Crystal basis

- Kashiwara (90's)
- Lusztig (90's) "canonical basis"

Sources: JC Jantzen "Lectures on Quantum Groups
J. Hong, S.J. Kang "Introduction to Quantum Groups and Crystal basis"

(also Chari - Pressley, Joseph, Lusztig books)

In sl_2 or $U_q(sl_2)$ operators e, f map a basis element of a representation (f.d. irrep or Verma module)

Want similar for other $U_q(\mathfrak{g})$ in order to obtain various info (eg branching rules) in terms of these data.

Can be obtained as combinatorial data.

$\mathcal{U}_q(\mathfrak{g})$: standard

- stick to simply laced case, algebra over $\mathbb{C}(q)$ (or $\mathbb{Q}(q)$)
- e_i, f_i $(i \in I)$

$$\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i$$
$$\Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i$$

<u>Notation</u> $e_i^{(k)} = \dfrac{e_i^k}{[k]!}$, $[k]! = [1] \cdots [k]$, $[r] = \dfrac{q^r - q^{-r}}{q - q^{-1}}$

and $f_i^{(k)} = \dfrac{f_i^k}{[k]!}$

M finite dimensional modules (can be extended to integrable)

$M = \bigoplus_\lambda M_\lambda$ decomposition in weight spaces (type 1)

$$M_\lambda = \{ v \in M \mid K_h \cdot v = q^{\lambda(h)} v \quad \forall h \}$$

$(\lambda \in P,\ h \in P^\vee$ or part of it$)$

Kashiwara operators on M.

$i \in I$, and $\mathcal{U}_q(sl_2) = \langle e_i, f_i, K_i^{\pm} \rangle \subset \mathcal{U}_q(\mathfrak{g})$

and $M = \bigoplus_j n_j M(j)$ as $\mathcal{U}_q(sl_2)$-module. Decomposition is given by considering $\ker(e_i) \cap M_\lambda$ as highest weight vectors. $M(j)$ type 1 module of dim $j+1$.

For $u \in M_\lambda$ write $u = \sum_{k \geq 0} f_i^{(k)} u_k$ with $u_k \in \ker e_i \cap M_{\lambda + k\alpha_i}$

(note: need $\lambda(h_i) + k \geq 0$)

<u>Lemma</u>: u_k's are uniquely determined.

<u>Pf</u> Let

$$0 = u_0 + f_i u_1 + \cdots + f_i^{(N)} u_N, \quad u_k \in M_{\lambda + k\alpha_i} \cap \ker e_i$$

<u>Claim</u> $u_0 = u_1 = \cdots = u_N = 0$

By induction wrt N.

$N = 0$: $0 = u_0$ ✓

For $N>0$ we get

$$0 = e_i \cdot 0 = \underbrace{e_i u_0}_{=0} + e_i f_i u_1 + \cdots + e_i f_i^{(N)} u_N$$

and use

$$e_i f_i^{(k)} = f_i^{(k-1)} \cdot \frac{K_i q^{1-k} - K_i^{-1} q^{k-1}}{q-q^{-1}} \qquad (\text{induction on } k)$$

$$\Rightarrow \quad 0 = \frac{q^{\lambda(h_i)+2} - q^{-\lambda(h_i)-2}}{q-q^{-1}} u_1 + \cdots + f_i^{(N-1)} \frac{q^{\lambda(h_i)+2N+1-N} - q^{-\lambda(h_i)-2N-1+N}}{q-q^{-1}} u_N$$

(use $\alpha_j(h_i) = a_{ij}$, so $K_i u_r = q^{(\lambda+r\alpha_i)(h_i)} u_r = q^{\lambda(h_i)+2r} u_r$)

Induction:

$$\underbrace{[\lambda(h_i)+2]}_{>0} u_1 = \cdots = \underbrace{[\lambda(h_i)+N+1]}_{>0} u_N = 0$$

$$\Rightarrow \quad u_1 = \cdots = u_N \quad \text{and} \quad 0 = u_0 . \quad \boxtimes$$

Def (Kashiwara operator) $\tilde{e}_i, \tilde{f}_i : M \to M$

$M_\lambda \ni u = \sum_{k \geq 0} f_i^{(k)} \cdot u_k$, $u_k \in M_{\lambda + k\alpha_i} \cap \ker e_i$

then

$$\tilde{e}_i u := \sum_{k\geq 1} f_i^{(k-1)} \cdot u_k, \quad \tilde{f}_i u = \sum_{k\geq 0} f_i^{(k+1)} u_k$$

(and extended linearly to $M = \bigoplus_\lambda M_\lambda$.)

Proposition

(i) $\tilde{e}_i M_\lambda = e_i M_\lambda \subset M_{\lambda+\alpha_i}$, $\tilde{f}_i M_\lambda = f_i M_\lambda \subset M_{\lambda-\alpha_i}$

(ii) $\tilde{e}_i, \tilde{f}_i$ commute with $U_q(\mathfrak{g})$-module homomorphisms.

Pf:

(i) show that $f_i u \in \tilde{f}_i M_\lambda$ and $\tilde{f}_i u \in f_i M_\lambda$ (and similarly for e_i)

(ii) follows from uniqueness of decomposition. $\boxtimes$

$$A_0 := \{ f \in \mathbb{C}(q) \mid f \text{ regular at } q=0 \}$$
$$= \{ f = \tfrac{g}{h} \mid g, h \in \mathbb{C}[q],\ h(0) \neq 0 \}$$

Def M $U_q(\mathfrak{g})$-module.
A free A_0-submodule $\mathcal{L}$ of M is a **crystal lattice** if

i) $\mathcal{L}$ generates M as vector space over $\mathbb{C}(q)$; $\mathcal{L} \otimes_{A_0} \mathbb{C}(q) \cong M$

ii) with $\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$ we have $\mathcal{L} = \bigoplus_\lambda \mathcal{L}_\lambda$

iii) $\tilde{e}_i \mathcal{L} \subset \mathcal{L}$, $\tilde{f}_i \mathcal{L} \subset \mathcal{L}$ $\forall i \in I$

Note: $\mathcal{L}_\lambda \otimes_{A_0} \mathbb{C}(q) \cong M_\lambda$ for weight λ.

J_0 maximal ideal in A_0 generated by q

$$A_0/J_0 \xrightarrow{\cong} \mathbb{C}, \quad f(q) + J_0 \longmapsto f(0)$$

$$\mathcal{L}/q\mathcal{L} = \mathcal{L}/J_0\mathcal{L} \xrightarrow{\cong} \mathbb{C} \otimes_{A_0} \mathcal{L}$$

($z \cdot f = z f(0)$ action of f on $\mathbb{C}$)

Def M $U_q(\mathfrak{g})$-module has a **crystal basis** if there exists a pair $(\mathcal{L}, \mathcal{B})$ so that

i) $\mathcal{L}$ crystal lattice for M

ii) $\mathcal{B}$ is a $\mathbb{C}$-basis for $\mathcal{L}/q\mathcal{L}$

iii) $\mathcal{B} = \bigcup_\lambda \mathcal{B}_\lambda$, $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}_\lambda / q\mathcal{L}_\lambda)$

iv) $\tilde{e}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$, $\tilde{f}_i \mathcal{B} \subset \mathcal{B} \cup \{0\}$

v) $\forall i \in I$ $\forall b, b' \in \mathcal{B}$ $\tilde{f}_i b = b' \iff \tilde{e}_i b' = b$.

NB $\tilde{e}_i, \tilde{f}_i$ preserve $\mathcal{L}$ and are linear (over $\mathbb{C}(q)$) it follows that
$$\tilde{e}_i (q\mathcal{L}) \subset q\mathcal{L} \implies \tilde{e}_i : \mathcal{L}/q\mathcal{L} \to \mathcal{L}/q\mathcal{L} \text{ well defined.}$$

Example
$\mathfrak{g} = \mathfrak{sl}_2$ $V(m)$ $m+1$-dim. module $eu = 0$, $Ku = q^m u$.
has basis
$$\{u, fu, f^{(2)}u, \ldots, f^{(m)}u\}$$

then $\mathcal{L} = \bigoplus_{k=0}^{m} A_0 (f^{(k)} u)$, $\mathcal{B} = \{ \bar{u}, \overline{fu}, \dots, \overline{f^{(m)}u} \}$

($\bar{v}$ is image of $v \in \mathcal{L}$ to $\bar{v} \in \mathcal{L}/q\mathcal{L}$)

<u>Check</u>: $\tilde{f} f^{(k)} u = f^{(k+1)} u$, $\tilde{e} f^{(k)} u = f^{(k-1)} u$ $(k \geq 1)$

(note $f \cdot f^{(k)} u = \frac{[k+1]!}{[k]!} f^{(k+1)} u = \underbrace{\frac{q^{k+1} - q^{-k-1}}{q^k - q^{-k}}}_{\frac{q^{2k+1} - q^{-1}}{q^{2k} - 1}} f^{(k+1)} u$)

$\frac{q^{2k+1} - q^{-1}}{q^{2k} - 1}$ is not specialisable.

<u>Def</u> M with crystal bases $(\mathcal{L}_1, \mathcal{B}_1)$, $(\mathcal{L}_2, \mathcal{B}_2)$ are isomorphic if there exists

$$\psi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2 \quad A_0\text{-linear isomorphism}$$

so that

- ψ commutes with the action of $\tilde{e}_i, \tilde{f}_i$ $\forall i$
- $\bar{\psi} : \mathcal{L}_1 / q\mathcal{L}_1 \longrightarrow \mathcal{L}_2 / q\mathcal{L}_2$ induces a bijection

  $\bar{\psi} : \mathcal{B}_1 \cup \{0\} \longrightarrow \mathcal{B}_2 \cup \{0\}$ commuting with $\tilde{e}_i, \tilde{f}_i$ on $\mathcal{B}_1, \mathcal{B}_2$

<u>Notation</u> Crystal graph : coloured graph

- nodes : elements from $\mathcal{B}$
- edge with colour $i \in I$ from b to b' if $\tilde{f}_i : b \longmapsto b'$.

<u>Example</u>

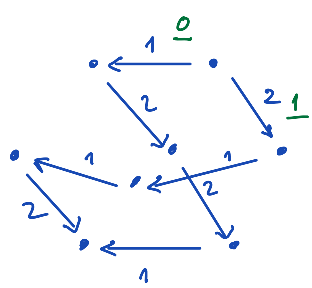
$V(m)$, $U_q(sl_2)$ $m+1$ nodes

$$\bullet \to \bullet \to \bullet \to \bullet \quad \cdots \quad \bullet \to \bullet \qquad (\text{only } 1 \text{ colour})$$

$U_q(sl_3)$, fundamental repr.

$$\bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet$$

$U_q(sl_3)$ adjoint repr



(Stein)

<u>Theorem</u> $U_q(\mathfrak{g})$, $\lambda \in P^+$. Then $V(\lambda)$ has a crystal basis $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ which is unique up to isomorphism.

Can actually take for v_λ h.w.

$\mathcal{L}(\lambda)_\mu$ A_0-span of non-zero vectors of the form

$\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda$ with $\lambda - (\alpha_{i_1} + \dots + \alpha_{i_r}) = \mu$.

and $\mathcal{L}(\lambda) = \bigoplus_\mu \mathcal{L}(\lambda)_\mu$

$$\mathcal{B}(\lambda)_\mu = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} v_\lambda + q\mathcal{L}(\lambda) \mid \lambda - \sum_{k=1}^{r} \alpha_{i_k} = \mu, \text{ non-zero} \}$$

$$\mathcal{B}(\lambda) = \bigcup \mathcal{B}(\lambda)_\mu .$$

How to calculate crystal graphs?

Thm M_i $(i=1,2)$ with crystal basis $(\mathcal{L}_i, \mathcal{B}_i)$ $(i=1,2)$

the $(\mathcal{L}_1 \otimes_{A_0} \mathcal{L}_2 , \mathcal{B}_1 \times \mathcal{B}_2)$ crystal basis for $M_1 \otimes M_2$ $C(q)$

and the action of Kashiwara operators is given by

$$\tilde{e}_i (b_1 \otimes b_2) = \begin{cases} (\tilde{e}_i b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes (\tilde{e}_i b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i (b_1 \otimes b_2) = \begin{cases} (\tilde{f}_i b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes (\tilde{f}_i b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

with

$$\varepsilon_i(b) = \max \{ k \geq 0 \mid \tilde{e}_i^k b \in \mathcal{B} \}$$

$$\varphi_i(b) = \max \{ k \geq 0 \mid \tilde{f}_i^k b \in \mathcal{B} \}$$

Example $\mathfrak{g} = \mathfrak{sl}_2$

$$\bar{u} \longrightarrow \overline{fu} \longrightarrow \overline{f^{(2)}u} \longrightarrow \cdots \longrightarrow \overline{f^{(m)}u}$$

$$\varepsilon(\overline{f^{(k)}u}) = k, \quad \varphi(\overline{f^{(k)}u}) = m-k$$

$\mathfrak{g} = \mathfrak{sl}_{n+1}$ $\quad 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \longrightarrow \cdots \xrightarrow{n} n+1$

$$\varepsilon_i(k) = \begin{cases} 0 & k \neq i+1 \\ 1 & k = i+1 \end{cases} \qquad \varphi_i(k) = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$$

Idea (for sl_n, gl_n) embed $V(\lambda)$ into $V^{\otimes N}$, V natural representation and consider appropriate part of

Find crystal basis in terms of Young tableaux.

Algorithm description

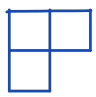
$\lambda \in P^+$, $\lambda = \sum_{i=1}^{n} \lambda_i \omega_i$, $\omega_i = \varepsilon_1 + \dots + \varepsilon_i$ (standard basis)

$$\lambda = \sum_{i=1}^{n} \Big(\sum_{j=i}^{n} \lambda_i \Big) \varepsilon_i = \sum_{i=1}^{n} a_i \varepsilon_i$$

and $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ corresponds to partition of $N = \sum_{i=1}^{n} a_i$

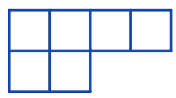
Example $\mathfrak{g} = sl_3$

$\rho = \alpha_1 + \alpha_2 = \omega_1 + \omega_2 = 2\varepsilon_1 + \varepsilon_2$



(2,1) of $3 = N$

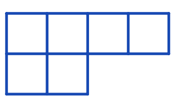
$2\rho = 2\omega_1 + 2\omega_2 = 4\varepsilon_1 + 2\varepsilon_2$



(4,2) of $6 = N$

Semistandard tableaux: filled with numbers corresponding to fundamental representation

strictly increasing ↓



$\longrightarrow$ increasing

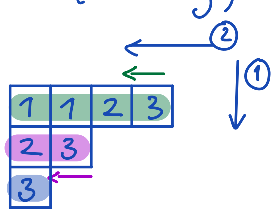
Ex
$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$$

Weight of semistandard tableau

$$\mathrm{wt}\big(\quad T \quad \big) = \sum_{i=1}^{n} \#(i \text{ in } T)\, \varepsilon_i$$

Ex $\mathrm{wt}\left(\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \right) = 2\varepsilon_1 + 3\varepsilon_2 + \varepsilon_3$ $(= \varepsilon_1 + 2\varepsilon_2$ for type A!)

Semistandard tableau $\longmapsto V^{\otimes N}$ (V fundamental representation)
via
ME (reading) Middle Eastern (reading



$\longmapsto$ $3 \otimes 2 \otimes 1 \otimes 1 \otimes 3 \otimes 2 \otimes 3$

Action of Kashiwara operators on semistandard tableaux given by action of N-fold tensor product using (iterated) tensor product representation. Then translate back to semistandard tableau.

<u>Example</u> $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = \rho = 2\varepsilon_1 + \varepsilon_2$ $\quad \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$

to get highest weight

$$\rho = \mathrm{wt}\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right) \quad \text{(no other possibility)}$$

$$\tilde{f}_1 \cdot \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad\qquad \tilde{f}_2 \cdot \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\mathrm{wt} = \varepsilon_1 + 2\varepsilon_2 = \rho - \alpha_1 \qquad\qquad \mathrm{wt} = 2\varepsilon_1 + \varepsilon_3 = \rho - \alpha_2$$

This is easy, because $\tilde{f}_i$ changes i to $i+1$ and then there is only semistandard tableau available.

$$\tilde{f}_2 \cdot \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \text{has 2 options; namely} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \text{ or } \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{aligned} \mathrm{wt} &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ &= \rho - \alpha_1 - \alpha_2 \; (=0) \end{aligned}$$

We have to go back to $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \longmapsto 2 \otimes 1 \otimes 2$

and apply $\tilde{f}_2$ to $2 \otimes 1 \otimes 2$ using the (iterated) tensor product rule.

$$\text{So } \tilde{e}_2(1 \otimes 2) = \begin{cases} (\tilde{e}_2 1) \otimes 2 = 0 & \overset{=0}{\varphi_2(1)} \geq \overset{=0}{\varepsilon_2(2)} \\ 1 \otimes \tilde{e}_2 2 = 0 & \varphi_2(1) < \varepsilon_2(2) \end{cases}$$

and $\varepsilon_2(1 \otimes 2) = 0$.

$$\left(\text{Similarly, } \tilde{f}_2(1 \otimes 2) = \begin{cases} \tilde{f}_2 1 \otimes 2 = 0 & \overset{=0}{\varphi_2(1)} > \overset{=1}{\varepsilon_2(2)} \\ 1 \otimes \tilde{f}_2 2 = 1 \otimes 3 & \varphi_2(1) \leq \varepsilon_2(2) \end{cases} \right)$$

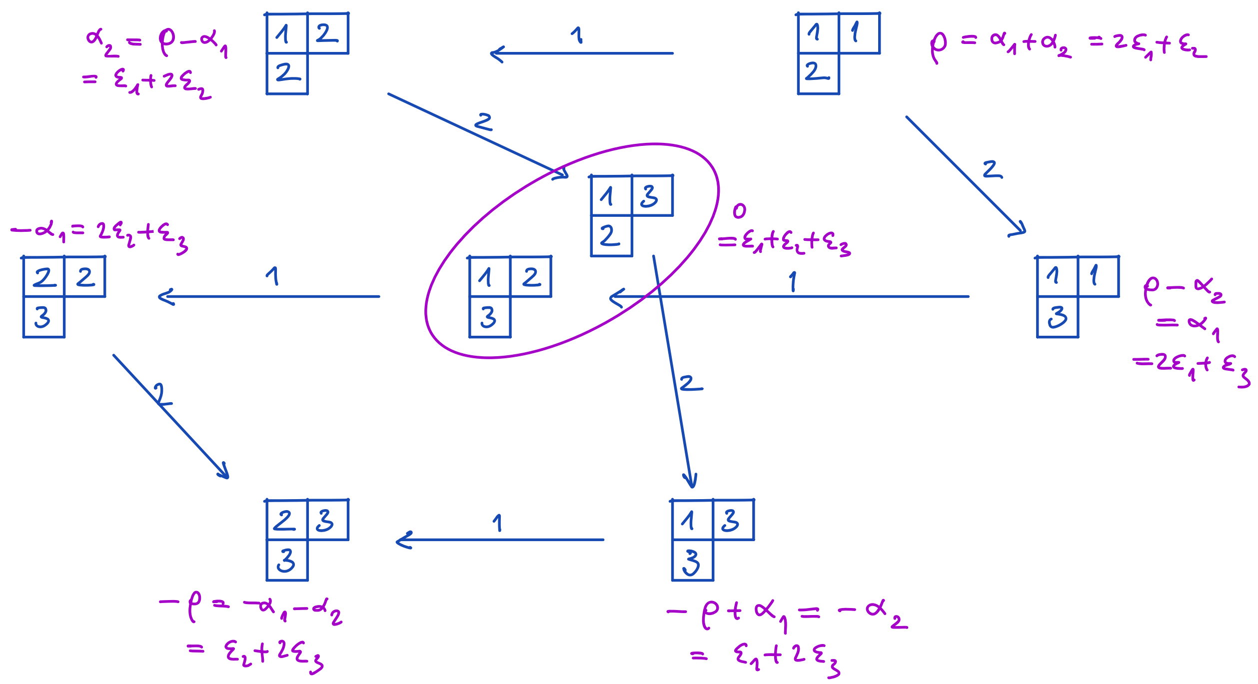
and $\varphi_2(1 \otimes 2) = 1$.

$$\text{So } \tilde{f}_2 \cdot 2 \otimes 1 \otimes 2 = \begin{cases} (\tilde{f}_2 \cdot 2) \otimes 1 \otimes 2 & \text{if } \overset{=1}{\varphi_2(2)} > \overset{=0}{\varepsilon_2(1 \otimes 2)} \\ 2 \otimes \tilde{f}_2 \cdot (1 \otimes 2) & \text{if } \varphi_2(2) \leq \varepsilon_2(1 \otimes 2) \end{cases}$$

$$\Rightarrow \tilde{f}_2 \cdot 2 \otimes 1 \otimes 2 = 3 \otimes 1 \otimes 2$$

and $2 \otimes 1 \otimes 3 \longmapsto \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$ by ME reading

Crystal graph of $V(\rho)$ for $U_q(sl_3)$



(subject to $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$)

(in certain computer algebra available).

Other links/remarks

- crystal basis can be uniformly lifted to $U_q^-(\mathfrak{g})$
- kind of $*$-structure making $\tilde{e}_i \leftrightarrow \tilde{f}_i$ "duals".
- global basis of $V(\lambda)$ as $U_q(\mathfrak{g})$ module

  $b \in B$, $G(b)$ global basis satisfying invariance under

  (switch roles $q \leftrightarrow q^{-1}$)

  invariant under $q \leftrightarrow q^{-1}$, $e_i \mapsto e_i$, $f_i \mapsto f_i$

  $K_h \leftrightarrow K_h^{-1}$ denote by $\overline{}$

 unique basis $G(\lambda)$ of $V(\lambda)$ with $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ if

  - $G(b) \equiv b \mod q\mathcal{L}(\lambda)$
  - $\overline{G(b)} = b$

gives a basis of $V(\lambda)$ over $A = \mathbb{C}[q, q^{-1}]$

Global basis will involve more terms, lose the charm of crystal basis.