Crystal basis theory for quantum symmetric pairs

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1 Crystal basis theory for quantum symmetric pairs

These are some rough notes that are made when preparing for the talk, they are far from polished and may contain some errors.

2 Introduction

The goal of this talk is to give an introduction to the crystal basis theory of quantum symmetric pairs. Crystal basis theory for quantum groups was developed by Kashiwara in the 90's, the rough intuition was that at $q = 0$, there should appear pleasant structure in the quantum group. This particular structure is best observed in the modules instead of the quantum group itself. The parameter q in quantum groups arose from statistical mechanical models, and $q = 0$ corresponds to the absolute zero in these models, motivated by this intuaition Kahiwara tought that the modules "crystallize" at $q = 0$. Hence the name *crystal basis.* Crystal basis theory in the classical sense have two key features: Firstly they form pleasant basis of $\mathcal{U}_q(\mathfrak{g})$ modules in a way that their structure becomes of a combinatorial nature. The second is that they are "cannonical". In this talk we will introduce crystal basis theory for quantum symmetric pairs of type $AIII$, and in particular look at the connections to ordinary crystal basis theory.

2.1 Structure of the talk

We will first introduce the quantum symmetric pairs of type $AIII$. Afterwards, we will introduce the crystal basis theory for quantum symmetric pairs and look at their crystal graphs. The main goal of the talk will be to highlight the connections to ordinary crystal basis theory. This talk will be based on [HW22] and ocasionaly we will use [HL02].

3 Quantum symmetric pairs of type AIII

We will be interested in quantum symmetric pairs related to the quantum group $\mathcal{U}_q(\mathfrak{su}(2r+1))$ of type A_{2r} over $\mathbb{Q}(p,q)$ with generators $E_i, F_i, K_i^{\pm}$, $i \in \mathbf{I} = \{(r-\frac{1}{2}), \dots (r+\frac{1}{2})\}$ subject to the relations

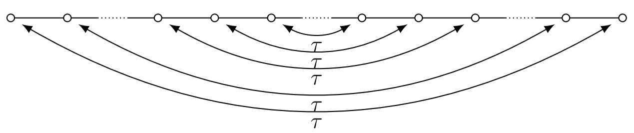
$$
\begin{aligned}
&K_i^{-1}K_i = K_iK_i^{-1} = 1,\\
&K_iK_j = K_jK_i,\\
&K_iE_jK_i^{-1} = q^{(\alpha_i,\alpha_j)}E_j,\\
&K_iF_jK_i^{-1} = q^{-(\alpha_i,\alpha_j)}F_j,\\
&E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i - K_i^{-1}}{q-q^{-1}},\\
&E_i^2E_j + (q+q^{-1})E_iE_jE_i + E_jE_i^2 \qquad |i-j| = 1,\\
&F_i^2F_j + (q+q^{-1})F_iF_jF_i + F_jF_i^2 \qquad |i-j| = 1,\\
&E_iE_j - E_jE_i = 0 \qquad |i-j| > 1,\\
&F_iF_j - F_jF_i = 0 \qquad |i-j| > 1.
\end{aligned}
$$

Here we consider p, q as transcendental parameters. Then $\mathcal{U}_q(\mathfrak{su}(2r+1))$ becomes a Hopf algebra by

$$
\begin{aligned}
&\triangle(E_i) = 1\otimes E_i + E_i \otimes K^{-1}, \quad &&\triangle(F_i) = F_i\otimes 1 + K_i\otimes F_i \quad \triangle(K_i) = K_i\otimes K_i,\\
&\epsilon(E_i) = \epsilon(F_i) = 0, &&\epsilon(K_i) = 1.
\end{aligned}
$$

Remark 3.1. This coproduct structure is a twisted version of the "standard" coproduct.

We can visualise the Satake diagram corresponding to the symmetric pair $(S(2r+1), S(U(r)\times U(r+1))$ as follows;



We introduce the coideal subalgebrs $\mathfrak{B}_{\mathbf{c}} = \mathcal{U}^{\iota}$. The algebra $\mathfrak{B}_{\mathbf{c}}$ is the subalgebra generated by

$$
\begin{aligned}
k_j^{\pm} &= (K_{j-\frac{1}{2}}K^{-1}_{-(j-\frac{1}{2})})^{\pm}\\
e_j &= E_{j-\frac{1}{2}} + (p)^{-\delta_{j,1}}F_{-(j-\frac{1}{2})}K^{-1}_{j-\frac{1}{2}}\\
f_j &= E_{-(j-\frac{1}{2})} + (p)^{\delta_{j,1}}K^{-1}_{-(j-\frac{1}{2})}F_{j-\frac{1}{2}} \qquad\qquad j \in \mathbf{I}^{\iota} = \{1,\dots,r\}.
\end{aligned}
$$

We will refer to $\mathcal{U}_1^\iota$ as the coideal subalgra $\mathcal{U}^\iota$ when $r = 1$. Note that $\mathcal{U}_1^\iota \hookrightarrow \mathcal{U}^\iota$ for general $r = 1$ by the map

$$e_1 \mapsto e_1, \qquad f_1 \mapsto f_1, \qquad k_1 \mapsto k_1.$$

Remark 3.2. The algebra $\mathfrak{B}_{\mathbf{c}}$ can be intrinsically characterized by generators and relations, see [HW22].

Remark 3.3. For each $1 \neq j \in \mathbf{I}^\iota$ the triple (e_i, k_i, f_i) is a $\mathfrak{sl}_2$ triple. Let us check the nontrivial condition $e_i f_i - f_i e_i = \frac{k_i - k_i^{-1}}{q-q^{-1}}$. We have

$$\begin{aligned}
&(E_{j-\frac{1}{2}} + F_{-(j-\frac{1}{2})}K^{-1}_{j-\frac{1}{2}})(E_{-(j-\frac{1}{2})} + K^{-1}_{-(j-\frac{1}{2})}F_{(j-\frac{1}{2})}) \\
&- (E_{-(j-\frac{1}{2})} + K^{-1}_{-(j-\frac{1}{2})}F_{(j-\frac{1}{2})})(E_{j-\frac{1}{2}} - F_{-(j-\frac{1}{2})}K^{-1}_{j-\frac{1}{2}}) \\
&= E_{j-\frac{1}{2}}K^{-1}_{-(j-\frac{1}{2})}F_{(j-\frac{1}{2})} + E_{-(j-\frac{1}{2})}F_{-(j-\frac{1}{2})}K^{-1}_{j-\frac{1}{2}} \\
&- K^{-1}_{-(j-\frac{1}{2})}F_{j-\frac{1}{2}}E_{j-\frac{1}{2}} + F_{-(j-\frac{1}{2})}K^{-1}_{j-\frac{1}{2}}E_{-(j-\frac{1}{2})} \\
&= \frac{1}{q-q^{-1}}\left(K^{-1}_{-(j-\frac{1}{2})}(K_{j-\frac{1}{2}} - K^{-1}_{j-\frac{1}{2}}) - K^{-1}_{j-\frac{1}{2}}(K_{-(j-\frac{1}{2})} - K^{-1}_{-(j-\frac{1}{2})})\right) \\
&= \frac{1}{q-q^{-1}}\left(k_i - k_i^{-1}\right).
\end{aligned}$$

4 ι-Crystal basis theory

Our first goal will be to study suitable analogs of the Kashiwara operators. Let $M \in \mathcal{O}_{\text{int}}$ be a integrable $\mathcal{U}^\iota$-module. For this we first note that, for each $1 \neq j \in \mathbf{I}^\iota$ the module M becomes a $U_q(\mathfrak{sl}_2)$ by the inclusion

$$U_q(\mathfrak{sl}_2) \to \mathcal{U}_1^\iota, \qquad E_i \to e_i, \quad F_i \to f_i, \quad k_i \to K_i.$$

Hence for $j \neq 1$ we might define the analogs of the Kashiwara operators $\tilde{f}_j, \tilde{e}_j \in \mathrm{End}(M)$. Sow we only have to worry about the case that $j = 1$. For this we use the representation theory of $\mathcal{U}^\iota$, that can be found in [HW22, Section 3]. By complete reducibility of $\mathcal{U}_1^\iota \hookrightarrow \mathfrak{B}_{\mathbf{c}}$ we can decompose

$$M = \bigoplus_{\boldsymbol{\lambda} \in P_1^\iota} L(\boldsymbol{\lambda}), \qquad\qquad L(\boldsymbol{\lambda}) \cong \bigoplus_{i=1}^{n_{\boldsymbol{\lambda}}} \mathbb{Q}(p,q) f^{(i)} v_{\boldsymbol{\lambda}},$$

into irreducible $\mathcal{U}_1^\iota$ modules, where $v_{\boldsymbol{\lambda}}$ is a highest weight vector of $L(\boldsymbol{\lambda})$. With respect to this decomposition, on can naturally define the Kashiwara opertaor $\tilde{f}_1$, $\tilde{e}_1$ With this description we can naturally introduce ι-Cystal basis. Let $\mathbf{A} = \{f/g \in \mathbb{Q}(q,p) : \lim_{q\to 0}(\lim_{p\to 0} f(q,p)/g(p,q) \text{ exists}\}$.

Remark 4.1. If $p = q$, then $\mathbf{A}$ coincides with $\mathbf{A}_0$ defined earlier by Erik. Furthermore this definition can be motivated by the fact that we again look at the quotient by $q\mathcal{L}$ which again should result in a free $\mathbb{Q}$ vector space.

Definition 4.2 (ι-crystal lattice)**.** Let M be a finite dimensional $\mathcal{U}^\iota-$module and $\mathcal{L}$ an $\mathbf{A}-$submodule of M. We say that $\mathcal{L}$ is a *quasi-ιcrystal lattice* of M if

(i) $\mathcal{L}$ is a free $\mathbf{A}-$module of rank $\dim_{\mathbb{Q}(p,q)} M$ and $\mathbb{Q}(p,q) \otimes_{\mathbf{A}} \mathcal{L} = M$,

(ii) $\mathcal{L} = \oplus_\lambda \mathcal{L}_\lambda$ where $\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$.

(iii) $\tilde{f}_i(\mathcal{L}) \subset \mathcal{L}$ and $\tilde{e}_i(\mathcal{L}) \subset \mathcal{L}$ for all $i \in \mathbf{I}^\iota$

$\diamondsuit$

Definition 4.3 (Quasi ι-crystal basis)**.** Let M be a finite dimensional $\mathcal{U}^\iota-$module, $\mathcal{L}$ an $\mathbf{A}-$submodule of M and $\mathcal{B}$ be an subset of $\mathcal{L}/q\mathcal{L}$. We say that $(\mathcal{L}, \mathcal{B})$ is a *quasi* $\iota-$*basis* if

(i) $\mathcal{L}$ is a quasi-ι crystal lattice of M,

(ii) $\mathcal{B}$ is a $\mathbb{Q}$ basis of $\mathcal{L}/q\mathcal{L}$,

(iii) $\mathcal{B} = \sqcup \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}_\lambda/q\mathcal{L}_\lambda)$,

(iv) $\tilde{f}_i(\mathcal{B}) \subset \mathcal{B} \sqcup \{0\}$ and $\tilde{e}_i(\mathcal{B}) \subset \mathcal{B} \sqcup \{0\}$ fo all $i \in \mathbf{I}^\iota$,

(v) for each $b, b' \in \mathcal{B}$ and $i \in \mathbf{I}^\iota$, one has $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i(b')$.

$\diamondsuit$

Definition 4.4. For a quasi ι-crsytal base $(\mathcal{L}, \mathcal{B})$ and $i \in \mathbf{I}^\iota$, we define three maps $\varphi_i : \mathcal{B} \to \mathbb{Z}_{\geqslant 0}$, $\epsilon_i : \mathcal{B} \to \mathbb{Z}_{\geqslant 0}$ and $wt^\iota : \mathcal{B} \to \Lambda^\iota$

$$\varphi_i(b) = \max\{n : \tilde{f}_i^n(b) \neq 0\}, \quad \epsilon_i(b) = \max\{n : \tilde{e}_i^n(b) \neq 0\}, \quad \mathrm{wt}^\iota(b) = \text{ if } b \in \mathcal{B}_\lambda.$$

$\diamondsuit$

Example 4.5. In the case that $r = 1$ any $\mathcal{U}^\iota$ irreducible module M can be described as

$$M = \bigoplus_{i=1}^{n} \mathbb{Q}(p,q) f_1^{(i)} v,$$

for some highest weight vector v, see [HW22, Thm 3.1.5]. Then

$$\mathcal{L} = \mathrm{span}_{\mathbf{A}}\{f^{(i)} v \,:\, 1 \leqslant i \leqslant n\}$$

is a ι-crystal lattice and

$$\{f^{(i)} v + \mathcal{L}/q\mathcal{L} \,:\, 1 \leqslant i \leqslant n\}$$

is the corresponding ι-crystal basis. $\lhd$

Definition 4.6 (ι-crystal graph)**.** Let $(\mathcal{L}, \mathcal{B})$ be a quasi $\iota-$crystal basis. The ι-*crystal graph* associated to $(\mathcal{L}, \mathcal{B})$ is the colored directed graph with vertex set $\mathcal{B}$ and edges $b \xrightarrow{i} b'$ if $\tilde{f}_i b = b'$. $\diamondsuit$

Example 4.7. In the case of $r = 1$ any crystal graph of a $\mathcal{U}^\iota$-module will have the structure

$$\boxed{0} \xrightarrow{\tilde{f}_1} \boxed{1} \xrightarrow{\tilde{f}_1} \boxed{2} \xrightarrow{\tilde{f}_1} \boxed{3} \cdots\cdots\to \boxed{\text{n-1}} \xrightarrow{\tilde{f}_1} \boxed{\text{n}}$$

$\lhd$

Remark 4.8. At this point we do not have existence and uniqueness, this is shown in [HW22]. What we do remark is that by completer reducibility each $\mathcal{U}_1^\iota$-module has a crystal base.

Theorem 4.9 ([HW22][Thm 6.2.9]). *Let $M \in \mathcal{O}_{int}$ be a $\mathcal{U}_1^\iota$ module. Then M has a crystal basis. Moreover there exists a isomorphism $M \to \bigoplus_{\boldsymbol{\lambda} \in P_1^\iota} L(\boldsymbol{\lambda}^{\oplus m_{\boldsymbol{\lambda}}})$ of $\mathcal{U}_1^\iota$ modules which induces an isomorphism*

$$(\mathcal{L}, \mathcal{B}) \to \left(\bigoplus_{\boldsymbol{\lambda} \in P_1^\iota} \mathcal{L}(\boldsymbol{\lambda}^{\oplus m_{\boldsymbol{\lambda}}}), \bigoplus_{\boldsymbol{\lambda} \in P_1^\iota} \mathcal{B}(\boldsymbol{\lambda}^{\oplus m_{\boldsymbol{\lambda}}}) \right),$$

that is unique up to highest weight vector.

Proof. One checks that direct sums of crystal basis are crystal basis and that a direct sum of crystal basis is a crystal basis if and only if all its components have crystal basis. The uniqueness follows by the fact that crystal lattices are unique up to highest weight vector. □

4.1 Tensor product rules and decompositions

Recall that $\mathcal{U}^\iota$ is a right coideal subalgebra, meaning that $\triangle(\mathcal{U}^\iota) \subset \mathcal{U}^\iota \otimes \mathcal{U}_q(\mathfrak{su}(2r+1))$. If M is a $\mathcal{U}^\iota$ module and N is a $\mathcal{U}_q(\mathfrak{su}(2r+1))$ module, then $M \otimes N$ becomes a $\mathcal{U}^\iota$ module via the coproduct $\triangle$.

Remark 4.10. Consider the one dimensional module $M = \mathbb{Q}(p, q)$ with the action of ϵ on M and let N be a $\mathcal{U}_q(\mathfrak{su}(2r+1))$ module. As vector spaces we have $\mathbb{Q}(p, q) \otimes N \cong N$ and the action on N is given by

$$(\epsilon \otimes 1) \circ \triangle = 1.$$

Meaning that this action corresponds to restricting the module N to the coideal subalgebra $\mathcal{U}^\iota$.

Remark 4.11. The philosophy of studying crystal basis, in the classical case and also in the ι-setting, will be the following. In the vector representation, the structure of the crystal basis is reasonably well understood as well as their tensor products. To study crystal basis in general, we recall that each representation is an irreducible factor of a high enough tensor power, for some details see the notes of Japsers talk from last time, we extract the information from the vector representation.

As mentioned one of the accomplishments of Watanabe is the characterisation of irreducible $\mathcal{U}^\iota$ modules, see [HW22, Cor 4.3.8]. In particular they are parameterized by $(a, b) \in \mathbb{Z}^r \times \mathbb{Z}^r_{\geqslant 0}$ with $a_i \geqslant b_i$ $i \in \mathbf{I}^\iota \backslash \{1\}$, using analogs of Verma modules.

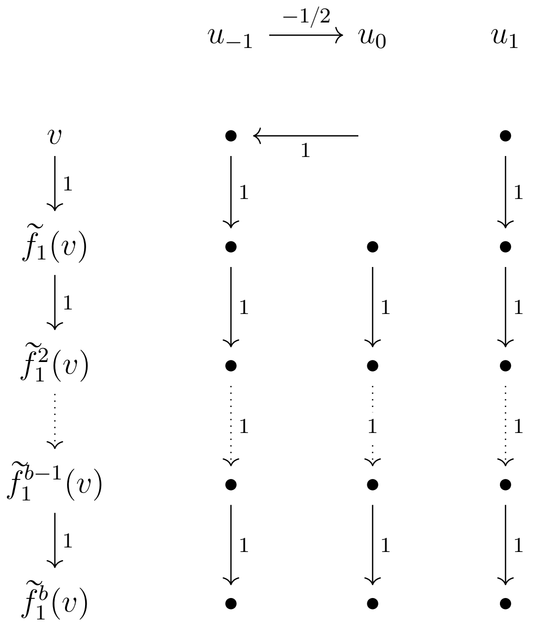
Proposition 4.12. *Let $a \in \mathbb{Z}$, $b \in Z_{\geqslant 0}$. Then we have the isomorphism*

$$L(a; b) \otimes \mathbf{V} \cong L(a+2; b+1) \otimes L(a-1; b) \oplus L(a-1; b-1)$$

of $\mathcal{U}_1^\iota$ modules. Moreover $L(a; b) \otimes \mathbf{L}, \mathcal{B}(a; b) \otimes \mathbf{B})$ is a quasi-ι crystal basis of $L(a; b) \otimes \mathbf{V}$.

Proof. The proof is a direct computation, it is quite lengthy and I do not think it is well suited for the talk. For the proof see [HW22, Prop 6.3.1] □

Example 4.13. If one carries out the computation, the crystal graph of $L(a;b)\otimes\mathbf{V}$ can be read of to be



where the vertices are interpreted as simple tensors in $\mathcal{B}(a;b)\otimes\mathbf{B}$. ◁

Proposition 4.14. *Let $a\in\mathbb{Z}$ and $b\in\mathbb{Z}_{\geqslant 0}$ then $(\mathcal{L}(a;b)\otimes\mathbf{L}^{\otimes n},\mathcal{B}(a;b)\otimes\mathbf{B}^{\otimes n})$ is a quasi$-\iota$ crystal basis for $L(a;b)\otimes\mathbf{V}^{\otimes n}$.*

Proof. By (4.12) it follows that $(\mathcal{L}(a;b)\otimes\mathbf{L},\mathcal{B}(a;b)\otimes\mathbf{B})$ is a quasi$-\iota$ crystal basis for $L(a;b)\otimes\mathbf{V}$. Again by (4.12) we note that

$$L(a;b)\otimes V^{\otimes(n-1)}\cong\bigotimes_i L(a_i;b_i)$$

and that

$$L(a;b)\otimes V^{\otimes(n)}\cong\bigotimes_i L(a_i;b_i)\otimes V.$$

Each of the $L(a_i;b_i)\otimes V$ has a quasi$-\iota$ crystal basis $(\mathcal{L}(a_i;b_i)\otimes\mathbf{L}^{\otimes n},\mathcal{B}(a_i;b_i)\otimes\mathbf{B}^{\otimes n})$. As a result of [HW22, Thm 6.2.9]

$$(\otimes_i\mathcal{L}(a_i;b_i)\otimes\mathbf{L}^{\otimes n},\otimes_i\mathcal{B}(a_i;b_i)\otimes\mathbf{B}^{\otimes n})$$

is a quasi$-\iota$ crystal basis for $L(a;b)\otimes\mathbf{V}^{\otimes n}$. □

Corollary 4.15. *Let M be a $\mathcal{U}_1^\iota$ module with quasi$-\iota$ crystal base $(\mathcal{L},\mathcal{B})$ and N be a $\mathcal{U}_q(\mathfrak{su}(3))$ module with crystal base $(\mathcal{L}',\mathcal{B}')$. Then $M\otimes N$ has a quasi$-\iota$ crystal basis $(\mathcal{L}\otimes\mathcal{L}',\mathcal{B}\otimes\mathcal{B}')$, where the Kashiwara operators act as*

$$\tilde{f}_1(b\otimes b')=\begin{cases}b\otimes\tilde{E}_{-1/2}(b') & if\ \epsilon_1(b)<\epsilon_{-1/2}(b')\\ \tilde{f}_1(b)\otimes b' & if\ \epsilon_1(b)\geqslant\epsilon_{-1/2}(b')\end{cases}$$

$$\tilde{e}_1((b\otimes b')=\begin{cases}b\otimes\tilde{F}_{-1/2}(b') & if\ \epsilon_1(b)\leqslant\epsilon_{-1/2}(b')\\ \tilde{e}_1(b)\otimes b' & if\ \epsilon_1(b)>\epsilon_{-1/2}(b')\end{cases}$$

Proof. Note that there exists a natural number N with an embedding $M \hookrightarrow \mathbf{V}^{\otimes n}$, as a irreducible component which preserves the crystal basis by uniqueness. By Proposition (4.12) it follows that $(\mathcal{L}\otimes\mathcal{L}', \mathcal{B}\otimes\mathcal{B}')$ is a crystal base. So it suffices to check this action on modules of the form $L(a_i, b_i)\otimes V$. The action on these modules follows from (4.12) or examine (4.13). ☐

Up until now all the calculations have been in rank 1. Luckily we do not have to do much work to extend the results for general r, by a rank 1 reduction.

Theorem 4.16. *Let M be a $\mathcal{U}^\iota$-module and N be a $\mathcal{U}_q(\mathfrak{su}(2r+1))$ module with crystal bases $(\mathcal{L}, \mathcal{B})$ and $(\mathcal{L}', \mathcal{B}')$. Then $(\mathcal{L}\otimes_{\mathbf{A}_0}\mathcal{L}', \mathcal{B}\times\mathcal{B}')$ is a crystal base for $M\otimes N$.*

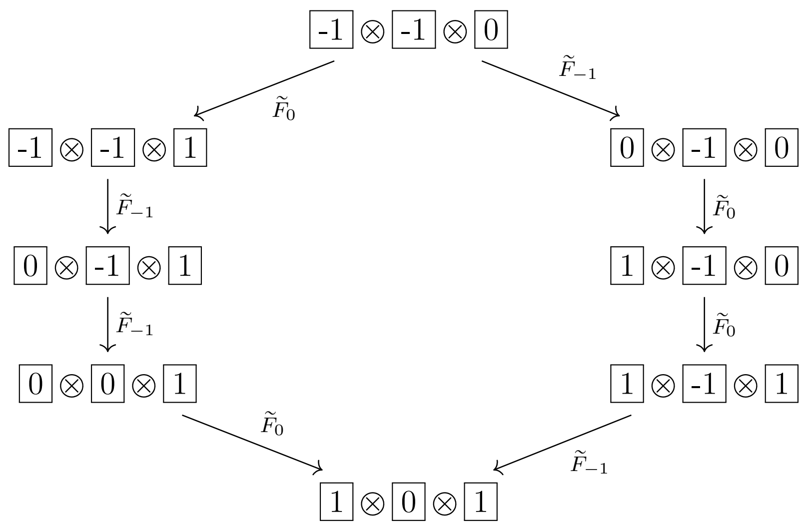
Proof. The strategy of the proof combines two of our tricks. The first trick is the rank 1 reduction and the second is the embedding into $\mathbf{V}^{\otimes n}$. We will first worry about the Kashiwara operators. Let $i \in \mathbf{I}^\iota$, we want to show that $\tilde{e}_i, \tilde{f}_i(u\otimes v) \in \mathcal{L}_N\otimes\mathcal{L}_M$. Consider a $u \in \mathcal{L}_M$ and $v \in \mathcal{L}_N$. We may assume that $u = f_i^{(k)}u_0$ and $v = v_0$ for some $u_0 \in \mathcal{L}_N \cap \ker e_i$ and $v_0 \in \mathcal{L}_N \cap \ker F_{1/2}$. Now $\mathcal{U}_1^\iota u \cong L(a;b)$ and $\mathcal{U}_q(\mathfrak{sl}_2)v \hookrightarrow \mathbf{V}^{\otimes m}$ for some positive integer m. By Proposition (4.12) and uniqueness of the Kashiwara operators we know that that $\tilde{f}_i u\otimes v \in \mathcal{L}_N\otimes\mathcal{L}_M$. Similarly one verifies that the crystal basis are invariant under the Kashiwara operators. The ι-crystal lattice conditions are easy to verify the "basis" conditions, as well as the weight space considerations. ☐

Corollary 4.17. *Let N be a $\mathcal{U}_q(\mathfrak{su}(2r+1))$ module with crystal basis $(\mathcal{L}', \mathcal{B}')$. Then $(\mathcal{L}', \mathcal{B}')$ is a quasi$-\iota$ crytal basis of N. Furthermore for $r = 1$ the action of the Kashiwara operators is given by*

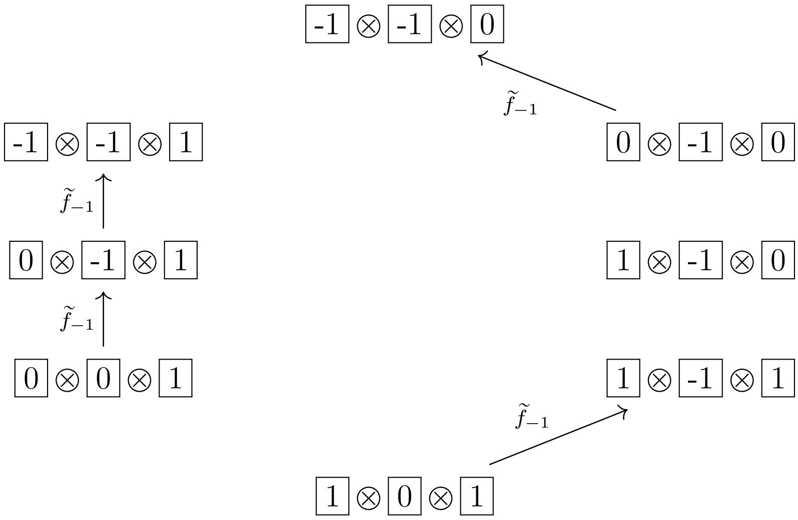
$$\tilde{f}_1(b) = \tilde{E}_{-1/2}(b), \qquad \tilde{e}_1(b) = \tilde{F}_{-1/2}(b), \qquad b \in \mathcal{B}'.$$

Proof. Apply Theorem (4.16) to the $\mathcal{U}^\iota$ module $\mathbb{Q}(p,q)$. The action of the Kashiwara operators can be read of by (4.15) ☐

Example 4.18 (The crystal graph of $L(\rho)$)**.** The crystal graph of $(\mathcal{L}(\rho), \mathcal{B}(\rho)$ can be visualised as



Using [HW22, Cor 6.3.7], we obtain the ι-crystal graph of $L(\rho)$ restricted to the coideal subalgebra $\mathcal{U}^\iota$



As in classical crystal basis theory, the connected components of the crystal graph correspond to the irreducible components of the module. As a result we see that we can identify the trivial module with the crystal $\boxed{1} \otimes \boxed{\text{-1}} \otimes \boxed{0}$. ◁

The upshot of the crystal basis theory is that the actions of the Kashiwara operators on the vector represenation are described in a combinatorial nature. As a result, for instance, finding the irreducible components of a module is reduced to a combinatorial problem of finding the highest weight vectors in the vector representation. Even more can be said about the branching rules via a translation to Young tablaux, the branching rules are made "explicit" by finding *Yamanouchi biwords.*

References

[HW22] Watanabe, *Crystal basis theory for a quantum symmetric pair* (U, U^ι).

[HW11] Watanabe, *Global crystal basis for integrable modules over a Quantum symmetric pair of type AIII.*

[HL02] Hong & Kang, *An introduction to quantum groups and crystal basis.*