respect to* X if ρ_1 and ρ_2 are quasi-compatible at all points $x \in X^{cl}$ that satisfy the following condition:

   The places λ_1 and λ_2 have a residue characteristic that is different from the residue characteristic of x, and ρ_1 and ρ_2 are unramified at x.
5. The representations ρ_1 and ρ_2 are *strongly quasi-compatible* if they are strongly quasi-compatible with respect to every model of K.

6.10 Remark. — Let K, E, λ_1, λ_2, ρ_1, and ρ_2 be as in the above definition.

1. If there is one model X of K such that ρ_1 and ρ_2 are quasi-compatible with respect to X, then ρ_1 and ρ_2 are quasi-compatible with respect to every model of K, since all models of K are birational to each other.
2. The notion of strong compatibility is *not* known to be stable under birational equivalence: if ρ_1 and ρ_2 are quasi-compatible with respect to some model X of K, then by definition there exists a non-empty open subset $U \subset X$ such that ρ_1 and ρ_2 are strongly quasi-compatible with respect to U. But there is no *a priori* reason to expect that ρ_1 and ρ_2 are strongly quasi-compatible with respect to X.

6.11 Definition. — Let K be a field. With a *system of Galois representations* of K we mean a triple $(E, \Lambda, (\rho_\lambda)_{\lambda \in \Lambda})$, where E is a number field; Λ is a set of finite places of E; and ρ_λ $(\lambda \in \Lambda)$ is a λ-adic Galois representation of K.

6.12 — In what follows, we often denote a system of Galois representations $(E, \Lambda, (\rho_\lambda)_{\lambda \in \Lambda})$ with ρ_Λ, leaving the number field E implicit. In contexts where there are multiple number fields the notation will make clear which number field is meant (*e.g.*, by denoting the set of finite places of a number field E' with Λ', etc...).

6.13 Definition. — Let K be a finitely generated field. Let E be a number field, and let Λ be a set of finite places of E. Let ρ_Λ be a system of Galois representations of K.

1. Let X be a model of K. The system ρ_Λ is called *(strongly) quasi-compatible with respect to* X if for all $\lambda_1, \lambda_2 \in \Lambda$ the representations ρ_{λ_1} and ρ_{λ_2} are (strongly) quasi-compatible with respect to X.
2. The system ρ_Λ is called *(strongly) quasi-compatible* if for all $\lambda_1, \lambda_2 \in \Lambda$ the representations ρ_{λ_1} and ρ_{λ_2} are (strongly) quasi-compatible.

6.14 Remark. — Remark 6.10 applies mutatis mutandis to the above definition: compatibility is stable under birational equivalence, but for strong compatibility we do not know this.

6.15 Lemma. — *Let* K *be a finitely generated field. Let* E *be a number field, and let* Λ *be a set of finite places of* E*. Let* ρ_Λ *be a system of Galois representations of* K*. Let* L *be a finitely generated extension of* K*. Let* ρ'_Λ *denote the system of Galois representations of* L *obtained by restricting the system* ρ_Λ *to* L*.*

1. *The system* ρ_Λ *is quasi-compatible if and only if the system* ρ'_Λ *is quasi-compatible.*
2. *If the system* ρ'_Λ *is strongly quasi-compatible, then the system* ρ_Λ *is strongly quasi-compatible.*

Proof. Without loss of generality we may and do assume that $\Lambda = \{\lambda_1, \lambda_2\}$. Let X be a model of K. Let Y be an X-scheme that is a model of L. Let $x \in X^{cl}$ be a closed point whose residue characteristic is different from the residue characteristic of λ_1 and λ_2.

For the remainder of the proof, we may and do assume that ρ_{λ_1} and ρ_{λ_2} are both unramified at x. Then ρ'_{λ_1} and ρ'_{λ_2} are both unramified at all points $y \in Y^{cl}_x$. If $y \in Y^{cl}_x$ is a closed point, and k denotes the residue extension degree $[\kappa(y) : \kappa(x)]$, then we have $F_{y,\rho_\lambda} = F^k_{x,\rho_\lambda}$ for all $\lambda \in \Lambda$. This leads to the following conclusions: (*i*) For every point $y \in Y^{cl}_x$, if ρ'_{λ_1} and ρ'_{λ_2} are quasi-compatible at y, then ρ_{λ_1} and ρ_{λ_2} are quasi-compatible at x; and (*ii*) if ρ_{λ_1} and ρ_{λ_2} are quasi-compatible at x, then ρ'_{λ_1} and ρ'_{λ_2} are quasi-compatible at all points $y \in Y^{cl}_x$. Together, these two conclusions complete the proof. □

(Note that I cannot prove the converse implication in point 2, for the following reason. Let $y \in Y^{cl}$ be a closed point whose residue characteristic is different from the residue characteristic of λ_1 and λ_2. If ρ'_{λ_1} and ρ'_{λ_2} are unramified at y, but ρ_{λ_1} and ρ_{λ_2} are not unramified at the image x of y in X, then I do not see how to prove that ρ'_{λ_1} and ρ'_{λ_2} are quasi-compatible at y.)

6.16 — Let K be a finitely generated field. Let E be a number field, and let Λ be a set of finite places of E. Let ρ_Λ be a system of Galois representations over K. Let $E' \subset E$ be a subfield, and let Λ' be the set of places λ' of E′ satisfying the following condition:

For all places λ of E, with $\lambda | \lambda'$, we have $\lambda \in \Lambda$.

For each $\lambda' \in \Lambda'$, the representation $\rho_{\lambda'} = \bigoplus_{\lambda|\lambda'} \rho_\lambda$ is naturally a λ'-adic Galois represenation of K. We thus obtain a system of Galois representations $\rho_{\Lambda'}$.

6.17 Lemma. — *Let* K, $E' \subset E$, Λ, Λ', ρ_Λ, *and* $\rho_{\Lambda'}$ *be as in* §*6.16. If* ρ_Λ *is a (strongly) quasi-compatible* *system of Galois representations, then* $\rho_{\Lambda'}$ *is a (strongly) quasi-compatible system of Galois representations.*

Proof. To see this, we may assume that $\Lambda' = \{\lambda'_1, \lambda'_2\}$ and Λ is the set of all places λ of E that lie above a place $\lambda' \in \Lambda'$. Let X be a model of K. Let $x \in X^{cl}$ be a closed point whose residue characteristic is different from the residue characteristic of λ'_1 and λ'_2. Assume that $\rho_{\lambda'_1}$ and $\rho_{\lambda'_2}$ are both unramified at x. Suppose that for all $\lambda_1, \lambda_2 \in \Lambda$, the representations ρ_{λ_1} and ρ_{λ_2} are quasi-compatible at x. (If ρ_Λ is a strongly quasi-compatible system, then this is automatic. If ρ_Λ is merely a quasi-compatible system, then this is true for $x \in U^{cl}$, for some non-empty open subset $U \subset X$.)

There exists an integer $n \geq 1$ such that $P(t) = P_{x,\rho_\lambda,n}(t)$ does not depend on $\lambda \in \Lambda$ (since we assumed that Λ is a finite set). We may then compute

$$P_{x,\rho_{\lambda'},n}(t) = \prod_{\lambda|\lambda'} \mathrm{Nm}^{E_\lambda}_{E'_{\lambda'}} P_{x,\rho_\lambda,n}(t) = \mathrm{Nm}^{E}_{E'} P(t).$$

We conclude that $P_{x,\rho_{\lambda'},n}(t)$ is a polynomial in $E'[t]$ that does not depend on $\lambda' \in \Lambda'$. □

6.18 — A counterpart to the previous lemma is as follows. Let K be a finitely generated field. Let E be a number field, and let Λ be a set of finite places of E. Let ρ_Λ be a system of Galois representations over K. Let $E \subset \tilde{E}$ be a finite extension, and let $\tilde{\Lambda}$ be the set of finite places $\tilde{\lambda}$ of $\tilde{E}$ that lie above places $\lambda \in \Lambda$.

Let $\lambda \in \Lambda$ be a finite place of E. Write $\tilde{E}_\lambda$ for $\tilde{E} \otimes_E E_\lambda$ and recall that $\tilde{E}_\lambda = \prod_{\tilde{\lambda}|\lambda} \tilde{E}_{\tilde{\lambda}}$. Consider the representation $\tilde{\rho}_\lambda = \rho_\lambda \otimes_{E_\lambda} \tilde{E}_\lambda$, and observe that it naturally decomposes as $\tilde{\rho}_\lambda = \bigoplus_{\tilde{\lambda}|\lambda} \tilde{\rho}_{\tilde{\lambda}}$, where $\tilde{\rho}_{\tilde{\lambda}} = \rho_\lambda \otimes_{E_\lambda} \tilde{E}_{\tilde{\lambda}}$. We assemble these Galois representations $\tilde{\rho}_{\tilde{\lambda}}$ in a system of Galois representations that we denote with $\tilde{\rho}_{\tilde{\Lambda}}$ or $\rho_\Lambda \otimes_E \tilde{E}$.

6.19 Lemma. — *Let* K, $E \subset \tilde{E}$, Λ, $\tilde{\Lambda}$, ρ_Λ, *and* $\tilde{\rho}_{\tilde{\Lambda}}$ *be as in §6.18. If* ρ_Λ *is a (strongly) quasi-compatible system of Galois representations, then* $\tilde{\rho}_{\tilde{\Lambda}}$ *is a (strongly) quasi-compatible system of Galois representations.*

Proof. Let X be a model of K and let $x \in X^{cl}$ be a closed point. Let $\tilde{\lambda} \in \tilde{\Lambda}$ be a place that lies above $\lambda \in \Lambda$, and let $n \geq 1$ be an integer. Then $P_{x,\rho_\lambda,n} = P_{x,\tilde{\rho}_{\tilde{\lambda}},n}$. □

6.20 — Let K be a finitely generated field. Let E be a number field, and let Λ be the set of all finite places of E. Let ρ_Λ be a quasi-compatible system of semisimple Galois representations of K. In proposition 8.3 we show that under a suitable condition the data of *one* λ-adic representation ρ_λ, with $\lambda \in \Lambda$, is sufficient to recover the number field E as subfield of E_λ.

We conclude this section by showing that the usual representation-theoretic constructions may be applied to quasi-compatible systems of Galois representations. Let μ be a partition of some integer m. With $\mathbb{S}^\mu$ we mean the Schur functor indexed by μ. Recall that if $\mu = (1, 1, \ldots, 1)$, then $\mathbb{S}^\mu = \bigwedge^m$.

6.21 LEMMA. — *Let* K *be a finitely generated field. Let* E *be a number field; and let* Λ *be a set of finite places of* E. *Let* ρ_Λ *and* ρ'_Λ *be two systems of Galois representations over* K. *Then one may naturally form the following systems of Galois representations:*

(a) *the dual:* $\check{\rho}_\Lambda = (\mathrm{E}, \Lambda, (\check{\rho})_{\lambda\in\Lambda})$*;*
(b) *the direct sum:* $\rho_\Lambda \oplus \rho'_\Lambda = (\mathrm{E}, \Lambda, (\rho_\lambda \oplus \rho'_\lambda)_{\lambda\in\Lambda})$*;*
(c) *the tensor product:* $\rho_\Lambda \otimes \rho'_\Lambda = (\mathrm{E}, \Lambda, (\rho_\lambda \otimes \rho'_\lambda)_{\lambda\in\Lambda})$*;*
(d) *the Schur constructions for some partition* μ, $(\mathbb{S}^\mu \rho_\Lambda) = (\mathrm{E}, \Lambda, (\mathbb{S}^\mu \rho)_{\lambda\in\Lambda})$*;*
(e) *the internal Hom:* $\underline{\mathrm{Hom}}(\rho_\Lambda, \rho'_\Lambda) = (\mathrm{E}, \Lambda, (\underline{\mathrm{Hom}}(\rho_\lambda, \rho'_\lambda))_{\lambda\in\Lambda})$.

If ρ_Λ *and* ρ'_Λ *are systems of Galois representations over* K *that are both quasi-compatible, then the constructions (a) through (e) form a quasi-compatible system of Galois representations.*

If ρ_Λ *and* $\rho_\Lambda \oplus \rho'_\Lambda$ *are quasi-compatible systems of Galois representations then* ρ'_Λ *is a quasi-compatible system of Galois representations.*

Proof. This lemma is an immediate consequence of the following lemma. □

6.22 LEMMA. — *Let* V *and* V′ *be finite-dimensional vector spaces over a field* K. *Let* g *and* g' *be endomorphisms of* V *and* V′ *respectively.*

1. *The coefficients of the characteristic polynomial* $\mathrm{c.p.}(g \oplus g' | \mathrm{V} \oplus \mathrm{V}')$ *are integral polynomial expressions in the coefficients of* $\mathrm{c.p.}(g|\mathrm{V})$ *and* $\mathrm{c.p.}(g'|\mathrm{V}')$.
2. *Let* μ *be some partition of an integer. Then the coefficients of* $\mathrm{c.p.}(\mathbb{S}^\mu g | \mathbb{S}^\mu \mathrm{V})$ *are integral polynomial expressions in the coefficients of* $\mathrm{c.p.}(g|\mathrm{V})$.
3. *The coefficients of* $\mathrm{c.p.}(g \otimes g' | \mathrm{V} \otimes \mathrm{V}')$ *are integral polynomial expressions in the coefficients of* $\mathrm{c.p.}(g|\mathrm{V})$ *and* $\mathrm{c.p.}(g'|\mathrm{V}')$.

Proof. 1. Note that $\mathrm{c.p.}(g \oplus g' | \mathrm{V} \oplus \mathrm{V}) = \mathrm{c.p.}(g|\mathrm{V}) \cdot \mathrm{c.p.}(g'|\mathrm{V}')$.

For the other to cases, we make some reductions. If suffices to prove the assertions in the case that $\mathrm{K} = \bar{\mathrm{K}}$. Since the diagonalisable endomorphisms are Zariski dense, we may also assume that g and g' are diagonalisable (*i.e.* their eigenspaces span V and V′).

2. Let $\alpha = (\alpha_i)_i$ be the eigenvalues of g; and write

$$\mathrm{c.p.}(g|\mathrm{V}) = \prod_i (\mathrm{X} - \alpha_i) = \sum_j a_j \mathrm{X}^j, \qquad \text{and} \qquad \mathrm{c.p.}(\mathbb{S}^\mu g | \mathbb{S}^\mu \mathrm{V}) = \sum_k b_k \mathrm{X}^k.$$

Let n be the dimension of $\mathbb{S}^\mu \mathrm{V}$. Note that $b_{n-k} = (-1)^k \cdot \mathrm{tr}(\bigwedge^k \mathbb{S}^\mu g)$. We have to show that b_{n-k} is an integral polynomial expression in the a_j.

The composition of two Schur functors decomposes as the direct sum of other Schur constructions:

$$\mathbb{S}^\lambda \mathbb{S}^\mu \mathrm{V} \cong \bigoplus_\nu (\mathbb{S}^\nu \mathrm{V})^{\oplus \mathrm{M}_{\lambda\mu\nu}}$$

where λ (resp. μ) is a partition of l (resp. m) and ν runs over the partitions of lm. Hence

$\bigwedge^k \mathbb{S}^\mu V$ is a direct sum of other Schur constructions. Since the trace is additive it suffices to show that b_{n-1} is an integral polynomial expression in the a_j.

Now $b_{n-1} = -\mathrm{tr}(\mathbb{S}^\mu g) = -S^\mu(\alpha)$, where S^μ is the Schur polynomial associated with μ (which is a symmetric polynomial). Since a_j is the j-th elementary symmetric polynomial expression in the α_i we conclude that b_{n-1} is an integral polynomial expression in the a_j.

Incidentally, the second Jacobi–Trudi identity gives the precise relation. If $\tilde{\mu}$ denotes the partition of length l, conjugate to μ, then

$$b_{n-1} = -\det(a_{\tilde{\mu}_i+j-i})_{i,j=1}^{l} = -\det\begin{pmatrix} a_{\tilde{\mu}_1} & a_{\tilde{\mu}_1+1} & \cdots & a_{\tilde{\mu}_1+l-1} \\ a_{\tilde{\mu}_2-1} & a_{\tilde{\mu}_2} & \cdots & a_{\tilde{\mu}_2+l-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\tilde{\mu}_l-l+1} & a_{\tilde{\mu}_l-l+2} & \cdots & a_{\tilde{\mu}_l} \end{pmatrix}.$$

3. Like before, write $\mathrm{c.p.}(g \otimes g' | V \otimes V') = \sum_k b_k X^k$, and let n denote the dimension of $V \otimes V'$. Note once more that $b_{n-k} = (-1)^k \cdot \mathrm{tr}(\bigwedge^k (g \otimes g'))$.

   Suppose that W and W′ are two finite-dimensional vector spaces over K with endomorphisms h and h' respectively. We use the well-known isomorphism
   $$\bigwedge^k (W \oplus W') \cong \bigoplus_{p+q=k} \bigwedge^p W \otimes \bigwedge^q W'$$
   to deduce that $\mathrm{tr}(\bigwedge^k (h \oplus h')) = \sum_{p+q=k} \mathrm{tr}(\bigwedge^p h) \cdot \mathrm{tr}(\bigwedge^q h')$. This shows by induction to k that $\mathrm{tr}(\bigwedge^k h')$ is an integral polynomial expression in the coefficients of $\mathrm{c.p.}(h|W)$ and $\mathrm{c.p.}(h \oplus h'|W \oplus W')$.

   Now observe that
   $$\bigwedge^2 (V \oplus V') \cong \bigwedge^2 V \oplus (V \otimes V') \oplus \bigwedge^2 V'$$
   and apply the above remarks with $(W, h) = (\bigwedge^2 V \oplus \bigwedge^2 V', \bigwedge^2 g \oplus \bigwedge^2 g')$ and $(W', h') = (V \otimes V', g \otimes g')$. Note that $b_{n-k} = (-1)^k \mathrm{tr}(\bigwedge^k h')$. We conclude by induction that b_{n-k} is an integral polynomial expression in the coefficients of $\mathrm{c.p.}(h|W)$ and $\mathrm{c.p.}(h \oplus h'|W \oplus W')$. But then b_{n-k} is an integral polynomial expression in the coefficients of $\mathrm{c.p.}(g|V)$ and $\mathrm{c.p.}(g't|V')$, by part 2 of this lemma. □

7 Examples of quasi-compatible systems of Galois representations

Readme. — Important: theorem 7.2.

In this section we show that abelian varieties give rise to quasi-compatible systems of Galois representations.

7.1 — Let K be a finitely generated field. Let M be a motive over K. Let $E \subset \mathrm{End}(M)$ be a number field. Let Λ be the set of finite places of E whose residue characteristic is different from char(K). Let ℓ be a prime number that is different from char(K). Then $H_\ell(M)$ is a module over $E \otimes \mathbb{Q}_\ell \cong \prod_{\lambda|\ell} E_\lambda$. Correspondingly, the Galois representation $H_\ell(M)$ decomposes as $H_\ell(M) \cong \bigoplus_{\lambda|\ell} H_\lambda(M)$, where $H_\lambda(M)$ is an E_λ-vector space. The λ-adic representations $H_\lambda(M)$, with $\lambda \in \Lambda$, form a system of Galois representations that we denote with $H_\Lambda(M)$. It is expected that $H_\Lambda(M)$ is a quasi-compatible system of Galois representations, and even a compatible system in the sense of Serre. (Indeed, this assertion is implied by the Tate conjecture.)

The following theorem is a slightly weaker version of a result proven by Shimura in §11.10.1 of [Shi67]. We present the proof by Shimura in modern notation, and with a bit more detail. The proof is given in §7.8, and relies on proposition 7.3, which is proposition 11.9 of [Shi67]. For similar discussions, see [Chi92], §11 of [Rib76], [Noo09], and [Noo13].

7.2 Theorem (§11.10.1 of [Shi67]). — *Let* K *be a finitely generated field. Let* A *be an abelian variety over* K*; and let* $E \subset \mathrm{End}(A) \otimes \mathbb{Q}$ *be a number field. Let* Λ *be the set of finite places of* E *whose residue characteristic is different from* char(K). *Then* $H^1_\Lambda(A)$ *is a strongly quasi-compatible system of Galois representations.*

Proof. See §7.8. □

7.3 Proposition (11.9 of [Shi67]). — *Let* E *be a number field. Let* $\mathscr{L}$ *be a set of prime numbers. Let* Λ *be the set of finite places of* E *that lie above a prime number in* $\mathscr{L}$. *For every prime number* $\ell \in \mathscr{L}$, *let* H_ℓ *be a finitely generated* E_ℓ*-module. (Recall that* $E_\ell = E \otimes \mathbb{Q}_\ell \cong \prod_{\lambda|\ell} E_\lambda$.*) Decompose* H_ℓ *as* $\prod_{\lambda|\ell} H_\lambda$.

Let R *be a finite-dimensional commutative semisimple* E*-algebra; and suppose that, for every prime number* $\ell \in \mathscr{L}$, *we are given* E*-algebra homomorphisms* $R \to \mathrm{End}_{E_\ell}(H_\ell)$. *Assume that for every* $r \in R$ *the characteristic polynomial* $\mathrm{c.p.}_{\mathbb{Q}_\ell}(r|H_\ell)$ *has coefficients in* $\mathbb{Q}$ *and is independent of* $\ell \in \mathscr{L}$. *Under these assumptions, for every* $r \in R$ *the characteristic polynomial* $\mathrm{c.p.}_{E_\lambda}(r|H_\lambda)$ *has coefficients in* E *and is independent of* $\lambda \in \Lambda$.

Proof. The assumptions on R imply that R is a finite product of finite field extensions K_i/E. Let ϵ_i be the idempotent of R that is 1 on K_i and 0 elsewhere. For $r \in R$, observe that

$$\mathrm{c.p.}_{\mathbb{Q}_\ell}(r|H_\ell) = \prod_i \mathrm{c.p.}_{\mathbb{Q}_\ell}(\epsilon_i r|\epsilon_i H_\ell),$$
$$\mathrm{c.p.}_{E_\lambda}(r|H_\lambda) = \prod_i \mathrm{c.p.}_{E_\lambda}(\epsilon_i r|\epsilon_i H_\lambda).$$

We conclude that we only need to prove the lemma for $R = K_i$, and $H_\ell = \epsilon_i H_\ell$, *i.e.*, that we can reduce to the case where R is a field.

Suppose R is a finite field extension of E, and choose an element $\pi \in \mathrm{R}$ that generates R as a field. Let $f^{\pi}_{\mathbb{Q}}$ be the minimum polynomial of π over $\mathbb{Q}$. Observe that $\mathrm{c.p.}_{\mathbb{Q}_\ell}(\pi|\mathrm{H}_\ell)$ is a divisor of a power of $f^{\pi}_{\mathbb{Q}}$ in $\mathbb{Q}_\ell[t]$. Since both are elements of $\mathbb{Q}[t]$ and $f^{\pi}_{\mathbb{Q}}$ is irreducible, we conclude that $\mathrm{c.p.}_{\mathbb{Q}_\ell}(\pi|\mathrm{H}_\ell)$ is equal to $(f^{\pi}_{\mathbb{Q}})^d$, for some positive integer d. Since π is semisimple, it follows that $\mathrm{H}_\ell \cong \mathbb{Q}_\ell[\pi]^d$ as $\mathbb{Q}_\ell[\pi]$-modules. Let H be the R-vector space R^d. By construction $\mathrm{H}_\ell \cong \mathrm{H} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ as $(\mathrm{R} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$-modules. Because $\mathrm{R} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong \mathrm{R} \otimes_{\mathrm{E}} \mathrm{E} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, this implies that $\mathrm{H}_\lambda \cong \mathrm{H} \otimes_{\mathrm{E}} \mathrm{E}_\lambda$ as $(\mathrm{R} \otimes_{\mathrm{E}} \mathrm{E}_\lambda)$-modules. For all $r \in \mathrm{R}$, we have $\mathrm{c.p.}_{\mathrm{E}_\lambda}(r|\mathrm{H}_\lambda) = \mathrm{c.p.}_{\mathrm{E}}(r|\mathrm{H})$, and therefore $\mathrm{c.p.}_{\mathrm{E}_\lambda}(r|\mathrm{H}_\lambda)$ has coefficients in E and is independent of $\lambda \in \Lambda$. □

7.4 LEMMA. — *Let* κ *be a finite field of characteristic* p. *Let* A *be an abelian variety over* κ. *Let* E *be a number field inside* $\mathrm{End}(\mathrm{A}) \otimes \mathbb{Q}$. *Let* Λ *be the set of finite places of* E *whose residue characteristic is different from* p. *Then* $\mathrm{H}^1_\Lambda(\mathrm{A})$ *is a quasi-compatible system of Galois representations.*

Proof. Note that $\mathrm{Spec}(\kappa)$ is the only model of κ. Let x denote the single point of $\mathrm{Spec}(\kappa)$. Let $\mathrm{E}[\mathrm{F}_x]$ be the subalgebra of $\mathrm{End}(\mathrm{A}) \otimes \mathbb{Q}$ generated by E and $\mathrm{F}_{\bar{\kappa}/\kappa}$. Note that $\mathrm{E}[\mathrm{F}_x]$ may naturally be viewed as the subalgebra of $\mathrm{End}(\mathrm{H}^1_\ell(\mathrm{A}))$ generated by E and F_{x,ρ_ℓ}. This algebra is semisimple by work of Weil. For every $r \in \mathrm{E}[\mathrm{F}_x]$ the characteristic polynomial $\mathrm{c.p.}(r|\mathrm{H}^1_\ell(\mathrm{A}))$ has coefficients in $\mathbb{Q}$, and is independent of ℓ, by theorem 2.2 of [KM74]. It follows from proposition 7.3 that $\mathrm{P}_{x,\rho_\lambda,1}(t)$ has coefficients in E and is independent of $\lambda \in \Lambda$. □

7.5 COROLLARY (theorem II.2.1.1 of [Rib76]). — *Let* K *be a finitely generated field. Let* A *be an abelian variety over* K. *Let* E *be a number field inside* $\mathrm{End}(\mathrm{A}) \otimes \mathbb{Q}$. *Let* ℓ *be a prime number different from* char(K). *Then* $\mathrm{H}^1_\ell(\mathrm{A})$ *is a free* E_ℓ*-module.*

Proof. Let X be a model of K, and let $x \in \mathrm{X}^{\mathrm{cl}}$ be a closed point whose residue characteristic is different from ℓ and such that A has good reduction at x. Specialise to x and apply lemma 7.4. □

7.6 LEMMA. — *Let* K *be a field. Let* $\mathrm{T} \hookrightarrow \mathrm{G} \xrightarrow{\alpha} \mathrm{A}$ *be a semiabelian variety over* K. *Let* E *be a number field inside* $\mathrm{End}(\mathrm{G}) \otimes \mathbb{Q}$. *Then* E *embeds naturally into* $\mathrm{End}(\mathrm{A}) \otimes \mathbb{Q}$ *and* $\mathrm{End}(\mathrm{T}) \otimes \mathbb{Q}$.

Proof. Let f be an endomorphism of G. Note that $\alpha \colon \mathrm{G} \to \mathrm{A}$ is the Albanese variety of G. By the universal property of the Albanese variety, the morphism $\alpha \circ f$ factors via A as $f' \circ \alpha$, for a unique $f' \in \mathrm{End}(\mathrm{A})$. This gives an embedding $\mathrm{E} \hookrightarrow \mathrm{End}(\mathrm{A}) \otimes \mathbb{Q}$. As a consequence, f preserves the kernel of α, and therefore $f|_{\mathrm{T}}$ is an endomorphism of T. This gives an embedding $\mathrm{E} \hookrightarrow \mathrm{End}(\mathrm{T}) \otimes \mathbb{Q}$. □

7.7 LEMMA. — *Let* X *be the spectrum of a discrete valuation ring. Let* $\eta = \mathrm{Spec}(\mathrm{K})$ *denote the generic point of* X, *and let* x *denote the special point of* X. *Let* A *be a semistable abelian variety over* η. *Let* E *be a number field inside* $\mathrm{End}(\mathrm{A}) \otimes \mathbb{Q}$. *Let* λ *be a finite place of* E *such that the residue characteristics of* λ *and* x *are different. Then* A *has good reduction at* x *if and only if* $\mathrm{H}^1_\lambda(\mathrm{A})$ *is unramified at* x.

Proof. This is a slight generalisation of the criterion of Néron–Ogg–Shafarevic, theorem 1 of [ST68]. It is clear that if A has good reduction at x, then $H^1_\lambda(A)$ is unramified at x. We focus on the converse implication. Let ℓ be the residue characteristic of λ. By theorem 1 of [ST68] it suffices to show that $H^1_\ell(A)$ is unramified at x. Let $H^1_\ell(A)^I$ denote the subspace of $H^1_\ell(A)$ that is invariant under inertia. Let G be the Néron model of A over X. Recall that $H^1_\ell(A)^I \cong H^1_\ell(G_x)$, by lemma 2 of [ST68]. It follows from the definition of the Néron model that E embeds into $\mathrm{End}(G) \otimes \mathbb{Q}$. Hence E embeds into $\mathrm{End}(G_x) \otimes \mathbb{Q}$, and we claim that $H^1_\ell(A)^I \cong H^1_\ell(G_x)$ is a free E_ℓ-module. (With E_ℓ we mean $E \otimes \mathbb{Q}_\ell \cong \prod_{\lambda|\ell} E_\lambda$.) Before proving the claim, let us see why it is sufficient for proving the lemma. By corollary 7.5 we know that $H^1_\ell(A)$ is a free E_ℓ-module. Thus $H^1_\ell(A)/H^1_\ell(A)^I$ is a free E_ℓ-module. We conclude that $H^1_\lambda(A)$ is unramified at x, if and only if $H^1_\ell(A)$ is unramified at x.

We will now prove the claim that $H^1_\ell(A)^I \cong H^1_\ell(G_x)$ is a free E_ℓ-module. Since A is semistable, the special fibre G_x is a semiabelian variety $T \hookrightarrow G_x \to B$. The semiabelian variety G_x is a special case of a 1-motive, and thus we have a short exact sequence

$$0 \to H^1_\ell(B) \to H^1_\ell(G_x) \to H^1_\ell(T) \to 0.$$

We also have $H^1_\ell(T) \cong \mathrm{Hom}(T, \mathbb{G}_m) \otimes \mathbb{Q}_\ell(-1)$, see variante 10.1.10 of [Del74b]. By lemma 7.6, the action of E on G_x gives an action of E on both T and B. Since $\mathrm{Hom}(T, \mathbb{G}_m) \otimes \mathbb{Q}$ is a free E-module, we know that $H^1_\ell(T)$ is a free E_ℓ-module. By corollary 7.5 we also know that $H^1_\ell(B)$ is free as E_ℓ-module. Therefore, $H^1_\ell(G_x) \cong H^1_\ell(A)^I$ is free as E_ℓ-module. □

7.8 *Proof* (of theorem 7.2). — Let X be a model of K; and let $x \in X^{cl}$ be a closed point. Let $\Lambda^{(x)}$ be the set of places $\lambda \in \Lambda$ that have a residue characteristic ℓ that is different from the residue characteristic of x. If there is a $\lambda \in \Lambda^{(x)}$ such that $H^1_\lambda(A)$ is unramified at x, then A has good reduction at x, by lemma 7.7. Assume that A has good reduction at x. We denote this reduction with A_x. It follows from lemma 7.4 that $P_{x,\rho_\lambda,1}(t)$ has coefficients in E and is independent of $\lambda \in \Lambda^{(x)}$. □

7.9 Remark. — Let K be a finitely generated field of characteristic 0. Let X be a K3 surface over K, and write M for $H^2(X)^{tra}$. Assume that $G_\ell(M)$ is connected for all prime numbers ℓ; and assume that $E = \mathrm{End}(M)$ is a CM field. Recently, Buskin showed that E consists of cycle classes of algebraic correspondences on $X \times X$ (see theorem 1.1 of [Bus15] and theorem 5.4 of [Ram08]). An imitation of the proof of theorem 7.2 also shows that $H_\Lambda(M)$ is a quasi-compatible system.

8 Isomorphisms of quasi-compatible systems of Galois representations

Readme. — Important: theorem 8.2; proposition 8.3.

The sole purpose of this section is to prove theorem 8.2 and proposition 8.3. Both results are consequences of proposition 8.1, and the rest of this section is devoted to its proof.

Proposition 8.3 is a slightly technical result to state, but it is extremely useful. In a broad sketch, the number field E occuring in the definition of a quasi-compatible system of Galois representations is usually the endomorphism algebra of a motive M. Proposition 8.3 will allow us to recover E (motivic information!) from one λ-adic realisation of M.

8.1 Proposition. — *Let* K *be a finitely generated field. Let* E *be a number field; and let* λ *be a finite place of* E*. For* $i = 1,2$*, let* ρ_i *be a* λ*-adic Galois representation of* K*. If* ρ_1 *and* ρ_2 *are semisimple, quasi-compatible, and* $G_\lambda(\rho_1 \oplus \rho_2)$ *is connected, then* $\rho_1 \cong \rho_2$*.*

8.2 Theorem. — *Let* K *be a finitely generated field. Let* E *be a number field; and let* Λ *be a set of finite places of* E*. Let* ρ_Λ *and* ρ'_Λ *be two quasi-compatible systems of Galois representations. Assume that* $G_\lambda(\rho_\lambda \oplus \rho'_\lambda)$ *is connected for all* $\lambda \in \Lambda$*. If there is a* $\lambda \in \Lambda$ *such that* $\rho_\lambda \cong \rho'_\lambda$*, then* $\rho_\Lambda \cong \rho'_\Lambda$*.*

Proof. This is an immediate consequence of proposition 8.1. □

8.3 Proposition. — *Let* K *be a finitely generated field. Let* E *be a number field, and let* Λ *be the set of finite places of* E *whose residue characteristic is different from* char(K)*. Let* $\mathscr{L}$ *be the set of prime numbers different from* char(K)*. Let* ρ_Λ *be a quasi-compatible system of semisimple Galois representations of* K*. Let* $\rho_{\mathscr{L}}$ *be the quasi-compatible system of Galois representations obtained by restricting to* $\mathbb{Q} \subset E$*, as in §6.16; in other words,* $\rho_\ell = \bigoplus_{\lambda|\ell} \rho_\lambda$*. Assume that* $G_\ell(\rho_\ell)$ *is connected for all* $\ell \in \mathscr{L}$*. Fix* $\lambda_0 \in \Lambda$*. Define the field* $E' \subset E$ *to be the subfield of* E *generated by elements* $e \in E$ *that satisfy the following condition:*

There exists a model X *of* K*, a point* $x \in X^{cl}$*, and an integer* $n \geq 1$*,*

such that $P_{x,\rho_{\lambda_0},n}(t) \in E[t]$ *and* e *is a coefficient of* $P_{x,\rho_{\lambda_0},n}(t)$*.*

Let ℓ *be a prime number that splits completely in* $E/\mathbb{Q}$*. If* $\mathrm{End}_{\mathrm{Gal}(\bar{K}/K),\mathbb{Q}_\ell}(\rho_\ell) \cong E \otimes \mathbb{Q}_\ell$*, then* $E = E'$*.*

Proof. We restrict our attention to a finite subset of Λ, namely $\Lambda_0 = \{\lambda_0\} \cup \{\lambda|\ell\}$. Let $U \subset X$ be an open subset such that for all $\lambda_1, \lambda_2 \in \Lambda$ the representations ρ_{λ_1} and ρ_{λ_2} are quasi-compatible at all $x \in U^{cl}$. For each $x \in U^{cl}$, let n_x be an integer such that $P_x(t) = P_{x,\rho_\lambda,n_x}(t) \in E[t]$ does not depend on $\lambda \in \Lambda_0$.

Let λ' be a place of E' above ℓ. Let λ_1 and λ_2 be two places of E that lie above λ'. We view ρ_{λ_1} and ρ_{λ_2} as λ'-adic representations. Since ℓ splits completely in $E/\mathbb{Q}$, the inclusions $\mathbb{Q}_\ell \subset E'_{\lambda'} \subset E_{\lambda_i}$ are isomorphisms. By definition of E' we have $P_x(t) \in E'[t]$. Therefore ρ_{λ_1} and ρ_{λ_2} are quasi-compatible λ'-adic representations; hence they are isomorphic by proposition 8.1. Let $\rho_{\lambda'}$ be the λ'-adic Galois representation $\bigoplus_{\lambda|\lambda'} \rho_\lambda$, as in §6.16. We conclude that $\mathrm{End}_{\mathrm{Gal}(\bar{K}/K),E'_{\lambda'}}(\rho_{\lambda'}) \cong \mathrm{Mat}_{[E:E']}(E'_{\lambda'})$, which implies $[E:E'] = 1$. □

8.4 — Let K be a finitely generated field, and let X be a model of K. There is a good notion of density for subsets of X^{cl}. This is described by Serre in [Ser65] and [Ser12], and by Pink in appendix B of [Pin97]. For the convenience of the reader, we list some features of these densities. Most of the following list is a reproduction of the statement of proposition B.7 of [Pin97]. Let $T \subset X^{cl}$ be a subset. If T has a density, we denote it with $\mu_X(T)$.

1. If $T \subset X^{cl}$ has a density, then $0 \leq \mu_X(T) \leq 1$.
2. The set X^{cl} has density 1.
3. If T is contained in a proper closed subset of X, then T has density 0.
4. If $T_1 \subset T \subset T_2 \subset X^{cl}$ such that $\mu_X(T_1)$ and $\mu_X(T_2)$ exist and are equal, then $\mu_X(T)$ exists and is equal to $\mu_X(T_1) = \mu_X(T_2)$.
5. If $T_1, T_2 \subset X^{cl}$ are two subsets, and three of the following densities exist, then so does the fourth, and we have
$$\mu_X(T_1 \cup T_2) + \mu_X(T_1 \cap T_2) = \mu_X(T_1) + \mu_X(T_2).$$
6. If $u : X \to X'$ is a birational morphism, then T has a density if and only if $u(T)$ has a density, and if this is the case, then $\mu_X(T) = \mu_{X'}(u(T))$.

8.5 — Chebotarev's density theorem also generalises to this setting. Let $Y \to X$ be a finite étale Galois covering of integral schemes of finite type over $\mathrm{Spec}(\mathbb{Z})$. Denote the Galois group with G. For each point $y \in Y^{cl}$ with image $x \in X^{cl}$ the inverse of the Frobenius endomorphism of $\kappa(y)/\kappa(x)$ determines an element $F_y \in G$. The conjugacy class of F_y only depends on x, and we denote it with $\mathscr{F}_x$.

8.6 Theorem. — *Let* $Y \to X$ *be a finite étale Galois covering of integral schemes of finite type over* $\mathrm{Spec}(\mathbb{Z})$ *with group* G. *For every conjugacy class* $C \subset G$, *the set* $\{x \in X^{cl} \mid \mathscr{F}_x = C\}$ *has density* $\frac{\#C}{\#G}$.
Proof. See proposition B.9 of [Pin97]. □

8.7 Definition (see also §3 of [Chi92]). — Let K be a finitely generated field, let X be a model of K, and let $x \in X^{cl}$ be a closed point. Let E be a number field, and let λ be a finite place of E. Let ρ be a semisimple λ-adic Galois representation of K. Assume that ρ is unramified at x. The algebraic subgroup $H_n \subset G_\lambda(\rho)$ generated by $F^n_{x,\rho}$ is well-defined up to conjugation. Note that H_n is a finite-index subgroup of H_1, and therefore the identity component of H_n does not depend on n. We denote this identity component with $T_x(\rho)$, and we call it the *Frobenius torus* at x. (The algebraic group $T_x(\rho)$ is indeed an algebraic torus, which means that $T_x(\rho)_{\bar{E}_\lambda} \cong \mathbb{G}_m^k$, for some $k \geq 0$.)

8.8 THEOREM. — *Let* K *be a finitely generated field. Let* E *be a number field, and let* λ *be a finite place of* E. *Let* ρ *be a semisimple* λ*-adic Galois representation of* K. *Assume that* $G_\lambda(\rho)$ *is connected. There is a non-empty Zariski open subset* $U \subset G_\lambda(\rho)$ *such that for every model* X *of* K, *and every closed point* $x \in X^{cl}$, *if* ρ *is unramified at* x, *and for some* $n \geq 1$ *the Frobenius element* $F_{x,\rho}^n$ *is conjugate to an element of* $U(E_\lambda)$, *then* $T_x(\rho)$ *is a maximal torus of* $G_\lambda(\rho)$.

Proof. See theorem 3.7 of [Chi92]. The statement in [Chi92] is for abelian varieties, but the proof is completely general. □

8.9 COROLLARY (3.8 of [Chi92]). — *Let* K *be a finitely generated field. Let* E *be a number field, and let* λ *be a finite place of* E. *Let* ρ *be a semisimple* λ*-adic Galois representation of* K. *Assume that* $G_\lambda(\rho)$ *is connected. Let* X *be a model of* K. *Let* $\Sigma \subset X^{cl}$ *be the set of points* $x \in X^{cl}$ *for which* ρ *is unramified at* x *and the Frobenius torus* $T_x(\rho)$ *is a maximal torus of* $G_\lambda(\rho)$. *Then* Σ *has density* 1.

8.10 LEMMA. — *Let* K *be a finitely generated field. Let* E *be a number field; and let* λ *be a finite place of* E. *For* $i = 1, 2$, *let* ρ_i *be a semisimple* λ*-adic Galois representation of* K. *Write* ρ *for* $\rho_1 \oplus \rho_2$. *Assume that* $G_\lambda(\rho)$ *is connected. If there is a model* X *of* K, *and a point* $x \in X^{cl}$ *such that* ρ *is unramified at* x, *and* $T_x(\rho)$ *is a maximal torus, and* $P_{x,\rho_1,n}(t) = P_{x,\rho_2,n}(t)$ *for some* $n \geq 1$, *then* $\rho_1 \cong \rho_2$ *as* λ*-adic Galois representations.*

Proof. Write T for $T_x(\rho)$. Observe that $P_{x,\rho_1,kn}(t) = P_{x,\rho_2,kn}(t)$ for all $k \geq 1$. Let H_n be the algebraic subgroup of $G_\lambda(\rho)$ that is generated by $F_{x,\rho}^n$. Recall that T is the identity component of H_n. Note that for some $k \geq 1$, we have $F_{x,\rho}^{kn} \in T(E_\lambda)$. Replace n by kn, so that we may assume that $F_{x,\rho}^n$ generates T as algebraic group.

The set $\{F_{x,\rho}^{kn} \mid k \geq 1\}$ is a Zariski dense subset of T. Since $P_{x,\rho_1,kn}(t) = P_{x,\rho_2,kn}(t)$, for all $k \geq 1$, lemma 1.7 implies that $\rho_1|_T \cong \rho_2|_T$. Because T is a maximal torus of $G_\lambda(\rho)$ and $G_\lambda(\rho)$ is connected, we find that $\rho_1 \cong \rho_2$ as representations of $G_\lambda(\rho)$, and hence as λ-adic Galois representations of K. □

8.11 *Proof* (of proposition 8.1). — Let X be a model of K. By corollary 8.9, the subset of points $x \in X^{cl}$ for which $T_x(\rho_1 \oplus \rho_2)$ is a maximal torus is a subset with density 1. By definition of compatibility, the subset of points $x \in X^{cl}$ at which ρ_1 and ρ_2 are quasi-compatible is also a subset with density 1. By §8.4.5 these subsets have non-empty intersection: there exists a point $x \in X^{cl}$ such that $T_x(\rho_1 \oplus \rho_2)$ is a maximal torus and ρ_1 and ρ_2 are quasi-compatible at x. Now proposition 8.1 follows from lemma 8.10. □

9 Abelian CM motives

Readme. — Important: theorem 9.4.

9.1 — A Hodge structure V is called a *CM Hodge structure* if the Mumford–Tate group $G_B(V)$ is commutative. Every CM Hodge structure is a direct sum of irreducible CM Hodge structures. Let V be an irreducible CM Hodge structure. There are two options: either $V \cong \mathbb{Q}(n)$ for some integer n; or $E = End(V)$ is a CM field and $\dim_E(V) = 1$.

9.2 Definition. — A motive M over a field K of characteristic 0 is called a *CM motive* if there is a field extension L/K such that $G_{mot,\omega}(M_L)$ is commutative for some (and hence every) fibre functor ω on Mot_L.

9.3 — Let M be CM motive over a field K of characteristic 0. Let $\sigma : K \hookrightarrow \mathbb{C}$ be a complex embedding. Since $G_\sigma(M) \subset G_{mot,\sigma}(M)$ we see that $H_\sigma(M)$ is a CM Hodge structure. If M is an abelian motive and $G_{mot,\sigma}(M)$ is connected, then theorem 5.2.1 shows that $End(M) = End(H_\sigma(M))$.

9.4 Theorem (see also corollary I.6.5.7 of [Sch88]). — *Let* M *be an abelian CM motive over a finitely generated field* K *of characteristic* 0. *Let* E *be a subfield of* End(M), *and let* Λ *be the set of finite places of* E. *Then the system* $H_\Lambda(M)$ *is a strongly quasi-compatible system of Galois representations.*
Proof. See §9.9. □

9.5 — Let E be a CM field (*cf.* our conventions in §0.14). Let $\Sigma(E)$ be the set of complex embeddings of E. The complex conjugation on E induces an involution $\sigma \mapsto \sigma^\dagger$ on $\Sigma(E)$. If T is a subset of $\Sigma(E)$, then we denote with $T^\dagger$ the image of T under this involution. Recall that a CM type $\Phi \subset \Sigma(E)$ is a subset such that $\Phi \cup \Phi^\dagger = \Sigma(E)$ and $\Phi \cap \Phi^\dagger = \varnothing$. Each CM type Φ defines a Hodge structure E_Φ on E of type $\{(0,1),(1,0)\}$, via

$$E_\Phi \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^{\Sigma(E)}, \qquad E_\Phi^{0,1} \cong \mathbb{C}^{\Phi^\dagger}, \qquad E_\Phi^{1,0} \cong \mathbb{C}^{\Phi}.$$

9.6 Half-twists. — The idea of half-twists originates from [Gee01], though we use the description in §7 of [Moo16]. Let V be a Hodge structure of weight n. The level of V, denoted m, is by definition $\max\{p-q \mid V^{p,q} \neq 0\}$. Suppose that End(V) contains a CM field E. Let $\Sigma(E)$ denote the set of complex embeddings $E \hookrightarrow \mathbb{C}$. Let $T \subset \Sigma(E)$ be the embeddings through which E acts on $\bigoplus_{p \geq \lfloor n/2 \rfloor} V^{p,q}$. Assume that $T \cap T^\dagger = \varnothing$. (Note that if $\dim_E(V) = 1$, then the condition $T \cap T^\dagger = \varnothing$ is certainly satisfied.)

Let $\Phi \subset \Sigma(E)$ be a CM type, and let E_Φ be the associated Hodge structure on E. If $T \cap \Phi = \varnothing$ and $m \geq 1$, then the Hodge structure $W = E_\Phi \otimes_E V$ has weight $n+1$ and level $m-1$. In that case we call W a *half-twist* of V. Note that under our assumption $T \cap T^\dagger = \varnothing$ we can certainly find a CM type with $T \cap \Phi = \varnothing$, so that there exist half-twists of V. For each CM type Φ with $T \cap \Phi = \varnothing$, there is an abelian variety A_Φ (well-defined up to isogeny), with $H^1_B(A_\Phi) \cong E_\Phi$. By construction we have $E \subset \mathrm{End}(H^1_B(A_\Phi))$ and $E \subset \mathrm{End}(W)$. Note that $V \cong \underline{\mathrm{Hom}}_E(H^1_B(A_\Phi), W)$. In the next paragraph we will see that this construction generalises to abelian motives.

9.7 — Let K be a finitely generated field of characteristic 0. Let M be an abelian motive over K. Assume that $G_\ell(M)$ is connected for all prime numbers ℓ. Assume that M is pure of weight n, and assume that End(M) contains a CM field E. Fix an embedding $\sigma\colon K \hookrightarrow \mathbb{C}$. Note that $H_\sigma(M)$ is a Hodge structure of weight n. Let $T \subset \Sigma(E)$ be the set of embeddings through which E acts on $\bigoplus_{p \geq \lfloor n/2 \rfloor} H_\sigma(M)^{p,q}$. Assume that $T \cap T^\dagger = \varnothing$.

Then there exists a finitely generated extension L/K, an abelian variety A over L and a motive N over L, such that $E \subset \mathrm{End}(H^1(A))$, and $E \subset \mathrm{End}(N)$, and such that $M_L \cong \underline{\mathrm{Hom}}_E(H^1(A), N)$. Indeed, choose a CM type $\Phi \subset \Sigma(E)$ such that $T \cap \Phi = \varnothing$. Over the complex numbers, put $N = H^1(A_\Phi) \otimes_E M$. Then $M_\sigma \cong \underline{\mathrm{Hom}}_E(H^1(A_\Phi), N)$, by theorem 5.2.1 and the construction above. The abelian variety A_Φ, the motive N, and the isomorphism $M_\sigma \cong \underline{\mathrm{Hom}}_E(H^1(A_\Phi), N)$ are defined over some finitely generated extension L of K, which proves the claim.

9.8 PROPOSITION. — *Let* K *be a finitely generated field of characteristic* 0. *Let* M *be an abelian motive of weight n over* K. *Assume that* $G_\ell(M)$ *is connected for all prime numbers* ℓ. *Assume that* End(M) *contains a* CM *field* E *such that* $\dim_E(M) = 1$. *Let* Λ *be the set of finite places of* E. *Then the system* $H_\Lambda(M)$ *is a strongly quasi-compatible system of Galois representations.*

Proof. Let m be the level of M, that is $\max\{p - q \mid H_\sigma(M)^{p,q} \neq 0\}$ for some (and hence every) complex embedding $\sigma\colon K \hookrightarrow \mathbb{C}$. We apply induction to m, and use half-twists as described above. If $m = 0$ then there is nothing to be done. Suppose that $m \geq 1$.

Let $\sigma\colon K \hookrightarrow \mathbb{C}$ be a complex embedding. Let $T \subset \Sigma(E)$ be the set of embeddings through which E acts on $\bigoplus_{p \geq \lfloor n/2 \rfloor} H_\sigma(M)^{p,q}$. Since $\dim_E(M) = 1$ we know that $T \cap T^\dagger = \varnothing$. Therefore there exists a finitely generated extension L/K, an abelian variety A over L, and a motive N over L such that $M_L \cong \underline{\mathrm{Hom}}_E(H^1(A), N)$. It follows from the discussion in §9.6 and §9.7 that the level of N is $m-1$, and $\dim_E(N) = 1$. By theorem 7.2 we know that $H^1_\Lambda(A)$ is a strongly quasi-compatible system, and by induction we may assume that $H^1_\Lambda(N)$ is a strongly quasi-compatible system. It follows from lemma 6.21 that $H_\Lambda(M_L) \cong \underline{\mathrm{Hom}}_E(H^1_\Lambda(A), H^1_\Lambda(N))$ is a quasi-compatible system of Galois representations over L, and we will now argue that it is even a strongly quasi-compatible system.

Let X be a model of K, and let $x \in X$ be a closed point. Let $\Lambda^{(x)}$ be the set of finite places of E whose residue characteristic is different from the residue characteristic of x. Fix $\lambda \in \Lambda^{(x)}$. We may assume that A is semistable over L (possibly replacing L with a finite field extension). Since A is a semistable CM abelian variety, we know that A has good reduction everywhere, and thus $H^1_\lambda(A)$ is unramified at x. Hence $H_\lambda(M_L)$ is unramified at x if and only if $H^1_\lambda(N)$ is unramified at x. Finally, lemma 6.15 shows that $H_\Lambda(M)$ is also a strongly quasi-compatible system of Galois representations over K. □

9.9 *Proof* (of theorem 9.4). — By lemma 6.15 we may replace K by a finitely generated extension and thus we may and do assume that $G_\ell(M)$ is connected for all prime numbers ℓ. Let $M = M_1 \oplus \ldots \oplus M_r$ be the decomposition of M into isotypical components. Observe that $E \subset \mathrm{End}(M_i)$ for $i = 1, \ldots, r$. By lemma 6.21 we see that it suffices to show that $H_\Lambda(M_i)$ is a strongly quasi-compatible system for $i = 1, \ldots, r$. Thus we may assume that $M \cong (M')^{\oplus k}$, where M' is an irreducible CM-motive. (Since we assumed that $G_\ell(M)$ is connected, we know that $M'_{\bar{K}}$ is also irreducible.) Hence $E' = \mathrm{End}(M')$ is a CM field, and $\dim_{E'}(M') = 1$. By assumption E acts on $(M')^{\oplus k}$, and thus we get a specific embedding $E \subset \mathrm{Mat}_k(E')$. We may find a field $\tilde{E} \subset \mathrm{Mat}_k(E')$ that contains the field E, and such that $[\tilde{E} : E'] = k$. Then $M = M' \otimes_{E'} \tilde{E}$. Let $\tilde{\Lambda}$ be the set of finite places of $\tilde{E}$. By proposition 9.8, the system $H_{\Lambda'}(M')$ is a strongly quasi-compatible system of Galois representations, and by lemma 6.19 we find that $H_{\tilde{\Lambda}}(M) = H_{\Lambda'}(M') \otimes_{E'} \tilde{E}$ is a strongly quasi-compatible system. We conclude that $H_\Lambda(M)$ is a strongly quasi-compatible system of Galois representations by lemma 6.17. □

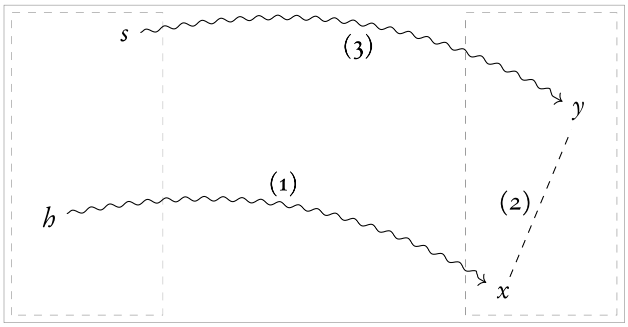
10 Deformations of abelian motives

Readme. — Important: theorem 10.1.

The goal of this section is to prove that the λ-adic realisations of an abelian motive form a quasi-compatible system of Galois representations. The proof relies heavily on the fact that every abelian motive naturally fits into a family of abelian motives over a Shimura variety of Hodge type.

10.1 Theorem. — *Let* M *be an abelian motive over a finitely generated field* K *of characteristic* 0*. Assume that* $G_\ell(M)$ *is connected for all prime numbers* ℓ*. Let* E *be a subfield of* End(M)*, and let* Λ *be the set of finite places of* E*. Then the system* $H_\Lambda(M)$ *is a quasi-compatible system of Galois representations.*

10.2 — As mentioned above, the proof of this theorem uses the fact that an abelian motive can be placed as fibre in a family of abelian motives over a Shimura variety of Hodge type. Lemma 10.4 summarises this result. Its proof uses the rather technical construction 10.3. Once we have the family of motives in place, we prove the main theorem of this section. The following picture aims to capture the intuition of the proof.



The picture is a cartoon of an integral model of a Shimura variety, and the motive M fits into a family $\mathscr{M}$ over the generic fibre, such that $M \cong \mathscr{M}_h$. We give a rough sketch of the strategy for the proof that explains the three steps in the picture: (1) We have a system of Galois representations $H_\Lambda(\mathscr{M}_h)$ and we want to show that it is quasi-compatible at x; (2) we replace x by an isogenous point y (in the sense of Kisin [Kis17]); and (3) we may assume that y lifts to a special point s. The upshot is that we have to show that the system $H_\Lambda(\mathscr{M}_s)$ is quasi-compatible at y. We will see that this follows from theorem 9.4.

10.3 Construction. — Fix an integer $g \in \mathbb{Z}_{\geq 0}$. Let $(G, X) \hookrightarrow (GSp_{2g}, \mathfrak{H}^{\pm})$ be a morphism of Shimura data, and let $h \in X$ be a morphism $\mathbb{S} \to G_{\mathbb{R}}$. In this paragraph we will construct an abelian scheme over an integral model of the Shimura variety $Sh_{\mathscr{K}}(G, X)$, where $\mathscr{K}$ is a certain compact open subgroup of $G(\mathbb{A}_f)$. Along the way, we make two choices, labeled (*i*) and (*ii*) so that we may refer to them later on.

For each integer $n \geq 3$, let $\mathscr{K}'_n$ denote the congruence subgroup of $GSp_{2g}(\hat{\mathbb{Z}})$ consisting of elements congruent to 1 modulo n. Write $\mathscr{K}_n$ for $\mathscr{K}'_n \cap G(\mathbb{A}_f)$. This gives a morphism of Shimura varieties

$$Sh_{\mathscr{K}_n}(G, X) \to Sh_{\mathscr{K}'_n}(GSp_{2g}, \mathfrak{H}^{\pm}).$$

By applying lemma 3.3 of [Noo96] with $p = 6$ we can choose n in such a way that it is coprime with p and such that this morphism of Shimura varieties is a closed immersion. (In [Noo96], Noot assumes that p is prime, but he does not use this fact in his proof.)

(*i*) Fix such an integer n, and write $\mathscr{K}$ for $\mathscr{K}_n$. Since $n > 3$, the subgroup $\mathscr{K}'_n$ is neat, hence $\mathscr{K}$ is neat, and therefore $Sh_{\mathscr{K}}(G, X)$ is smooth. As is common, we denote with $\mathscr{A}_{g,1,n}/\mathbb{Z}[1/n]$ the moduli space of principally polarised abelian varieties of dimension g with a level-n structure. Recall that $\mathscr{A}_{g,1,n}$ is smooth over $\mathbb{Z}[1/n]$.

We have a closed immersion of Shimura varieties

$$\mathrm{Sh}_{\mathscr{K}}(\mathrm{G},\mathrm{X}) \hookrightarrow \mathscr{A}_{g,1,n,\mathbb{C}}.$$

Let $\mathrm{F}' \subset \mathbb{C}$ be the reflex field of (G,X). Let $\mathscr{S}_{\mathscr{K}}(\mathrm{G},\mathrm{X})$ be the Zariski closure of $\mathrm{Sh}_{\mathscr{K}}(\mathrm{G},\mathrm{X})$ in $\mathscr{A}_{g,1,n}$ over $\mathscr{O}_{\mathrm{F}'}[1/n]$. There exists an integer multiple N_0 of n such that $\mathscr{S}_{\mathscr{K}}(\mathrm{G},\mathrm{X})_{\mathscr{O}_{\mathrm{F}'}[1/\mathrm{N}_0]}$ is smooth.

For a prime number p, let $\mathscr{K}_p$ be $\mathscr{K} \cap \mathrm{G}(\mathbb{Q}_p)$, and let $\mathscr{K}^p$ be $\mathscr{K} \cap \mathrm{G}(\mathbb{A}_{\mathrm{f}}^p)$. The set of prime numbers for which $\mathscr{K} \neq \mathscr{K}_p\mathscr{K}^p$ is finite. Write N_1 for the product of those prime numbers. Let p be a prime number that does not divide N_1. The group $\mathscr{K}_p$ is called *hyperspecial* if there is a reductive model $\mathscr{G}/\mathbb{Z}_p$ of $\mathrm{G}/\mathbb{Q}$ such that $\mathscr{K}_p = \mathscr{G}(\mathbb{Z}_p)$. The set of prime numbers for which $\mathscr{K}_p$ is not hyperspecial is finite. Write N_2 for the product of those prime numbers. Let N be the integer $\mathrm{N}_0 \cdot \mathrm{N}_1 \cdot \mathrm{N}_2$.

The point $h \in \mathrm{X}$ is a complex point of $\mathscr{S}_{\mathscr{K}}(\mathrm{G},\mathrm{X})$. After replacing F' by a finite extension $\mathrm{F} \subset \mathbb{C}$ we may assume that the generic fibre of the irreducible component $\mathscr{S} \subset \mathscr{S}_{\mathscr{K}}(\mathrm{G},\mathrm{X})_{\mathscr{O}_{\mathrm{F}}[1/\mathrm{N}]}$ that contains the point h is geometrically irreducible.

(*ii*) Choose such a field $\mathrm{F} \subset \mathbb{C}$. In the following paragraphs we will consider the closed immersion of Shimura varieties $\mathscr{S} \hookrightarrow \mathscr{A}_{g,1,n}$ as a morphism of schemes over $\mathscr{O}_{\mathrm{F}}[1/\mathrm{N}]$.

10.4 Lemma. — *Let* M *be an abelian motive over a finitely generated field* K *characteristic* 0*. Fix a complex embedding* $\sigma\colon \mathrm{K} \hookrightarrow \mathbb{C}$*. There exist*

- » *fields* $\mathrm{F} \subset \mathrm{L} \subset \mathbb{C}$, *with* F *a number field,* L *finitely generated, and* $\sigma(\mathrm{K}) \subset \mathrm{L}$*;*
- » *a smooth irreducible component* $\mathscr{S}$ *of an integral model of a Shimura variety over* F, *such that the generic fibre* $\mathscr{S}_{\mathrm{F}}$ *is geometrically irreducible;*
- » *an abelian scheme* $f\colon \mathscr{A} \to \mathscr{S}$*;*
- » *an idempotent motivated cycle* γ *in* $\mathrm{End}((\mathbb{T}^{a,b}\mathrm{R}^1 f_{\mathbb{C},*}\mathbb{Q})(m))$, *for certain integers* a, b, *and* m*;*
- » *a family of abelian motives* $\mathscr{M}/\mathscr{S}_{\mathrm{L}}$, *such that* $\mathscr{M}/\mathscr{S}_{\mathbb{C}} \cong \mathrm{Im}(\gamma)$*;*
- » *an isomorphism* $\mathrm{M}_{\mathrm{L}} \cong \mathscr{M}_h$, *for some point* $h \in \mathscr{S}(\mathrm{L})$.

Proof. Since M is an abelian motive, there exists a principally polarised complex abelian variety A such that $\mathrm{M}_\sigma \in \langle \mathrm{A} \rangle^{\otimes}$. Write V for $\mathrm{H}_\sigma(\mathrm{M})$. Observe that $\mathrm{G}_{\mathrm{B}}(\mathrm{V})$ is naturally a quotient of $\mathrm{G}_{\mathrm{B}}(\mathrm{A})$. Write G for $\mathrm{G}_{\mathrm{B}}(\mathrm{A})$, and let $h\colon \mathbb{S} \to \mathrm{G}_{\mathbb{R}}$ be the map that defines the Hodge structure on $\mathrm{H}_{\mathrm{B}}(\mathrm{A})$. Let X be the $\mathrm{G}(\mathbb{R})$-orbit of h in $\mathrm{Hom}(\mathbb{S},\mathrm{G}_{\mathbb{R}})$. Let g be $\dim(\mathrm{A})$. The pair (G,X) is a Shimura datum, and by construction we get a morphism of Shimura data $(\mathrm{G},\mathrm{X}) \hookrightarrow (\mathrm{GSp}_{2g},\mathfrak{H}^{\pm})$. Now run construction 10.3, choosing (*i*) an integer n; (*ii*) a number field $\mathrm{F} \subset \mathbb{C}$; and producing a closed immersion of Shimura varieties $\mathscr{S} \hookrightarrow \mathscr{A}_{g,1,n}$ over $\mathscr{O}_{\mathrm{F}}[1/\mathrm{N}]$.

It follows from construction 10.3, that the Hodge structure V gives rise to a variation of Hodge structure $\mathscr{V}$ on $\mathscr{S}_{\mathbb{C}}$ such that the fibre of $\mathscr{V}$ above h is V, and such that h is a Hodge generic point of $\mathscr{S}_{\mathbb{C}}$ with respect to the variation $\mathscr{V}$. The embedding $\mathscr{S} \hookrightarrow \mathscr{A}_{g,1,n}$ gives a natural abelian scheme $f\colon \mathscr{A} \to \mathscr{S}$. The point h is also a Hodge generic point with respect to f. Observe that $\mathrm{A} = \mathscr{A}_h$.

Recall that $\mathrm{V} \in \langle \mathrm{H}^1_{\mathrm{B}}(\mathscr{A}_h)\rangle^{\otimes}$, which means that there exist integers a, b, and m and some projector γ_h on $\mathbb{T}^{a,b}\mathrm{H}^1_{\mathrm{B}}(\mathscr{A}_h)(m)$ whose image is isomorphic to V. Since h is a Hodge generic point of $\mathscr{S}_{\mathbb{C}}$, the projector γ_h spreads out to a projector γ on $(\mathbb{T}^{a,b}\mathrm{R}^1 f_{\mathbb{C},*}\mathbb{Q})(m)$, and $\mathscr{V}_{\mathscr{S}_{\mathbb{C}}} \cong \mathrm{Im}(\gamma)$.

By theorem 5.2.1, the projector γ is motivated, and thus we obtain a family of abelian motives $\mathscr{M}/\mathscr{S}_{\mathbb{C}}$ whose Betti realisation is $\mathscr{V}_{\mathscr{S}_{\mathbb{C}}}$. In particular $\mathscr{M}_h \cong \mathrm{M}_\sigma$. Finally, the point h, the projector γ, and the family of motives $\mathscr{M}$ are all defined over a finitely generated subfield $\mathrm{L} \subset \mathbb{C}$ that contains F and $\sigma(\mathrm{K})$. □

10.5 — We will now start the proof of theorem 10.1. We retain the assumptions and notation of construction 10.3 and lemma 10.4. Write S for $\mathscr{S}_{\mathrm{L}}$. Let $\mathscr{V}'$ be the variation of Hodge structure $\mathrm{R}^1 f_{\mathbb{C},*}\mathbb{Q}$ over $\mathrm{S}(\mathbb{C})$, and write $\mathscr{V}$ for the image of γ in $(\mathbb{T}^{a,b}\mathscr{V}')(m)$; it is a variation of Hodge structure that is the Betti realisation of $\mathscr{M}/\mathrm{S}(\mathbb{C})$. Because h is a Hodge generic point, the field E is a subfield of $\mathrm{End}(\mathscr{V})$. Let $(e_i)_i$ be a basis of E as $\mathbb{Q}$-vector space.

Let ℓ be a prime number. Let $\mathscr{V}'_\ell$ be the lisse ℓ-adic sheaf $\mathrm{R}^1 f_*\mathbb{Q}_\ell$ over $\mathscr{S}$. By theorem 5.2.2, the projector γ on $(\mathbb{T}^{a,b}\mathscr{V}')(m)$ induces a projector on $(\mathbb{T}^{a,b}\mathscr{V}'_{\ell,\mathrm{S}})(m)$ over S that spreads out to a projector γ_ℓ on $(\mathbb{T}^{a,b}\mathscr{V}'_\ell)(m)$ over the entirety of $\mathscr{S}$. Let $\mathscr{V}_\ell$ denote the image of γ_ℓ. Note that $\mathscr{V}_{\ell,\mathrm{S}}$ is the ℓ-adic realisation of $\mathscr{M}/\mathrm{S}$.

By theorem 5.2.2 we see that $\mathrm{E}_\ell = \mathrm{E} \otimes \mathbb{Q}_\ell$ is a subfield of $\mathrm{End}(\mathscr{V}_{\ell,\mathrm{S}})$. Since S is the generic fibre of $\mathscr{S}$, we see that $\mathrm{E}_\ell \subset \mathrm{End}(\mathscr{V}_\ell)$. This has two implications, namely (*i*) we obtain classes $e_{i,\ell} \in \mathrm{End}(\mathscr{V}_\ell)$ that form a $\mathbb{Q}_\ell$-basis for E_ℓ; and (*ii*) because $\mathrm{E}_\ell = \mathrm{E} \otimes \mathbb{Q}_\ell \cong \prod_{\lambda|\ell} \mathrm{E}_\lambda$, the lisse ℓ-adic sheaf $\mathscr{V}_\ell$ decomposes as a sum $\bigoplus_{\lambda|\ell} \mathscr{V}_\lambda$ of lisse λ-adic sheaves.

10.6 — Let p be a prime number that does not divide N, so that $\mathscr{K}$ decomposes as $\mathscr{K}_p\mathscr{K}^p$, and $\mathscr{K}_p$ is hyperspecial. Let $\mathbb{F}_q/\mathbb{F}_p$ be a finite field. Let $x \in \mathscr{S}(\mathbb{F}_q)$ be a point. Kisin defines the *isogeny class of* x in §1.4.14 of [Kis17]. It is a subset of $\mathscr{S}(\bar{\mathbb{F}}_q)$.

Let y be a point in $\mathscr{S}(\bar{\mathbb{F}}_q)$ that is isogenous to x. Proposition 1.4.15 of [Kis17] implies that there is an isomorphism of Galois representations $\mathscr{V}'_{\ell,x} \cong \mathscr{V}'_{\ell,y}$ such that $\gamma_{\ell,x} \in \mathrm{End}((\mathbb{T}^{a,b}\mathscr{V}'_{\ell,x})(m))$ is mapped to $\gamma_{\ell,y} \in \mathrm{End}((\mathbb{T}^{a,b}\mathscr{V}'_{\ell,y})(m))$, and such that $e_{i,\ell,x}$ is mapped to $e_{i,\ell,y}$. This implies that $\mathscr{V}_{\ell,x} \cong \mathscr{V}_{\ell,y}$ as $\mathrm{E}[\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)]$-modules. We conclude that $\mathscr{V}_{\lambda,x} \cong \mathscr{V}_{\lambda,y}$ as λ-adic Galois representations.

10.7 — We need one more key result by Kisin [Kis17]. Theorem 2.2.3 of [Kis17] states that for every point $x \in \mathscr{S}(\bar{\mathbb{F}}_q)$, there is a point $y \in \mathscr{S}(\bar{\mathbb{F}}_q)$ that is isogenous to x and such that y is the reduction of a special point in S.

10.8 — We are now set for the attack on theorem 10.1. Let λ_1 and λ_2 be two finite places of E. Let ℓ_1 and ℓ_2 be the residue characteristics of λ_1 respectively λ_2. Let X be the Zariski closure of h in $\mathscr{S}$. Note that X is a model for the residue field of h. Let $U \subset X$ be the Zariski open locus of points $x \in X$ such that the residue characteristic p of x does not divide $N \cdot \ell_1 \cdot \ell_2$. To prove theorem 10.1, it suffices to show that $H_{\lambda_1}(M)$ and $H_{\lambda_2}(M)$ are quasi-compatible at all points $x \in U^{cl}$. Fix a point $x \in U^{cl}$. Let $\mathbb{F}_q$ be the residue field of x. We want to show that $\mathscr{V}_{\lambda_1,x}$ and $\mathscr{V}_{\lambda_2,x}$ are quasi-compatible. This means that we have to show that the characteristic polynomials of the Frobenius endomorphisms of $\mathscr{V}_{\lambda_1,x}$ and $\mathscr{V}_{\lambda_2,x}$ are equal, possibly after replacing the Frobenius endomorphism by some power. Equivalently, we may pass to a finite extension of $\mathbb{F}_q$. This is what we will now do.

As mentioned in §10.7, theorem 2.2.3 of [Kis17] shows that there exists a point $y \in \mathscr{S}(\bar{\mathbb{F}}_q)$ such that y is isogenous to x and such that y is the reduction of a special point $s \in S$. The point y is defined over a finite extension of $\mathbb{F}_q$. As explained in the previous paragraph, we may replace $\mathbb{F}_q$ with a finite extension. Thus we may and do assume that y is $\mathbb{F}_q$-rational.

We want to prove that $\mathscr{V}_{\lambda_1,x}$ and $\mathscr{V}_{\lambda_2,x}$ are quasi-compatible. By our remarks in §10.6 we may as well show that $\mathscr{V}_{\lambda_1,y}$ and $\mathscr{V}_{\lambda_2,y}$ are quasi-compatible. In other words, we may show that $H_{\lambda_1}(\mathscr{M}_s)$ and $H_{\lambda_2}(\mathscr{M}_s)$ are quasi-compatible at y. Recall that s is a special point in S. Therefore $\mathscr{M}_s$ is an abelian CM motive, and we conclude by theorem 9.4 that $H_{\lambda_1}(\mathscr{M}_s)$ and $H_{\lambda_2}(\mathscr{M}_s)$ are quasi-compatible at y. This completes the proof of theorem 10.1.

10.9 Remark. — Laskar [Las14] has obtained similar results. Let $\mathscr{L}$ denote the set of prime numbers, and let M be an abelian motive over a number field K. Let ω be a fibre functor on Mot_K. Laskar needs the following condition: Assume that $G_{\mathrm{mot},\omega}(M)^{\mathrm{ad}}$ does not have a factor whose Dynkin diagram has type D_k. Then theorem 1.1 of [Las14] implies that the system $H_{\mathscr{L}}(M)$ is a compatible system in the sense of Serre (that is, one may take $n = 1$ in definition 6.9) after replacing K by a finite extension. If $G_{\mathrm{mot},\omega}(M)^{\mathrm{ad}}$ does have a factor whose Dynkin diagram has type D_k, then Laskar also obtains results, but I do not see how to translate them into our terminology. See [Las14] for more details.

The result by Laskar and the result above (theorem 10.1) are related, but neither is a formal consequence of the other. Theorem 10.1 does not place conditions on M and it takes endomorphisms by a field $E \subset \mathrm{End}(M)$ into account. This last fact is crucial in the proof of the second main result of this thesis (theorem 17.4).

Motives of K3 type

11 Basics

Readme. — This section consists of several basic definitions and well-known or easy results. We define Hodge structures of K3 type and recall Zarhin's description of their Mumford–Tate groups (theorem 11.2). We define motives of K3 type and define the representation type of such a motive (definition 11.7) and the distinguished embedding associated with such a motive (definition 11.8). We define the determinantal motive (definition 11.9) and compute its structure (lemma 11.10). The determinantal motive will play a small role in section 16, see lemma 16.11 and §16.10.

11.1 Definition. — A Hodge structure V is said to be *of K3 type* if V is polarisable, pure of weight 0, and $\dim_{\mathbb{C}} V^{-1,1} = 1$, and $\dim_{\mathbb{C}} V^{-n,n} = 0$ for $n > 1$.

11.2 Theorem. — *Let* V *be an irreducible Hodge structure of K3 type.*

1. *The endomorphism algebra* E *of* V *is a field.*
2. *The field* E *is a* TR *(totally real) field or a* CM *field.*
3. *If* E *is* TR, *then* $\dim_E(V) \geq 3$.
4. *Let* $\tilde{\phi}$ *be an* E*-bilinear form if* E *is* TR, *resp. a skew-hermitian form if* E *is* CM, *such that* $\mathrm{tr}_{E/\mathbb{Q}} \circ \tilde{\phi}$ *is a polarisation on* V. *Let* E^0 *be the maximal totally real subfield of* E. *The Mumford–Tate group of* V *is*

$$G_B(V) \cong \begin{cases} \mathrm{Res}^{E}_{\mathbb{Q}} \mathrm{SO}(V, \tilde{\phi}), & \text{if E is TR;} \\ \mathrm{Res}^{E^0}_{\mathbb{Q}} \mathrm{U}(V, \tilde{\phi}), & \text{if E is CM.} \end{cases}$$

(Remark: with $\mathrm{SO}(V, \tilde{\phi})$ *we mean the special orthogonal group over* E, *and analogously,* $\mathrm{U}(V, \tilde{\phi})$ *means the unitary group over* E^0.*)*

Proof. These results are mostly due to Zarhin [Zar83].

1. See theorem 1.6.a of [Zar83].

2. See theorem 1.5 of [Zar83].
3. This is observed by Van Geemen, in lemma 3.2 of [Gee08].
4. This is a combination of theorems 2.2 and 2.3 of [Zar83].

(We note that [Zar83] deals with Hodge groups, but because our Hodge structure has weight 0, the Mumford–Tate group and the Hodge group coincide.) □

11.3 Remark. — Let V be an irreducible Hodge structure of K3 type. Let E be the endomorphism algebra of V. Let E^0 be the maximal TR subfield of E. By theorem 11.2.4 we know that $G_B(V) = \mathrm{Res}_{\mathbb{Q}}^{E^0} G$ for some algebraic group G over E^0.

We may regard V as E^0-linear representation of G and as $\mathbb{Q}$-linear representation representation of $G_B(V)$. If follows from proposition 1.2 that the E^0-linear representation V is invariant under all automorphisms of G. Observe that

$$\begin{aligned} V \otimes_{\mathbb{Q}} \mathbb{C} &\cong \bigoplus_{\tau : E^0 \hookrightarrow \mathbb{C}} V \otimes_{E^0,\tau} \mathbb{C} \\ G_B(V) \otimes_{\mathbb{Q}} \mathbb{C} &\cong \bigoplus_{\tau : E^0 \hookrightarrow \mathbb{C}} G \otimes_{E^0,\tau} \mathbb{C} \end{aligned}$$

Hence the complex representation $V \otimes_{\mathbb{Q}} \mathbb{C}$ is invariant under all automorphisms of $G_B(V) \otimes_{\mathbb{Q}} \mathbb{C}$, and consequently the $\mathbb{Q}$-linear representation V is invariant under all automorphisms of $G_B(V)$.

11.4 Definition. — Let K be a finitely generated field of characteristic 0. A motive M over K is said to be *of K3 type* if M is pure of weight 0, and $\dim_K \mathrm{Fil}^1 H_{dR}(M) = 1$, and $\dim_K \mathrm{Fil}^2 H_{dR}(M) = 0$.

By the comparison theorem between $H_\sigma(M)$ and $H_{dR}(M)$ (see §2.4.1), this is equivalent to requiring that $H_\sigma(M)$ is a Hodge structure of K3 type for one (and hence every) embedding $\sigma : K \hookrightarrow \mathbb{C}$.

11.5 Example. — Let K be finitely generated field of characteristic 0. We give some examples of abelian motives of K3 type.

1. Let A be an abelian surface over K. The motive $H^2(A)(1)$ is a motive of K3 type.
2. Let X be a K3 surface over K. The motive $H^2(X)(1)$ is a motive of K3 type.
3. Let X be a cubic fourfold over K. The motive $H^4(X)(2)$ is a motive of K3 type.

All these examples are abelian motives; see théorème 0.6.3 of [And96b].

11.6 Lemma. — *Let* K *be a finitely generated field of characteristic* 0. *Let* M *be an irreducible abelian motive of K3 type over* K. *Then* $H_\sigma(M)$ *is invariant under all automorphisms of* $G_\sigma(M)$. *Assume that* $G_\ell(M)$ *is connected for all prime numbers* ℓ. *Let* ω *be a fibre functor on* Mot_K.

1. *Then* M *is invariant under all automorphisms of* $G_{mot,\omega}(M)$*, where we view* M *as representation of* $G_{mot,\omega}(M)$ *via the fibre functor* ω.

Assume that furthermore MTC(M) *is true.*

2. *Then the* ℓ*-adic Galois representation* $H_\ell(M)$ *is invariant under all automorphisms of* $G_\ell(M)$.

Let E^0 *be the maximal* TR *subfield of* End(M)*, and let* λ *be a finite place of* E^0.

3. *Then the* λ*-adic Galois representation* $H_\lambda(M)$ *is invariant under all automophisms of* $G_\lambda(M)$.

Proof. (As we will see in theorem 14.1, MTC(M) is true for all abelian motives of K3 type.) Continuing remark 11.3, we see that $H_\sigma(M)$ is invariant under all automorphisms of $G_\sigma(M)$. Assume that $G_\ell(M)$ is connected for all prime numbers ℓ. Since M is an abelian motive this implies that $G_{mot,\omega}(M)$ is connected, see also lemma 2.12. By theorem 5.2.1 we find that M, viewed as representation of $G_{mot,\omega}(M)$ via the fibre functor ω, is invariant under all automorphisms of $G_{mot,\omega}(M)$. Given that MTC(M) holds, we find that $H_\ell(M)$ is invariant under all automorphisms of $G_\ell(M)$. Finally, using the decompositions $H_\ell(M) = \bigoplus_{\lambda|\ell} H_\lambda(M)$ and $G_\ell(M) = \prod_{\lambda|\ell} \mathrm{Res}^{E^0_\lambda}_{\mathbb{Q}_\ell} G_\lambda(M)$ we also see that the λ-adic Galois representation $H_\lambda(M)$ is invariant under all automophisms of $G_\lambda(M)$. □

11.7 Definition. — Let M be a geometrically irreducible abelian motive of K3 type over a finitely generated field K of characteristic 0. Let $E = \mathrm{End}(M_{\bar{K}})$ be the geometric endomorphism algebra of M (*cf.* §2.14). Recall from theorem 11.2.2 that E is either a TR field or a CM field. Write n for $\dim_E(M_{\bar{K}})$. The *representation type* of M (over K) is defined to be the formal symbol

$$\begin{cases} (O, n) & \text{if E is a TR field,} \\ (U, n) & \text{if E is a CM field.} \end{cases}$$

Note: in §12.3 we refine the representation type $(O, 4)$ into two separate cases.

11.8 Definition. — Let K be a finitely generated field of characteristic 0; and let M be a an irreducible motive of K3 type over K. Let E be the endomorphism algebra of M. Since $\mathrm{Fil}^1 H_{dR}(M) \cong K$, we obtain a natural embedding $E \hookrightarrow K$. We call this embedding the *distinguished embedding* associated with M.

11.9 Definition. — Let K be a finitely generated field of characteristic 0. Let M be an irreducible abelian motive of K3 type over K. Assume that $G_\ell(M)$ is connected for all prime numbers ℓ, and that M has representation type (U, n). Let E be the endomorphism algebra of M. The *determinantal motive* M^{det} is defined to be $\bigwedge^n_E M$.

11.10 Lemma. — *Let* K *be a finitely generated field of characteristic* 0*. Let* M *be an irreducible abelian motive of K3 type over* K*. Assume that* $G_\ell(M)$ *is connected for all prime numbers* ℓ*, and that* M *has* *representation type* (U, n). *Then the motive* $\mathrm{M}^{\mathrm{det}}$ *is an irreducible abelian motive of K3 type with representation type* $(\mathrm{U}, 1)$. *We have an isogeny* $\mathrm{G}_\ell(\mathrm{M})^{\mathrm{ab}} \to \mathrm{G}_\ell(\mathrm{M}^{\mathrm{det}})$.

Proof. Let E be the endomorphism algebra of M, and let E^0 be the maximal TR subfield of E. Note that $\mathrm{H}_{\mathrm{dR}}(\mathrm{M}^{\mathrm{det}}) = \bigwedge_{\mathrm{E}}^n \mathrm{H}_{\mathrm{dR}}(\mathrm{M})$. Therefore $\mathrm{Fil}^2\mathrm{H}_{\mathrm{dR}}(\mathrm{M}^{\mathrm{det}})$ is trivial, and we claim that $\mathrm{Fil}^1\mathrm{H}_{\mathrm{dR}}(\mathrm{M}^{\mathrm{det}})$ has dimension 1. The easiest way to see this might be by choosing a complex embedding $\sigma\colon \mathrm{K} \hookrightarrow \mathbb{C}$. Let $\tau\colon \mathrm{E} \hookrightarrow \mathrm{K}$ be the distinguished embedding associated with M. Then $\mathrm{H}_\sigma(\mathrm{M}) \otimes_{\mathrm{E},\sigma\tau} \mathbb{C}$ has dimension n, and $\mathrm{Fil}^1\mathrm{H}_{\mathrm{dR}}(\mathrm{M}^{\mathrm{det}})$ corresponds with $\bigwedge^n(\mathrm{H}_\sigma(\mathrm{M}) \otimes_{\mathrm{E},\sigma\tau} \mathbb{C}) \cong \mathbb{C}$ via the comparison isomorphism between singular cohomology and algebraic de Rham cohomology, see §2.4.1. This shows that $\mathrm{M}^{\mathrm{det}}$ is an abelian motive of K3 type over K.

By definition, $\mathrm{E} \subset \mathrm{End}(\mathrm{M}^{\mathrm{det}})$, and on the other hand $\dim_{\mathrm{E}}(\mathrm{M}^{\mathrm{det}}) = 1$. Therefore $\mathrm{M}^{\mathrm{det}}$ is irreducible, and has representation type $(\mathrm{U}, 1)$ with $\mathrm{End}(\mathrm{M}^{\mathrm{det}}) = \mathrm{E}$. Recall that $\mathrm{Z}_\sigma(\mathrm{M}) \otimes \mathbb{Q}_\ell \cong \mathrm{Z}_\ell(\mathrm{M})$, by theorem 5.6. Note that $\mathrm{G}_\ell(\mathrm{M}^{\mathrm{det}})$ is a quotient of $\mathrm{G}_\ell(\mathrm{M})$. It is a commutative algebraic group over $\mathbb{Q}_\ell$ of rank $[\mathrm{E}^0 : \mathbb{Q}]$. The absolute rank of $\mathrm{Z}_\ell(\mathrm{M})$ is also $[\mathrm{E}^0 : \mathbb{Q}]$. Since $\mathrm{G}_\ell(\mathrm{M})$ is reductive (theorem 5.4) we get an isogeny $\mathrm{G}_\ell(\mathrm{M})^{\mathrm{ab}} \to \mathrm{G}_\ell(\mathrm{M}^{\mathrm{det}})$. □

12 The hyperadjoint motive of an abelian motive of K3 type

README. — Important: §12.4.

In this section we apply the construction of the hyperadjoint object (see section 4) to abelian motives of K3 type. The goal of this section is to describe the structure of the resulting motives. We summarise the results in §12.4. The case where the representation type is $(\mathrm{O}, 4)$ requires extra care, see §12.3.

The hyperadjoint motive M^{ha} precisely captures the semisimple part of the motivic Galois group $\mathrm{G}_{\mathrm{mot},\omega}(\mathrm{M})$. Let us take a small step back to see why this is useful. Later on we want to prove the Mumford–Tate conjecture for the sum $\mathrm{M}_1 \oplus \mathrm{M}_2$ of two abelian motives of K3 type, M_1 and M_2. By theorem 5.6 we already know the Mumford–Tate conjecture on centres for the motive $\mathrm{M}_1 \oplus \mathrm{M}_2$. The hyperadjoint construction allows us to focus on the remaining part: the semisimple parts of $\mathrm{G}_\sigma(\mathrm{M}_1 \oplus \mathrm{M}_2)$ and $\mathrm{G}_\ell^\circ(\mathrm{M}_1 \oplus \mathrm{M}_2)$.

12.1 — Let K be a finitely generated field of characteristic 0. Let M be an irreducible abelian motive of K3 type over K. Assume $\mathrm{G}_\ell(\mathrm{M})$ is connected for all prime numbers ℓ. Let E be the endomorphism algebra of M, and let E^0 be the maximal TR subfield of E. Fix a complex embedding $\sigma\colon \mathrm{K} \hookrightarrow \mathbb{C}$.

Let G be the algebraic group over E^0 such that $\mathrm{G}_{\mathrm{mot},\sigma}(\mathrm{M}) \cong \mathrm{Res}_{\mathbb{Q}}^{\mathrm{E}^0}\mathrm{G}$ (*cf.* theorem 11.2.4). Recall from remarks 4.7.3 and 4.7.4 that $\mathrm{M}^{\mathrm{ha}} \cong \mathrm{Lie}(\mathrm{G}_{\mathrm{mot},\sigma}(\mathrm{M})^{\mathrm{ad}})$. If the representation type of M is $(\mathrm{U}, 1)$, then G^{ad} is trivial. In particular $\mathrm{G}_{\mathrm{mot},\sigma}(\mathrm{M})^{\mathrm{ad}}$ is then trivial, and $\mathrm{M}^{\mathrm{ha}} = 0$. If the representation type of M is neither $(\mathrm{U},1)$ nor $(\mathrm{O},4)$, then the group G^{ad} is E^0-simple. Indeed, the complex Lie algebras $\mathfrak{sl}_n$ $(n \geq 2)$ and $\mathfrak{so}_n$ $(n \geq 5)$ are simple Lie algebras. (Recall that $\mathfrak{so}_3 \cong \mathfrak{sl}_2$.) However, $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. We will investigate the case $(\mathrm{O},4)$ in detail in §12.3.

Assume that the representation type of M is neither $(\mathrm{U},1)$ nor $(\mathrm{O},4)$. By the above remarks we see that M^{ha} is an irreducible abelian motive. We distinguish between the cases (O,n) and (U,n).

» Suppose that M has representation type (O,n), $n \neq 4$. Recall that in this case $\mathrm{E} = \mathrm{E}^0$. We claim that the inclusion $\mathrm{M}^{\mathrm{ha}} \subset \underline{\mathrm{End}}(\mathrm{M})$ factors as $\mathrm{M}^{\mathrm{ha}} \subset \underline{\mathrm{End}}_{\mathrm{E}}(\mathrm{M}) \subset \underline{\mathrm{End}}(\mathrm{M})$. Consider the E-Lie algebra Lie(G), and let $\mathrm{Lie}(\mathrm{G})_{(\mathbb{Q})}$ denote the $\mathbb{Q}$-Lie algebra obtained by forgetting the E-structure on Lie(G). Then there is a canonical isomorphism $\mathrm{Lie}(\mathrm{G})_{(\mathbb{Q})} = \mathrm{Lie}(\mathrm{Res}^{\mathrm{E}}_{\mathbb{Q}}\mathrm{G})$, which shows that $\mathrm{M}^{\mathrm{ha}} \subset \underline{\mathrm{End}}_{\mathrm{E}}(\mathrm{M})$. This proves the claim.

  Since M is self-dual we have $\underline{\mathrm{End}}_{\mathrm{E}}(\mathrm{M}) \cong \mathrm{M} \otimes_{\mathrm{E}} \mathrm{M} \cong \mathrm{Sym}^2_{\mathrm{E}}\mathrm{M} \oplus \bigwedge^2_{\mathrm{E}} \mathrm{M}$. Recall that $\mathfrak{so}_n$ consists of the anti-symmetric matrices in $\mathrm{Mat}_n(\mathbb{C})$. Therefore we find that $\mathrm{M}^{\mathrm{ha}} \cong \bigwedge^2_{\mathrm{E}} \mathrm{M}$. Observe that $\mathrm{End}(\mathrm{M}^{\mathrm{ha}}) \cong \mathrm{E}$, and note that if $n = 3$, then $\mathrm{M} \cong \mathrm{M}^{\mathrm{ha}}$.

» Suppose that M has representation type (U,n), with $n \neq 1$. We denote with $\mathbb{1}$ the unit motive. Recall the notation $\mathrm{M}^{(i)}$ from section 4.

  Observe that $\mathrm{Lie}(\mathrm{G}_{\mathrm{mot},\sigma}(\mathrm{M})) \cong \mathrm{Lie}(\mathrm{Z}_{\mathrm{mot},\sigma}(\mathrm{M})) \oplus \mathrm{Lie}(\mathrm{G}_{\mathrm{mot},\sigma}(\mathrm{M})^{\mathrm{ad}})$. In other words, we have $\mathrm{M}^{(1)} \cong (\mathbb{1} \otimes_{\mathbb{Q}} \mathrm{E}^0) \oplus \mathrm{M}^{(2)}$, and $\mathrm{M}^{(2)} = \mathrm{M}^{\mathrm{ha}}$. The real Lie algebra $\mathfrak{u}_n$ consists of the skew-hermitian matrices in $\mathrm{Mat}_n(\mathbb{C})$. This shows that $\bigwedge^2_{\mathrm{E}^0} \mathrm{M} \cong \bigwedge^2_{\mathrm{E}} \mathrm{M} \oplus (\mathbb{1} \otimes \mathrm{E}^0) \oplus \mathrm{M}^{\mathrm{ha}}$. Observe that $\mathrm{End}(\mathrm{M}^{\mathrm{ha}}) \cong \mathrm{E}^0$.

12.2 — Assume that the representation type of M is $(\mathrm{U},2)$. As above, let E be the endomorphism algebra of M; and let E^0 be the maximal totally real subfield of E. The dimension of M^{ha} over E^0 is 3. A computation shows that $\dim_{\mathrm{K}} \mathrm{Fil}^1\mathrm{H}_{\mathrm{dR}}(\mathrm{M}^{\mathrm{ha}}) = 1$, and $\dim_{\mathrm{K}} \mathrm{Fil}^2\mathrm{H}_{\mathrm{dR}}(\mathrm{M}^{\mathrm{ha}}) = 0$. Therefore M^{ha} is an abelian motive of K3 type with representation type $(\mathrm{O},3)$.

12.3 — Let us now turn our attention to the case where M has representation type $(\mathrm{O},4)$. Recall from above the group G over E such that $\mathrm{G}_{\mathrm{mot},\sigma}(\mathrm{M}) \cong \mathrm{Res}^{\mathrm{E}}_{\mathbb{Q}}\mathrm{G}$. There are two cases: either G is E-simple, or it is not. In both cases, there is an étale extension E'/E of degree 2, and an algebraic group G' over E' such that G is isomorphic to a quotient of $(\mathrm{Res}^{\mathrm{E}'}_{\mathrm{E}}\mathrm{G}')/\langle -1 \rangle$ by a finite central subgroup of order 2. The group G' over E' is a form of SL_2 over E'. As before, we have $\mathrm{M}^{\mathrm{ha}} \cong \bigwedge^2_{\mathrm{E}} \mathrm{M}$; but the difference is that $\mathrm{End}(\mathrm{M}^{\mathrm{ha}}) \cong \mathrm{E}'$. We will now take a closer look at the two cases mentioned above.

1. Suppose that G is E-simple. Then E' is a quadratic field extension of E, and M^{ha} is an irreducible abelian motive.

   An example of this case is the motive $\mathrm{M} = \mathrm{H}^2(\mathrm{A})(1)^{\mathrm{tra}}$, where A is an absolutely simple abelian surface such that $\mathrm{F} = \mathrm{End}(\mathrm{A}_{\bar{\mathrm{K}}}) \otimes \mathbb{Q}$ is a real quadratic extension of $\mathbb{Q}$. In this case $\mathrm{E} = \mathbb{Q}$, and the geometric Picard number of A is 2, so $\dim(\mathrm{M}) = 4$. The field F does not act on M but it does act on M^{ha}. So in this example $F = E'$.

2. Suppose that G is not E-simple. Then E' is isomorphic to $E \times E$, and M^{ha} has two irreducible components M_1 and M_2. Note that $End(M_1) \cong E \cong End(M_2)$. Note also that $M_1 \not\cong M_2$, for otherwise $End(M^{ha}) \cong Mat_2(E)$.

   Recall that $\dim_E M^{ha} = 6$. A computation shows that $\dim_K Fil^1 H_{dR}(M^{ha}) = 2$, and $\dim_K Fil^2 H_{dR}(M^{ha}) = 0$. Therefore M_1 and M_2 are abelian motives of K3 type with representation type $(O, 3)$.

   An example of this case is the motive $M = H^2(A)(1)^{tra}$, where A is the product of two elliptic curves $X_1 \times X_2$, such that $X_{1,\bar{K}}$ is not isogenous to $X_{2,\bar{K}}$ and $End(X_{1,\bar{K}}) = \mathbb{Z} = End(X_{2,\bar{K}})$. In this case $E = \mathbb{Q}$, and the geometric Picard number of A is 2, so $\dim(M) = 4$. Observe that $M(-1) = H^1(X_1) \otimes H^1(X_2)$. A computation shows that $M^{ha} = H^1(X_1)^{ha} \oplus H^1(X_2)^{ha}$ and in fact, for $i = 1, 2$, the motive $H^1(X_i)^{ha}$ is isomorphic to $H^2(X_i \times X_i)(1)^{tra}$ which is a motive of K3 type.

If we want to distinguish between these two cases, then we say that M has representation type $(O, 4)_1$ in the case where M^{ha} has 1 irreducible component (and G is E-simple); and we say that M has representation type $(O, 4)_2$ in the case where M^{ha} has 2 irreducible components (and G is not E-simple). The mnemonic is that M^{ha} has i components when M is of representation type $(O, 4)_i$.

12.4 — The following table summarises the above description of M^{ha}.

Rep. type	$End(M^{ha})$	*Notes*
$(U, 1)$	0	$M^{ha} = 0$
$(U, 2)$	E^0	M^{ha} is of K3 type, with rep. type $(O, 3)$
(U, n), $n > 2$	E^0	$M^{ha} \oplus \bigwedge^2_E M \oplus (\mathbb{1} \otimes E^0) \cong \bigwedge^2_{E^0} M$
$(O, 3)$	E	$M = M^{ha}$
$(O, 4)_1$	E'	E'/E a quadratic field extension
$(O, 4)_2$	$E \times E$	$M^{ha} = M_1 \oplus M_2$, with M_i of K3 type and rep. type $(O, 3)$
(O, n), $n > 4$	E	$M^{ha} \cong \bigwedge^2_E M$

13 The Kuga–Satake construction

Readme. — We recall the Kuga–Satake construction. Briefly, the Kuga–Satake construction attaches an abelian variety to every Hodge structure of K3 type. This construction generalises to the motivic setting.

13.1 — The Kuga–Satake construction was first described in [KS67]. Deligne gave a representation-theoretic description in §§3–4 of [Del72] and exhibited the motivic aspects of the construction. Another detailed treatment is in §4 of [Huy16].

13.2 — We will now recapitulate some facts about tensor algebras, Clifford algebras, and spin representations. The main reference is §9 of [Bou07]. One may also consult §4.1 of [Huy16].

Let K be a field of characteristic 0, and let V be a finite-dimensional K-vector space. Let T(V) denote the *tensor algebra* $\bigoplus_{i\geq 0} V^{\otimes i}$ with its natural $\mathbb{Z}$-grading. For $v = v_1 \otimes v_2 \otimes \cdots \otimes v_k \in T(V)$ we denote with $v^\star$ the element $v_k \otimes \cdots \otimes v_2 \otimes v_1$. This construction extends to an involution $v \mapsto v^\star$ on T(V).

13.3 — Let q be a non-degenerate quadratic form on the vector space V. For $v \in V$, we view $q(v) \in K$ as element of T(V) via the canonical identification $K \cong V^{\otimes 0}$. The *Clifford algebra* $Cl(V,q)$ is the algebra $T(V)/I_q$, where I_q is the two-sided ideal generated by $v \otimes v - q(v)$, $(v \in V)$. Since all the elements of I_q are in even degrees, the algebra $Cl(V,q)$ inherits a $\mathbb{Z}/2\mathbb{Z}$-grading

$$Cl(V,q) \cong Cl^+(V,q) \oplus Cl^-(V,q).$$

Note that $Cl^+(V,q)$ is a sub-algebra of $Cl(V,q)$.

If dim(V) is even, then $Cl(V,q)$ is a central simple algebra over K, and $Cl^+(V,q)$ is a central simple algebra over a quadratic étale K-algebra. On the other hand, if dim(V) is odd, then $Cl(V,q)$ is a central simple algebra over a quadratic étale K-algebra, and $Cl^+(V,q)$ is a central simple algebra over K.

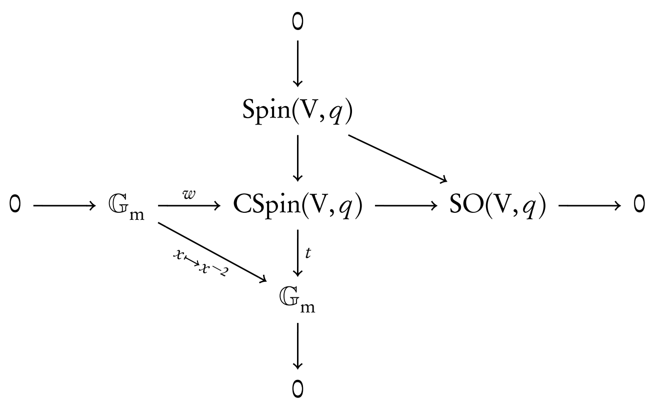
13.4 — The involution $v \mapsto v^\star$ on T(V) preserves the ideal I_q, and since $(v \otimes w)^\star = w^\star \otimes v^\star$, this induces an involution $v \to v^\star$ on the algebra $Cl(V,q)$. The map $v \mapsto v \cdot v^\star$ has image in K, and the induced map $N\colon \mathbb{G}_{m,Cl(V,q)} \to \mathbb{G}_m$ is called the *(spinorial) norm*.

The *Clifford group* $CSpin(V,q)$ is by definition $\{g \in Cl(V,q)^\star \mid gVg^{-1} \subset V\}$. It turns out that $CSpin(V,q)$ is a connected reductive group. The *spin group* $Spin(V,q)$ is the kernel of the norm map $N\colon CSpin(V,q)^\star \to \mathbb{G}_m$. The Clifford group $CSpin(V,q)$ has a natural representation on V, where $g \in CSpin(V,q)$ acts on V via $v \mapsto gvg^{-1}$. In $Cl(V,q)$ we have

$$q(gvg^{-1}) = (gvg^{-1}) \cdot (gvg^{-1}) = g \cdot q(v) \cdot g^{-1} = q(v) \cdot gg^{-1} = q(v)$$

and therefore this representation is orthogonal: we get a representation $CSpin(V,q) \to O(V,q)$. Since $CSpin(V,q)$ is connected, the image of $CSpin(V,q)$ lies in $SO(V,q)$. The kernel of this representation is $w\colon \mathbb{G}_m \hookrightarrow CSpin(V,q)$.

13.5 — Let t denote the inverse of the norm map $\mathrm{N}\colon \mathrm{CSpin}(\mathrm{V},q) \to \mathbb{G}_{\mathrm{m}}$; so for $g \in \mathrm{CSpin}(\mathrm{V},q)$ we have $t(g) = \mathrm{N}(g)^{-1}$. Recall diagram 3.2.1 of [Del72].



13.6 — The Clifford group $\mathrm{CSpin}(\mathrm{V},q)$ has two natural representations on $\mathrm{Cl}^{+}(\mathrm{V},q)$.

1. Via left multiplication:

$$\mathrm{spin}\colon \mathrm{CSpin}(\mathrm{V},q) \to \mathrm{GL}(\mathrm{Cl}^{+}(\mathrm{V},q)), \quad \mathrm{spin}(g) = (v \mapsto gv)$$

   We denote this representation with $\mathrm{Cl}^{+}(\mathrm{V},q)_{\mathrm{spin}}$.

2. Via conjugation:

$$\mathrm{ad}\colon \mathrm{CSpin}(\mathrm{V},q) \to \mathrm{GL}(\mathrm{Cl}^{+}(\mathrm{V},q)), \quad \mathrm{ad}(g) = (v \mapsto gvg^{-1})$$

   We denote this representation with $\mathrm{Cl}^{+}(\mathrm{V},q)_{\mathrm{ad}}$. This representation is the composition of $\mathrm{CSpin}(\mathrm{V},q) \to \mathrm{SO}(\mathrm{V},q)$ with the natural representation of $\mathrm{SO}(\mathrm{V},q)$ on $\mathrm{Cl}^{+}(\mathrm{V},q)$.

13.7 — We recall from §3 of [Del72] the following four fundamental isomorphisms of representations of $\mathrm{CSpin}(\mathrm{V},q)$.

$$\mathrm{Cl}^{+}(\mathrm{V},q)_{\mathrm{ad}} \cong \underline{\mathrm{End}}_{\mathrm{Cl}^{+}(\mathrm{V},q)}(\mathrm{Cl}^{+}(\mathrm{V},q)_{\mathrm{spin}}), \qquad (\text{§3.3.3 of [Del72]}) \tag{13.7.1}$$

$$\mathrm{Cl}^{+}(\mathrm{V},q)_{\mathrm{ad}} \cong \bigwedge^{2\star}\mathrm{V}, \qquad (\text{§3.3.4 of [Del72]}) \tag{13.7.2}$$

Assume that K is algebraically closed, and assume that $\dim(\mathrm{V})$ is odd. Let W_q be a simple $\mathrm{Cl}^{+}(\mathrm{V},q)$-module. It is the *spin representation* of $\mathrm{CSpin}(\mathrm{V},q)$.

$$\mathrm{Cl}^{+}(\mathrm{V},q)_{\mathrm{spin}} \cong \mathrm{W}_q^{\oplus 2^n}, \qquad (\text{§3.4.1 of [Del72]}) \tag{13.7.3}$$

$$\mathrm{Cl}^{+}(\mathrm{V},q)_{\mathrm{ad}} \cong \underline{\mathrm{End}}(\mathrm{W}_q) \cong \mathrm{W}_q^{\otimes 2}, \qquad (\text{§3.4.2 of [Del72]}) \tag{13.7.4}$$

13.8 Kuga–Satake construction. — Let (V, ϕ) be a polarised Hodge structure of K3 type. The Hodge structure on V is given by a morphism $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$. Since ϕ is a polarisation on h, and V is of even weight, the image of h lies in $\mathrm{SO}(V, \phi)_{\mathbb{R}}$; see also theorem 11.2. The Kuga–Satake construction shows that there is a unique lift of h to a morphism $\tilde{h}\colon \mathbb{S} \to \mathrm{CSpin}(V, \phi)_{\mathbb{R}}$ such that $\tilde{h}$ endows $\mathrm{Cl}^{+}(V, \phi)_{\mathrm{spin}}$ with a polarisable Hodge structure of type $\{(0,1),(1,0)\}$, see lemme 4.3 of [Del72]. Thus there is a complex abelian variety A (up to isogeny) such that $H^1_B(A) \cong \mathrm{Cl}^{+}(V, \phi)_{\mathrm{spin}}$. We write V^{KS} for the abelian motive $H^1(A)$ over $\mathbb{C}$. Observe that $V \in \langle H^1_B(A)\rangle^{\otimes}$.

This construction also works in families. Let S be a smooth complex variety, and let $(\mathscr{V}/S, \phi)$ be a polarised variation of Hodge structure of K3 type. Then there exists a finite étale cover $S' \to S$ and an abelian scheme $\mathscr{A}/S'$ such that $H^1_B(\mathscr{A})/S' \cong \mathrm{Cl}^{+}(\mathscr{V}_{S'}, \phi_{S'})_{\mathrm{spin}}$. See proposition 5.7 of [Del72].

13.9 Remark. — It is expected that every Hodge class in the Betti realisation of a motive is motivated (*cf.* conjecture 2.7.1 and theorem 5.2.1). It is therefore expected that every motive of K3 type over $\mathbb{C}$ is an abelian motive: If M is a motive of K3 type, then the Kuga–Satake construction shows that there is a complex abelian variety A such that $H_B(M) \in \langle H^1_B(A)\rangle^{\otimes}$. In other words, there is an integer n, and an idempotent endomorphism γ of $H^{2n}_B(A)(n)$ such that $H_B(M) \cong \mathrm{Im}(\gamma)$. It is expected that γ is motivated, and thus $M \in \langle H^1(A)\rangle^{\otimes}$.

So far we know this for the examples listed in example 11.5. Other examples include $H^2(X)(1)$, where X is a surface with $p_g = 1$ that is dominated by a product of curves, or a deformation of such a surface.

14 The Mumford–Tate conjecture for abelian motives of K3 type

Readme. — Important: theorem 14.1

In this section we prove the Mumford–Tate conjecture for abelian motives of K3 type over a finitely generated field of characteristic 0. We will do this by applying several results and techniques of [Moo16]. (Indeed, roughly speaking, the main theorem of [Moo16] shows that the Mumford–Tate conjecture is true for fibres in a non-isotrivial family of motives of K3 type.)

14.1 Theorem. — *Let* M *be an abelian motive of K3 type over a finitely generated field* K *of characteristic* 0*. Then the Mumford–Tate conjecture for* M *is true.*

14.2 — The proof of theorem 14.1 will consist of applying results from [Moo16]. The main difference with [Moo16] is that we make the a priori assumption that our motive M is an abelian motive. This saves us from a lot of the hard work that happens in [Moo16].

First of all, we remark that we may apply theorem 2.6 and corollary 2.7 of [Moo16]. Indeed, these results rely on theorem 2.4 of [Moo16], which is a slight generalisation of theorem 3.18 of [Pin98]. Theorem 3.18 of [Pin98] applies in our situation, because (*i*) the ℓ-adic realisations $H_\ell(M)$ form a quasi-compatible system of Galois representations, by theorem 10.1; and (*ii*) quasi-compatible systems admit a good notion of Frobenius tori (definition 8.7).

Without loss of generality we may and do assume that M is irreducible. Indeed, suppose that $M = M' \oplus M''$. Then precisely one of the summands (say M') is of K3 type. Therefore $H_\sigma(M'')$ is a sum of copies of the trivial Hodge structure. Hence $G_\sigma(M'')$ is trivial, and since M'' is an abelian motive we conclude that $G^\circ_\ell(M'')$ is trivial. Thus proving theorem 14.1 for M amounts to proving it for M', and we may assume that M is irreducible. We distinguish two cases, based on the representation type of M.

14.3 — Assume that the representation type of M is (U, n). We mimick §7.7 of [Moo16]. Let E be the endomorphism algebra of M. By §9.7 there exists a finitely generated extension L/K and abelian varieties A and B over L such that $M_L \cong \underline{\mathrm{Hom}}_E(H^1(A), H^1(B))$. By lemma 3.4 we may assume that $L = K$. Korollar 1 of Satz 4 of Faltings's [Fal83] (see also [Fal84]) shows that $H_\ell(M) \cong \underline{\mathrm{Hom}}_{E_\ell}(H^1_\ell(A), H^1_\ell(B))$ does not contain any Tate classes. We make three remarks.

1. Observe that $\mathrm{End}_E(H^1(A)) \cong E$. This implies

$$\mathrm{End}_{\mathrm{Mot}_K,E}(M) \cong \mathrm{End}_{\mathrm{Mot}_K,E}(H^1(B)), \qquad \mathrm{End}_{\mathrm{Gal}(\bar{K}/K),E_\ell}(H_\ell(M)) \cong \mathrm{End}_{\mathrm{Gal}(\bar{K}/K),E_\ell}(H^1_\ell(B)).$$

2. Another application of Satz 4 of [Fal83] gives $\mathrm{End}_{\mathrm{Mot}_K,E}(H^1(B)) \otimes \mathbb{Q}_\ell \cong \mathrm{End}_{\mathrm{Gal}(\bar{K}/K),E_\ell}(H^1_\ell(B))$.
3. By theorem 2.6 of [Moo16] we know that $\mathrm{End}_{\mathrm{Gal}(\bar{K}/K),E_\ell}(H_\ell(M))$ is a commutative semisimple algebra containing E_ℓ; and therefore $\mathrm{End}_{\mathrm{Gal}(\bar{K}/K),E_\ell}(H_\ell(M)) = \mathrm{End}_{\mathrm{Gal}(\bar{K}/K)}(H_\ell(M))$.

We conclude that $\mathrm{End}_{\mathrm{Gal}(\bar{K}/K)}(H_\ell(M)) = E_\ell$. Then MTC(M) follows from corollary 2.7 of [Moo16]. This completes the proof of theorem 14.1 if M has representation type (U, n).

14.4 — Assume that the representation type of M is (O, n). This case is more involved than the previous case, but the global idea is the same. In [Moo16], Moonen sets up a version of the Kuga–Satake construction that is relative to the endomorphism algebra End(M). We will not give all the details, but suffice to say that we can apply proposition 5.2 of [Moo16] to our situation. Indeed, since M is an abelian motive, there is a finitely generated extension L/K such that equation 5.2.1 in proposition 5.2 of [Moo16] is satisfied by M_L. By lemma 3.4 we may assume that $L = K$. The proof of proposition 5.2 of [Moo16] goes through verbatim in our situation if one replaces every occurence of 'algebraic cycle' with 'motivated cycle' and every occurence of the Tate conjecture with our conjecture 2.7.2.a (Tate = motivated).

15 Hodge–Tate maximality

Readme. — Important: theorem 15.2.

The goal of this section is to prove theorem 15.2. This result is the ℓ-adic analogue of proposition 6.2 of [CM15] by Cadoret and Moonen, which says the following: *Let* V *be a Hodge structure of K3 type. If* V' *is a polarisable Hodge structure such that* $V \in \langle V' \rangle^{\otimes}$, *and if the natural morphism* $f\colon G_B(V') \twoheadrightarrow G_B(V)$ *is an isogeny, then* f *is an isomorphism.* We will have to adapt parts of the Hodge-theoretic setup in [CM15] to our situation.

The proof of theorem 15.2 uses p-adic Hodge theory. Since prime numbers in p-adic Hodge theory are usually denoted with p we also use p instead of ℓ.

15.1 — Let p be a prime number, and let K_v be a p-adic field. Let C_v be the completion of $\bar{K}_v$. Note that the Galois group $\mathrm{Gal}(\bar{K}_v/K_v)$ acts on C_v. For $i \in \mathbb{Z}$, let $C_v(i)$ denote the 1-dimensional C_v-vector space on which $\mathrm{Gal}(\bar{K}_v/K_v)$ acts by twisting the action on C_v with χ^i, where χ is the p-adic cyclotomic character. The ring $B_{\mathrm{HT},K_v} = \bigoplus_{i\in\mathbb{Z}} C_v(i)$ is the graded quotient of the filtered ring B_{dR,K_v}, see Fontaine [Fon94].

Let K be a finitely generated subfield of K_v. Let M be a pure motive of weight n over K. By theorem III.4.1 of [Fal88] (see also §2.4.3) we get an isomorphism

$$(15.1.1)\qquad \bigoplus_{i\in\mathbb{Z}} \big(C_v(-i) \otimes_K \mathrm{gr}^i H_{\mathrm{dR}}(M)\big) \cong C_v \otimes_{\mathbb{Q}_p} H_p(M).$$

of graded vector spaces that is compatible with the action of $\mathrm{Gal}(\bar{K}_v/K_v)$. This isomorphism may be reformulated as

$$B_{\mathrm{HT},K_v} \otimes_K \mathrm{gr}\, H_{\mathrm{dR}}(M) \cong B_{\mathrm{HT},K_v} \otimes_{\mathbb{Q}_p} H_p(M).$$

(Historically, this "graded" isomorphism was known before the isomorphism given in §2.4.3.)

Suppose that M is of K3 type, and let E^0 be the maximal TR subfield of End(M). Assume that $G_\ell(M)$ is connected for all prime numbers ℓ. Recall that $H_p(M)$ decomposes as $\bigoplus_{\pi|p} H_\pi(M)$, where π runs over the finite places of E^0 that lie above p. Since M is of K3 type there is a unique place π of E^0 lying above p such that $C_v \otimes_{\mathbb{Q}_p} H_\pi$ has a non-trivial Hodge–Tate decomposition. Call this place π the *distinguished* place above p associated with M.

In the following theorem the notation $\langle M' \rangle^{\otimes_{E^0}}$ means the Tannakian subcategory generated by M' inside the category of motives with an action by E^0.

15.2 Theorem. — *Let* K *be a finitely generated field of characteristic* 0. *Let* M *be an irreducible abelian motive of K3 type over* K. *Let* E *be the endomorphism algebra of* M, *and let* E^0 *be the maximal*

TR subfield of E. *Let* M′ *be an abelian motive over* K *such that* E^0 *acts on* M′ *and such that* $M \in \langle M' \rangle^{\otimes_{E^0}}$. *Assume that* $G_\ell(M')$ *is connected for all prime numbers* ℓ. *Let* p *be a prime number that is totally split in* E^0, *and let* π *be the distinguished place above* p *associated with* M. *Assume that the projection homomorphism* $f : G_\pi(M') \twoheadrightarrow G_\pi(M)$ *is an isogeny. Then* f *is an isomorphism.*

15.3 — The proof of theorem 15.2 is analogous to the proof of proposition 6.2 of [CM15]. We first give the analogue of §2.2 of [CM15], and then present the proof of theorem 15.2 in §15.5.

Let G be a connected reductive group over a field K of characteristic 0. Let $K \subset F$ be a field extension, S an algebraic group over F, and $h : S \to G_F$ a homomorphism. The we say that h is *maximal* (definition 2.1 of [CM15]) if there is no non-trivial isogeny of connected K-groups $G' \to G$ such that h lifts to a homomorphism $S \to G'_F$. (If F is algebraically closed, then the maximality of h depends only on its G(F)-conjugacy class.)

15.4 — (Analogue of §2.2 of [CM15].) As in §15.1, let p be a prime number, let K_v be a p-adic field, and let C_v be the completion of $\bar{K}_v$. Let G be a connected reductive group over K_v. Let $\mathscr{C}$ be a conjugacy class of cocharacters $\mathbb{G}_{m,C_v} \to G_{C_v}$. Let $\pi_1(G)$ denote the fundamental group of G as defined by Borovoi in [Bor98]. This is a finitely generated $\mathbb{Z}$-module with a continuous action of $\Gamma = \mathrm{Gal}(\bar{K}/K)$. If $(X^*, R, X_*, \check{R})$ is the root datum of $G_{\bar{K}_v}$ and $Q(\check{R}) = \langle \check{R} \rangle \subset X_*$ is the coroot lattice, then $\pi_1(G) \cong X_*/Q(\check{R})$.

The conjugacy class $\mathscr{C}$ of cocharacters corresponds to an orbit $\mathscr{C} \subset X_*$ under the Weyl group W. As the induced W-action on $\pi_1(G)$ is trivial, any two elements in $\mathscr{C}$ have the same image in $\pi_1(G)$; call it $[\mathscr{C}] \in \pi_1(G)$.

If G′ is a connected reductive K-group and $f : G' \to G$ is an isogeny, then the map induced by f identifies $\pi_1(G')$ with a $\mathbb{Z}[\Gamma]$-submodule of finite index in $\pi_1(G)$. Conversely, every such submodule comes from an isogeny of connected K-groups that is unique up to isomorphism over G. A conjugacy class $\mathscr{C}$ as above lifts to G′ if and only if $[\mathscr{C}] \in \pi_1(G')$. The cocharacters in $\mathscr{C}$ are maximal (in the sense of the previous paragraph) if and only if $[\mathscr{C}]$ generates $\pi_1(G)$ as a $\mathbb{Z}[\Gamma]$-module.

15.5 *Proof* (of theorem 15.2). — Embed K into a p-adic field K_v. Let C_v be the completion of $\bar{K}_v$. Recall that a grading on a vector space V determines a cocharacter of GL(V). Recall that p is totally split in E^0, and therefore $G_\pi(M)$ is an algebraic group over $\mathbb{Q}_p$. The isomorphism in 15.1.1 gives cocharacters

$$\mu : \mathbb{G}_{m,C_v} \to GL(C_v \otimes_{\mathbb{Q}_p} H_\pi(M)), \qquad \mu' : \mathbb{G}_{m,C_v} \to GL(C_v \otimes_{\mathbb{Q}_p} H_\pi(M')),$$

By §1.4 of [Ser79], we find that the image of the cocharacters μ (resp. μ') is contained in $G_\pi(M) \otimes_{\mathbb{Q}_p} C_v$ (resp. $G_\pi(M') \otimes_{\mathbb{Q}_p} C_v$); and we have $\mu = f_{C_v} \circ \mu'$.

The remainder of the proof is now completely analogous to the proof of proposition 6.2 of [CM15]. What follows is a copy of their arguments, adapted to our notation. Put $G = G_\pi(M)$. First suppose that M is of type (O, n), so that G is absolutely simple. If $n = 2k+1$ is odd (resp. $n = 2k$ is even), then the root system of G_{C_v} is of type B_k (resp. D_k). We follow the notation of [Bou81], planches II and IV. In the even case the calculation that follows goes through without changes if $k = 2$. With respect to the basis $\varepsilon_1, \dots, \varepsilon_k$ for $\mathbb{R}^k = X_*(G) \otimes \mathbb{R}$, we have $X_*(G) = \mathbb{Z}^k$, and the coroot lattice $Q(\check{R})$ consists of the vectors $(m_1, \dots, m_k) \in \mathbb{Z}^k$ for which $\sum m_j$ is even. On the other hand, the cocharacter μ corresponds to the vector $(1, 0, \dots, 0)$; its image in $\pi_1(G) = X_*(G)/Q(\check{R}) \cong \mathbb{Z}/2\mathbb{Z}$ is therefore the non-trivial class. By what was explained in §15.4 this implies the assertion.

Next suppose that M is of type (U, n), and we have $G_{C_v} \cong GL_n$ in such a way that μ is conjugate to the cocharacter $\mathbb{G}_m \to GL_n$ given by $z \mapsto \mathrm{diag}(z, 1, \dots, 1)$. It is straightforward to check that the corresponding class in $\pi_1(GL_n) \cong \mathbb{Z}$ is a generator, and again by §15.4 this implies the assertion. □

Products of abelian motives of K3 type

16 An intermediate result

Readme. — This section makes the final preparations for the proof of the Mumford–Tate conjecture for products of abelian motives of K3 type, which we prove in section 17. In this section we (*i*) summarise the results of the previous parts; (*ii*) outline the strategy of the proof; and (*iii*) prove the main ingredient, which is the following statement.

16.1 Theorem. — *Let* K *be a finitely generated field of characteristic* 0. *Let* M_1 *and* M_2 *be two irreducible abelian motives of K3 type. Assume that* $G_\ell(M_1 \oplus M_2)$ *is connected for all prime numbers* ℓ. *For* $i = 1,2$, *assume that the representation type of* M_i *is neither* $(U,1)$, *nor* $(U,2)$, *nor* $(O,4)_2$. *Recall that there is a natural inclusion* $G_\ell(M_1^{ha} \oplus M_2^{ha}) \subset G_\ell(M_1^{ha}) \times G_\ell(M_2^{ha})$. *Then*

$$M_1 \cong M_2 \iff \exists \ell : G_\ell(M_1^{ha} \oplus M_2^{ha}) \subsetneq G_\ell(M_1^{ha}) \times G_\ell(M_2^{ha}).$$

16.2 Remark. — The condition on the representation type of M_1 and M_2 is necessary.

- If M_i has representation type $(U,1)$, then $G_\ell(M_i^{ha})$ is trivial, and therefore the statement $G_\ell(M_1^{ha} \oplus M_2^{ha}) \subsetneq G_\ell(M_1^{ha}) \times G_\ell(M_2^{ha})$ always fails.
- For $i = 1,2$, let E_i be the endomorphism algebra of M_i. If M_i has representation type $(U,2)$, then it is not possible to recover $M_i^{det} = \det_{E_i} M_i$ from M_i^{ha}. We will see below that this is possible if M_i has representation type (U,n) with $n > 2$.

  For example, for $i = 1,2$ let A_i be the product of two elliptic curves $Y_0 \times Y_i$ such that Y_0 is an elliptic curve with trivial endomorphism algebra, and Y_i is an elliptic curve with CM. Suppose that $M_i = H^2(A_i)(1)^{tra}$. Then M_i is an abelian motive of K3 type with representation type $(U,2)$. A computation shows that $M_i^{ha} \cong H^2(Y_0 \times Y_0)(1)^{tra}$, whereas $M_i^{det} \cong H^2(Y_i \times Y_i)(1)^{tra}$.

  In this example, we find $G_\ell(M_1^{ha} \oplus M_2^{ha}) \cong G_\ell(M_i^{ha}) \cong PGL_{2,\mathbb{Q}}$. But if Y_1 and Y_2 are non-isogenous elliptic curves, then $M_1 \not\cong M_2$.

» If M_i has representation type $(O,4)_2$, then M_i^{ha} is not irreducible, but rather is the sum of two irreducible abelian motives of K3 type with representation type $(O,3)$. (See section 12.) Suppose that M_2 has representation type $(O,4)_2$, and suppose that $M_2^{ha} \cong M_0 \oplus M_1$. Then clearly $M_1 \not\cong M_2$, but $G_\ell(M_1^{ha} \oplus M_2^{ha}) \cong G_\ell(M_2^{ha})$ since $M_1^{ha} \in \langle M_2^{ha} \rangle^{\otimes}$.

To construct an abelian motive of K3 type with representation type $(O,4)_2$, take two non-isogenous elliptic curves Y_1 and Y_2 such that $\mathrm{End}(Y_1) \cong \mathbb{Z} \cong \mathrm{End}(Y_2)$. Take $M = H^2(Y_1 \times Y_2)(1)^{tra}$. A computation shows that $M^{ha} \cong H^2(Y_1^2)(1)^{tra} \oplus H^2(Y_2^2)(1)^{tra}$.

16.3 — Let us recapitulate the essential gist of the previous sections, which is contained in theorem 14.1, theorem 10.1 and proposition 8.3. Let M be an irreducible abelian motive of K3 type over a finitely generated field of characteristic 0. Assume that $G_\ell(M)$ is connected for all prime numbers ℓ. Then we know that:

1. The Mumford–Tate conjecture is true for M. (See theorem 14.1.)
2. Let E be the endomorphism algebra of M, and let Λ be the set of finite places of E. The system $H_\Lambda(M)$ is a quasi-compatible system of Galois representations. (See theorem 10.1.)
3. The field E may be recovered from $H_\lambda(M)$, for any $\lambda \in \Lambda$, as the subfield of $\mathrm{End}(H_\lambda(M))$ generated by the coefficients of the polynomials $P_{x,H_\lambda(M),n_x}$. (See proposition 8.3.)
4. Assume that M^{ha} is irreducible. (That is, assume that M does not have representation type $(U,1)$ or $(O,4)_2$.) Let E' be the endomorphism algebra of M^{ha}, and let Λ' be the set of finite places of E'. The system $H_{\Lambda'}(M^{ha})$ is a quasi-compatible system of Galois representations. (See theorem 10.1.)
5. The field E' may be recovered from $H_{\lambda'}(M^{ha})$, for any $\lambda' \in \Lambda'$, as the subfield of $\mathrm{End}(H_{\lambda'}(M^{ha}))$ generated by the coefficients of the polynomials $P_{x,H_{\lambda'}(M^{ha}),n_x}$. (See proposition 8.3.)

16.4 — The proof of the theorem 16.1 above constitutes the remainder of this section. The groups $G_\ell(M_i^{ha})$ are non-trivial, since M_i does not have representation type $(U,1)$. Hence the forward implication ($\Longrightarrow$) is trivial and it is of course the converse implication that is the beef of the theorem. The proof roughly goes as follows.

1. Assume that $G_\ell(M_1^{ha} \oplus M_2^{ha}) \subsetneq G_\ell(M_1^{ha}) \times G_\ell(M_2^{ha})$.
2. Show that $H_\ell(M_1^{ha}) \cong H_\ell(M_2^{ha})$.
3. Show that $G_\ell(M_1 \oplus M_2) \twoheadrightarrow G_\ell(M_i)$ is an isogeny.
4. Show that $G_\ell(M_1 \oplus M_2) \twoheadrightarrow G_\ell(M_i)$ is an isomorphism.
5. Show that $H_\ell(M_1)$ and $H_\ell(M_2)$ are isomorphic as representations of $G_\ell(M_1 \oplus M_2)$.
6. Use a result of André to conclude that $M_1 \cong M_2$.

16.5 — Let us now embark on the proof of theorem 16.1. Let ℓ be a prime such that the natural inclusion $G_\ell(M_1^{ha} \oplus M_2^{ha}) \subset G_\ell(M_1^{ha}) \times G_\ell(M_2^{ha})$ is a strict inclusion. For $i = 1,2$, let E_i be the endomorphism algebra of M_i, and let E'_i be the endomorphism algebra of M_i^{ha}. Since the Mumford–Tate conjecture is true for M_i by theorem 14.1, we know that $H_\ell(M_i^{ha}) \cong \mathrm{Lie}(G_\ell(M_i)^{ad})$, see lemma 5.8.

We claim that there must be a summand of $H_\ell(M_1^{ha})$ that is isomorphic to a summand of $H_\ell(M_2^{ha})$ as representations of $G_\ell(M_1^{ha} \oplus M_2^{ha})$ and therefore as Galois representations. Indeed, the endomorphism algebra of $H_\ell(M_i^{ha})$ is $E'_i \otimes \mathbb{Q}_\ell$ (since we know the Mumford–Tate conjecture for M_i^{ha}), and therefore the summands of $H_\ell(M_i^{ha})$ are precisely the representations $H_{\lambda'_i}(M_i^{ha})$ where λ'_i runs over the finite places of E'_i that lie above ℓ. By proposition 1.2, the summand $H_{\lambda'_i}(M_i^{ha})$ is invariant under all automorphisms of $G_{\lambda'_i}(M_i^{ha})$. Hence our assumption that $G_\ell(M_1^{ha} \oplus M_2^{ha}) \subsetneq G_\ell(M_1^{ha}) \times G_\ell(M_2^{ha})$, together with Goursat's lemma for Lie algebras shows that there exist finite places λ'_i of E'_i (lying above ℓ) and an isomorphism $\phi\colon H_{\lambda'_1}(M_1^{ha}) \to H_{\lambda'_2}(M_2^{ha})$ in $\mathrm{Rep}_{\mathbb{Q}_\ell}(\mathrm{Gal}(\bar{K}/K))$. The isomorphism ϕ induces an isomorphism $\psi\colon \mathrm{End}_{\mathrm{Gal}}(H_{\lambda'_1}(M_1^{ha})) \to \mathrm{End}_{\mathrm{Gal}}(H_{\lambda'_2}(M_2^{ha}))$. Observe that $\mathrm{End}_{\mathrm{Gal}}(H_{\lambda'_i}(M_i^{ha})) \cong E'_{i,\lambda'_i}$, is commutative, and therefore the isomorphism ψ does not depend on the choice of ϕ. By proposition 8.3 we recover $\lambda'_i\colon E'_i \hookrightarrow E'_{i,\lambda'_i} \cong \mathrm{End}_{\mathrm{Gal}}(H_{\lambda'_i}(M_i^{ha}))$. Therefore ψ induces a canonical isomorphism $E'_1 = E'_2$ that identifies λ'_1 with λ'_2. Write E' for $E'_1 \cong E'_2$, and write λ' for $\lambda'_1 = \lambda'_2$. We conclude that $H_{\lambda'}(M_1^{ha}) \cong H_{\lambda'}(M_2^{ha})$ as λ'-adic representations.

16.6 — Write Λ' for the set of finite places of E'. We assumed that $G_\ell(M_1 \oplus M_2)$ is connected for all prime numbers ℓ. Because $H_{\lambda'}(M_1^{ha})$ and $H_{\lambda'}(M_2^{ha})$ are semisimple and quasi-compatible, theorem 8.2 shows that $H_{\Lambda'}(M_1^{ha})$ and $H_{\Lambda'}(M_2^{ha})$ are isomorphic quasi-compatible systems of Galois representations. Therefore we have achieved point 2 of §16.4. In particular, the projection maps $G_\ell(M_1^{ha} \oplus M_2^{ha}) \twoheadrightarrow G_\ell(M_i^{ha})$ are isomorphisms.

By lemma 4.8 the projection maps $G_\ell(M_1^{ha} \oplus M_2^{ha}) \to G_\ell(M_i^{ha})$ factor via the quotient map $G_\ell(M_1^{ha} \oplus M_2^{ha}) \twoheadrightarrow G_\ell((M_1 \oplus M_2)^{ha})$. By lemma 5.8 we know that the group $G_\ell((M_1 \oplus M_2)^{ha})$ is equal to $G_\ell(M_1 \oplus M_2)^{ad}$ and $G_\ell(M_i^{ha}) = G_\ell(M_i)^{ad}$. Hence the projections $G_\ell(M_1 \oplus M_2)^{ad} \twoheadrightarrow G_\ell(M_i)^{ad}$ are isomorphisms.

16.7 — Our next goal is to show that M_1 and M_2 have the same representation type. Observe that all simple factors of $G_\ell(M_1^{ha})_{\bar{\mathbb{Q}}_\ell}$ have the same Dynkin type, and likewise all simple factors of $G_\ell(M_1^{ha})_{\bar{\mathbb{Q}}_\ell}$ have the same Dynkin type. Therefore, it makes sense to speak of *the* Dynkin type of $G_\ell(M_1^{ha})$, and *the* Dynkin type of $G_\ell(M_2^{ha})$. The table on the following page lists the Dynkin types corresponding to the various representation types. In the rightmost column we point out the relevant exceptional isomorphisms between Dynkin diagrams of low rank.

Rep. type	*condition*	*Dynkin type*	*remark*
$(\mathrm{O},3)$		A_1	$\mathrm{B}_1 \cong \mathrm{A}_1$
$(\mathrm{O},4)$		A_1	$\mathrm{D}_2 \cong \mathrm{A}_1 \oplus \mathrm{A}_1$
$(\mathrm{O},6)$		A_3	$\mathrm{D}_3 \cong \mathrm{A}_3$
$(\mathrm{O},2k+1)$	$k \geq 2$	B_k	
$(\mathrm{O},2k)$	$k \geq 4$	D_k	
$(\mathrm{U},k+1)$	$k \geq 1$	A_k	

If the Dynkin types of $\mathrm{G}_\ell(\mathrm{M}_1^{\mathrm{ha}})$ and $\mathrm{G}_\ell(\mathrm{M}_2^{\mathrm{ha}})$ differ, then the assumption $\mathrm{G}_\ell(\mathrm{M}_1^{\mathrm{ha}} \oplus \mathrm{M}_2^{\mathrm{ha}}) \subsetneq \mathrm{G}_\ell(\mathrm{M}_1^{\mathrm{ha}}) \times \mathrm{G}_\ell(\mathrm{M}_2^{\mathrm{ha}})$ fails. Thus, in order to show that M_1 and M_2 have the same representation type, we only need to consider two cases: (*i*) the case that M_1 and M_2 both have Dynkin type A_1; and (*ii*) the case that M_1 and M_2 both have Dynkin type A_3.

16.8 — Assume that M_1 and M_2 both have Dynkin type A_1. Recall that we excluded representation type $(\mathrm{U},2)$ and $(\mathrm{O},4)_2$ by assumption. Suppose that M_1 has representation type $(\mathrm{O},3)$, and suppose that M_2 has representation type $(\mathrm{O},4)_1$. Then a small computation shows that $\dim_{\mathrm{K}} \mathrm{Fil}^1 \mathrm{H}_{\mathrm{dR}}(\mathrm{M}_1^{\mathrm{ha}}) = 1$, while $\dim_{\mathrm{K}} \mathrm{Fil}^1 \mathrm{H}_{\mathrm{dR}}(\mathrm{M}_2^{\mathrm{ha}}) = 2$. By p-adic Hodge theory (see §2.4.3) we conclude that $\mathrm{H}_p(\mathrm{M}_1^{\mathrm{ha}}) \ncong \mathrm{H}_p(\mathrm{M}_2^{\mathrm{ha}})$, for all prime numbers p. This contradicts the fact that $\mathrm{H}_{\Lambda'}(\mathrm{M}_1^{\mathrm{ha}}) \cong \mathrm{H}_{\Lambda'}(\mathrm{M}_2^{\mathrm{ha}})$ that we obtained in §16.5. We conclude that this case does not occur.

16.9 — Assume that M_1 and M_2 both have Dynkin type A_3. Suppose that M_1 has representation type $(\mathrm{O},6)$ and suppose that M_2 has representation type $(\mathrm{U},4)$. Recall that $\mathrm{E}_i = \mathrm{End}(\mathrm{M}_i)$ and recall that we already showed $\mathrm{End}(\mathrm{M}_1^{\mathrm{ha}}) = \mathrm{E}' = \mathrm{End}(\mathrm{M}_2^{\mathrm{ha}})$. In this case $\mathrm{E}_1 = \mathrm{E}' = \mathrm{E}_2^0 \subset \mathrm{E}_2$ where E_2^0 is the maximal TR subfield of E_2. Let λ' be a place of E' that is inert in E_2/E'. Then $\mathrm{G}_{\lambda'}(\mathrm{H}_{\lambda'}(\mathrm{M}_1^{\mathrm{ha}}))$ is an inner form of $\mathrm{PGL}_{4,\mathrm{E}'_{\lambda'}}$, while $\mathrm{G}_{\lambda'}(\mathrm{H}_{\lambda'}(\mathrm{M}_2^{\mathrm{ha}}))$ is an outer form of $\mathrm{PGL}_{4,\mathrm{E}'_{\lambda'}}$. This leads to a contradiction, so this case does not occur either. We conclude that M_1 and M_2 must have the same representation type.

16.10 — In this paragraph we show that the projections $\mathrm{G}_\ell(\mathrm{M}_1 \oplus \mathrm{M}_2) \twoheadrightarrow \mathrm{G}_\ell(\mathrm{M}_i)$ are isogenies. If M_1 and M_2 have representation type (O,n), then $\mathrm{G}_\ell(\mathrm{M}_1 \oplus \mathrm{M}_2)$ is semisimple, and hence §16.6 shows that $\mathrm{G}_\ell(\mathrm{M}_1 \oplus \mathrm{M}_2) \twoheadrightarrow \mathrm{G}_\ell(\mathrm{M}_i)$ is an isogeny. The case that M_1 and M_2 have representation type (U,n) is more involved, and we consider it next.

Assume that M_1 and M_2 have representation type (U,n). Observe that $n \geq 3$, because by assumption the representation type of M_1 and M_2 is neither $(\mathrm{U},1)$ nor $(\mathrm{U},2)$. Recall that E_1 and E_2 are quadratic extensions of E'. Since $n \geq 3$, a place λ' of E' is split in E_i/E' if and only if $\mathrm{G}_{\lambda'}(\mathrm{H}_{\lambda'}(\mathrm{M}_i^{\mathrm{ha}}))$ is an inner form of $\mathrm{PGL}_{n,\mathrm{E}'_{\lambda'}}$. Since $\mathrm{H}_{\Lambda'}(\mathrm{M}_1^{\mathrm{ha}}) \cong \mathrm{H}_{\Lambda'}(\mathrm{M}_2^{\mathrm{ha}})$ we conclude that

$G_{\lambda'}(H_{\lambda'}(M_1^{ha})) \cong G_{\lambda'}(H_{\lambda'}(M_2^{ha}))$; and therefore λ' is split in E_1/E' if and only if λ' is split in E_2/E'. Recall that a Galois extension of number fields is determined up to isomorpism by its set of splitting primes (Korollar VII.13.10 of [Neu06]). We conclude that E_1 and E_2 are isomorphic as field extensions of E'.

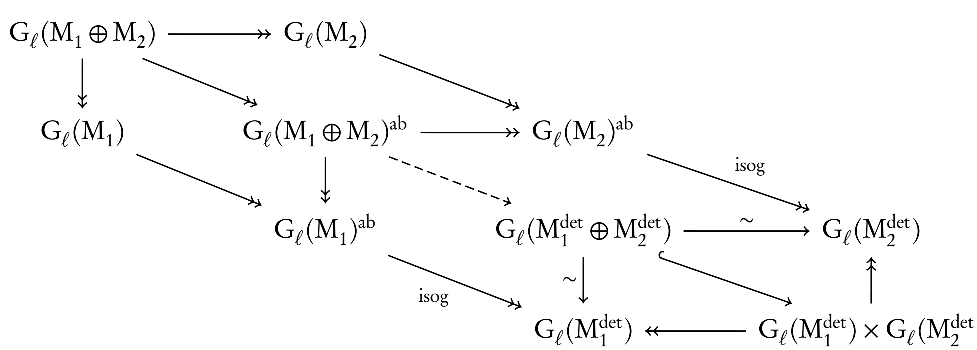
Let τ_i be the distinguished embedding $E_i \hookrightarrow K$ associated with M_i (*cf.* definition 11.8). We will now show that τ_1 and τ_2 have the same image in K, thus providing a canonical isomorphism $E_1 \cong E_2$. Embed K into a p-adic field K_v. Let B_{dR,K_v} be the p-adic period ring in the sense of Fontaine [Fon94], associated with K_v. Then p-adic Hodge theory (see §2.4.3) gives an isomorphism of filtered modules with $\mathrm{Gal}(\bar{K}_v/K_v)$-action

$$B_{dR,K_v} \otimes_K H_{dR}(M_i^{ha}) \cong B_{dR,K_v} \otimes_{\mathbb{Q}_p} H_p(M_i^{ha}).$$

In §16.5 we concluded that $H_p(M_1^{ha})$ is isomorphic to $H_p(M_2^{ha})$ as Galois representation together with the action of E'. Therefore $B_{dR,K_v} \otimes_K H_{dR}(M_1^{ha})$ is isomorphic to $B_{dR,K_v} \otimes_K H_{dR}(M_2^{ha})$ in a way that is compatible with the action of E'.

Note that E' acts on $B_{dR,K_v} \otimes_K \mathrm{Fil}^1 H_{dR}(M_i^{ha})$ via $E' \subset E_i \xhookrightarrow{\tau_i} K \subset B_{dR,K_v}$. The above remarks about p-adic Hodge theory show that $\tau_1|_{E'} = \tau_2|_{E'}$. Since E_1 and E_2 are isomorphic quadratic extensions of E', the image of τ_1 coincides with the image of τ_2. Write E for the image of τ_1 and τ_2 in K. We identify E_i with E via τ_i, and we write τ for the inclusion $E \subset K$. In conclusion, $E \cong \mathrm{End}(M_i)$, and τ is the distinguished embedding $E \hookrightarrow K$ associated with both M_i.

Recall from lemma 11.10 that $M_i^{det} = \bigwedge_E^n M_i$ is an abelian motive of K3 type with representation type $(U, 1)$ and $G_\ell(M_i)^{ab} \to G_\ell(M_i^{det})$ is an isogeny. By construction we have $E = \mathrm{End}(M_i^{det})$, and τ is the distinguished embedding associated with M_i^{det}. Hence $M_1^{det} \cong M_2^{det}$, by lemma 16.11 below. Consider the following diagram



The groups $G_\ell(M_1 \oplus M_2)^{ab}$ and $G_\ell(M_1^{det} \oplus M_2^{det})$ are the image of $G_\ell(M_1 \oplus M_2)$ in respectively $G_\ell(M_1)^{ab} \times G_\ell(M_2)^{ab}$ and $G_\ell(M_1^{det}) \times G_\ell(M_2^{det})$. Therefore the dashed arrow exists. Since the arrows labeled 'isog' are isogenies the dashed arrow is also an isogeny. Finally, this shows that the projection maps $G_\ell(M_1 \oplus M_2)^{ab} \twoheadrightarrow G_\ell(M_i)^{ab}$ are isogenies.

By §16.6 the maps $G_\ell(M_1 \oplus M_2)^{ad} \to G_\ell(M_i)^{ad}$ are isogenies as well, and hence we determine that all in all, the projection maps $G_\ell(M_1 \oplus M_2) \twoheadrightarrow G_\ell(M_i)$ are isogenies, for all prime numbers ℓ.

16.11 Lemma. — *Let* K *be a finitely generated field of characteristic* 0. *Let* M_1 *and* M_2 *be two irreducible abelian motives of K3 type over* K. *Assume that* $G_\ell(M_1 \oplus M_2)$ *is connected and assume that* M_1 *and* M_2 *both have representation type* $(U, 1)$. *Let* E_i *be the endomorphism algebra of* M_i. *If* E_1 *and* E_2 *have the same image* $E \subset K$ *under the distinguished embeddings (cf. definition 11.8) associated with* M_1 *and* M_2, *then* $M_1 \cong M_2$.

Proof. Fix a complex embedding $\sigma: K \hookrightarrow \mathbb{C}$. Since M_1 and M_2 are abelian motives, and since $G_{mot,\sigma}(M_1 \oplus M_2)$ is connected (see lemma 2.12), it suffices to show that $H_\sigma(M_1) \cong H_\sigma(M_2)$, by theorem 5.2.1.

Recall that

$$H_\sigma(M_i) \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\tau: E_i \hookrightarrow \mathbb{C}} H_\sigma(M_i) \otimes_{E_i,\tau} \mathbb{C},$$

and observe that the $H_\sigma(M_i) \otimes_{E_i,\tau} \mathbb{C}$ are complex vector spaces of dimension 1. There is one embedding $\tau: E_i \hookrightarrow \mathbb{C}$ that is distinguished, namely the composition of σ with the distinguished embedding $E_i \hookrightarrow K$ associated with M_i. For this τ we have $H_\sigma(M_i)^{-1,1} \cong H_\sigma(M_i) \otimes_{E_i,\tau} \mathbb{C}$.

Analogously, observe that $E \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\tau: E \hookrightarrow \mathbb{C}} \mathbb{C}^{(\tau)}$. One of the embeddings $\tau: E \hookrightarrow \mathbb{C}$ is distinguished, namely the composition of σ with $E \subset K$. This defines a Hodge structure of K3 type on E, by declaring $E^{-1,1} = \mathbb{C}^{(\tau)}$. This Hodge structure is isomorphic to $H_\sigma(M_i)$ by transport of structure. In particular, $H_\sigma(M_1) \cong H_\sigma(M_2)$, and therefore $M_1 \cong M_2$. □

16.12 — Observe that by now we have deduced $\mathrm{End}(M_1) = \mathrm{End}(M_2)$ in all cases, except for the case that M_1 and M_2 have representation type $(O, 4)_1$. Let us turn our attention to that case. Note that we do know $\mathrm{End}(M_1^{ha}) = E' = \mathrm{End}(M_2^{ha})$, and E' is a quadratic extension of $E_i = \mathrm{End}(M_i)$.

After replacing K by a finite extension, we may and do assume that K contains a subfield $\tilde{E}' \subset K$ that is a normal closure of E'. Let $\Sigma' = \Sigma(E')$ denote the set of embeddings $E' \hookrightarrow \tilde{E}'$. Let p be a prime that is totally split in $\tilde{E}'$. Let C_p denote the completion of $\bar{\mathbb{Q}}_p$, and fix an embedding $K \hookrightarrow C_p$. Via the composed embedding $\tilde{E}' \subset K \hookrightarrow C_p$ we get a correspondence between the embeddings $(E' \hookrightarrow \tilde{E}') \in \Sigma'$ and the places of E' above p.

For $i = 1, 2$, there are precisely two places/embeddings $\pi: E' \hookrightarrow \tilde{E}'$ such that $H_\pi(M_i^{ha}) \otimes C_p$ has a non-trivial Hodge–Tate decomposition; namely, the two places/embeddings that lie above the distinguished embedding $E_i \subset E' \hookrightarrow \tilde{E}' \subset K$. Let $S_i \subset \Sigma'$ be the set of these two places/embeddings, associated with M_i^{ha}. Recall that the Galois group $\mathrm{Gal}(\tilde{E}'/\mathbb{Q})$ acts transitively on Σ'. The subfield of $\tilde{E}'$ that is fixed by the stabiliser of S_i is precisely the field E_i (viewed as subfield of $\tilde{E}'$ via the distinguished embedding).

Since $H_{\Lambda'}(M_1^{ha}) \cong H_{\Lambda'}(M_2^{ha})$ we see that $H_\pi(M_1^{ha}) \otimes C_p$ has a non-trivial Hodge–Tate decomposition if and only if $H_\pi(M_2^{ha}) \otimes C_p$ has a non-trivial Hodge–Tate decomposition. We conclude that $S_1 = S_2$, and therefore $E_1 = E_2$.

16.13 — It is therefore justified to write E for $\mathrm{End}(M_1) = \mathrm{End}(M_2)$ in all cases. We denote the maximal TR subfield of E by E^0. With Λ (resp. Λ^0) we mean the the set of finite places of E (resp. E^0). We have now finished the first three steps of the proof, as listed in §16.4.

For the next two steps we argue as follows: Let p be a prime number that is totally split in E^0. Fix an embedding $K \hookrightarrow C_p$. Let π_i be the distinguished place of E^0 above p associated with M_i. (It is the place $\pi_i : E^0 \hookrightarrow \mathbb{Q}_p \subset C_p$ equal to the composed distinguished embedding $E^0 \hookrightarrow K \hookrightarrow C_p$; and it is the place for which $H_{\pi_i}(M_i) \otimes C_p$ has a non-trivial Hodge–Tate decomposition.) Since $H_p(M_1^{ha}) \cong H_p(M_2^{ha})$, we find that $\pi_1 = \pi_2$. By theorem 15.2, the isogeny $G_\pi(M_1 \oplus M_2) \twoheadrightarrow G_\pi(M_i)$ is an isomorphism. This means that the Galois representations $H_\pi(M_1)$ and $H_\pi(M_2)$ are two faithful representations of $G_\pi(M_1 \oplus M_2)$. By lemma 11.6 they are isomorphic as π-adic Galois representations, and by theorem 8.2 we conclude that $H_{\Lambda^0}(M_1) \cong H_{\Lambda^0}(M_2)$. In particular, the Galois representations $H_\ell(M_1)$ and $H_\ell(M_2)$ are isomorphic for all prime numbers ℓ.

The final step of the proof is to show that $M_1 \cong M_2$. This follows from the following theorem, whose statement and proof is inspired by theorem 1.6.1.4 of [And96a].

16.14 THEOREM. — *Let* M_1 *and* M_2 *be two abelian motives of K3 type over a finitely generated field* K *of characteristic* 0. *Assume that* $G_\ell(M_1 \oplus M_2)$ *is connected for all prime numbers* ℓ. *If there is a prime number* ℓ *such that* $H_\ell(M_1) \cong H_\ell(M_2)$ *as Galois representations, then* $M_1 \cong M_2$ *as motives.*

Proof. Observe that the theorem is true for M_1 and M_2, if and only if it is true for $M_1 \oplus \mathbb{1}$ and $M_2 \oplus \mathbb{1}$. If $N = \dim(M_1) = \dim(M_2)$ is even, replace M_i by $M_i \oplus \mathbb{1}$; so that we may assume that $N = 2n + 1$ is odd. Since $G_\ell(M_1 \oplus M_2)$ is connected for all prime numbers ℓ, we know that $G_{\mathrm{mot},\ell}(M_1 \oplus M_2)$ is invariant under base change by field extensions of K. Thus we may replace K by a finitely generated extension if needed.

We will now employ the Kuga–Satake construction, see section 13. Because we have assumed that $G_\ell(M_1 \oplus M_2)$ is connected, we know that

$$G_{\mathrm{mot},\sigma}(M_1 \oplus M_2) = G_\sigma(M_1 \oplus M_2) = G_B((M_1 \oplus M_2)_\sigma),$$

by lemma 2.12 and theorem 5.2.2. Therefore, the functor $H_B(\_)$ induces an equivalence between the subcategory $\langle (M_1 \oplus M_2)_\sigma \rangle^\otimes$ of motives and the subcategory $\langle H_\sigma(M_1 \oplus M_2) \rangle^\otimes$ of Hodge structures. Hence all the Hodge-theoretic constructions that follow are motivic over $\mathbb{C}$. After passing to a suitable finitely generated extension of K, all these steps are also motivic over K.

For $i = 1,2$, let ϕ_i be a polarisation on M_i. Let Cl_i^+ denote the even Clifford algebra $Cl^+(M_i,\phi_i)$. For the notation $Cl_{i,ad}^+$ and $Cl_{i,spin}^+$ we refer to §13.6. By §13.8 we know that $Cl_{i,spin}^+$ is the motive in degree 1 associated with an abelian variety. Therefore we will suggestively write A_i for the motive $Cl_{i,spin}^+$.

By construction, there is a map $\mathrm{Gal}(\bar{K}/K) \to \mathrm{CSpin}(M_i,\phi_i)(\mathbb{Q}_\ell)$, corresponding with the Galois representation of $H_\ell(A_i) = H_\ell(Cl_{i,spin}^+)$. Let W_i be the spin representation of $\mathrm{CSpin}(M_i,\phi_i)_{\bar{\mathbb{Q}}_\ell}$, which inherits a Galois representation (with coefficients in $\bar{\mathbb{Q}}_\ell$) by composition with the map $\mathrm{Gal}(\bar{K}/K) \to \mathrm{CSpin}(M_i,\phi_i)(\mathbb{Q}_\ell)$ from the previous sentence. Recall that W_i is self-dual of dimension 2^n.

We recall the fundamental isomorphisms of §13.7. There are isomorphisms of motives

$$Cl_{i,ad}^+ \cong \underline{\mathrm{End}}_{Cl_i^+}(A_i)$$
$$Cl_{i,ad}^+ \cong \bigwedge^{2\star} M_i$$

and isomorphisms of Galois representations

$$H_\ell(A_i)_{\bar{\mathbb{Q}}_\ell} \cong W_i^{\oplus 2^n}$$
$$H_\ell(Cl_{i,ad}^+)_{\bar{\mathbb{Q}}_\ell} \cong \underline{\mathrm{End}}_{\bar{\mathbb{Q}}_\ell}(W_i) \cong W_i^{\otimes 2}.$$

It follows from these isomorphisms that a Galois-equivariant isomorphism $H_\ell(M_1) \to H_\ell(M_2)$, yields Galois-equivariant isomorphisms

$$\bigwedge^{2\star} H_\ell(M_1) \to \bigwedge^{2\star} H_\ell(M_2), \qquad H_\ell(Cl_{1,ad}^+) \to H_\ell(Cl_{2,ad}^+), \qquad H_\ell(Cl_{1,ad}^+)_{\bar{\mathbb{Q}}_\ell} \to H_\ell(Cl_{2,ad}^+)_{\bar{\mathbb{Q}}_\ell},$$

and thus $W_1^{\otimes 2} \to W_2^{\otimes 2}$. By applying sublemma 7.3.2 of [And96a] with the group $G_\ell^\circ(W_1 \oplus W_2)$, we find $W_1 \cong W_2$ as representations of $G_\ell^\circ(W_1 \oplus W_2)$, and hence (possibly after a finite extension of K) also as Galois representations. Fix an isomorphism $W_1 \to W_2$. By the four isomorphisms listed above, this induces an isomorphism $H_\ell(A_1)_{\bar{\mathbb{Q}}_\ell} \to H_\ell(A_2)_{\bar{\mathbb{Q}}_\ell}$. Therefore there exists an isomorphism $H_\ell(A_1) \to H_\ell(A_2)$. By Faltings' theorem (Korollar 1 of Satz 4 of [Fal83], see also [Fal84]), we find $A_1 \cong A_2$.

Finally, observe from the Kuga–Satake construction that $A_i^{ha} \cong M_i^{ha}$, and therefore we know $\mathrm{MTC}(A_i)$, by theorem 14.1 and proposition 5.9. This gives $\mathrm{MTC}(A_1 \times A_2)$, since $A_1 \cong A_2$. Recall that $M_i \in \langle A_i \rangle^{\otimes}$, and therefore $M_1 \oplus M_2 \in \langle A_1 \oplus A_2 \rangle^{\otimes}$. This implies $\mathrm{MTC}(M_1 \oplus M_2)$ and together with the assumption $H_\ell(M_1) \cong H_\ell(M_2)$ we find $H_\sigma(M_1) \cong H_\sigma(M_2)$. By theorem 5.2 and our assumption that $G_\ell(M_1 \oplus M_2)$ is connected (see also lemma 2.12) we conclude that $M_1 \cong M_2$. □

17 The Mumford–Tate conjecture for products of abelian motives of K3 type

Readme. — Important: theorem 17.4.

We define a Tannakian subcategory of Mot_K that contains all abelian motives of K3 type, and we prove the Mumford–Tate conjecture for motives in this subcategory. As a corollary, we deduce the Mumford–Tate conjecture for products of K3 surfaces.

17.1 — Let K be a finitely generated field of characteristic 0. If L/K is a finitely generated field extension, let $\mathscr{C}_L$ denote the collection of motives M^{ha}, where M is an abelian motive of K3 type over L and $G_\ell(M)$ is connected for all prime numbers ℓ. Let $\mathscr{M}_K^{K3}$ be the full subcategory of abelian motives M over K for which M_L^{ha} is a sum of motives in $\mathscr{C}_L$, for some finitely generated field extension L/K.

17.2 Lemma. — *The category $\mathscr{M}_K^{K3}$ is a Tannakian subcategory of* Mot_K.

Proof. By lemma 4.8 we see that $\mathscr{M}_K^{K3}$ is closed under direct sums. Also, if M is a motive in $\mathscr{M}_K^{K3}$, and M′ is a motive in $\langle M\rangle^{\otimes}$, then remark 4.7.2 shows that M'^{ha} is a quotient (and thus a direct summand) of M^{ha}. Hence $\langle M\rangle^{\otimes} \subset \mathscr{M}_K^{K3}$, and we conclude that $\mathscr{M}_K^{K3}$ is closed under tensor products, duals, and subquotients; in other words, it is a Tannakian subcategory of Mot_K. □

17.3 Lemma. — *Let* K *be a finitely generated field of characteristic* 0. *The following motives are elements of* $\mathscr{M}_K^{K3}$.

1. H(A), *where* A *is an abelian surface over* K.
2. H(E), *where* E *is an elliptic curve over* K.
3. H(X), *where* X *is a K3 surface over* K.
4. H(X), *where* X *is a cubic fourfold over* K.

Proof.
1. It suffices to show that $H^1(A) \in \mathscr{M}_K^{K3}$. Assume that $G_\ell(H^1(A))$ is connected for all prime numbers ℓ. Note that $G_\ell(H^1(A))$ surjects onto $G_\ell(H^2(A))$ with finite kernel. Therefore $H^1(A)^{ha} \cong H^2(A)^{ha}$. We are done, since $H^2(A)(1)$ is a motive of K3 type.
2. Note that $H^1(E \times E) \cong H^1(E)^{\oplus 2}$, and the result follows from the previous point.
3. Recall from example 11.5 that $H^2(X)$ is an abelian motive.
4. Recall from example 11.5 that $H^4(X)$ is an abelian motive of K3 type. □

17.4 Theorem. — *Let* K *be a finitely generated field of characteristic* 0. *Let* M *be a motive in* $\mathscr{M}_K^{K3}$. *Then* MTC(M) *is true.*

Proof. By proposition 5.9 it suffices to prove $\mathrm{MTC}(M^{ha})$. By lemma 3.4, we may replace K with a finitely generated extension, and therefore we may assume that $G_\ell(M)$ is connected for all prime numbers ℓ and that M^{ha} is a sum of motives in $\mathscr{C}_K$. Let $(M_i)_{i\in I}$ be a finite collection of motives of K3 type such that $G_\ell(M_i)$ is connected for all prime numbers ℓ, and such that $M^{ha} \cong \bigoplus_{i\in I} M_i^{ha}$. By §12.4 we may assume that for all $i \in I$, the representation type of M_i is neither $(U,1)$, nor $(U,2)$, nor $(O,4)_2$. Finally, we may assume that the M_i are pairwise non-isomorphic.

Observe that $MTC(M^{ha})$ is implied by $MTC(\bigoplus_{i\in I} M_i)$. We will now prove $MTC(\bigoplus_{i\in I} M_i)$. By theorem 16.1, we know that if $i, j \in I$ are two different indices, then $G_\ell(M_i \oplus M_j) \cong G_\ell(M_i) \times G_\ell(M_j)$. Recall that $G_\ell(M) \hookrightarrow \prod_{i\in I} G_\ell(M_i)$, with surjective projections on to the factors $G_\ell(M_i)$. By the lemma in step 3 on pages 790–791 of [Rib76] we conclude that $G_\ell(\bigoplus_{i\in I} M_i) \cong \prod_{i\in I} G_\ell(M_i)$. Now consider the following diagram.

$$\begin{array}{ccc} G_\ell(\bigoplus_{i\in I} M_i) & \xrightarrow{\simeq} & \prod_{i\in I} G_\ell(M_i) \\ \big\downarrow & & \big\downarrow \simeq \\ G_\sigma(\bigoplus_{i\in I} M_i) \otimes \mathbb{Q}_\ell & \hookrightarrow & \prod_{i\in I} G_\sigma(M_i) \otimes \mathbb{Q}_\ell \end{array}$$

The vertical arrows exist by theorem 5.2.2, and the arrow on the right is an isomorphism since we know $MTC(M_i)$, by theorem 14.1. We conclude that $MTC(\bigoplus_{i\in I} M_i)$ holds. Therefore $MTC(M^{ha})$ and $MTC(M)$ are true. □

17.5 COROLLARY. — *Let* K *be a finitely generated field of characteristic* 0. *Let* X/K *be a product of elliptic curves, abelian surfaces, K3 surfaces, and cubic fourfolds. Then the Mumford–Tate conjecture is true for* $H^i(X)$, *for all* $i \geq 0$.

Bibliography

[And96a] Yves André. “On the Shafarevich and Tate conjectures for hyper-Kähler varieties”. *Mathematische Annalen* 305.2 (1996), pp. 205–248.

[And96b] Yves André. “Pour une théorie inconditionnelle des motifs”. *Institut des Hautes Études Scientifiques. Publications Mathématiques* 83 (1996), pp. 5–49.

[Bor98] Mikhail Borovoi. “Abelian Galois cohomology of reductive groups”. *Memoirs of the American Mathematical Society* 132.626 (1998), pp. viii+50.

[Bou07] Nicolas Bourbaki. *Éléments de mathématique. Algèbre. Chapitre 9*. Reprint of the 1959 original. Springer-Verlag, Berlin, 2007, p. 211.

[Bou12] Nicolas Bourbaki. *Éléments de mathématique. Algèbre. Chapitre 8. Modules et anneaux semi-simples*. Second revised edition of the 1958 edition. Springer, Berlin, 2012, pp. x+489.

[Bou81] Nicolas Bourbaki. *Éléments de mathématique. Groupes et algèbres de Lie. Chapitres 4, 5 et 6*. Masson, Paris, 1981, p. 290.

[Bus15] Nikolay Buskin. *Every rational Hodge isometry between two K3 surfaces is algebraic*. 2015. arXiv: `1510.02852v1`.

[Chi92] Wên Chên Chi. “ℓ-adic and λ-adic representations associated to abelian varieties defined over number fields”. *American Journal of Mathematics* 114.2 (1992), pp. 315–353.

[CM15] Anna Cadoret and Ben Moonen. *Integral and adelic aspects of the Mumford–Tate conjecture*. 2015. arXiv: `1508.06426`.

[Del+82] Pierre Deligne et al. *Hodge cycles, motives, and Shimura varieties*. Vol. 900. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1982, pp. ii+414.

[Del72] Pierre Deligne. “La conjecture de Weil pour les surfaces K3”. *Inventiones Mathematicae* 15 (1972), pp. 206–226.

[Del74a] Pierre Deligne. “La conjecture de Weil. I”. *Institut des Hautes Études Scientifiques. Publications Mathématiques* 43 (1974), pp. 273–307.

[Del74b] Pierre Deligne. “Théorie de Hodge. III”. *Institut des Hautes Études Scientifiques. Publications Mathématiques* 44 (1974), pp. 5–77.

[Fal83] Gerd Faltings. “Endlichkeitssätze für abelsche Varietäten über Zahlkörpern”. *Inventiones Mathematicae* 73.3 (1983), pp. 349–366.

[Fal84] Gerd Faltings. “Complements to Mordell”. In: *Rational points (Bonn, 1983/1984)*. Aspects Math., E6. Friedr. Vieweg, Braunschweig, 1984, pp. 203–227.

[Fal88] Gerd Faltings. “p-adic Hodge theory”. *Journal of the American Mathematical Society* 1.1 (1988), pp. 255–299.

[Fal89] Gerd Faltings. “Crystalline cohomology and p-adic Galois-representations”. In: *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*. Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 25–80.

[Fon94] Jean-Marc Fontaine. “Le corps des périodes p-adiques”. *Astérisque* 223 (1994). With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988), pp. 59–111.

[Gee01] Bert van Geemen. “Half twists of Hodge structures of CM-type”. *Journal of the Mathematical Society of Japan* 53.4 (2001), pp. 813–833.

[Gee08] Bert van Geemen. “Real multiplication on K3 surfaces and Kuga–Satake varieties”. *Michigan Mathematical Journal* 56.2 (2008), pp. 375–399.

[Gro66] Alexander Grothendieck. “On the de Rham cohomology of algebraic varieties”. *Institut des Hautes Études Scientifiques. Publications Mathématiques* 29 (1966), pp. 95–103.

[Huy16] Daniel Huybrechts. *Lectures on K3 surfaces*. Vol. 158. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2016, pp. xi+485.

[Kis17] Mark Kisin. “Mod p points on Shimura varieties of abelian type”. *Journal of the American Mathematical Society* (2017). Published electronically: January 11, 2017.

[KM74] Nicholas Katz and William Messing. "Some consequences of the Riemann hypothesis for varieties over finite fields". *Inventiones Mathematicae* 23 (1974), pp. 73–77.

[KS67] Michio Kuga and Ichirô Satake. "Abelian varieties attached to polarized K3-surfaces". *Mathematische Annalen* 169 (1967), pp. 239–242.

[Las14] Abhijit Laskar. "ℓ-independence for a system of motivic representations". *Manuscripta Mathematica* 145.1-2 (2014), pp. 125–142.

[LP92] Michael Larsen and Richard Pink. "On ℓ-independence of algebraic monodromy groups in compatible systems of representations". *Inventiones Mathematicæ* 107.3 (1992), pp. 603–636.

[Moo16] Ben Moonen. "On the Tate and Mumford–Tate conjectures in codimension one for varieties with $h^{2,0} = 1$". *Duke Mathematical Journal* (2016), pp. 739–799.

[Neu06] Jürgen Neukirch. *Algebraische Zahlentheorie*. Berlin: Springer-Verlag, 2006.

[Noo09] Rutger Noot. "Classe de conjugaison du Frobenius d'une variété abélienne sur un corps de nombres". *Journal of the London Mathematical Society. Second Series* 79.1 (2009), pp. 53–71.

[Noo13] Rutger Noot. "The system of representations of the Weil–Deligne group associated to an abelian variety". *Algebra & Number Theory* 7.2 (2013), pp. 243–281.

[Noo96] Rutger Noot. "Models of Shimura varieties in mixed characteristic". *Journal of Algebraic Geometry* 5.1 (1996), pp. 187–207.

[Pin97] Richard Pink. "The Mumford–Tate conjecture for Drinfeld-modules". *Kyoto University. Research Institute for Mathematical Sciences. Publications* 33.3 (1997), pp. 393–425.

[Pin98] Richard Pink. "ℓ-adic algebraic monodromy groups, cocharacters, and the Mumford–Tate conjecture". *Journal für die Reine und Angewandte Mathematik* 495 (1998), pp. 187–237.

[Poh68] Henry Pohlmann. “Algebraic cycles on abelian varieties of complex multiplication type”. *Annals of Mathematics. Second Series* 88 (1968), pp. 161–180.

[Ram08] José Ramón Marí. “On the Hodge conjecture for products of certain surfaces”. *Universitat de Barcelona. Collectanea Mathematica* 59.1 (2008), pp. 1–26.

[Rib76] Kenneth Ribet. “Galois action on division points of Abelian varieties with real multiplications”. *American Journal of Mathematics* 98.3 (1976), pp. 751–804.

[Rib90] Kenneth Ribet. “Book Review: Abelian l-adic representations and elliptic curves”. *American Mathematical Society. Bulletin. New Series* 22.1 (1990), pp. 214–218.

[Sch88] Norbert Schappacher. *Periods of Hecke characters*. Vol. 1301. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1988, pp. xvi+160.

[Ser12] Jean-Pierre Serre. *Lectures on* $\mathrm{N}_X(p)$. Vol. 11. Chapman & Hall/CRC Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2012, pp. x+163.

[Ser13] Jean-Pierre Serre. *Œuvres/Collected papers. IV. 1985–1998*. Springer Collected Works in Mathematics. Springer, Heidelberg, 2013, pp. viii+694.

[Ser65] Jean-Pierre Serre. “Zeta and L-functions”. In: *Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)*. Harper & Row, New York, 1965, pp. 82–92.

[Ser79] Jean-Pierre Serre. “Groupes algébriques associés aux modules de Hodge-Tate”. In: *Journées de Géométrie Algébrique de Rennes. (Rennes, 1978), Vol. III*. Vol. 65. Astérisque. Soc. Math. France, Paris, 1979, pp. 155–188.

[Ser98] Jean-Pierre Serre. *Abelian l-adic representations and elliptic curves*. Vol. 7. Research Notes in Mathematics. With the collaboration of Willem Kuyk and John Labute, Revised reprint of the 1968 original. A K Peters, Ltd., Wellesley, MA, 1998, p. 199.

[SGA4-3] *Théorie des topos et cohomologie étale des schémas. Tome 3*. Lecture Notes in Mathematics, Vol. 305. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J.L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat. Springer-Verlag, Berlin-New York, 1973, pp. vi+640.

[Shi67] Goro Shimura. “Algebraic number fields and symplectic discontinuous groups”. *Annals of Mathematics. Second Series* 86 (1967), pp. 503–592.

[ST68] Jean-Pierre Serre and John Tate. “Good reduction of abelian varieties”. *Annals of Mathematics. Second Series* 88 (1968), pp. 492–517.

[Tan90] Sergey Tankeev. “Surfaces of K3 type over number fields and the Mumford–Tate conjecture”. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 54.4 (1990), pp. 846–861.

[Tan95] Sergey Tankeev. “Surfaces of K3 type over number fields and the Mumford–Tate conjecture. II”. *Rossiĭskaya Akademiya Nauk. Izvestiya. Seriya Matematicheskaya* 59.3 (1995), pp. 179–206.

[UY13] Emmanuel Ullmo and Andrei Yafaev. “Mumford–Tate and generalised Shafarevich conjectures”. *Annales Mathématiques du Québec* 37.2 (2013), pp. 255–284.

[Vas08] Adrian Vasiu. “Some cases of the Mumford–Tate conjecture and Shimura varieties”. *Indiana University Mathematics Journal* 57.1 (2008), pp. 1–75.

[Zar83] Yuri Zarhin. “Hodge groups of K3 surfaces”. *Journal für die Reine und Angewandte Mathematik* 341 (1983), pp. 193–220.

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Finally, I thank my heavenly Father. The ancient giant Atlas is said to lift the skies on his shoulders; but my awesome God carries me, and all of his creation, in the palm of his hand.

Soli Deo Gloria.

Samenvatting

Zij K een eindig voortgebracht lichaam van karakteristiek 0. In dit proefschrift spelen motieven, in de zin van André [And96b], een sleutelrol. Zij M een motief over K. Als $\sigma : K \hookrightarrow \mathbb{C}$ een complexe inbedding van K is, dan noteren we met $H_\sigma(M)$ de Hodge-realisatie van M_σ. De $\mathbb{Q}$-vectorruimte $H_\sigma(M)$ is op canonieke wijze voorzien van een Hodgestructuur. Als ℓ een priemgetal is, dan noteren we met $H_\ell(M)$ de ℓ-adische étale-realisatie van M. De $\mathbb{Q}_\ell$-vectorruimte $H_\ell(M)$ is op canonieke wijze voorzien van een representatie van de Galoisgroep $\mathrm{Gal}(\bar{K}/K)$. Zij $\bar{\sigma} : \bar{K} \hookrightarrow \mathbb{C}$ een inbedding, en schrijf σ voor de compositie van $\bar{\sigma}$ met de inclusie $K \subset \bar{K}$. Zij ℓ een priemgetal. De vergelijkingsstelling van Artin geeft een canoniek isomorfisme $H_\sigma(M) \otimes \mathbb{Q}_\ell \cong H_\ell(M)$ van $\mathbb{Q}_\ell$-vectorruimte.

Dit proefschrift heeft twee nauw verwante hoofdthemas. Het eerste betreft de vraag in welke mate de Galoisrepresentaties op de vectorruimten $H_\ell(M)$ gemeenschappelijke structuur hebben wanneer men ℓ laat variëren over de priemgetallen. Serre heeft de notie van een *compatibel systeem van Galoisrepresentaties* ingevoerd. Het is een gevolg van Deligne's bewijs van de Weil-vermoedens dat de Galoisrepresentaties $H_\ell(M)$ een dergelijk compatibel systeem vormen wanneer M een motief is van de vorm $H^i(X)$, waarbij X een gladde projectieve variëteit is over K.

Neem aan dat het motief M een abels motief is. Zij E een deellichaam van End(M), en zij Λ de verzameling van eindige plaatsen van E. Dankzij werk van Deligne en André is de E-actie op $H_\ell(M)$ Galois-equivariant, omdat M een abels motief is. Dit maakt $H_\ell(M)$ tot een vrij moduul over $E \otimes \mathbb{Q}_\ell = \prod_{\lambda|\ell} E_\lambda$, hetgeen een decompositie $H_\ell(M) = \bigoplus_{\lambda|\ell} H_\lambda(M)$ van Galoisrepresentaties geeft. We noteren met $H_\Lambda(M)$ het systeem van de λ-adische Galoisrepresentaties $H_\lambda(M)$ waarbij λ varieert over Λ.

In dit proefschrift wordt de notie van een *quasicompatibel systeem van Galoisrepresentaties* geïntroduceerd, een lichte verzwakking van de compatibiliteitseis van Serre. Het eerste hoofdresultaat van dit proefschrift luidt als volgt:

> Stelling. *Zij* M *een abels motief over een eindig voortgebracht lichaam van karakteristiek* 0. *Zij* E *een deellichaam van* End(M), *en zij* Λ *de verzameling van eindige plaatsen van* E. *Dan is het systeem* $H_\Lambda(M)$ *een quasicompatibel systeem van Galoisrepresentaties.*

Het tweede hoofdthema betreft de vraag in welke mate het vergelijkingsisomorfisme van Artin compatibel is met de Hodgestructuur op $H_\sigma(M)$ en de Galoisrepresentatie op $H_\ell(M)$. Het Mumford–Tate vermoeden maakt deze vraag precies.

Om het Mumford–Tate vermoeden te formuleren hebben we eerst meer definities en notaties nodig. De Hodgestructuur op $H_\sigma(M)$ wordt volledig bepaald door een representatie $\mathbb{S} \to \mathrm{GL}(H_\sigma(M))_\mathbb{R}$, waarbij $\mathbb{S}$ de Deligne-torus $\mathrm{Res}_\mathbb{R}^\mathbb{C}\mathbb{G}_m$ weergeeft. De *Mumford–Tate groep* $G_\sigma(M)$ is de kleinste algebraïsche ondergroep $G \subset \mathrm{GL}(H_\sigma(M))$ over $\mathbb{Q}$, zodanig dat $G_\mathbb{R}$ het beeld van $\mathbb{S}$ bevat.

Met $G_\ell(M)$ noteren we de Zariski-afsluiting van het beeld van $\mathrm{Gal}(\bar{K}/K)$ in $\mathrm{GL}(H_\ell(M))$. Dit is een algebraïsche groep over $\mathbb{Q}_\ell$. We schrijven $G_\ell^\circ(M)$ voor de samenhangscomponent van de eenheid van $G_\ell(M)$. Het Mumford–Tate vermoeden zegt dat het vergelijkingsisomorfisme van Artin de groep $G_\sigma(M)_{\mathbb{Q}_\ell}$ identificeerd met $G_\ell^\circ(M)$.

Een motief M over het eindig voortgebrachte lichaam K is van *K3-type* als voor een (en dus alle) inbeddingen $\sigma\colon K \hookrightarrow \mathbb{C}$ de Hodgestructuur $H_\sigma(M)$ van K3-type is, wat wil zeggen dat $H_\sigma(M)$ van gewicht 0 is, en $H_\sigma(M)^{p,q} = 0$ voor $p < -1$, en $\dim H_\sigma(M)^{-1,1} = 1$. K3-oppervlakken geven een natuurlijke klasse van voorbeelden van motieven van K3-type: Als X een K3-oppervlak over K, dan is het motief $H^2(X)(1)$ een motief van K3-type, en André heeft laten zien dat $H^2(X)(1)$ ook een abels motief is.

Dit brengt ons bij het tweede hoofdresultaat van dit proefschrift.

STELLING. *Zij* K *een eindig voortgebracht lichaam van karakteristiek* 0. *Zij* $n > 0$ *een geheel getal, en laten* $M_1, \ldots, M_n$ *abelse motieven van K3-type over* K *zijn. Dan is het Mumford–Tate vermoeden waar voor* $M_1 \oplus \ldots \oplus M_n$.

Als onmiddellijk gevolg van deze stelling leidt men af dat het Mumford–Tate vermoeden geldt voor willekeurige eindige producten van K3-oppervlakken over K.

Curriculum vitae

Op 4 september 1990 zag Johan Commelin het levenslicht in Ditsobotla, Zuid-Afrika. Enkele weken later zette hij voet op Nederlandse bodem om met zijn ouders in Zevenhuizen (Zuid-Holland) te gaan wonen. Nadat hij leerde lezen en schrijven op basisschool De Eendracht vertrok hij op zesjarige leeftijd naar Zimbabwe; zijn vader ging daar als tropenarts werken in een lokaal ziekenhuis. In Zimbabwe heeft Johan op drie basisscholen gezeten: eerst een halfjaar op Petra School in Bulawayo, vervolgens een jaar op St. Davids Primary School in Bonda, om ten slotte nog twee jaar naar de kostschool Highveld in Rusape te gaan. In deze periode leerde Johan wat een priemgetal is, en dat er zelfs een even priemgetal bestaat! Op tienjarige leeftijd keerde Johan met zijn ouders en broers en zus terug naar Nederland, om twee jaar in Waddinxveen te wonen. In die tijd ging hij naar de Eben–Haëzer basisschool in Boskoop. Het laatste halfjaar van de basisschool heeft hij in Wezep aan de Oranjeschool voltooid.

Na deze roerige periode met vele verhuizingen volgde een periode van relatieve rust. Tijdens zijn tienerjaren heeft Johan in Wezep gewoond, en volgde hij een gymnasiumcurriculum aan de Pieter Zandt Scholengemeenschap te Kampen. In de bovenbouw ontdekte hij de twee liefdes van zijn leven: zijn toekomstige vrouw, Grietje, die behalve Grieks en Latijn erg weinig vakken met hem gemeen had; en wiskunde. Met veel plezier heeft hij twee jaar lang deelgenomen aan trainingen voor de Nederlandse Wiskunde Olympiade.

In 2008 begon Johan aan een studie wiskunde in Leiden en een studie theologie in Apeldoorn. Na drie jaar heeft hij de studie theologie afgebroken en zich volledig gericht op zijn master wiskunde. In het najaar van 2010 trad Johan Commelin in het huwelijk met Grietje Troost en een jaar later ontvingen zij op derde kerstdag hun oudste dochter: Hannah. Samen vertrokken ze voor een halfjaar naar Papua Nieuw Guinea, waar Grietje onderzoek verrichtte voor haar masterscriptie. Op deze exotische locatie werkte Johan aan zijn masterscriptie onder begeleiding van dr. Robin de Jong. Eenmaal terug in Nederland zijn Johan en Grietje in Ede gaan wonen, waar later Boaz en Judith geboren werden.

Sinds 2013 is Johan als promovendus verbonden aan de Radboud Universiteit Nijmegen. Daar heeft hij onder begeleiding van prof. dr. Ben Moonen onderzoek gedaan naar abelse motieven en het Mumford–Tate vermoeden. Het resultaat van dit onderzoek kunt u in dit boekje lezen.

Vanaf augustus 2017 gaat Johan als onderzoeker werken in de vakgroep van prof. dr. Carel Faber aan de Universiteit Utrecht.

Colophon. — This thesis is typeset using the microtypographic capabilities of pdfLaTeX and the `microtype` package. The text is set in 10pt Garamond with 15pt leading, and the height of text block is 36 times the leading. An attempt has been made to align the baselines of the main text to the grid of 36 lines on the text block. The page design—inspired by the Van de Graaf canon—is constructed as follows: (*i*) the margins and the text block adhere to the following ratios: $(m_{\text{top}} : h_{\text{text}} : m_{\text{bottom}}) = (1:9:2) = (m_{\text{inner}} : w_{\text{text}} : m_{\text{outer}})$; and (*ii*) the width of the page is equal to the height of the text block. Consequently $(w_{\text{page}} : h_{\text{page}}) = (3:4) = (w_{\text{text}} : h_{\text{text}})$.
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