Sur les polynômes symétriques et l'algèbre de Vagna Tcherednique

(reimen muss man ja...)

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• Plan: DAHA^s, ortho./sym. pol^s, example A_1 (Dec relevante

Cases: (I) $R' = R^\vee$, $L = P$, $L' = P^\vee$. ~~Sur les algèbres de Hecke doublement affines et leur action sur les polynômes symétriques associés à un système de racines.~~

(II) $R' = R$, $S = S'$, $L = L' = P^\vee$ (R reduced)

Assume $k \in \mathbb{Z}_{\geq 0}$. $K = \mathbb{Q}(q, \tau_i)$

Recall: • braid group $\mathcal{B} = \langle T(w) \mid T(vw) = T(v)T(w)$ if $\ell(vw) = \ell(v) + \ell(w) \rangle$ $(w \in W)$, $W = W_R \ltimes t(L')$

$= \langle T_i, U_j \mid i \in I, j \in J \rangle / \text{braid rel}^s$ $T_i = T(s_i)$, $U_j = T(u_j)$

$= \langle T_i, Y^{\lambda'} \mid i \in I_R, \lambda' \in L' \rangle / \text{rel}^s$ $Y^{\lambda'} = T(t(\lambda'))$. $u_j = u(\pi_j')$ (minimal elt.) shortest elt. of $t(\pi_j')W_R$

• AHA $\mathcal{H} = K[\mathcal{B}] / (T - \tau_i = T_i^{-1} - \tau_i^{-1} \mid i \in I) =:$ Hecke

$= K\{T(w) Y^{\lambda'} \mid w \in W_R, \lambda' \in L'\}$ as K-module. $X^\mu \in \mathrm{End}_K K[L]$ mult. by e^μ, $\mu \in L$

• Basic rep. $\beta: \mathcal{H} \hookrightarrow \mathrm{End}_K(\underbrace{K[L]}_{A})$, $T_i \mapsto \tau_i s_i + b_i(X)(1 - s_i)$

$U_j \mapsto u_j$

$Y^{\lambda'} \mapsto q^{-\langle \lambda', r_k'(\mu) \rangle} e^\mu + \text{LOT}$ ($e^\mu \mapsto$)

$r_k'(\mu) = u'(\mu)(-\rho_k)$ see last time

Thm (Черeдник): define the DAHA $\tilde{\mathcal{H}} = \mathcal{H}\ell$ by $\mathrm{im}(\beta) + \{X^\lambda : e^\mu \mapsto e^{\mu+\lambda} \mid \lambda \in L\}$. Paarsgewijs versch. want $\mu \mapsto r_k'(\mu)$ inj!

proof similar as for $\mathcal{H}$: check X-analogue of Lusztig and verify image of basis under β remains K-lin. indep.

Then $\tilde{\mathcal{H}} = K[\tilde{\mathcal{B}}]/\text{Hecke}$, where the DBG is $\tilde{\mathcal{B}} = \langle T(w), X^\lambda, Y^{\lambda'} \mid w \in W_R, \lambda \in L, \lambda' \in L' \rangle / \text{rel}^s$

$= \langle \mathcal{B}, X^\Lambda \rangle$ where $\Lambda = L \oplus \mathbb{Z}(\frac{1}{\#J} \cdot c)$.

Thm: There are PBW-esque K-bases of $\tilde{\mathcal{H}}$
pf as for $\mathcal{H}$: $\{Y^{\lambda'} T(w) X^\mu\}$, $\{X^\mu T(w) Y^{\lambda'}\}$.

Not: Let $\tilde{\mathcal{H}}'$ be the dual DAHA, obtenue par $R \longleftrightarrow R'$ & $L \longleftrightarrow L'$.

Thm: The map $\omega: \tilde{\mathcal{H}}' \to \tilde{\mathcal{H}}$, $X^{\lambda'} T(w) Y^\mu \mapsto X^{-\mu} T(w^{-1}) Y^{-\lambda'}$ is an anti-iso. of K-alg^s.
pf. check Hecke rel^s manually.

Ex: $R = A_1$, $L = L' = \mathbb{Z}\frac{\alpha}{2} = \mathbb{Z}\pi_1'$, $J = I = \{0, 1\}$, $U = YT_1^{-1} = U^{-1}$

$\tilde{\mathcal{H}} = \langle T_1, X, Y \mid T_1 Y^{-1} T_1 = Y,\ T_1 X T_1 = X^{-1},\ UXU = q^{1/2} X^{-1}$ &
$(T_1 - \tau)(T_1 + \tau^{-1}) = 0 \rangle$

Let $x := e^{\alpha/2} \in \mathbb{A} = K[x,x^{-1}]$. $\mathcal{H}$-action is:

- X is mult. by x.
- U acts via $u_1 = t(\frac{\alpha}{2})s_1$: $(e^{n\frac{\alpha}{2}} \mapsto q^{n/2} e^{-n\alpha/2})$: $f(x)\mapsto f(q^{1/2}x^{-1})$.
- T_1 acts by $b(\tau;x^2) + c(\tau,x^2)s_1$: $f(x)\mapsto \frac{\tau-\tau^{-1}}{1-x^2}f(x) + \frac{\tau x^2-\tau^{-1}}{x^2-1}f(x^{-1})$

$Y = UT_1$.

Recall: $\forall \lambda\in L$, $\exists! E_\lambda \in \mathbb{A}^*$ s.t.

i) $E_\lambda 1 = e^\lambda + LOT$

ii) $(E_\lambda, e^\mu) = 0 \ \forall \mu<\lambda$.

$(f,g) = ct(fg^*\Delta_{S,k})$

$\boxed{\text{Dually, } E'_\mu \in K[L'] = \mathbb{A}'}$

Pf: $(-,-)$ nondeg. on $\langle e^\mu \mid \mu\le\lambda\rangle$

Have $\forall f \in \mathbb{A}'$ that $f(Y)E_\lambda = f(-r_{k'}(\lambda))E_\lambda$; the E_λ form an orthog. K-basis diagonalising the action of $K[L'](Y) \circlearrowleft K[L]$

$Y^{\lambda'}$'s act via q-powers on $K[L]$. Moreover: $E_\lambda(r'_k(\mu))E'_\mu(-\rho_{k'}) = E_\lambda(-\rho_k)E'_\mu$

$Y^{\lambda'} e^\mu = q^{\langle\lambda', r_k(\mu)\rangle} e^\mu + LOT$, r_k injective

Thm: Let $0\neq f\in K[L]$ be a simultaneous eigenfunction of all $Y^{\lambda'}$ w/ eigenvalue $\varphi\in (L')^*$. Then f is a scalar multiple of some E_μ, and $\varphi(\lambda') = q^{-\langle\lambda', r_{k'}(\mu)\rangle}$

Pf: use that the E_μ are an eigenbasis w/ distinct Y-eigenvalues

Not: For $\lambda\in L_{++}$ dom, let $m_\lambda := \sum_{\mu\in W_R\cdot\lambda} e^\mu \in K[L]^{W_R}$ note $m_\lambda(Y)$ maps $K[L']^{W_R}$ to itself.

On $K[L]$, define $\langle f,g\rangle = \frac{1}{\#W_R} ct(fg^* \nabla_{S,k})$

Variation of $(-,-)$: $\nabla = \Delta\Delta^c$

Jasper: write $(-,-) = \int$ over cpx. torus then $\langle-,-\rangle$ = same $\int$ but restricted

Thm: $\forall\lambda\in L_{++}$, $\exists! P_\lambda \in K[L]^{W_R}$ s.t.

Symmetric polynomials

1) $P_\lambda = m_\lambda + LOT \in K\{m_\mu \mid L_{++}\ni\mu<\lambda\}$

2) $\langle P_\lambda, m_\mu\rangle = 0\ \forall L_{++}\ni \mu<\lambda$.

Proof verbatim as for E_μ!

$\forall f\in K[L']^{W_R}$, $f(Y)P_\lambda = f(-\lambda-\rho_k')P_\lambda$. ~~and they di~~

Ex: 0) $k\equiv 0 \Rightarrow P_\lambda = m_\lambda$.

1) Case I, $k\equiv 1$, $P_\lambda = \chi_\lambda = \frac{\sum_{w\in W_R}(-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\prod_{\alpha\in R^+}(e^{\alpha/2}-e^{-\alpha/2})}$ ← Weyl formula

char. of hwr of Lie alg. assoc. to R. (e.g. A1; get $V(n)$ for $\mathfrak{sl}_2$, $\lambda = \frac{n\alpha}{2}$)

2) Case III: Koornwinder polynomials (Askey-Wilson voor $n=1$).

Thm (Tom): $P_\lambda(\mu'+\rho_k')P_{\mu'}(\rho_k) = P_\lambda(\rho_k)P_{\mu'}(\lambda+\rho_{k'})$.

Build P_λ from E_λ. Massive shortcuts.

For $\varepsilon: W_R \to \mathbb{C}^\times$ lin char (triv or sign for ADE, otherwise two more)

Def. de ε-symmetriser $U_\varepsilon \in \mathcal{H}$ by

$$U_\varepsilon := (\tau_{w_0}^\varepsilon)^{-1} \sum_{w\in W_R} \tau_w^\varepsilon T(w),$$

where $\tau_i^\varepsilon := \begin{cases} \tau_i & \varepsilon(s_i)=1 \\ -\tau_i^{-1} & \varepsilon(s_i)=-1 \end{cases}$, $\tau_w^\varepsilon := \tau_{i_1}^\varepsilon \cdots \tau_{i_p}^\varepsilon$ if $w = s_{i_1}\cdots s_{i_p}$.

all well defined

~~Then $U_{triv}E_\lambda = P_\lambda$. More generally~~

since $U_\varepsilon E_{W_R\lambda}$'s are scalar mult^s ‖ since if $s_i\lambda=\lambda$ AND $\varepsilon(s_i)=-1$, ~~then~~ $U_\varepsilon E_\lambda \equiv 0$.

Let $\lambda \in L'_{++}$, assume $\varepsilon|_{\mathrm{Stab}_{W_R}(\lambda)} = \mathrm{triv}$. (e.g. λ reg., ε = sign)

Not: set $P_\lambda^\varepsilon := \tau_{w_0}\left(\sum_{w\in \mathrm{Stab}_{W_R}(\lambda)} \tau_w^2\right)^{-1} U_\varepsilon E_\lambda = e^{w_0\lambda} + LOT$ / P_λ^{sign} = skew-sym. Macdo. pol^s

Then $f(Y)P_\lambda^\varepsilon = f(-\lambda-\rho_k')P_\lambda^\varepsilon \quad \forall f\in K[L']^{W_R}$ $\quad \varepsilon = $ $\boxed{P_\lambda^{triv} = P_\lambda}$.

pf. $Z(\mathcal{H}) = K[L']^{W_R} \ni f$ so $f(Y)$ commutes with all $T(w\in W_R)$ hence with U_ε.

→ Dual results

Def Define the creation operators in $\mathrm{End}_K(K[L'])$:

1) for $i\in I_R$: $\underline{\alpha}_i := T_i - b_i(Y^{-\alpha_i})$ ($\underline{x}_i = \ldots$)

$b_i(Y^{-\alpha_i}) = \omega^{-1}(b_i(X^{\alpha_i}))$ $\quad \underline{\alpha}_0 := \omega^{-1}(T_0) - b_0(qY^\varphi)$ $\quad T_0 = Y^{\varphi^\vee}T(s_\varphi)^{-1}$

$= \omega^{-1}(b(\tau_i;\tau_i';X^{\alpha_i}))$ $\quad = T(s_0)$,

2) for $j\in J$; $\underline{\beta}_j := \omega^{-1}(U_j^{-1})$ NB $\underline{\beta}_0 = \mathrm{id}$. $\quad \alpha_0 = -\varphi + c$

Then $Y^\lambda \underline{\alpha}_i = \underline{\alpha}_i Y^{s_i\lambda}$; $Y^\lambda\underline{\alpha}_0 = q^{-\langle\lambda,\varphi\rangle}\underline{\alpha}_0 Y^{s_\varphi\lambda}$; $Y^\lambda \underline{\beta}_j = q^{-\langle\lambda,\pi_j'\rangle}\underline{\beta}_j Y^{v_j^{-1}\lambda}$

Thm: $\mu'\in L'$, $u(\mu') \overset{= t(\mu')v(\mu')^{-1},\ \checkmark\ \text{kotske ett}}{=} u_j s_{i_1}\cdots s_{i_p}$. Then

$$E_{\mu'} = \tau_{v(\mu')}^{-1} \underline{\beta}_j\underline{\alpha}_{i_1}\cdots\underline{\alpha}_{i_p}(1 = E_0').$$

deep result; long proof

to get leading term right. $\quad$ E_λ dual

A1 $\quad \beta_1 = \omega^{-1}(U^{-1}) = \omega^{-1}(T_1Y^{-1}) = XT_1 = XUY$.

$\alpha_1 = T_1 - b(Y^{-1}) = UY - b(Y^{-1})$

Not: $n\in\mathbb{N}$, $0\le r\le n$. The q-binomial coefficient is

$$\binom{n}{r}_q := \frac{(q;q)_n}{(q;q)_r(q;q)_{n-r}} = \frac{(1-q^n)\cdots(1-q^{n-r+1})}{(1-q^r)\cdots(1-q)}.$$

← Pochhammer

$E_n := E_{n\frac{\alpha}{2}}$

Prop: For $n \geq 0$: ($\underline{\alpha}_1 E_{n+1} = q^{-k/2} E_{-n-1}$ & $\beta_1 E_{-n} = q^{k/2} E_{n+1}$

Hence,) $\begin{cases} E_{n+1} = q^{-k/2} \beta (\underline{\alpha\beta})^n (1) \\ E_{-n} = (\underline{\alpha\beta})^n (1) \end{cases}$

Indeed, $E_{-n} = \binom{k+n}{n}_q^{-1} \sum_{r+s=n} \binom{k+r-1}{r}_q \binom{k+s}{s}_q x^{r-s}$

and analogously for E_{n+1}.

$\Longrightarrow$ $E_0 = 1$; $E_1 = x$; $E_{-1} = x^{-1} + \frac{1-q^k}{1-q^{k+1}} x$ (indeed $x < x^{-1}$ for $\frac{\alpha}{2} < -\frac{\alpha}{2}$) Heckman ordering

etc.

Prop ~~Affx~~ For $n \geq 0$:

$P_n(x) = E_{-n}$ with $k+n-1$ and $k+s-1$

$= \binom{k+n-1}{n}_q^{-1} \frac{(q^k;q)_n}{(q;q)_n} x^n {}_2\varphi_1\left(\begin{matrix} q^{-n} & q^k \\ & q^{1-n-k} \end{matrix} ; q, x^{-2} q^{1-k}\right)$

Rogers's q-ultraspherical pol^s.

$$\boxed{= E_{-n}(x) + q^k \frac{1-q^n}{1-q^{k+n}} E_n(x).}$$

$\Longrightarrow$ $P_0 = 1$, $P_1 = x^{-1} + x = m_1$,

$P_3 = x^{-3} + x^3 + \frac{(1-q^3)(1-q^k)}{(1-q)(1-q^{k+2})} P_1$

Erik K. 25-04-'23 | Crystal basis

(Lusztig: 'canonical basis')

$\mathfrak{g}$ $\frac{1}{2}$simple, (take $\mathfrak{sl}_n$ or even $\mathfrak{gl}_n$), $U_q\mathfrak{g}$.

For $\mathfrak{sl}_2$: $V(m)$ irrep. (type 1) of $\dim = m+1$, $u \in V(m)$ hwv $\rightsquigarrow$ $fu, f^2u, \ldots, f^m u$ (e, f, K, K^{-1} have usual relations) and $ef^n u = f^{n-1} u$, $eu = 0 = f^{m+1} u$. Crystal bases generalise this to other $\mathfrak{g}$.

Assume ADE. $U_q\mathfrak{g}$ as algebra over $\mathbb{Q}(q)$ or $\mathbb{C}(q)$.