I. G. MACDONALD'S *Affine Hecke Algebras and Orthogonal Polynomials*: A SUMMARY

EDWARD BERENGOLTZ

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Korteweg–de Vries Instituut voor Wiskunde

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Avec toute l'algèbre du monde on n'est souvent qu'un sot lorsqu'on ne sait pas autre chose. Peut-être dans dix ans la société tirera-t-elle de l'avantage des courbes que des songe-creux d'algébristes auront carrées laborieusement. J'en félicite d'avance la postérité.

— FREDERICK II (THE GREAT) OF PRUSSIA (1712–86), letter 93 to Voltaire

Introduction

A thorough perusal of the first three chapters of Macdonald's impressive book [Macdon] has revealed that the basics of the theory of (affine) root systems, as expected, are a rather lengthy string of lemmas and auxiliary results through which the reader should carefully and diligently manœuvre lest he drown. As such, I'm writing this abridgement of sorts to keep track of the important results (skipping intermediate lemmas and proofs, for I aim not to copy the book verbatim) and make sure the most pertinent elements of the story remain unravelled for the weary reader who may wish to consult the book later on (i.e., myself).

At the end of each chapter, we'll exhibit some of its content in the examples A_1 and/or C_2 for concreteness.

A fine root system Weyl you wait

1.1 Notational nonsense

The first part is mainly introducing notation. Let us immediately fix our base field to be $\mathbb{R}$ and E a real affine space. That is, it carries a faithful and transitive action by *translations* of a real vector space V by

$$v \cdot x =: x + v, \qquad v \in V, x \in E.$$

For any $x, y \in E$ there exists a unique $v \in V$ such that $y = x + v$ and we write $y - x$ for this v. A map $f\colon E \longrightarrow E'$ is called *affine-linear* if there exists a 'derivative' $Df \in \mathrm{Hom}_{\mathbb{R}}(V, V')$ such that $f(x+v) = f(x) + Df(v)$ for all $v \in V$, $x \in E$. Let

$$F := \{\text{affine-linear maps } E \longrightarrow \mathbb{R}\}$$

and

$$D\colon F \longrightarrow V^*, \quad f \longmapsto (Df\colon V \longrightarrow \mathbb{R})$$

be the ($\mathbb{R}$-linear) derivative map, whose kernel F° comprises precisely the constant functions.

NOTATION 1.1.1. We henceforth fix $n := \dim_{\mathbb{R}} V > 0$ and equip V with an inner product $\langle -, - \rangle$. This identifies E with $\mathbb{A}^n_{\mathbb{R}}$ as affine space and gives it a metric by $d(x,y) = \sqrt{\langle y-x, y-x\rangle}$ for $x, y \in E$ with $y - x$ as above. The space F becomes $(n+1)$-dimensional.

Importantly, we identify V with its dual V^* via $\langle -, - \rangle$, viz. by identifying linear functionals on V with the elements of V to whose inner product they correspond. In particular, for $f \in F$ we write $f(x+v) = f(x) + \langle Df, v\rangle$.

Moreover, we equip F with a positive-semidefinite symmetric bilinear form, also written $\langle -, - \rangle$ by $\langle f, g\rangle := \langle Df, Dg\rangle$, which vanishes on F°.

As usual, for $v \in V$ we define

$$v^\vee := \frac{2v}{\langle v, v\rangle}$$

in V (or V^* if you will) and the same formula also defines $f^\vee = \frac{2f}{\langle f,f\rangle}$ for $f \in F \setminus F^0$. Whilst all definitions so far have been nothing new, we give the next one an environment because it's so ubiquitous.

Definition 1.1.2. Let $f \in F \setminus F^\circ$. The **reflection in the affine hyperplane $f^{-1}(0) \subset E$** is the isometry of E given by

$$s_f\colon E \longrightarrow E, \quad x \longmapsto x - f^\vee(x)Df = x - f(x)Df^\vee.$$

Such reflections, indeed any isometry $\iota\colon E \longrightarrow E$, are affine-linear and act on $F \ni g$ by precomposition, i.e., for $f \in F$ and $x \in E$ we have

$$(\iota \cdot f)(x) := f(\iota^{-1}x).$$

In particular, $s_f \cdot g = g - \langle g, f\rangle f^\vee$.

Definition 1.1.3. For $v \in V$, the **translation by v** is the isometry

$$t(v)\colon E \longrightarrow E, \quad x \longmapsto x + v.$$

Clearly, $t(v)(x+u) = x + u + v = t(v)(x) + \langle Dt(v), u\rangle$, whence we see that the derivative of a translation is the constant function $V \longrightarrow \mathbb{R}, \quad u \longmapsto 1$.

Notation 1.1.4. We write c for this constant function 1 on V.[1]

Unravelling the action on F we see that $t(v) \cdot f = f - \langle Df, v\rangle\, c$ and for any isometry ι,

$$\iota \circ t(v) \circ \iota^{-1} = t(D\iota(v)).$$

1.2 Affine roots and alcoves

Throughout, the letter R will be used for [finite] root systems, whereas S is reserved for affine root systems. Elements of the former use Greek letters and of the latter, Latin.

Definition 1.2.1. An **affine root system** is a subset $S \subset F \setminus F^0$ such that

(A1) $\mathbb{R}S = F$,

(A2) For all $a, b \in S$ we have that $s_a(b) = b - \langle a, b\rangle a^\vee$ again lies in S,

(A3) For all $a, b \in S$, the numbers $\langle a^\vee, b\rangle$ are integers,

(A4) The **affine Weyl group** W_S, generated by all reflections s_a for $a \in S$, acts properly on E.[2]

The rank of S is n. As with root systems, if $a \in S$ and $pa \in S$ then $p \in \{\pm 1, \pm\frac{1}{2}, \pm 2\}$. An affine root a is called *indivisible* if $\pm\frac{1}{2} \notin S$. Notice that $W_S = W_{S^\vee}$. An isomorphism between two affine root systems[3] is a bijection obtained as the restriction of some isometry of the ambient Euclidean spaces and two affine root systems are called *similar* if they are isomorphic up to a global nonzero scalar.

[1] Happily, the speed of light is also $c = 1$!
[2] Viz. for all compacta $K_1, K_2 \subset E$, the number of elements $w \in W_S$ such that $wK_1 \cap K_2 \neq \varnothing$ is finite.
[3] At this point we resist the temptation to abbreviate these objects by referring to ARS'es.

Definition 1.2.2. An affine root system S is called

i) **reduced** if every affine root is indivisible, i.e., the only multiple in S of any $s \in S$ is $\pm a$,

ii) **irreducible** if $S \neq S_1 \sqcup S_2$ for any nonempty subsets S_i that are orthogonal with respect to $\langle -, - \rangle$.

As it turns out, any affine root system can be described as a root system 'with translations'. More precisely:

Theorem 1.2.3. *Let R be an irreducible root system inside a vector space V. For $\alpha \in R$ and $r \in \mathbb{Z}$, define $a_{\alpha,r} \in F$ by $\alpha + rc$. In other words, for $v \in V$,*

$$a_{\alpha,r}(v) := \langle \alpha, v \rangle + r.$$

Then the set

$$S = S(R) := \left\{ a_{\alpha,r} \;\middle|\; \alpha \in R \text{ and } r \in \begin{cases} \mathbb{Z} & \text{if } \frac{1}{2}\alpha \notin R \\ 2\mathbb{Z}+1 & \text{if } \frac{1}{2}\alpha \in R \end{cases} \right\}$$

is a reduced, irreducible affine root system.

Conversely, any reduced, irreducible affine root system is similar to $S(R)$ or $S(R)^\vee$ for some irreducible (though not necessarily reduced!) root system R.

Explicit expressions for affine coroots are nice in the root part;

Notation 1.2.4. Observe that for $a_{\alpha,r} \in S(R)$ we have

$$a_{\alpha,r}^\vee = \frac{2a_{\alpha,r}}{\langle Da_{\alpha,r}, Da_{\alpha,r} \rangle} = \frac{2\alpha}{\langle D\alpha, D\alpha \rangle} + \frac{2r}{\langle D\alpha, D\alpha \rangle} c = a_{\alpha^\vee, 2r/\langle D\alpha, D\alpha \rangle}.$$

Let S be an irreducible (not necessarily reduced) affine root system. One can show that the complement of the union of the affine hyperplanes associated to the affine roots is open in E and its connected components are *alcoves*. The affine Weylgroup acts on these faithfully and transitively. Their closures are n-simplices as expected

Notation 1.2.5. We henceforth fix a distinguished Weyl alcove C. If $S = S(R)$, choose it inside a fixed Weyl chamber of R for consistency with the forthcoming. We thus obtain a *basis* of S of size $n+1$, comprising those indivisible affine roots $a \in S$ whose affine hyperplanes $a^{-1}(0)$ go through a wall of C and that are positive everywhere inside C.

This basis of *simple affine roots* is written $\{a_i \mid i \in I\}$ for an index set I of size $n+1$.

For $i \neq j$, we have $\langle a_i, a_j \rangle \leqslant 0$ and hence the *Cartan integers* $\langle a_i^\vee, a_j \rangle$ are also nonpositive in that case (and equal to 2 if $i = j$).

As expected, we can now define the positive and negative roots of S by

$$S^\pm := \{a \in S \mid \pm a(x) > 0 \text{ for all } x \in C\}, \tag{1.2.1}$$

such that $S^- = -S^+$ and $S = S^+ \sqcup S^-$. An affine root $a \in S^\pm$ can be written

$$a = \sum_{i \in I} n_i a_i \qquad \text{with } \pm n_i \in \mathbb{Z}_{\geqslant 0}.$$

Let us now move towards identifying E and V. To do so, set $\alpha_i := Da_i \in V$ for each $i \in I$.[4] Since there are $n+1$ of these, there is a unique linear dependence $0 = \sum_i m_i \alpha_i$ with $m_i \in \mathbb{Z}_{>0}$, at least one of which equals 1. There exists at least one vertex x_i of C with $m_i = 1$ and such that $\{\alpha_j \mid j \neq i\}$ forms a basis of the root system $D(S) \subset V^* \cong V$.

NOTATION 1.2.6. We henceforth fix one such vertex x_0, or, more saliently, we fix one special index $0 \in I$ with $m_0 = 1$ and thus identify E with V by fixing x_0 as the origin. In particular we have $a_i = \alpha_i$ for all $0 \neq i \in I$.

We denote $s_i := s_{a_i}$ for all i. These simple reflections generate W_S, which is thus a Coxeter group on the generating set of simples.[5]

Macdonald proceeds with the famous classification theorem using (affine) Dynkin diagrams, which we skip here.

1.3 From finite to affine

If $S = S(R)$, then set $I_R := I \setminus \{0\}$ such that $\{\alpha_i \mid i \in I_R\}$ is a basis of simple roots for R and $a_i = \alpha_i$.[6] In particular, $D\alpha = \alpha$ for all $\alpha \in R$. Remains to determine $a_0 \in S$.

NOTATION 1.3.1. Henceforth fix $\varphi \in R$ to be the highest root, say $\varphi = \sum_{i \in I_R} m_i \alpha_i$ with the m_i nonnegative and their sum maximised. Then

$$a_0 = -\varphi + c$$

completes the simple affine roots, for then indeed

$$\sum_{i \in I} m_i a_i = c$$

is constantly 1. Define $\alpha_0 = Da_0 = -\varphi$. We assume $|\varphi|^2 = 2$ (so $\varphi^\vee = \varphi$) is the long root length.

We assume R to be **reduced and irreducible** now in view of the resulting Weyl groups. As usual, let P be the weight lattice

$$P = \{v \in V \mid \langle \alpha^\vee, v \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\},$$

and Q, the root lattice $\mathbb{Z}R$ therein. Similarly define $P^\vee, Q^\vee$. In view of dualities of various objects in the forthcoming, we outline what shall be known as 'the three cases'. The notation introduced therewith shall recur throughout, so we give it a green bar.

[4]Of course, this notation is confusing if $S = S(R)$ for some R, so let us keep S abstract for now.
[5]To wit, $s_i^2 = 1$ for all i and $(s_i s_j)^{m_{ij}} = 1$ for all finite such orders m_{ij}, where $i \neq j$.
[6]Macdonald uses I_0 for I_R but that feels ambiguous.

NOTATION 1.3.2. We define three pairs (R, R'), (S, S'), and (L, L') of (reduced and irreducible) root systems, (irreducible) affine root systems, and lattices[7]inside V, respectively, as follows. In the first two cases, R can be any root system with aforementioned requirements.

(I) $R' = R^\vee$ and $S = S(R)$ and $S' = S(R^\vee)$ and $L = P$ and $L' = P^\vee$.

(II) $R' = R$ and $S = S(R)^\vee = S'$ and $L = P^\vee = L'$.

(III) $R = C_n$ and $R' = R$ and $S = S(R)^\vee \cup S(R) = S'$ and $L = Q^\vee = L'$.

In each case, define the assignment $(\ )' \colon R \longrightarrow R'$ mapping α to α', being α if $R' = R$ and $\alpha^\vee$ if $R' = R^\vee$. This can be extended to S.

Moreover, define $\psi \in R$ to be such that ψ' is the highest root of R', viz. $\psi = \varphi$ if $R' = R$ and ψ is the highest short root if $R' = R^\vee$. Because we have normalised φ to 2, it follows from Notation 1.2.4 that in case I we have $a_0' = -\psi^\vee + c$ and in the other cases $a_0' = -\varphi + c$.

In any case, both $\langle \lambda, \alpha' \rangle$ and $\langle \lambda', \alpha \rangle$ lie in $\mathbb{Z}$ for all $\alpha \in R$, $\lambda \in L$ and $\lambda' \in L'$.

DEFINITION 1.3.3. Let e be the **exponent** of the finite group $\Omega' := L'/Q^\vee$, i.e., the least common multiple of the orders of all elements, unless $R = R'$ is of type B_n or C_{2n}, in which case it is set to 1.

NOTATION 1.3.4. Set $c_0 := e^{-1}c \in F^\circ$.

We define a new lattice inside F by $\Lambda := L \oplus \mathbb{Z}c_0$. Note it carries an obvious action of the extended affine Weyl group (to be defined anon; q.v. Definition 2.1.3).

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We postpone the definition of what Macdonald calls a W-labelling and what Eric and Heckman call multiplicity functions to when it is actually needed in Chapter 2.

1.4 Examples

Throughout this document, we shall restrict ourselves to types A_1 and $B_2 \cong C_2$ vis-à-vis examples, for these provide ample intuition (and anything higher-dimensional induces headache).

EXAMPLE 1.4.1. Let $R = A_1 = \{\pm\alpha\} \subset \mathbb{R}$ with $\alpha_1 = \alpha = \varphi = \sqrt{2}$ and $\alpha^\vee = \alpha$. We view the (co)weight lattice as $P = P^\vee = 2^{-1/2}\mathbb{Z} \subset \mathbb{R}$ on the nose and the (co)root lattice is $2^{1/2}\mathbb{Z}$. Therefore, Ω' has order $e = 2$. The Weyl chamber is $\mathbb{R}_{>0}$ and the Weyl group is $W_R = \{1, s_1\}$ with $s_1 = -\operatorname{id}$ (the longest element).

We have $S = \{\pm\alpha + rc \mid r \in \mathbb{Z}\}$ and $a_0 = -\alpha + c$. We have that for $x \in \mathbb{R}$ (see Notation 2.1.2 ahead),

$$a_0(x) = -\langle \alpha, x \rangle + 1 = -\sqrt{2}x + 1,$$

[7]Elements of these lattices shall always be denoted λ and λ', respectively.

which is positive for x in the Weyl chamber if and only if $x < \frac{1}{2}\sqrt{2}$, so the alcove is the interval $C = (0, \frac{1}{2}\sqrt{2})$. The positive affine roots are all $\pm\alpha + rc$ where $r \geqslant 0$ for the plus sign and $r \geqslant 1$ for the minus sign.

EXAMPLE 1.4.2. Let $R = C_2 = \{(2\varepsilon, 0), (0, 2\varepsilon), (\varepsilon, \varepsilon') \mid \varepsilon, \varepsilon' = \pm 1\} \subset \mathbb{R}^2$ and pick the basis $\alpha_1 = (1, -1)$ and $\alpha_2 = (0, 2)$. We see that $\alpha_1^\vee = \alpha_1$ and $\alpha_2^\vee = \frac{1}{2}\alpha_2$. One sees that

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$s_1 s_2$ has order 4 and so the longest element of $W_R \cong D_4$ is $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 = -\operatorname{id}$. Clearly, $Q^\vee = \mathbb{Z}R^\vee$ whilst $Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$ is 'missing' half the points on any horizontal line of odd y-coordinate compared to $Q^\vee$. An element $\lambda \in P$ is of the form (a, b) with $a, a - b \in \mathbb{Z}$, wherefore P equals $\mathbb{Z}(1, 1) \oplus \mathbb{Z}(1, 0)$. Similarly, the coweight lattice is of the form $\lambda' = (a, b) \in P^\vee$ with $2a, a - b \in \mathbb{Z}$ and so $P^\vee = \mathbb{Z}(\frac{1}{2}, \frac{1}{2}) \oplus \mathbb{Z}(1, 0)$ has 'twice the points' on half the lines compared to P.

The Weyl chamber is the positive cone walled by the hyperplanes spanned by $(1, 1)$ and $(1, 0)$. The highest root is $\varphi = 2\alpha_1 + \alpha_2 = (2, 0)$. As Macdonald remarks, this conflicts with our normalisation $|\varphi|^2 = 2$, but we shall not worry about this. The Weyl alcove C contains all those (x, y) in the Weyl chamber such that

$$a_0(x, y) = -2x + 1 > 0 \quad \Longleftrightarrow \quad x < \frac{1}{2}.$$

Its closure is thus the triangle with vertices the origin, $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2})$.

In principle, we allow any of the three cases for (S, S') and so on.

(I) If $R' = R^\vee$, then $\psi \in R$ is the highest short root $\alpha_1 + \alpha_2 = (1, 1)$ and so $a_0' = -\psi^\vee + c$. As $\psi^\vee = \psi$, the Weyl alcove is given by all those (x, y) in the Weyl chamber such that $-x - y + 1 > 0$, i.e., that lie below the line $y = 1 - x$. The (closure of the) alcove C' is thus the triangle with vertices the origin, $(\frac{1}{2}, \frac{1}{2})$ and $(1, 0)$ (i.e., two copies of C). We have $\Omega' = L'/Q^\vee$ of order and exponent 2.

We have $S = S(R)$ and $S' = S(R^\vee)$. Now $R^\vee = B_2$ on the nose if we define $B_2 = \{(\varepsilon, 0), (0, \varepsilon), (\varepsilon, \varepsilon') \mid \varepsilon, \varepsilon' = \pm 1\}$ with basis $\beta_1 = \alpha_1$ (now a long rather than a short root) and $\beta_2 = \alpha_2^\vee$ (now a short root).

(II) If $R' = R$ and $L = L' = P^\vee$, we have $a_0' = a_0 = -\varphi^\vee + c$ and so $C = C'$. Whilst Ω' is still order 2, its exponent is (manually set to) 1.

(III) The reducible case is somewhat cumbersome; we remark that $\Omega' = 1$ itself now.

EXAMPLE 1.4.3. Now, for case III, let S be of type $(C_1^\vee, C_1)$ so that $R = R' = \{\pm\alpha\}$ with $|\alpha|^2 = 2$ as for A_1 and

$$S = \{\pm\alpha + \tfrac{r}{2}c, \pm 2\alpha + rc \mid r \in \mathbb{Z}\}.$$

We have $a_0 = -\alpha + \frac{1}{2}c$ and $a_1 = \alpha$ (This is from Section 6.4; in Section 1.4, Macdonald says $a_1 = \frac{\alpha}{2}$ instead.) with lattices $L = L' = Q^\vee = \mathbb{Z}\alpha$. *Note bene* the difference with A_1. The action on $V = \mathbb{R}$ is $a_0(x) = \frac{1}{2} - \langle \alpha, x\rangle = \frac{1}{2} - x$ and $a_1(x) = x$. No idea what Macdonald is doing here since $\alpha \neq 1$ by choice of norm; but then for A_1 he also didn't bother with $\sqrt{2}$'s. Anyway it should be $a_0(x) = \frac{1}{2} - \sqrt{2}\,x$ and $a_1(x) = \sqrt{2}\,x$. The Weyl alcove is $(0, \frac{1}{2})$ only if you believe the action on V.

An expedition to the Weyl group

2.1 A lengthy disquisition

With notation as prior, $S = S(R)$ is now a reduced and irreducible affine root system with affine Weyl group W_S generated by the simple reflections s_i, where $i \in I = I_R \sqcup \{0\}$.

NOTATION 2.1.1. The Weyl group of R is written $W_R \subset \mathrm{O}(V)$.[1]Its longest element is written w_0.

We continue identifying V with its dual so that a root $\alpha \in R$ corresponds to the functional $\langle \alpha, - \rangle$ and

$$S = \{\alpha + rc \mid \alpha \in R \text{ and } r \in \mathbb{Z}\}.$$

NOTATION 2.1.2. For utmost clarity, an affine root $a = \alpha + rc$ acts on $x \in V$ by

$$a(x) = \langle \alpha, x \rangle + rc(x) = \langle \alpha, x \rangle + r.$$

Using Definition 1.1.2, the corresponding reflection thus acts by (recall that, under our identifications, the 'derivative' D is the identity on R)

$$s_{\alpha+rc}(x) = x - (\alpha + rc)(x)D\alpha^\vee = x - (r + \langle \alpha, x \rangle)\alpha^\vee.$$

It is easy to show by direct computation that for any $\alpha \in R$ we have

$$s_\alpha \circ s_{\alpha+c} = s_{-\alpha+c}s_\alpha = t(\alpha^\vee).$$

Thus $t(Q^\vee)$ is a subgroup of W_S. This subgroup, being isomorphic to $Q^\vee$, inherits an obvious action of W_R by postcomposition, whence we conclude that

$$W_S = W_R \ltimes t(Q^\vee).$$

We can extend these translations to all coweights (except in case III).

DEFINITION 2.1.3. The **extended affine Weyl group $W(R, L')$** is

$$W = W(R, L') := W_R \ltimes t(L').^{[2]}$$

[1]Again, Macdonald uses W_0, which is fine, but let us be consistent with the index set's nomenclature.

Clearly $W_S \trianglelefteq W$ and W/W_S is Ω'. Dually, one may define $W' = W(R', L)$ for which the analogues of all forthcoming results hold. One easily sees that W permutes S (and the alcoves) by virtue of the following expression (which we label as a theorem only to make it stand out).

THEOREM 2.1.4. *Let $w \in W$, written $w = vt(\lambda')$ for $v \in W_R$ and $\lambda' \in L'$. For any $a = \alpha + rc \in S$ we have*

$$wa = va - \underbrace{\langle \lambda', \alpha \rangle}_{\in \mathbb{Z}} c,$$

meaning for any $x \in V$

$$(wa)(x) = \langle v\alpha, x \rangle + r - \langle \lambda', \alpha \rangle .$$

$$\ast\ast\ast$$

Let us now, for $w \in W$, count the number of positive affine roots that w makes negative.

DEFINITION 2.1.5. The **length $\ell(w)$** of $w \in W$ is defined to be $|S(w)|$, where

$$S(w) := S^+ \cap w^{-1}S^- = \{a \in S \mid a(x) > 0 \text{ and } (wa)(x) < 0 \text{ for all } x \in C\}.$$

Of course, $\ell(w^{-1}) = \ell(w)$ for $S(w^{-1}) = -wS(w)$. Moreover, clearly $\ell(s_i) = 1$ for all $i \in I$.

NOTATION 2.1.6. We define

$$\Omega := \{u \in W \mid \ell(u) = 0\}.$$

One can show $\Omega \cong \Omega'$ and $W = W_S \rtimes \Omega$; we shall encounter this group later.

The length satisfies a number of properties. In general, $\ell(vw) \leqslant \ell(v) + \ell(w)$ and equality is equivalent to four particular conditions on the sets $S(w)$ and so on. These are important for proofs but morally boil down to 'the positive roots turned negative by vw are those $a \in S^+$ such that either wa is already negative or wa is positive but vwa is negative,' which is probably not super enlightening. What one can say is that $S(v) = S(w)$ if and only if $vw^{-1} \in \Omega$.

NOTATION 2.1.7. We define two characteristic functions for future usage. Let

$$\sigma := \mathbb{1}_{S^+} - \mathbb{1}_{S^-} : S \longrightarrow \{\pm 1\} \quad \text{and} \quad \chi := \mathbb{1}_{R^-} : R \longrightarrow \{0, 1\}.$$

Notice from Notation 2.1.2 and Equation (1.2.1) that

$$S^+ = \{\alpha + rc \mid \alpha \in R \text{ and } r \geqslant \chi(\alpha)\}.$$

The length is nicely behaved when composing with simple reflections, namely for $w \in W$ and $i \in I$ we have

$$\ell(s_i w) = \ell(w) + \sigma(w^{-1}a_i) \quad \text{and} \quad \ell(ws_i) = \ell(w) + \sigma(wa_i). \tag{2.1.1}$$

Using induction on the length, one can hence derive existence of the following.

[2]Eric writes W for W_R, W^a for W_S and W^e for W in [Opdam].

DEFINITION 2.1.8. Let $w \in W$ and $p := \ell(w)$. A **reduced form** of w is a (nonunique) expression of the form

$$w = us_{i_1} \cdot \ldots \cdot s_{i_p}$$

for certain $u \in \Omega$ and $i_j \in I$, with $1 \leqslant j \leqslant p$.

With this notation, we can show easily that $S(w) = \{b_j\}_{j=1}^{p}$ with $b_j := s_{i_p} \cdot \ldots \cdot s_{i_{j+1}} a_{i_j}$.

Finally, the general length formula of any $w \in W_R \ltimes t(L')$ is nice to have. It depends on the order of the two factors of the semidirect product.

THEOREM 2.1.9. *Let $\lambda' \in L'$ and $v \in W_R$. Then*

$$\ell(vt(\lambda')) = \sum_{\alpha \in R^+} \left|\langle \lambda', \alpha \rangle + \chi(v\alpha)\right| \quad \text{and} \quad \ell(t(\lambda')v) = \sum_{\alpha \in R^+} \left|\langle \lambda', \alpha \rangle - \chi(v^{-1}\alpha)\right|.$$

In particular, $\ell(t(\lambda')) = \sum_{\alpha \in R^+} \langle \lambda', \alpha \rangle$ and $\ell(v) = \sum_{\alpha \in R^+} \chi(v\alpha)$.

As an interesting corollary, for a simple root α_i we have

$$\ell(t(\alpha_i^\vee)) = \langle \alpha_i^\vee, \alpha_i \rangle + \sum_{\alpha_i \neq \alpha \in R^+} \langle \alpha_i^\vee, \alpha \rangle = 2.$$

2.2 Vees and yous and omega again

NOTATION 2.2.1. For $\lambda' \in L'$, let λ'_+ be the unique dominant weight in its W_R-orbit and λ'_-, the antidominant weight.

DEFINITION 2.2.2 (Cherednik)**.** Let $\lambda' \in L'$.

i) Let $\boldsymbol{v(\lambda')}$ be the *shortest* element of W_R such that $v(\lambda')\lambda' = \lambda'_-$.

ii) Let $\boldsymbol{u(\lambda')}$ be the *shortest* element of the coset $t(\lambda')W_R$ of W. Explicitly, it is given by $u(\lambda') := t(\lambda')v(\lambda')^{-1}$.

We see $t(\lambda') = u(\lambda')v(\lambda')$, whereas $t(\lambda'_-) = v(\lambda')u(\lambda')$. Their lengths are additive. In fact, ℓ is additive on all elements of a coset of the form $u(\lambda')W_R$. These elements' $S(-)$ sets are concretely computable, as follows.

THEOREM 2.2.3. *Let $\lambda' \in L'$ and $a = \alpha + rc \in S$.*

i) $S(v(\lambda')) = \{\beta \in R^+ \mid \langle \lambda', \beta \rangle > 0\}$,

ii) $a \in S(u(\lambda')) \iff \alpha \in R^-$ *and* $1 \leqslant r \leqslant \chi(v(\lambda')^{-1}\alpha) + \langle \lambda', v(\lambda')^{-1}\alpha \rangle$,

iii) $a \in S(u(\lambda')^{-1}) \iff \chi(\alpha) \leqslant r \leqslant -\langle \lambda', \alpha \rangle$.
Equivalently, using Notation 2.1.2, $a(\lambda') < 0$.

***

So far we have not discussed the fundamental weights; time to rectify this transgression!

NOTATION 2.2.4. For $i \in I_R$, let $\pi'_i \in L'$ be the fundamental weights for $R^\vee$, i.e., $\langle \pi'_i, \alpha_j \rangle = \delta_{ij}$ and set $\pi'_0 := 0$.

Dually, define π_i to be the fundamental coweights for R', i.e., $\left\langle \pi_i, \alpha'_j \right\rangle = \delta_{ij}$ and $\pi_0 := 0$.

Recall m_i and ψ from Notations 1.3.1 and 1.3.2 analogously to define m'_i for $i \in I$ (with $m'_0 = 1$).

Using these we can give an explicit description of Ω, the group of length-0 elements.

NOTATION 2.2.5.

i) For $i \in I$, set $u_i := u(\pi'_i)$.

ii) For $i \in I$, set $v_i := v(\pi'_i)$ and $w_i := v(\pi_i)$.

iii) Let[3]

$$J := \{j \in I \mid \pi'_j \in L' \text{ and } m_j = 1\} \quad \text{and} \quad J' := \{k \in I \mid \pi_k \in L \text{ and } m'_k = 1\}.$$

The π'_j for $j \in J$ are called *minuscule weights*. Of course, $m_j = \left\langle \pi'_j, \varphi \right\rangle$.

Observe that $0 \in J, J'$ and $u_0 = v_0 = w_0 = 1$.

THEOREM 2.2.6. *The $\{u_j \mid j \in J\}$ form a group that coincides with Ω.*

We can turn J into an abelian group by declaring that $J \longrightarrow \Omega, \quad j \longmapsto u_j$ be an isomorphism. The v_j then also obey the relations $v_j v_k = v_{j+k} = v_k v_j$ for $j, k \in J$. One can show that for any $j \in J$, we have $u_j a_0 = a_j$, whence $u_j a_i = a_{i+j}$ now for any $i \in I$. This defines an additive action of J on I and moreover $v_j \alpha_i = \alpha_{i-j}$.[4] Given this, then for any $i \in I$ and $j, k \in J$ we have

$$\pi'_{k+j} = \pi'_j + v_j^{-1} \pi'_k \quad \text{and} \quad \pi'_{i+j} = m_i \pi'_j + v_j^{-1} \pi'_i.$$

REMARK 2.2.7. For $j \in J$, set w_{0j} to be the longest element of $\mathrm{Stab}_{W_R}(\pi'_j) = \left\langle s_{\alpha_1}, \ldots, \widehat{s_{\alpha_j}}, \ldots, s_{\alpha_n} \right\rangle$.[5] (This equality is obvious from the definition of the fundamental coweights.) Then the shortest element of W_R sending π'_j to $(\pi'_j)_-$ (see Notation 2.2.1) must be $v_j = w_0 w_{0j}$.

2.3 Order, order!

Let us define the partial dominance ordering on the lattice L'. To do so, we first need the Bruhat ordering on the Weyl group and its extension.

DEFINITION 2.3.1. Let G be a Coxeter group. The **Bruhat ordering** $\leqslant$ on G is defined by

$$g \leqslant h \quad \Longleftrightarrow \quad \begin{array}{c}\text{some (not necessarily connected) substring of some} \\ \text{reduced expression for } h \text{ is a reduced expression for } g.\end{array}$$

[3]Cf. Eric's O^* in [Opdam], which in our notation corresponds to $J \setminus \{0\}$. The seemingly superfluous requirement that the fundamental (co)weights lie in the appropriate lattice is to ensure all three cases are covered.

[4]Recall α_0 from Notation 1.3.1.

[5]In Eric's notation, w_{0j} is w_{λ_j} where λ_j is our π'_j.

Apply this to W_S and extend it to $W = \Omega \ltimes W_S$ separately on each coset of W_S, viz. by declaring that for $u, u' \in \Omega$ and $w, w' \in W_S$ we have $uw \leqslant u'w'$ if and only if $u = u'$ and $w \leqslant w'$.

NOTATION 2.3.2. Let $L'_{++} := \{\lambda' \in L' \mid \langle \lambda', \alpha_i \rangle \geqslant 0 \text{ for all } i \in I_R\}$ be the dominant elements.

As usual, set $Q_+^\vee = \sum_{i \in I_R} \mathbb{Z}_{\geqslant 0} \alpha_i^\vee$. The dominance ordering on L'_{++} is the usual one:

$$\lambda' \geqslant \mu' \quad \Longleftrightarrow \quad \lambda' - \mu' \in Q_+^\vee.$$

Quite some work goes into showing that the following extension is sensible.

DEFINITION 2.3.3 (Heckman)**.** Let $\lambda', \mu' \in L'$. We declare

$$\lambda' \geqslant \mu' \quad \Longleftrightarrow \quad \begin{array}{l} \textit{either} \text{ we have that } \lambda'_+ > \mu'_+ \text{ in } L'_{++}, \\ \textit{or } \lambda'_+ = \mu'_+ \text{ and } v(\lambda') \leqslant v(\mu') \text{ in } W_R. \end{array}$$

The appropriate analogue holds in L as well.

As Eric observes in [Opdam, f. 8], the last inequality is not a typographical error. With this ordering, λ'_- is the dominant element of the orbit $W_R\lambda'$! We cherrypick some results.

THEOREM 2.3.4. *Let $\lambda' \in L'$.*

(i) *For any $i \in I$ we have*

$$a_i(\lambda') > 0 \quad \Longleftrightarrow \quad s_i\lambda' > \lambda.$$

(ii) *Let $v, w \in W$. Then $v \leqslant w$ implies that $v(0) \leqslant w(0)$ in L'.*

(iiia) *Let $v(\lambda') = s_{i_1} \cdot \ldots \cdot s_{i_p}$ be a reduced form. For $0 \leqslant j \leqslant p$, set $\lambda'_j := s_{i_{j+1}} \cdot \ldots \cdot s_{i_p}\lambda'$. Then*

$$\lambda'_- = \lambda'_0 > \lambda'_1 > \ldots > \lambda'_p = \lambda'.$$

(iiib) *Let $v(-\lambda')^{-1} = s_{j_q} \cdot \ldots \cdot s_{j_1}$ be a reduced form. For $0 \leqslant k \leqslant q$, set $\mu'_k := s_{j_{k+1}} \cdot \ldots \cdot s_{i_q}\lambda'$. Then*

$$\lambda'_+ = \mu'_0 < \mu'_1 < \ldots < \mu'_q = \lambda'.$$

2.4 Labellings and other things

DEFINITION 2.4.1. A **multiplicity function** of S is a map $k\colon S \longrightarrow \mathbb{R}$ that is constant on W-orbits.

If R is simply laced (of type ADE), then any such k must be constant. If $R \neq R^\vee$, then k assumes at most two values (for there are long and short roots). In case III (see Notation 1.3.2), there are five possible values. If k takes an argument from $R \subset S$, we write it as a subscript to be consistent with Eric. We define a dual function k' on S' on a case-by-case basis:

(I) For $a' = \alpha^\vee + rc \in S'$, set $k'(a') := k(\alpha + rc)$,

(II) Simply set $k' := k$,

(III) [something specific and ugly].

Let us define what Bourbaki would call the *demi-somme des racines positives* weighted by multiplicity.

NOTATION 2.4.2. Set

$$\rho_{k'} := \frac{1}{2} \sum_{\alpha \in R^+} k'_{\alpha^\vee} \alpha \quad \text{and} \quad \rho'_k := \frac{1}{2} \sum_{\alpha \in R^+} k_{(\alpha')^\vee} \alpha'.$$

One easily observes (perhaps recalling Notation 2.1.7) that $w^{-1} \in W_R$ maps either of these two expressions to themselves with an additional factor $\sigma(w\alpha)$ in each term. In particular, $s_i = s_{\alpha_i}$ for $i \in I_R$ simply subtracts $k(\alpha_i^\vee)\alpha_i$ from either expression with the appropriate placement of apostrophes. Therefore, if k (resp. k') does not assume the value 0, then ρ'_k (resp. $\rho_{k'}$) are stabilised only by $1 \in W_R$.

NOTATION 2.4.3. Akin to Definition 2.2.2ii), for $\lambda \in L$ we define $u'(\lambda)$ be the *shortest* element of the coset $t(\lambda)W_R$ of W. (Note that $W_{R'} = W_R$ on the nose in all cases.)

NOTATION 2.4.4. For $\lambda' \in L'$, let

$$r'_k(\lambda') := u(\lambda')(-\rho'_k).$$

For $\lambda \in L$, let

$$r_{k'}(\lambda) := u'(\lambda)(-\rho_{k'}).$$

These can be explicitly computed.

NOTATION 2.4.5. Let $\eta\colon \mathbb{R} \longrightarrow \pm 1$ be given by $\mathbb{1}_{>0} - \mathbb{1}_{\leqslant 0}$.

Then actually

$$r'_k(\lambda') = \lambda' + \frac{1}{2} \sum_{\alpha \in R^+} \eta(\langle \lambda', \alpha \rangle) k_{(\alpha')^\vee} \alpha' \quad \text{and} \quad r_{k'}(\lambda) = \lambda + \frac{1}{2} \sum_{\alpha \in R^+} \eta(\langle \lambda, \alpha' \rangle) k'_{\alpha^\vee} \alpha.$$

One can show that $r'_k\colon L' \longrightarrow V$ thus defines an injective map whose image excludes elements of the form $s_i r'_k(\lambda')$ for any $\lambda' \in L'$ that are fixed by s_i for some $i \in I$. One can actually be slightly more precise: for any $i \in I$ we have

$$s_i r'_k(\lambda') = r'_k(s_i\lambda') + \begin{cases} 0 & \text{if } s_i\lambda' \neq \lambda', \\ k_{(\alpha'_i)^\vee}\alpha'_i & \text{if } s_i\lambda' = \lambda'. \end{cases} \tag{2.4.1}$$

Finally, r'_k commutes with the action of Ω on L'.

2.5 Examples

EXAMPLE 2.5.1. We use the same notation as in Example 1.4.1. Let $w = t(\lambda') \in W = W_S$, where $\lambda' = n \cdot 2^{-1/2} \in P$, for $n \in \mathbb{Z}$. Then

$$\begin{aligned} S(w) &= \{\pm\alpha + rc \in S \mid \pm x + r > 0 \text{ and } \pm x + r < \pm\lambda' \text{ for all } x \in C\} \\ &= \{\alpha + rc \mid r \geqslant 0 \text{ and } r < n\} \cup \{-\alpha + rc \mid r > 0 \text{ and } r \leqslant -n\}, \end{aligned}$$

whence we conclude that the allowed values of r (for precisely one affine root $\pm\alpha + rc$) are

$$r \in \begin{cases} \varnothing & \lambda' = 0, \\ \{0, 1, \dots, n-1\} & \lambda' > 0, \\ \{1, 2, \dots, |n|\} & \lambda' < 0. \end{cases}$$

In any case, the length is $|n|$, which evidently agrees with the length formula Theorem 2.1.9. For $v = s_1 t(\lambda') \in W$ we get

$$S(v) = \{\alpha + rc \mid r \geqslant 0 \text{ and } r \leqslant n\} \cup \{-\alpha + rc \mid r > 0 \text{ and } r < -n\}$$

by similar analysis and so the length is now $|n+1|$, which agrees with the formula for $\ell(s_i w)$ involving σ.

The new generator for W_S compared to W_R is

$$s_0 = s_{-\alpha+c} = t(\alpha^\vee)s_1 = t(\alpha/2)s_1.$$

For $\lambda' = n\alpha/2$, we have $\lambda'_\pm = \pm|n|\alpha/2$, such that $v(\lambda')$ is trivial if and only if $n \leqslant 0$. The unique fundamental weight is $\pi'_1 = 2^{-1/2} = \alpha/2 \in P^\vee$ so $v_1 = s_1$. The 'minuscule indices' are $J = \{0, 1\} = I$ and so we see

$$\Omega = \{1, 2^{-1/2}\} = 2^{1/2}\mathbb{Z} \big/ 2^{-1/2}\mathbb{Z} = P^\vee / Q^\vee = \Omega',$$

as expected. The dominant weights are $L'_{++} = \mathbb{Z}_{\geqslant 0}\pi'_1$.

Finally, for $k(\alpha) = k = k'(\alpha^\vee)$, we have $\rho = \rho'_k = \rho_{k'} = \frac{1}{2}k\alpha$ so that for $\lambda' = n\alpha/2$ we get

$$r_{k'}(\lambda') = u(\lambda')(-\rho) = t(\lambda')v(\lambda')^{-1}(-\rho) = \begin{cases} t(\lambda')s_1(-\rho) = \lambda' + \rho = (n+k)\alpha/2 & n > 0 \\ t(\lambda')(-\rho) = \lambda' - \rho = (n-k)\alpha/2 & n \leqslant 0. \end{cases}$$

Indeed, Equation (2.4.1) applied to $\lambda' = \alpha/2$ is true, as

$$s_1 r_{k'}(\alpha/2) = s_1(1+k)\alpha/2 = (-1-k)\alpha/2 = r_{k'}(s_1\alpha/2)$$

and, since $s_0\alpha/2 = \alpha/2 + s_1\alpha/2 = 0 \neq \alpha/2$,

$$s_0 r_{k'}(\alpha/2) = \alpha/2 + s_1 r_{k'}(\alpha/2) = -k\alpha_2 = -\rho = t(0)v(0)^{-1}(-\rho) = r_{k'}(0)(-\rho) = r_{k'}(s_0\alpha/2)(-\rho).$$

On to C_2.

EXAMPLE 2.5.2. Now to C_2 again from Example 1.4.2 and let us stick to case I. We see that s_φ is the reflection in the x-axis and so equals

$$s_\varphi = -s_2 = s_1 s_2 s_1 s_2^2 = s_1 s_2 s_1.$$

The coweight lattice is spanned by $\pi'_1 := (1, 0)$ and $\pi'_2 := (\frac{1}{2}, \frac{1}{2})$, as one easily checks. (The fundamental coweights, spanning P, are $\pi_1 = (1, 0)$ and $\pi_2 = (1, 1)$.) As such, then,

$$s_0 = s_{-\varphi+c} = t(\varphi^\vee)s_\varphi = t(\pi'_1)s_1 s_2 s_1.$$

In $W_{S'}$, we have $s_\psi = -s_1$ (because $\psi \perp \alpha_1$) and so $s_0' = s_{-\psi^\vee + c} = t(\psi)s_\psi = t(\pi_2)s_2s_1s_2$.

Take $w = t(\pi_2')s_1 \in W$ as an example. The length formula predicts

$$\begin{aligned}\ell(w) &= \sum_{\alpha\in R^+} \left|\langle \pi_2', \alpha\rangle - \chi(s_1\alpha)\right| \\ &= \underbrace{|-\chi(-\alpha_1)|}_{\alpha=\alpha_1} + \underbrace{|1-\chi(-2\alpha_1+2\alpha_1+\alpha_2)|}_{\alpha=2\alpha_1+\alpha_2=\varphi} + \underbrace{|1-\chi(-\alpha_1+2\alpha_1+\alpha_2)|}_{\alpha=\alpha_1+\alpha_2=\psi} + \underbrace{|1-\chi(2\alpha_1+\alpha_2)|}_{\alpha=\alpha_2} \\ &= 1+1+1+1 = 4.\end{aligned}$$

To find these elements of $S(w)$, we need a reduced expression for w.

We have $\varphi = 2\alpha_1 + \alpha_2$ and so $J = \{0, 2\}$. We see that the shortest element of W_R mapping π_2' to the antidominant element in its orbit, being minus itself, is $w_0 w_{02}$ by Remark 2.2.7, where $w_0 = -\mathrm{id} = s_2s_1s_2s_1$, the longest element overall, and $w_{02} = s_1$. Therefore $v_2 = s_2s_1s_2$, which one easily verifies to effect $\pi_2' \longmapsto -\pi_2' = (\pi_2')_-$. The nontrivial element of Ω is therefore

$$u_2 = u(\tfrac{1}{2}, \tfrac{1}{2}) = t(\pi_2')v(\pi_2')^{-1} = t(\pi_2')s_2s_1s_2.$$

Indeed, we see that (using the same summation order as above)

$$\ell(u_2) = \ell(t(\pi_2')) - \ell(v_2) = (0+1+1+1) - (0+1+1+1) = 0,$$

as expected. Therefore, a reduced expression of our w above is $w = t(\pi_2')s_1 = u_2s_2s_1s_2s_1$, which shows the length is indeed 4. We conclude that

$$S(w) = \{b_1, \ldots, b_4\} = \{a_2, a_1+a_2, 2a_1+a_2, a_1\} = R^+.^{[6]}$$

Now that we have u_2, one we can also explicitly verify that $u_2a_0 = a_2$ (and vice versa) and $u_2a_1 = a_1$.

Finally, since $\psi = \alpha_1 + \alpha_2 = \beta_1 + 2\beta_2$, we see that $J' = \{0, 1\}$ and the nontrivial element of Ω' is $u(\pi_1)$. The entire analysis is analogous to the apostropheless case.

EXAMPLE 2.5.3. The goup W_S is generated by $s_0\colon x \longmapsto x - (\frac{1}{2} + \langle \alpha, x\rangle)\alpha^\vee = 1 - x$ (again, this cannot be right; since $\alpha = \sqrt{2}$, this should be $2^{-1/2} - x$) and of course $s_1(x) = -x$. Since $L' = Q^\vee$, the extended affine Weyl group W is just W_S.

[6]Apparently, there are no 'strictly' affine roots in this set; if nonetheless $\alpha + rc \in S(w)$, then $r \geqslant \chi(\alpha)$ and also $w(\alpha + rc) = s_1\alpha + (r - \langle \pi_2', \alpha\rangle)c$ must be in S^- and so we should have $r - \langle \pi_2', \alpha\rangle < \chi(s_1\alpha)$. If $\alpha \in R^+$ then $\langle \pi_2', \alpha\rangle$ is either 0 or 1. If it is 0 then $s_1\alpha = -\alpha$ so $r < 1$ and $r \geqslant 0$ allows for no translations. If it is 1 then $\alpha \neq \alpha_1$ and so $s_1\alpha \in R^+$ and we have $r - 1 < 0$ and $r \geqslant 0$. The situation for $\alpha \in R^-$ is similar.

Jack of all braids, master of none

3.1 Generators and relations

Definition 3.1.1. The **braid group $\mathfrak{B}(W)$** associated to W is the group with presentation

$$\mathfrak{B} = \mathfrak{B}(W) = \langle T(w) \mid T(vw) = T(v)T(w) \text{ for all } v, w \in W \text{ s.t. } \ell(vw) = \ell(v) + \ell(w) \rangle .$$

Let us define some special generators.

Notation 3.1.2. We set $T_i := T(s_i)$ for $i \in I$ and $U_j := T(u_j)$ for $j \in J$.

A warning: the assignment $w \longmapsto T(w)$ is not a group homomorphism. For example, $T_i^2 \neq 1$ since $0 = \ell(s_i^2) \neq 2\ell(s_i) = 2$. (Of course, $T(1) = 1$, though.)

Let $s_i, s_j \in W$ be such that $s_i s_j$ has finite order m_{ij}. Then we know $s_i s_j s_i \cdot \ldots = s_j s_i s_j \cdot \ldots$ with m_{ij} simple reflections on either side and this is a reduced form. Recalling the bit beneath Theorem 2.2.6, for $j, k \in J$ we have $u_j u_k = u_{j+k}$ with all factors having length zero. Finally, we knew that for $i \in I$ and $j \in J$,

$$u_j s_i = s_{u_j a_i} u_j = s_{i+j} u_j$$

is a reduced expression of length 1. Concluding, we obtain the following relations between the generators of $\mathfrak{B}$.

Notation 3.1.3 (Braid relations)**.** With notation as above, we have

(a) $T_i T_j T_i \cdot \ldots = T_j T_i T_j \cdot \ldots$ for all $i \neq j \in I$ with $m_{ij} < \infty$,

(b) $U_j U_k = U_{j+k}$ for all $j, k \in J$,

(c) $U_j T_i U_j^{-1} = T_{i+j}$ for all $i \in I$ and $j \in J$.

Actually, these are precisely all of the relations.

Theorem 3.1.4. *$\mathfrak{B}$ is generated by the T_i, U_j, for $i \in I, j \in J$, subject to the relations* (a), (b) *&* (c).

The assignment T behaves controlledly on reduced expressions. First of all, we can immediately use Equation (2.1.1) to see that for all $w \in W$ and $i \in I$ we have $T(ws_i) = T(w)T_i^{\sigma(wa_i)}$ and $T(s_i w) = T_i^{\sigma(w^{-1}a_i)} T(w)$. By induction, we obtain:

Theorem 3.1.5. *Let $v, w \in W$ and let $v^{-1}w = u_j s_{i_1} \cdot \ldots \cdot s_{i_p}$ be a reduced form. For $1 \leqslant j \leqslant p$, set $b_j := u_j s_{i_1} \cdot \ldots \cdot s_{i_{j-1}} a_{i_j}$ and $\varepsilon_j := \sigma(vb_j)$. Then*

$$T(v)^{-1}T(w) = U_j T_{i_1}^{\varepsilon_1} \cdot \ldots \cdot T_{i_p}^{\varepsilon_p}.$$

Theorem 2.1.9 immediately shows that ℓ is additive on $t(L'_{++})$. Hence we may do the following.

Notation 3.1.6. For $\lambda' \in L'_{++}$, set $Y^{\lambda'} := T(t(\lambda'))$. Then the set of all such elements lies in the centre of $\mathfrak{B}$.

For any $\lambda \in L'$, pick $\mu', \nu' \in L'_{++}$ such that $\lambda' = \mu' - \nu'$ and define $Y^{\lambda'} := Y^{\mu'}(Y^{\nu'})^{-1}$, which is well-defined by the above.

This defines $Y^{L'} = \{Y^{\lambda'} \mid \lambda' \in L'\}$ as abelian subgroup in $\mathfrak{B}$, isomorphic to L'.

The Y's associated to the minuscule weights are written $Y'_j := Y^{\pi'_j}$.

Earlier results then give the following, albeit after some labour.

Theorem 3.1.7. *Let $\lambda' \in L'$ and $i \in I_R$ such that $\langle \lambda', \alpha_i \rangle$ is either 0 or 1. In the first case, $s_i\lambda' = \lambda'$ and then $T_i Y^{\lambda'} = Y^{\lambda'} T_i$. In the second, $s_i \lambda' = \lambda' - \alpha_i^\vee$ and $T_i Y^{\lambda' - \alpha_i^\vee} = Y^{\lambda'} T_i^{-1}$.*

Remark 3.1.8. It is indeed true that $\langle \lambda', \alpha_i \rangle = 1$ if and only if $s_i\lambda' = \lambda' - \alpha_i^\vee$. From right to left is clear, whereas from left to right we see $1 = \langle \lambda', \alpha_i \rangle = -\langle s_i\lambda', \alpha_i \rangle$, from which obtain that $\langle -s_i\lambda' + \lambda', \alpha_i \rangle = 2 = \langle \alpha_i^\vee, \alpha_i \rangle$. From this we obtain that $-s_i\lambda' + \lambda' - \alpha_i^\vee \in \alpha_i^\perp$ and hence it is fixed by s_i. But it is also mirrored by s_i and therefore zero. ❏

We can give some more expressions for some of our distinguished elements entirely in terms of our basic generators.

Theorem 3.1.9. *We have*

$$T_0 = Y^{\varphi^\vee} T(s_\varphi)^{-1}$$

and

$$U_j = Y'_j T(v_j)^{-1}$$

for all $j \in J$. Moreover, if $\lambda' \in L'$ and $u(\lambda') = u_j s_{i_1} \cdot \ldots \cdot s_{i_q}$ is a reduced form, then

$$Y^{\lambda'} = U_j T_{i_1}^{\varepsilon_1} \cdot \ldots \cdot T_{i_q}^{\varepsilon_q} T(v(\lambda'))$$

for certain $\varepsilon_j \in \{\pm 1\}$, where $1 \leqslant j \leqslant q$.

Finally, if we define $\mathfrak{B}_R$ (again, Macdonald uses $\mathfrak{B}_0$) to be the subgroup of $\mathfrak{B}$ generated by the T_i for $i \in I_R$ only, then T_0 and the U_j can be replaced by $Y^{L'}$, as follows.

Theorem 3.1.10. *$\mathfrak{B}$ is generated by $\mathfrak{B}_R$ and $Y^{L'}$ subject to the relations in Theorem 3.1.7.*

3.2 Don't $\mathfrak{B}$ a braid; fear not the tilde

Recall $\Lambda = L \oplus \mathbb{Z}(e^{-1}c)$ from Notation 1.3.4. Its elements are viewed as functions on V in analogy with Notation 2.1.2 (with an extra factor e^{-1} in the latter term) and the extended affine Weyl group acts on it analogously to the action in Theorem 2.1.4 (with the appropriate e^{-1}).

NOTATION 3.2.1. Turn Λ into a multiplicative group X^Λ comprising multipliable formal symbols X^f, where $f \in \Lambda$, by declaring that

$$\Lambda \longrightarrow X^\Lambda, \quad f \longmapsto X^f$$

be an isomorphism. Denote $X^L = \{X^\lambda \mid \lambda \in L\}$.

DEFINITION 3.2.2 (Cherednik)**.** The **double braid group** $\widetilde{\mathfrak{B}}$ is the group generated by $\mathfrak{B}$ and X^Λ subject to the relations

i) For all $i \in I$ and $f \in \Lambda$ such that $f(\alpha_i') = 0$ or 1,[1]

$$T_i X^f X_i^\varepsilon = X^{s_i f},$$

with $\varepsilon = -1$ if $f(\alpha_i') = 0$ and $\varepsilon = 1$ otherwise,

ii) For all $j \in J$ and $f \in \Lambda$,

$$U_j X^f U_j^{-1} = X^{u_j f}.$$

We can enrich Theorem 3.1.10 to give generators and relations for $\widetilde{\mathfrak{B}}$, as follows.

NOTATION 3.2.3. Define the element $q_0 := X^{c_0} \in \widetilde{\mathfrak{B}}$. It is not difficult to check that it commutes with all T_i and U_j and is therefore central. Also, let $q := X^c = q_0^e$.

The relations in the next theorem are not actually all independent, but we omit the details.

THEOREM 3.2.4. *The double braid group $\widetilde{\mathfrak{B}}$ is generated by $\mathfrak{B}_R$, X^L, $Y^{L'}$ and q_0 subject to the following relations:*

(a) $T_i^\varepsilon Y^{-\lambda'} T_i = Y^{-s_i\lambda'}$ *for all $i \in I_R$ and $\lambda' \in L'$ such that $\langle \lambda', \alpha_i \rangle$ is either 0 — in which case $\varepsilon = -1$ — or 1, in which case $\varepsilon = 1$,*

(b) $T_i X^\lambda T_i^\varepsilon = X^{s_i\lambda}$ *with the same conditions as* (a) *except with* (L', R) *replaced by* (L, R')*,*

(c) $T_0 X^\lambda T_0 = q^{-1} X^{s_\varphi \lambda}$ *for all $\lambda \in L$ with $\langle \lambda, \varphi' \rangle = -1$,*[2]

(d) $T_0 X^\lambda T_0^{-1} = X^\lambda$ *for all $\lambda \in L$ with $\langle \lambda, \varphi' \rangle = 0$,*

(e) $U_j X^\lambda U_j^{-1} = q^{-\langle \lambda, v_j \pi_j' \rangle} X^{v_j^{-1}\lambda}$ *for all $\lambda \in L$ and $j \in J$.*

The reader may now wish to recall Notation 2.2.4 and 2.2.2.

[1]Macdonald writes $\langle f, \alpha_i' \rangle$ but, interpreting f as a function on V, our notation makes more sense and is consistent with the action of affine roots on V, which is also written as evaluation.

[2]We suspect the apostrophe is for notational consistency amongst expressions of the form $\langle L, -' \rangle$, since in Notation 1.3.1 we set $\varphi = \varphi^\vee$.

NOTATION 3.2.5. For $k \in J'$, set $X_k := X^{\pi_k}$ as the dual counterpart to Y'_j from Notation 3.1.6, $j \in J$.

Cherednik computed the commutator

$$[X_k, Y'_j]^{-1} = X_k^{-1}(Y'_j)^{-1}X_kY'_j = q^{\langle \pi'_j, \pi_k \rangle} T(w_k^{-1})T(v_j w_k^{-1})T(v_j)$$

for any $j \in J$ and $k \in J'$.

***

NOTATION 3.2.6. We can define the dual double braid group $\widetilde{\widetilde{\mathfrak{B}}}'$, generated by $\mathfrak{B}$ and $X^{\Lambda'}$ where, of course, $\Lambda' := L' \oplus \mathbb{Z}c_0$.

The counterparts of Notation 3.1.2 (or actually Theorem 3.1.9) are

$$T'_i := T_i, \quad T'_0 := Y^{(\psi')^\vee} T(s_\psi)^{-1} \quad \text{and} \quad U'_k := Y^{\pi_k} T(w_k)^{-1}$$

for $i \in I_R$ and $k \in J'$. (See Notation 1.3.2 for ψ.)

We know from the definition of L and L' in all three cases (Notation 1.3.2) that they are 'the same', possibly up to taking $\vee$'s. This is formalised as follows.

THEOREM 3.2.7. *There exists an anti-isomorphism* $\omega \colon \widetilde{\widetilde{\mathfrak{B}}}' \xrightarrow{\sim} \widetilde{\widetilde{\mathfrak{B}}}$ *such that*

- *i)* $\omega(X^{\lambda'}) = Y^{-\lambda'}$ *for all* $\lambda' \in L'$,
- *ii)* $\omega(Y^{\lambda}) = X^{-\lambda}$ *for all* $\lambda \in L$,
- *iiia)* $\omega(T_i) = T_i$ *for all* $i \in I_R$,
- *iiib)* $\omega(T'_0) = T_0^* := T(s_\psi)^{-1} X^{-(\psi')^\vee}$,
- *iv)* $\omega(U'_k) = V_k := T(w_k)^{-1} X_k^{-1}$ *for all* $k \in J'$,
- *v)* $\omega(q_0) = q_0$.

These V_k and T_0^* satisfy the anti-analogues of the braid relations 3.1.3:

$$V_k V_l = V_{k+l} \quad \text{and} \quad V_k^{-1} T_i V_k = \begin{cases} T_{i+k} & i+k \neq 0, \\ T_0^* & i+k = 0, \end{cases}$$

for all $i \in I_R$ and $k, l \in J'$.

The proof of the theorem is done separately for each of the three cases and boils down to showing that T_0^* and the V_k satisfy the appropriate relations in Theorem 3.2.4. As far as labours go in this book, it's rather Herculean. Like the Keryneian hind, we therefore run like the clappers. 'Aha!' exclaims the reader, 'into the next chapter?' Yes, but first the examples.

3.3 Examples

EXAMPLE 3.3.1. Continuing Example 2.5.1, we see $\mathfrak{B}$ is generated by T_0, T_1 and U_1 (since $U_0 = T(1_W) = 1_{\mathfrak{B}}$) subject to the braid relations. Explicitly, we see that $s_0 s_1 = s_{-\alpha+c} s_\alpha = t(\alpha^\vee)$ has infinite order, so the braid relations reduce to:

(a) empty statement,

(b) $U_1^2 = 1$,

(c) $U_1 T_0 U_1 = T_1$.

Therefore, given a word in these generators we can move any and all U_1's to the left, swapping T_0 with T_1 as they are passed. There are no relations amongst these latter two, so any element of $\mathfrak{B}$ is written $T_0^{i_1} T_1^{i_2} T_0^{i_3} \cdot \ldots$ or U_1 times such an element. Alternatively, we can get rid of all T_0's using (c) so the element is written as a word in T_1 and U_1 subject to (b).

Moreover, Theorem 3.1.9 says that $Y'_1 = T(t(\pi'_1)) = U_1 T(v_1) = U_1 T_1$ and $T_0 = Y^\alpha T_1^{-1} = T(t(\alpha)) T_1^{-1}$. Since $\ell(t(\alpha)) = \langle \alpha, \alpha \rangle = 2 = 1 + 1 = 2 \langle \pi'_1, \alpha \rangle = 2\ell(t(\pi'_1))$ using Theorem 2.1.9, we get that $Y^\alpha = (Y'_1)^2 = U_1 T_1 U_1 T_1 = T_0 T_1$, so $T_0 = T_0 T_1 T_1^{-1} = T_0$, which says nothing. Of course, we have $Y'_0 = 1$.

We see $\mathfrak{B}_R$ is the free group on one generator T_1 and $Y^{L'}$ is generated by $Y'_1 = U_1 T_1$. They satisfy the relation $T_1 (Y'_1)^{-1} = Y'_1 T_1^{-1}$ and indeed both are equal to U_1. Checking Theorem 3.1.10, then, we should be able to retrieve the two nontrivial braid relations. Indeed, this single relation gives both, for $U_1^2 = Y'_1 T_1^{-1} T_1 (Y'_1)^{-1} = 1$ and

$$U_1 T_0 U_1 = T_1 (Y'_1)^{-1} T_0 T_1 (Y'_1)^{-1} = T_1 (Y'_1)^{-1} (Y'_1)^2 T_1^{-1} T_1 (Y'_1)^{-1} = T_1.$$

The double braid group has an additional generating set X^Λ, where $\Lambda = 2^{-1/2}\mathbb{Z} \oplus 2^{-1}\mathbb{Z}c$. It is alternatively generated by U_1, T_1, $X_1 := X^{\pi_1}$ and Y'_1 with the appropriate relations. (Mind that $\pi_1 = \pi'_1$ generates both L and L'; see Example 1.4.1.) The relations in Theorem 3.2.4 are

(a) $T_1 (Y'_1)^{-1} T_1 = Y'_1$,

(b) $T_1 X_1 T_1 = X_1^{-1}$,

(c) $T_0 X_1^{-1} T_0 = q^{-1} X^{s_\varphi(-\pi_1)} = q^{-1} X_1$,

(d) empty statement,

(e) $U_1 X_1 U_1 = q^{-\langle \pi_1, -\pi'_1 \rangle} X^{s_1^{-1}\pi_1} = q^{1/2} X_1^{-1}$.

Note that $e = 2$ so $q^{1/2} = q_0$.

EXAMPLE 3.3.2. The braid group for our C_2 example is generated by T_0, T_1, T_2, U_2 subject to the braid relations. Again, $s_0 s_1 = t(\varphi^\vee) s_1 s_2$ has infinite order, as does $s_0 s_2$, and $m_{12} = 4$. Therefore, using the identities for u_2 from Example 2.5.2,

(a) $T_1 T_2 T_1 T_2 = T_2 T_1 T_2 T_1$,

(b) $U_2^2 = 1$,

(c) $U_2 T_0 U_2 = T_2$ and $U_2 T_1 U_2 = T_1$.

We moreover have

$$T_0 = Y^{\varphi^\vee} T(s_\varphi)^{-1} = Y^{\pi_1'} T(s_1 s_2 s_1)^{-1} = Y^{\pi_1'} T_1^{-1} T_2^{-1} T_1^{-1},$$

for $\ell(s_1 s_2 s_1) = 3 = \ell(s_1) + \ell(s_2) + \ell(s_1)$, as one verifies, and

$$U_2 = Y_2' T(v_2)^{-1} = Y_2' T_2^{-1} T_1^{-1} T_2^{-1}$$

for the same reason. Thus, $\mathfrak{B}$ is generated by $T_1, T_2, Y^{\pi_1'}$ and Y_2' subject to the following relations:

i) $T_1 Y^{\pi_1' - \alpha_1^\vee} = T_1 (Y_2')^2 Y^{-\pi_1'} = Y^{\pi_1'} T_1^{-1}$,

ii) $T_1 Y_2' = Y_2' T_1$,

iii) $T_2 Y^{\pi_1'} = Y^{\pi_1'} T_2$,

iv) $T_2 Y^{\pi_2' - \alpha_2^\vee} = T_2 Y^{\pi_1'} (Y_2')^{-1} = Y_2' T_2^{-1}$.

We should be able to retrieve the braid relations from these. For example, (b) follows from

$$\begin{aligned} U_2^2 &= Y_2' T_2^{-1} T_1^{-1} T_2^{-1} Y_2' T_2^{-1} T_1^{-1} T_2^{-1} \\ &\overset{\text{iv)}}{=} T_2 Y^{\pi_1'} (Y_2')^{-1} T_1^{-1} T_2^{-1} T_2 Y^{\pi_1'} (Y_2')^{-1} T_1^{-1} T_2^{-1} \\ &\overset{\text{i)}}{=} T_2 Y^{\pi_1'} (Y_2')^{-1} (Y_2')^2 Y^{-\pi_1'} T_1 (Y_2')^{-1} T_1^{-1} T_2^{-1} \\ &\overset{\text{ii)}}{=} T_2 Y^{\pi_1'} Y^{-\pi_1'} Y_2' (Y_2')^{-1} T_1 T_1^{-1} T_2^{-1} \\ &= 1, \end{aligned}$$

where the second to last equality also used commutativity of $Y^{L'}$.

Aha! Affine Hecke algebras!

4.1 More generators and relations

If but we could turn the braid group into an algebra... and we can!

NOTATION 4.1.1. Henceforth fix forever:

- A real number $q \in (0,1)$,
- Real numbers $\tau_i \in \mathbb{R}_{>0}$ for each $i \in I$, such that $\tau_i = \tau_j$ whenever s_i and s_j are conjugate in W,
- A subfield $K \subseteq \mathbb{R}$ containing all τ_i and $q_0 := q^{1/e}$.

In particular, q and q_0 will from now on refer to the above rather than the double braid group element from Notation 3.2.3. (Of course, the two are related.)

DEFINITION 4.1.2. The **Hecke algebra** $\mathfrak{H}$ of W over K is the K-algebra

$$\mathfrak{H} = K[\mathfrak{B}] \Big/ \left((T_i - \tau_i)(T_i + \tau_i^{-1}) \;\Big|\; i \in I \right).$$

The basis elements of the group algebra of $\mathfrak{B}$ are written as elements of $\mathfrak{B}$ (rather than using e's or δ's). Its unit element is 1.

By previous results, $\mathfrak{H}$ is generated as K-algebra by the T_i and U_j (with $i \in I$ and $j \in J$) subject to the braid relations 3.1.3 as well as the *Hecke relations*

NOTATION 4.1.3.

(d) $(T_i - \tau_i)(T_i + \tau_i^{-1}) = 0$ for all $i \in I$. Equivalently, $T_i - \tau_i = T_i^{-1} - \tau_i^{-1}$.

The Hecke relations modify the previous result that $T(s_i w) = T_i^{\sigma(w^{-1}a_i)} T(w)$ in the Hecke algebra, namely

$$T_i T(w) = T(s_i w) + \chi(w^{-1}a_i)(\tau_i - \tau_i^{-1})T(w) \quad \text{and} \quad T(w)T_i = T(ws_i) + \chi(wa_i)(\tau_i - \tau_i^{-1})T(w).$$

***

Let $t, u \in \mathbb{R}^\times$ be parameters and x, a formal indeterminate.

NOTATION 4.1.4. We define

$$b(x) = b(t,u;x) := \frac{t - t^{-1} + (u - u^{-1})x}{1 - x^2}$$

and

$$c(x) = c(t,u;x) := \frac{tx - (tx)^{-1} + u - u^{-1}}{x - x^{-1}}.$$

A priori, consider these as formal rational functions in x.

The following results are crucial for the forthcoming and follow by direct computation.

PROPOSITION 4.1.5.

i) $c(t,u;x) = c(t^{-1}, u^{-1}; x^{-1})$,

iia) $b(t,t;x) = \frac{t-t^{-1}}{1-x}$,

iib) $c(t,t;x) = \frac{t^{-1}-tx}{1-x}$,

iii) $c(x) = t - b(x) = t^{-1} + b(x^{-1})$,

iva) $c(x) + c(x^{-1}) = t + t^{-1}$,

ivb) $b(x) + b(x^{-1}) = t - t^{-1}$,

v) $c(x)c(x^{-1}) - b(x)b(x^{-1}) = 1$.

The generalisation of Theorem 3.1.7 to $\mathfrak{H}$ is an important result called the Lusztig relation. We need some new letters first because when do we not?

NOTATION 4.1.6. Let $i \in I_R$ and set

$$\upsilon_i := \begin{cases} \tau_i & \text{if } \langle L', \alpha_i \rangle = \mathbb{Z}, \\ \tau_0 & \text{if } \langle L', \alpha_i \rangle = 2\mathbb{Z}. \end{cases}$$

The latter possibility *only* occurs in case (III) (q.v. 1.3.2) for α_i the unique long root.

Fix $i \in I_R$ (the dependence will be left out of notation) and define, for $j \in \mathbb{Z}$,

$$\widetilde{u}_j := \begin{cases} \tau_i - \tau_i^{-1} & \text{if } j \text{ even}, \\ \upsilon_i - \upsilon_i^{-1} & \text{if } j \text{ odd}. \end{cases}$$

The tilde, absent in Macdonald, I added to avoid any confusion with the elements of Ω.

(Case (III) haunting the notation is a recurring theme.) Recall Notation 3.1.6.

THEOREM 4.1.7 (Lusztig). *Let $\lambda' \in L'$ and $i \in I_R$. Then*

$$Y^{\lambda'} T_i - T_i Y^{s_i \lambda'} = b(\tau_i, \upsilon_i; Y^{-\alpha_i^\vee})(Y^{\lambda'} - Y^{s_i \lambda'}). \tag{4.1.1}$$

We can deduce the following explicit formulæ from Lusztig's result.

COROLLARY 4.1.8.

i) If $\langle \lambda', \alpha_i \rangle = \begin{cases} r > 0 \\ r < 0 \\ 0 \end{cases}$, *then* (4.1.1) *equals* $\begin{cases} \sum_{j=0}^{r-1} \widetilde{u}_j Y^{\lambda' - j\alpha_i^\vee} \\ \sum_{j=1}^{-r} \widetilde{u}_j Y^{\lambda' + j\alpha_i^\vee} \\ 0 \end{cases}$, *respectively,*

iia) $(T_i - \tau_i)Y^{\lambda'} - Y^{s_i\lambda'}(T_i - \tau_i) = -c(\tau_i, \upsilon_i; Y^{-\alpha_i^\vee})(Y^{\lambda'} - Y^{s_i\lambda'})$,

iib) $(T_i + \tau_i^{-1})Y^{\lambda'} - Y^{s_i\lambda'}(T_i + \tau_i^{-1}) = c(\tau_i, \upsilon_i; Y^{-\alpha_i^\vee})(Y^{\lambda'} - Y^{s_i\lambda'})$.

Considerable effort proves the following theorem, showing that the relations are more or less 'precisely enough' compared to those in $\mathfrak{B}$.

THEOREM 4.1.9. *The set $\{T(w) \mid w \in W\}$ forms a basis for $\mathfrak{H}$ as K-module.*

Generalising Theorem 3.1.10, we similarly have the following.

THEOREM 4.1.10. *The set $\{T(w)Y^{\lambda'} \mid w \in W_R$ and $\lambda' \in L'\}$ forms a basis for $\mathfrak{H}$ as K-module. The same is true for all $Y^{\lambda'}T(w)$.*

4.2 Time for a representation

NOTATION 4.2.1. Let $A' := K[L']$ as group algebra, with basis $\{e^{\lambda'} \mid \lambda' \in L'\}$ behaving in the expected manner. We have an action of W_R on A' by $we^{\lambda'} = e^{w\lambda'}$ for $w \in W_R$, extended K-linearly. The invariants for this action are denoted $A'_R := (A')^{W_R}$.[1] Analogously define $A = K[L]$ and $A_R = A^{W_R}$.

Elements $f \in A'$ will be written $f = \sum_{\lambda'} f_{\lambda'} e^{\lambda'}$ with almost all $f_{\lambda'} \in K$ equal to 0. We then define

$$f(Y) := \sum_{\lambda'} f_{\lambda'} Y^{\lambda'},$$

which span a commutative subalgebra $A'(Y)$ inside $\mathfrak{H}$ isomorphic to A'.

COROLLARY 4.2.2. *By Lusztig, for all $f \in A'$ and $i \in I_R$, we have*

$$f(Y)T_i - T_i(s_i f)(Y) = b(\tau_i, \upsilon_i; Y^{-\alpha_i^\vee})(f(Y) - (s_i f)(Y))$$

in $A'_R(Y)$.

THEOREM 4.2.3. *The centre of the Hecke algebra is $Z(\mathfrak{H}) = A'_R(Y)$.*

***

We include the construction of the basic representation from first principles. Skip ahead to the next asterism for the final result.

[1] As usual, Macdonald uses the subscript 0 here.

NOTATION 4.2.4. Set $\mathfrak{H}_R$ to be the K-subalgebra of $\mathfrak{H}$ generated by the $T(w)$ for $w \in W_R$.

Defined on the basis from Theorem 4.1.10, the map

$$\mathfrak{H} \longrightarrow A' \underset{K}{\otimes} \mathfrak{H}_R, \quad Y^{\lambda'} T(w) \longmapsto e^{\lambda'} \otimes T(w)$$

is an isomorphism of K-modules. Thus, if M is a left $\mathfrak{H}_R$-module, then we identify

$$\mathrm{Ind}_{\mathfrak{H}_R}^{\mathfrak{H}}(M) = \mathfrak{H} \underset{\mathfrak{H}_R}{\otimes} M \xrightarrow{\sim} A' \underset{K}{\otimes} M$$

via $f(Y)T(w) \otimes x \longmapsto f \otimes T(w) \cdot x$ for $f \in A'$, $w \in W_R$ and $x \in M$. Thus, using Lusztig, for any $i \in I_R$, the induced $\mathfrak{H}_R$-action on $A' \otimes_K M$ is given on pure tensors by

$$T_i \cdot (f \otimes x) = s_i f \otimes (T_i \cdot x) + b(\tau_i, \upsilon_i; e^{-\alpha_i^\vee})(f - s_i f) \otimes x.$$

Now *fix* the $\mathfrak{H}_R$-module $M = K$, spanned over K by some element x, with action $T_i \cdot x := \tau_i x$. The induced representation is then identified with A' with action

$$T_i \cdot f = \tau_i s_i f + (f - s_i f) b(\tau_i, \upsilon_i; e^{-\alpha_i^\vee})$$

for $i \in I_R$. This defines a K-algebra representation (q.v. Notation 4.2.6 ahead for the X inside b)

$$\mathfrak{H}_R \longrightarrow \mathrm{End}_K(A'), \quad T_i \longmapsto \tau_i s_i + b(\tau_i, \upsilon_i; X^{-\alpha_i^\vee})(\mathrm{id} - s_i) \tag{4.2.1}$$

that turns out to be faithful. The full action of $\mathfrak{H}$ will appear shortly.

* * *

We can view L inside F, so that $\mu \in L$ acts on $x \in V$ by evaluation $\mu(x) = \langle \mu, x \rangle$ and the action of W on F by precomposition restricts to L as follows.

NOTATION 4.2.5. For $w = t(\lambda')v \in W$ with $\lambda' \in L'$ and $v \in W_R$, and for $x \in V$, we have

$$(w \cdot \mu)(x) = \left\langle \mu, w^{-1} x \right\rangle = \left\langle \mu, v^{-1}(x - \lambda') \right\rangle,$$

so that $w \cdot \mu = v\mu - \langle v\mu, \lambda' \rangle c$.

Recall q from Notation 4.1.1.

NOTATION 4.2.6. For $f = \mu + rc \in F$ with $\mu \in L$, define its action on A (from Notation 4.2.1) as follows. Define $e^f := q^r e^\mu \in A$ and let $X^f \in \mathrm{End}_K A$ be defined by multiplication by e^f.

In general, for any $\lambda' \in L'$ or $\mu \in L$, let $X^{\lambda'} \in \mathrm{End}_K(A')$ and $X^\mu \in \mathrm{End}_K(A)$ be given by multiplication by $e^{\lambda'}$ and e^μ, respectively.

Finally, with this notation to hand, the W_R-action on A from Notation 4.2.1 can be extended to W as follows.

NOTATION 4.2.7. Let $w = t(\lambda')v \in W$ with $v \in W_R$. Then for $\mu \in L$,

$$w \cdot e^\mu = e^{w\mu} = q^{-\langle v\mu, \lambda' \rangle} e^{v\mu}.$$

This action is in fact faithful.

Now comes a rather annoying bit of notation entirely due to the existence of case III.

NOTATION 4.2.8. Hearkening back to Notation 4.1.1, define for $i \in I$ numbers $\tau_i' \in \mathbb{R}_{>0}$ by

- For all i we set $\tau_i' := \tau_i$ in cases I and II.
- For all $0 \neq i \neq n$ we set $\tau_i' := \tau_i$ in case III.

I.e., τ_0', τ_n' in case III are new.

These new numbers satisfy an appropriate version of the Hecke relations 4.1.3.

NOTATION 4.2.9. In case III only, set

$$T_0' := X^{-a_0} T_0^{-1} \quad \text{and} \quad T_n' := X^{-a_n} T_n^{-1}$$

(and $T_i' := T_i$ for $0 \neq i \neq n$). Then

$$(T_i' - \tau_i')(T_i' + \tau_i'^{-1}) = 0$$

for all $i \in I$.

We now define a bunch of operators on A to appear frequently in the forthcoming.

NOTATION 4.2.10. For $i \in I$, set

$$b_i := b(\tau_i, \tau_i'; e^{a_i}) \quad \text{and} \quad c_i := c(\tau_i, \tau_i'; e^{a_i}).$$

For $\varepsilon = \pm 1$, use these expressions to define operators (as in Notation 4.2.6)

$$b_i(X^\varepsilon) := b(\tau_i, \tau_i'; X^{\varepsilon a_i}) \quad \text{and} \quad c_i(X^\varepsilon) := c(\tau_i, \tau_i'; X^{\varepsilon a_i}).$$

Set $1 := \mathrm{id} \in \mathrm{End}_K A$ and identify elements of W with their action on A according to Notation 4.2.7.

REMARK 4.2.11. A warning: whilst the b_i and c_i commute amongst each other, the $b_i(X^\varepsilon)$ and so on do not commute with W. Indeed, one can easily show that

$$b_i(X)s_i = s_i b_i(X^{-1})$$

and similarly for c_i. More generally, for any $w \in W$ and $\mu \in L$, we have

$$wX^\mu w^{-1} = X^{w\mu}$$

as operators.

In view of the induced representation (4.2.1), one can prove the following.

THEOREM 4.2.12 (Cherednik). *There exists a representation $\beta\colon \mathfrak{H} \longrightarrow \mathrm{End}_K A$ such that for all $i \in I$, we have*

$$\beta(T_i) = \tau_i s_i + b_i(X)(1 - s_i)$$

and for all $j \in J$ we have

$$\beta(U_j) = u_j.$$

Moreover, the set $\{X^\mu \beta(T(w)) \mid \mu \in L, w \in W\}$ is K-linearly independent in $\mathrm{End}(A)$ *and hence β is faithful. It is called the* ***basic representation*** *of $\mathfrak{H}$.*

By virtue of the faithfulness of this basic representation, we *henceforth identify each $h \in \mathfrak{H}$ with the operator $\beta(h)$*. For instance, we shall write

$$T_i = \tau_i s_i + b_i(X)(1 - s_i) \quad \text{and} \quad U_j = u_j.$$

Proposition 4.1.5 (together with Remark 4.2.10) then yield more useful results.

PROPOSITION 4.2.13. *Let $i \in I$, $\varepsilon = \pm 1$ and $\mu \in L$. Then*

i) $T_i - \tau_i = c_i(X)(s_i - 1)$,

ii) $T_i + \tau_i^{-1} = (1 + s_i)c_i(X^{-1})$,

iii) $T_i^\varepsilon = \varepsilon b_i(X^\varepsilon) + c_i(X)s_i$,

iv) $T_i X^\mu - X^{s_i\mu} T_i = b_i(X)(X^\mu - X^{s_i\mu})$.

From this, explicit computation reveals that the T_i and X^μ satisfy the $\widetilde{\mathfrak{B}}$ relations from Theorem 3.2.4.

4.3 More identities in the affine Hecke algebra

Any 'function' $f \in A'_R$ gives rise to a central operator $f(Y) \in Z(\mathfrak{H})$ that maps A_R into itself. For functions living in A, now, we want to look at what they do 'to leading order', as follows.

NOTATION 4.3.1. Let $f \in A = K[L]$ and write

$$f = \sum_{\mu \leqslant \lambda} f_\mu e^\mu$$

for some $\lambda \in L$ dominating (q.v. Definition 2.3.3) the (finitely many) μ's with $f_\mu \neq 0$. We shall write

$$f = f_\lambda e^\lambda + \text{LOT}$$

to disregard the lower-order terms.

Recall the map η from Notation 2.4.5. To leading order, we can compute the action of $\mathfrak{H}$ on A explicitly. For simples; let $i \in I_R$ and $\lambda \in L$, then

$$T_i^{-1} e^\lambda = \tau_i^{-\eta(\langle \lambda, \alpha_i' \rangle)} e^{s_i\lambda} + \text{LOT}.$$

To be able to present the generalisation to arbitrary $T(w)$, we need more notation as usual. Recall Notation 4.1.1.

NOTATION 4.3.2.

- Define a function κ on the simple roots of R by $\kappa_i := \kappa(\alpha_i)$ such that $\tau_i = q^{\kappa_i/2}$.
- Extend κ to all of R by setting $\kappa_\alpha := \kappa_i$ for $\alpha \in W_R\alpha_i$.
- Similarly define κ' on R' using the τ_i'. Also define κ_0, κ_0' in this manner.
- For $w \in W_R$ and $\lambda \in L$, set

$$f(w, \lambda) := \frac{1}{2} \sum_{\alpha \in R^+} \eta(-\langle \lambda, \alpha' \rangle)\chi(w\alpha)\kappa_\alpha.$$

By applying the previous result to a reduced form, we get:

THEOREM 4.3.3. *Let $w \in W_R$ and $\lambda \in L$. Then*

$$T(w^{-1})^{-1}e^{\lambda} = q^{f(w,\lambda)}e^{w\lambda} + \text{LOT}.$$

NOTATION 4.3.4. If $w = s_{i_1} \cdot \ldots \cdot s_{i_p} \in W_R$ is a reduced form, let

$$\tau_w := \tau_{i_1} \cdot \ldots \cdot \tau_{i_p} = q^{\frac{1}{2}\sum_{\alpha \in R^+} \chi(w\alpha)\kappa_\alpha},$$

which is well-defined.

PROPOSITION 4.3.5. *On $1_A \in A$, the operator $T(w)$, where $w \in W_R$, acts by τ_w.*

Of course, we want to generalise this to all of W (equivalently, all of $\mathfrak{H}$), which requires... you guessed it. The spanner in the works is, as usual, case III. Let $S_1 = \{a \in S \mid \frac{1}{2}a \notin S\}$, which equals S except in case III. In general, it equals the union of all orbits Wa_i, for $i \in I$.

NOTATION 4.3.6. Henceforth fix a multiplicity function k on S_1, which we for convenience assume never to hit 0, by

$$k(a) := \tfrac{1}{2}(\kappa_i + \kappa_i') \quad \text{and} \quad k(2a) := \tfrac{1}{2}(\kappa_i - \kappa_i')$$

if $a \in Wa_i$. Define its dual labelling k' as in Section 2.4.

In cases I and II, $\tau_i = \tau_i'$ and so $k(a) = \kappa_i$ for all $a \in S$ (and $2a$ is moot).

sips tea

NOTATION 4.3.7. Let $a \in S_1$ such that $a = wa_i$ for some $w \in W$ and $i \in I$. Define

$$\tau_a := \tau_i \quad \text{and} \quad \tau_a' = \tau_i'$$

in accordance with Notation 4.2.8. Using these, define

$$b_a = b_{a,k} := b(\tau_a, \tau_a'; e^a) \quad \text{and} \quad c_a = c_{a,k} := c(\tau_a, \tau_a'; e^a)$$

and the corresponding operators $b_a(X), c_a(X) \in \mathrm{End}(A)$, as the analogues of Notation 4.2.10. Indeed, $b_a = wb_i$ and $c_a = wc_i$ evidently.

burns mouth

NOTATION 4.3.8. Let $w \in W$. With $S_1(w)$ as in Definition 2.1.5, let

$$c(w) = c_{S,k}(w) := \prod_{a \in S_1(w)} c_a.$$

sips tea more carefully

NOTATION 4.3.9. Let Φ be the field of fractions of A. Define a K-subalgebra hereof by

$$A[c] := A[c_a \mid a \in S].$$

(Note that any $a \in S \setminus S_1$ is of the form $2b$ for some $b \in S_1$, which defines c_a.)

With this to hand, we can say, for instance that

THEOREM 4.3.10. *Let $u, v \in W$. Then, as operators on A,*

$$T(u)^{-1}T(v) = \sum_{W \ni w \leqslant u^{-1}v} f_w(X)w$$

for certain $f_w \in A[c]$ such that $f_{u^{-1}v} = c(v^{-1}u)$. Furthermore, for $\lambda' \in L'$, we have

$$Y^{\lambda'} = c(u(\lambda')^{-1})(X)u(\lambda')T(v(\lambda')) + \sum_{\substack{w \in W \\ w(0) < \lambda'}} g_w(X)w$$

and

$$Y^{-\lambda'} = T(v(\lambda')^{-1})c(u(\lambda'))(X)u(\lambda')^{-1} + \sum_{\substack{w \in W \\ w(0) < \lambda'}} h_w(X)w^{-1}$$

for certain $g_w, h_w \in A[c]$.

In particular, if λ' is antidominant (i.e., $w_0\lambda' \in L'_{++}$),

$$Y^{\lambda'} = c(t(-\lambda'))(X)t(\lambda') + \dots,$$

whence we can derive the a complicated formula that becomes useful in the next chapter.

NOTATION 4.3.11. For λ' (anti)dominant, let

$$m_{\lambda'} := \sum_{\mu' \in W_R\lambda'} e^{\mu'},$$

which lies in A'_R and hence $m_{\lambda'}(Y)_R := m_{\lambda'}(Y)\big|_{A'_R}$ lands in A'_R again. Similarly define m_μ for $\mu \in L_{++}$ (anti)dominant.

THEOREM 4.3.12. *Let λ' be antidominant. Then*

$$m_{\lambda'}(Y)_R = \sum_{w \in W_R^{\lambda'}} (wc(t(-\lambda')))(X)t(w\lambda') + \sum_{\mu' \in \Sigma^R(\lambda')} g_{\mu'}(X)t(\mu'),$$

where $g_{\mu'} \in A[c]$, $W_R^{\lambda'}$ is transversal[2]*to the isotropy group* $\mathrm{Stab}_{W_R}(\lambda')$, *and $\Sigma^R(\lambda') = \Sigma(\lambda') - W_R\lambda'$, with $\Sigma(\lambda')$ the saturation*[3]*of $\{\lambda'\}$ in L'.*

In the cases to be treated, $\Sigma^R(\lambda')$ will be contained inside $\{0\}$ and so the precise definition does not matter for now.

*
* *

Recalling Notation 4.2.5, we can view elements $f \in A$ (or, analogously, A') as functions on V.

[3]Meaning it intersects each coset of the isotropy group in W_R in exactly one element.

[3]We have omitted saturated sets; $\Sigma(\lambda')$ is the smallest subset of L' containing λ', such that for all $\sigma' \in \Sigma(\lambda')$, $\alpha \in R$ and $r \in \mathbb{Z}$ with $0 \leqslant r \leqslant \langle \sigma', \alpha \rangle$, the entire string $\lambda' - r\alpha^\vee$ lies in $\Sigma(\lambda')$. (In particular, $s_\alpha\sigma'$ does, and so this saturation carries a W_R-action.)

NOTATION 4.3.13. For $f = \sum f_\lambda e^\lambda \in A$ and $x \in V$, define

$$f(x) := \sum_{\lambda \in L} f_\lambda q^{\langle \lambda, x \rangle}.$$

If $h = \frac{f}{g} \in \Phi$, define $h(x) := \frac{f(x)}{g(x)}$ wherever $g(x) \neq 0$.

EXAMPLE 4.3.14. As an important example, for $i \in I$ we have $c_i(x) = c(\tau_i, \tau_i'; e^{a_i})(x)$, whose denominator is $q^{\langle a_i, x \rangle} - q^{\langle -a_i, x \rangle}$. This is nonzero as long as $\langle a_i, x \rangle = a_i(x) \neq 0$.

There are several more results in section 4.5, which we skip at least for now.

NOTATION 4.3.15. For $a \in S_1$, define

$$G_a := \tau_a + b_a(X^{-1})(s_a - 1) = c_a(X^{-1}) + b_a(X^{-1}) s_a$$

as operators on A. In particular, let

$$G_i := G_{a_i} = s_i T_i.$$

One easily verifies (using Remark 4.2.11) that $w G_a w^{-1} = G_{wa}$ and $G_a^{-1} = c_a(X) - b_a(X^{-1}) s_a$ for all $w \in W$.

One may easily verify that if $W \ni w = u_j s_{i_1} \cdot \ldots \cdot s_{i_p}$ is a reduced expression, and $b_r := s_{i_p} \cdot \ldots \cdot s_{i_{r+1}} a_{i_r}$ (for $1 \leqslant r \leqslant p$, these are precisely the elements of $S_1(w)$), we have

$$T(w) = w G_{b_1} \cdot \ldots \cdot G_{b_p}.$$

THEOREM 4.3.16. *Let $a \in S_1$ such that $\alpha = Da \in R^+$. Then for any $\mu \in L$,*

$$G_a e^\mu = \tau_a^{-\eta(\langle \mu, \alpha^\vee \rangle)} e^\mu + \text{LOT}.$$

Hence, if $w \in W$ is such that $Da \in R^+$ for all $a \in S(w)$, then

$$w^{-1} T(w) e^\mu = \tau(w, \mu) e^\mu + \text{LOT},$$

with $\tau(w, \mu)$ defined below.

Compare the following to Notation 4.3.2.

NOTATION 4.3.17.

- For $a \in S_1$ in the W-orbit of a_i, define $\kappa_a := \kappa_i$. In other words,

$$\tau_a = q^{\kappa_a / 2}.$$

- For $w \in W$ and $\mu \in L$, define $\tau(w, \mu) := \prod_{a \in S_1(w)} \tau_a^{-\eta(\langle \mu, Da^\vee \rangle)}$.
- Similarly, set $f(w, \mu) := \frac{1}{2} \sum_{a \in S_1(w)} \eta(-\langle \mu, Da^\vee \rangle) \kappa_a$, such that

$$\tau(w, \mu) = q^{f(w, \mu)}.$$

Recall $r_{k'}$ from Notation 2.4.4. For $\lambda' \in L'_{++}$ dominant and $\mu \in L$ arbitrary, one can show that

$$f(t(\lambda'),\mu) = \langle \lambda', \mu - r_{k'}(\mu) \rangle .$$

From this one can deduce the action of $Y^{L'}$ on A; for any $\lambda' \in L'$,

$$Y^{\lambda'} e^{\mu} = q^{-\langle \lambda', r_{k'}(\mu) \rangle} e^{\mu} + \text{LOT}. \tag{4.3.1}$$

Consequently, for $f \in A'$ and $\mu \in L$ we have

$$f(Y)e^{\mu} = f(-r_{k'}(\mu))e^{\mu} + \text{LOT}. \tag{4.3.2}$$

If, in fact, $f \in A'_R$ and μ is dominant, then (see Notation 4.3.11)

$$f(Y)m_{\mu} = f(-\mu - \rho_{k'})m_{\mu} + \text{LOT}. \tag{4.3.3}$$

4.4 Double affine, double the fun

Recall the double braid group Definition 3.2.2 as well as Notations 4.2.8 and 4.2.9.

Definition 4.4.1 (Cherednik)**.** The **double affine Hecke algebra** $\widetilde{\mathfrak{H}}$ (DAHA) is

$$\widetilde{\mathfrak{H}} = K[\widetilde{\mathfrak{B}}] \Big/ \Big((T'_i - \tau'_i)(T'_i + \tau_i'^{-1}) \;\Big|\; i \in I \Big) .$$

This quotient is simply the Hecke relations (d) from Notation 4.1.3 except in case III for $i = 0, n$, where the apostrophes mean something.

By virtue of Theorem 3.2.4, the DAHA is generated by the AHA $\mathfrak{H}$ and X^L (q.v. Notation 3.2.1) as K-algebra, subject to the relations in the theorem. The correct analogue of Proposition 4.2.13iv) to $\Lambda \supset L$ is that for all $i \in I$ and $f \in \Lambda$,

$$T_i X^f - X^{s_i f} T_i = b_i(X)(X^f - X^{s_i f}). \tag{4.4.1}$$

Hence, Theorem 4.2.12 extends as follows.

Theorem 4.4.2 (Cherednik). *The representation β extends to a faithful representation (also written β because why not) $\widetilde{\mathfrak{H}} \longrightarrow \operatorname{End}_K A$, such that for all $\mu \in L$,*

$$\beta(X^{\mu}) = X^{\mu}$$

as operators (i.e., multiplication by e^{μ}).

We also get the familiar statements that the sets

$$\{T(w)X^{\mu} \mid w \in W, \mu \in L\}, \quad \{X^{\mu}T(w) \mid w \in W, \mu \in L\},$$
$$\{Y^{\lambda'}T(w)X^{\mu} \mid w \in W_R, \mu \in L, \lambda' \in L'\} \quad \text{and} \quad \{X^{\mu}T(w)Y^{\lambda'} \mid w \in W_R, \mu \in L, \lambda' \in L'\}$$

each form K-bases of $\widetilde{\mathfrak{H}}$ as vector space.

Now recall Notation 3.2.6 and Theorem 3.2.7.

Notation 4.4.3. Let $\widetilde{\mathfrak{H}}'$ be the dual DAHA, defined as follows.

- In cases I and II, it is obtained by swapping R with R' and L with L'.
- In case III, it is obtained by swapping τ_0' and τ_n'.[4]

Of course, the analogues of the above bases for $\widetilde{\mathfrak{H}}$ are bases for this dual.

Theorem 4.4.4. *The map*

$$\omega\colon \widetilde{\mathfrak{H}}' \longrightarrow \widetilde{\mathfrak{H}}, \quad X^{\lambda'}T(w)Y^{\mu} \longmapsto X^{-\mu}T(w^{-1})Y^{-\lambda'},$$

where $w \in W_R$, $\mu \in L$ and $\lambda' \in L'$, is an anti-isomorphism of K-algebras.

This now follows easily by checking the new Hecke relations compared to $\widetilde{\mathfrak{B}}'$ and $\widetilde{\mathfrak{B}}$.

By virtue of faithfulness, we henceforth identify $\widetilde{\mathfrak{H}}$ with $\beta(\widetilde{\mathfrak{H}}) \subset \operatorname{End}(A)$ as we did for $\mathfrak{H}$. Also recall from Notation 4.3.13 that we view A as functions on V.

Notation 4.4.5 (Cherednik)**.** Define K-linear maps as follows

$$\vartheta\colon \widetilde{\mathfrak{H}} \longrightarrow K, \quad h \longmapsto h(1_A)(-\rho_k') \quad \text{and} \quad \vartheta'\colon \widetilde{\mathfrak{H}}' \longrightarrow K, \quad h' \longmapsto h(1_{A'})(-\rho_{k'}).$$

Let $\widetilde{\mathfrak{H}} \ni h = f(X)T(w)g(Y^{-1})$ for some $f \in A$, $g \in A'$ and $w \in W_R$. By (4.3.2) and the absence of lower-order terms for $\mu = 0$, we know

$$g(Y^{-1})(1_A) = g(-r_{k'}(0))1_A = g(-\rho_{k'})1_A.$$

By Proposition 4.3.5, $T(w)1_A = \tau_w 1_A$. Hence

$$\vartheta(h) = f(-\rho_k')\tau_w g(\rho_{k'})$$

as operators on A. From this we get that

Proposition 4.4.6. $\vartheta' = \vartheta \circ \omega$.

Notation 4.4.7 (Cherednik)**.** Let $h \in \widetilde{\mathfrak{H}}$ and $h' \in \widetilde{\mathfrak{H}}'$. Define two 'commutators'

$$[h, h'] := \vartheta'(\omega^{-1}(h)h') = \left(\omega^{-1}(h)h'\right)(1_A)(-\rho_{k'}) \quad \text{and} \quad [h', h] := \vartheta(\omega(h')h).$$

Actually, they are equal on the nose by virtue of the proposition above.

For any $\gamma \in \widetilde{\mathfrak{H}}$ we have

$$[\gamma h, h'] = \vartheta'(\omega^{-1}(\gamma h)h') = \vartheta'(\omega^{-1}(h)\omega^{-1}(\gamma)h) = [h, \omega^{-1}(\gamma)h'].$$

We can therefore extend this pairing to arbitrary functions.

Notation 4.4.8. Let $f \in A$ and $f' \in A'$. Define

$$[f, f'] := [f(X), f'(X)] = \vartheta'(f(Y^{-1})f'(X)) = \left(f(Y^{-1}f'\right)(-\rho_{k'}).$$

If, dually, $[f', f] := \left(f'(Y^{-1})f\right)(-\rho_k')$, then the two are equal, and the pairing is thus symmetric.

[4]Macdonald says τ_0 and τ_n'; presumably the missing apostrophe is a typo.

4.5 Examples

Example 4.5.1 (A_1 lite)**.** Let us consider A_1 with $\tau_0 = 1 = \tau_1$ and $K = \mathbb{R}$ to keep things overly simple. Recall from Example 3.3.1 that the braid relations on the three generators T_0, T_1 and U_1 were $U_1^2 = U_1 T_0 U_1 T_1^{-1} = 1$. Because $\tau_i = \tau_i^{-1}$ for both $i \in I$, the Hecke relations read

$$T_i = T_i^{-1}.$$

Write $P^\vee = 2^{-1/2}\mathbb{Z} =: d\mathbb{Z}$ for convenience ($d = \pi_1'$), let $\lambda' = nd$ (with $n \in \mathbb{Z}$) and recall $s_0 = t(\alpha^\vee)s_1$ with $\alpha^\vee = \alpha = \alpha_1 = 2d$ and also $\alpha_0 = -\varphi = -2d$. Lusztig's relations are

$$Y^{nd}T_0 - T_0 Y^{(2-n)d} = b(1,1;Y^{-2d})(Y^{nd} - Y^{(2-n)d}) = 0,$$

because of Proposition 4.1.5iia), and similarly

$$Y^{nd}T_1 - T_1 Y^{-nd} = 0.$$

Since $\tau_i = \tau_i'$ for both i, we find that $b_i = 0$ and $c_i = 1$. Correspondingly, in the basic representation β, the T_i act as s_i and U_1 as $u_1 = u(d)$. This trivially agrees with all the formulæ.

We have $T_i^{-1}e^\lambda = e^{s_i\lambda}$ for all $\lambda \in L$ on the nose, so there are no lower-order terms.

Our simple choice of τ_1 does mean that $\kappa_1 = \kappa(\alpha)$ should be such that $1 = q^{\kappa_1/2}$. Since $q \neq 1$, we see $\kappa \equiv 0$ and so the results of the last section do not apply.

For all $a \in S$ we have $\tau_a = 1 = \tau_a'$ and so again $b_a = 0$ and $c_a = 1$, wherefore $G_a = 1$ (the constant). Indeed, this agrees with $G_{a_i} = s_i T_i = s_i^2$ and $T(w) = w$ as operators.

The double braid group is given in Example 3.3.1. In its $\mathbb{R}$-linearisation, we apply the Hecke relations $T_i = T_i^{-1}$ for both i again to get the DAHA.

The previous examples in rank 1 that I worked out myself agree with Chapter 6 of Macdonald, except for the action of α. (We both normalise $|\alpha|^2 = 2$ yet Macdonald's a_0 acts on $x \in V = \mathbb{R}$ by $1 - x$, rather than my $1 - \sqrt{2}\,x$.) Let us follow his calculation of the Hecke algebras.

Example 4.5.2 (A_1 proper)**.** Recall from Example 2.5.1 that $v_1 = s_1$, so that $s_0 = t(\pi_1')^2 s_1 = u_1 s_1 u_1$. Moreover, $u_1^2 = 1$ since $\Omega = \{1, u_1\}$. Therefore, $\tau_0 = \tau_1 =: \tau$ and $K = \mathbb{Q}(q^{1/2}, \tau)$ is the 'minimal' field we can take. From Example 3.3.1, the braid group was generated by T_1 and $U := U_1$ with relation $U^2 = 1$. Therefore, $\mathfrak{H}$ is the K-algebra generated by T_1 and U with $U^2 = 1$ and $(T_1 - \tau)(T_1 + \tau^{-1}) = 0$. (The Hecke relation for T_0 is redundant, being the U-conjugate of this.) Moreover, we must have $\tau = q^{k/2}$ for the multiplicity function k.

The double affine Hecke algebra is the K-algebra generated by the double braid group $\widetilde{\mathfrak{B}}$ subject to the same Hecke relation above. Explicitly, using Theorem 3.2.4, it is generated by T_1, $X := X_1$ and $Y := Y_1'$ with

$$T_1 Y^{-1} T_1 = Y, \quad T_1 X T_1 = X^{-1}, \quad UXU = q^{1/2}X^{-1} \quad \text{and} \quad (T_1 - \tau)(T_1 + \tau^{-1}) = 0,$$

where we already have $U = YT_1^{-1} = U^{-1} = T_1 Y^{-1}$ by virtue of Theorem 3.1.9. We also have $T_0 = UT_1U$ as before and $q^{1/2} \in Z(\widetilde{\mathfrak{H}})$. Because $\pi_1' = \frac{\alpha}{2}$, we have $L = P = P^\vee = L' = \mathbb{Z}\frac{\alpha}{2}$ and so $A = K[L] = K[x, x^{-1}]$ for $x := e^{\alpha/2}$. Let

$$b(X) = b(\tau, \tau; X^2) = \frac{\tau - \tau^{-1}}{1 - X^2} \quad \text{and} \quad c(X) = c(\tau, \tau; X^2) = \frac{\tau X^2 - \tau^{-1}}{X^2 - 1}.$$

For $f \in A$, the action of $\widetilde{\mathfrak{H}}$ is as follows. The generator $X = X^{\pi_1'}$ acts by (left) multiplication with $e^{\pi_1'} = x$. Similarly, U acts by $u_1 = t(\frac{\alpha}{2})s_1$ on the left; a monomial $e^{\lambda'} = e^{n\alpha/2}$ is mapped by U to $q^{n/2}e^{-\lambda}$ using Notation 4.2.7. Therefore,

$$(Uf)(x) = f(q^{1/2}x^{-1}). \tag{4.5.1}$$

Since, for $i = 1$, say, we have $b_1 = b(\tau, \tau'; e^{\alpha}) = b(\tau, \tau; x^2) = b(X)$ 'evaluated at $X = x$' by Notation 4.2.10 (and similarly for c), Proposition 4.2.13*iii)* tells us that

$$T_1 f = \big(b(X) + c(X)s_1\big)f. \tag{4.5.2}$$

We use this to consider already the shift operators from Section 5.6 (q.v.). As elements of $\operatorname{End}_K A$, we have $s_1 = c(X)^{-1}(T_1 - b(X))$. Since $\omega(T_1) = \omega(T(s_1)) = T_1$, applying ω to the identity $s_1 X = X^{-1}s_1$, we get

$$Y^{-1}(T_1 - b(Y^{-1}))c(Y^{-1})^{-1} = (T_1 - b(Y^{-1}))c(Y^{-1})^{-1}Y.$$

Thus, $\boldsymbol{\alpha}_1 = T_1 - b(Y^{-1}) = UY - b(Y^{-1})$ satisfies $Y^{-1}\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_1 Y$ (since Y commutes with $c(Y^{-1})^{-1}$). Similarly,

$$\boldsymbol{\beta}_1 = \omega^{-1}(U^{-1}) = \omega^{-1}(T_1 Y^{-1}) = XT_1 = XUY,$$

such that (using the third relation above)

$$Y^{-1}\boldsymbol{\beta}_1 = Y^{-1}XUY = q^{1/2}Y^{-1}UX^{-1}Y = q^{1/2}\boldsymbol{\beta}_1 Y,$$

as desired.

In for a penny, in for a polynomial

5.1 Some awful triangles

Let S be an irreducible affine root system, $S_1 = \{a \in S \mid \frac{1}{2}a \notin S\}$ and assume it falls under one of the three cases (in particular, not BC_n). Consider $S^+ \subset \Lambda^+ := L \oplus \mathbb{Z}_{\geqslant 0}c_0 \subsetneq \Lambda \supset S$. If $\Lambda \ni f = \mu + rc_0$, recall from Notation 4.2.6 we wrote $e^f = q^{r/e}e^\mu$.

NOTATION 5.1.1. For $a \in S$, let $t_a \in \mathbb{R}_{>0}$ be such that $t_a = t_b$ whenever $a \in Wb$. This determines a multiplicity function k on S by defining, for $a \in S_1$,

$$q^{k(a)} := t_a\sqrt{t_{2a}} \quad \text{and} \quad q^{k(2a)} := \begin{cases} \sqrt{t_{2a}} & 2a \in S, \\ 1 & 2a \notin S. \end{cases}$$

We then have $\kappa(a) = k(a) + k(2a)$ and $\kappa'(a) = k(a) - k(2a)$, where $\tau_a = q^{\kappa(a)/2} = \sqrt{t_a t_{2a}}$ and $\tau'_a = q^{\kappa'(a)/2} = \sqrt{t_a}$, cf. Notation 4.3.2. In cases I and II, the apostrophes remain meaningless.

NOTATION 5.1.2. For $a \in S$, set

$$\Delta_a = \Delta_{a,k} := \frac{1 - q^{k(2a)}e^a}{1 - q^{k(a)}e^a}.$$

A simple calculation shows that for $a \in S_1$,

$$\Delta_a\Delta_{2a} = \frac{1 - e^{2a}}{(1 - q^{k(a)}e^a)(1 + q^{k(2a)}e^a)},$$

with inverse $\tau_a c_a = \tau_a c(\tau_a, \tau'_a; e^a)$.

DEFINITION 5.1.3. The **weight function** is the product

$$\Delta = \Delta_{S,k} := \prod_{a \in S^+} \Delta_a.$$

Similarly define $\Delta' = \Delta_{S',k'}$.

Viewed as element of $\left(\mathbb{R}[t_a, \sqrt{t_{2a}} \mid a \in S^+]\right)[\![e^{a_i} \mid i \in I]\!]$ and taking S^+ as subset of Λ^+, we can expand this as

$$\Delta = \sum_{\substack{\lambda \in L \\ r \geqslant 0}} u_{\lambda + rc} q^r e^\lambda,$$

for certain coefficients $u_{\lambda+rc} \in \mathbb{R}[t_a, \sqrt{t_{2a}} \mid a \in S^+] =: \widetilde{\mathbb{R}}$. Then the multiplication $f\Delta$, where $f \in A$, makes sense.

Definition 5.1.4. For $f = \sum_{\lambda \in L} f_\lambda e^\lambda \in A$, the **constant term** of $f\Delta$ is

$$\mathrm{ct}(f\Delta) = \sum_{r \geqslant 0} \left(\sum_{\lambda \in L} u_{\lambda+rc} f_{-\lambda} \right) q^r \in \widetilde{\mathbb{R}}[\![q]\!].$$

Notation 5.1.5. Let

$$\Delta_1 := \frac{\Delta}{\mathrm{ct}(\Delta)} =: \sum_{\mu \in L} v_\mu(q,t) e^\mu$$

for certain functions v such that $v_0(q,t) \equiv 1$. With much effort, one proves that $v_\mu(q,t) = v_{-\mu}(q^{-1}, t^{-1})$ are rational functions in q and $t_a, \sqrt{t_{2a}}$.

There are some case-specific results that we largely skip.

Notation 5.1.6. For $n \in \mathbb{N} \cup \{\infty\}$, define

$$(x;q)_n := \prod_{i=0}^{n-1} (1 - xq^i).$$

It is called the *q-Pochhammer symbol* and has loads of properties.[1]For instance, for all $y \in \mathbb{R}$,

$$\frac{(x;q)_\infty}{(q^y x;q)_\infty} \xrightarrow{q \uparrow 1} (1-x)^y.$$

We can use these to express Δ in the three cases. In case I, for instance, we have (cf. Eric's δ_k)

$$\Delta_I = \prod_{\alpha \in R^+} \frac{(e^\alpha;q)_\infty (qe^{-\alpha};q)_\infty}{(q^{k(\alpha)}e^\alpha;q)_\infty (q^{k(\alpha)+1}e^{-\alpha};q)_\infty} \xrightarrow{q\uparrow 1} \prod_{\alpha \in R} (1-e^\alpha)^{k(\alpha)}. \tag{5.1.1}$$

* * *

Let K be the 'minimal' field of interest, generated over $\mathbb{Q}$ by the τ_a and τ'_a (where $a \in S$) as well as $q_0 = q^{1/e}$. Recall Notation 4.2.1.

Notation 5.1.7.

- Define the star involution on $K \ni x$ by $x^*(q_0, \tau_a, \tau'_a) := x(q_0^{-1}, \tau_a^{-1}, \tau_a'^{-1})$. (And of course $x^* = x$ if $x \in \mathbb{Q}$.)
- Define the star involution on A by mapping $f = \sum_\lambda f_\lambda e^\lambda$, where of course $f_\lambda \in K$, to $f^* := \sum_\lambda f_\lambda^* e^{-\lambda}$.

[1]See www.en.wikipedia.org/wiki/Q-Pochhammer_symbol.

On basis elements, therefore, $(e^\lambda)^* = e^{-\lambda}$.

If the values of k on R are nonnegative integers, then Δ is a finite product, hence an element of A, and its star is $q^{-N(k)}\Delta$, where $N(k)$ is some modification of $\sum_{\alpha\in R^+} k(\alpha)^2$ depending on the case.

NOTATION 5.1.8 (Cherednik)**.** We define a sesquilinear (with respect to the star) scalar product on A via

$$(f,g) = (f,g)_k := \mathrm{ct}(fg^*\Delta).$$

Normalise it by

$$(f,g)_1 := \mathrm{ct}(fg^*\Delta_1) = \frac{(f,g)}{(1,1)};$$

this is K-valued and Hermitian, meaning $(f,g)_1 = (g,f)_1^* \in K$. Moreover, both are nondegenerate, as $(f,f) \neq 0$ for all $0 \neq f \in A$.

Similarly, define a product $(-,-)'$ on A' using Δ'.

Recall that $\{T(w)f(X) \mid w \in W \,\&\, f \in A\}$ forms a K-basis of $\widetilde{\mathfrak{H}} \subset \mathrm{End}_K A$.

PROPOSITION 5.1.9. *Every $F \in \mathfrak{H}$ has an adjoint with respect to $(-,-)$, denoted F^*, and $(T(w)f(X))^* = f^*(X)T(w)^{-1}$. In particular, $T_i^* = T_i^{-1}$ and $U_j^* = u_j^{-1}$ for all $i \in I$ and $j \in J$.*

NOTATION 5.1.10.

- Consider the finite root sytem $S_0 = \{a \in S \mid a(0) = 0\}$ and $S_0^+ := S_0 \cap S^+$. If $S = S(R)$, the former seems to be just R.
- Set
$$\Delta^0 = \Delta^0_{S,k} := \prod_{a\in S_0^+} \Delta_{-a,k} \quad \text{and} \quad \nabla = \nabla_{S,k} := \Delta_{S,k}\Delta^0_{S,k}.$$

This ∇ is W_R-invariant, as one can easily show that $\frac{s_i\Delta^0}{\Delta^0} = \frac{\Delta}{s_i\Delta}$ for all $i \in I_R$.

NOTATION 5.1.11 (Macdonald)**.**

- For $g = \sum_{\mu\in L} g_\mu e^\mu \in A$, define
  - (i) $\overline{g} := \sum_\mu g_{-\mu}e^\mu = \sum_\mu g_\mu e^{-\mu}$ (then $\overline{\nabla} = \nabla$),
  - (ii) $g^0 := \sum_\mu g_\mu^* e^\mu = \overline{g}^*$.
- Define another symmetric scalar product on A (and its analogue on A') by
$$\langle f,g\rangle = \langle f,g\rangle_k := \frac{1}{\#W_R}\mathrm{ct}(f\overline{g}\nabla) \quad \text{and} \quad \langle f,g\rangle_1 := \frac{\langle f,g\rangle}{\langle 1,1\rangle}.$$
- For $w \in W_R$, define, recalling Definition 2.1.5 and Notation 5.1.1,
$$k(w) := \sum_{a\in S(w)} k(a) \quad \text{and} \quad W_R(q^k) = \sum_{w\in W_R} q^{k(w)} = \sum_{w\in W_R} t_w := \sum_{w\in W_R}\prod_{a\in S(w)} t_a.$$

Recalling Notation 2.4.2, we have $W_R(q^k) = (\Delta^0_{S,k}(-\rho_k'))^{-1} = W_R(q^{k'})$.

PROPOSITION 5.1.12. *For all $f, g \in A_R$ we have*

$$(f, g) = W_R(q^k) \langle f, g^0 \rangle .$$

5.2 Orthogonal polynomials

At last we come to the *orthogonal polynomials*, discovered in various stages by Opdam, Macdonald and Cherednik, that simultaneously diagonalise the Dunkl–Cherednik operators and form an orthogonal eigenbasis of A. Recall Notation 4.3.1.

THEOREM 5.2.1. *For all $\lambda \in L$, there exists a* unique *$E_\lambda \in A$, such that*

(i) $E_\lambda = e^\lambda + \text{LOT}$,

(ii) $(E_\lambda, e^\mu) = 0$ *for all $\mu < \lambda$.*

Dually, define $E'_{\mu'} \in A'$ for $\mu' \in L'$.

As a special case, note that $E_0 = E'_0 = 1$. Using Proposition 5.1.9 and Equation (4.3.2), we have for all $f \in A'$ and $\lambda > \mu \in L$ that

$$(f(Y)E_\lambda, e^\mu) = (E_\lambda, f^*(Y)e^\mu) = 0,$$

whence Theorem 5.2.1(ii) gives the first statement of the following.

THEOREM 5.2.2. *For all $f \in A'$ we have $f(Y)E_\lambda = f(-r_{k'}(\lambda))E_\lambda$. Moreover, $\{E_\lambda \mid \lambda \in L\}$ forms an orthogonal K-basis of A with respect to $(-,-)$ that diagonalises the action of $A'(Y)$ on A. The dual result applies to the $E'_{\mu'}$. The two are related by*

$$E_\lambda(r'_k(\mu'))E'_{\mu'}(-\rho_{k'}) = E_\lambda(-\rho'_k)E'_{\mu'}(r_{k'}(\lambda))$$

for all $\lambda \in L$ and $\mu' \in L'$.

The orthogonal polynomials do not form an orthonormal basis, though; the goal is to compute $(E_\lambda, E_\lambda)_1$. One easily shows that $E_\lambda(-\rho'_k) \not\equiv 0$, so we can define the normalised polynomials below.

NOTATION 5.2.3. For $\lambda \in L$, $\mu' \in L'$, set

$$\widetilde{E_\lambda} := \frac{E_\lambda}{E_\lambda(-\rho'_k)} \quad \text{and} \quad \widetilde{E'_{\mu'}} := \frac{E'_{\mu'}}{E'_{\mu'}(-\rho_{k'})}.$$

Recall the Definition 2.1.3 and its dual as well as Notations 4.3.7 and 4.3.9. Of course, the dualised versions of the results below also hold.

PROPOSITION 5.2.4. *Let $\lambda, \mu \in L$.*

(i) *As operators on A', we have*

$$Y^\lambda = \sum_{W' \ni w \leqslant t(\lambda')} w g_w(X) \quad \textit{and} \quad Y^{-\lambda} = \sum_{W' \ni w \leqslant t(\lambda')} f_w(X) w^{-1}$$

for certain $f_w, g_w \in A[c]$.

(ii) *Moreover:*

$$e^{\lambda}\widetilde{E_\mu} = \sum_{\substack{W' \ni w \leqslant t(\lambda') \\ w(r_{k'}(\mu)) = r_{k'}(w\mu)}} f_w(r_{k'}(\mu))\widetilde{E_{w\mu}} \quad \text{and} \quad e^{-\lambda}\widetilde{E_\mu} = \sum_{\substack{W' \ni w \leqslant t(\lambda') \\ w^{-1}(r_{k'}(\mu)) = r_{k'}(w^{-1}\mu)}} g_w(w^{-1}(r_{k'}(\mu)))\widetilde{E_{w^{-1}\mu}}.$$

We shortcircuit the proof of the 1-norm of the orthogonal polynomials and skip to the result. Recall Notations 2.4.3 and (the dual of) 4.3.8.

NOTATION 5.2.5. For $\lambda \in L$, let

$$\varphi_\lambda^{\pm} := c_{S', \pm k'}(u'(\lambda)^{-1}) = \prod_{\substack{a' \in S_1'^{+} \\ a'(\lambda) < 0}} c_{a', \pm k'}.$$

The equality follows from Theorem 2.2.3iii) and $S_1'^{+} = \{a' \in S'^{+} \mid \frac{1}{2}a' \notin S'\}$.

Now recall Notation 4.3.4 and note that the dual of Definition 2.2.2i) is the same on the nose since $W_R = W_{R'}$ on the nose (so there is no v-analogue of Notation 2.4.3). Once you're done recalling, behold the final result.[2]

THEOREM 5.2.6. *For all $\lambda \in L$, we have*

$$E_\lambda(-\rho_k') = \tau_{v(\lambda)}^{-1}\varphi_\lambda^{-}(r_{k'}(\lambda)) \quad \text{and} \quad (E_\lambda, E_\lambda)_1 = \varphi_\lambda^{+}(r_{k'}(\lambda))\varphi_\lambda^{-}(r_{k'}(\lambda)).$$

The dual results of course hold for the $E'_{\mu'}$.

***

We saw that the orthogonal polynomials diagonalise $A'(Y)$. They turn out to be the only elements of A to do so.

THEOREM 5.2.7. *Let $0 \neq f \in A$ be a simultaneous eigenfunction of all $Y^{\lambda'}$ with eigenvalue $g \in (L')^*$. Then f is a scalar multiple of some E_μ and the eigenvalues are $g(\lambda') = q^{-\langle \lambda', r_{k'}(\mu)\rangle} \in K$.*

We are interested in finding out how $\mathfrak{H} \subset \mathrm{End}_K A$ acts on the symmetric polynomials. We first state some auxiliary results. Recall Notation 4.1.6.

LEMMA 5.2.8. *Let $\lambda \in L$, $i \in I_R$ and set $b_i' := b(\tau_i, \upsilon_i; e^{a_i'})$.*

i) *If $\langle \lambda, \alpha_i' \rangle > 0$ then $(T_i - b_i'(r_{k'}(\lambda)))E_\lambda = \tau_i^{-1}E_{s_i\lambda}$.*

ii) *If $\lambda = s_i\lambda$ then the above holds with 0 instead of τ_i^{-1} and moreover $E_\lambda = s_iE_\lambda$.*

Recall Notation 4.2.4.

NOTATION 5.2.9. For $\lambda \in L_{++}$, consider the $\mathfrak{H}$-submodule $A_\lambda := K\{E_\mu \mid \mu \in W_R\lambda\}$ of A.

[2]Initially, Macdonald assumes $\mathrm{Stab}_{W'}(\rho_{k'})$ to be trivial in order to derive this, but then states it holds for any k.

THEOREM 5.2.10. *For any dominant λ, this A_λ is an irreducible $\mathfrak{H}$-module and equal to $\mathfrak{H}_R E_\lambda$. If λ is moreover regular, which is to say $\#W_R = |W_R\lambda|$, it is free of rank 1 as $\mathfrak{H}_R$-module.*

5.3 Symmetric polynomials

Next, we repeat a W_R-invariant analogue of the previous construction to produce the *symmetric polynomials*. Recall Notations 4.3.11 and 5.1.11.

THEOREM 5.3.1. *For all $\lambda \in L_{++}$, there exists a* unique *$P_\lambda \in A_R$,*

(i) $P_\lambda = m_\lambda + \text{LOT}$,

(ii) $\langle P_\lambda, m_\mu \rangle = 0$ *for all* $L_{++} \ni \mu < \lambda$.

Dually, define $P'_{\mu'} \in A'_R$ for $\mu' \in L'_{++}$.

The lower-order terms in (i) are now referring to $K\{m_\mu \mid L_{++} \ni \mu < \lambda\}$. It's easy to see that $P^0_\lambda = P_\lambda$ for $m^0_\lambda = m_\lambda$. Similarly to what we did for the orthogonal polynomials — now using that $f(Y)(A_R) \subset A_R$ for all $f \in A'_R$ —, Equation (4.3.3) shows that

$$\langle f(Y)P_\lambda, m_\mu \rangle_1 = \langle P_\lambda, (f^*(Y)m_\mu)^0 \rangle_1 = 0$$

if $\mu < \lambda$. The analogue of Theorem 5.2.2 is then the following, with the last statement due to Koornwinder.

THEOREM 5.3.2. *For all $f \in A'_R$ we have $f(Y)P_\lambda = f(-\lambda - \rho_{k'})P_\lambda$. Moreover, the symmetric polynomials are orthogonal with respect to $\langle -, - \rangle$ and diagonalise the action of $A'_R(Y)$ on A_R. The dual result applies to the $P'_{\mu'}$. The two are related by*

$$P_\lambda(\mu' + \rho'_k)P'_{\mu'}(\rho_{k'}) = P_\lambda(\rho'_k)P'_{\mu'}(\lambda + \rho_{k'})$$

for all $\lambda \in L_{++}$ and $\mu' \in L'_{++}$.

Again, we want to compute the polynomials' norm in $\langle -, - \rangle_1$.

NOTATION 5.3.3. For $\lambda \in L_{++}$, $\mu' \in L'_{++}$, set

$$\widetilde{P_\lambda} := \frac{P_\lambda}{P_\lambda(\rho'_k)} \quad \text{and} \quad \widetilde{P'_{\mu'}} := \frac{P'_{\mu'}}{P'_{\mu'}(\rho_{k'})}.$$

We once more skip to the final result. Recall Notation 4.3.8 again.

NOTATION 5.3.4. Let $\lambda \in L_{++}$ and set $c'_\lambda := c_{S',k'}(t(\lambda))$. (Of course, there is a dual hereof.)

THEOREM 5.3.5. *For all $\lambda \in L_{++}$, we have*

$$P_\lambda(\rho'_k) = c'_\lambda(\rho_{k'}) \quad \text{and} \quad \langle P_\lambda, P_\lambda \rangle_1 = c'_\lambda(-\lambda - \rho_{k'})c'_\lambda(\rho_{k'}).$$

The dual results of course hold for the $P'_{\mu'}$.

There is an alternative characterisation for triangle enthusiasts. First, we need more triangles, cf. Notations 5.1.2 and 5.1.10.

NOTATION 5.3.6. Let $\Delta^{\pm}_{S,k} := \prod_{\substack{a \in S^+ \\ Da \in R^{\pm}}} \Delta_a$ and $\Delta^{\pm}_{S',k'}$ analogously.

PROPOSITION 5.3.7. *We have*

$$P_\lambda(\rho'_k) = q^{-\langle \lambda, \rho'_k \rangle} \frac{\Delta^+_{S',k'}(\lambda + \rho_{k'})}{\Delta^+_{S',k'}(\rho_{k'})} \quad \text{and} \quad \langle P_\lambda, P_\lambda \rangle_1 = \frac{\Delta^+_{S',k'}(\lambda + \rho_{k'})\Delta^-_{S',-k'}(-\lambda - \rho_{k'})}{\Delta^+_{S',k'}(\rho_{k'})\Delta^-_{S',-k'}(-\rho_{k'})}.$$

* * *

We conclude with some special cases.

(a) If $k \equiv 0$ then $\Delta = \Delta^0 = \nabla = 1$, so $P_\lambda = m_\lambda$.

Proof. If $\mu < \lambda$ in L_{++}, then

$$\langle m_\lambda, m_\mu \rangle = \frac{1}{\# W_R} \mathrm{ct} \left(\sum_{\nu \in W_R \lambda} e^\nu \cdot \sum_{\xi \in W_R \mu} e^{-\xi} \right) = 0,$$

as $W_R \mu \cap W_R \lambda = \varnothing$. To see this; if $\lambda = w\mu$ for some $w \in W_R$, then [Bour456, Prop. VI.6.18] says $\mu \geqslant w\mu = \lambda$. ❑

(b) Suppose we are in case I; $S = S(R)$ reduced and $k \equiv 1$ on R. Then one can show, using the obvious fact that $\prod_{\alpha \in R} e^{-\alpha/2} = 1$, that

$$\nabla = \prod_{\alpha \in R} (1 - e^\alpha) = \prod_{\alpha \in R} (e^{\alpha/2} - e^{-\alpha/2}) = \delta \bar{\delta},$$

where as usual

$$\delta = \prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w \in W_R} (-1)^{\ell(w)} e^{w\rho}$$

is the *Weyl denominator*. (Here, $\rho = \rho_k = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$.) For $\lambda \in L_{++}$, let

$$\chi_\lambda := \delta^{-1} \sum_{w \in W_R} (-1)^{\ell(w)} e^{w(\lambda + \rho)}$$

be the character of the Lie algebra associated with R of the highest-weight representation for λ. Then $\chi_\lambda = m_\lambda + \mathrm{LOT}$ and these characters are orthonormal, so that $P_\lambda = \chi_\lambda$.

In the particular case $S = S(A_{n-1})$, the P_λ are Macdonald's symmetric polynomials.

(c) Case II is similar except that χ_λ now belongs to the Lie algebra of $R^\vee$ and $k^\vee = k' \equiv 1$.

(d) Expectedly, case III for a particular k (sometimes 1, sometimes 0) again has $\nabla = \delta \bar{\delta}$ for the Weyl denominator associated to C_n. In general, in case III, the P_λ are the Koornwinder polynomials and, if $S = S(C_1^\vee, C_1)$, the Askey–Wilson ones.

5.4 Symmetrisers and intertwiners

In the proof of Proposition 5.1.9, it was shown that $s_i^* = c_i(X)c_i(X^{-1})^{-1}s_i$, which we shall need from now on.

NOTATION 5.4.1. Fix $\varepsilon\colon W_R \longrightarrow \mathbb{C}^\times$ to be a linear character of W_R. (In particular, $\varepsilon(s_i) = \pm 1$ for all $i \in I_R$ and it is constant on conjugacy classes.) If R is of ADE type, then $\varepsilon = \text{triv}$ or sign; otherwise there are two other possibilities.

Set $s_i^\varepsilon := \begin{cases} s_i & \text{if } \varepsilon(s_i) = 1, \\ s_i^* & \text{if } \varepsilon(s_i) = -1. \end{cases}$ For $W_R \ni w = s_{i_1} \cdot \ldots \cdot s_{i_p}$ any reduced form, $w^\varepsilon := s_{i_1}^\varepsilon \cdot \ldots \cdot s_{i_p}^\varepsilon$ is well-defined. Recall Notation 4.3.4 again.

Recall that $wX^\mu = X^{w\mu}w$ for all $\mu \in L$ and $w \in W_R$, so that $w^\varepsilon X^\mu = X^{w\mu}w^\varepsilon$, because $X^{s_i\mu}$ commutes with $c_i(X)c_i(X^{-1})^{-1}$.

NOTATION 5.4.2. For $i \in I_R$, set $\tau_i^\varepsilon := \begin{cases} \tau_i & \text{if } \varepsilon(s_i) = 1, \\ -\tau_i^{-1} & \text{if } \varepsilon(s_i) = -1 \end{cases}$ and $\tau_w^\varepsilon = \tau_{i_1}^\varepsilon \cdot \ldots \cdot \tau_{i_p}^\varepsilon$, independent of the reduced form of $w \in W_R$.

Recall the longest element w_0, e.g. Notation 2.1.1.

DEFINITION 5.4.3. Define the ***ε*-symmetriser U_ε** as the element of $\mathfrak{H} \subset \operatorname{End}_K A$ given by

$$U_\varepsilon := \left(\tau_{w_0}^\varepsilon\right)^{-1} \sum_{w \in W_R} \tau_w^\varepsilon T(w).$$

As special cases, we have

$$U^+ := U_{\text{triv}} = \tau_{w_0}^{-1} \sum_{w \in W_R} \tau_w T(w) \quad \text{and} \quad U^- := U_{\text{sign}} = (-1)^{\ell(w_0)} \tau_{w_0} \sum_{w \in W_R} (-1)^{\ell(w)} \tau_w^{-1} T(w).$$

These symmetrisers kill $T_i - \tau_i^\varepsilon$ and are the only operators to do so.

THEOREM 5.4.4. *Let $i \in I_R$.*

(i) *We have $(T_i - \tau_i^\varepsilon)U_\varepsilon = 0 = U_\varepsilon(T_i - \tau_i^\varepsilon)$.*

(ii) *Conversely, if $h \in A(X) \cdot \mathfrak{H}_R$ is such that $h(T_i - \tau_i^\varepsilon) = 0$ for all $i \in I_R$ (resp., $(T_i - \tau_i^\varepsilon)h = 0$), then in fact $h = f(X)U_\varepsilon$ for some $f \in A$ (resp., $h = U_\varepsilon f(X)$).*

The symmetrisers also behave nicely with respect to $(-,-)$. First, if $W_R \ni w = s_{i_1} \cdot \ldots \cdot s_{i_p}$ is a reduced form, recall from Section 2.1 that the elements of $S(w)$ were the $\beta_r = s_{i_p} \cdot \ldots \cdot s_{i_{r+1}} \alpha_{i_r}$ for $1 \leqslant r \leqslant p$. Define the dual $b'_{a',k'}$ of the b's in Notation 4.3.7. One can then show that for any $x \in r_{k'}(L)$, the expression

$$F_w(x) := \prod_{r=1}^{p} \left(T_{i_r} - b'_{\beta_r}(x)\right)$$

is well-defined. Recall the first point of Notation 5.1.10.

PROPOSITION 5.4.5.

a) $U_\varepsilon = F_{w_0}(x)$ *for* $x = \rho_{\varepsilon k'} := \frac{1}{2}\sum_{\alpha\in R^+} \varepsilon(s_\alpha)k'(\alpha')\alpha$.

b) $U_\varepsilon = V_\varepsilon c_+(X^{-\varepsilon})$, *where* $V_\varepsilon := \varepsilon(w_0)\sum_{w\in W_R}\varepsilon(w)w^\varepsilon$ *and* $c_+(X^{-\varepsilon}) := \prod_{a\in S_0^+} c_{a,k}(X^{-\varepsilon(s_a)})$.

c) $U_\varepsilon^2 = \left(\tau_{w_0}^\varepsilon\right)^{-1} W_R(t^\varepsilon)U_\varepsilon$, *where* $W_R(t^\varepsilon) := \sum_{w\in W_R}\left(\tau_w^\varepsilon\right)^2$.

d) $U_\varepsilon^* = U_\varepsilon$, *so in particular, for all* $f,g\in A$ *we get* $(U_\varepsilon f, U_\varepsilon g) = \left(\tau_{w_0}^\varepsilon\right)^{-1} W_R(t^\varepsilon)(U_\varepsilon f, g)$.

***

We want a Y-analogue of $w^\varepsilon X^\mu = X^{w\mu}w^\varepsilon$. Using Propositions 4.1.5, the Hecke relation, and 4.2.13, we see that, for any $i\in I$, the operator $T_i - b_i(X)$ is selfadjoint with respect to $(-,-)$, with the analogue holding for the dualised operator, and that $s_i = (T_i - b_i;(X))c_i(X^{-1})^{-1} \in \mathrm{End}_K(A')$. Therefore, we may compute its adjoint with respect to $(-,-)'$, not to be confused with s_i^*, which is the $(-,-)$-adjoint.

NOTATION 5.4.6. The dual adjoint of s_i is written

$$(s_i^*)' = (T_i - b_i'(X))c_i'(X)^{-1} = c_i'(X^{-1})^{-1}(T_i - b_i'(X)).$$

In analogy with Notation 5.4.1, define

$$(s_i^\varepsilon)' := \begin{cases} s_i & \text{if } \varepsilon(s_i) = 1, \\ (s_i^*)' & \text{if } \varepsilon(s_i) = -1 \end{cases} = (T_i - b_i'(X))c_i'(X^{-\varepsilon(s_i)})^{-1} = c_i'(X^{\varepsilon(s_i)})^{-1}(T_i - b_i'(X)).$$

This extends to reduced forms of any $w\in W_R$ by $(w^\varepsilon)' = (s_{i_1}^\varepsilon)'\cdot\ldots\cdot(s_{i_p}^\varepsilon)'$.

Recall the anti-isomorphism from Theorem 4.4.4.

DEFINITION 5.4.7. The ***Y*-intertwiners** are defined on the simple roots $i\in I_R$ by

$$\eta_i^\varepsilon := \omega((s_i^\varepsilon)') = c_i'(Y^{\varepsilon(s_i)})^{-1}(T_i - b_i'(Y^{-1})) = (T_i - b_i'(Y^{-1}))c_i'(Y^{-\varepsilon(s_i)})^{-1}$$

and on reduced forms of $w\in W_R$ by $\eta_w = \eta_{i_1}^\varepsilon\cdot\ldots\cdot\eta_{i_p}^\varepsilon$.

The point is that $\eta_w^\varepsilon Y^{\lambda'} = Y^{w\lambda'}\eta_w^\varepsilon$ for all $w\in W_R$ and $\lambda'\in L'$. Now recall Notation 2.2.1.

PROPOSITION 5.4.8. *Let* $\lambda\in L$ *and* $i\in I_R$ *such that* $\langle\lambda,\alpha_i'\rangle\neq 0$. *Let* $\pm$ *denote the parity of this last expression. Then*

$$\eta_i^\varepsilon E_\lambda = \tau_i^{\mp} c_i'(\pm\varepsilon(s_i)r_{k'}(\lambda))^{\mp} E_{s_i\lambda}.$$

Moreover, if $\xi_\lambda^\varepsilon := \tau_{v(\lambda)}c_{S',\varepsilon k'}(v(\lambda))(r_{k'}(\lambda))$ *with* $(\varepsilon k')(\alpha') = \varepsilon(s_\alpha)k'(\alpha')$, *then* $\eta_{v(\lambda)}^\varepsilon E_\lambda = \left(\xi_\lambda^\varepsilon\right)^{-1} E_{\lambda_-}$.

In analogy with Proposition 5.4.5b), define

$$V_\varepsilon' := \varepsilon(w_0)\sum_{w\in W_R}\varepsilon(w)(w^\varepsilon)' \quad\text{and}\quad c_+'(X^{-\varepsilon}) := \prod_{a\in S_0^+} c_{a',k'}(X^{-\varepsilon(s_a)}),$$

so that $U_\varepsilon = U_\varepsilon' = V_\varepsilon c_+'(X^{-\varepsilon})$.

Notation 5.4.9.

- Let $\mathfrak{V}_\varepsilon := \omega(V'_\varepsilon) = \varepsilon(w_0) \sum_{w \in W_R} \varepsilon(w)\eta^\varepsilon_w$.
- Set $-\varepsilon$ to be the character $(-1)^{\ell(-)}\varepsilon(-)$ of W_R, such that for example $-\text{triv} = \text{sign}$.

Then from the definition we get $(\eta^\varepsilon_i)^* = \eta_i^{-\varepsilon}$. Relevance of this last bit is not plain.

5.5 Symmεtric polynomials with a twist

Notation 5.5.1. For $\lambda \in L$, set $F^\varepsilon_\lambda := U_\varepsilon E_\lambda$.

It is not difficult to show that if $\lambda = s_i\lambda$ for some $i \in I_R$ such that $\varepsilon(s_i) = -1$, then $F^\varepsilon_\lambda = 0$. If, on the other hand, $\langle \lambda, \alpha'_i \rangle > 0$, then

$$F^\varepsilon_{s_i\lambda} = \varepsilon(s_i)\tau_i c'_i(\varepsilon(s_i) r_{k'}(\lambda)) F^\varepsilon_\lambda.$$

Thus, we may assume we have a dominant λ, recalling Notation 5.2.9, for then $\dim(U_\varepsilon A_\lambda) \leqslant 1$. In view of the situation for $\lambda = s_i\lambda$, we **henceforth assume that**

$$\varepsilon|_{\mathrm{Stab}_{W_R}(\lambda)} \equiv 1.$$

This is satisified for e.g. $\varepsilon = \text{triv}$ and, if λ is moreover regular, for $\varepsilon = \text{sign}$. Observe that Theorem 5.4.4(i) then implies that this isotropy subgroup fixes F^ε_λ and, for $\varepsilon = \text{triv}$, all of W_R does. Let

$$W_{R_\lambda} = \mathrm{Stab}_{W_R}(\lambda) \quad \text{and} \quad W_R^\lambda = \{v(-\mu)^{-1} \mid \mu \in W_R\lambda\}.^{[3]}$$

We proceed with the ε-twisted analogues of Theorems 5.3.1 and 5.3.2.

Notation 5.5.2. For $\lambda \in L_{++}$, define

$$P^\varepsilon_\lambda := \tau_{w_0} W_{R_\lambda}(\tau^2)^{-1} F^\varepsilon_\lambda = e^{w_0\lambda} + \text{LOT},$$

with $W_{R_\lambda}(\tau^2) := \sum_{w \in W_{R_\lambda}} \tau^2_w$.

The following results in particular imply that $P^{\text{triv}}_\lambda = P_\lambda$ as one might have hoped.

Proposition 5.5.3. *Let $\lambda \in L_{++}$.*

i) $P^\varepsilon_\lambda = \sum_{\mu \in W_R\lambda} \varepsilon(v(\mu))\xi_\mu^{-\varepsilon}$, *with the ξ as in Proposition 5.4.8 except with $-\varepsilon$ from Notation 5.4.9.*

ii) *Let $f \in A'_R$. Then $f(Y)P^\varepsilon_\lambda = f(-\lambda - \rho_{k'})P^\varepsilon_\lambda$.*

iii) $\dfrac{(P^\varepsilon_\lambda, P^\varepsilon_\lambda)}{(P_\lambda, P_\lambda)} = \dfrac{\xi^{-\varepsilon}_\lambda}{\xi^{\text{sign}}_\lambda}$.

iv) *If $\varepsilon = \text{sign}$ and λ is also dominant, then $v(\lambda) = w_0$ and so* iii) *becomes* $\prod_{\alpha \in R^+} \dfrac{c_{\alpha^\vee, k'}(\lambda + \rho_{k'})}{c_{\alpha^\vee, -k'}(\lambda + \rho_{k'})}$.

[3] Each coset wW_{R_λ} has a unique shortest element, which is also the shortest element of W_R mapping $\lambda'_+ \longmapsto \lambda'$. In analogy with Cherednik's v, it is denoted $\overline{v}(w\lambda)$ by Macdonald but he immediately shows that it equals $v(-w\lambda)^{-1}$.

$$\ast\ast\ast$$

It's about to get nasty again. Assume S is irreducible and falls under the three cases, again write S_1 for the indivisible roots and S_0 for the 'finite roots' as in Notation 5.1.10. Let $S_{01}^+ := S_0 \cap S_1 \cap S^+$. We shall consider three multiplicity functions on S: k as before, an ℓ satisfying

$$\ell(a) = \begin{cases} 1 & \text{if } s_a \text{ is conjugate in } W \text{ to some } s_i \text{ with } i \neq 0 \text{ and } \varepsilon(s_i) = -1, \\ 0 & \text{otherwise,} \end{cases}$$

and $k + \ell$ defined pointwise. On S_0, we have $\varepsilon k\colon a \longmapsto \varepsilon(s_a)k(a)$ as prior. Maybe see Notations 5.1.1 again as well as 4.3.7 and 4.1.4.

NOTATION 5.5.4.

- For $a \in S_1$, define $\delta_a = \delta_{a,k}$ by

  $$\begin{cases} q^{k(a)/2}e^{a/2} - q^{-k(a)/2}e^{-a/2} = (e^{a/2} - e^{-a/2})c_{a,k} & \text{if } 2a \notin S, \\ (q^{k(a)/2}e^{a/2} - q^{-k(a)/2}e^{-a/2})(q^{k(2a)/2}e^{a/2} - q^{-k(2a)/2}e^{-a/2}) = (e^{a} - e^{-a})c_{a,k} & \text{if } 2a \in S. \end{cases}$$

  It is clear that $\delta_a^* = -\delta_a$.

- Then, let $\delta_{\varepsilon,k} := \prod_{\substack{a \in S_{01}^+ \\ \ell(a) = 1}} \delta_{a,k}$.

- Similarly define $\delta_{a'}$ for $a' \in S_1'$ as well as $\delta'_{\varepsilon,k'}$ using $S_{01}'^+$.

Recalling Notations 5.1.3 and 5.1.10, one can show that $\delta_{\varepsilon,k}\delta_{\varepsilon,k}^*\Delta_{S,k} = \nabla_{S,k+\ell}\big/\Delta_{S,\varepsilon k}^0$. Recalling Notations 5.1.1, 5.1.8 and 5.1.11, one can also show that, for all $f, g \in A_R$,

$$(f,g)_{k+\ell} = \frac{W_R(q^{k+\ell})}{W_R(q^{\varepsilon k})}(\delta_{\varepsilon,k}f, \delta_{\varepsilon,k}g)_k.$$

Using an auxiliary result on how $\delta_{\varepsilon,k}$ multiplies with $T_i - \tau_i$, one shows that $U_\varepsilon(A) = \delta_{\varepsilon,k}(A_R)$.

Now note that the symmetric polynomials implicitly depend on $\langle -,- \rangle$, which depends on $\nabla_{S,k}$. We therefore write $P_{\lambda,k}$ to indicate which multiplicity function is used.

NOTATION 5.5.5. Let $n(k,\ell) := \frac{1}{2}\sum_{a \in S_0^+} k(a)\ell(a)$ and $\rho_\ell := \frac{1}{2}\sum_{a \in S_{01}^+} u_a \ell(a) a$, where u_a is 1 if $2a \notin S$ and 2 if $2a \in S$.

Furthermore, for $f \in A$, let $|f|_k^2 = \langle f, f \rangle_k$.

Here is a first result and two consequences, showing how, in a sense, ε shifts the symmetric polynomials by ℓ.

PROPOSITION 5.5.6. *For all $\lambda \in L_{++}$, we have*

$$P^\varepsilon_{\lambda+\rho_\ell,k} = \varepsilon(w_0)q^{n(k,\ell)/2}\delta_{\varepsilon,k}P_{\lambda,k+\ell}.$$

COROLLARY 5.5.7. *We have $P^\varepsilon_{\rho_\ell,k} = \varepsilon(w_0)q^{n(k,\ell)/2}\delta_{\varepsilon,k}$ and so $P_{\lambda,k+\ell} = P^\varepsilon_{\lambda+\rho_\ell,k}\big/P^\varepsilon_{\rho_\ell,k}$.*

Recall Notation 5.3.6. The following lemma provides the inductive step for an important theorem.

LEMMA 5.5.8. *For all $\lambda \in L_{++}$, we have*

$$\frac{|P_{\lambda,k+\ell}|^2_{k+\ell}}{|P_{\mu,k}|^2_k} = \frac{W_R(q^k)}{W_R(q^{\varepsilon k})} \cdot \frac{\xi^{-\varepsilon}_{\lambda+\rho_\ell}}{\xi^{\text{sign}}_{\lambda+\rho_\ell}} = \frac{\Delta^+_{S',k'+\ell'}(\mu+\rho_{k'})\Delta^-_{S',-k'-\ell'}(-\mu-\rho_{k'})}{\Delta^+_{S',k'}(\mu+\rho_{k'})\Delta^-_{S',-k'}(-\mu-\rho_{k'})},$$

with $\mu := \lambda + \rho_\ell$.

THEOREM 5.5.9. *[Norm formula] For all $\lambda \in L_{++}$, we have*

$$|P_{\lambda,k}|^2_k = \Delta^+_{S',k'}(\lambda+\rho_{k'})\Delta^-_{S',-k'}(-\lambda-\rho_{k'}).$$

Finally, a significant amount of insufferably boring effort goes into showing the following. Set

$$S'(\lambda) := \{a' \in S'^+ \mid \chi(Da') + \langle \lambda, Da' \rangle > 0\}$$

for any $\lambda \in L$.

THEOREM 5.5.10. *We have*

$$(E_\lambda, E_\lambda)_k = \prod_{a' \in S'(\lambda)} (\Delta_{a',k'}\Delta_{a',-k'})(r_{k'}(\lambda)).$$

5.6 Shifting into different $G_\varepsilon$ars

This section gives another proof of Lemma 5.5.8 using shift operators. Recall Notation 5.5.4.

DEFINITION 5.6.1. The **shift operators** are defined as

$$G_\varepsilon := \delta_{\varepsilon,k}(X)^{-1}\delta'_{\varepsilon,k'}(Y^{-1}) \quad \text{and} \quad \widehat{G_\varepsilon} := \delta'_{\varepsilon,k'}(Y)\delta_{\varepsilon,k}(X^{-1}).$$

They both map A_R into itself and behave as adjoints with respect to $\langle -,- \rangle$ except that they shift the associated multiplicity function. That is, for $f, g \in A_R$ we have

$$\langle G_\varepsilon f, g^0 \rangle_{k+\ell} = q^{\widetilde{n(k,\ell)}} \left\langle f, (\widehat{G_\varepsilon} g)^0 \right\rangle_k,$$

where $\widetilde{n(k,\ell)} := \sum_{a \in S^+_{01}} k(a)\ell(a)$, cf. Notation 5.5.5. The nomenclature stems from the following result, which shows that they shift the symmetric polynomials (up to a scalar), cf. Proposition 5.5.6.

THEOREM 5.6.2. *Let $\lambda \in L_{++}$. Then*

$$G_\varepsilon P_{\lambda+\rho_\ell,k} = d_{k,\ell}(\lambda)P_{\lambda,k+\ell} \qquad \& \qquad \widehat{G_\varepsilon} P_{\lambda,k+\ell} = \widehat{d_{k,\ell}}(\lambda)P_{\lambda+\rho_\ell,k},$$

where

$$d_{k,\ell}(\lambda) := q^{\widetilde{n(k,\ell)}/2}\delta'_{\varepsilon,-k'}(\lambda+\rho_{k'+\ell'}) \qquad \& \qquad \widehat{d_{k,\ell}}(\lambda) := \varepsilon(w_0)q^{-\widetilde{n(k,\ell)}/2}\delta'_{\varepsilon,k'}(\lambda+\rho_{k'+\ell'}).$$

$$\ast\ast\ast$$

Finally, we shall see how the orthogonal polynomials can be created from $E_0 = 1 = E_0'$ by repeatedly applying certain kinds of operators. First, recall that S and W act on V according to Notation 2.1.2 and W acts on F (see Section 1.1) by precomposition with the inverse as in Theorem 2.1.4. To $\lambda \in L$, say, we associate the function $f_\lambda = \langle \lambda, - \rangle$ on V, so that, if $w \in W \setminus W_R$, there is a difference between $f_{w\lambda}$ and wf_λ. Therefore, recalling Notation 1.3.1, we have $s_0 = s_{-\varphi+c} = t(\varphi)s_\varphi$, so that $s_0\lambda = \varphi + s_\varphi\lambda$ as element of L and $s_0 \cdot \lambda = s_\varphi\lambda + \langle \lambda, \varphi \rangle\, c$ as element of F. For $i \in I_R$, however, $s_i\lambda = s_i \cdot \lambda$.

Recalling Notation 4.2.6, we have $X^{a_0} = qX^{-\varphi}$ as operators on A.[4] Also, from Equation (4.4.1), we get that

$$(T_i - b_i(X^{a_i}))X^\lambda = X^{s_i \cdot \lambda}(T_i - b_i(X^{a_i}))$$

for all $\lambda \in L$ and $i \in I$. Applying ω^{-1} from Theorem 4.4.4, we thus see that, for $i \neq 0$,

$$Y^{-\lambda}(T_i - b_i(Y^{-a_i})) = (T_i - b_i(Y^{-a_i}))Y^{-s_i\lambda},$$
$$Y^{-\lambda}(\omega^{-1}(T_0) - b_0(qY^\varphi)) = q^{\langle \lambda, \varphi \rangle}(\omega^{-1}(T_0) - b_0(qY^\varphi))Y^{-s_\varphi\lambda}.$$

Definition 5.6.3. For $i \in I_R$ we define the **$\boldsymbol{\alpha}$-creation operators**

$$\boldsymbol{\alpha}_i := T_i - b_i(Y^{-a_i}) \quad \text{and} \quad \boldsymbol{\alpha}_0 := \omega^{-1}(T_0) - b_0(qY^\varphi)$$

as elements of $\mathrm{End}_K(A')$.

Then the equations above reduce to

$$Y^\lambda \boldsymbol{\alpha}_i = \boldsymbol{\alpha}_i Y^{s_i\lambda} \quad \text{and} \quad Y^\lambda \boldsymbol{\alpha}_0 = q^{-\langle \lambda, \varphi \rangle} \boldsymbol{\alpha}_0 Y^{s_\varphi\lambda}.$$

By Theorem 5.2.2, then, we get $Y^\lambda \boldsymbol{\alpha}_i E'_{\mu'} = q^{-\left\langle s_i\lambda, r'_k(\mu') \right\rangle} \boldsymbol{\alpha}_i E'_{\mu'}$ for all $\mu' \in L'$, and something similar for $i = 0$. Recall Equations (2.4.1) in combination with Theorem 5.2.7 (these two applied to the previous equation prove that $\boldsymbol{\alpha}_i E'_{\mu'}$ is a scalar multiple of $E'_{s_i\mu'}$; it remains to determine the scalar), as well as Definition 2.3.3.

Proposition 5.6.4. *Let $i \in I_R$, $\mu' \in L'$ and assume $s_i\mu' > \mu'$. Then $\boldsymbol{\alpha}_i E'_{\mu'} = \tau_i^{-1} E'_{s_i\mu'}$. Similarly, if $s_0\mu' > \mu'$, then $\boldsymbol{\alpha}_0 E'_{\mu'} = \tau_{v(s_0\mu')}\tau^{-1}_{v(\mu')} E'_{s_0\mu'}$.*

Recalling Notation 2.2.5 if necessary, we define a second set of operators.

Definition 5.6.5. For all $j \in J$, define the **$\boldsymbol{\beta}$-creation operators** by

$$\boldsymbol{\beta}_j := \omega^{-1}(U_j^{-1}).$$

By Theorem 3.2.4(e), we have for $\lambda \in L$ that $U_j^{-1} X^{-\lambda} U_j = X^{u_j^{-1} \cdot (-\lambda)} = q^{\left\langle \lambda, \pi'_j \right\rangle} X^{v_j\lambda}$, so that $Y^\lambda \boldsymbol{\beta}_j = q^{-\left\langle \lambda, \pi'_j \right\rangle} \boldsymbol{\beta}_j Y^{v_j\lambda}$. A fully analogous argument proves the following.

Proposition 5.6.6. *For all $j \in J$ and $\mu' \in L'$, we have $\boldsymbol{\beta}_j E'_{\mu'} = \tau_{v(u_j\mu')}\tau_{v(\mu')} E'_{u_j\mu'}$.*

Thus, by Definition 2.1.8 and Theorem 2.2.6, we can derive the promised result explaining the creation operators' name.

[4]Except in case III, but we ignore that case in this final section.

THEOREM 5.6.7. *Let $\mu' \in L'$ and suppose $u(\mu') = u_j s_{i_1} \cdot \ldots \cdot s_{i_p}$ is a reduced form. Then*

$$E'_{\mu'} = \tau^{-1}_{v(\mu')} \boldsymbol{\beta}_j \boldsymbol{\alpha}_{i_1} \cdot \ldots \cdot \boldsymbol{\alpha}_{i_p}(1).$$

Dually, we of course have $\boldsymbol{\alpha}'_i$ and $\boldsymbol{\beta}'_j$ that build E_λ from 1.

5.7 Example: the grand A_1 finale

5.7.1 The E_n

We continue Example 4.5.2, in which the creation operators had already been computed. First, recall Notation 5.1.6 for $n < \infty$. We have to define an analogue of the usual binomial coefficient as polynomial in q, where the q stands for Gauß.

DEFINITION 5.7.1. Let $n \in \mathbb{N}$ and $0 \leqslant r \leqslant n$. Then the ***q*-binomial coefficient** is

$$\binom{n}{r}_q := \frac{(q;q)_n}{(q;q)_r (q;q)_{n-r}} = \frac{(1-q^{n-r+1}) \cdot \ldots \cdot (1-q^n)}{(1-q) \cdot \ldots \cdot (1-q^r)}$$

as element of $\mathbb{Z}[q]$.

Obviously, $\binom{n}{r}_q = \binom{n}{n-r}_q$ and it turns out that $\lim_{q \uparrow 1} \binom{n}{r}_q = \binom{n}{r}$.

PROPOSITION 5.7.2 (*q*-binomial theorem). *For any indeterminate x, we have*

$$(x;q)_n = \sum_{r=0}^{n} (-1)^r q^{r(r-1)/2} \binom{n}{r}_q x^r \qquad \& \qquad (x;q)_n^{-1} = \sum_{r=0}^{\infty} \binom{n+r-1}{r}_q x^r.$$

Recall Equation (5.1.1) for Δ, to be applied to $S = S(A_1)$. (Ignore the limit $q \uparrow 1$.) **Assume**

$k = k(\alpha) \in \mathbb{Z}_{>0}$. For us, $x = e^{\alpha/2}$, so we obtain

$$\begin{aligned}
\Delta &= \prod_{i=0}^{\infty} \frac{(1-x^2q^i)(1-x^{-2}q^{i+1})}{(1-x^2q^{i+k})(1-x^{-2}q^{i+k+1})} \\
&= \prod_{i=0}^{k-1}(1-x^2q^i)(1-x^{-2}q^{i+1}) \\
&= \prod_{i=k}^{2k-1}(1-x^2q^{i-k})\prod_{j=0}^{k-1}(1-x^{-2}q^{j+1}) \\
&= (q^{-k}x^2;q)_{2k}\prod_{i=0}^{k-1}(1-x^2q^{i-k})^{-1}\prod_{j=0}^{k-1}(-x^{-2}q^{j+1})(1-x^2q^{-j-1}) \\
&= (-1)^k x^{-2k} q^{1+\ldots+k}(q^{-k}x^2;q)_{2k}\prod_{i=0}^{k-1}\frac{(1-x^2q^{-1})\cdot\ldots\cdot(1-x^2q^{-k})}{(1-x^2q^{-k})\cdot\ldots\cdot(1-x^2q^{-1})} \\
&= (-1)^k x^{-2k} q^{k(k+1)/2}(q^{-k}x^2;q)_{2k} \\
&= (-1)^k x^{-2k} q^{k(k+1)/2}\sum_{r=0}^{2k}(-1)^r q^{r(r-1)/2}\binom{2k}{r}_q q^{-rk}x^{2r} \\
&= (-1)^k x^{-2k} q^{k(k+1)/2}\sum_{r=-k}^{k}(-1)^{r+k} q^{r(r-1)/2}q^{rk}q^{(k^2-k)/2}\binom{2k}{r+k}_q q^{-rk}q^{-k^2}x^{2r}x^{2k} \\
&= \sum_{r=-k}^{k}(-1)^r q^{r(r-1)/2}\binom{2k}{r+k}_q x^{2r}.
\end{aligned}$$

Therefore, we can explicitly compute $(f,g) = \mathrm{ct}(fg^*\Delta)$ on A if we want to. Recall from Example 2.5.1 the expression for $r_k(n\alpha/2)$, with $n \in \mathbb{Z}$. For $E_n := E_{n\alpha/2} \in A = K[x,x^{-1}]$, we know it is a monic polynomial of degree n and moreover $Y = Y_1' = Y^{\alpha/2}$ acts, through Notation 4.3.13, by

$$YE_n = f(-r_k(n\alpha/2))E_n = q^{\langle \alpha/2, -r_k(n\alpha/2)\rangle}E_n = \begin{cases} q^{\langle \alpha/2, -(n+k)\alpha/2)\rangle}E_n = q^{-(n+k)/2}E_n & n > 0 \\ q^{\langle \alpha/2, (k-n)\alpha/2)\rangle}E_n = q^{(k-n)/2}E_n & n \leqslant 0, \end{cases}$$

where $f = x \in A$. By Notation 5.1.7 and Theorem 5.1.9, the adjoint for this action of Y is clearly Y^{-1}, which acts by the inverses of the powers of q. Therefore, the polynomials are indeed orthogonal: for example, if $n, m > 0$ and $n \neq m$, then

$$(E_n, E_m) = q^{(n+k)/2}(YE_n, E_m) = q^{(n+k)/2}(E_n, Y^{-1}E_m) = \underbrace{q^{(n-m)/2}}_{\neq 1}(E_n, E_m) \overset{!}{=} 0.$$

PROPOSITION 5.7.3. *Let $n > 0$. The only monomials x^m that can appear in E_n are those with $m = n - 2i$ for $0 \leqslant i \leqslant n-1$ and in E_{-n}, those with $m = n - 2i$ but now $0 \leqslant i \leqslant n$.*

Proof. By the ordering Definition 2.3.3 (which we now denote by $\preceq$ to avoid confusion), we know that if $n \geqslant 0$, then $m \prec n$ if and only if $|m| < n$; and if $n < 0$, then $m \prec n$ if and only if $n < m \leqslant |n| = -n$. This restricts the possible monomials x^m that can appear in E_n apart from x^n. Remains to prove that only the m with *even* difference with n can appear; the claimed ranges for i then work out.

Because $(E_n, x^m) = \mathrm{ct}(E_n x^{-m}\Delta) = 0$ for all $m \prec n$, write (in the case that $n > 0$; the other case is similar)

$$E_n = \sum_{i=-n+1}^{n} e_i x^i \qquad \text{with } e_n = 1.$$

Then the inner product is

$$0 = \sum_r (-1)^r q^{r(r-1)/2} \binom{2k}{k+r}_q e_{m-2r}$$

ranging over $-k \leqslant r \leqslant k$ as well as $-n+1 \leqslant m-2r \leqslant n$ or, equivalently $\frac{m-n}{2} \leqslant r \leqslant \frac{n+m-1}{2}$. The next step is best illustrated using examples.

For $n = 1$ and $m = 0$, we see that the equation reduces to $e_0 = 0$, since $\binom{2k}{k}_q \neq 0$ in K. Therefore, $E_1 = x$, sans constant term. For $n = 2$ and $m = 0$, assuming k to be sufficiently large, only $r = -1, 0$ are allowed and so

$$-q\binom{2k}{k-1}_q e_2 + \binom{2k}{k}_q e_0 = 0, \text{ whence}$$

$$\begin{aligned} e_0 &= q\frac{(1-q^{k+2})\cdot\ldots\cdot(1-q^{2k})}{(1-q)\cdot\ldots\cdot(1-q^{k-1})} \cdot \frac{(1-q)\cdot\ldots\cdot(1-q^{k})}{(1-q^{k+1})\cdot\ldots\cdot(1-q^{2k})} \\ &= q\frac{1-q^k}{1-q^{k+1}}. \end{aligned}$$

If $m = 1$, we see that $r = 0, 1$ contribute, so that

$$\binom{2k}{k}_q e_1 - \binom{2k}{k+1}_q e_{-1} = \frac{(1-q^{k+2})\cdot\ldots\cdot(1-q^{2k})}{(1-q)\cdot\ldots\cdot(1-q^{k})}((1-q^{k+1})e_1 - (1-q^k)e_{-1}) = 0.$$

Finally, if $m = -1$, we have $r = -1, 0$ again but now the equation is

$$-q\binom{2k}{k-1}_q e_1 + \binom{2k}{k}_q e_{-1} = \frac{(1-q^{k+2})\cdot\ldots\cdot(1-q^{2k})}{(1-q)\cdot\ldots\cdot(1-q^{k})}(-q(1-q^k)e_1 + (1-q^{k+1})e_{-1}) = 0.$$

Since the common prefactor in both equations is not zero in K, we have both

$$e_1 = \frac{1-q^k}{1-q^{k+1}} e_{-1} \quad \text{and} \quad e_1 = \frac{1-q^{k+1}}{q(1-q^k)} e_{-1}.$$

Then either both are zero, or, setting the quotient to 1, one solves for $k = -\frac{1}{2}$, which is a contradiction. (Notice we have now fully computed E_2.) Since $\frac{m-n}{2} \geqslant -\frac{3}{2}$ and $\frac{n+m-1}{2} \leqslant 1$, our assumption on k's being sufficiently large actually meant $k \geqslant 1$, which we assumed from the onset anyway. The arguments for higher n are similar. ❑

Recall we found $\boldsymbol{\alpha}_1 = UY - b(Y^{-1})$ and $\boldsymbol{\beta}_1 = XUY$ in Example 4.5.2.

PROPOSITION 5.7.4. *Let $n \geqslant 0$. Then*

$$\boldsymbol{\alpha}_1 E_{n+1} = q^{-k/2} E_{-n-1} \quad \textit{and} \quad \boldsymbol{\beta}_1 E_{-n} = q^{k/2} E_{n+1}.$$

Proof. First, we have $Y\boldsymbol{\alpha}_1 E_{n+1} = \boldsymbol{\alpha}_1 Y^{-1} E_{n+1} = q^{(n+k+1)/2}\boldsymbol{\alpha}_1 E_{n+1}$, so by Theorem 5.2.7 (in accordance with the action of Y computed above), we find that $\boldsymbol{\alpha}_1 E_{n+1}$ is a scalar times E_{-n-1}. Now

$$\boldsymbol{\alpha}_1 E_{n+1} = UYE_{n+1} - b(Y^{-1})E_{n+1}.$$

The second term has $b(Y^{-1}) = (q^{k/2} - q^{-k/2})(1 + Y^{-2} + Y^{-4} + \dots)$ so E_{n+1} is an eigenvector and hence the coefficient of x^{-n-1} lies outside the range of allowed monomials. The first term is

$$\begin{aligned} UYE_{n+1} &= q^{-(n+k+1)/2} Y T_1^{-1}(x^{n+1} + \mathrm{LOT}) \\ &= q^{-(n+k+1)/2} Y(q^{-k/2}x^{-n-1} + \mathrm{LOT}) \\ &= q^{-(n+k+1)/2} q^{-(-n-1-k)/2} q^{-k/2}(x^{-n-1} + \mathrm{LOT}), \end{aligned}$$

where we used Theorem 4.3.3 (computing $f(s_1, (n+1)\alpha/2)$ from Notation 4.3.2 easily) and Equation (4.3.1). The coefficient of x^{-n-1} is therefore $q^{-k/2}$. This is therefore the scalar we were looking for, proving the first statement. The second follows by an analogous computation. ❑

COROLLARY 5.7.5. *For any $n \geqslant 0$, we have*

$$E_{n+1} = q^{-k/2}\boldsymbol{\beta}_1(\boldsymbol{\alpha}_1\boldsymbol{\beta}_1)^n(1) \quad \text{and} \quad E_{-n} = (\boldsymbol{\alpha}_1\boldsymbol{\beta}_1)^n(1).$$

Proof. Clearly $q^{-k/2}\boldsymbol{\beta}_1 E_0 = E_1$ and moreover

$$\boldsymbol{\alpha}_1\boldsymbol{\beta}_1 E_0 = \boldsymbol{\alpha}_1 q^{k/2} E_1 = q^{k/2} q^{-k/2} E_{-1} \quad \text{and} \quad q^{-k/2}\boldsymbol{\beta}_1\boldsymbol{\alpha}_1\boldsymbol{\beta}_1 E_0 = q^{-k/2}\boldsymbol{\beta}_1 E_{-1} = E_2,$$

and so forth. ❑

Next, we can explicitly calculate the orthogonal polynomials. For no apparent reason, introduce the following.

NOTATION 5.7.6. Let

$$f = f_k(x,z) := \frac{1}{(xz;q)_k (x^{-1}z;q)_{k+1}} = \sum_{n=0}^{\infty} f_n(x) z^n = \sum_{r,s=0}^{\infty} \binom{k+r-1}{r}_q \binom{k+s}{s}_q x^{r-s} z^{r+s},$$

where the last step is the q-binomial theorem, which also guarantees there are no negative powers of z. We can read off $f_n(x) = \sum_{r+s=n} \binom{k+r-1}{r}_q \binom{k+s}{s}_q x^{r-s}$. Similarly, set

$$g = \frac{x}{(xz;q)_{k+1}(qx^{-1}z;q)_k} = \sum_{n=0}^{\infty} g_n(x) z^n = \sum_{n=0}^{\infty} \underbrace{\left(\sum_{r+s=n} \binom{k+r-1}{r}_q \binom{k+s}{s}_q q^r x^{s-r+1} \right)}_{g_n(x)} z^n.$$

LEMMA 5.7.7. *We have*

$$T_1 f(x,z) = q^{k/2} f(q^{1/2}x^{-1}, q^{1/2}z) \quad \text{and} \quad T_1 g(x,z) = q^{-(k+1)/2} f(q^{1/2}x^{-1}, q^{1/2}z).$$

Proof. Using Equation (4.5.2),

$$\begin{aligned}
T_1 f(x,z) &= (b(X) + c(X)s_1) f(x,z) \\
&= q^{k/2} \frac{1-q^{-k}}{1-x^2} f(x,z) + q^{k/2} \frac{x^2 - q^{-k}}{x^2-1} f(x^{-1}, z) \\
&= \frac{q^{k/2}}{1-x^2} \left(\frac{1-q^{-k}}{(xz;q)_k (x^{-1}z;q)_{k+1}} - \frac{x^2 - q^{-k}}{(x^{-1}z;q)_k (xz;q)_{k+1}} \right) \\
&= \frac{q^{k/2}}{1-x^2} \cdot \frac{(1-q^{-k})(1-xzq^k) - (x^2-q^{-k})(1-x^{-1}zq^k)}{(1-xz)\cdot \ldots \cdot (1-xzq^k)(1-x^{-1}z)\cdot \ldots \cdot (1-x^{-1}zq^k)} \\
&= \frac{q^{k/2}}{1-x^2} \cdot \frac{(1-x^2)(1-x^{-1}z)}{(1-xz)\cdot \ldots \cdot (1-xzq^k)(1-x^{-1}z)\cdot \ldots \cdot (1-x^{-1}zq^k)} \\
&= \frac{q^{k/2}}{(qx^{-1}z;q)_k (xz;q)_{k+1}} \\
&= q^{k/2} f(q^{1/2}x^{-1}, q^{1/2}z),
\end{aligned}$$

as desired. The argument for g is analogous. ❏

COROLLARY 5.7.8. *Let $n \in \mathbb{Z}_{\geqslant 0}$. The orthogonal polynomials are given by*

$$E_{-n} = \binom{k+n}{n}_q^{-1} f_n(x) \quad \text{and} \quad E_{n+1} = \binom{k+n}{n}_q^{-1} g_n(x).$$

Proof. Recall $Y = UT_1$, so that by Equation (4.5.1),

$$Yf(x,z) = q^{k/2} u f(q^{1/2}x^{-1}, q^{1/2}z) = q^{k/2} f(x, q^{1/2}z) = \sum_{n=0}^{\infty} q^{(m+k)/2} f_n(x) z^n.$$

We see that $f_n(x)$ are eigenfunctions of Y with the 'correct' eigenvalue and so by arguments employed earlier, they are scalar multiples of E_{-n}. The monomial x^{-n} appears in f_n with coefficient

$$\sum_{\substack{r+s=n \\ r-s=-n}} \binom{k+r-1}{r}_q \binom{k+s}{s}_q = \binom{k-1}{0}_q \binom{k+n}{n}_q = \binom{k+n}{n}_q,$$

giving the desired equality. The case with g is similar. ❏

Let's compute some of these. As an addendum to Definition 5.7.1, we set $\binom{n}{s}_q$ to be 0 if $s < 0$. It is clearly 1 if $s = 0$.

EXAMPLE 5.7.9.

- As a sanity check,

$$E_0 = \binom{k}{0}_q^{-1} f_0(x) = \sum_{r=0}^{\infty} \binom{k+r-1}{r}_q \binom{k-r}{-r}_q x^{2r} = 1.$$

- Similarly, we have

$$E_1 = \binom{k}{0}_q^{-1} g_0(x) = \sum_{r=0}^{\infty} \binom{k+r-1}{r}_q \binom{k-r}{-r}_q q^r x = x.$$

- Next, something happens for

$$E_{-1} = \binom{k+1}{1}_q^{-1} f_1(x) = \binom{k+1}{1}_q^{-1} \sum_{r=0}^{\infty} \binom{k+r-1}{r}_q \binom{k-r+1}{1-r}_q x^{2r-1}.$$

In the sum, only $r = 0, 1$ are allowed, so we get

$$\begin{aligned} E_{-1} &= \binom{k+1}{1}_q^{-1} \left(\binom{k-1}{0}_q \binom{k+1}{1}_q x^{-1} + \binom{k}{1}_q \binom{k}{0}_q x^1 \right) \\ &= x^{-1} + \frac{(1-q)(1-q^k)}{(1-q^{k+1})(1-q)} x \\ &= x^{-1} + \frac{1-q^k}{1-q^{k+1}} x. \end{aligned}$$

Note that indeed $\alpha/2 \prec -\alpha/2$, since $1_{W_R} = v(-\alpha/2) < s_1 = v(\alpha/2)$ by Example 2.5.1, so that E_{-1} is x^{-1} plus terms of lower order (in accordance with Proposition 5.7.3).

- We already know E_2 from this Prop.; on to E_3. Only $r = 0, 1, 2$ contribute to $g_2(x)$, so that

$$\begin{aligned} E_3 &= \binom{k+2}{2}_q^{-1} \left(\binom{k-1}{0}_q \binom{k+2}{2}_q x^3 + \binom{k}{1}_q \binom{k+1}{1}_q qx + \binom{k+1}{2}_q \binom{k}{0}_q q^2 x^{-1} \right) \\ &= x^3 + q \frac{(1-q^2)(1-q^k)}{(1-q)(1-q^{k+2})} x + q^2 \frac{1-q^k}{1-q^{k+2}} x^{-1}. \end{aligned}$$

- For later purposes, let's take a look at two more.

$$\begin{aligned} E_{-2} &= x^{-2} + \binom{k+2}{2}_q^{-1} \left(\binom{k}{1}_q \binom{k+1}{1}_q + \binom{k+1}{2}_q x^2 \right) \\ &= x^{-2} + \frac{(1-q^2)(1-q^k)}{(1-q)(1-q^{k+2})} + \frac{1-q^k}{1-q^{k+2}} x^2 \qquad \text{and} \\ E_{-3} &= x^{-3} + \binom{k+3}{3}_q^{-1} \left(\binom{k}{1}_q \binom{k+2}{2}_q x^{-1} + \binom{k+1}{2}_q \binom{k+1}{1}_q x + \binom{k+2}{2}_q x^3 \right) \\ &= x^{-3} + \frac{(1-q^3)(1-q^k)}{(1-q)(1-q^{k+3})} x^{-1} + \frac{(1-q^3)(1-q^k)(1-q^{k+1})}{(1-q)(1-q^{k+2})(1-q^{k+3})} x + \frac{1-q^3}{1-q^{k+3}} x^3. \end{aligned}$$

5.7.2 The P_n

Clearly, $A_R = K[x, x^{-1}]^{s_1} = K[x + x^{-1}]$. The scalar product $\langle -, - \rangle$ restricted to A_R (where the bar does nothing) is $\langle f, g \rangle = \frac{1}{2}\mathrm{ct}(fg\nabla)$ for $f, g \in A_R$, where

$$\nabla = \prod_{\alpha \in R} \frac{(e^\alpha; q)_\infty}{(q^{k(\alpha)} e^\alpha; q)_\infty} = \prod_{i=0}^{k-1} (1 - x^2 q^i)(1 - x^{-2} q^i) = (x^2; q)_k (x^{-2}; q)_k.$$

By Proposition 5.1.12 and the fact that $S(1_{W_R}) = \varnothing$ while $S(s_1) = \{\alpha\}$, we have

$$\begin{aligned}
\langle 1,1\rangle_k &= (1+q^k)^{-1}(1,1)_k \\
&= (1+q^k)^{-1}\mathrm{ct}\left(\sum_{r=-k}^{k}(-1)^r q^{r(r-1)/2}\binom{2k}{r+k}_q x^{2r}\right) \\
&= (1+q^k)^{-1}\binom{2k}{k}_q \\
&= \frac{(1-q^{k+1})\cdot\ldots\cdot(1-q^{2k})}{(1-q)\cdot\ldots\cdot(1-q^k)(1+q^k)} \\
&= \frac{(1-q^{k+1})\cdot\ldots\cdot(1-q^{2k-1})}{(1-q)\cdot\ldots\cdot(1-q^{k-1})} \\
&= \binom{2k-1}{k-1}_q.
\end{aligned}$$

Again, for $\lambda = n\alpha/2 \in L$, let $P_n := P_{\lambda,k} = x^n + x^{-n} + \textsc{lot}$. As with the orthogonal polynomials (now using Theorem 5.3.2 and once more Notation 4.3.13), we have that

$$(Y+Y^{-1})P_n = f(-n\tfrac{\alpha}{2}-\rho_k)P_n = \left(q^{\left\langle\frac{\alpha}{2},-(n+k)\frac{\alpha}{2}\right\rangle} + q^{-\left\langle\frac{\alpha}{2},-(n+k)\frac{\alpha}{2}\right\rangle}\right)P_n = \left(q^{-(n+k)/2}+q^{(n+k)/2}\right)P_n$$

for $f = x + x^{-1} \in A_R$. Now let $Z := (Y+Y^{-1})\big|_{A_R}$. Then, as operator on A_R, we have $T_1 = \tau_1 = \tau$ (see Theorem 4.2.12), so, using that $Y = UT_1 = \tau U$ and $Y^{-1} = T_1^{-1}U$, the Hecke relation and Proposition 4.2.13ii),

$$Z = (\tau + T_1^{-1})U = (T_1+\tau^{-1})U = (1+s_1)c(X^{-1})u = c(X^{-1})u + c(X)s_1 u.$$

Furthermore, let

$$F_k(x,z) := \frac{1}{(xz;q)_k(x^{-1}z;q)_k} = \sum_{n=0}^{\infty}F_n(x)z^n = \sum_{n=0}^{\infty}\left(\sum_{r+s=n}\binom{k+r-1}{r}_q\binom{k+s-1}{s}_q x^{r-s}\right)z^n,$$

cf. Notation 5.7.6. Observe that this object is W_R-invariant, i.e., invariant under $x \longmapsto x^{-1}$.
Lemma 5.7.10. *We have*

$$ZF_k(x,z) = \tau F_k(x,q^{1/2}z) + \tau^{-1}F_k(x,q^{-1/2}z). \tag{5.7.1}$$

Proof. We have

$$\begin{aligned}
(x-x^{-1})ZF_k(x,z) &= (x-x^{-1})\frac{\tau x^{-2}-\tau^{-1}}{x^{-2}-1}F_k(q^{1/2}x^{-1},z) + (x-x^{-1})\frac{\tau x^2-\tau^{-1}}{x^2-1}F_k(q^{1/2}x,z) \\
&= (\tau^{-1}x-\tau x^{-1})F_k(q^{-1/2}x,z) + (\tau x-\tau^{-1}x^{-1})F_k(q^{1/2}x,z).
\end{aligned}$$

The equality (5.7.1) is equivalent to the one obtained by multiplying both sides by the element $A_k(x,z) := q^{(k-1)/2}z(q^{-1/2}xz;q)_{k+1}(q^{-1/2}x^{-1}z;q)_{k+1} \in K[x,z]^\times$, which is what we shall prove using the equality just derived.

Define, for convenience, the following:

$$\alpha := 1-xzq^{-1/2}, \quad \beta := 1-x^{-1}zq^{-1/2}, \quad \gamma := 1-xzq^{k-1/2} \quad \text{and} \quad \delta := 1-x^{-1}zq^{k-1/2}.$$

The left-hand side is

$$\begin{aligned}
A_k(x,z)ZF_k(x,z) &= \frac{A_k(x,z)}{x-x^{-1}}\Big((q^{k/2}x - q^{-k/2}x^{-1})F_k(q^{1/2}x,z) + (q^{-k/2}x - q^{k/2}x^{-1})F_k(q^{-1/2}x,z)\Big) \\
&= \frac{q^{-1/2}z}{x-x^{-1}}\Bigg((q^k x - x^{-1})\frac{(q^{-1/2}xz;q)_{k+1}(q^{-1/2}x^{-1}z;q)_{k+1}}{(q^{1/2}xz;q)_k(q^{-1/2}x^{-1}z;q)_k} \\
&\qquad + (x - q^k x^{-1})\frac{(q^{-1/2}xz;q)_{k+1}(q^{-1/2}x^{-1}z;q)_{k+1}}{(q^{-1/2}xz;q)_k(q^{1/2}x^{-1}z;q)_k}\Bigg) \\
&= \frac{1}{x-x^{-1}}\big((\beta-\gamma)\alpha\delta + (\delta-\alpha)\beta\gamma\big). \qquad (5.7.2)
\end{aligned}$$

Meanwhile, the right-hand side becomes

$$\begin{aligned}
A_k(x,z)\Big(\tau F_k(x,q^{1/2}z) + \tau^{-1}F_k(x,q^{-1/2}z)\Big) &= A_k(x,z)\Bigg(\frac{q^{k/2}}{(q^{1/2}xz;q)_k(q^{1/2}x^{-1}z;q)_k} \\
&\qquad + \frac{q^{-k/2}}{(q^{-1/2}xz;q)_k(q^{-1/2}x^{-1}z;q)_k}\Bigg) \\
&= q^{-1/2}z(q^k\alpha\beta + \gamma\delta) \\
&= \frac{1}{x-x^{-1}}\big((\delta-\gamma)\alpha\beta + (\beta-\alpha)\gamma\delta\big). \qquad (5.7.3)
\end{aligned}$$

Clearly, (5.7.2) and (5.7.3) are equal, proving (5.7.1). ❑

This, combined with the expression for ZP_n, implies that F_n is a scalar multiple of P_n. The scalar is the coefficient of $x^n + x^{-n}$, which is $\binom{k+n-1}{n}_q$ by inspection.

THEOREM 5.7.11. $P_n(x) = \binom{k+n-1}{n}_q^{-1} F_n(x)$.

REMARK 5.7.12. These are the continuous q-ultraspherical polynomial of Rogers.[5] Specifically, if we view $x = e^{i\theta}$ as an indeterminate on S^1, then

$$F_n(x) = C_n(\cos\theta; q^k \mid q) = \frac{(q^k;q)_n}{(q;q)_n} x^n {}_2\varphi_1\left(\begin{matrix} q^{-n} & q^k \\ & q^{1-n-k}\end{matrix}; q, x^{-2}q^{1-k}\right),$$

where we use the q-hypergeometric series

$${}_i\varphi_j\left(\begin{matrix} a_1 & \cdots & a_i \\ b_1 & \cdots & b_j\end{matrix}; q, z\right) := \sum_{\ell=0}^{\infty} \frac{(a_1;q)_\ell \cdot \ldots \cdot (a_i;q)_\ell}{(b_1;q)_\ell \cdot \ldots \cdot (b_j;q)_\ell (q;q)_\ell} \left((-1)^\ell q^{\binom{\ell}{2}}\right)^{1+j-i} z^\ell.$$

For $i = 2$ and $j = 1$, this simplifies to

$${}_2\varphi_1\left(\begin{matrix} a_1 & a_2 \\ & b \end{matrix}; q, z\right) = \sum_{\ell=0}^{\infty} \frac{(a_1;q)_\ell (a_2;q)_\ell}{(b;q)_\ell (q;q)_\ell} z^\ell.$$

By [Macdon, f. 106], we have

$$\Delta_k^+ = \frac{(x^2;q)_\infty}{(q^k x^2;q)_\infty} \quad \text{and} \quad \Delta_{-k}^- = \frac{(qx^{-2};q)_\infty}{(q^{1-k}x^{-2};q)_\infty}.$$

[5] Q.v. `www.dlmf.nist.gov/18.28#v`.

Therefore, Theorem 5.5.9 becomes, employing Notation 4.3.13,

$$
\begin{aligned}
|P_n|^2 &= \frac{(x^2;q)_\infty}{(q^k x^2;q)_\infty}\left(\tfrac{(n+k)\alpha}{2}\right) \cdot \frac{(qx^{-2};q)_\infty}{(q^{1-k}x^{-2};q)_\infty}\left(\tfrac{-(n+k)\alpha}{2}\right) \\
&= \prod_{r=0}^{\infty} \frac{1-q^r x^2}{1-q^{k+r}x^2}\bigg|_{\frac{(n+k)\alpha}{2}} \frac{1-q^{r+1}x^{-2}}{1-q^{r+1-k}x^{-2}}\bigg|_{\frac{-(n+k)\alpha}{2}} \\
&= \prod_{r=0}^{\infty} \frac{1-q^{r+\langle 2\alpha/2,(n+k)\alpha/2\rangle}}{1-q^{k+r+\langle 2\alpha/2,(n+k)\alpha/2\rangle}} \cdot \frac{1-q^{r+1+\langle -2\alpha/2,-(n+k)\alpha/2\rangle}}{1-q^{-k+r+1+\langle -2\alpha/2,-(n+k)\alpha/2\rangle}} \\
&= \prod_{r=0}^{\infty} \frac{1-q^{r+n+k}}{1-q^{2k+r+n}} \cdot \frac{1-q^{r+1+n+k}}{1-q^{1+n+r}} \\
&= \frac{\prod_{r=0}^{\infty}(1-q^{r+n+k})/(1-q^{1+n+r})}{\prod_{r=k}^{\infty}(1-q^{r+n+k})/(1-q^{1+n+r})} \\
&= \prod_{r=0}^{k-1} \frac{1-q^{r+n+k}}{1-q^{1+n+r}} \\
&= \frac{(1-q^{n+k})\cdot\ldots\cdot(1-q^{2k+n-1})(1-q)\cdot\ldots\cdot(1-q^k)}{(1-q^{n+1})\cdot\ldots\cdot(1-q^{n+k})(1-q)\cdot\ldots\cdot(1-q^k)} \\
&= \binom{2k+n-1}{k}_q \binom{n+k}{k}_q^{-1}.
\end{aligned}
$$

PROPOSITION 5.7.13. *For all $n \geqslant 0$ (viz. $\lambda \in L_{++}$) we have*

$$P_n = E_{-n} + q^k \frac{1-q^n}{1-q^{k+n}} E_n.$$

Proof. For $n = 0$, indeed $1 = 1 + 0$. Notice that the right-hand side is an eigenvector for $(Y + Y^{-1})$ with the expected eigenvalue $q^{(k+n)/2} + q^{-(k+n)/2}$ and hence a scalar multiple of P_n, the scalar being the coefficient of $x^n + x^{-n}$. The coefficient of x^{-n} in E_{-n} is 1 and it is 0 in E_n by Proposition 5.7.3. That of x^n in E_{-n} we can compute with Corollary 5.7.8 to be

$$\binom{k+n}{n}_q^{-1}\binom{k+n-1}{n}_q\binom{k-1}{0} = \frac{1-q^k}{1-q^{k+n}}.$$

Hence the total coefficient of x^n is

$$\frac{1-q^k}{1-q^{k+n}} + q^k\frac{1-q^n}{1-q^{k+n}} = 1$$

and so the total coefficient of $x^n + x^{-n}$ is 1, whence we conclude the claim. ❑

EXAMPLE 5.7.14. We have computed a few E_n so let's see.

- Most easily (apart from $P_0 = 1$...),

$$P_1 = E_{-1} + q^k\frac{1-q}{1-q^{k+1}}E_1 = x^{-1} + \frac{1-q^k}{1-q^{k+1}}x + q^k\frac{1-q}{1-q^{k+1}}x = x^{-1} + x,$$

  as expected.

- Next,

$$\begin{aligned}
P_2 &= x^{-2} + \frac{(1-q^2)(1-q^k)}{(1-q)(1-q^{k+2})} + \frac{1-q^k}{1-q^{k+2}}x^2 + q^k\frac{1-q^2}{1-q^{k+2}}\left(x^2 + q\frac{1-q^k}{1-q^{k+1}}\right)\\
&= x^{-2} + x^2 + \frac{(1-q^2)(1-q^k)(1-q^{k+1})}{(1-q)(1-q^{k+1})(1-q^{k+2})} + q^{k+1}\frac{(1-q)(1-q^2)(1-q^k)}{(1-q)(1-q^{k+1})(1-q^{k+2})}\\
&= x^{-2} + x^2 + \frac{(1-q^2)(1-q^k)}{(1-q)(1-q^{k+1})(1-q^{k+2})}\Big((1-q^{k+1}) + q^{k+1}(1-q)\Big)\\
&= x^{-2} + x^2 + \frac{(1-q^2)(1-q^k)}{(1-q)(1-q^{k+1})}.
\end{aligned}$$

- And finally, we can compute

$$\begin{aligned}
P_3 &= x^{-3} + \frac{(1-q^3)(1-q^k)}{(1-q)(1-q^{k+3})}x^{-1} + \frac{(1-q^3)(1-q^k)(1-q^{k+1})}{(1-q)(1-q^{k+2})(1-q^{k+3})}x + \frac{1-q^3}{1-q^{k+3}}x^3\\
&\quad + q^k\frac{1-q^3}{1-q^{k+3}}\left(x^3 + q\frac{(1-q^2)(1-q^k)}{(1-q)(1-q^{k+2})}x + q^2\frac{1-q^k}{1-q^{k+2}}x^{-1}\right)\\
&= x^{-3} + x^3 + \left(\frac{(1-q^3)(1-q^k)(1-q^{k+2})}{(1-q)(1-q^{k+2})(1-q^{k+3})} + q^{k+2}\frac{(1-q)(1-q^3)(1-q^k)}{(1-q)(1-q^{k+2})(1-q^{k+3})}\right)x^{-1}\\
&\quad + \left(\frac{(1-q^3)(1-q^k)(1-q^{k+1})}{(1-q)(1-q^{k+2})(1-q^{k+3})} + q^{k+1}\frac{(1-q^2)(1-q^3)(1-q^k)}{(1-q)(1-q^{k+2})(1-q^{k+3})}\right)x\\
&= x^{-3} + x^3 + \frac{(1-q^3)(1-q^k)}{(1-q)(1-q^{k+2})}(x^{-1}+x).
\end{aligned}$$

Finally, we compute the symmetrisers' and shift operators' actions. Notations 5.4.1 and 5.4.2 become the following. We set ε = sign and omit it from notation if we consider the trivial character. Then $s_1 = s_1$ (gasp) and

$$s_1^\varepsilon = s_1^* = c(X)c(X^{-1})^{-1}s_1 = \frac{(\tau X^2 - \tau^{-1})(X^{-2}-1)}{(X^2-1)(\tau X^{-2} - \tau^{-1})}s_1 = \frac{(\tau+\tau^{-1})(1-X^2) + \tau^{-1}(X^2 - X^{-2})}{(\tau+\tau^{-1})(1-X^2) + \tau(X^2-X^{-2})}s_1.$$

Similarly, $\tau^\varepsilon = -\tau^{-1}$ and the symmetrisers are

$$U^+ = \tau^{-1}(1+\tau T_1) = \tau^{-1} + T_1 \quad \text{and} \quad U^- = -\tau(1-\tau^{-1}T_1) = T_1 - \tau.$$

Indeed, U^+ kills $T_1 - \tau$ whilst U^- does $T_1 - \tau^\varepsilon$ since this is precisely the Hecke relation. Clearly, W_{R_λ} is trivial for any $\lambda \neq 0$ so we set $F_n^\pm = U^\pm E_n$ for any $n \neq 0$ and, if $n > 0$, then Notation 5.5.2 becomes

$$P_n^\pm = s_1 F_n^\pm = s_1(T_1 \pm \tau^\mp)E_n = \begin{cases} s_1(1+s_1)c(X^{-1})E_n = (1+s_1)\frac{\tau x^{-2}-\tau^{-1}}{x^{-2}-1}E_n & \text{if } +,\\ s_1 c(X)(s_1-1)E_n = c(X^{-1})(1-s_1)E_n & \text{if } -,\end{cases}$$

by Proposition 4.2.13. To do: figure out why $P_n^+ = P_n = E_{-n} + \text{blah}\cdot E_n$. Using the formula with the ξ^ε doesn't seem to work either.

From Notation 5.5.4, we have $\delta := \delta_{\alpha,k} = (x - x^{-1})c_{\alpha,k} = (x-x^{-1})c(X) = \tau x - \tau^{-1}x^{-1}$. In the present case,

$$S_{01}^+ = \{\pm\alpha + rc \in S \mid r = 0 \geqslant \chi(\pm\alpha)\} = \{\alpha\} = R^+$$

and $\ell(\alpha) = 1$ if $\varepsilon = $ sign but 0 if $\varepsilon = $ triv (same for $-\alpha$). Therefore $\delta_{\text{triv},k} = 1$ and $\delta_{\text{sign},k} = \delta$. Indeed, the shift operators G_{triv} are expected to be the identity from Theorem 5.6.2 since $\ell \equiv 0$ for that character and indeed they are by definition. Henceforth, fix $\varepsilon = $ sign, so that $\ell \equiv 1$.

Evaluating δ at $Y^{-1} = T_1^{-1}U$ as operator on A_R (where, recall, T_1 acts by τ), we get

$$\begin{aligned}\delta(Y^{-1})\Big|_{A_R} &= \tau T_1^{-1}U - \tau^{-1}UT_1 = (\tau T_1^{-1} - 1)U = \tau(T_1^{-1} - \tau^{-1})U \\ &= \tau(T_1 - \tau)U = \tau c(X)(s_1 - 1)U = \tau\frac{\delta(X)}{X - X^{-1}}(s_1 - 1)U.\end{aligned}$$

Then by definition,

$$G := G_\varepsilon = \delta(X)^{-1}\delta(Y^{-1}),$$

which acts on $f \in A_R$ by

$$Gf = \frac{\tau}{x - x^{-1}}(s_1uf - uf) = \frac{\tau}{x - x^{-1}}(s_1u - us_1)f.$$

Only the $\tau = q^{k/2}$ involves k, so $\widetilde{G} := \tau^{-1}G$ does not. Using Equation (4.5.1), therefore

$$\widetilde{G}f(x) = \frac{1}{x - x^{-1}}(f(q^{1/2}x) - f(q^{1/2}x^{-1})) = \frac{1}{x - x^{-1}}(f(q^{1/2}x) - f(q^{-1/2}x)).$$

Take $f = F_k(x,z)$. Then, first of all, $F_k(q^{1/2}x,z) - F(q^{-1/2}x,z)$ is equal to

$$\begin{aligned}&\prod_{r=0}^{k-1}\frac{1}{(1 - xzq^{r+1/2})(1 - x^{-1}zq^{r-1/2})} - \prod_{s=0}^{k-1}\frac{1}{(1 - xzq^{s-1/2})(1 - x^{-1}zq^{s+1/2})} \\ &= \frac{(1 - xzq^{-1/2})(1 - x^{-1}zq^{k-1/2}) - (1 - x^{-1}zq^{-1/2})(1 - xzq^{k-1/2})}{\prod_{r=0}^{k}(1 - xzq^{r-1/2})(1 - x^{-1}zq^{r-1/2})} \\ &= \Big((1 - xzq^{-1/2})(1 - x^{-1}zq^{k-1/2}) - (1 - x^{-1}zq^{-1/2})(1 - xzq^{k-1/2})\Big)F_{k+1}(x, q^{-1/2}z).\end{aligned}$$

We claim that $\widetilde{G}F_k(x,z) = q^{-1/2}(q^k - 1)zF_{k+1}(x, q^{-1/2}z)$. It suffices to prove that

$$\frac{(1 - xzq^{-1/2})(1 - x^{-1}zq^{k-1/2}) - (1 - x^{-1}zq^{-1/2})(1 - xzq^{k-1/2})}{x - x^{-1}} = q^{-1/2}(1 - q^k)z.$$

The left-hand numerator is clearly equal to $z(x - x^{-1})(q^{k-1/2} - q^{-1/2})$, proving this equality. Book says $(1 - q^k)$ but I believe this is WRONG; you seem to get this extra minus sign which you really need to make sure GP = dP below. We continue working with my version. Of course,

$$\widetilde{G}F_k(x,z) = \sum_{n\geqslant 0}\widetilde{G}F_{n,k}(x)z^n = \sum_{n\geqslant 0}(q^k - 1)F_{n,k+1}(x)q^{-(n+1)/2}z^{n+1},$$

whence

$$\begin{aligned}\widetilde{G}P_{n,k} &= \binom{k+n-1}{n}_q^{-1}\widetilde{G}F_{n,k} \\ &= \binom{k+n-1}{n}_q^{-1}q^{-n/2}(q^k - 1)F_{n-1,k+1} \\ &= \binom{(k+1)+(n-1)-1}{n-1}_q^{-1}q^{-n/2}(q^n - 1)F_{n-1,k+1} \\ &= (q^{n/2} - q^{-n/2})P_{n-1,k+1}.\end{aligned}$$

Indeed, this is what we wanted from Theorem 5.6.2, as $\rho_\ell = \frac{1}{2}\ell(\alpha)\alpha = \alpha/2$, so that

$$G_{\mathrm{sign}} P_{\lambda+\rho_\ell,k} = GP_{n+1,k} = \tau\widetilde{G}P_{n+1,k} = q^{k/2}(q^{(n+1)/2} - q^{-(n+1)/2})P_{n,k+1},$$

whilst (when replacing $k \longmapsto -k$, in c this has the effect $\tau \longmapsto \tau^{-1}$ or, equivalently, $x \longmapsto x^{-1}$)

$$\begin{aligned} d_{k,\ell}(\lambda)P_{\lambda,k+\ell} &= q^{\widetilde{n(k,1)}/2}\delta_{\mathrm{sign},-k}\big((n+k+1)\alpha/2\big)P_{n,k+1} \\ &= q^{k/2}(x - x^{-1})c_{\alpha,-k}\big((n+k+1)\alpha/2\big)P_{n,k+1} \\ &= q^{k/2}(x - x^{-1})\frac{\tau^{-1}x^{(n+k+1)/2} - \tau x^{-(n+k+1)/2}}{x - x^{-1}}P_{n,k+1} \\ &= q^{k/2}(q^{(n+1)/2} - q^{-(n+1)/2})P_{n,k+1}, \end{aligned}$$

as desired, where the last equality used Notation 4.3.13 again. We really need that minus sign! Maybe Macdonald forgot to substitute $-k$ for k in this?

Having done this, let $\vartheta := s_1 u - us_1$ and $\Phi_{k+1} := (x - x^{-1})^{-1}\nabla_{k+1}$ as operators. By orthogonality of the symmetric polynomials, we then have, for $m \neq n$ (say both positive to avoid zero),

$$0 = \langle P_{m-1,k+1}, P_{n-1,k+1}\rangle_{k+1} = \left\langle \widetilde{G}P_{m,k}, P_{n-1,k+1}\right\rangle_{k+1} = \mathrm{ct}(\vartheta(P_{m,k})\Phi_{k+1}P_{n-1,k+1}),$$

or, alternatively, $\mathrm{ct}(P_{m,k}\vartheta(\Phi_{k+1}P_{n-1,k+1})) = 0$, where we used that the bar does nothing on A_R and ϑ^* is a multiple of itself lalalalalallalalala

References

[Bour456] NICOLAS BOURBAKI, *Groupes et algèbres de Lie: Chapitres 4 à 6*, Éléments de mathématique, Masson, Paris, 1981.

[Macdon] IAN GRANT MACDONALD, *Affine Hecke Algebras and Orthogonal Polynomials*, Cambridge Tracts in Mathematics 157, Cambridge University Press, 2003.

[Opdam] ERIC M. OPDAM, *Lecture Notes on Dunkl Operators for Real and Complex Reflection Groups*, MSJ Memoirs 8, Mathematical Society of Japan, 2000.