Commutative triples in group representations and special functions

Lecture notes for course NWI-WM116B Advanced Topics in Representation Theory (2022/2023) and for winter school "Harmonic analysis and special functions in mathematical physics" (31/7-4/8 2023, Córdoba, Argentina)

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Chapter 1

Introduction

In these lecture notes the reader can find some basic results on the interaction of group representations and special functions. This is a vast area with many applications in e.g. mathematical physics, number theory, combinatorics, etc., and these notes should be viewed as an introduction to the subject. We recall some topics of special functions and orthogonal polynomials, and we have taken some time in order to derive some details of the important aspects of the related representation theory of groups. In particular, the part on compact groups in Chapter 3 is rather detailed, and it aims to understand a few aspects of orthogonal polynomials in relation to the compact group $\mathrm{SU}(2)$.

In Chapter 4 we develop the beginning of Gelfand pairs, and we apply this in particular to recast some of the results of Chapter 3. We also look into the spherical functions on $\mathrm{SL}(2,\mathbb{R})$. However, since $\mathrm{SL}(2,\mathbb{R})$ is a non-compact matrix group, the representation theory is rather more complicated than in the compact case. So we sometimes only state some of the results, and then we focus on particular techniques in order to allow an association between representation theory and special functions.

In a sense, one of the origins is the work of Wigner, and his claim of "the unreasonable effectiveness of mathematics". Wigner's view has been recorded in Talman's book [72], see also Miller [60]. See also Koornwinder [56, §8] for historic comments, and for this one can also consult the papers [53], [54] by Koornwinder. There is an enormous amount of literature on this topic, and even restricting to one-variable functions there is a huge list of papers available. To mention a few, we mention the classic books by Vilenkin [79], and their successors in the three volume set of Klimy and Vilenkin [80], [81], [82], see als [38]. See also Barut and Rączka [4] and Wawrzyńczyk [85].

Since there is quite some attention to the theory of groups and its structure and representation theory including that of Lie algebras, there are also several general references in this direction, see [7], [8],[32], [36], [39], [57], [40], [67], [84], [87]. Other books focussing on groups, representation theory and special functions are [17], [21], [22]. Books focussing on Gelfand pairs are van Dijk [17] and Folland [23]. Needless to say, this is an incomplete list.

In Chapter 3 we introduce explicitly the matrix entries of representations of the compact group $\mathrm{SU}(2)$, which we consider as a subgroup of 2×2 complex matrices. In Chapter 3 we also develop some general tools in the representation theory of compact (matrix) groups, such as the Schur orthogonality relations. Not all statements of this type will have a complete proof, e.g. the completeness of the matrix entries in $C(G)$ or $L^2(G)$, which is known as the Peter-Weyl Theorem 3.3.14 is assumed and not proved. (But there is an exercise indicating the proof.) We find in Chapter 3 an intimate relation between special functions of hypergeometric type and in the following table we give a kind of dictionary between the properties of the Legendre polynomials and the corresponding group or representation theoretic property. This is a baby example of the intimate relationship between the two topics, and is mentioned to indicate the flavour of the results that can be obtained.

Legendre polynomials	*related group property*
orthogonality	Schur orthogonality
3-term recurrence	tensor product decomposition
differential equation	matrix entries are eigenfunctions of Casimir operator
linearisation $P_nP_m = \sum_k \alpha_k P_k$ with nonnegative coefficients α_k	α_k are squares of Clebsch-Gordan coefficients
addition formula	homomorphism of representation
product formula	spherical functions

In Chapter 4 we study more generally Gelfand pairs, and we look at explicit cases. Some of the results of Chapter 3 are reformulated in terms of Gelfand pairs, and these correspond to compact Gelfand pairs. In general we study spherical functions and the corresponding product formulas and the spherical Fourier transform. As an example of a non-compact Gelfand pair we study the spherical functions on $\mathrm{SL}(2,\mathbb{R})$ by characterising them as eigenfunctions of a suitable second-order differential equation arising from the Casimir element in the centre of the universal enveloping algebra. For this one needs to be able to do the radial part calculation, which was introduced by Harish-Chandra, see [10]. We do some of these calculations explicitly for easy cases.

In Chapter 5 we extend the notion of Gelfand pair to the notion of a commutative triple, and we consider the associated matrix spherical functions. A commutative triple is defined in terms of commutativity of a suitable convolution algebra, and it can be characterised in terms of a multiplicity freeness condition. The corresponding spherical functions, or matrix spherical functions, have been studied already by Godement [25] in the 1950s and Tirao [74] in the 1970s in even greater generality, see [24] for more references. We discuss a special case to emphasise the ideas and the way of calculating the associated special functions. Especially, the radial part of the Casimir operator gives rise to matrix valued differential operator. We consider the special case of the group situation for $\mathrm{SU}(n+1)$ in more detail, especially for the case $n = 1$. This part is based on the analysis on the symmetric pair $(\mathrm{SU}(2)\times\mathrm{SU}(2), \mathrm{diag})$ as given in [48], [49] and following the more general theory of [63]. The study for the related pair $(\mathrm{SO}(4), \mathrm{SO}(3))$ is given in [61]. Finally, the matrix differential operators arising in this way have implications for physical models, so-called spin integrable systems, see e.g. Burić and Schomerus [9] for an example and further references.

The relation between spherical functions and more generally matrix spherical functions on compact groups on the one hand and orthogonal polynomials and vector valued orthogonal polynomials on the other hand goes back a long time. The general theory goes back a long time, see e.g. [24], [25], [31], [74], [87] and references give there. For the explicit relation to special function, probably the first instance is Koornwinder [55], see also [48], [49] for a follow-up. Another example is due to Grünbaum, Pacharoni and Tirao [27], where matrix spherical functions are studied for the symmetric space $(\mathrm{SU}(3), \mathrm{U}(2))$ exploiting the Casimir operators of order 2 and 3. Since then there are studies of more general examples of matrix spherical functions related to Gelfand pairs, see e.g. Heckman and van Pruijssen [29], van Pruijssen [63], as well as Pezzini and van Pruijssen [62], which gives a classification of commutative triples in semisimple Lie groups. For multivariable analogues, there is a general theory and example in [50] for type A, as well as in [46] and [58] related to type BC.

There are various extensions possible of the relations as described in these notes. First of all, a lot of the connection described can be extended to the relation between quantum groups and special functions of basic hypergeometric type, see e.g. [11], [37]. Secondly, there is a very important interplay between group theory and special functions of many variables, we refer to e.g. [31], [30], in the scalar case. For the compact setting there are some links to multivariable matrix polynomials, see [50], [46], [58] for explicit examples, and [63] and [62] more general theory. However, in the non-compact setting there are not many examples available, but see [64] and [66] for examples. Thirdly, there is the extension to multivariable special functions of basic hypergeometric type in relation to (double) affine Hecke algebras, see e.g. [59]. Initially, the plan was to also include some more details on the relation to the general theory matrix-valued orthogonal polynomials, we refer to [65].

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Chapter 2

Prerequisites

2.1 Orthogonal polynomials

In this section we recall some facts for orthogonal polynomials on the real line.

Assume that μ is a Borel measure on the real line $\mathbb{R}$ so that all moments are finite, i.e. for any $n \in \mathbb{N}$ the function $x \mapsto x^n$ is integrable with respect to μ. We also assume that μ is not a finite discrete measure, but this case can also be considered in this context. Put

$$m_n = \int_{\mathbb{R}} x^n \, d\mu(x), \qquad n \in \mathbb{N},$$

for the n-th moment of the measure μ. Note that $m_n \in \mathbb{R}$. Moreover, without loss of generality we can assume that μ is a probability measure, i.e. we assume $m_0 = 1$. This is not essential.

In particular, it follows that all polynomials are integrable with respect to the measure μ. Defining the inner product for two (suitable) functions on $\mathbb{R}$ by

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)} \, d\mu(x)$$

we see that we get the Hilbert space of square integrable functions, where we identify two functions μ-a.e., and we denote this Hilbert space by $L^2(\mu)$. We can apply the Gram-Schmidt process to the sequence $(x^n)_{n=0}^{\infty}$ in the Hilbert space $L^2(\mu)$, and we find an orthonormal sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials, i.e. p_n is a polynomial of degree n and the orthogonality relations

$$\int_{\mathbb{R}} p_n(x)\, p_m(x) \, d\mu(x) = \delta_{m,n} \tag{2.1.1}$$

is valid. Recall that $\delta_{m,n} = 0$ if $m \neq n$ and $\delta_{m,n} = 1$ if $m = n$. Note the moments m_n are real, and so the polynomials p_n have real coefficients, since the moments is all that is needed in the Gram-Schmidt process. In particular, this means that in (2.1.1) we can do without complex conjugation, since $\overline{p_n(x)} = p_n(x)$ for $x \in \mathbb{R}$. Note that in the Gram-Schmidt process there is a choice in normalisation, and we choose it such that leading coefficient of p_n is positive. The orthogonality relation (2.1.1) means that the polynomial p_n is orthogonal to any polynomial of degree less than n. This in particular gives uniqueness up to a constant in any degree. Finally, note that $p_0(x) = 1$ because we assume $m_0 = 1$.

Exercise 2.1.1. The polynomials are completely determined in terms of the moments. Prove that

$$q_n(x) = \det \begin{pmatrix} m_0 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_{n+2} \\ \vdots & \vdots & & \ddots & \vdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{pmatrix}$$

is a polynomial of degree n and satisfies

$$\int_{\mathbb{R}} q_n(x)\, x^m \, d\mu(x) = 0, \qquad \forall\, m \in \{0, 1, 2, \cdots, n-1\}.$$

Conclude that q_n is a multiple of p_n, and determine this multiple.

Theorem 2.1.2 (Three-term recurrence)**.** *Let μ be a probability measure on $\mathbb{R}$ with finite moments, and let $(p_n)_{n\in\mathbb{N}}$ be the sequence of orthonormal polynomials, then there exist sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ with $a_n > 0$, $b_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, so that*

$$x\, p_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) \qquad \forall\, x$$

with the convention $a_{-1} = 0$ and $p_{-1}(x) = 0$.

Proof. Observe that $x\, p_n(x)$ is a polynomial of degree $n+1$, so that we can write

$$x\, p_n(x) = \sum_{k=0}^{n+1} c_k p_k(x)$$

for some coefficients c_k. Using the orthogonality we see that

$$c_k = \int_{\mathbb{R}} x\, p_n(x)\, p_k(x)\, d\mu(x) = \int_{\mathbb{R}} p_n(x) \bigl(x p_k(x)\bigr)\, d\mu(x) = 0 \quad \text{for } k+1 < n,$$

since p_n is orthogonal to any polynomial of degree less than n and $xp_k(x)$ has degree $k+1$. This gives the three-term structure, and to find the results for the coefficients we observe

$$c_{n+1} = \int_{\mathbb{R}} x\, p_n(x)\, p_{n+1}(x)\, d\mu(x), \qquad c_{n-1} = \int_{\mathbb{R}} x\, p_n(x)\, p_{n-1}(x)\, d\mu(x),$$

are the same expressions up to shifting n. Moreover, since we have normalised p_n to have positive leading coefficient, we see that $a_n > 0$. Finally,

$$b_n = \int_{\mathbb{R}} x\, (p_n(x))^2\, d\mu(x)$$

is real, which also follows from the remark that the orthogonal polynomials have real coefficients. $\square$

It is actually true that the three-term recurrence charactises orthogonality. This is the following converse, which is known as Favard's Theorem.

Theorem 2.1.3 (Favard)**.** *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of polynomials with degree p_n equal to n for all $n \in \mathbb{N}$. Assume there exist sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ with $a_n > 0$, $b_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, so that*

$$x\, p_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) \qquad \forall\, x$$

with the convention $a_{-1} = 0$ and $p_{-1}(x) = 0$, $p_0(x) = 1$. Then there exists a probability measure μ with finite moments such that the sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ is orthonormal with respect to the measure μ.

Favard's Theorem 2.1.3 can be found in many texts, see [1], [12], [33], and there are many possible proofs. One of the proofs is based on the spectral theorem for self-adjoint operators, see e.g. [44], [69].

Exercise 2.1.4. We use the notation as in Theorem 2.1.2. Consider the $N \times N$-matrix

$$J_N = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots & 0 \\ a_0 & b_1 & a_1 & 0 & \cdots & 0 \\ 0 & a_1 & b_2 & a_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{N-3} & b_{N-2} & a_{N-2} \\ 0 & \cdots & \cdots & 0 & a_{N-2} & b_{N-1} \end{pmatrix}$$

J_N is an example of a tridiagonal matrix.

(i) Note that $J_N^* = J_N$, so that its spectrum, i.e. the set of eigenvalues, is contained in $\mathbb{R}$. Show that the dimension of an eigenspace equals 1.

(ii) Show that

$$J_N \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \\ p_{N-2}(x) \\ p_{N-1}(x) \end{pmatrix} = x \begin{pmatrix} p_0(x) \\ p_1(x) \\ p_2(x) \\ \vdots \\ p_{N-2}(x) \\ p_{N-1}(x) \end{pmatrix} - a_{N-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ p_N(x) \end{pmatrix}$$

(iii) Conclude that the zeros of p_N correspond to the eigenvalues of J_N, and that p_N has simple real zeros. Hint: either prove that $\det(J_n - x) = C p_N(x)$ with $C = (-1)^N \prod_{i=0}^{N-1} a_i \neq 0$ or use Exercise 2.1.5 to see that all zeros of p_N are simple.

Exercise 2.1.5. We assume that μ is a probability measure on $\mathbb{R}$ with finite moments, and that $(p_n)_{n\in\mathbb{N}}$ is the corresponding sequence of orthonormal polynomials.

(i) Show that $f(x) = \int_{\mathbb{R}} K_n(x,y)\ f(y)\, d\mu(y)$ for any function f that is a polynomial up to degree n for the kernel $K_n(x,y) = \sum_{k=0}^n p_k(x)p_k(y)$.

(ii) The kernel $K_n(x,y)$ can be evaluated explicitly by the Christoffel-Darboux formula;

$$K_n(x,y) = a_n \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}.$$

Prove this by multiplying the three-term recurrence with $p_n(y)$, replace the roles of x and y and subtract the recursions.

(iii) Determine the kernel at the diagonal, i.e. $K_n(x,x)$, and use this to see that the zeros of each p_n are simple.

Exercise 2.1.6. Another way to normalise a sequence of polynomials is by requesting that they are monic, i.e. the leading coefficient equals 1.

(i) Assume that μ is a probability measure on $\mathbb{R}$ with finite moments, and that $(p_n)_{n\in\mathbb{N}}$ is the corresponding sequence of orthonormal polynomials. Let $P_n(x) = p_n(x)/\mathrm{lc}(p_n)$, where $\mathrm{lc}(p_n)$ denotes the leading coefficient of p_n, be the corresponding monic orthogonal polynomials. Show that the sequence $(P_n)_{n\in\mathbb{N}}$ satisfies a three-term recurrence relation and relate the coefficients to the coefficients for the polynomials $(p_n)_{n\in\mathbb{N}}$. Determine

$$\int_{\mathbb{R}} P_n(x)P_m(x)\, d\mu(x)$$

Formulate the analogue of Favard's Theorem 2.1.3 for the monic polynomials $(P_n)_{n\in\mathbb{N}}$.

(ii) Assume that $(P_n)_{n\in\mathbb{N}}$ are monic orthogonal polynomials, and let $a \in \mathbb{R}\setminus\{0\}$, $b \in \mathbb{R}$. Show that $Q_n(x) = a^{-n}P_n(ax+b)$ gives a sequence $(Q_n)_{n\in\mathbb{N}}$ of monic orthogonal polynomials. What can you say about the orthogonality measure for $(Q_n)_{n\in\mathbb{N}}$ in terms of the orthogonality measure for the $(P_n)_{n\in\mathbb{N}}$?

2.2 Hypergeometric functions

We recall some facts about hypergeometric functions, and related orthogonal polynomials. We use it to fix notation. For completeness, some general references are e.g. [1], [12], [33], [43], [42], [73].

We start with the Γ-function, due to Euler in 1729. It is defined as a holomorphic function in the right half plane:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\, dt, \qquad \Re(z) > 0. \tag{2.2.1}$$

Then $\Gamma(z+1) = z\Gamma(z)$ for $\Re(z) > 0$ (Show this!), and in particular $\Gamma(n+1) = n!$ since $\Gamma(1) = 1$. So we can view the Γ-function as the extension of the factorial to a holomorphic function in the right half plane.

Exercise 2.2.1. (i) Argue that $z \mapsto \int_1^\infty t^{z-1}e^{-t}\,dt$ is an entire function.

(ii) Show that for any $\delta > 0$, $A > 0$ we have $0 < \delta \le \Re(z) \le A$ that

$$\int_0^1 t^{z-1}e^{-t}\,dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!\,(n+z)}$$

(iii) Derive the Mittag-Leffler expansion

$$\Gamma(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!\,(n+z)} + \int_1^\infty t^{z-1}e^{-t}\,dt$$

for $z \in \mathbb{C} \setminus -\mathbb{N}$. Conclude that the Γ-function has simple poles at $n \in -\mathbb{N}$ and that the residue at $-n$ equals $(-1)^n/n!$.

A closely related integral is the beta-integral

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt, \qquad \Re(a) > 0, \Re(b) > 0. \tag{2.2.2}$$

It can be evaluated in many ways to yield the result

$$B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}, \qquad \Re(a) > 0, \Re(b) > 0, \tag{2.2.3}$$

see e.g. [1], [33], [73].

Exercise 2.2.2. (i) Show that $\int_0^\infty x^{2z-1}e^{-x^2}\,dx = \frac{1}{2}\Gamma(z)$ for $\Re(z) > 0$.

(ii) Consider integral

$$I(a,b) = \int_0^\infty \int_0^\infty x^{2a-1}y^{2b-1}e^{-(x^2+y^2)}\,dxdy$$

and conclude that $I(a,b) = \frac{1}{4}\Gamma(a)\Gamma(b)$.

(iii) Use polar coordinates to write

$$I(a,b) = \int_0^\infty r^{2a+2b-1}e^{-r^2}\,dr \int_0^{\frac{1}{2}\pi} (\cos\phi)^{2a-1}(\sin\phi)^{2b-1}\,d\phi.$$

Recognise the first integral as a Γ-function using (i), and the second as a beta-integral (use the transformation $t = \cos^2\phi$).

We introduce the general notation for hypergeometric series;

$$
\begin{gathered}
{}_rF_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r)_k}{(b_1, \cdots b_s)_k} \frac{z^k}{k!}, \\
(a_1, \cdots, a_r)_k = \prod_{i=1}^{r} (a_i)_k, \qquad (a)_k = \prod_{i=0}^{k-1} (a+i) = \frac{\Gamma(a+k)}{\Gamma(a)}
\end{gathered}
\tag{2.2.4}
$$

whenever the series is well-defined. The term $(a)_k$ is called a Pochhammer symbol or a rising factorial. Note that in particular for $a_i \in -\mathbb{N}$ the series terminates. In case one of the $b_j \in -\mathbb{N}$, the series is not well-defined. However, in case $b_j = -N$, $N \in \mathbb{N}$, we still consider such a hypergeometric series if only one of the a_i is of the form $a_i = -n$ for $n \in \{0, 1, 2, \cdots, N\}$. In that case we consider the series as a terminating series, terminating after the $k = n$ term.

Exercise 2.2.3. Show that the radius of convergence of (2.2.4) for general parameters is ∞ for $r < s + 1$, 1 for $r = s + 1$ and 0 for $r > s + 1$.

The special case $r = s = 0$ gives the exponential function, and the case $r = 1$, $s = 0$ gives the binomial formula, i.e.

$$
{}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; z\right) = (1-z)^{-a}, \qquad |z| < 1. \tag{2.2.5}
$$

In case the series terminates, i.e. $a = -n$, $n \in \mathbb{N}$, this is Newton's binomium. Note that (2.2.5) can be considered as an analytic continuation of the hypergeometric series on the left hand side to $\mathbb{C} \setminus [1, \infty)$.

Exercise 2.2.4. Show that the first order differential equation

$$
(1-z)\frac{df}{dz}(z) = a\, f(z)
$$

has both sides of (2.2.5) as solutions. Conclude (2.2.5) from the existence and uniqueness statement for solutions of initial value problems.

The special case $r = 2$, $s = 1$ is very well-known, it is known as the Gauss hypergeometric function:

$$
{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k\, k!} z^k. \tag{2.2.6}
$$

This function is a solution to the hypergeometric differential equation

$$
z(1-z)\frac{d^2w}{dz^2}(z) + \big(c - (a+b+1)z\big)\frac{dw}{dz}(z) - abw(z) = 0, \tag{2.2.7}
$$

i.e. $w(z) = {}_2F_1(a, b; c; z)$ satisfies (2.2.7).

Exercise 2.2.5. (i) Show that ${}_2F_1(a, b; c; z)$ is indeed a solution to (2.2.7).

(ii) Show that for generic parameters the function

$$z^{1-c}\,{}_2F_1\left(\begin{matrix} a-c+1, b-c+1 \\ 2-c \end{matrix}; z\right)$$

solves (2.2.7) as well. Conclude that ${}_2F_1(a, b; c; z)$ is the unique analytic solution to (2.2.7) up to scalar multiplication.

(iii) Show that the change of coordinates $u = 1 - z$ in (2.2.7) gives again a hypergeometric differential operators, but with $(a, b, c) \mapsto (a, b, a + b - c + 1)$. Conclude that

$${}_2F_1\left(\begin{matrix} a, b \\ a+b-c+1 \end{matrix}; 1-z\right), \qquad (1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a, c-b \\ c-a-b+1 \end{matrix}; 1-z\right)$$

are also solutions to (2.2.7).

(iv) Obtain solutions of the form $\sum_{n=0}^{\infty} a_n z^{\mu-n}$ for the hypergeometric differential equation (2.2.7), which corresponds to solutions around ∞. Show that these solutions are

$$(-z)^{-a}\,{}_2F_1\left(\begin{matrix} a, 1-c+a \\ 1-b+a \end{matrix}; \frac{1}{z}\right), \qquad (-z)^{-b}\,{}_2F_1\left(\begin{matrix} b, 1-c+b \\ 1-a+b \end{matrix}; \frac{1}{z}\right).$$

Next we take $\Re(b) > 0$ and $\Re(c - b) > 0$, and we observe that using (2.2.5) for $|z| < 1$ in

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n \int_0^1 t^{n+b-1}(1-t)^{c-b-1}\,dt,$$

where interchanging the series and integral is allowed because of uniform comvergence. Now the integral is a beta integral and can be evaluated by (2.2.3). Collecting the terms we recognise a ${}_2F_1$-series, and this gives Euler's integral representation

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt, \qquad \Re(c) > \Re(b) > 0 \quad (2.2.8)$$

and for $|z| < 1$. Note that the right hand side gives an analytic extension of the ${}_2F_1$-series to $\mathbb{C} \setminus [1, \infty)$. An immediate consequence of Euler's integral representation (2.2.8) and the beta integral is that we can evaluate the ${}_2F_1$-series at $z \to 1$ as long as the integral converges. We obtain Gauss's summation theorem

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \Re(c-a-b) > 0. \quad (2.2.9)$$

(The condition $\Re(c - a - b) > 0$ is also exactly the condition for Raabe's test to make the ${}_2F_1$-series at $z = 1$ absolutely convergent. The condition $\Re(c) > \Re(b) > 0$ is removed using analytic continuation, an argument we skip, see [1].) Note that for terminating series the condition is not necessary, and we obtain the Chu-Vandermonde summation

$${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; 1\right) = \frac{(c-b)_n}{(c)_n}, \qquad n \in \mathbb{N}. \quad (2.2.10)$$

Exercise 2.2.6. Prove Pfaff's transformation formula

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; z\right) = (1-z)^{-a}\, {}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix} ; \frac{z}{z-1}\right) $$

by replacing t by $1-s$ in Euler's integral representation (2.2.8) for the ${}_2F_1$-series.

The Jacobi polynomial $P_n^{(\alpha,\beta)}$ of degree n is defined by

$$ P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, \alpha+\beta+n+1 \\ \alpha+1 \end{matrix} ; \frac{1-x}{2}\right). \tag{2.2.11} $$

For $\alpha, \beta > -1$ we have the orthogonality relations

$$ \begin{aligned} &\int_{-1}^{1} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\, (1-x)^{\alpha}(1+x)^{\beta}\, dx = \\ &\qquad \delta_{m,n} \frac{2^{\alpha+\beta+1}(n+\alpha+\beta+1)_n \Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\,\Gamma(2n+\alpha+\beta+2)} \end{aligned} \tag{2.2.12} $$

So we see that the Jacobi polynomials are the orthogonal polynomials with respect to the beta-distribution, see (2.2.2), up to a change of variables. Before giving some additional properties and subfamilies of the Jacobi polynomials, we present a proof of the orthogonality relation (2.2.12). Again, there are many different approaches to the proof. We consider

$$ \begin{aligned} &\int_{-1}^{1} (1+x)^m P_n^{(\alpha,\beta)}(x)\, (1-x)^{\alpha}(1+x)^{\beta}\, dx \\ =&\frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(\alpha+\beta+n+1)_k}{(\alpha+1)_k k!} 2^{-k} \int_{-1}^{1} (1-x)^{\alpha+k}(1+x)^{\beta+m}\, dx \\ =&\frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k(\alpha+\beta+n+1)_k}{(\alpha+1)_k k!} 2^{-k} 2^{\alpha+\beta+k+m+1} \frac{\Gamma(\alpha+k+1)\Gamma(\beta+m+1)}{\Gamma(\alpha+\beta+m+k+2)} \\ =&\frac{(\alpha+1)_n}{n!} 2^{\alpha+\beta+m+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+m+1)}{\Gamma(\alpha+\beta+m+2)} \sum_{k=0}^{n} \frac{(-n)_k(\alpha+\beta+n+1)_k}{k!(\alpha+\beta+m+2)_k} \end{aligned} $$

where the integral is a beta-integral after a change of variables. Now the remaining sum is

$$ {}_2F_1\left(\begin{matrix} -n, \alpha+\beta+n+1 \\ \alpha+\beta+m+2 \end{matrix} ; 1\right) = \frac{(m-n+1)_n}{(\alpha+\beta+m+2)_n} $$

by the Chu-Vandermonde summation (2.2.10). Finally, note that

$$ (m-n+1)_n = 0, \qquad m \in \{0, 1, 2, \cdots, n-1\}. $$

This shows that the Jacobi polynomial of degree n is orthogonal to all polynomials of lower degree, so the Jacobi polynomials are the orthogonal polynomials with respect to the beta integral. This proves the case $n \neq m$ of (2.2.12).

Exercise 2.2.7. Give a proof of the case $m = n$ of (2.2.12). Use the calculation above for $m = n$, and note that $P_n^{(\alpha,\beta)}(-x)$ is a multiple of $P_n^{(\beta,\alpha)}(x)$.

Since the Jacobi polynomials are expressed in terms of a hypergeometric function ${}_2F_1$, they satisfy a differential equation following from (2.2.7). Explicitly,

$$(1-z^2)\frac{d^2y}{dz^2}(z) + (\beta - \alpha - (\alpha+\beta+2)z)\frac{dy}{dz}(z) = -n(n+\alpha+\beta+1)y(z). \qquad (2.2.13)$$

voor $y(z) = P_n^{(\alpha,\beta)}(z)$. Finally, since the Jacobi polynomials are orthogonal polynomials, they satisfy a three-term recurrence. Note that the Jacobi polynomials are not normalised to be orthonormal. The recurrence is

$$\begin{aligned} xP_n^{(\alpha,\beta)}(x) &= A_n P_{n+1}^{(\alpha,\beta)}(x) + B_n P_n^{(\alpha,\beta)}(x) + C_n P_{n-1}^{(\alpha,\beta)}(x), \qquad (2.2.14)\\ A_n &= \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}\\ B_n &= \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\\ C_n &= \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} \end{aligned}$$

Exercise 2.2.8. Derive the result of (2.2.14). First derive A_n by comparing leading coefficients, next derive C_n be rewriting the Jacobi polynomials to their orthonormal versions using (2.2.12) and Theorem 2.1.2. Finally, use a special value for x to determine B_n.

The special case $(\alpha, \beta) = (0, 0)$ is the Legendre polynomial, and

$$U_n(x) = \frac{(n+1)!}{(\frac{3}{2})_n} P_n^{(\frac12,\frac12)}(x) = (n+1)\ {}_2F_1\left(\begin{matrix} -n, n+2 \\ \frac{3}{2} \end{matrix} ; \frac{1-x}{2}\right) \qquad (2.2.15)$$

are the Chebyshev polynomials of the second kind, which also can be written as

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \qquad (2.2.16)$$

The Chebyshev polynomials of the first kind can be written as

$$T_n(\cos\theta) = \cos(n\theta), \qquad (2.2.17)$$

which are Jacobi polynomials as well;

$$T_n(x) = {}_2F_1\left(\begin{matrix} -n, n \\ \frac{1}{2} \end{matrix} ; \frac{1-x}{2}\right) = \frac{n!}{(1/2)_n} P_n^{(-\frac12,-\frac12)}(x). \qquad (2.2.18)$$

The relations between Chebyshev polynomials and Jacobi polynomials for special values for (α, β) can be proved by showing that the trigonometric versions are also orthogonal with respect to the same weight from the beta integral.

Exercise 2.2.9. Prove that (2.2.15) and (2.2.16) are indeed equal. Also show that (2.2.17) and (2.2.18) are equal.

Finally, we discuss one particular example of a set of orthogonal polynomials with respect to a finite discrete measure. If we have a finite discrete orthogonality measure on the real line, then we get only finitely many orthogonal polynomials. If the support of the orthogonality measure consists of $N+1$ points, then we have only orthogonal polynomials up to degree N. Almost all of the general statements on orthogonal polynomials as in Section 2.1 have an analogous statement in the case of finite discrete orthogonal polynomials.

The *Krawtchouk polynomials* $K_n(x;p,N)$ are polynomials depending on a parameter $p \in (0,1)$ and an integer $N \in \mathbb{N}$. The Krawtchouk polynomials are orthogonal on the finite set $\{0,1,\cdots,N\}$. The orthogonality relations are

$$\sum_{x=0}^{N} \binom{N}{x} p^x (1-p)^{N-x} K_m(x;p,N) K_n(x;p,N) = \delta_{mn} \left(\frac{1-p}{p}\right)^n \binom{N}{n}^{-1}, \tag{2.2.19}$$

for $n, m \in \mathbb{N}$ so that $n, m \leq N$. We see that the Krawtchouk polynomials are orthogonal with respect to the binomial distribution. The explicit expression as hypergeometric series is given

$$K_n(x;p,N) = {}_2F_1\left(\begin{matrix}-n,-x\\ -N\end{matrix};\frac{1}{p}\right), \qquad x,n \in \{0,1,\cdots,N\}. \tag{2.2.20}$$

Note that (2.2.20) shows that $K_n(x;p,N) = K_x(n;p,N)$.

2.3 Linear groups

We denote by $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ the vectorspace of $n \times n$ matrices with real or complex entries with the standard topologies. By $\mathrm{GL}(n,\mathbb{C}) \subset M_n(\mathbb{C})$ and $\mathrm{GL}(n,\mathbb{R}) \subset M_n(\mathbb{R})$ we denote the invertible matrices. This gives the induced topology on $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{GL}(n,\mathbb{R})$.

Definition 2.3.1. *A set G in $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$ is a linear group if $x, y \in G$ imply $xy \in G$ and $x^{-1} \in G$. We assume G to be closed in $\mathrm{GL}(n,\mathbb{C})$ or $\mathrm{GL}(n,\mathbb{R})$.*

Remark 2.3.2. We give a number of examples, and this will also fix notation.

$$\begin{aligned}
\mathrm{GL}(n,\mathbb{C}) &= \{g \in M_n(\mathbb{C}) \mid \det(g) \neq 0\}\\
\mathrm{SL}(n,\mathbb{C}) &= \{g \in M_n(\mathbb{C}) \mid \det(g) = 1\}\\
\mathrm{GL}(n,\mathbb{R}) &= \{g \in M_n(\mathbb{R}) \mid \det(g) \neq 0\}\\
\mathrm{SL}(n,\mathbb{R}) &= \{g \in M_n(\mathbb{R}) \mid \det(g) = 1\}
\end{aligned}$$

gives the general linear group GL of size $n \times n$ over $\mathbb{R}$ or $\mathbb{C}$. The special linear group SL then arises as a subgroup of the corresponding general linear group by restricting to the matrices of determinant 1. Some other important examples are the unitary and special unitary groups

$$\begin{aligned}
\mathrm{U}(n) &= \{g \in M_n(\mathbb{C}) \mid g^*g = gg^* = 1\}\\
\mathrm{SU}(n) &= \{g \in M_n(\mathbb{C}) \mid g^*g = gg^* = 1, \det(g) = 1\}
\end{aligned}$$

the orthogonal and special orthogonal groups over $\mathbb{R}$ and $\mathbb{C}$;

$$\begin{aligned} \mathrm{O}(n,\mathbb{C}) &= \{g \in M_n(\mathbb{C}) \mid g^t g = g g^t = 1\} \\ \mathrm{SO}(n,\mathbb{C}) &= \{g \in M_n(\mathbb{C}) \mid g^t g = g g^t = 1, \det(g) = 1\} \\ \mathrm{O}(n,\mathbb{R}) &= \{g \in M_n(\mathbb{R}) \mid g^t g = g g^t = 1\} \\ \mathrm{SO}(n,\mathbb{R}) &= \{g \in M_n(\mathbb{R}) \mid g^t g = g g^t = 1, \det(g) = 1\} \end{aligned}$$

Here g^t denotes the transpose of the matrix, g^* the adjoint of a matrix (so that $g^t = g^*$ for $g \in M_n(\mathbb{R})$), and det denotes the determinant of a matrix g. We use the abbreviated notation $\mathrm{O}(n) = \mathrm{O}(n,\mathbb{R})$, $\mathrm{SO}(n) = \mathrm{SO}(n,\mathbb{R})$. Note that we use 1 to denote the real or complex number 1 as well as the identity matrix. Its meaning should be clear from the context. We have listed the classical groups apart from the symplectic group over $\mathbb{R}$ and $\mathbb{C}$, and some of these groups have extensions to groups over the octonions $\mathbb{H}$.

Exercise 2.3.3. (i) Show that the group $\mathrm{SU}(2)$ can be written as

$$\mathrm{SU}(2) = \{g = \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \mid |a|^2 + |c|^2 = 1,\ a, c \in \mathbb{C}\}$$

and show that $SU(2)$ is isomorphic as a group to the group of elements of norm 1 in the quaternions $\mathbb{H}$, i.e. the unit sphere in $\mathbb{H}$.

(ii) Show that $\mathrm{SO}(2)$ can be written as

$$\mathrm{SO}(2) = \{\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \mid t \in \mathbb{R}\}$$

and conclude that $\mathrm{SO}(2)$ is an abelian group (or commutative group), which is isomorphic to $\mathbb{T}$, the group of complex numbers of modulus 1, i.e. $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. The notation $\mathbb{T}$ stands for 'torus'.

We recall that we can take the exponential of a matrix. For $X \in M_n(\mathbb{R})$ or for $X \in M_n(\mathbb{C})$ we define

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k \tag{2.3.1}$$

The series converges in the norm defined by $\|X\| = \sqrt{\mathrm{Tr}\,(X^*X)}$, so that $\|X\|^2 = \sum_{i,j=1}^{n} |X_{i,j}|^2$ when writing X as a matrix $X = (X_{i,j})$. Here Tr denotes the trace of a matrix. So with this norm we consider $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ and $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ as the standard normed vector spaces. Note that these are inner product spaces for $\langle X, Y\rangle = \mathrm{Tr}\,(Y^*X)$.

Proposition 2.3.4. *The matrix exponential satisfies the following properties:*

(i) If D is a diagonal matrix $D = \mathrm{diag}(d_1, \cdots, d_n)$, then $\exp(D)$ is a diagonal matrix, $\exp D = \mathrm{diag}(e^{d_1}, \cdots, e^{d_n})$. In particular, $\exp(0) = 1$, the exponent of the zero matrix is the identity matrix.

(ii) If N is a nilpotent matrix, i.e. $N^p = 0$ for some $p \in \mathbb{N}$, then $\exp(N) = \sum_{k=0}^{p-1} \frac{1}{k!} N^k$ *is a finite sum.*

(iii) If g is an invertible matrix, then $\exp(gXg^{-1}) = g\exp(X)g^{-1}$.

(iv) $(\exp(X))^t = \exp(X^t)$ *and* $(\exp(X))^* = \exp(X^*)$

(v) If X and Y commute, i.e. $XY = YX$, then $\exp(X+Y) = \exp(X)\exp(Y)$ *and* $\exp(X)$ *and* $\exp(Y)$ *commute.*

(vi) Write $X = S + N$, where S is diagonalisable and N is nilpotent and $SN = NS$, i.e. write X in its Jordan normal form, then $\exp(X) = \exp(S)\exp(N)$.

(vii) $\det(\exp(X)) = e^{\mathrm{Tr}\,(X)}$, *and* $\exp(X)$ *is invertible with* $(\exp(X))^{-1} = \exp(-X)$

(viii) The function $\mathbb{R} \ni t \mapsto g(t) = \exp(tX)$ is the unique solution of the first order initial value problem $\frac{dg}{dt}(t) = Xg(t)$ (and of $\frac{dg}{dt}(t) = g(t)X$) with initial value $g(0) = 1$.

Exercise 2.3.5. Prove Proposition 2.3.4.

Remark 2.3.6. Proposition 2.3.4 indicates how to calculate the exponential of a matrix; $\exp(X)$ By the Jordan decomposition we can conjugate X to a semisimple, i.e. diagonal, matrix S plus a nilpotent matrix N, such that $SN = NS$, or $gXg^{-1} = S + N$. Now $\exp(S) = \mathrm{diag}(e^{s_1}, \cdots, e^{s_n})$ is just the diagonal matrix with the exponential of the diagonal entries of S. And $\exp(N)$ is a finite sum, since N is nilpotent, i.e. $N^k = 0$ for some $k \in \mathbb{N}$. Then

$$\exp(X) = g^{-1}\exp(S)\exp(N)g.$$

As an example, we calculate $\exp(t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$. Note that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}^{-1} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

so that

$$\exp(t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}^{-1} \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

In this case, this can also be proved directly by using

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{2k} = (-1)^k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{2k+1} = (-1)^k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and using the definition

$$\exp(t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Proposition 2.3.7. *For $X \in M_n(\mathbb{C})$ we have that $\exp(tX) \in \mathrm{SU}(n)$ for all $t \in \mathbb{R}$ if and only if X satisfies $X^* = -X$ and $\mathrm{Tr}\,(X) = 0$.*

Proof. Assume that $\exp(tX) \in \mathrm{SU}(n)$ for all $t \in \mathbb{R}$, then we have $\exp(tX)^* = \exp(tX)^{-1}$ for all $t \in \mathbb{R}$. By Proposition 2.3.4 we have $\exp(tX^*) = \exp(-tX)$ for all t. Differentiating with respect to t and putting $t = 0$ gives $X^* = -X$. Since we also have $1 = \det(\exp(tX)) = e^{t\mathrm{Tr}\,(X)}$ for all t, we similarly get $\mathrm{Tr}\,(X) = 0$.

The converse follows directly from Proposition 2.3.4. □

Introducing the notation

$$\mathfrak{su}(n) = \{X \in M_n(\mathbb{C}) \mid X^* + X = 0,\ \mathrm{Tr}\,(X) = 0\} \tag{2.3.2}$$

we see that $\mathfrak{su}(n)$ is vector space over $\mathbb{R}$ (and not over $\mathbb{C}$). Moreover, $\mathfrak{su}(n)$ is closed under the commutator bracket, i.e. for $X, Y \in \mathfrak{su}(n)$ we have $[X, Y] = XY - YX \in \mathfrak{su}(n)$. The vector space $\mathfrak{su}(n)$ is an example of a (real) Lie algebra, see Appendix A. Note that $\dim_{\mathbb{R}}\mathfrak{su}(n) = 2\sum_{k=1}^{n-1}(n-k) + n - 1 = n^2 - 1$ by counting the off-diagonal freedom and the freedom on the diagonal.

Proposition 2.3.8. *The exponential map* $\exp\colon \mathfrak{su}(n) \to \mathrm{SU}(n)$ *is surjective*

Note that the map is not injective, since any element $\mathrm{diag}(2\pi i k_1, \cdots, 2\pi i k_n)$, $k_i \in \mathbb{Z}$, $\sum_{i=1}^{n} k_i = 0$, in $\mathfrak{su}(n)$ gets mapped to the identity by the exponential map. It follows from Proposition 2.3.8 that $\mathrm{SU}(n)$ is (path-)connected. Since $[0, 1] \ni t \mapsto \exp(tX)$ connects the identity and $\exp(X) \in \mathrm{SU}(n)$. Moreover, $\mathrm{SU}(n)$ is even simply connected, see e.g. [40, Prop. 1.136], [67, §3.4] for a proof.

Proof. Any unitary matrix is diagonalisable, hence conjugated to a diagonal matrix of the form $\mathrm{diag}(e^{i\theta_1}, \cdots, e^{i\theta_n})$ and the product of the diagonal being equal to 1 for real θ_i. Now take $X = \mathrm{diag}(i\theta'_1, \cdots, i\theta'_n)$ so that $\theta'_i \equiv \theta_i \mod 2\pi$ and $\sum_{i=1}^{n} \theta'_i = 0$. Then conjugate back, and use Proposition 2.3.4. □

Exercise 2.3.9. Introduce

$$\mathfrak{so}(n) = \{X \in M_n(\mathbb{R}) \mid X^t + X = 0\}$$

(i) Show that $\mathfrak{so}(n)$ is a real vector space and calculate its dimension. Show that for $X, Y \in \mathfrak{so}(n)$ we also have $[X, Y] \in \mathfrak{so}(n)$, and conclude that $\mathfrak{so}(n)$ is a (real) Lie algebra.

(ii) Show that $\mathbb{R} \ni t \mapsto \exp(tX) \in \mathrm{SO}(n)$ if and only if $X \in \mathfrak{so}(n)$, i.e. prove the analogue of Proposition 2.3.7 for the special orthogonal group.

(iii) Show that $\exp\colon \mathfrak{so}(n) \to \mathrm{SO}(n)$ is surjective, i.e. prove the analogue of Proposition 2.3.8 for the special orthogonal group. Conclude that $\mathrm{SO}(n)$ is connected.

$\mathrm{SO}(n)$ is not simply connected; for $n = 2$ we have that the fundamental group is $\pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$ and for $n \geq 3$ we have $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2\mathbb{Z}$, see e.g. [7, Ch. 7], [40, §I.17], [67, §3.4].

Exercise 2.3.10. For $g \in \mathrm{SU}(n)$, $X \in \mathfrak{su}(n)$, define $\mathrm{Ad}(g)X = gXg^{-1}$, which is known as the adjoint representation.

(i) Show that $\mathrm{Ad}(g)\colon \mathfrak{su}(n) \to \mathfrak{su}(n)$ is a linear invertible map satisfying $\mathrm{Ad}(g_1g_2) = \mathrm{Ad}(g_1)\mathrm{Ad}(g_2)$ for $g_1, g_2 \in \mathrm{SU}(n)$.

(ii) Show that for $Y \in \mathfrak{su}(n)$ we have

$$\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp(tY))X = [Y, X] = YX - XY.$$

2.4 Haar measure

A famous theorem of Haar states that a locally compact topological group G, i.e. a topological space G whose topology is locally compact and such that the group operations are continuous, has a left invariant measure unique up to a positive constant[1], see e.g. Cohn [13, Ch. 9]. We refer to [13, Ch. 9] for all unproved statements and examples in this section. Since all the groups considered in these lecture notes meet this criterion, we have a left Haar measure assigning a measure to a set in the Borel σ-algebra. Any non-empty open subset has positive measure, and the Haar measure is a regular Borel measure. We usually denote this by dg, and by d_lg if we need to stress that it is a left Haar measure. This means that for an integrable function f we have

$$\int_G f(g)\, dg = \int_G f(g_1g)\, dg \qquad \forall\, g_1 \in G. \tag{2.4.1}$$

For the groups considered we need not invoke the full generality of Haar's theorem, but more direct proofs of the existence exist, see e.g. [7, §I.5], [40, §VIII.2], [67, Ch. 5].

Similarly, such a group G also has a right Haar measure, say d_rg, unique up to a positive constant such that

$$\int_G f(g)\, d_rg = \int_G f(gg_1)\, d_rg \qquad \forall\, g_1 \in G.$$

In case the total measure of G is finite under the left Haar measure, we normalise it to be 1. This happens exactly for the case of compact groups G. In case the left and right Haar measure coincide, we call the group G *unimodular*.

More generally, the difference between left and right Haar measure is indicated by the modular function $\Delta\colon G \to \mathbb{R}_{>0}$ satisfying

$$\int_G f(gh^{-1})\, d_lg = \Delta(h) \int_G f(g)\, d_lg \tag{2.4.2}$$

for $h \in G$ and integrable $f\colon G \to \mathbb{C}$. Then $\Delta\colon G \to \mathbb{R}_{>0}$ is a continuous homomorphism, and the left and right Haar measure coincide if and only if the modular function is a constant equal to 1.

The main examples of unimodular groups are abelian groups and compact groups.

[1]The uniqueness up to a constant was not proved by Haar, but by von Neumann.

Exercise 2.4.1. Show that abelian groups are unimodular using the definition. Show that compact groups are unimodular using that the only compact subgroup of $\mathbb{R}_{>0}$ equals $\{1\}$.

The modular function relates the integral of f and the integral of f composed with the inverse. In general, we have

$$\int_G f(g)\,\Delta(g^{-1})\,d_l g = \int_G f(g^{-1})\,d_l g. \tag{2.4.3}$$

Most of the groups in these lecture notes are unimodular, so that for unimodular G (2.4.3) reduces to

$$\int_G f(g)\,d_l g = \int_G f(g^{-1})\,d_l g. \tag{2.4.4}$$

We can prove (2.4.4) directly, since the right hand side gives rise to a right invariant measure. By unimodularity it differs by a constant from the left hand side. Now take an integrable function such that $f(g^{-1}) = f(g)$ to determine the constant as 1. (Note we can take f to be the identity in case G is compact.)

The notation $L^2(G)$, $L(G)$, etc., is the corresponding space of square integrable or integrable functions on G with respect to the left Haar measure.

Remark 2.4.2. (i) The Haar measure on $\mathbb{R}$ and $\mathbb{T}$ are the Lebesgue measures, and $\mathbb{R}$ and $\mathbb{T}$ are unimodular groups.

(ii) $\mathrm{GL}(n,\mathbb{R})$ is unimodular, and the left and right Haar measure is given by

$$\int_{\mathrm{GL}(n,\mathbb{R})} f(X)\frac{1}{|\det(X)|^n}\,dX$$

where $X = (x_{1,1},\cdots,x_{n,n})$ are the n^2 coordinate functions for the matrix $X \in \mathrm{GL}(n,\mathbb{R})$, and $dX = \prod_{i,j=1}^n dx_{i,j}$ the corresponding Lebesgue measure.

(iii) An example of a non-unimodular group is the group of (orientation preserving) affine transformations of the real line $\mathbb{R}$, which is

$$G = \{\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix} \mid a > 0, b \in \mathbb{R}\}$$

which corresponds to a semidirect product of the multiplicative group $\mathbb{R}_{>0}$ and the additive group $\mathbb{R}$. Then the left Haar measure is explicitly

$$\int_G f(g)\,d_l g = \int_0^\infty \int_{\mathbb{R}} f(\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix})\,\frac{db\,da}{a^2}$$

and the right Haar measure is

$$\int_G f(g)\,d_r g = \int_0^\infty \int_{\mathbb{R}} f(\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix})\,\frac{db\,da}{a}$$

Exercise 2.4.3. Show that the examples given in Remark 2.4.2 are indeed left and right Haar measures in these cases. Determine the modular function for the third example.

Next we make the Haar measure on SU(2) explicit. Making the Haar measure explicit requires a choice of parametrisation of the matrices in SU(2). Note that in Exercise 2.3.3 we have established that SU(2) is the same as $S^3 \subset \mathbb{R}^4$ after identifying $\mathbb{H} \cong \mathbb{R}^4$. Recall the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and put σ_0 for the 2×2-identity matrix.

Exercise 2.4.4. (i) Show that $\Phi \colon S^3 \to \mathrm{SU}(2)$ given by

$$\Phi(x_1, x_2, x_3, x_4) = x_1\sigma_0 + ix_2\sigma_3 - ix_3\sigma_2 + ix_4\sigma_1 = \begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \tag{2.4.5}$$

is the required homeomorphism between S^3 and SU(2). In particular, SU(2) is a compact topological group.

(ii) Define polar coordinates on $\mathbb{R}^4$ by using the usual $\mathbb{R}^2$ polar coordinates in the variables x_1, x_2 and x_3, x_4, i.e.

$$(x_1, x_2, x_3, x_4) = (r_1 \cos\phi, r_1 \sin\phi, r_2 \cos\psi, r_2 \sin\psi), \qquad r_1, r_2 \geq 0,\ 0 \leq \phi, \psi < 2\pi.$$

Next we express r_1, r_2 in polar coordinates, $(r_1, r_2) = (r \sin\theta, r\cos\theta)$, with $r \geq 0$ and $0 \leq \theta \leq \frac{\pi}{2}$, then

$$(x_1, x_2, x_3, x_4) = (r \sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta\cos\psi, r\cos\theta\sin\psi),$$

with

$$r \geq 0, \quad 0 \leq \phi, \psi < 2\pi, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

Calculate the corresponding Jacobian as $r^3 \cos\theta \sin\theta$.

(iii) Label the unit sphere S^3 in spherical coordinates by

$$S^3 = \Big\{(r, \theta, \phi, \psi) \mid r = 1,\ 0 \leq \theta \leq \tfrac{\pi}{2},\ 0 \leq \phi, \psi < 2\pi\Big\},$$

and show that

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} e^{i\phi} \sin\theta & -e^{-i\psi}\cos\theta \\ e^{i\psi}\cos\theta & e^{-i\phi}\sin\theta \end{pmatrix} \mid 0 \leq \theta \leq \tfrac{\pi}{2},\ 0 \leq \phi, \psi < 2\pi \right\}.$$

(iv) We define an integral on SU(2) as follows. Identify the function $f\colon \mathrm{SU}(2) \to \mathbb{C}$ with a complex valued function F on S^3 by $f(g) = F(\theta, \phi, \psi)$ with

$$g = \begin{pmatrix} e^{i\phi}\sin\theta & -e^{-i\psi}\cos\theta \\ e^{i\psi}\cos\theta & e^{-i\phi}\sin\theta \end{pmatrix}.$$

To simplify notations we make this identification implicitly. Now we define the measure on SU(2) by

$$\int_{\mathrm{SU}(2)} f(g)\, dg = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} f(\begin{pmatrix} e^{i\phi}\sin\theta & -e^{-i\psi}\cos\theta \\ e^{i\psi}\cos\theta & e^{-i\phi}\sin\theta \end{pmatrix}) \cos\theta\, \sin\theta\, d\theta\, d\phi\, d\psi, \tag{2.4.6}$$

for all continuous functions f. Show that this is the (left and right) Haar measure on SU(2) by following the steps:

(a) Extend $\Phi\colon \mathbb{R}^4 \to M_2(\mathbb{C})$ by the same formula, and show that it is a continuous injection and that the inverse $\Phi^{-1}\colon \mathrm{Im}\,\Phi \to \mathbb{R}^4$ is continuous. Show that $\det(\Phi(\mathbf{x})) = \|\mathbf{x}\|^2$.

(b) Show that $\mathrm{Im}\,\Phi$ is closed under multiplication and transport left multiplication L_g by an element of $g \in \mathrm{SU}(2)$ to $\mathbb{R}^4$;

$$L_g\colon \mathbb{R}^4 \to \mathbb{R}^4, \qquad \mathbf{x} \mapsto \Phi^{-1}(g\Phi(\mathbf{x}))$$

and show that L_g is an isometry, and gives an orthogonal transformation. Using that SU(2) is connected, show that $L_g \in \mathrm{SO}(4)$.

(c) Conclude that the measure in (2.4.6) is left invariant for L_g since the Lebesgue measure on $\mathbb{R}^4$ and its restriction to S^3 is rotation invariant.

(d) Do the same for right multiplication, or use the fact that compact groups are unimodular, to conclude that the measure in (2.4.6) is also right invariant.

Note that the measure on SU(2) in Exercise 2.4.4 is normalised;

$$\int_{\mathrm{SU}(2)} dg = 1.$$

2.5 Representations

Given a linear group G, so embedded in $GL(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$ and hence coming with a topology, the group G has continuous group operations. By convention the Hilbert spaces in these lecture notes are assumed to be separable.

Definition 2.5.1. *Let V be finite dimensional complex vectorspace, and let $B(V)$ denote the space of linear maps of V to itself. In case V is a (complex) Hilbert space, $B(V)$ denotes the* *space of bounded linear operators. We say that V is a representation of the group G if there exists a homomorphism $\pi\colon G \to B(V)$ such that*

$$G \times V \to V, \qquad (g, v) \mapsto \pi(g)v$$

is continuous.

We call π the representation, and if we want to stress the dependence on V, we also call (π, V) the representation. Usually, V is called the representation space. A *finite-dimensional representation* is given by (π, V) with V finite-dimensional. In case V is one-dimensional, the corresponding homomorphism $\pi\colon G \to \mathbb{C}$ is called a *character*.

So we assume that the representation space V has an inner product. We call the representation π (or (π, V)) *unitary* if $\pi(g)$ is unitary for all $g \in G$. A closed subspace W is an *invariant subspace* if $\pi(g)W \subset W$ for all $g \in G$. In case the only invariant subspaces of (π, V) are the trivial subspace V and $\{0\}$, we call the representation *irreducible*. Note that any one-dimensional representation is irreducible.

In case π is unitary, the orthocomplement $W^\perp$ of an invariant subspace W is an invariant subspace. Indeed,

$$\langle \pi(g)v, w\rangle = \langle v, \pi(g)^* w\rangle = \langle v, \pi(g^{-1})w\rangle = 0$$

for $v \in W^\perp$ and any $w \in W$, since W is an invariant subspace. This means that $\pi(g)v \in W^\perp$. So in case of a unitary representation we can write V as a (possibly infinite) direct sum of invariant subspaces.

For unitary representations, the continuity condition in Definition 2.5.1 can be weakened.

Lemma 2.5.2. *If $\pi\colon G \to B(V)$ is a homomorphism, such that $\pi(g)$ is unitary for each $g \in G$. Then (π, V) is a representation of G if and only if $G \ni g \mapsto \pi(g)v \in V$ is a continuous map for all $v \in V$.*

Proof. In case (π, V) is a representation, then the map $G \times V \to V$, $(g, v) \mapsto \pi(g)v$ is continuous. Fixing $v \in V$ we see that the map $G \to V$, $g \mapsto \pi(g)v$ is continuous.

Conversely, note that

$$\|\pi(g)v - \pi(h)w\| \le \|\pi(g)(v-w)\| + \|\pi(g)w - \pi(h)w\| \le \|v - w\| + \|\pi(g)w - \pi(h)w\|$$

using $\|\pi(g)\| = 1$. So $G \times V \to V$, $(g, v) \mapsto \pi(g)v$ is continuous if $g \mapsto \pi(g)v$ is continuous for all $v \in V$. $\square$

Given a representation (π, V), then we can define the *contragredient representation*, also known as dual representation, (π^*, V^*) by

$$(\pi^*(g)v^*)(v) = v^*(\pi(g^{-1})v), \qquad v^* \in V^*,\ v \in V, \tag{2.5.1}$$

in the dual space V^* of V.

In case we have two representations (π_1, V_1) and (π_2, V_2) of the linear group G, then we have two constructions to create more representations. The first is the direct sum representation $\pi_1 \oplus \pi_2$ in the direct sum space $V_1 \oplus V_2$ defined by

$$\big(\pi_1 \oplus \pi_2(g)\big)\, v_1 \oplus v_2 = (\pi_1(g)v_1) \oplus (\pi_2(g)v_2) \tag{2.5.2}$$

Note that $v \mapsto (v, 0)$ gives the embedding $V_1 \subset V_1 \oplus V_2$, and this is an invariant subspace for $\pi_1 \oplus \pi_2$. Similarly, V_2 is an invariant subspace. The second construction is the tensor product representation $\pi_1 \otimes \pi_2$ in the tensor product space $V_1 \otimes V_2$ defined by

$$\big(\pi_1 \otimes \pi_2(g)\big)\, v_1 \otimes v_2 = (\pi_1(g)v_1) \otimes (\pi_2(g)v_2) \tag{2.5.3}$$

Exercise 2.5.3. (i) Show that $(\pi_1 \oplus \pi_2, V_1 \oplus V_2)$ and $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ are representations of G.

(ii) Show that if (π, V) is unitary, then so is (π^*, V^*).

(iii) Assume that (π_1, V_1) and (π_2, V_2) are unitary representations of G. Show that $(\pi_1 \oplus \pi_2, V_1 \oplus V_2)$ and $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ are unitary representations of G.

Example 2.5.4. Assume that the group G is unimodular, then the space of square integrable functions with respect to the Haar measure is a Hilbert space (after identifying functions a.e.). The space $L^2(G)$ carries two natural unitary representations. The *left regular representation* is defined by

$$\lambda\colon G \to B(L^2(G)), \qquad (\lambda(g)f)(x) = f(g^{-1}x),\ f \in L^2(G)$$

which is an isometry by

$$\begin{aligned}\langle \lambda(g)f_1, \lambda(g)f_2\rangle &= \int_G (\lambda(g)f_1)(x)\overline{(\lambda(g)f_2)(x)}\, dx \\ = \int_G f_1(g^{-1}x)\overline{f_2(g^{-1}x)}\, dx &= \int_G f_1(x)\overline{f_2(x)}\, dx = \langle f_1, f_2\rangle\end{aligned}$$

and since $\lambda(g)$ is surjective, we see that $\lambda(g)$ is a unitary. Now the continuity follows from Lemma 2.5.2 and standard estimates for the Haar measure, see e.g. [13], [84, §2.4, §2.6].

Similarly, the right regular representation

$$\rho\colon G \to B(L^2(G)), \qquad (\rho(g)f)(x) = f(xg),\ f \in L^2(G)$$

is a unitary representation of G on $L^2(G)$. One of the main problems is to decompose the left and right regular representation into irreducible representations. Note that we can also view $L^2(G)$ as a $G \times G$-representation by

$$((\lambda \times \rho)(g_1, g_2)f)(x) = f(g_1^{-1}xg_2)$$

since the action of $\lambda(g_1)$ and $\rho(g_2)$ commute (by associativity of the group multiplication). This is known as the *biregular representation*.

Definition 2.5.5. *Let (π_1, V_1), (π_2, V_2) be a representations of G. In case $T\colon V_1 \to V_2$ is a (bounded) linear operator satisfying $T\pi_1(g) = \pi_2(g)T$ for all $g \in G$, then T is an intertwiner or intertwining operator. The set of all intertwiners is denoted by $B_G(V_1, V_2)$, and this is a (closed with respect to the norm topology) vector space in the space of (bounded) linear operators of V_1 into V_2.*

Intertwiner operators are also called G-equivariant maps, which stresses the role of the group involved.

In case there is a (bounded) invertible intertwiner $T\colon V_1 \to V_2$ we call π_1 and π_2 *equivalent representations*. In case, π_1 and π_2 are unitary representations, we say that they are *unitarily equivalent* if the intertwiner is an isometry with an isometric inverse. The *unitary dual* of the group G is the set of irreducible unitary representations of G up to unitary equivalence, and the unitary dual of G is typically denoted by $\hat{G}$. The unitary dual $\hat{G}$ comes with a topology, but we will not use this.

Example 2.5.6. Recalling Fourier theory and realising that $\mathbb{T} \ni e^{i\theta} \mapsto e^{in\theta}$, $n \in \mathbb{Z}$, gives all unitary representations up to equivalence, we see that $\hat{\mathbb{T}} \cong \mathbb{Z}$. Similarly, by Fourier theory we have $\mathbb{R} \ni x \mapsto e^{i\lambda x}$, $\lambda \in \mathbb{R}$, gives all unitary representations of $\mathbb{R}$. So $\hat{\mathbb{R}} \cong \mathbb{R}$. More generally, for an abelian group G, the dual $\hat{G}$ is again an abelian group, but for general groups G this is no longer the case.

One of the most important results on intertwiners is known as Schur's Lemma.

Lemma 2.5.7 (Schur)**.** *Assume that (π_1, V_1) and (π_2, V_2) are finite-dimensional irreducible representations of G. Then $B_G(V_1, V_2)$ is trivial if π_1 and π_2 are not equivalent; $\pi_1 \not\cong \pi_2$ and $B_G(V_1, V_2) \cong \mathbb{C}$ if π_1 and π_2 are equivalent; $\pi_1 \cong \pi_2$.*

Proof. Assume $T\colon V_1 \to V_2$ is an intertwiner. Then the kernel $\mathrm{Ker}(T)$ and the image $\mathrm{Im}(T)$ are invariant subspaces of V_1 and V_2. Here we use the finite-dimensionality to see that the subspaces are closed. Since the representations are irreducible, we have $\mathrm{Ker}(T) = \{0\}$ or $\mathrm{Ker}(T) = V_1$. In the last case $T = 0$, and in the first case T is injective. Similarly, $\mathrm{Im}(T) = \{0\}$, i.e. $T = 0$, or $\mathrm{Im}(T) = V_2$, i.e. T is surjective. So either T is the zero map, or T is an isomorphism. So in case $\pi_1 \not\cong \pi_2$, we have $B_G(V_1, V_2) = \{0\}$, proving the first statement.

In case $\pi_1 \cong \pi_2$, we have to show that that $B_G(V_1, V_2)$ is one-dimensional. Assume that T and S are intertwiners, then S and T are isomorphisms, so that $S^{-1}T\colon V_1 \to V_1$ is a well-defined linear map. Then $S^{-1}T$ intertwines the action of π_1 with itself;

$$S^{-1}T\pi_1(g) = S^{-1}\pi_2(g)T = \pi_1(g)S^{-1}T.$$

Now take an eigenvector v of $S^{-1}T$ for eigenvalue λ. Since the identity obviously intertwines π_1 with itself, we see that $S^{-1}T - \lambda$ is an intertwiner. Since $\mathrm{Ker}(S^{-1}T - \lambda)$ is non-trivial since it contains v, it follows by the first part of the proof that $\mathrm{Ker}(S^{-1}T - \lambda) = V_1$, hence $T = \lambda S$. $\square$

Exercise 2.5.8. (i) Show that any finite-dimensional unitary representation is equivalent to a direct sum of finitely many irreducible representations. This is called *complete reducibility.*

(ii) Let $\pi \colon G \to B(V)$ be a finite dimensional representation. Identify $V \otimes V^*$ with the space $\mathrm{End}(V) = B(V)$ of linear maps of the finite dimensional Hilbert space by letting $v \otimes v^*$ correspond to the linear map $V \ni w \mapsto v^*(w)v \in V$, which is a rank one operator. Show that, under this identification,

$$(\pi \otimes \pi^*)(g)T = \pi(g)T\pi(g^{-1}), \qquad T \in \mathrm{End}(V),\ g \in G,$$

and moreover, $(\pi \otimes \pi^*)(g)T = \pi(g)T\pi(g)^*$ in case we additionally assume π to be a unitary representation.

(iii) Show that the trivial representation of G occurs in the tensor product representation $\pi \otimes \pi^*$.

(iv) Show that the trivial representation in the decomposition occurs precisely once if (π, V) is an irreducible representation.

Exercise 2.5.9. Let G be an abelian group. Show that every irreducible finite dimensional representation of G is one-dimensional. Hint: use Schur's Lemma 2.5.7.

Schur's Lemma 2.5.7 can be generalised in several ways, one is to not necessarily finite-dimensional representations in Hilbert spaces. However, this requires some input from functional analysis. Lemma 2.5.10 gives a possible generalisation.

Lemma 2.5.10. *Let H be a Hilbert space and $\mathcal{S} \subset B(H)$ be a collection of operators so that the only closed subspace invariant for S is either $\{0\}$ or H. Assume that $T \in B(H)$ is a self-adjoint operator commuting with each element of $\mathcal{S}$, $TS = ST\ \forall S \in \mathcal{S}$. Then A is a multiple of the identity. Similarly, if T is a normal operator commuting with all $S \in \mathcal{S}$, then T is a multiple of the identity.*

Exercise 2.5.11. The proof of Lemma 2.5.10 requires the Spectral Theorem.

(i) Assume T is self-adjoint operator, and assume $\lambda_1 \neq \lambda_2 \in \sigma(T)$ for real elements in the spectrum $\sigma(T) \subset \mathbb{R}$ of T. Pick two bounded continuous functions f and g on $\sigma(T)$ so that $f \neq 0$, $g \neq 0$ and $fg = 0$ on $\sigma(T)$. Show that $f(T)$ commutes with $S\ \forall S \in \mathcal{S}$.

(ii) Show that the closure of $f(T)H$ is an invariant subspace for $\mathcal{S}$, which is not $\{0\}$ nor H and conclude that $\sigma(T)$ consists of a single point, and derive the result for a self-adjoint T.

(iii) Prove the result for a normal T, i.e. $T^*T = TT^*$, by writing T as a linear combination of two self-adjoint operators.

Chapter 3

SU(2) and hypergeometric polynomials

We develop the link between hypergeometric functions and actions of 2×2-matrices on functions in two variables in Section 3.1, and by explicit calculations we already see hypergeometric sums appearing. In Section 3.2 we recall the compact group SU(2), which we study in various ways. In particular, we introduce the exponential function and the related Lie algebra, several coordinate systems and corresponding integrals. Then the results of Section 3.1 are recast in terms of representations of SU(2) of which the matrix entries are expressed in terms of terminating hypergeometric functions. Next, in Section 3.3 we study generalities of representations of compact matrix groups. This is done in some detail, and in particular we prove the Schur orthogonality relations for matrix entries of finite-dimensional irreducible representations explicitly. We apply this to characters, and we state the Peter-Weyl Theorem 3.3.14 without proof. In the next Section 3.4 we combine the previous results, and we connect the representations of SU(2) to Chebyshev polynomials of the second kind, Jacobi polynomials, Krawtchouk polynomials and Legendre polynomials.

3.1 Matrices and actions on functions

We consider a complex 2×2-matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$$

where $M_2(\mathbb{C})$ denotes the vector space of all complex 2×2-matrices. Consider a function $f \colon \mathbb{C}^2 \to \mathbb{C}$ of 2-variables $(x, y) \in \mathbb{C}^2$. We now define a new function $T \cdot f \colon \mathbb{C}^2 \to \mathbb{C}$ by

$$(T \cdot f)(x, y) = f(ax + cy, bx + dy). \tag{3.1.1}$$

Lemma 3.1.1. *Let*

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be elements of $M_2(\mathbb{C})$, then

$$S \cdot (T \cdot f) = (ST) \cdot f$$

for any function $f\colon \mathbb{C}^2 \to \mathbb{C}$.

Proof. Identify f with a function F of a vector variable;

$$f(x,y) = F(\begin{pmatrix} x \\ y \end{pmatrix}).$$

Then $(T \cdot f)(x,y)$ identifies with the function G of a vector variable;

$$G(\begin{pmatrix} x \\ y \end{pmatrix}) = F(T^t \begin{pmatrix} x \\ y \end{pmatrix}) = F(\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}) = F(\begin{pmatrix} ax+cy \\ bx+dy \end{pmatrix})$$

where T^t denotes the transpose of the matrix T.

So $S \cdot (T \cdot f)$ corresponds to

$$G(S^t \begin{pmatrix} x \\ y \end{pmatrix}) = F(T^t S^t \begin{pmatrix} x \\ y \end{pmatrix}) = F((ST)^t \begin{pmatrix} x \\ y \end{pmatrix})$$

which corresponds to $(ST) \cdot f$. $\square$

So the crucial property is that taking transpose is an antimultiplicative map; $(TS)^t = S^t T^t$ for $S, T \in M_2(\mathbb{C})$.

Exercise 3.1.2. Present a direct proof using the explicit expressions for the matrices T and S and TS.

Next we look for classes of functions as in Lemma 3.1.1 which are being preserved by the action of any matrix $T \in M_2(\mathbb{C})$. It is clear that if f is a polynomial in 2 variables x and y, then so is $T \cdot f$ for any $T \in M_2(\mathbb{C})$. This is still an infinite dimensional space, and we look for finite-dimensional subspaces which are invariant, i.e. preserved by any $T \in M_2(\mathbb{C})$. Recall that a polynomial $f\colon \mathbb{C}^2 \to \mathbb{C}$ is *homogeneous of degree* N for a degree $N \in \mathbb{N}$ when

$$f(\lambda x, \lambda y) = \lambda^N f(x,y), \qquad \forall (x,y) \in \mathbb{C}^2,\ \forall \lambda \in \mathbb{C}.$$

We denote by V_N the subspace of homogeneous polynomials of degree N. Then $\dim_{\mathbb{C}} V_N = N+1$ and we find that

$$f_k(x,y) = x^k y^{N-k}, \qquad k \in \{0, 1, \cdots, N\}$$

span the space V_N.

Lemma 3.1.3. *Let* $f \in V_N$, $T \in M_2(\mathbb{C})$, *then* $T \cdot f \in V_N$.

Proof. Let F the function of the vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2$ as in the proof of Lemma 3.1.1, then we see that f is homogeneous of degree N if and only if

$$F(\lambda \begin{pmatrix} x \\ y \end{pmatrix}) = \lambda^N F(\begin{pmatrix} x \\ y \end{pmatrix})$$

for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2$ and all scalars $\lambda \in \mathbb{C}$. Since the multiplication by the scalar λ commutes with the action of T^t on $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2$, it follows that

$$F(T^t\lambda \begin{pmatrix} x \\ y \end{pmatrix}) = \lambda^N F(T^t \begin{pmatrix} x \\ y \end{pmatrix})$$

so that $T \cdot f$ is homogeneous polynomial of degree N. Thus, $T \cdot f \in V_N$. □

Let us calculate the explicit action;

$$\begin{aligned}
(T \cdot f_k)(x, y) &= f_k(ax + cy, bx + dy) = (ax + cy)^k (bx + dy)^{N-k} \\
&= \sum_{r=0}^{k} \binom{k}{r} (ax)^r (cy)^{k-r} \sum_{s=0}^{N-k} \binom{N-k}{s} (bx)^s (dy)^{N-k-s} \\
&= \sum_{r=0}^{k} \sum_{s=0}^{N-k} \binom{k}{r} \binom{N-k}{s} a^r b^s c^{k-r} d^{N-k-s}\, x^{s+r} y^{N-s-r}
\end{aligned}$$

where

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}).$$

Note that in particular, for a diagonal matrix T we have

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \cdot f_k = a^k d^{N-k} f_k, \tag{3.1.2}$$

i.e. the f_k's are eigenfunctions for the action of the diagonal matrices. We can rewrite the matrix entries for the action of such a matrix explicitly;

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f_k &= \sum_{p=0}^{N} T^N_{p,k}(a, b, c, d)\, f_p, \\
T^N_{p,k}(a, b, c, d) &= \sum_{\substack{r=0 \\ r+s=p}}^{k} \sum_{s=0}^{N-k} \binom{k}{r} \binom{N-k}{s} a^r b^s c^{k-r} d^{N-k-s}
\end{aligned} \tag{3.1.3}$$

By a straightforward inspection we find Lemma 3.1.4.

Lemma 3.1.4. *$T^N_{p,k}$ is a polynomial in the variables (a, b, c, d), and the polynomial satisfies the homogeneity properties:*

$$\begin{aligned}
T^N_{p,k}(\lambda a, \lambda b, \lambda c, \lambda d) &= \lambda^N T^N_{p,k}(a, b, c, d) \\
T^N_{p,k}(\lambda a, \lambda b, c, d) &= \lambda^p T^N_{p,k}(a, b, c, d) \\
T^N_{p,k}(a, b, \lambda c, \lambda d) &= \lambda^{N-p} T^N_{p,k}(a, b, c, d) \\
T^N_{p,k}(\lambda a, b, \lambda c, d) &= \lambda^k T^N_{p,k}(a, b, c, d) \\
T^N_{p,k}(a, \lambda b, c, \lambda d) &= \lambda^{N-k} T^N_{p,k}(a, b, c, d)
\end{aligned}$$

for all $(a,b,c,d) \in \mathbb{C}^4$ *and all* $\lambda \in \mathbb{C}$.

We rewrite

$$T^N_{p,k}(a,b,c,d) = T^N_{p,k}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in order to exhibit explicit symmetries.

Lemma 3.1.5. *The matrix entries defined by* (3.1.3) *satisfy*

$$T^N_{p,k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = T^N_{p,N-k}\begin{pmatrix} b & a \\ d & c \end{pmatrix} = T^N_{N-p,N-k}\begin{pmatrix} d & c \\ b & a \end{pmatrix}$$

and

$$\binom{N}{k} T^N_{p,k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \binom{N}{p} T^N_{k,p}\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Proof. The first symmetries follow from the definition as in

$$(ax+cy)^k(bx+dy)^{N-k} = \sum_{p=0}^{N} T^N_{p,k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^p y^{N-p}$$

and we rewrite the left hand side under flipping $k \leftrightarrow N-k$, $a \leftrightarrow b$, $c \leftrightarrow d$. Comparing the coefficients gives the first equality of the first symmetry. Similarly, flipping $k \leftrightarrow N-k$, $a \leftrightarrow d$, $b \leftrightarrow c$, $x \leftrightarrow y$ and comparing coefficients gives the second equality of the first symmetry.

Multiplying with $\binom{N}{k} u^k v^{N-k}$ and summing over $k = 0$ up to N, and using the binomial theorem, we get

$$(axu + cyu + bxv + dyv)^N = \sum_{p=0}^{N}\sum_{k=0}^{N} \binom{N}{k} T^N_{p,k}\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^p y^{N-p} u^k v^{N-k} \tag{3.1.4}$$

and the left hand side is invariant under $x \leftrightarrow u$, $y \leftrightarrow v$, $b \leftrightarrow c$. Expanding gives the last symmetry. $\square$

3.2 SU(2) and its representations

We next want to show that we can use the set-up as in Section 3.1 te generate the representations of SU(2).

The diagonal subgroup K of SU(2) is given by

$$k(\phi) = \begin{pmatrix} e^{i\frac{1}{2}\phi} & 0 \\ 0 & e^{-i\frac{1}{2}\phi} \end{pmatrix} \in \mathrm{SU}(2), \qquad K = \{k(\phi) \mid 0 \le \phi < 4\pi\} \tag{3.2.1}$$

so that K is a compact abelian group, which is homeomorphic to the unit circle $\mathbb{T}$ in $\mathbb{C}$ by $k(\phi) \mapsto e^{i\frac{1}{2}\phi}$. By Fourier analysis we know that all characters, i.e. continuous group homomorphisms of K to $\mathbb{T}$, are labeled by $\mathbb{Z}$;

$$\chi_n\colon K \to \mathbb{T}, \qquad k(\phi) \mapsto e^{in\frac{1}{2}\phi}, \qquad n \in \mathbb{Z}. \tag{3.2.2}$$

We view $K \cong \mathrm{U}(1)$, the group of 1×1-unitary matrices.

We also consider the group of real 2×2 orthogonal matrices of determinant 1, i.e. SO(2). We introduce coordinates

$$a(\phi) = \begin{pmatrix} \cos(\frac{1}{2}\phi) & -\sin(\frac{1}{2}\phi) \\ \sin(\frac{1}{2}\phi) & \cos(\frac{1}{2}\phi) \end{pmatrix} \in \mathrm{SU}(2), \qquad A = \{a(\phi) \mid 0 \leq \phi < 4\pi\} \tag{3.2.3}$$

so that $A = \mathrm{SO}(2)$.

Exercise 3.2.1. Show that $g \in \mathrm{SU}(2)$ can be written as

$$g = k(\theta)a(\phi)k(\psi), \qquad 0 \leq \theta < 2\pi,\ 0 \leq \psi < 4\pi,\ 0 \leq \phi \leq \pi$$

and that (θ, ϕ, ψ), $0 < \phi < \pi$, form a coordinate system for the (open) subset of SU(2) where every matrix entry of $g \in \mathrm{SU}(2)$ is non-zero.

In the parametrisation of Exercise 3.2.1 we define

$$\int_{\mathrm{SU}(2)} f(g)\, dg = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{4\pi} \int_0^{\pi} f(k(\theta)a(\phi)k(\psi))\, \sin\phi \, d\phi\, d\theta\, d\psi. \tag{3.2.4}$$

Note that $\int_{\mathrm{SU}(2)} dg = 1$.

Exercise 3.2.2. Show that (2.4.6) and (3.2.4) correspond to each other.

Since $\mathrm{SU}(2) \subset M_2(\mathbb{C})$ we can consider the construction of Section 3.1 for the restriction to SU(2). For this it is convenient to relabel and renormalise a bit. First of all, we put $N = 2\ell$ for $\ell \in \frac{1}{2}\mathbb{N}$, since the notation then corresponds to the physicist's *spin* ℓ representation of SU(2).

We renormalise the basis of the space V_ℓ of homogeneous polynomials of degree 2ℓ; we take

$$e^\ell_n(x,y) = \binom{2\ell}{\ell - n}^{\frac{1}{2}} x^{\ell-n} y^{\ell+n}, \qquad n \in \{-\ell, \ldots, \ell\}. \tag{3.2.5}$$

and we define V_ℓ as a Hilbert space by putting $\langle e^\ell_n, e^\ell_m \rangle = \delta_{n,m}$. We view V_ℓ as a finite-dimensional complex Hilbert space. Now we translate the results of Section 3.1 using the relabeling to find the action of a 2×2-matrix on the basis vector e^ℓ_n;

$$\binom{2\ell}{\ell - n}^{\frac{1}{2}} (ax+cy)^{\ell-n}(bx+dy)^{\ell+n} = \sum_{m=-\ell}^{\ell} \binom{2\ell}{\ell - m}^{\frac{1}{2}} t^\ell_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^{\ell-m} y^{\ell+m}, \tag{3.2.6}$$

for a 2×2- matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In particular, e^ℓ_n is an eigenvector under the diagonal matrices and we see that e^ℓ_n realises a one-dimensional representation of the diagonal subgroup K. Indeed,

$$k(\phi)\cdot e^\ell_n = e^{i\frac{1}{2}\phi(\ell-n)}e^{-i\frac{1}{2}\phi(\ell+n)}e^\ell_n = e^{-i\phi n}e^\ell_n = \chi_{-2n}(k(\phi))\, e^\ell_n \tag{3.2.7}$$

Note that $-2n\in\mathbb{Z}$ for $n\in\{-\ell,-\ell+1,\cdots,\ell\}$. This means that V_ℓ splits as a K-representation as a direct sum of $2\ell+1$ one-dimensional K-representations;

$$V_\ell|_K \cong \bigoplus_{n=-\ell}^{\ell} \chi_{-2n} \tag{3.2.8}$$

or χ_m occurs in V_ℓ if and only if $|m|\leq 2\ell$ and $2\ell-m\in 2\mathbb{Z}$. .

From Lemma 3.1.5 we get the following symmetry relations:

$$t^\ell_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = t^\ell_{n,m}\begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad \text{and} \qquad t^\ell_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = t^\ell_{-m,-n}\begin{pmatrix} d & c \\ b & a \end{pmatrix}. \tag{3.2.9}$$

Lemma 3.2.3. *For $g\in \mathrm{SU}(2)$ we have $t^\ell_{m,n}(g^{-1}) = \overline{t^\ell_{n,m}(g)}$ for all $n,m\in\{-\ell,\cdots,\ell\}$.*

Lemma 3.2.3 implies that the map $\mathrm{SU}(2)\to B(V_\ell)$ given by $g\mapsto (t^\ell(g))_{m,n})^\ell_{m,n=-\ell}$ is unitary. Having the decomposition as in Exercise 3.2.1, it suffices to know the matrix $t^\ell(a(\phi))$. Indeed, for $g=k(\theta)a(\phi)k(\psi)$

$$\begin{aligned} t^\ell(g)_{m,n} &= \langle t^\ell(g)\cdot e^\ell_n, e^\ell_m\rangle = \langle t^\ell(a(\phi))t^\ell(k(\psi))e^\ell_n, t^\ell(k(-\theta))e^\ell_m\rangle \\ &= \chi_{-2n}(k(\psi))\overline{\chi_{-2m}(k(-\theta))}\langle t^\ell(a(\phi))e^\ell_n, e^\ell_m\rangle = e^{-in\psi-im\theta}\langle t^\ell(a(\phi))e^\ell_n, e^\ell_m\rangle, \end{aligned}$$

so it suffices to calculate the matrix entries of the representation t^ℓ when restricted to the subgroup A.

Proof. Take

$$g=\begin{pmatrix} \alpha & -\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} \qquad \Longrightarrow \qquad g^{-1}=\begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ -\gamma & \alpha \end{pmatrix}$$

since $|\alpha|^2+|\gamma|^2=1$. So the statement is

$$t^\ell_{m,n}\begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ -\gamma & \alpha \end{pmatrix} = \overline{t^\ell_{n,m}\begin{pmatrix} \alpha & -\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix}} = t^\ell_{n,m}\begin{pmatrix} \overline{\alpha} & -\gamma \\ \overline{\gamma} & \alpha \end{pmatrix}$$

since (3.2.6) implies that the coefficients of $t^\ell_{n,m}$ are real. This follows by the first symmetry of (3.2.9). ☐

Rewriting (3.1.1) gives, after some work,

$$t^\ell_{m,n}(g) = \binom{2\ell}{\ell-n}^{\frac{1}{2}}\binom{2\ell}{\ell-m}^{-\frac{1}{2}}\sum_{i=\max\{0,m+n\}}^{\min\{\ell+m,l+n\}}\binom{\ell-n}{\ell+m-i}\binom{\ell+n}{i}a^{i-m-n}c^{m+l-i}b^{l+n-i}d^i,$$

for $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Note that there are four cases to consider:

1. $m+n \geq 0$ and $m \leq n$;
2. $m+n \geq 0$ and $m \geq n$;
3. $m+n \leq 0$ and $m \leq n$;
4. $m+n \leq 0$ and $m \geq n$.

By the symmetry relations in (3.2.9) it suffices to compute the matrix elements $t^\ell_{m,n}(g)$ for only one of these cases. So assume that $m+n \geq 0$ and $m \leq n$, then

$$\begin{aligned} t^\ell_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \left(\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}\right)^{\frac{1}{2}} \sum_{i=m+n}^{\ell+m} \frac{(\ell-n)!(\ell+n)!\, a^{i-m-n} c^{m+\ell-i} b^{\ell+n-i} d^i}{(\ell+m-i)!(i-m-n)!i!(\ell+n-i)!} \\ &= \left(\frac{(\ell-m)!(\ell+m)!}{(\ell-n)!(\ell+n)!}\right)^{\frac{1}{2}} \sum_{j=0}^{\ell-n} \frac{(\ell-n)!(\ell+n)!\, a^j c^{\ell-n-j} b^{\ell-m-j} d^{j+m+n}}{(\ell-n-j)!j!(j+m+n)!(\ell-m-j)!} \end{aligned}$$

We use the identities

$$(k-i)! = \frac{k!}{(-1)^i(-k)_i}, \qquad (i+k)! = k!(k+1)_i,$$

then

$$\begin{aligned} t^\ell_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \left(\frac{(\ell+n)!(\ell+m)!}{(\ell-m)!(\ell-n)!}\right)^{\frac{1}{2}} \frac{b^{\ell-m}c^{\ell-n}d^{m+n}}{(m+n)!} \sum_{j=0}^{\ell-n} \frac{(n-\ell)_j(m-\ell)_j}{(m+n+1)_j j!}\left(\frac{ad}{bc}\right)^j \\ &= \left(\frac{(\ell+n)!(\ell+m)!}{(\ell-m)!(\ell-n)!}\right)^{\frac{1}{2}} \frac{b^{\ell-m}c^{\ell-n}d^{m+n}}{(m+n)!} \, {}_2F_1\left(\begin{matrix} -(\ell-n), -(\ell-m) \\ m+n+1 \end{matrix}; \frac{ad}{bc}\right) \end{aligned} \tag{3.2.10}$$

where for convenience we assume that $bc \neq 0$. Using Pfaff's transformation, see Exercise 2.2.6, we obtain

$$\begin{aligned} &t^\ell_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \left(\frac{(\ell+n)!(\ell+m)!}{(\ell-m)!(\ell-n)!}\right)^{\frac{1}{2}} \frac{b^{n-m}d^{m+n}(bc-ad)^{\ell-n}}{(m+n)!} \, {}_2F_1\left(\begin{matrix} -(\ell-n), \ell+n+1 \\ m+n+1 \end{matrix}; \frac{ad}{ad-bc}\right) \end{aligned} \tag{3.2.11}$$

assuming $ad-bc \neq 0$, i.e. the matrix to be invertible. The ${}_2F_1$-series terminates and can be expressed as a Jacobi polynomial (2.2.11);

$${}_2F_1\left(\begin{matrix} -(\ell-n), \ell+n+1 \\ m+n+1 \end{matrix}; \frac{ad}{ad-bc}\right) = \frac{(\ell-n)!}{(1+m+n)_{\ell-n}} P^{(m+n,n-m)}_{\ell-n}\left(\frac{bc+ad}{bc-ad}\right).$$

So we have proved Theorem 3.2.4.

Theorem 3.2.4. *For $m+n \geq 0$ and $m \leq n$,*

$$t^{\ell}_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{(\ell+n)!(\ell-n)!}{(\ell+m)!(\ell-m)!}\right)^{\frac{1}{2}} b^{n-m} d^{m+n} (bc-ad)^{\ell-n} P^{(m+n,n-m)}_{\ell-n}\left(\frac{bc+ad}{bc-ad}\right).$$

In particular, for an element in SU(2) we get

$$t^{\ell}_{m,n}\begin{pmatrix} \alpha & -\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} = \left(\frac{(\ell+n)!(\ell-n)!}{(\ell+m)!(\ell-m)!}\right)^{\frac{1}{2}} (-1)^{\ell-m} \overline{\alpha}^{m+n} \overline{\gamma}^{n-m} P^{(m+n,n-m)}_{\ell-n}\left(|\gamma|^2 - |\alpha|^2\right). \quad (3.2.12)$$

and the other cases follow by symmetry (3.2.9).

Next we want to show that the representation V_ℓ is an irreducible representation for SU(2). If we assume that $W \subset V_\ell$ a subspace is, such that W is invariant under the action of the elements $g \in \mathrm{SU}(2)$, then W has a basis consisting of a subset of e^{ℓ}_n. Indeed, for $w \in W$ we can write

$$w = \sum_{n=-\ell}^{\ell} w_n \, e^{\ell}_n$$

with not all w_n equal to zero. Now we take $w_k \neq 0$, and we claim that $e^{\ell}_k \in W$. We act on w diagonally and next we integrate out to find

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta} \begin{pmatrix} e^{\frac{1}{2}i\theta} & 0 \\ 0 & e^{-\frac{1}{2}i\theta} \end{pmatrix} \cdot w \, d\theta = w_k e^{\ell}_k$$

by (3.2.7), since

$$\begin{pmatrix} e^{\frac{1}{2}i\theta} & 0 \\ 0 & e^{-\frac{1}{2}i\theta} \end{pmatrix} \cdot w = \sum_{n=-\ell}^{\ell} w_n \, e^{-in\theta} e^{\ell}_n.$$

Proposition 3.2.5. *If $W \subset V_\ell$ is invariant for the action of* SU(2), *i.e. $g \cdot w \in W$ for all $w \in W$ and all $g \in \mathrm{SU}(2)$, then $W = \{0\}$ or $W = V_\ell$.*

Proof. Assume $W \neq \{0\}$, then there exists a $0 \neq w \in W$ and hence there is an $e^{\ell}_k \in W$. Now take

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{su}(2), \qquad \exp(tX) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \in \mathrm{SU}(2).$$

and consider

$$\exp(tX) \cdot e^{\ell}_k(x,y) = \binom{2\ell}{\ell-k}^{\frac{1}{2}} (\cos(t)x + \sin(t)y)^{\ell-k} (-\sin(t)x + \cos(t)y)^{\ell+k} \in W$$

In particular, taking the derivative with respect to t gives

$$(\ell-k)\binom{2\ell}{\ell-k}^{\frac{1}{2}} (\cos(t)x+\sin(t)y)^{\ell-k-1} (-\sin(t)x+\cos(t)y)^{\ell+k} (-\sin(t)x+\cos(t)y)$$

$$+(\ell+k)\binom{2\ell}{\ell-k}^{\frac{1}{2}} (\cos(t)x+\sin(t)y)^{\ell-k} (-\sin(t)x+\cos(t)y)^{\ell+k-1} (-\cos(t)x-\sin(t)y)$$

as an element of W (using that W is a subspace). Putting $t = 0$ gives

$$(\ell - k)\binom{2\ell}{\ell - k}^{\frac{1}{2}} x^{\ell-k-1}y^{\ell+k+1} - (\ell + k)\binom{2\ell}{\ell - k}^{\frac{1}{2}} x^{\ell-k+1}y^{\ell+k-1}$$
$$= \sqrt{(\ell + k + 1)(\ell - k)}e^{\ell}_{k+1} - \sqrt{(\ell + k)(\ell - k + 1)}e^{\ell}_{k-1}$$

as an element of W. Integrating over the diagonal subgroup as before, shows that $e^{\ell}_{k+1} \in W$ whenever $k + 1 \in \{-\ell, \cdots, \ell\}$ and $e^{\ell}_{k-1} \in W$ whenever $k - 1 \in \{-\ell, \cdots, \ell\}$.

Proceeding in this way we see that all basis elements of V_ℓ are in W, and $W = V_\ell$. □

Remark 3.2.6. Note that the action of the Lie algebra element X in the proof of Proposition 3.2.5 gives an action on homogeneous polynomials as a differential operator;

$$X \cdot f = \frac{d}{dt}\Big|_{t=0} \exp(tX) \cdot f = (y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})f$$

So we have proved the following result.

Theorem 3.2.7. *The space of homogeneous polynomials of degree* 2ℓ*,* $\ell \in \frac{1}{2}\mathbb{N}$*, is an irreducible unitary representation for* SU(2) *of dimension* $2\ell + 1$*.*

We denote the corresponding finite dimensional irreducible representation by (π_ℓ, V_ℓ) or π_ℓ. Moreover, any irreducible unitary representation is equivalent to a representation V_ℓ. We will not prove this statement, but it will be motivated in Theorem 3.4.1. Note that the irreducible representation is determined by its dimension $2\ell + 1$, so we can conclude from Theorem 3.2.7 that $\widehat{\mathrm{SU}(2)} \cong \{2\ell + 1 \mid \ell \in \frac{1}{2}\mathbb{N}\}$ is the unitary dual of SU(2).

Exercise 3.2.8. Show that with

$$Y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathfrak{su}(2), \qquad Z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathfrak{su}(2)$$

the elements can be similarly realised as first order partial differential operators:

$$Y \cdot f = (ix\frac{\partial}{\partial x} - iy\frac{\partial}{\partial y})f, \qquad Z \cdot f = (iy\frac{\partial}{\partial x} + ix\frac{\partial}{\partial y})f.$$

Check the commutators $[X, Y] = 2Z$, $[Y, Z] = 2X$, $[Z, X] = 2Y$ by using the first order partial differential operators.

3.3 Generalities on representations of compact groups

We let G be a compact (infinite) subgroup of $n \times n$-matrices, and we let dg the invariant measure. The main example in Chapter 3 is $G = \mathrm{SU}(2)$. This measure is both left and right invariant, and we can use integration over G to obtain several results. As a first example we obtain complete reducibility of finite-dimensional representations of the compact group G as a consequence of Lemma 3.3.1.

Lemma 3.3.1. *Let V be finite-dimensional vector space and $\pi\colon G \to B(V)$ be a representation. Then there exists an inner product on V making π a unitary representation.*

Using Exercise 2.5.8(i) we see that any finite-dimensional representation of a compact group is completely reducible.

Corollary 3.3.2. *Let $\pi\colon G \to B(V)$ be a finite-dimensional representation of a compact group G. Then π is completely reducible, i.e. $\pi \cong \bigoplus_{i=1}^N \pi_i$ with π_i irreducible representations of G for all i.*

Proof of Lemma 3.3.1. Equip V with an arbitrarily chosen inner product

$$\langle \cdot, \cdot \rangle_a \colon V \times V \to \mathbb{C}$$

by e.g. choosing a basis and declaring the basis orthonormal. Now define

$$\langle v, w \rangle = \int_G \langle \pi(g)v, \pi(g)w \rangle_a \, dg, \qquad v, w \in V.$$

Then this defines an inner product. Indeed, it is obviously sesquilinear, and $\langle v, v \rangle \geq 0$. In case $\langle v, v \rangle = 0$, the integrand has to $\langle \pi(g)v, \pi(g)v \rangle_a$ be zero almost everywhere. By continuity it is zero everywhere, so $v = 0$.

This inner product makes the representation unitary. Since the representation is finite-dimensional it suffices to show that $\langle \pi(h)v, \pi(h)w \rangle = \langle v, w \rangle$ for all $h \in G$. Now

$$\begin{aligned} \langle \pi(h)v, \pi(h)w \rangle &= \int_G \langle \pi(g)\pi(h)v, \pi(g)\pi(h)w \rangle_a \, dg = \int_G \langle \pi(gh)v, \pi(gh)w \rangle_a \, dg \\ &= \int_G \langle \pi(g)v, \pi(g)w \rangle_a \, dg = \langle v, w \rangle \end{aligned}$$

since the left invariant Haar measure is also right invariant for a compact group. □

By Lemma 3.3.1 we can assume without loss of generality that a finite-dimensional representation of a compact group G is unitary.

Definition 3.3.3. *Let G be a matrix group (not necessarily compact) and let $\pi\colon G \to B(H)$ be a unitary representation in a (not necessarily finite-dimensional) Hilbert space. A continuous function $G \to \mathbb{C}$ of the form*

$$g \mapsto \langle \pi(g)v, w \rangle$$

for $v, w \in H$ is called a matrix entry or a matrix coefficient of the representation π.

Note that this is indeed a continuous function by the continuity requirement on the representation from Definition 2.5.1 and since taking an inner product gives a continuous linear map $G \to \mathbb{C}$, i.e. a functional, on a Hilbert space. Moreover, any continuous functional is of this form by the Riesz representation theorem.

For G a compact group, the continuity means that any matrix coefficient is in the Hilbert space $L^2(G)$. We aim to show that these matrix coefficients satisfy very interesting orthogonality relations in Theorem 3.3.6. For this we start with Lemma 3.3.4.

Lemma 3.3.4. *Let $\pi\colon G \to B(V)$, $\sigma\colon G \to B(W)$ be finite-dimensional unitary representations of the compact group G. Pick $v \in V$, $w \in W$, then $T_{v,w} = T\colon V \to W$ defined by*

$$T(x) = \int_G \langle \pi(g)x, v\rangle\, \sigma(g^{-1})w\, dg, \qquad x \in V,$$

is an intertwiner, i.e. $T \in B_G(V, W)$.

Proof. We calculate for $h \in G$

$$\begin{aligned} T(\pi(h)x) &= \int_G \langle \pi(g)\pi(h)x, v\rangle\, \sigma(g^{-1})w\, dg = \int_G \langle \pi(gh)x, v\rangle\, \sigma(g^{-1})w\, dg \\ &= \int_G \langle \pi(g)x, v\rangle\, \sigma(hg^{-1})w\, dg = \int_G \langle \pi(g)x, v\rangle\, \sigma(h)\sigma(g^{-1})w\, dg \\ &= \sigma(h) \int_G \langle \pi(g)x, v\rangle\, \sigma(g^{-1})w\, dg = \sigma(h)T(x) \end{aligned}$$

using that π and σ are homomorphisms and that dg is right invariant. □

Theorem 3.3.5. *Let $\pi\colon G \to B(V)$, $\sigma\colon G \to B(W)$ be finite-dimensional unitary irreducible representations of the compact group G. Then either any matrix coefficient of π is orthogonal to any matrix coefficient of σ with respect to the inner product of $L^2(G)$, or $\pi \cong \sigma$.*

Proof. Take $v_1, v_2 \in V$, $w_1, w_2 \in W$ and consider

$$\int_G \langle \pi(g)v_1, v_2\rangle\, \overline{\langle \sigma(g)w_1, w_2\rangle}\, dg = \int_G \langle \pi(g)v_1, v_2\rangle\, \langle \sigma(g^{-1})w_2, w_1\rangle\, dg = \langle T_{v_2,w_2}(v_1), w_1\rangle.$$

If there is any matrix coefficient of π non-orthogonal in $L^2(G)$ to a coefficient σ it means that the intertwiner $T_{v_2,w_2}\colon V \to W$ is non-zero, and by Lemma 2.5.7 we see that T_{v_2,w_2} is an isomorphism, so that $\pi \cong \sigma$. □

In case $\pi \cong \sigma$ for irreducible unitary representations, we can just as well assume that $\pi = \sigma$, $W = V$ and then T_{v_2,w_2} is a multiple of the identity, cf the proof of Lemma 2.5.7. This gives

$$\begin{aligned} &\int_G \langle \pi(g)v_1, v_2\rangle\, \overline{\langle \pi(g)w_1, w_2\rangle}\, dg = \int_G \langle \pi(g)v_1, v_2\rangle\, \langle \pi(g^{-1})w_2, w_1\rangle\, dg \\ &= \langle T_{v_2,w_2}(v_1), w_1\rangle = c(v_2, w_2)\langle v_1, w_1\rangle, \qquad v_1, v_2, w_1, w_2 \in V. \end{aligned} \tag{3.3.1}$$

Theorem 3.3.6 (Schur orthogonality relations)**.** *Let $\pi\colon G \to B(V)$, $\sigma\colon G \to B(W)$ be finite-dimensional unitary irreducible representations of the compact group G. Then*

$$\int_G \langle \pi(g)v_1, v_2\rangle\, \overline{\langle \sigma(g)w_1, w_2\rangle}\, dg = \delta_{\sigma,\pi} \frac{1}{\dim_{\mathbb{C}} V} \langle v_1, w_1\rangle \langle w_2, v_2\rangle$$

where $\delta_{\sigma,\pi} = 1$ if $\sigma \cong \pi$ and $\delta_{\sigma,\pi} = 0$ if $\sigma \ncong \pi$.

Proof. By Theorem 3.3.5 it suffices to consider the case $\sigma = \pi$, and then we can use (3.3.1). Using (2.4.4) we rewrite

$$\int_G \langle \pi(g)v_1, v_2\rangle\, \overline{\langle \pi(g)w_1, w_2\rangle}\, dg = \int_G \langle \pi(g^{-1})v_1, v_2\rangle\, \overline{\langle \pi(g^{-1})w_1, w_2\rangle}\, dg$$
$$= \int_G \langle v_1, \pi(g)v_2\rangle\, \overline{\langle w_1, \pi(g)w_2\rangle}\, dg = \int_G \langle \pi(g)w_2, w_1\rangle\, \overline{\langle \pi(g)v_2, v_1\rangle}\, dg$$

i.e. we find the same expression as before but with $v_1 \leftrightarrow w_2$ and $v_2 \leftrightarrow w_1$. So

$$c(v_2, w_2)\langle v_1, w_1\rangle = c(w_1, v_1)\langle w_2, v_2\rangle$$

and so $c(\cdot,\cdot)$ is some multiple of the inner product; $c(w_1, v_1) = C\langle w_1, v_1\rangle$ and it remains to find the constant C.

Now we take an orthonormal basis $\{e_1, \cdots, e_d\}$ of V, $d = \dim_{\mathbb{C}} V$. Since the representation π is unitary, we have

$$\|v\|^2 = \sum_{i=1}^{d} |\langle \pi(g)v, e_i\rangle|^2$$

and integrating over G yields, using the normalisation $\int_G dg = 1$,

$$\|v\|^2 = \sum_{i=1}^{d} \int_G |\langle \pi(g)v, e_i\rangle|^2\, dg = \sum_{i=1}^{d} C\langle v, v\rangle\langle e_i, e_i\rangle = C\, d\, \|v\|^2$$

so that $C = d^{-1}$. □

We apply the Schur orthogonality relations of Theorem 3.3.6 to characters of representations.

Definition 3.3.7. *Let $\pi\colon G \to B(V)$ be a finite dimensional representation of the compact group G, then the character of π is a function $\chi_\pi\colon G \to \mathbb{C}$ defined by $\chi_\pi(g) = \mathrm{Tr}\,(\pi(g))$.*

Exercise 3.3.8. (i) Let $\pi\colon G \to B(V)$ be a finite dimensional representation. Show that χ_π is a *class function*, i.e. $\chi_\pi(ghg^{-1}) = \chi_\pi(h)$ for all $g, h \in G$.

(ii) Let $\pi\colon G \to B(V)$ be a finite dimensional unitary representation. Show that $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$.

(iii) Let $\pi\colon G \to B(V)$, $\sigma\colon G \to B(W)$ be finite dimensional representations. Show that $\chi_{\pi\oplus\sigma} = \chi_\pi + \chi_\sigma$, $\chi_{\pi\otimes\sigma} = \chi_\pi \cdot \chi_\sigma$ as functions on G. Recall that for V and W inner products spaces, the map $\langle\cdot,\cdot\rangle\colon V \oplus W \times V \oplus W \to \mathbb{C}$ $\langle v \oplus w, u \oplus z\rangle = \langle v, u\rangle_V + \langle w, z\rangle_W$ makes $V \oplus W$ in an inner product space. Similarly, $\langle\cdot,\cdot\rangle\colon V \otimes W \times V \otimes W \to \mathbb{C}$ $\langle v \otimes w, u \otimes z\rangle = \langle v, u\rangle_V \langle w, z\rangle_W$ makes $V \otimes W$ in an inner product space.

(iv) Let $\pi\colon G \to B(V)$, $\sigma\colon G \to B(W)$ be finite dimensional unitary irreducible representations. Show that

$$\int_G \chi_\pi(g)\overline{\chi_\sigma(g)}\, dg = \begin{cases} 1, & \sigma \cong \pi \\ 0, & \text{otherwise} \end{cases}$$

By $\hat{G}$ we denote the set of irreducible representations of the compact group G up to equivalence by intertwiners. Without loss of generality we can assume that the irreducible representations are unitary, and $\hat{G}$ is known as the *unitary dual* of G, see Section 2.5. If we now take $\pi \in \hat{G}$, and an orthonormal basis $\{e_1, \cdots, e_{d_\pi}\}$, $d_\pi = \dim_{\mathbb{C}} V_\pi$ is the dimension of the representation π, and we define

$$f^\pi_{i,j}\colon G \to \mathbb{C}, \qquad f^\pi_{i,j}(g) = \sqrt{d_\pi}\, \langle \pi(g)e_j, e_i\rangle$$

then Theorem 3.3.6 states that

$$\big\{ f^\pi_{i,j} \mid \pi \in \hat{G}, i, j \in \{1, \cdots, d_\pi\}\big\} \tag{3.3.2}$$

is an orthonormal set in $L^2(G)$. These functions can be considered as matrix entries of representations of G.

Remark 3.3.9. Let $\pi\colon G \to B(V)$ be a finite dimensional (not necessarily unitary) representation of G. Let V^* be the dual space of linear maps $V \to \mathbb{C}$. For any pair $(v, v^*) \in V \times V^*$ the function

$$g \mapsto v^*(\pi(g)v), \qquad G \to \mathbb{C}$$

is a matrix entry. Since the map associating to the pair (v, v^*) the corresponding matrix entry is bilinear, we can identify the matrix entries with $V \otimes V^*$, which we can indentify with $\mathrm{End}(V)$ by $v \otimes v^*\colon w \to v^*(w)v$. In case V is a Hilbert space, the Riesz representation theorem shows that all matrix entries are given by inner products, i.e. $g \mapsto \langle \pi(g)v, w\rangle$ with $v^*\colon V \to \mathbb{C}$ given by $v^*(x) = \langle x, w\rangle$, which leads us back to Definition 3.3.3.

Exercise 3.3.10. Let $\pi\colon G \to B(V)$ be an irreducible finite-dimensional unitary representation of G. Take $v, w \in V$ and consider the matrix entry $\pi_{w,v}\colon G \to \mathbb{C}$, $g \mapsto \langle \pi(g)v, w\rangle$. In Exercise 3.3.12 we indicate a proof of the fact that any irreducible unitary representation of the compact group G is finite dimensional as a consequence of the Peter-Weyl Theorem 3.3.14.

(i) Show that the action of the left regular representations, see Example 2.5.4, on the matrix entry gives

$$\lambda(g)\pi_{w,v} = \pi_{\pi(g)w,v}$$

(ii) Conclude that the span $S = \mathrm{Span}\{\pi_{w,v} \mid w \in V\}$ is a finite dimensional invariant subspace of $L^2(G)$ for the regular representation.

(iii) Show that the left regular representation λ restricted to S is equivalent to π by checking that $T\colon V \to S$, $Tw = \pi_{w,v}$ gives the isomorphic intertwiner.

(iv) Let $M_\pi \subset L^2(G)$ be the space of matrix entries for the representation π. Show that it is an invariant subspace of the left regular representation. The corresponding representation is equivalent to $\dim_{\mathbb{C}} V$ copies of π.

We next study the class functions in M_π. Note that $\chi_\pi \in M_\pi$ is a class function. Moreover, it is the only one up to a scalar.

Proposition 3.3.11. *Let π be a finite-dimensional irreducible representation of G. Let $f \in M_\pi$ be a class function, then f is a multiple of χ_π.*

The proof is part of the general structure in Exercise 3.3.12.

Exercise 3.3.12. In this exercise we consider a finite dimensional irreducible representation $\pi\colon G \to B(V)$ of the compact group G, and compare to the Exercise 2.5.8. Using this we construct the following three actions of $G \times G$;

(a) for $f \in M_\pi$ define $((s,t)\cdot f)(g) = f(t^{-1}gs)$, $(s,t) \in G \times G$,

(b) for $T \in \operatorname{End} V$ define $(s,t)\cdot T = \pi(s)T\pi(t^{-1})$, $(s,t) \in G \times G$,

(c) for $x \in V$, $\phi \in V^*$ define $(s,t)\cdot(x \otimes \phi) = (\pi(s)x \otimes \pi^*(t)\phi)$, $(s,t) \in G \times G$.

(i) Show that (a), (b), (c) define representations of $G \times G$.

(ii) Show that these three representations of $G \times G$ are equivalent. Hint: consider $x \otimes \phi \mapsto (g \mapsto \phi(\pi(g)x))$ and $x \otimes \phi \mapsto (v \mapsto \phi(v)x)$.

(iii) Restrict to the diagonal $(t,t) \subset G \times G$ to conclude that $M_\pi \cong \operatorname{End} V \cong V \otimes V^*$ as representations of G.

(iv) Show that under this identification a class function in M_π corresponds to a G-intertwiner of V in $\operatorname{End} V$.

(v) Use Schur's Lemma 2.5.7 to conclude that the dimension of class functions in M_π is one.

Corollary 3.3.13. *The representation $\pi\colon G \to B(V)$ is completely determined by its character.*

The Peter-Weyl Theorem 3.3.14 states that the set in (3.3.2) is actually an orthonormal basis of $L^2(G)$.

Theorem 3.3.14 (Peter-Weyl Theorem)**.** *The set of* (3.3.2) *is an orthonormal basis for the Hilbert space $L^2(G)$. In particular,*

$$L^2(G) = \bigoplus_{\pi \in \hat{G}} M_\pi$$

where we take the Hilbert space direct sum. Moreover, the matrix entries of the representations of G are dense in $C(G)$ with respect to the supremum norm.

We skip the proof of the remaining statements of Theorem 3.3.14, see e.g. [8, §4], [22, §6.4], [23, §5.2], [67, §6,2], [84, §2.8]. Theorem 3.3.14 gives a solution for compact groups to the general problem of decomposition of the left regular representation, which can be viewed as an important problem in harmonic analysis on Lie groups.

Exercise 3.3.15. In this exercise we indicate a proof of the density of the matrix entries of irreducible unitary representations of the compact group G in $L^2(G)$. Let $U \subset L^2(G)$ be the closure of the linear space spanned by the matrix entries of irreducible finite dimensional representations of G.

(i) Show that $U^\perp$ is closed under left and right translation and mapping f to $g \mapsto \overline{f(g^{-1})}$. (Hint: show that U has these properties, and that the orthogonal complement preserves this as well. Use that G is compact, hence unimodular.)

(ii) Assume that $U^\perp$ is non-empty, show that there exists a non-zero continuous function f in $U^\perp$. (Hint: take a non-zero element h of $U^\perp$ and take a convolution product with a sequence of functions leading to sequence of continuous functions approximating h.)

(iii) Show that, by replacing f by a suitable other candidate, we may assume that $f \in U^\perp$ is a non-zero continuous function satisfying $f(hgh^{-1}) = f(g)$, $f(g) = \overline{f(g^{-1})}$ and $0 \neq f(e) \in \mathbb{R}$.

(iv) Define $k(x,y) = f(x^{-1}y)$ as a function $k\colon G \times G \to \mathbb{C}$, and show that

$$T\colon L^2(G) \to L^2(G), \qquad (Tf)(x) = \int_G k(x,y) f(y)\, dy$$

defines a non-zero self-adjoint Hilbert-Schmidt operator on $L^2(G)$.

(v) By the spectral theorem for Hilbert-Schmidt operators there exists an eigenvalue $0 \neq \mu \in \sigma_p(T) \subset \mathbb{R}$ with finite dimensional eigenspace $V_\mu \subset L^2(G)$. Show that V_μ is invariant under the left-regular representation λ.

(vi) Pick an irreducible invariant subspace $\{0\} \neq W_\mu \subset V_\mu$ with an orthonormal basis $\{f_1, \cdots, f_n\}$. Show that $f \in U^\perp$ leads to the contradiction $0 = \int_G |f_i(g)|^2\, dg = 1$ by calculating the inner product of f with the matrix entry $\langle \lambda(g) f_i, f_i\rangle$.

Having the decomposition of the left regular representation, we consider the corresponding decomposition of an arbitrary function. For a continuous function $f\colon G \to \mathbb{C}$ we define the operator

$$\mathcal{F}(f)(\pi) = \hat{f}(\pi) = \int_G f(g)\pi(g^{-1})\, dg\colon V_\pi \to V_\pi \tag{3.3.3}$$

for $\pi \in \hat{G}$ acting in the representation space V_π, which is a Hilbert space. We view (3.3.3) as the Fourier transform of f at the element $\pi \in \hat{G}$. Then we have the following analogue of the Plancherel theorem as a consequence of the Peter-Weyl Theorem 3.3.14, see [22, Thm. 6.4.2].

Theorem 3.3.16. *For $f \in L^2(G)$ we have*

$$f(g) = \sum_{\pi \in \hat{G}} d_\pi \operatorname{Tr}\big(\mathcal{F}(f)(\pi)\pi(g)\big)$$

as identity in $L^2(G)$. Moreover, for $f_1, f_2 \in L^2(G)$,

$$\int_G f_1(g)\overline{f_2(g)}\, dg = \sum_{\pi \in \hat{G}} d_\pi \operatorname{Tr}\big(\mathcal{F}(f_1)(\pi)\big(\mathcal{F}(f_2)(\pi)\big)^*\big)$$

so that $\|f\|^2_{L^2(G)} = \sum_{\pi \in \hat{G}} d_\pi \|\mathcal{F}(f)(\pi)\|^2$, where the norm on the right hand side is given by the Hilbert-Schmidt norm; $\|T\|^2 = \operatorname{Tr}(T^*T)$.

Theorem 3.3.16 is a rather general theorem, which has an appropriate analogue for more general groups, see e.g. [18]. Typically we use Theorem 3.3.16 and its analogues for classes of functions on the group G involved.

Proof. For each $\pi \in \hat{G}$ we pick an orthonormal basis $\{e_i = e_i^\pi \mid i = 1, \cdots, d_\pi\}$, where d_π is the dimension of the representation space V_π of π. Then for a continuous $f\colon G \to \mathbb{C}$ we have using (3.3.3)

$$\begin{gathered}\langle \mathcal{F}(f)(\pi)\pi(g)e_j^\pi, e_i^\pi\rangle = \int_G f(h)\langle \pi(h^{-1})\pi(g)e_j^\pi, e_i^\pi\rangle\, dh = \\ \sum_{k=1}^{d_\pi} \int_G f(h)\langle \pi(g)e_j^\pi, e_k^\pi\rangle\langle e_k^\pi, \pi(h)e_i^\pi\rangle\, dh = \frac{1}{d_\pi}\sum_{k=1}^{d_\pi}\langle f, f_{k,i}^\pi\rangle\, f_{k,j}^\pi(g)\end{gathered} \tag{3.3.4}$$

using the notation (3.3.2). So

$$d_\pi \operatorname{Tr}\big(\mathcal{F}(f)(\pi)\pi(g)\big) = \sum_{k,i=1}^{d_\pi}\langle f, f_{k,i}^\pi\rangle\, f_{k,i}^\pi(g).$$

Using the fact that (3.3.2) yields an orthonormal basis for the Hilbert space $L^2(G)$ by Theorem 3.3.6 and the Peter-Weyl Theorem 3.3.14, the first statement follows.

For the last statement, we note that

$$\operatorname{Tr}\big(\mathcal{F}(f_1)(\pi)\big(\mathcal{F}(f_2)(\pi)\big)^*\big) = \sum_{i,k=1}^{d_\pi}\langle \mathcal{F}(f_1)(\pi)e_i^\pi, e_k^\pi\rangle\overline{\langle \mathcal{F}(f_2)(\pi)e_i^\pi, e_k^\pi\rangle}$$

and the terms in the summand can be evaluated using (3.3.4) for $g = e$ and observing that $f_{i,j}^\pi(e) = \sqrt{d_\pi}\delta_{i,j}$. So $\operatorname{Tr}\big(\mathcal{F}(f_1)(\pi)\big(\mathcal{F}(f_2)(\pi)\big)^*\big) = d_\pi^{-1}\sum_{i,jk=1}^{d_\pi}\langle f_1, f_{i,k}^\pi\rangle\overline{\langle f_2, f_{i,k}^\pi\rangle}$. Then the last statement follows using (3.3.2) is an orthonormal basis for the Hilbert space $L^2(G)$. $\square$

3.4 Representations of SU(2) and orthogonal polynomials

Note that from Section 3.2 we have for each spin $\ell \in \frac{1}{2}\mathbb{N}$ a representation $\pi_\ell \colon \mathrm{SU}(2) \to B(V_\ell)$, which, by Proposition 3.2.5, is an irreducible representation and by Lemma 3.2.3 it is a unitary representation, see Theorem 3.2.7. Hence, for each dimension $2\ell + 1$ we have an irreducible unitary representation π_ℓ.

3.4.1 Characters of SU(2) and Chebyshev polynomials

Let $\chi_\ell(g) = \mathrm{Tr}\,(\pi_\ell(g))$, $g \in \mathrm{SU}(2)$, be the corresponding character, then χ_ℓ is a class function. Since any element $g \in \mathrm{SU}(2)$ is conjugated in SU(2) to a diagonal matrix with entries its eigenvalues, which are in $\mathbb{T}$ (since g is unitary) and have product 1 (since $\det(g) = 1$), we only need to calculate $\chi_\ell(g)$ –or any class function– for a diagonal matrix.

$$\begin{aligned}\chi_\ell \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} &= \sum_{k=-\ell}^{\ell} \langle \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \cdot e^\ell_k, e^\ell_k \rangle = \sum_{k=-\ell}^{\ell} e^{-2ik\theta} \\ &= \frac{e^{i(2\ell+1)\theta} - e^{-i(2\ell+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(2\ell+1)\theta}{\sin\theta} = U_{2\ell}(\cos\theta)\end{aligned}$$

where U_n is the Chebyshev polynomial of the second kind, see (2.2.16). This explicit calculation is a very special case of a much more general result, known as Weyl's character formula, see e.g. [32], [78].

The expressions (2.4.6), (3.2.4) are not well-suited for the Haar measure for central functions. Instead, we use the classical result of Weyl, see [7, Ch. IV, (1.11)], [21, Thm. 11.2.1], for the general case. For the case of SU(2) it simplifies to

$$\int_{\mathrm{SU}(2)} f(g)\,dg = \frac{1}{\pi}\int_0^\pi f\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \sin^2\theta\,d\theta \tag{3.4.1}$$

for a central (continuous) function $f \colon \mathrm{SU}(2) \to \mathbb{C}$. Then the Schur orthogonality relations for characters, see Exercise 3.3.8, correspond to the orthogonality relations for the Chebyshev polynomials of the second kind

$$\int_{-1}^{1} U_n(x)U_m(x)\sqrt{1-x^2}\,dx = \int_0^\pi U_n(\cos\theta)U_m(\cos\theta)\,\sin^2\theta\,d\theta = \frac{1}{2}\pi\delta_{m,n}. \tag{3.4.2}$$

Since the Chebyshev polynomials are dense in $L^2([-1,1], \sqrt{1-x^2}\,dx)$ we have no other central functions than χ_ℓ on SU(2).

Theorem 3.4.1. *The representations π_ℓ, $\ell \in \frac{1}{2}\mathbb{N}$, form a complete set of irreducible unitary representations of* SU(2) *(up to isomorphism).*

The Plancherel Theorem 3.3.16 restricted to central functions is saying that we can decompose a central function as a series of Chebyshev polynomials.

So we can rephrase Theorem 3.4.1 as $\widehat{\mathrm{SU}(2)} \cong \frac{1}{2}\mathbb{N}$. Since all dimensions are different, the dimension of an irreducible unitary representation determines the irreducible unitary representation of SU(2). We formulate a few consequences of Theorem 3.4.1.

Corollary 3.4.2. *Any finite dimensional unitary representation of* SU(2) *is isomorphic to its contragredient representation.*

This follows from Theorem 3.4.1 for irreducible unitary representations, since the dimension of the contragredient representations equals the dimension of the representation. Theorem 3.4.1 also gives an explicit description of the tensor product of two irreducible representations.

Corollary 3.4.3 (Clebsch-Gordan decomposition)**.** *For* $\ell_1, \ell_2 \in \frac{1}{2}\mathbb{N}$ *we have*

$$\pi_{\ell_1} \otimes \pi_{\ell_2} \cong \bigoplus_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \pi_\ell$$

Proof. It suffices to check the trigonometric identity

$$U_{2\ell_1}(\cos\theta)U_{2\ell_2}(\cos\theta) = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} U_{2\ell}(\cos\theta)$$

and to appeal to Corollary 3.3.13 and Exercise 3.3.8. □

Exercise 3.4.4. Prove the trigonometric identity using (2.2.16) and a finite telescoping sum.

With respect to the standard basis the matrix entries of the intertwiner

$$C\colon V_{\ell_1} \otimes V_{\ell_2} \to \bigoplus_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} V_\ell$$

are known as Clebsch-Gordan coefficients, and we give a description in Section 3.4.4.

3.4.2 Schur orthogonality and Jacobi polynomials

Recall that the functions $t^\ell_{m,n}\colon \mathrm{SU}(2) \to \mathbb{C}$ are matrix entries for irreducible unitary representations π_ℓ;

$$t^\ell_{m,n}(g) = \langle \pi_\ell(g)e^\ell_n, e^\ell_m\rangle.$$

By Theorem 3.2.4 we have

$$t^\ell_{m,n}\begin{pmatrix} e^{i\phi}\sin\theta & -e^{-i\psi}\cos\theta \\ e^{i\psi}\cos\theta & e^{-i\phi}\sin\theta \end{pmatrix} =$$

$$(-1)^{\ell+m}\left(\frac{(\ell+n)!(\ell-n)!}{(\ell+m)!(\ell-m)!}\right)^{\frac{1}{2}} e^{i(m-n)\psi}e^{-i(n+m)\phi}\sin^{n+m}(\theta)\cos^{n-m}(\theta)P^{(m+n,n-m)}_{\ell-n}(\cos(2\theta)).$$

Using the symmetries (3.2.9) and the expression (2.4.6) for the Haar measure on SU(2) we see that the orthogonality relations

$$\int_{\mathrm{SU}(2)} t^{\ell_1}_{m_1,n_1}(g)\overline{t^{\ell_2}_{m_2,n_2}(g)}\,dg = \frac{1}{2\ell_1+1}\delta_{\ell_1,\ell_2}\delta_{n_1,n_2}\delta_{m_1,m_2}$$

for $(m_1,n_1)\neq(m_2,n_2)$ are directly related to the orthogonality of the exponential functions. Now assume $(m,n)=(m_1,n_1)=(m_2,n_2)$ and $m+n\geq 0$, $m-n\geq 0$,

$$\int_{\mathrm{SU}(2)} t^{\ell_1}_{m,n}(g)\overline{t^{\ell_2}_{m,n}(g)}\,dg = (-1)^{\ell_1+\ell_2}\left(\frac{(\ell_1+n)!(\ell_1-n)!(\ell_2+n)!(\ell_2-n)!}{(\ell_1+m)!(\ell_1-m)!(\ell_2+m)!(\ell_2-m)!}\right)^{\frac{1}{2}}$$
$$\times\int_0^{\frac{\pi}{2}} P^{(m+n,n-m)}_{\ell_1-n}(\cos(2\theta))\,P^{(m+n,n-m)}_{\ell_2-n}(\cos(2\theta))\,\cos^{2n-2m+1}(\theta)\sin^{2n+2m+1}(\theta)d\theta.$$

Substituting $x=\cos(2\theta)$, so $1-x=2\sin^2\theta$ and $1+x=2\cos^2\theta$, we find

$$\begin{aligned}\frac{1}{2\ell_1+1}\delta_{\ell_1,\ell_2} &= (-1)^{\ell_1+\ell_2}\left(\frac{(\ell_1+n)!(\ell_1-n)!(\ell_2+n)!(\ell_2-n)!}{(\ell_1+m)!(\ell_1-m)!(\ell_2+m)!(\ell_2-m)!}\right)^{\frac{1}{2}}\\ &\times 2^{-2n-1}\int_{-1}^{1} P^{(m+n,n-m)}_{\ell_1-n}(x)P^{(m+n,n-m)}_{\ell_2-n}(x)\,(1-x)^{n+m}(1+x)^{n-m}\,dx.\end{aligned}\tag{3.4.3}$$

We conclude that we have obtained the orthogonality relations (2.2.12) from the Schur orthogonality relations for the Jacobi polynomials $P^{(\alpha,\beta)}_k$ with parameters $(\alpha,\beta)\in\mathbb{N}\times\mathbb{N}$.

Exercise 3.4.5. Show that the orthogonality relations of (3.4.3) give (2.2.12) for $\alpha,\beta\in\mathbb{N}$.

The Plancherel Theorem 3.3.16 then shows that we expand any function on SU(2) in terms of Fourier series times Jacobi polynomials.

3.4.3 Unitarity and Krawtchouk polynomials

The representations π_ℓ, $\ell\in\frac{1}{2}\mathbb{N}$, are unitary representations by Theorem 3.4.1. In particular, it means that the columns and rows of the matrix entries with respect to an orthonormal basis are orthogonal;

$$\sum_{k=-\ell}^{\ell} t^{\ell}_{m,k}(g)\overline{t^{\ell}_{n,k}(g)} = \delta_{m,n},\qquad g\in\mathrm{SU}(2).\tag{3.4.4}$$

In order to match this to orthogonal polynomials with a finite discrete orthogonality measure we need to pick $g\in\mathrm{SU}(2)$ and to transform the hypergeometric series in the expression of Theorem 3.2.4.

We claim that (3.4.4) will give the orthogonality relations for the Krawtchouk polynomials as in (2.2.20). Using the transformation

$${}_2F_1\left(\begin{matrix}-n,-m\\c\end{matrix};z\right)=\frac{(c)_{n+m}}{(c)_n(c)_m}\,{}_2F_1\left(\begin{matrix}-n,-m\\1-c-n-m\end{matrix};1-z\right),\qquad n,m\in\mathbb{N}$$

on the expression for the matrix entry in (3.2.10) we get for $m+n \geq 0$

$$t^{\ell}_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\binom{2\ell}{\ell-m}\binom{2\ell}{\ell-n} \right)^{\frac{1}{2}} b^{\ell-m}c^{\ell-n}d^{m+n}\,{}_2F_1\left(\begin{matrix} -\ell+n, -\ell+m \\ -2\ell \end{matrix} ; \frac{bc-ad}{bc} \right)$$

Now the ${}_2F_1$ is of the right form for the Krawtchouk polynomials (2.2.20). Take $g = a(\phi)$, then for $m+n \geq 0$

$$t^{\ell}_{m,n}(a(\phi)) = \left(\binom{2\ell}{\ell-m}\binom{2\ell}{\ell-n} \right)^{\frac{1}{2}}$$
$$\times (-1)^{\ell-m} \sin^{2\ell-m-n}(\frac{1}{2}\phi) \cos^{m+n}(\frac{1}{2}\phi) K_{\ell-m}(\ell-n; \sin^2(\frac{1}{2}\phi), 2\ell)$$

Now we plug this expression in (3.4.4) and we use the symmetries (3.2.9) to obtain the orthogonality relations (2.2.19) for the Krawtchouk polynomials.

Corollary 3.4.6. *The unitarity of the irreducible unitary representations of* SU(2) *give the orthogonality relations for the Krawtchouk polynomials.*

Exercise 3.4.7. Show that taking column orthogonality $\sum_{k=-\ell}^{\ell} t^{\ell}_{k,m}(g)\overline{t^{\ell}_{k,n}(g)} = \delta_{m,n}$ of the unitary matrix $\pi_\ell(g)$ instead of row orthogonality gives the same orthogonality. Explain this using the self-duality of the Krawtchouk polynomials, i.e. $K_n(x;p,N) = K_x(n;p,N)$.

3.4.4 Clebsch-Gordan coefficients and Hahn and dual Hahn polynomials

Note that we can take the intertwiner of Section 3.4.1

$$C\colon V_{\ell_1} \otimes V_{\ell_2} \to \bigoplus_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} V_\ell \tag{3.4.5}$$

implementing the Clebsch-Gordan decomposition as a unitary matrix with respect to the basis $e^{\ell_1}_{n_1} \otimes e^{\ell_2}_{n_2}$, $n_1 \in \{-\ell_1, \cdots, \ell_1\}$, $n_2 \in \{-\ell_2, \cdots, \ell_2\}$, on the left hand side and the basis e^{ℓ}_n, $\ell \in \{|\ell_1-\ell_2|, \cdots, \ell_1+\ell_2\}$, $n \in \{-\ell, \cdots, \ell\}$, on the right hand side. Then considering the action of the diagonal matrices in SU(2) gives

$$e^{\ell_1}_{n_1} \otimes e^{\ell_2}_{n_2} = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sum_{\substack{n_1=-\ell_1 \\ n_1+n_2=n}}^{\ell_1} \sum_{n_2=-\ell_2}^{\ell_2} C^{\ell_1,\ell_2,\ell}_{n_1,n_2,n}\, e^{\ell}_n. \tag{3.4.6}$$

The coefficients in (3.4.6) are known as the *Clebsch-Gordan coefficients*, and the Clebsch-Gordan coefficients can be identified with terminating ${}_3F_2$-series. The Clebsch-Gordan coefficients are also known as Wigner's coefficients or as $3j$-coefficients. Note that (3.4.6) does not determine the Clebsch-Gordan coefficients completely, but up to a phase factor only depending on ℓ.

Exercise 3.4.8. The Clebsch-Gordan coefficients satisfy a number of recursions, one of them being

$$\sqrt{(\ell-n)(\ell+n+1)}C^{\ell_1,\ell_2,\ell}_{n_1,n_2,n} = \sqrt{(\ell_1-n_1+1)(\ell_1+n_1)}C^{\ell_1,\ell_2,\ell}_{n_1-1,n_2,n} + \sqrt{(\ell_2-n_2+1)(\ell_2+n_2)}C^{\ell_1,\ell_2,\ell}_{n_1,n_2-1,n}.$$

Prove this by acting on (3.4.6) by a suitable element of the Lie algebra of $\mathfrak{su}(2)$ or its complexification $\mathfrak{sl}(2,\mathbb{C})$.

In order to sketch the proof of this derivation we follow Koornwinder [52]. We realise the space $V_{\ell_1}\otimes V_{\ell_2}$ as the space of polynomials in (x,y,u,v), homogeneous of degree $2\ell_1$ in (x,y) and homogeneous of degree $2\ell_2$ in (u,v). Then we identify

$$e^{\ell_1}_{n_1}\otimes e^{\ell_2}_{n_2} \mapsto \binom{2\ell_1}{\ell_1-n_1}^{\frac12}\binom{2\ell_2}{\ell_2-n_2}^{\frac12} x^{\ell_1-n_1}y^{\ell_1+n_1}u^{\ell_2-n_2}v^{\ell_2+n_2}, \tag{3.4.7}$$

cf. (3.2.5). The next step is to find the basis elements in this realisation corresponding to e^{ℓ}_n.

Lemma 3.4.9. *For suitable constants $a_{\ell_1,\ell_2,\ell}$ the functions*

$$\phi^{\ell_1,\ell_2,\ell}_n(x,y,u,v) = a_{\ell_1,\ell_2,\ell}(xv-yu)^{\ell_1+\ell_2-\ell}t^{\ell}_{\ell_2-\ell_1,n}\begin{pmatrix} x & y\\ u & v\end{pmatrix}$$

form an orthonormal basis for $V_{\ell_1}\otimes V_{\ell_2}$ in this realisation. Moreover, for the action of $\begin{pmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{pmatrix}\in \mathrm{SU}(2)$ we have

$$\phi^{\ell_1,\ell_2,\ell}_n(\alpha x-\bar\beta y, \beta x+\bar\alpha y, \alpha u-\bar\beta v, \beta u+\bar\alpha v) = \sum_{m=-\ell}^{\ell} t^{\ell}_{m,n}\begin{pmatrix}\alpha & \beta\\ -\bar\beta & \bar\alpha\end{pmatrix}\phi^{\ell_1,\ell_2,\ell}_m(x,y,u,v) \tag{3.4.8}$$

Note that $xv-yu$ is precisely the determinant of the matrix in which the matrix entry is evaluated.

Proof. Lemma 3.1.4 shows that $\phi^{\ell_1,\ell_2,\ell}_n(x,y,u,v)$ is homogeneous of degree $2\ell_1$ in (x,y) and of degree $2\ell_2$ in (u,v). The fact that SU(2) acts in the right way follows from $t^{\ell}_{m,n}$ being the matrix entries of the unitary representation π^ℓ of SU(2) and π^ℓ being a homomorphism. This shows that the $\phi^{\ell_1,\ell_2,\ell}_n$ are orthonormal for a suitable constant that only depends on ℓ by Schur's Lemma 2.5.7 and Corollary 3.4.3. Since the tensor product decomposition is multiplicity free by Corollary 3.4.3, and counting dimensions we see that we have an orthonormal basis. □

Exercise 3.4.10. Check the details of the proof of Lemma 3.4.9, and in particular prove (3.4.8)

The factor $a_{\ell_1,\ell_2,\ell}$ is determined up to a phase factor, which only depends on ℓ given ℓ_1 and ℓ_2. So we pick the special choice $n = \ell$, and then Theorem 3.2.4 gives

$$\phi_\ell^{\ell_1,\ell_2,\ell}(x,y,u,v) = a_{\ell_1,\ell_2,\ell}(xv - yu)^{\ell_1+\ell_2-\ell} \binom{2\ell}{\ell_1 - \ell_2 + \ell}^{\frac{1}{2}} y^{\ell_1-\ell_2+\ell} v^{\ell_2-\ell_1+\ell}$$

and we normalise $a_{\ell_1,\ell_2,\ell}$ by

$$\langle \phi_\ell^{\ell_1,\ell_2,\ell}, e^{\ell_1}_{\ell_1} \otimes e^{\ell_2}_{\ell-\ell_1} \rangle > 0.$$

Exercise 3.4.11. Explain why the factor $a_{\ell_1,\ell_2,\ell}$ is determined up to a phase factor, which only depends on ℓ given ℓ_1 and ℓ_2 using Schur's Lemma 2.5.7.

Lemma 3.4.12. *With this normalisation*

$$a_{\ell_1,\ell_2,\ell} = (-1)^{\ell_1+\ell_2-\ell} \sqrt{\frac{(2\ell+1)\ (2\ell_1)!\,(2\ell_2)!}{(\ell_1+\ell_2-\ell)!\,(\ell_1+\ell_2+\ell+1)!}}$$

Proof. Expand $\phi_\ell^{\ell_1,\ell_2,\ell}(x,y,u,v)$ using Newton's binomium to get

$$\begin{aligned}\phi_\ell^{\ell_1,\ell_2,\ell}(x,y,u,v) &= a_{\ell_1,\ell_2,\ell} \binom{2\ell}{\ell_1-\ell_2+\ell}^{\frac{1}{2}} \\ &\times \sum_{k=0}^{\ell_1+\ell_2-\ell} (-1)^{\ell_1+\ell_2-\ell-k} \binom{\ell_1+\ell_2-\ell}{k} x^k y^{2\ell_1-k}\, u^{\ell_1+\ell_2-\ell-k} v^{\ell_2-\ell_1+\ell+k}\end{aligned}$$

and this gives an expansion in terms of $e^{\ell_1}_{\ell_1-k} \otimes e^{\ell_2}_{\ell-\ell_1+k}$ using (3.4.7). This then gives the statement on the sign of $a_{\ell_1,\ell_2,\ell}$. The explicit value then follows by taking squared norms, so that

$$1 = |a_{\ell_1,\ell_2,\ell}|^2 \binom{2\ell}{\ell_1-\ell_2+\ell} \sum_{k=0}^{\ell_1+\ell_2-\ell} \binom{\ell_1+\ell_2-\ell}{k}^2 \binom{2\ell_1}{k}^{-1} \binom{2\ell_2}{\ell_1+\ell_2-\ell-k}^{-1}$$

and rewriting the binomial coefficients in terms of Pochhammer symbols, the sum can be rewritten as a factor times ${}_2F_1(\ell-\ell_1-\ell_2, \ell_2-\ell_1+\ell+1, -2\ell_1; 1)$. This can be evaluated by the Chu-Vandermonde sum (2.2.10), and this leads to the explicit value. □

Now we see that in this realisation the basis transition in (3.4.6) with this realisation gives rise to real Clebsch-Gordan coefficients. So we find from (3.4.6) that the inverse relation in the explicit realisation gives

$$\phi_n^{\ell_1,\ell_2,\ell}(x,y,u,v) = \sum_{\substack{n_1=-\ell_1 \\ n_1+n_2=n}}^{\ell_1} \sum_{n_2=-\ell_2}^{\ell_2} C^{\ell_1,\ell_2,\ell}_{n_1,n_2,n} \binom{2\ell_1}{\ell_1-n_1}^{\frac{1}{2}} \binom{2\ell_2}{\ell_2-n_2}^{\frac{1}{2}} x^{\ell_1-n_1} y^{\ell_1+n_1} u^{\ell_2-n_2} v^{\ell_2+n_2}, \tag{3.4.9}$$

and this formula can be considered as a generating function for the Clebsch-Gordan coefficients. Indeed, the left hand side can be written as a simple power $(xv - yu)^{\ell_1+\ell_2-\ell}$ times a hypergeometric series using Theorem 3.2.4. Plugging in the expression for the matrix entry as in (3.2.11) gives an explicit expression for the left hand side. Expanding the ${}_2F_1$-series and using Newton's binomium we find the right hand side as a double sum. Comparing the powers of the monomials on the left hand side gives an expression for the Clebsch-Gordan coefficient. Taking into account the conditions on the expressions for the matrix entries in terms of hypergeometric series in Section 3.1 we find the following result.

Theorem 3.4.13. *For $\ell_1 - \ell_2 \le n \le \ell_2 - \ell_1 \le \ell \le \ell_1 + \ell_2$, $-\ell_1 \le n_1 \le \ell_2$, $n_1 + n_2 = n$ we have*

$$\begin{aligned} C^{\ell_1,\ell_2,\ell}_{n_1,n_2,n} =& \sqrt{\frac{(2\ell+1)\,(\ell_2-n_2)!\,(\ell_2+n_2)!\,(\ell+n)!\,(\ell_2-\ell_1+\ell)!}{(\ell_1-n_1)!\,(\ell_1+n_1)!\,(\ell-n)!\,(\ell_1-\ell_2+\ell)!\,(\ell_1+\ell_2+\ell+1)!\,(\ell_1+\ell_2-\ell)!}} \\ &\times \frac{(-1)^{\ell_1-n_1}(2\ell_1)!}{(\ell_2-\ell_1+n)!}\, {}_3F_2\left(\begin{matrix} \ell_2-\ell_1-\ell, \ell_2-\ell_1+\ell+1, n_1-\ell_1 \\ \ell_2-\ell_1+n+1, -2\ell_1 \end{matrix}; 1\right) \end{aligned}$$

Exercise 3.4.14. Check the details of the proof sketched for Theorem 3.4.13.

Similarly, we find expressions for the other cases, or alternatively, one can derive symmetry relations for the Clebsch-Gordan coefficients and observe that Theorem 3.4.13 in combination with the symmetries give all cases. Usually the symmetries are easier to write down in terms of so-called Regge symbols.

Exercise 3.4.15. Show the following symmetries for the Clebsch-Gordan coefficients:

$$\begin{aligned} C^{\ell_1,\ell_2,\ell}_{n_1,n_2,n} &= C^{\ell_2,\ell_1,\ell}_{-n_2,-n_1,-n} \\ &= C^{\frac12(\ell_1+\ell_2-n),\frac12(\ell_1+\ell_2+n),\ell}_{\frac12(\ell_2-\ell_1+n_1-n_2),\frac12(\ell_2-\ell_1-n_1+n_2),\ell_2-\ell_1} \\ &= C^{\frac12(\ell_1+\ell_2+n),\frac12(\ell_1+\ell_2-n),\ell}_{\frac12(\ell_1-\ell_2+n_1-n_2),\frac12(\ell_1-\ell_2-n_1+n_2),\ell_1-\ell_2}. \end{aligned}$$

Use (3.4.9) and the explicit expression in Lemma 3.4.9 with the symmetries of Lemma 3.1.5.

Now the Clebsch-Gordan coefficients are (real) entries of a unitary matrix, so that we obtain the orthogonality relations

$$\sum_{\substack{n_1=-\ell_1 \\ n_1+n_2=n}}^{\ell_1} \sum_{n_2=-\ell_2}^{\ell_2} C^{\ell_1,\ell_2,\ell}_{n_1,n_2,n} C^{\ell_1,\ell_2,\ell'}_{n_1,n_2,n} = \delta_{\ell,\ell'}$$

and plugging in Theorem 3.4.13 gives the following orthogonality relations for the Hahn polynomials

$$\begin{aligned} \sum_{x=0}^{N} Q_k(x;\alpha,\beta;N) Q_l(x;\alpha,\beta;N) \frac{(\alpha+1)_x(\beta+1)_{N-x}\,N!}{x!\,(N-x)!\,(\alpha+\beta+2)_N} = \delta_{k,l} \\ \times \frac{k!\,(N-k)!\,(\beta+1)_k(N+\alpha+\beta+2)_k(k+\alpha+\beta+1)}{N!\,(\alpha+1)_k(\alpha+\beta+2)_k(2k+\alpha+\beta+1)}, \qquad \alpha,\beta > -1. \end{aligned} \tag{3.4.10}$$

under the identification $N = 2\ell_1$, $\alpha = \ell_2 - \ell_1 + n$, $\beta = \ell_2 - \ell_1 - n$, $x = \ell_1 - n_1$, $k = \ell_1 - \ell_2 + \ell$, $l = \ell_1 - \ell_2 + \ell'$, where the Hahn polynomials are polynomials of degree $k \in \{0, 1, \cdots, N\}$ in x defined by

$$Q_k(x; \alpha, \beta; N) = {}_3F_2\left(\begin{matrix} -k, -x, k+\alpha+\beta+1 \\ \alpha+1, -N \end{matrix}; 1\right) \tag{3.4.11}$$

Exercise 3.4.16. Prove (3.4.10) by filling in the details of the argument sketched. Derive similarly that the orthogonality

$$\sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} C^{\ell_1,\ell_2,\ell}_{p,n-p,n} C^{\ell_1,\ell_2,\ell}_{m,n-m,n} = \delta_{m,n}$$

give the orthogonality relations

$$\begin{aligned}
&\sum_{k=0}^{N} R_x(k; \alpha, \beta; N) R_y(k; \alpha, \beta; N) \frac{N!\,(\alpha+1)_k(\alpha+\beta+2)_k(2k+\alpha+\beta+1)}{k!\,(N-k)!\,(\beta+1)_k(N+\alpha+\beta+2)_k(k+\alpha+\beta+1)} \\
&= \delta_{x,y} \frac{x!\,(N-x)!\,(\alpha+\beta+2)_N}{(\alpha+1)_x(\beta+1)_{N-x}\,N!} \\
&R_x(k; \alpha, \beta; N) = Q_k(x; \alpha, \beta; N) = {}_3F_2\left(\begin{matrix} -k, -x, k+\alpha+\beta+1 \\ \alpha+1, -N \end{matrix}; 1\right)
\end{aligned}$$

under the same identification as in (3.4.10). Here $R_x(k; \alpha, \beta; N)$ is a polynomial of degree x in $k(k+\alpha+\beta+1)$, known as the dual Hahn polynomial.

We have been somewhat sketchy in the calculations. For a more detailed analysis that the unitarity of the intertwiner gives the orthogonality relations for the Hahn polynomials and the dual Hahn polynomials, see [80, Ch. 8], [52], [76].

Remark 3.4.17. The other classical orthogonal polynomials related to this interpretation is the two different ways to decompose the three-fold tensor product $(\pi_{\ell_1} \otimes \pi_{\ell_2}) \otimes \pi_{\ell_3} \cong \pi_{\ell_1} \otimes (\pi_{\ell_2} \otimes \pi_{\ell_3})$, and the corresponding overlap coefficients, also known as $6j$-coefficients, are Racah polynomials, which can be written as ${}_4F_3$-series, see [80, Ch. 8], [76]. Iterating the number of irreducible spin representations gives rise to multivariable orthogonal polynomials, see [76], [77]. In order to keep track of the three-fold tensor product decomposition, we refine the notation of (3.4.6) and its inverse by

$$e^{(\ell_1\ell_2)\ell}_n = \sum_{\substack{n_1=-\ell_1 \\ n_1+n_2=n}}^{\ell_1} \sum_{n_2=-\ell_2}^{\ell_2} C^{\ell_1,\ell_2,\ell}_{n_1,n_2,n}\, e^{\ell_1}_{n_1} \otimes e^{\ell_2}_{n_2}.$$

So decomposing the three-fold tensor product by first decomposing $\pi_{\ell_1} \otimes \pi_{\ell_2}$ and next decomposing the resulting sum $\sum_{\ell_{12}=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \pi_{\ell_{12}} \otimes \pi_{\ell_3} = \sum_{\ell_{12}=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sum_{\ell=|\ell_{12}-\ell_3|}^{\ell_{12}+\ell_3} \pi_\ell$ gives

$$e^{((\ell_1\ell_2)\ell_{12}\ell_3)\ell}_n = \sum_{n_{12},n_3} C^{\ell_{12},\ell_3,\ell}_{n_{12},n_3,n}\, e^{(\ell_1\ell_2)\ell_{12}}_{n_{12}} \otimes e^{\ell_3}_{n_3} = \sum_{\substack{n_1,n_2,n_3 \\ n_1+n_2+n_3=n}} C^{\ell_1,\ell_2,\ell_{12}}_{n_1,n_2,n_{12}} C^{\ell_{12},\ell_3,\ell}_{n_{12},n_3,n}\, e^{\ell_1}_{n_1} \otimes e^{\ell_2}_{n_2} \otimes e^{\ell_3}_{n_3}$$

using ℓ_{12} as an intermediate coupling. Decomposing in the other way gives

$$e_n^{(\ell_1(\ell_2\ell_3)\ell_{23})\ell} = \sum_{n_1,n_{23}} C^{\ell_1,\ell_{23},\ell}_{n_1,n_{23},n}\, e^{\ell_1}_{n_1} \otimes e^{(\ell_2\ell_3)\ell_{23}}_{n_{23}} = \sum_{\substack{n_1,n_2,n_3\\ n_1+n_2+n_3=n}} C^{\ell_2,\ell_3,\ell_{23}}_{n_2,n_3,n_{23}} C^{\ell_1,\ell_{23},\ell}_{n_1,n_{23},n}\, e^{\ell_1}_{n_1} \otimes e^{\ell_2}_{n_2} \otimes e^{\ell_3}_{n_3}$$

The overlap coefficients $\langle e_n^{((\ell_1\ell_2)\ell_{12}\ell_3)\ell}, e_{n'}^{(\ell_1(\ell_2\ell_3)\ell_{23})\ell'}\rangle$ are Racah coefficients. The overlap coefficient is 0 if $n \neq n'$ and $\ell \neq \ell'$. It turns out the inner product is independent of n in case $n = n'$.

Exercise 3.4.18. Define $\langle e_n^{((\ell_1\ell_2)\ell_{12}\ell_3)\ell}, e_{n'}^{(\ell_1(\ell_2\ell_3)\ell_{23})\ell'}\rangle = \delta_{\ell,\ell'}\delta_{n,n'}U^{\ell_1,\ell_2,\ell_{12}}_{\ell_3,\ell,\ell_{23}}$, where $U^{\ell_1,\ell_2,\ell_{12}}_{\ell_3,\ell,\ell_{23}}$ are known as the Racah coefficients. Give an explicit expression for the Racah coefficient as a sum of products of Clebsch-Gordan coefficients and state the orthogonality relations for the Racah coefficients.

3.4.5 Legendre polynomials as spherical functions

Let $K \cong \mathrm{U}(1)$ be the diagonal subgroup of SU(2), see (3.2.1), then

$$\begin{aligned} t^\ell_{m,n}(k(\phi)gk(\theta)) &= \langle \pi_\ell(k(\phi)gk(\theta))e^\ell_n, e^\ell_m\rangle = \langle \pi_\ell(g)\pi_\ell(k(\theta))e^\ell_n, \pi_\ell(k(\phi)^{-1})e^\ell_m\rangle \\ &= e^{-in\theta}e^{-im\psi}\langle \pi_\ell(g)e^\ell_n, e^\ell_m\rangle = e^{-in\theta}e^{-im\psi}t^\ell_{m,n}(g) \end{aligned} \tag{3.4.12}$$

using the notation of (3.2.1). By the decomposition of Exercise 3.2.1, we see that it suffices to know $t^\ell_{m,n}(a(\psi))$. In particular, we see that for each irreducible unitary representation there is at most a one-dimensional space of bi-K-invariant functions on SU(2), where a function $f\colon \mathrm{SU}(2) \to \mathbb{C}$ is *bi-K-invariant* if f is both *left K-invariant* and *right K-invariant*, i.e.

$$f(kg) = f(g), \quad \text{and} \quad f(gk) = f(g), \qquad \forall k \in K,\ \forall g \in \mathrm{SU}(2).$$

Indeed, in case $\ell \in \mathbb{N}$ this is spanned by $t^\ell_{0,0} \in M_{\pi_\ell}$, and in case $\ell \in \frac{1}{2} + \mathbb{N}$, there is no bi-$K$-invariant element in M_{π_ℓ}. A pair (G, K) of groups, with K a compact subgroup of G, such that for each irreducible unitary representation π of G the restriction of $\pi|_K$ contains the trivial representation of K at most once, is called a *Gelfand pair*, which we study in more detail in Chapter 4. We see that $(\mathrm{SU}(2), K \cong \mathrm{U}(1))$ is an example of a (compact) Gelfand pair, see also Example 4.1.10. In Chapter 4 the name *spherical function* is explained as well. Note that the Schur orthogonality relations of Theorem 3.3.6 and (3.2.4) correspond to the orthogonality of the Legendre polynomials $P_n = P_n^{(0,0)}$, see Section 3.4.2.

We now use this interpretation for the Legendre polynomials to derive two results for the Legendre polynomials. First we take $\ell_1, \ell_2 \in \mathbb{N}$ and we use the Clebsch-Gordan decomposition of Corollary 3.4.3 and (3.4.6) to write

$$t^{\ell_1}_{0,0}(g)t^{\ell_2}_{0,0}(g) = \langle \pi_{\ell_1}(g)e^{\ell_1}_0 \otimes \pi_{\ell_2}(g)e^{\ell_2}_0, e^{\ell_1}_0 \otimes e^{\ell_2}_0\rangle =$$

$$\sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sum_{\substack{n_1=-\ell_1\\ n_1+n_2=0}}^{\ell_1} \sum_{n_2=-\ell_2}^{\ell_2} \sum_{\ell'=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \sum_{\substack{n_1'=-\ell_1\\ n_1'+n_2'=0}}^{\ell_1} \sum_{n_2'=-\ell_2}^{\ell_2} C^{\ell_1,\ell_2,\ell}_{n_1,n_2,0}\overline{C^{\ell_1,\ell_2,\ell'}_{n_1',n_2',0}}\,\langle \pi_\ell(g)e^\ell_0, e^{\ell'}_0\rangle =$$

$$\sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} t^\ell_{0,0}(g)\Big(\sum_{\substack{n_1=-\ell_1\\ n_1+n_2=0}}^{\ell_1} \sum_{n_2=-\ell_2}^{\ell_2} \sum_{\substack{n_1'=-\ell_1\\ n_1'+n_2'=0}}^{\ell_1} \sum_{n_2'=-\ell_2}^{\ell_2} C^{\ell_1,\ell_2,\ell}_{n_1,n_2,0}\overline{C^{\ell_1,\ell_2,\ell}_{n_1',n_2',0}}\Big)$$

Now the term in parentheses can be written as $|\langle Cv, e^\ell_0\rangle|^2 \geq 0$, with

$$v = \sum_{\substack{n_1=-\ell_1\\ n_1+n_2=0}}^{\ell_1} \sum_{n_2=-\ell_2}^{\ell_2} e^{\ell_1}_{n_1} \otimes e^{\ell_2}_{n_2}.$$

So we have obtained Corollary 3.4.19.

Corollary 3.4.19. *The coefficients $P_{\ell_1}(x)P_{\ell_2}(x) = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} c_\ell(\ell_1,\ell_2)P_\ell(x)$ in the linearisation formula for the Legendre polynomials are non-negative; $c_\ell(\ell_1,\ell_2) \geq 0$.*

Remark 3.4.20. Note that the Legendre polynomials being symmetric, i.e. $P_n(-x) = (-1)^n P_n(x)$, we have $c_\ell(\ell_1,\ell_2) = 0$ if $\ell_1+\ell_2-\ell \notin 2\mathbb{Z}$. The explicit value of the coefficients can be calculated from the explicit value of the Clebsch-Gordan coefficients, but have already been calculated in the 19th century, see references in Askey [2, Lecture 5].

Exercise 3.4.21. Note that the case $\ell_1 = 1$ corresponds to the three-term recurrence for the Legendre polynomials. Calculate the explicit three term recurrence relation for the Legendre polynomials, and match this to the case $\alpha = \beta = 0$ of (2.2.14). Show that $t^1_{0,0}(g)t^{\ell_2}_{m,n}(g)$ for suitable choice of $g \in \mathrm{SU}(2)$ leads to the three term recurrence relations for the Jacobi polynomials with $(\alpha,\beta) \in \mathbb{N}^2$.

Exercise 3.4.22. Renormalising Corollary 3.4.19 we have an example of a set of orthogonal polynomials p_n normalised by $p_n(1) = 1$ and $p_n(x)p_m(x) = \sum_{k=|n-m|}^{n+m} c_k(n,m)p_k(x)$ with $c_k(n,m) \geq 0$. The normalisation leads to $\sum_{k=|n-m|}^{n+m} c_k(n,m) = 1$. We define a product on $\ell^1(\mathbb{N})$ by $\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} c_k(n,m)\delta_k$, where δ_k is the sequence with all zeroes, except a 1 on the k-th entry of the sequence. Show that $\ell^1(\mathbb{N})$ with this product is a Banach algebra, i.e. $\|a*b\| \leq \|a\|\|b\|$ with $\|a\| = \sum_{n\in\mathbb{N}} |a_n|$. Note that $c_k(n,m) = c_k(m,n)$, prove moreover that $c_k(n,m)(\int_{\mathbb{R}}(p_k(x))^2\,d\mu(x))^{-1}$ is symmetric in (n,m,k). Here μ is the orthogonality measure for the orthogonal polynomials $(p_n)_{n\in\mathbb{N}}$. See [70] for more information on this connection.

As a next application we first note that for $\ell \in \mathbb{N}$ we have, by (3.2.12), the expression

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sin(\theta_1/2) & -\cos(\theta_1/2) \\ \cos(\theta_1/2) & \sin(\theta_1/2) \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2}i\phi} & 0 \\ 0 & e^{-\frac{1}{2}i\phi} \end{pmatrix} \begin{pmatrix} \sin(\theta_2/2) & \cos(\theta_2/2) \\ -\cos(\theta_2/2) & \sin(\theta_2/2) \end{pmatrix},$$
$$t^\ell_{0,0}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = P_\ell\Big(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\cos(\phi)\Big). \tag{3.4.13}$$

Calculating the matrix entries and using the homomorphism property we derive

$$\begin{aligned} &P_\ell\Big(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\cos(\phi)\Big) \\ &= \sum_{k=-\ell}^{\ell} t^\ell_{0,k}\begin{pmatrix} \sin(\theta_1/2) & -\cos(\theta_1/2) \\ \cos(\theta_1/2) & \sin(\theta_1/2) \end{pmatrix} t^\ell_{k,0}\begin{pmatrix} \sin(\theta_2/2) & \cos(\theta_2/2) \\ -\cos(\theta_2/2) & \sin(\theta_2/2) \end{pmatrix} e^{-ik\phi}. \end{aligned} \tag{3.4.14}$$

by using the homomorphism property of the representations and (3.4.13). Indeed, writing the decomposition of (3.4.13) as $g = g_1 d g_2$, we get

$$\begin{aligned} t^\ell_{0,0}(g) =& \langle \pi_\ell(g_1 d g_2) e^\ell_0, e^\ell_0\rangle = \sum_{k=-\ell}^{\ell} \langle \pi_\ell(g_2)e^\ell_0, e^\ell_k\rangle \langle \pi_\ell(g_1)e^\ell_k, e^\ell_0\rangle \langle \pi_\ell(d)e^\ell_k, e^\ell_k\rangle \\ =& \sum_{k=-\ell}^{\ell} t^\ell_{0,k}(g_1) t^\ell_{k,k}(d) t^\ell_{k,0}(g_2). \end{aligned}$$

using the fact that for a diagonal $d \in \mathrm{SU}(2)$ we have $t^\ell_{m,n}(d) = 0$ for $m \neq n$. Now using (3.4.14), Theorem 3.2.4 and the symmetries (3.2.9) we can determine all elements explicitly. We can take the terms for k and $-k$ together. This then proves the addition formula for the Legendre polynomials.

Corollary 3.4.23. *The addition formula for the Legendre polynomials is*

$$\begin{aligned} &P_\ell\Big(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\cos(\phi)\Big) = P_\ell(\cos(\theta_1))P_\ell(\cos(\theta_2)) + \\ &2\sum_{k=1}^{\ell} \frac{(-1)^k}{4^k}\frac{(m+1)_k}{(m-k+1)_k} \sin^k(\theta_1)\, \sin^k(\theta_2)\, P^{(k,k)}_{\ell-k}(\cos(\theta_1))P^{(k,k)}_{\ell-k}(\cos(\theta_2))\cos(k\phi). \end{aligned}$$

Remark 3.4.24. Note that the addition formula for the Legendre polynomials can be viewed as an expansion of $P_\ell\Big(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\cos(\phi)\Big)$, considered as a polynomial of $x = \cos(\phi)$ in terms of Chebyshev polynomials $T_k(\cos(\phi)) = \cos(k\phi)$ of the first kind. So we can use the orthogonality of the Chebyshev polynomials to obtain explicit expressions for

$$\int_{-1}^{1} P_\ell\Big(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)x\Big) T_k(x)\, \frac{dx}{\sqrt{1-x^2}}$$

and the case $k = 0$ corresponds to the so-called product formula to which we come back in Theorem 4.1.16 and we state the explicit form in Corollary 4.2.3.

Exercise 3.4.25. Give an explicit evaluation for the case $k = 0$ of the integral in Remark 3.4.24.

3.4.6 The Casimir element and the hypergeometric differential operator

The elements of the universal enveloping algebra can be considered as differential operators on functions on the group, see Appendix A. Moreover, the Casimir element is a quadratic central element in the universal enveloping algebra $U(\mathfrak{sl}(2,\mathbb{C}))$, where $\mathfrak{sl}(2,\mathbb{C})$ is the complexification of the Lie algebra $\mathfrak{su}(2)$, see Section A.3. Taking as the basis $\mathfrak{su}(2)$ of traceless skew-adjoint 2×2-matrices the elements

$$J_0 = \begin{pmatrix} \frac{1}{2}i & 0 \\ 0 & -\frac{1}{2}i \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & \frac{1}{2}i \\ \frac{1}{2}i & 0 \end{pmatrix}.$$

The commutation relations are

$$[J_0, J_+] = -J_-, \qquad [J_0, J_-] = J_+, \qquad [J_-, J_+] = J_0.$$

And the Casimir operator then is $\Omega = -J_0^2 - J_-^2 - J_+^2$, and acts as a multiple $c_\ell = \ell(\ell+1)$ in the irreducible representation π_ℓ. see Section A.3. In particular, it follows that Ω, when viewed as a second order partial differential operator has $g \mapsto t^\ell_{m,n}(g)$ as eigenfunctions of eigenvalue c_ℓ, i.e.

$$\Omega \cdot t^\ell_{m,n} = c_\ell \, t^\ell_{m,n}. \tag{3.4.15}$$

On the other hand, with respect the decomposition 3.2.1, usually called the KAK-decomposition or Cartan decomposition, and (3.4.12) we know that $t^\ell_{m,n}(a(\phi))$ completely determines the function $t^\ell_{m,n}$. So we want to rewrite (3.4.15) as differential operator with respect to the variable ϕ only, i.e. $R(\Omega)\cdot t^\ell_{m,n}(a(\phi)) = c_\ell \, t^\ell_{m,n}(a(\phi))$, where the operator $R(\Omega)$ is a differential operator in ϕ only, the so-called radial part of the Casimir operator Ω.

From now on $f\colon \mathrm{SU}(2) \to \mathbb{C}$ is a sufficiently smooth function satisfying

$$f(k(\psi)a(\phi)k(\theta)) = e^{-in\theta}e^{-im\psi} f(a(\phi)) \tag{3.4.16}$$

cf. (3.4.12), and we calculate $R(\Omega)f$ when restricted to $a(\phi)$ as a differential operator with respect to the variable ϕ.

Theorem 3.4.26. *For f a smooth function satisfying* (3.4.16)*, the radial part of the Casimir operator is given by the following differential operator on $\phi \mapsto f(a(\phi))$*

$$R(\Omega) = -\frac{d^2}{d^2\phi} - \frac{\cos\phi}{\sin\phi}\frac{d}{d\phi} + \frac{n^2+m^2-2nm\cos\phi}{\sin^2\phi}.$$

The idea of proof in the general non-compact case goes back to Harish-Chandra (1923–1983), see [10]. We adapt the idea for the simpler case of the compact group SU(2). First, note that $\exp(tJ_0) = k(t)$ and $\exp(tJ_+) = a(t)$, see (3.2.1), (3.2.3). These two properties make it relatively easy to deal with $-(J_0)^2$ and $-(J_+)^2$ from the Casimir operator in the radial part calculation. However, this does not apply to $-(J_-)^2$, and we need a completely different approach.

Exercise 3.4.27. Show that

$$\exp(tJ_-) = \begin{pmatrix} \cos(\frac{1}{2}t) & i\sin(\frac{1}{2}t) \\ i\sin(\frac{1}{2}t) & \cos(\frac{1}{2}t) \end{pmatrix}$$

We start by characterising the Lie algebra $\mathfrak{k}$ of K as the fixed points of the conjugation θ given as $\theta(X) = JXJ$, with $J = \mathrm{diag}(1, -1)$. Then the fixed points are the diagonal matrices, and we view θ as a Lie algebra automorphism of $\mathfrak{su}(2)$ as well as of its complexification $\mathfrak{sl}(2, \mathbb{C})$. Note that θ is also defined as a group automorphism of SU(2), and then its fixed points are exactly the diagonal subgroup K.

Now J_+ is the generator of the abelian subgroup A, and we diagonalise its adjoint action on $\mathfrak{sl}(2, \mathbb{C})$.

Lemma 3.4.28. $\mathrm{ad}\,(J_+)J_+ = 0$, $\mathrm{ad}\,(J_+)(iJ_0 + J_-) = i(iJ_0 + J_-)$, *and* $\mathrm{ad}\,(J_+)(-iJ_0 + J_-) = -i(-iJ_0 + J_-)$.

Exercise 3.4.29. Prove Lemma 3.4.28 by verification, or calculate the action of $\mathrm{ad}\,J_+$ in the basis $\{J_0, J_+, J_-\}$ and diagonalise the corresponding 3×3-matrix.

It turns out that the eigenvectors for the action of $\mathrm{ad}\,(J_+)$ for non-zero eigenvalues are also eigenvectors for the action of $\mathrm{Ad}(a(\phi))$.

Lemma 3.4.30. $\mathrm{Ad}(a(\phi))(iJ_0 + J_-) = e^{i\phi}(iJ_0 + J_-)$, $\mathrm{Ad}(a(\phi))(-iJ_0 + J_-) = e^{-i\phi}(-iJ_0 + J_-)$.

Exercise 3.4.31. Prove Lemma 3.4.30 by direct calculation, or by use of Lemma 3.4.28, or by calculating the action of $\mathrm{Ad}(a(\phi))$ in the basis $\{J_0, J_+, J_-\}$ and diagonalise the corresponding 3×3-matrix.

The main trick is now to write the eigenfunctions of $\mathrm{Ad}(a(\phi))$ as a linear combination of elements of $\mathfrak{k}$ and conjugation of such elements under $a(\phi)$. For this we use the notation $X^a = \mathrm{Ad}(a^{-1})X = a^{-1}Xa$ for a group element a.

Proposition 3.4.32. *Put* $U = \mathrm{diag}(-1, 1)$, $V = -U$, *so that* $U, V \in \mathfrak{k}$, *then*

$$iJ_0 + J_- = \frac{e^{i\phi}}{1 - e^{2i\phi}}\big(U^{a(\phi)} - e^{i\phi}U\big), \qquad -iJ_0 + J_- = \frac{e^{-i\phi}}{1 - e^{-2i\phi}}\big(V^{a(\phi)} - e^{-i\phi}V\big)$$

We assume that ϕ is generic, and that we don't divide by zero.

Exercise 3.4.33. Verify the statements of Proposition 3.4.32 directly by calculating $U^{a(\phi)}$ and verifying the explicit 2×2-matrix identity. First check that

$$U^{a(\phi)} = \begin{pmatrix} -\cos(\phi) & \sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}.$$

Proof. Note that $U = (iJ_0 + J_-) + \theta(iJ_0 + J_-) \in \mathfrak{k}$ since it is θ-invariant by construction. Now $\theta(J_0) = J_0$, $\theta(J_-) = -J_-$, so that $U = (iJ_0 + J_-) - (-iJ_0 + J_-) = 2iJ_0 = \mathrm{diag}(-1, 1)$. Now

$$\begin{aligned} U^{a(\phi)} &= (iJ_0 + J_-)^{a(\phi)} + \big(\theta(iJ_0 + J_-)\big)^{a(\phi)} = e^{-i\phi}(iJ_0 + J_-) - (-iJ_0 + J_-)^{a(\phi)} \\ &= e^{-i\phi}(iJ_0 + J_-) - e^{i\phi}(-iJ_0 + J_-). \end{aligned}$$

Eliminating $(-iJ_0 + J_-)$ from these two equations gives the result. Applying the same procedure gives the other statement, or we can eliminate $(iJ_0 + J_-)$ and get the other statement. □

Now we can either determine the Casimir operator using the basis J_+, $iJ_0 + J_-$, $-iJ_0 + J_-$ or we can just rewrite

$$\Omega = -J_+^2 + iJ_+ - (iJ_0 + J_-)(-iJ_0 + J_-)$$

using the commutation relations and $\Omega = -J_+^2 - J_-^2 - J_0^2$.

Theorem 3.4.34. *The Casimir can be written*

$$\Omega = -J_+^2 - \frac{\cos\phi}{\sin\phi} J_+ - \frac{1}{\sin^2\phi}\Big(J_0^2 + (J_0^{a(\phi)})^2 - 2\cos\phi J_0^{a(\phi)} J_0\Big).$$

The expression for the Casimir element is known as the infinitesimal Cartan decomposition of Ω. Note that the element J_- has vanished from the expression, but that we have to add conjugation by elements of the abelian group A.

Proof. We use Proposition 3.4.32 and

$$[U, V^{a(\phi)}] = \begin{pmatrix} 0 & 2\sin(\phi) \\ -2\sin(\phi) & 0 \end{pmatrix} = -4\sin(\phi) J_+$$

and $U = 2iJ_0$, $V = -2iJ_0$, to obtain

$$(iJ_0 + J_-)(-iJ_0 + J_-) = \frac{-1}{\sin^2\phi}\Big(2\cos\phi J_0^{a(\phi)} J_0 - J_0^2 - (J_0^{a(\phi)})^2\Big) + \frac{e^{i\phi}}{\sin\phi} J_+.$$

Using this in the expression for Ω and simplifying the coefficient of J_+ gives the result. □

Proof of Theorem 3.4.26. If we consider $f|_A$ as a function of $a(\phi)$ then

$$(J_+ \cdot f)(a(\phi)) = \frac{d}{dt}\Big|_{t=0} f(a(\phi)\exp(tJ_+)) = \frac{d}{dt}\Big|_{t=0} f(a(\phi + t)) = \frac{d}{d\phi}\big(\phi \mapsto f(a(\phi))\big)$$

and similarly $(J_+^2 \cdot f)(a(\phi)) = \frac{d^2}{d\phi^2}\big(\phi \mapsto f(a(\phi))\big)$. If we now assume that f satisfies (3.4.16), then

$$\begin{aligned} J_0 \cdot f(a(\phi)) &= \frac{d}{dt}\Big|_{t=0} f(a(\phi)\exp(tJ_0)) = \frac{d}{dt}\Big|_{t=0} f(a(\phi)k(t)) \\ &= \frac{d}{dt}\Big|_{t=0} f(a(\phi))e^{-int} = -in\, f(a(\phi)). \end{aligned}$$

and

$$J_0^{a(\phi)} \cdot f(a(\phi)) = \frac{d}{dt}\Big|_{t=0} f(a(\phi)\exp(t(J_0)^{a(\phi)})) = \frac{d}{dt}\Big|_{t=0} f(a(\phi)a(\phi)^{-1}\exp(tJ_0)a(\phi))$$
$$= \frac{d}{dt}\Big|_{t=0} f(k(t)a(\phi)) = \frac{d}{dt}\Big|_{t=0} e^{-imt} f(a(\phi)) = imf(a(\phi))$$

Proceeding for products by iteration, see Appendix A, gives $(J_0^{a(\phi)})^2 \cdot f(a(\phi)) = -m^2 f(a(\phi))$, $(J_0)^2 \cdot f(a(\phi)) = -n^2 f(a(\phi))$ and $J_0^{a(\phi)} J_0 \cdot f(a(\phi)) = -mnf(a(\phi))$. Collecting the terms proves Theorem 3.4.26. □

Note that the proof requires that all terms involving the conjugation with $a(\phi)$ in Theorem 3.4.34 need to occur on the left hand side.

Corollary 3.4.35. *The matrix entries $t^\ell_{m,n}$ satisfy the second order differential equation*

$$\Big(-\frac{d^2}{d^2\phi} - \frac{\cos\phi}{\sin\phi}\frac{d}{d\phi} + \frac{n^2+m^2-2nm\cos\phi}{\sin^2\phi}\Big) t^\ell_{m,n}(a(\phi)) = \ell(\ell+1)\, t^\ell_{m,n}(a(\phi))$$

Using the expression in Section 3.4.2 for $t^\ell_{m,n}(a(\phi))$ in terms of elementary factors times a Jacobi polynomial, we can conjugate the result of Corollary 3.4.35 in order to obtain a second order differential equation for the Jacobi polynomials. The result is that the Jacobi polynomials are eigenfunctions to the hypergeometric differential operator, see (2.2.7), after a suitable coordinate change.

Exercise 3.4.36. Check the above statement and show that we obtain the hypergeometric differential operator (2.2.7) for the Jacobi polynomials for non-negative integer values for the parameters (α, β) of the Jacobi polynomials.

Chapter 4

Gelfand pairs

In this chapter we introduce Gelfand pairs in terms of certain convolution algebras being commutative in Section 4.1. We introduce spherical functions in terms of characters of such a commutative algebra, and the product formula for spherical functions is derived. Then we view some of the results of Chapter 3 in terms of two Gelfand pairs, leading to a product formula for Legendre polynomials and to a product formula for Chebyshev polynomials of the second kind. In Section 4.4 we state some results without proof on Gelfand pairs. In Section 4.5 we discuss the case of $\mathrm{SL}(2,\mathbb{R})$, which we view as part of the Gelfand pair $(\mathrm{SL}(2,\mathbb{R}), \mathrm{SO}(2))$. We determine the spherical functions by establishing the spherical functions as eigenfunctions to a suitable differential operator.

4.1 Convolution product and Gelfand pairs

Consider the matrix group G with left Haar measure dg, then for continuous functions $f_1, f_2 \colon G \to \mathbb{C}$ with compact support we define the convolution product

$$f_1 * f_2(x) = \int_G f_1(xg) f_2(g^{-1})\, dg = \int_G f_1(g) f_2(g^{-1}x)\, dg, \qquad f_1 * f_2 \colon G \to \mathbb{C}. \tag{4.1.1}$$

In general, $f_1 * f_2 \neq f_2 * f_1$. However, for an abelian group G the convolution product is commutative.

Exercise 4.1.1. Prove the statement that for an abelian group G the convolution product is commutative.

Exercise 4.1.2. Define for a unitary representation $\pi \colon G \to B(H)$ the operator

$$\pi(f) = \int_G f(g)\pi(g)\, dg, \qquad f \in C_0(G).$$

Show that this is a well defined operator. Assume moreover that G is unimodular. Show that for $f_1, f_2 \in C_0(G)$ we have

$$\pi(f_1 * f_2) = \pi(f_1)\pi(f_2).$$

Exercise 4.1.3. Let G be a compact group, and let $\pi, \sigma \in \hat{G}$, so that σ and π are finite dimensional unitary representations. Take $\pi\colon G \to B(V)$, $\sigma\colon G \to B(W)$ and take vectors $v_1, v_2 \in V$, and $w_1, w_2 \in W$.

(i) Calculate the convolution product of the matrix entries $g \mapsto \langle \pi(g)v_1, v_2\rangle$ and $g \mapsto \langle \sigma(g)w_1, w_2\rangle$. Hint: use the homomorphism property of π and Theorem 3.3.6.

(ii) Show that characters of irreducible unitary representations of G satisfy

$$\chi_\pi * \chi_\sigma = \begin{cases} \frac{1}{\dim \pi}\chi_\pi, & \pi \cong \sigma \\ 0 & \text{otherwise} \end{cases}, \qquad \pi, \sigma \in \hat{G}$$

and conclude that $\xi_\pi = \dim \pi\ \chi_\pi$ satisfies

$$\xi_\pi * \xi_\sigma = \delta_{\pi,\sigma}\xi_\pi.$$

(iii) Let $\nu\colon G \to B(H)$ be an arbitrary unitary representation of G. Conclude that $\nu(\xi_\pi)$, $\pi \in \hat{G}$ is an orthogonal projection. Show moreover that $\nu(\xi_\pi) \in \mathrm{End}_G(H)$ using ξ_π being a class function. The range of the projection is the isotypical subspace of $\pi \in \hat{G}$ in H, i.e. this is the closed subspace of H for which we have that any two irreducible representations occurring in this subspace are equivalent.

The convolution product then gives an (associative) algebra structure on the space of integrable functions on G, which in general is non-commutative. Just as for the convolution on $\mathbb{R}$ or $\mathbb{R}^n$, we can apply Fubini's theorem to conclude

$$\|f_1 * f_2\|_1 \leq \|f_1\|_1\, \|f_2\|_1$$

where $\|f\|_1 = \int_G |f(g)|\, dg$ is the L^1-norm. Since we know that $L^1(G)$ (after identifying functions a.e. with respect to the Haar measure) is a Banach space, we have obtained the following lemma.

Lemma 4.1.4. *$L^1(G)$ with the convolution product is a Banach algebra.*

Exercise 4.1.5. Assume that G is a unimodular group, i.e. the left Haar measure is also right invariant. Show that $L^1(G)$ is a Banach $*$-algebra with involution given by

$$f^*(g) = \overline{f(g^{-1})}.$$

Let $K \subset G$ be a subgroup of G, which is assumed to be compact. Then a function $f\colon G \to \mathbb{C}$ is *bi-K-invariant* if

$$f(k_1 g k_2) = f(g), \qquad \forall k_1, k_2 \in K\ \forall g \in G. \tag{4.1.2}$$

Note that this means that f is fixed under the action of the biregular representation $\lambda \times \rho$ restricted to $K \times K$.

It follows from (4.1.1) that if f_1, f_2 are bi-K-invariant continuous functions on G with compact support, then the convolution product $f_1 * f_2$ is a bi-K-invariant function as well.

Definition 4.1.6. *Let G be a matrix group with left Haar measure dg with a compact subgroup K. The pair (G,K) is a Gelfand pair if the algebra of bi-K-invariant continuous functions with compact support is commutative under convolution. In case G is compact, (G,K) is a compact Gelfand pair.*

We denote the the algebra of bi-K-invariant continuous functions with compact support by $C_c(G/\!/K)$, it is also notated $C_c(K\backslash G/K)$ in the literature. Note that we have a projection

$$P\colon C_c(G)\to C_c(G/\!/K), \qquad f\mapsto \big(x\mapsto \int_K\int_K f(k_1xk_2)\,dk_1\,dk_2\big) \tag{4.1.3}$$

where the integrations take place with respect to the normalised Haar measure of the compact group K. So we average out the left and right K-action to obtain a bi-K-invariant function.

Exercise 4.1.7. Show that $Pf=\xi_1 *_K f *_K \xi_1$, with the notation as in Exercise 4.1.3. The notation $*_K$ emphasises that the convolution product is with respect to K, and $1\in\hat{K}$ denoting the trivial representation.

In particular, for (G,K) a Gelfand pair the space $L^1(G/\!/K)$ of bi-K-invariant integrable functions (with equivalence almost everywhere) forms a commutative Banach $*$-algebra. Indeed, the space of bi-K-invariant functions in $L^1(G)$ forms a closed subspace, and, if f is bi-K-invariant, then so is $g\mapsto\overline{f(g^{-1})}$.

Example 4.1.8. If G is abelian, then we can take the trivial subgroup $K=\{e\}$. Since the convolution product is commutative for an abelian group, this gives a Gelfand pair.

The following criterion is used to establish our main examples of Gelfand pairs.

Proposition 4.1.9. *Assume that G is a matrix group with compact subgroup K, assume that there exists a continuous involutive automorphism $\theta\colon G\to G$ so that*

$$\forall g\in G \qquad g^{-1}\in K\theta(g)K,$$

then (G,K) is a Gelfand pair.

Example 4.1.10. (i) Let $G=\mathrm{SU}(2)$, and assume that $\theta\colon \mathrm{SU}(2)\to\mathrm{SU}(2)$, $\theta(g)=JgJ$, $J=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Since $J=J^*=J^{-1}$ we see that θ is an involutive automorphism. We put $K=\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(1))\cong\mathrm{U}(1)$ the diagonal subgroup of $\mathrm{SU}(2)$ as in (3.2.1), then K is just the group of fixed points of θ; $K=\{g\in G\mid\theta(g)=g\}=G^\theta$. By Exercise 3.2.1 we see that it suffices to check the condition of Proposition 4.1.9 for $g=a(\phi)$. But $a(\phi)^{-1}=a(-\phi)=\theta(a(\phi))$, so that Proposition 4.1.9 applies, and $(G,K)=(\mathrm{SU}(2),\mathrm{U}(1))$ is a Gelfand pair.

(ii) Let $G=\mathrm{SL}(2,\mathbb{R})$, and assume that $\theta\colon\mathrm{SL}(2,\mathbb{R})\to\mathrm{SL}(2,\mathbb{R})$ be defined by $\theta(g)=(g^t)^{-1}$. Then we take $K=\mathrm{SO}(2)$, i.e. $K=G^\theta$. We claim that any $g\in\mathrm{SL}(2,\mathbb{R})$ can be written as an element of the form $k_1a(t)k_2$ with $k_1,k_2\in K$ and $a(t)=\begin{pmatrix}\exp(t)&0\\0&\exp(-t)\end{pmatrix}$. We leave the proof to the reader, which is similar to Exercise 3.2.1. As in (i) we now see that $a(t)^{-1} = \theta(a(t))$, so that the condition of Proposition 4.1.9 is satisfied and $(\mathrm{SL}(2,\mathbb{R}), \mathrm{SO}(2))$ is a Gelfand pair.

(iii) Let K be a compact group, such as SU(2) or more generally SU(n), then $G = K \times K$ is equipped with an involutive automorphism $\theta((g_1, g_2)) = (g_2, g_1)$. Then K, diagonally embedded in G, is again the fixed point group of θ, and we can write $G = KAK$ with $A = \{(g, g^{-1}) \mid g \in K\}$. Again we leave the proof of this decomposition to the reader, and in the same way the condition of Proposition 4.1.9 is satisfied and $(K \times K, K)$ is a Gelfand pair. Note that the $G = KAK$-decomposition in this case is not unique.

Actually, all three examples in Example 4.1.10 are examples of so-called symmetric pairs or symmetric spaces, and these are always Gelfand pairs, see e.g. [87, §8.1]. Moreover, in all three examples of Example 4.1.10 the group G is unimodular, i.e. the left Haar measure is also right invariant. This is true in general; if (G, K) is a Gelfand pair, then G is unimodular. This is due to Berg [5, p. 136], see also e.g. [17, §6.1], [21, Prop. I.1], [87, §8.1], and the argument is in Exercise 4.1.11. However, there are also many cases of Gelfand pairs which do not arise as symmetric pairs, see [17].

Proof of Proposition 4.1.9. For a function $f\colon G \to \mathbb{C}$ we define

$$f^\vee(g) = f(g^{-1}), \qquad f^\theta(g) = f(\theta(g)).$$

So that for a bi-K-invariant function f, the condition implies $f^\vee = f^\theta$.

Now we check for general functions (continuous with compact support)

$$f_1^\theta * f_2^\theta(x) = \int_G f_1(\theta(xg)) f_2(\theta(g^{-1}))\, dg = \int_G f_1(\theta(x)\theta(g)) f_2(\theta(g)^{-1})\, dg$$

since θ is an automorphism. Next we observe that the integral $f \mapsto \int_G f(\theta(g))\, dg$ is left invariant, since θ is an automorphism and dg is left invariant. By uniqueness, up to a positive constant, we see that $\int_G f(\theta(g))\, dg = c \int_G f(g)\, dg$. Since θ is an involution, i.e. $\theta^2 = \mathbf{1}$, we see that $c = 1$. So, we find

$$f_1^\theta * f_2^\theta(x) = \int_G f_1(\theta(x)\theta(g)) f_2(\theta(g)^{-1})\, dg = \int_G f_1(\theta(x)g) f_2(g^{-1})\, dg = (f_1 * f_2)^\theta(x)$$

We do the same calculation for composition with the inverse, cf. Exercise (4.1.5),

$$f_1^\vee * f_2^\vee(x) = \int_G f_1^\vee(xg) f_2^\vee(g^{-1})\, dg = \int_G f_1(g^{-1}x^{-1}) f_2(g)\, dg = f_2 * f_1(x^{-1}) = (f_2 * f_1)^\vee(x)$$

Since $f^\vee = f^\theta$ for bi-K-invariant functions, we see that the convolution algebra of bi-K-invariant functions is commutative. □

Exercise 4.1.11. Recall the modular function and its properties in Section 2.4. Assume (G, K) is a Gelfand pair, and we want to show that G is unimodular. We follow the proof of Berg [5].

(i) Show that the modular function $\Delta\colon G \to \mathbb{R}$ is bi-K-invariant. (Hint: compare Exercise 2.4.1.)

(ii) Show that it is sufficient to show that

$$\int_G f(g)\, dg = \int_G f(g^{-1})\, dg$$

for all compactly supported continuous bi-K-invariant functions f. (Hint: use (i) and (2.4.3).)

(iii) Now take f any compactly supported continuous bi-K-invariant function, and pick a compactly supported bi-K-invariant function ψ so that $\psi = 1$ on $\mathrm{supp}(f) \cup \mathrm{supp}(g \mapsto f(g^{-1}))$. Show that $\int_G f(g)\, dg = f * \psi(e)$ and $\int_G f(g^{-1})\, dg = \psi * f(e)$. Show that (G, K) a Gelfand pair implies the unimodularity of G.

Definition 4.1.12. *A spherical function for a Gelfand pair (G, K) is a continuous bi-K-invariant function $\varphi\colon G \to \mathbb{C}$ so that the map*

$$C_c(G/\!/K) \to \mathbb{C}, \qquad f \mapsto \chi(f) = \int_G f(g)\varphi(g^{-1})\, dg$$

is a character of the convolution algebra $C_c(G/\!/K)$, i.e. $\chi(f_1 * f_2) = \chi(f_1)\chi(f_2)$ for all $f_1, f_2 \in C_c(G/\!/K)$.

Note that we don't require φ to be compactly supported. The convolution product is well defined if one of the continuous functions is compactly supported. Note that

$$\chi(f) = \int_G f(g)\varphi(g^{-1})\, dg = f * \varphi(e).$$

Remark 4.1.13. The Banach algebra $L^1(G/\!/K)$ is a commutative Banach algebra, so that any character is a linear functional on $L^1(G/\!/K)$ of norm 1, see e.g. [14, Ch. VII, Cor. 8.3]. In particular, this is a pairing with a $\varphi \in L^\infty(G)$, $\|\varphi\|_\infty = 1$, which we can assume without loss of generality to be bi-K-invariant. So any character χ of $L^1(G/\!/K)$ can be written as $\chi(f) = \int_G f(g)\varphi(g^{-1})\, dg$. Then writing

$$\begin{aligned}\chi(f_1 * f_2) = \int_G (f_1 * f_2)(g)\varphi(g^{-1})\, dg &= \int_G \int_G f_1(gx) f_2(x^{-1})\varphi(g^{-1})\, dg dx \\ &= \int_G \int_G f_1(g^{-1}x^{-1}) f_2(x)\varphi(g)\, dg dx = \int_G f_2(x)(f_1 * \varphi)(x^{-1})\, dx,\end{aligned}$$

using the unimodularity of G, and $\chi(f_1)\chi(f_2) = \chi(f_1)\int_G f_2(x)\varphi(x^{-1})\, dx$, we see that equality gives $(f_1 * \varphi)(x^{-1}) = \chi(f_1)\varphi(x^{-1})$ almost everywhere. Hence $(f_1 * \varphi) = \chi(f_1)\varphi$ a.e., so any character of $L^1(G/\!/K)$ gives rise to a spherical function almost everywhere. If there exists a continuous $f_1 \in L^1(G/\!/K)$ with $\chi(f) \neq 0$, we see that φ is continuous, since a convolution product with a continuous function is continuous.

Example 4.1.14. Take the Gelfand pair $(G, K) = (\mathbb{R}, \{e\})$, then we can take $\varphi(x) = e^{\lambda x}$ and check that this is a spherical function. Indeed, writing the group structure additively and realising that the Haar measure is the Lebesgue measure we get

$$\int_{\mathbb{R}} (f_1 * f_2)(x) e^{-\lambda x}\, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} f_1(x-y) f_2(y) e^{-\lambda x}\, dy\, dx =$$
$$\int_{\mathbb{R}} \int_{\mathbb{R}} f_1(x-y) e^{-\lambda(x-y)}\, dx\, f_2(y) e^{-\lambda y} dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f_1(x) e^{-\lambda x}\, dx\, f_2(y) e^{-\lambda y} dy =$$
$$\int_{\mathbb{R}} f_1(x) e^{-\lambda x}\, dx \int_{\mathbb{R}} f_2(y) e^{-\lambda y} dy$$

So we conclude that $\varphi(x) = e^{\lambda x}$ is a spherical function for the Gelfand pair $(G, K) = (\mathbb{R}, \{e\})$.

Exercise 4.1.15. Let G be an abelian matrix group. Determine the spherical functions for the Gelfand pair $(G, \{e\})$.

Theorem 4.1.16. *Let (G, K) be a Gelfand pair with G unimodular. Let $\varphi\colon G \to \mathbb{C}$ a continuous (non-zero) bi-K-invariant function. Then φ is spherical function if and only if for all $g, h \in G$ we have*

$$\varphi(g)\varphi(h) = \int_K \varphi(gkh)\, dk$$

As before, the integration is with respect to the normalised Haar measure of K. The formula of Theorem 4.1.16 is usually referred to as the *product formula* or as the *product formula for spherical functions*.

In case of Example 4.1.14, Theorem 4.1.16 gives the standard addition formula for the exponential function; $e^{x+y} = e^x e^y$.

Corollary 4.1.17. *Let (G, K) be a Gelfand pair with G unimodular and $\varphi\colon G \to \mathbb{C}$ a (non-zero) spherical function. Then $\varphi(e) = 1$.*

As noted before, the unimodularity of G follows from the Gelfand pair requirement.

Proof. Take $h = e$ and use

$$\varphi(g)\varphi(e) = \int_K \varphi(gk)\, dk = \varphi(g)$$

since φ is right K-invariant. Since $\varphi \neq 0$, there exists a $g \in G$ with $\varphi(g) \neq 0$. Hence, $\varphi(e) = 1$. □

Proof of Theorem 4.1.16. For $f_1, f_2 \in C_c(G)$ and $\varphi \in C(G)$ we have

$$\begin{aligned}
&(f_1 * f_2) * \varphi(e) - (f_1 * \varphi)(e)(f_2 * \varphi)(e) \\
&= \int_G \int_G f_1(gx) f_2(x^{-1}) \varphi(g^{-1})\, dx\, dg - \int_G f_1(g)\varphi(g^{-1})\, dg \int_G f_2(x)\varphi(x^{-1})\, dx \\
&= \int_G \int_G f_1(gx) f_2(x^{-1}) \varphi(g^{-1})\, dx\, dg - \int_G f_1(g)\varphi(g^{-1})\, dg \int_G f_2(x)\varphi(x^{-1})\, dx \\
&= \int_G \int_G f_1(g) f_2(x^{-1}) \varphi(xg^{-1})\, dx\, dg - \int_G f_1(g)\varphi(g^{-1})\, dg \int_G f_2(x)\varphi(x^{-1})\, dx
\end{aligned}$$

Since G is unimodular, we can replace x by x^{-1} in the first integral to get

$$(f_1 * f_2) * \varphi(e) - (f_1 * \varphi)(e)(f_2 * \varphi)(e) = \int_G \int_G f_1(g) f_2(x) \big(\varphi(x^{-1}g^{-1}) - \varphi(g^{-1})\varphi(x^{-1})\big)\, dx\, dg$$

Replacing f_1, f_2 by the projections $Pf_1, Pf_2 \in C_c(G/\!/K)$ we get

$$\begin{aligned}
&(Pf_1 * Pf_2) * \varphi(e) - (Pf_1 * \varphi)(e)(Pf_2 * \varphi)(e) \\
&= \int_G \int_G (Pf_1)(g)(Pf_2)(x) \big(\varphi(x^{-1}g^{-1}) - \varphi(g^{-1})\varphi(x^{-1})\big)\, dx\, dg \\
&= \int_G \int_G \int_K \int_K f_1(k_1 g k_2) \int_K \int_K f_2(k_3 x k_4) \big(\varphi(x^{-1}g^{-1}) - \varphi(g^{-1})\varphi(x^{-1})\big)\, dx\, dg\, dk_1\, dk_2\, dk_3\, dk_4 \\
&= \int_G \int_G f_1(g) f_2(x) \\
&\times \left(\int_K \int_K \int_K \int_K \big(\varphi(k_4 x^{-1} k_3 k_2 g^{-1} k_1) - \varphi(k_2 g^{-1} k_1)\varphi(k_4 x^{-1} k_3)\big)\, dk_1\, dk_2\, dk_3\, dk_4 \right) dx\, dg
\end{aligned}$$

By bi-K-invariance of φ the term in parentheses equals

$$\int_K \int_K \big(\varphi(x^{-1}k_3 k_2 g^{-1}) - \varphi(g^{-1})\varphi(x^{-1})\big)\, dk_2\, dk_3 = \int_K \big(\varphi(x^{-1}kg^{-1}) - \varphi(g^{-1})\varphi(x^{-1})\big)\, dk$$

using the left invariance and normalisation of K.

So φ is a spherical function if and only if $(Pf_1 * Pf_2) * \varphi(e) - (Pf_1 * \phi)(e)(Pf_2 * \varphi)(e) = 0$ for all $f_1, f_2 \in C_c(G)$ if and only if the term in parentheses is zero for all $x, g \in G$, which is equivalent to $\int_K \big(\varphi(x^{-1}kg^{-1}) - \varphi(g^{-1})\varphi(x^{-1})\big)\, dk = 0$ for all $g, x \in G$. This is equivalent to the statement of the theorem. □

4.2 Examples of Gelfand pairs and spherical functions

In this section we present examples of Gelfand pairs and the associated spherical functions, which we can relate to some of the results of Chapter 3. There are many more examples available, see e.g. [17], [21].

4.2.1 SU(2) and Legendre polynomials as spherical functions

Recall that $(\mathrm{SU}(2), \mathrm{U}(1))$ is a Gelfand pair, see Example 4.1.10(i). Then we know by the Peter-Weyl Theorem 3.3.14 that the continuous functions are spanned by the matrix entries of irreducible unitary representations of SU(2). The bi-K-invariant matrix entries arise for irreducible unitary representations π of SU(2) which have U(1)-vectors, i.e.

$$V_\pi^{\mathrm{U}(1)} = \{v \in V_\pi \mid \pi(k)v = v,\ \forall k \in \mathrm{U}(1)\}$$

is non-trivial. We can also rephrase this by saying that the irreducible unitary representation π of SU(2) upon restriction to U(1) contains the trivial representation (of U(1)), i.e. $[\pi|_{\mathrm{U}(1)} \colon 1] > 0$.

Remark 4.2.1. In general, for a linear group G and a compact subgroup K and any representation (π, H_π) of G we can consider π as a representation of K by restriction. The restricted representation $\pi|_K \colon K \to B(H_\pi)$ is in general no longer irreducible even if π is an irreducible G-representation. Since K is a compact group, $\pi|_K$ completely decomposes in terms of irreducible representations, so $\pi|_K \cong \bigoplus_{\sigma\in\hat{K}} n_\sigma \sigma$ where $n_\sigma \in \mathbb{N}$ is the multiplicity of the occurrence of σ in $\pi|_K$. We put $n_\sigma = [\pi|_K \colon \sigma]$. Note that by Schur's Lemma 2.5.7 we have $n_\sigma = \dim \mathrm{Hom}_K(V_\sigma, H_\pi)$, the dimension of the space of K-intertwiners of the representation space V_σ into the G-representation H_π. Restriction and induction of representations, see Section 4.5.2 for an example, are related by Frobenius reciprocity, see e.g. [39], [40].

Indeed, if $v, w \in V_\pi^{\mathrm{U}(1)}$ then the corresponding matrix entry is bi-K-invariant function, since

$$\langle \pi(k_1 g k_2)v, w\rangle = \langle \pi(g)\pi(k_2)v, \pi(k_1^{-1})w\rangle = \langle \pi(g)v, w\rangle.$$

Now by (3.4.12), a non-trivial $\mathrm{U}(1)$-fixed vector occurs in the irreducible unitary representation π_ℓ of spin ℓ if and only $\ell \in \mathbb{N}$, and in that case $\dim_{\mathbb{C}} V_\pi^{\mathrm{U}(1)} = 1$ and spanned by e_0^ℓ. Hence, using the Peter-Weyl Theorem 3.3.14 we see that the space of bi-K-invariant functions on $\mathrm{SU}(2)$ is spanned by $t_{0,0}^\ell \colon \mathrm{SU}(2) \to \mathbb{C}$, $\ell \in \mathbb{N}$.

Lemma 4.2.2. *For $\ell \in \mathbb{N}$ the function $t_{0,0}^\ell \colon \mathrm{SU}(2) \to \mathbb{C}$ is a spherical function for the Gelfand pair $(\mathrm{SU}(2), \mathrm{U}(1))$.*

Proof. By Theorem 3.3.6 we have

$$\begin{aligned} t_{m,n}^{\ell_1} * t_{0,0}^\ell(e) &= \int_G \langle \pi_{\ell_1}(g)e_n^{\ell_1}, e_m^{\ell_1}\rangle \langle \pi_\ell(g^{-1})e_0^\ell, e_0^\ell\rangle\, dg = \int_G \langle \pi_{\ell_1}(g)e_n^{\ell_1}, e_m^{\ell_1}\rangle \overline{\langle \pi_\ell(g)e_0^\ell, e_0^\ell\rangle}\, dg \\ &= \delta_{\ell,\ell_1}\delta_{n,0}\delta_{m,0}\frac{1}{2\ell+1} \end{aligned}$$

so that this gives the candidate for the associated character. In order to see that this is indeed a character we use Exercise 4.1.3, stating that in this special case

$$t_{m_1,n_1}^{\ell_1} * t_{m_2,n_2}^{\ell_2}(s) = \frac{1}{2\ell_1+1}\delta_{\ell_1,\ell_2}\delta_{n_1,m_2} t_{m_1,n_2}^{\ell_1}$$

Now this gives

$$t_{m_1,n_1}^{\ell_1} * t_{m_2,n_2}^{\ell_2} * t_{0,0}^\ell(e) = \frac{1}{2\ell_1+1}\delta_{\ell_1,\ell_2}\delta_{n_1,m_2} t_{m_1,n_2}^{\ell_1} * t_{0,0}^\ell(e) = \frac{\delta_{\ell_1,\ell_2}\delta_{\ell_1,\ell}\delta_{n_1,m_2}\delta_{m_1,0}\delta_{n_2,0}}{(2\ell_1+1)(2\ell+1)}$$

and

$$\big(t_{m_1,n_1}^{\ell_1} * t_{0,0}^\ell(e)\big)\big(t_{m_2,n_2}^{\ell_2} * t_{0,0}^\ell(e)\big) = \frac{\delta_{\ell,\ell_1}\delta_{\ell,\ell_2}\delta_{m_1,0}\delta_{n_1,0}\delta_{m_2,0}\delta_{n_2,0}}{(2\ell+1)^2}$$

and since these expressions are equal, we have established that $t_{0,0}^\ell$ is a spherical function. $\square$

Note that we have proved much more than necessary, since we only need to prove the character property on bi-K-invariant functions. So we could have restricted to the matrix entries for which $m_1 = n_1 = m_2 = n_2 = 0$.

Corollary 4.2.3. *The Legendre polynomials satisfy the product formula*

$$P_\ell(\cos(\theta_1))P_\ell(\cos(\theta_2)) = \frac{1}{2\pi}\int_0^{2\pi} P_\ell\Big(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\cos(\phi)\Big)\, d\phi.$$

Exercise 4.2.4. Give a proof of Corollary 4.2.3 using Lemma 4.2.2, Theorem 4.1.16 and the results of Chapter 3, see Exercise 3.4.25.

Remark 4.2.5. Actually, Corollary 4.2.3 can be obtained immediately from the addition formula of Corollary 3.4.23 by integrating out the constant term. So the product formula can be considered as the constant term in an addition formula.

4.2.2 SU(2) × SU(2) and Chebyshev polynomials as spherical functions

From Example 4.1.10(iii) we know that $(G,K) = (\mathrm{SU}(2)\times\mathrm{SU}(2), \mathrm{SU}(2))$ is a Gelfand pair. We describe the spherical functions as characters of SU(2), giving an alternative approach to Section 3.4.1. Since we have already established the link to Chebyshev polynomials in Section 3.4.1, the treatise here is more sketchy. It is one of the motivations to understand the generalisation to matrix-valued or vector-valued spherical functions.

To see this, we first identify G/K with SU(2);

$$\mathrm{SU}(2)\times\mathrm{SU}(2)/\mathrm{SU}(2) \ni (g_1,g_2)\mathrm{SU}(2) \mapsto g_1g_2^{-1} \in \mathrm{SU}(2).$$

Exercise 4.2.6. Show more generally that for a compact group K the quotient space $K\times K/K$ (again K diagonally embedded) can be identified with K.

So we think of a right SU(2)-invariant function f on $\mathrm{SU}(2)\times\mathrm{SU}(2)$ as a function F on SU(2) by

$$f((g_1,g_2)) = F(g_1g_2^{-1})$$

Now f being left-SU(2)-invariant, then shows

$$f((g,g)(g_1,g_2)) = f((g_1,g_2)) \quad\Longleftrightarrow\quad F(gg_1g_2^{-1}g^{-1}) = F(g_1g_2^{-1}).$$

We conclude that the bi-SU(2)-invariant functions on $\mathrm{SU}(2)\times\mathrm{SU}(2)$ can be identified with the class functions on SU(2). By the Peter-Weyl Theorem 3.3.14 all bi-SU(2)-invariant functions on $\mathrm{SU}(2)\times\mathrm{SU}(2)$ are given by characters. So we can expect by Corollary 3.3.13 that the bi-SU(2)-invariant functions on $\mathrm{SU}(2)\times\mathrm{SU}(2)$ are labeled by $\widehat{\mathrm{SU}(2)}$.

Proposition 4.2.7. *The irreducible unitary representations of* $\mathrm{SU}(2) \times \mathrm{SU}(2)$ *are given by the exterior tensor product of two irreducible unitary representations of* $\mathrm{SU}(2)$*, i.e.*

$$\pi_{\ell_1} \times \pi_{\ell_2}(g_1, g_2)(v_1 \otimes v_2) = (\pi_{\ell_1}(g_1)v_1 \otimes \pi_{\ell_2}(g_2)v_2)$$

for $(g_1, g_2) \in \mathrm{SU}(2)\times\mathrm{SU}(2)$*,* $\pi_{\ell_i} \colon \mathrm{SU}(2) \to B(V_{\ell_i})$*,* $i = 1, 2$*, irreducible unitary representations of* $\mathrm{SU}(2)$*, and* $v_i \in V_{\ell_i}$*,* $i = 1, 2$*.*

Moreover, $\pi_{\ell_1} \times \pi_{\ell_2}$ *has a* $\mathrm{SU}(2)$*-fixed vector (i.e. with respect to the diagonal) if and only if* $\ell_1 = \ell_2 = \ell$*, and in that case* $\dim_{\mathbb{C}}(V_\ell \otimes V_\ell)^{\mathrm{SU}(2)} = 1$*. The corresponding matrix entry for the normalised eigenvector can be identified with the normalised character*

$$G \ni (g_1, g_2) \mapsto \frac{1}{2\ell+1}\chi_\ell(g_1^{-1}g_2).$$

Sketch of proof. We work in a slightly more general setting, so K is a compact group (infinite subgroup of matrices) and we take $G = K \times K$. Note that in particular G is also a compact group.

For any two irreducible unitary representations of K, the exterior product is an irreducible unitary representation of G. To see that any irreducible unitary representation of G is an exterior product requires a bit more work, see [84, Prop. 2.3.7], which we assume for this proof. Now assume that $\pi \times \sigma$ is an arbitrary irreducible unitary representation of G, where $\pi \colon K \to B(V)$, $\sigma \colon K \to B(W)$ are irreducible unitary representations of K. Now define the space $U = \mathrm{Hom}\,(V^*, W)$ of linear maps from the dual of V to W. Then we set $\rho \colon G \to B(U)$ by

$$\rho(g_1, g_2)T = \sigma(g_2) \circ T \circ \pi^*(g_1)^{-1}, \qquad T \colon V^* \to W$$

using the contragredient representation for π. Since σ and π are representations it follows that ρ is a representation of G. We put the Hilbert-Schmidt norm on U, i.e. $\langle T, S\rangle = \mathrm{Tr}_{V^*}(S^*T)$ for $T, S \in U$. This makes U a (finite-dimensional) Hilbert space. Note that there are now several meanings for the notations involving $*$; one for the contragredient representation and one for the adjoint of an operator on a Hilbert space. Being careful with the notation and using that for a unitary representation, the adjoint is just evaluating in the inverse element, we get

$$\begin{aligned}\langle \rho(g_1, g_2)T, \rho(g_1, g_2)S\rangle &= \mathrm{Tr}\left((\sigma(g_2) \circ S \circ \pi^*(g_1)^{-1})^*\sigma(g_2) \circ T \circ \pi^*(g_1)^{-1}\right) \\ &= \mathrm{Tr}\left(\pi^*(g_1)S^*\sigma(g_2^{-1})\sigma(g_2)T\pi^*(g_1)^{-1}\right) = \mathrm{Tr}\left(\pi^*(g_1)S^*T\pi^*(g_1)^{-1}\right) \\ &= \mathrm{Tr}\left(\pi^*(g_1)^{-1}\pi^*(g_1)S^*T\right) = \mathrm{Tr}\left(S^*T\right) = \langle T, S\rangle\end{aligned}$$

and we conclude that ρ is a unitary representation of G.

Define the linear map $B \colon V \otimes W \to U$ by

$$B(v \otimes w) \colon V^* \to W, \qquad B(v \otimes w) \colon \phi \mapsto \phi(v)w$$

We claim that B intertwines the action of G on $V \otimes W$ and U;

$$\begin{aligned}&B(\pi(g_1)v \otimes \sigma(g_2)w) \colon \phi \mapsto \phi(\pi(g_1)v)\sigma(g_2)w \\ &= \sigma(g_2)\big((\pi^*(g_1^{-1})\phi)(v)w\big) = \Big(\sigma(g_2) \circ B(v \otimes w) \circ \pi^*(g_1^{-1})\Big)(\phi)\end{aligned}$$

cf. Exercise 3.3.12. Since $V \otimes W$ is an irreducible unitary representation of the compact group G, and B is non-zero, it follows from Schur's Lemma 2.5.7 (and its proof) that B is injective. Since $\dim_{\mathbb{C}} V \otimes W = \dim_{\mathbb{C}} U$ it follows that B is surjective. So we see that the exterior representation $\pi \times \sigma$ of G is equivalent to the representation ρ of G.

We have gone through this identification in some detail, because it is easier to look at K-fixed vectors in the representation ρ. Indeed, a K-fixed vector in U, means that there exists a non-zero $T\colon V^* \to W$ so that for all $g \in K$ we have

$$T = \rho(g,g)T = \sigma(g) \circ T \circ \pi^*(g)^{-1}$$

or $T \in \mathrm{Hom}_K(V^*, W)$. Since W and V, hence V^*, are irreducible unitary representations of K we see that V^* and W are equivalent by Schur's Lemma 2.5.7. So without loss of generality we can take $W = V^*$ and $\sigma = \pi^*$. In that case $\mathrm{Hom}_K(W,W)$ is spanned by the identity $\mathbf{1}\colon W \to W$. Taking the corresponding matrix entry

$$\langle \rho(g_1,g_2)\mathbf{1}, \mathbf{1}\rangle = \mathrm{Tr}_W(\mathbf{1}^*\sigma(g_2)\mathbf{1}\sigma(g_1^{-1})) = \mathrm{Tr}_W(\sigma(g_2 g_1^{-1})) = \chi_\sigma(g_2 g_1^{-1}).$$

In particular, $\langle \rho(e,e)\mathbf{1}, \mathbf{1}\rangle = \chi_\sigma(e) = \dim_{\mathbb{C}} W$, so that normalising we get the result.

Specialising to $K = \mathrm{SU}(2)$ and noting that $(\pi_\ell)^* \cong \pi_\ell$ by Corollary 3.4.2 gives the result. □

The proof of Proposition 4.2.7 has been written down in the more general situation in order not to obscure the role of the contragredient representation.

Now that we know that the bi-K-invariant functions on $G = K \times K$ can identified with the class functions on K, we can consider the convolution product of two characters on G using the result of Exercise 4.1.3. Indeed, for two irreducible unitary representations $\pi, \sigma \in \hat{K}$ we consider

$$\begin{aligned}
(\chi_\pi *_G \chi_\sigma)((s_1,s_2)) &= \int_K\int_K \chi_\pi((s_1g_1, s_2g_2))\, \chi_\sigma((g_1^{-1}, g_2^{-1}))\, dg_1 dg_2 \\
&= \int_K\int_K \chi_\pi(s_1g_1g_2^{-1}s_2^{-1})\, \chi_\sigma(g_1^{-1}g_2)\, dg_1dg_2 = \int_K\int_K \chi_\pi(s_2^{-1}s_1g_1)\, \chi_\sigma(g_2^{-1}g_1^{-1}g_2)\, dg_1dg_2 \\
&= \int_K\int_K \chi_\pi(s_2^{-1}s_1g_1)\, \chi_\sigma(g_1^{-1})\, dg_1dg_2 = \int_K \chi_\pi(s_2^{-1}s_1g)\, \chi_\sigma(g^{-1})\, dg \\
&= \frac{\delta_{\pi,\sigma}}{\dim\pi}\chi_\pi(s_2^{-1}s_1) = \frac{\delta_{\pi,\sigma}}{\dim\pi}\chi_\pi((s_1,s_2)),
\end{aligned}$$

where we use a subscript on the convolution product to keep track on which group we take the convolution product, we view the characters both as functions on G and on K. Now we have Lemma 4.2.8, which can be proved as Lemma 4.2.2.

Lemma 4.2.8. *For K a compact matrix group, and the corresponding Gelfand pair (G,K) with $G = K \times K$ and $K \cong$ diag, the spherical functions are labeled by $\hat{K}$. The explicit expression is*

$$\varphi_\pi\colon G \to \mathbb{C}, \qquad \varphi_\pi((g_1,g_2)) = \frac{1}{\dim\pi}\chi_\pi(g_1g_2^{-1})$$

where the right hand side denotes the character χ_π on K.

Now that we have identified the spherical functions with characters, we can relate the product formula for spherical functions to an identity for characters of K.

Proposition 4.2.9. *For an irreducible representation of K, the corresponding character χ satisfies*

$$\chi(g)\chi(h) = \chi(e)\int_K \chi(gkhk^{-1})\,dk$$

In particular, for $\ell \in \frac{1}{2}\mathbb{N}$,

$$U_{2\ell}(\cos\xi)U_{2\ell}(\cos\eta) = (2\ell+1)\int_0^1 U_{2\ell}\big(\cos(\xi)\cos(\eta) + (2u-1)\sin(\xi)\sin(\eta)\big)\,du.$$

Note that the product formula for the Chebyshev polynomials of the second kind can be rewritten as

$$U_n(x)U_n(y) = \frac{n+1}{2}\int_{-1}^1 U_n\big(xy + u\sqrt{1-x^2}\sqrt{1-y^2}\big)\,du. \tag{4.2.1}$$

Proof. We identify the characters with spherical functions on G as in Lemma 4.2.8, and use the product formula of Theorem 4.1.16. We denote the dimension of the representation by $d = \chi(e)$. This gives

$$d^{-2}\chi(g_1g_2^{-1})\chi(h_1h_2^{-1}) = \varphi((g_1,g_2))\varphi((h_1,h_2)) = \int_K \varphi\big((g_1,g_2)(k,k)(h_1,h_2)\big)\,dk =$$
$$\int_K \varphi\big((g_1kh_1, g_2kh_2)\big)\,dk = d^{-1}\int_K \chi\big(g_1kh_1h_2^{-1}k^{-1}g_2^{-1})\big)\,dk = d^{-1}\int_K \chi\big(g_2^{-1}g_1kh_1h_2^{-1}k^{-1})\big)\,dk$$

and use $\chi(g_1g_2^{-1}) = \chi(g_2^{-1}g_1)$, and replace $g_2^{-1}g_1$ by g and $h_1h_2^{-1}$ by h and multiply by d^2.

Now we take $K = \mathrm{SU}(2)$, $\ell \in \frac{1}{2}\mathbb{N} \cong \hat{K}$, and we take

$$g = \begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi}\end{pmatrix}, \qquad h = \begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta}\end{pmatrix},$$

so that the left hand side corresponds to $U_{2\ell}(\cos\xi)U_{2\ell}(\cos\eta)$. It remains to calculate the right hand side. For this, observe that the character is a class function, so that for $g \in \mathrm{SU}(2)$ it only depends on the eigenvalues, and then the argument of the Chebyshev polynomial is just half the trace of g; $\chi_\ell(g) = U_{2\ell}(\frac{1}{2}\mathrm{Tr}\,g)$. Now

$$\frac{1}{2}\mathrm{Tr}\,(\begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi}\end{pmatrix}\begin{pmatrix} \alpha & -\overline{\gamma} \\ \gamma & \overline{\alpha}\end{pmatrix}\begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{-i\eta}\end{pmatrix}\begin{pmatrix} \overline{\alpha} & \overline{\gamma} \\ -\gamma & \alpha\end{pmatrix}) = \cos(\xi+\eta)|\alpha|^2 + \cos(\xi-\eta)|\gamma|^2$$

Now we take the integration according to (2.4.6), which gives for the right hand side

$$
\begin{aligned}
&(2\ell+1)\int_{\mathrm{SU}(2)} \chi_\ell(gkhk^{-1})\,dk\\
&=(2\ell+1)\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} U_{2\ell}\big(\cos(\xi+\eta)\sin^2\theta+\cos(\xi-\eta)\cos^2\theta\big)\cos\theta\,\sin\theta\,d\theta\,d\phi\,d\psi\\
&=2(2\ell+1)\int_0^{\frac{\pi}{2}} U_{2\ell}\big(\cos(\xi+\eta)\sin^2\theta+\cos(\xi-\eta)\cos^2\theta\big)\cos\theta\,\sin\theta\,d\theta\\
&=(2\ell+1)\int_0^1 U_{2\ell}\big(\cos(\xi)\cos(\eta)+(2u-1)\sin(\xi)\sin(\eta)\big)\,du
\end{aligned}
$$

by putting $u=\cos^2\theta$ and using the addition formulas for the cosine. □

4.3 Positive definite functions and spherical functions

We have observed in the examples in Section 4.2 that the spherical functions are written as matrix entries with respect to K-fixed vectors. In this section we study more general matrix entries with respect to K-fixed vectors, where such matrix entries are seen as positive definite functions. To positive definite functions we assign unitary representations for which they arise as matrix entries. Note that in this section we don't require (G,K) to be a Gelfand pair.

Definition 4.3.1. *A function $f\colon G\to\mathbb{C}$ is a positive definite function, or a function of positive type, if for all $N\in\mathbb{N}$ and for all $(g_i)_{i=1}^N$ in G and for all $(c_i)_{i=1}^N$ in $\mathbb{C}$ we have*

$$\sum_{i,j=1}^N f(g_i^{-1}g_j)c_i\overline{c_j}\geq 0.$$

Note that Definition 4.3.1 means that the matrix $\big(f(g_i^{-1}g_j)\big)_{i,j=1}^N$ is positive definite, and in particular self-adjoint.

Exercise 4.3.2. (i) Show that for f a positive definite function on G we have $f(g^{-1})=\overline{f(g)}$ and $|f(g)|\leq f(e)$.

(ii) Let $\pi\colon G\to B(V)$ be a unitary (not necessarily finite dimensional) representation of G. Take $v\in V$ and show that $g\mapsto\langle\pi(g)v,v\rangle$ is a positive definite function.

Now we are interested in bi-K-invariant functions, and we look at positive definite bi-K-invariant functions. First we show that for a general class, the example of Exercise 4.3.2(ii) is generic.

Proposition 4.3.3. *Let $\varphi\colon G\to\mathbb{C}$ be a non-zero continuous, bi-K-invariant, positive definite function. Then there exists a unitary representation $\pi_\varphi\colon G\to B(H_\varphi)$, H_φ a Hilbert space and a vector $u\in H_\varphi^K=\{v\in H_\varphi\mid \pi_\varphi(k)v=v,\forall k\in K\}$ so that $\varphi(g)=\langle\pi_\varphi(g)u,u\rangle$. Moreover, u is a cyclic vector, i.e. the closure of the linear span of elements from $\pi_\varphi(G)u$ equals H_φ.*

The construction is explicit, and resembles the classical GNS-construction in C*-algebras, where GNS stands for Gelfand-Naimark-Segal, see e.g. [14, Ch. VII, 5.14].

Proof. We define V_φ as the space of finite linear combinations of actions of the left regular representation acting on φ, i.e. V_φ consists of functions of the form $g \mapsto \sum_{i=1}^N c_i \varphi(g_i^{-1}g)$ for any $N \in \mathbb{N}$, with $g_i \in G$ and $c_i \in \mathbb{C}$. Then V_φ is a vector space, and we equip V_φ with the sesquilinear form $\langle \cdot, \cdot \rangle \colon V_\varphi \times V_\varphi \to \mathbb{C}$ given by

$$\langle \sum_{i=1}^N c_i \varphi(g_i^{-1}g), \sum_{j=1}^M d_j \varphi(h_j^{-1}g) \rangle = \sum_{i=1}^N \sum_{j=1}^M c_i \overline{d_j} \varphi(g_i^{-1}h_j).$$

Note that the Cauchy-Schwarz inequality also holds in this setting, and it follows that the space N_φ of elements $v \in V_\varphi$ with $\langle v, v \rangle = 0$ forms a linear subspace. Hence V_φ / N_φ is a pre-Hilbert space and we denote by H_φ its completion.

The action by the left regular representation

$$\lambda(h)\big(\sum_{i=1}^N c_i \varphi(g_i^{-1}g)\big) = \sum_{i=1}^N c_i \varphi(g_i^{-1}h^{-1}g) = \sum_{i=1}^N c_i \varphi((hg_i)^{-1}g)$$

naturally preserves V_φ, and moreover $\langle \lambda(h)v, \lambda(h)w \rangle = \langle v, w \rangle$ by construction for $v, w \in V_\varphi$. So, in particular, N_φ is preserved by $\lambda(h)$, and so the action descends to a homomorphism by unitary operators on H_φ which is denoted by π_φ.

Next note that $\varphi \in V_\varphi$ and that $\langle \varphi, \varphi \rangle = \varphi(e) > 0$, since otherwise $\varphi = 0$ by Exercise 4.3.2(i). Let u be its image in H_φ, then u is cyclic by construction. □

Exercise 4.3.4. Prove the remaining statements of the proof of Proposition 4.3.3, i.e. the realisation of φ as a matrix entry and the continuity of the representation π_φ. (For the last statement you can use that the left regular action also is continuous on other $L^p(G)$ spaces, $1 \leq p < \infty$.)

Proposition 4.3.5. *Let $\varphi \colon G \to \mathbb{C}$ be a non-zero continuous, bi-K-invariant, positive definite function. Assume that $\pi \colon G \to B(H)$ is a representation with a cyclic K-invariant vector $\xi \in H^K$ so that $\varphi(g) = \langle \pi(g)\xi, \xi \rangle$. Then there exists $T \in B_G(H_\varphi, H)$ which is an isometric isomorphism with $Tu = \xi$, with the notation as in Proposition 4.3.3.*

Exercise 4.3.6. In this exercise we sketch the proof of Proposition 4.3.5. We construct T first on V_φ.

(i) For $v \colon g \mapsto \sum_{i=1}^N c_i \varphi(g_i^{-1}g)$, $v \in V_\varphi$ define $Tv = \sum_{i=1}^N c_i \pi(g_i)\xi$. Show that T is linear on V_φ, and that $\langle Tv, Tw \rangle = \langle v, w \rangle$.

(ii) Show that T descends to an isometry $T \colon H_\varphi \to H$. Show that T is surjective, and conclude that T is an isometric isomorphism.

(iii) Show that T is an intertwiner.

Definition 4.3.7. *Let $P^K(G)$ denote the set of continuous, bi-K-invariant, positive definite functions on G.*

Exercise 4.3.8. Show that $P^K(G)$ is a convex cone, i.e. closed under addition and multiplication by a non-negative (real) scalar. For non-zero $\varphi, \psi \in P^K(G)$ and $\lambda > 0$ discuss the relation between the representations (π_φ, H_φ), (π_ψ, H_ψ), $(\pi_{\varphi+\psi}, H_{\varphi+\psi})$ and $(\pi_{\lambda\varphi}, H_{\lambda\varphi})$.

Definition 4.3.9. *A non-zero element $\varphi \in P^K(G)$ is called extremal if any decomposition of the form $\varphi = \psi_1 + \psi_2$ with $\psi_1, \psi_2 \in P^K(G)$ implies that ψ_1 and ψ_2 are non-negative multiples of φ.*

Theorem 4.3.10. *Let $\varphi\colon G \to \mathbb{C}$ be a non-zero element of $P^K(G)$. The representation (π_φ, H_φ) of G is irreducible if and only if $\varphi \in P^K(G)$ is extremal.*

In the proof given below, there are some brief statements for which Exercise 4.3.11 provides some more information.

Proof. Assume (π_φ, H_φ) is irreducible, and write $\varphi = \psi_1 + \psi_2$, $\psi_1, \psi_2 \in P^K(G)$. We have to show that ψ_i is a multiple of φ. For elements $v(g) = \sum_{i=1}^N c_i\varphi(g_i^{-1}g)$, $w(g) = \sum_{j=1}^M d_j\varphi(h_j^{-1}g)$ of V_φ we define the sesquilinear form $B(v,w) = \sum_{i=1}^N \sum_{j=1}^M c_i\overline{d_j}\psi_1(g_i^{-1}h_j)$, then it follows $|B(v,w)| \leq \|v\|\|w\|$. In particular, B descends to a sesquilinear form on H_φ with the same estimate. Moreover, B is invariant for the action of π_φ, i.e. $B(\pi_\varphi(h)v, \pi_\varphi(h)w) = B(v,w)$ for all $v, w \in H_\varphi$. Since B is sesquilinear form with $|B(v,w)| \leq \|v\|\|w\|$ for all $v, w \in H_\varphi$, it follows that there exists a positive (hence self-adjoint) bounded operator $T\colon H_\varphi \to H_\varphi$ such that $B(v,w) = \langle Tv, w\rangle$. Then T is an intertwiner. By the generalisation of Schur's Lemma as in Lemma 2.5.10, we conclude that $T = \lambda\mathbf{1}$, with $\lambda \geq 0$, since π_φ is irreducible. Plugging this back in the definition of B we find $\psi_1 = \lambda\varphi$, hence φ is extremal.

Conversely, assume that φ is an extremal point of $P^K(G)$. We need to show that H_φ is an irreducible representation. Assume $H_\varphi = H_1 \oplus H_2$ is the direct sum of invariant closed subspaces. Then we have orthogonal projections $P_1, P_2 \in B_G(H_\varphi)$ on H_1, H_2 with $P_i^2 = P_i = P_i^*$, $P_1 + P_2 = \mathbf{1}$, $P_1P_2 = P_2P_1 = 0$. Put $u_i = P_iu$ and $\psi_i(g) = \langle \pi_\varphi(g)u_i, u_i\rangle$, then $\psi_i \in P^K(G)$ and $\varphi = \psi_1 + \psi_2$. Since φ is extremal, $\psi_1 = \lambda\varphi$ so that $\langle \pi(g)u, u_1 - \lambda u\rangle = 0$ as $\lambda \geq 0$. Since u is a cyclic vector, we see $u_1 = \lambda u$. Since P_1 is a projection we have $\lambda = 0$ or $\lambda = 1$, and consequently $H_1 = \{0\}$ or $H_1 = H_\varphi$. So H_φ is an irreducible representation for G. □

Exercise 4.3.11. In this exercise some more details and verifications of the proof of Theorem 4.3.10 have to be filled in.

(i) Give an explicit proof of $|B(v,w)| \leq \|v\|\|w\|$ on V_φ and also on H_φ.

(ii) Show that $B(\pi_\varphi(h)v, \pi_\varphi(h)w) = B(v,w)$ for all $v, w \in H_\varphi$ and for all $h \in G$.

(iii) Look up the existence of such a positive bounded operator T in a book on functional analysis. Show that T is an intertwiner.

(iv) Show that P_1 and P_2 are intertwiners for the action of G.

(v) Show that $\varphi = \psi_1 + \psi_2$ by first showing $\psi_i(g) = \langle \pi_\varphi(g)u, u_i\rangle$.

(vi) Show that $\lambda = 0$ or $\lambda = 1$ and prove that either $H_1 = \{0\}$ or $H_1 = H_\varphi$.

We can now prove a relatively simple statement on the irreducibility of such G-representations. The proof of the statement of Theorem 4.3.12 is sketched in Exercise 4.3.13

Theorem 4.3.12. *Let (π, H) be a unitary representation of G with a cyclic K-fixed vector u. If $\dim_{\mathbb{C}} H^K = 1$, then (π, H) is irreducible.*

Exercise 4.3.13. The proof of Theorem 4.3.12 follows the ideas of the proof of Theorem 4.3.10. Let $V \subset H$ be a closed invariant subspace of H.

(i) Let P be the orthonormal projection on V, and put $v = Pu$. Show that $v \in H^K$. Conclude that $v = \lambda u$ with $\lambda = 0$ or $\lambda = 1$.

(ii) Show that if $v = 0$, then $V = \{0\}$ using the cyclicity of u.

(iii) Show that if $v = u$, then $V = H$.

4.4 Characterisation of Gelfand pairs and the spherical transform

In this section we sketch (or skip) the proofs, since it requires several techniques from functional analysis about cyclic representations, the GNS-construction and the Gelfand transform for commutative Banach algebras. The Banach algebra is the commutative Banach algebra $L^1(G/\!/K)$ of bi-K-invariant integrable functions on G. We allow for non-compact groups G and infinite-dimensional representations of G, which are assumed to be unitary.

Theorem 4.4.1. *Let (G, K) be a pair with G a closed (infinite) subgroup of the real or complex matrices and compact subgroup K. Then (G, K) is a Gelfand pair if and only if for any irreducible unitary representation $\pi\colon G \to B(H)$, the space of K-invariant elements is at most one-dimensional, i.e. $H^K = \{v \in H \mid \pi(k)v = v \,\forall\, k \in K\}$ has $\dim_{\mathbb{C}} H^K \leq 1$.*

For a proof of Theorem 4.4.1 we refer to Faraut [21] and van Dijk [17, Prop. 6.3.1]. An approach to the proof of Theorem 4.4.1 is sketched in Exercise 4.4.2. Essentially, we already know that $\dim_{\mathbb{C}} H^K$ cannot be greater than 1 by Theorem 4.3.12. We can rephrase this as follows: for a Gelfand pair (G, K) the restriction of any irreducible unitary representation of G to the compact subgroup K contains the trivial representation of K at most once. It turns out that this condition is equivalent to (G, K) being a Gelfand pair, see [17], [21].

Exercise 4.4.2. We give a sketch of the proof of one implication in Theorem 4.4.1. So given (G, K) is a Gelfand pair and an irreducible unitary representation (π, H) of G. We need to prove that H^K is at most one-dimensional. We use representations of Banach $*$-algebras, see e.g. [14, Ch. VII, §8].

(i) Recall from Exercises 4.1.2, 4.1.5 and Lemma 4.1.4 that $L^1(G)$ is a Banach $*$-algebra. Deduce that $L^1(G/\!/K)$ is also a Banach $*$-algebra and that for $f \in L^1(G/\!/K)$ the operator $\pi(f)$ preserves H^K.

(ii) So we have $\pi\colon L^1(G/\!/K) \to B(H^K)$. Consider $V \subset H^K$ a closed invariant subspace for this action. Conclude that $V^\perp \subset H^K$ is also an invariant subspace for π as a representation of $L^1(G/\!/K)$. Show that $H_1 = \{\pi(f)v \mid v \in V, f \in L^1(G)\}$ is an invariant subspace for the action of G. Prove that $V^\perp \subset H_1^\perp$.

(iii) Note that H_1 is dense in H, and conclude $V^\perp = \{0\}$. Conclude that $L^1(G/\!/K)$ acts irreducibly on H^K.

(iv) Use the fact that irreducible representations of commutative Banach $*$-algebras are one-dimensional to finish the proof.

We denote by $\hat{G}_{\mathrm{sph}} \subset \hat{G}$ the space of irreducible unitary representations π in a Hilbert space H_π (up to equivalence) for which the space of K-invariant vectors is one-dimensional, i.e. $\dim_{\mathbb{C}} H_\pi^K = 1$. The irreducible unitary representations of $\hat{G}_{\mathrm{sph}}$ are known as the *spherical representations* for the Gelfand pair (G, K).

Exercise 4.4.3. Show that in the case of Sections 4.2.1 the spherical dual is $\widehat{\mathrm{SU}(2)}_{\mathrm{sph}} = \{\ell \in \frac{1}{2}\mathbb{N} \mid \ell \in \mathbb{N}\}$. Show that in case of the example in Section 4.2.2 the spherical representations are given by the pairs $(\ell, \ell) \in \frac{1}{2}\mathbb{N} \times \frac{1}{2}\mathbb{N}$ for $\ell \in \frac{1}{2}\mathbb{N}$. Note that in both cases $\hat{G}_{\mathrm{sph}}$ is discrete, since $\hat{G}$ is discrete.

Exercise 4.4.4. Assume $\varphi\colon G \to \mathbb{C}$ is a non-zero continuous positive definite function. Then φ is a spherical function for the Gelfand pair (G, K) if and only φ is an extremal element of $P^K(G)$ and $\varphi(e) = 1$. In this exercise we prove one implication of this statement. We use the notation as in Proposition 4.3.3.

(i) Assume φ is a non-zero spherical function, then $\varphi(e) = 1$ by Corollary 4.1.17. Show that for $f \in L^1(G/\!/K)$ we have $\pi(f)u = \chi(f)u$, with u as Proposition 4.3.3. Prove this by showing that in H_φ we have

$$\langle \pi(f)u, \pi(g)u\rangle = \langle \chi(f)u, \pi(g)u\rangle, \qquad \forall\, g \in G.$$

Next use that u is a cyclic vector.

(ii) Keeping the assumption of (i), let $Q\colon H_\varphi \to H_\varphi$ be the orthogonal projection on the subspace of K-fixed vectors H_φ^K. Show that

$$Qv = \int_K \pi(k)\, v\, dk$$

Now take $v = \pi(f)u$ for $f \in L^1(G)$. Show that $Q\pi(f)u = \pi(Pf)u$, with P as in (4.1.3). Use (i) to conclude that H_φ^K is one-dimensional and use Theorem 4.3.12 and Theorem 4.3.10.

In case $\pi \in \hat{G}_{\mathrm{sph}}$ acting in the Hilbert space H we take $v \in H^K$, $\|v\| = 1$, and define the spherical function

$$\varphi_\pi \colon G \to \mathbb{C}, \qquad \varphi_\pi(g) = \langle \pi(g)v, v\rangle \tag{4.4.1}$$

arising as a matrix entry of the irreducible unitary representation $\pi \in \hat{G}_{\mathrm{sph}}$. Note that the function φ_π in (4.4.1) is independent of the choice of v. It turns out that these are indeed spherical functions in the sense of Definition 4.1.12.

Then there is an analogue of the Plancherel Theorem 3.3.16 for the spherical functions, as follows.

Theorem 4.4.5. *The space $\hat{G}_{\mathrm{sph}}$ is a measure space with measure m such that the mapping*

$$\mathcal{F} \colon C_c(G/\!/K) \to L^2(\hat{G}_{\mathrm{sph}}, m), \qquad f \mapsto \mathcal{F}(f)(\pi) = \int_G f(g)\varphi_\pi(g^{-1})\, dg$$

extends to an isometric isomorphism $\mathcal{F} \colon L^2(G/\!/K) \to L^2(\hat{G}_{\mathrm{sph}}, m)$ with inversion formula

$$f(x) = \int_{\hat{G}_{\mathrm{sph}}} \mathcal{F}(f)(\pi)\, \varphi_\pi(x)\, dm(\pi)$$

See [21, §IV], [17, §6.4] for a proof of Theorem 4.4.5.

Note that Theorem 4.4.1 and 4.4.5 are abstractly established theorems, and that it is not straightforward to determine the spherical functions and the Plancherel measure. For compact Gelfand pairs, Theorem 4.4.5 can be derived from Theorem 3.3.16 restricted to bi-K-invariant functions. So in that case, the space $\hat{G}_{\mathrm{sph}}$ is discrete, and the Plancherel measure is a discrete measure.

Exercise 4.4.6. Assume (G, K) is a compact Gelfand pair, i.e. G is compact. Use the Peter-Weyl Theorem 3.3.14 to make Theorem 4.4.5 explicit.

Essentially, in explicit cases the problem is to determine the spherical functions φ_π and the Plancherel measure m on $\hat{G}_{\mathrm{sph}}$ explicitly, see e.g. [17], [21] for more examples.

4.5 The case of $\mathrm{SL}(2, \mathbb{R})$ and differential operator

The group $\mathrm{SL}(2, \mathbb{R})$ is a non-compact matrix group, and its unitary representations have been classified. We consider the subgroup $\mathrm{SO}(2)$, see Example 4.1.10(ii), which is a compact subgroup, and the pair $(\mathrm{SL}(2, \mathbb{R}), \mathrm{SO}(2))$ is a Gelfand pair.

In this section we want to show how to use a differential operator to find the spherical functions. In order to do so rigourously would be hard in the short span of these notes, so we will give the main steps and only indicate analytic difficulties.

First we observe that we have a Lie algebra corresponding to $\mathrm{SL}(2, \mathbb{R})$ which is

$$\mathfrak{sl}(2, \mathbb{R}) = \{X \in M_2(\mathbb{R}) \mid \mathrm{Tr}\ X = 0\}$$

the Lie algebra of traceless matrices. We still have the exponential map $\exp\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$, but it is no longer surjective. The Lie algebra is a real Lie algebra, and $\dim_{\mathbb{R}}\mathfrak{sl}(2,\mathbb{R}) = 3$ and we can take as a basis

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.5.1}$$

By a straightforward calculation we have the commutators

$$[H,X] = 2Y - X, \quad [H,Y] = Y, \quad [X,Y] = 2H. \tag{4.5.2}$$

The complexification of $\mathfrak{sl}(2,\mathbb{R})$ is $\mathfrak{sl}(2,\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{sl}(2,\mathbb{R})$, which is the Lie algebra of complex traceless 2×2-matrices.

Exercise 4.5.1. Show that the complexification of $\mathfrak{su}(2)$, see Appendix A.1, is $\mathfrak{sl}(2,\mathbb{C})$, i.e. $\mathfrak{sl}(2,\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{su}(2)$.

Note that the exponential map of X, H and Y generate subgroups of $\mathrm{SL}(2,\mathbb{R})$. We denote these subgroups

$$\begin{aligned} K &= \mathrm{SO}(2) = \left\{ k(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \mid t \in [0,2\pi) \right\}, \\ A &= \left\{ a(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t > 0 \right\}, \\ N &= \left\{ n(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}. \end{aligned}$$

The notation is to reflect that K is a compact subgroup, A is an abelian subgroup and N is the exponential of a nilpotent element. The multiplication map

$$K \times A \times N \to G, \quad (k,a,n) \mapsto g = kan \tag{4.5.3}$$

is a surjective diffeomorphism, see e.g. [40, §VI.6]. This is known as the *Iwasawa decomposition.* For this particular matrix group, this a reformulation of the QR-factorisation of a real 2×2-matrix.

Exercise 4.5.2. To prove the Iwasawa decomposition, take

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$$

and compare with

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t\cos\phi & xt\cos\phi - t^{-1}\sin\phi \\ t\sin\phi & xt\sin\phi + t^{-1}\cos\phi \end{pmatrix}$$

Show that (a,c) in polar coordinates $(t\cos\phi, t\sin\phi)$ and $x = \frac{ab+cd}{a^2+c^2}$ gives the required decomposition.

Exercise 4.5.3. Let $C = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$. Let $\mathrm{SU}(1,1)$ be the group of 2×2-complex matrices of determinant one preserving the inner product with signature $(1,1)$. Show that

$$\mathrm{SU}(1,1) = \left\{ g = g(\alpha,\gamma) = \begin{pmatrix} \alpha & \overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\gamma|^2 = 1 \right\}$$

and prove that $C\mathrm{SL}(2,\mathbb{R})C^{-1} = \mathrm{SU}(1,1)$. Determine the corresponding subgroups of the Iwasawa decomposition of Exercise 4.5.2 and show that in this case

$$K = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \mid t \in \mathbb{R} \right\}, \qquad A = \left\{ \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

We extend the definition of Remark 3.2.6 to C^∞-functions on $G = \mathrm{SL}(2,\mathbb{R})$, so we define

$$(X \cdot f)(g) = \frac{d}{dt}\Big|_{t=0} f(g \exp(tX)), \qquad f\colon \mathrm{SL}(2,\mathbb{R}) \to \mathbb{C}, \quad X \in \mathfrak{sl}(2,\mathbb{R}) \tag{4.5.4}$$

realising $X \in \mathfrak{sl}(2,\mathbb{R})$ as a first order differential operator on f, which commutes with the left regular representation. In particular, we want to understand these first-order differential operators (or left invariant vector fields) acting on functions which are matrix entries of representations of G. Then we can show that

$$(X \cdot (Y \cdot f))(g) - (Y \cdot (X \cdot f))(g) = ([X,Y] \cdot f)(g)$$

Hence, it gives a representation of the Lie algebra $\mathfrak{g}$, and by complexification also of $\mathfrak{g}_\mathbb{C}$, see Appendix A.2. Hence, we see that we get the action of the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ as differential operators of C^∞-functions on G.

4.5.1 The derived representation and the Casimir operator

We will generally discuss the case of G a (infinite) closed subgroup of the $n\times n$-matrices over $\mathbb{R}$ or $\mathbb{C}$. We assume that $\pi\colon G \to B(H)$ is a unitary representation of G in a Hilbert space H. The case to be kept in mind is $G = \mathrm{SL}(2,\mathbb{R})$. Let $\mathfrak{g}$ be the corresponding Lie algebra, which we can view as a linear subspace of $n\times n$-matrices over $\mathbb{R}$ or $\mathbb{C}$, so that $\exp\colon \mathfrak{g} \to G$ is a local diffeomorphism. We assume that G and $\mathfrak{g}$ are such that $\mathfrak{g}$ is a real Lie algebra, and by $\mathfrak{g}_\mathbb{C}$ we denote its complexification which is a complex Lie algebra. In the special case to be kept in mind, we have $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{g}_\mathbb{C} = \mathfrak{sl}(2,\mathbb{C})$. Recall, see Appendix A, the notation of a Lie algebra representation.

Let $\pi\colon G \to B(H)$ be a unitary representation of G in the Hilbert space H, then $d\pi(X)$, $X \in \mathfrak{g}$, defined by

$$\langle d\pi(X)v, w\rangle = \frac{d}{dt}\Big|_{t=0} \langle \pi(\exp(tX))v, w\rangle$$

gives a representation of the Lie algebra $\mathfrak{g}$ on the Hilbert space H. In case H is an infinite dimensional representation, this is an analytic issue and care needs to be taken since the operators $d\pi(X)$ are generally unbounded. The solution is to restrict the vectors v, w to suitable dense subspaces of H, known as C^∞-vectors or vectors from the Gårding subspace, see e.g. [39]. To see that

$$[d\pi(X), d\pi(Y)] = d\pi([X,Y])$$

we need the Campbell-Baker-Hausdorff formula, see e.g. [7], [36], [40]. Then we also get a representation of the corresponding complexification $\mathfrak{g}_\mathbb{C}$ of the Lie algebra $\mathfrak{g}$. Then the representation $d\pi\colon \mathfrak{g}_\mathbb{C} \to B(H)$ gives a $*$-representation of $\mathfrak{g}_\mathbb{C}$ as in Appendix A.

Using the fact that a representation of such a matrix group G in a Hilbert space gives a representation of the Lie algebra $\mathfrak{g}_\mathbb{C}$, we also find a representation, also denoted $d\pi$ of $U(\mathfrak{g}_\mathbb{C})$ on the Hilbert space. Again, there are analytic difficulties in case H is an infinite-dimensional Hilbert space.

Let us now assume that the representation of G in H is irreducible. Now, by an analogue of Schur's Lemma 2.5.7, we can assume that the elements of the centre $Z(U(\mathfrak{g}_\mathbb{C}))$ then act as a scalar. This is true if the Hilbert space is finite-dimensional, see Lemma 2.5.7 and its extension to the Hilbert space setting in Lemma 2.5.10. If we now assume that the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ has a Casimir element Ω, see Appendix A, then we, under certain conditions, can assume that $d\pi(\Omega) = \gamma(\pi)\mathbf{1}$ for a constant $\gamma(\pi) \in \mathbb{C}$ depending on π.

Proposition 4.5.4. *Let $\pi\colon G \to B(H)$ be a unitary representation of G, then for suitable $v, w \in H$ the matrix entry $\pi_{w,v}\colon g \mapsto \langle \pi(g)v, w\rangle$ is an eigenfunction for the Casimir operator;*

$$\Omega \cdot \pi_{w,v} = \gamma(\pi)\, \pi_{w,v}.$$

The Casimir element $\Omega \in Z(U(\mathfrak{g}_\mathbb{C}))$ generally gives a 2nd order partial differential operator. Note that Proposition 4.5.4 states that the matrix entries are eigenfunctions to this 2nd order partial differential operator, which usually is an elliptic PDE. Then regularity theory for elliptic partial differential operators gives results on such matrix entries, see e.g. [10], [39]. One can show that the suitable vectors actually span a dense subspace.

Very sketchy proof. Take $X \in \mathfrak{g}_\mathbb{C}$, then

$$\begin{aligned}(X \cdot \pi_{w,v})(g) &= \frac{d}{dt}\Big|_{t=0} \langle \pi(g\exp(tX))v, w\rangle = \frac{d}{dt}\Big|_{t=0} \langle \pi(\exp(tX))v, \pi(g^{-1})w\rangle \\ &= \langle d\pi(X)v, \pi(g^{-1})w\rangle = \langle \pi(g) d\pi(X)v, w\rangle\end{aligned}$$

Since $d\pi$ is a representation of the universal enveloping algebra, we can extend this to the universal enveloping algebra and $(\Omega \cdot \pi_{w,v})(g) = \langle \pi(g) d\pi(\Omega)v, w\rangle$ So $d\pi(\Omega)$ acting by a scalar gives the result. □

So in order to find information on the bi-K-invariant functions in this case we look for matrix entries that are spherical functions φ_π, $\pi \in \hat{G}_{\mathrm{sph}}$. Then by Proposition 4.5.4 we have to look for eigenfunctions of the 2nd order differential operator Ω acting on bi-K-invariant functions. Recall the KAK-decomposition of $\mathrm{SL}(2,\mathbb{R})$ as in Example 4.1.10(ii), which implies that a bi-K-invariant function is completely determined by its restriction to A, so we only need to determine $\varphi|_A\colon A \to \mathbb{C}$.

In order to find the differential operator for $\varphi|_A$ we need to calculate the differential operator $R(\Omega)$ acting on functions on A such that

$$(\Omega \cdot \varphi)|_A = R(\Omega) \cdot (\varphi|_A) \tag{4.5.5}$$

The differential operator $R(\Omega)$ is known as the *radial part* of the Casimir operator. The calculation of the radial part can be done for a much bigger subalgebra of the universal enveloping algebra, and this goes back to Harish-Chandra, see e.g. Casselman and Miličić, [10], [39], and this can be done in much greater generality. We follow [10].

In case of $\mathrm{SL}(2,\mathbb{R})$ the expression for the Casimir element in $U(\mathfrak{sl}(2,\mathbb{C}))$ is

$$\Omega = H^2 - H - Y \cdot \theta(Y), \quad \theta(Z) = -Z^t \tag{4.5.6}$$

with the notation of (4.5.1) and Z an arbitrary element. This can be obtained from Section A.2, but one can also directly check that Ω in (4.5.6) commutes with X, H and Y of (4.5.1).

Exercise 4.5.5. Check directly that the Casimir element of (4.5.6) commutes with X, Y and H using the commutation relations (4.5.2).

Note that the fixed points of the complexification of $\mathfrak{sl}(2,\mathbb{R})$ under θ can be calculated easily as

$$\{Z \in \mathfrak{sl}(2,\mathbb{C}) \mid \theta(Z) = Z\} = \mathbb{C} \cdot X$$

which precisely forms the complexification $\mathfrak{k}$ of the Lie algebra of K.

Let $a \in A$ be arbitrary, and assume that we want to calculate the action of $(Z \cdot \varphi)(a)$ for $Z \in \mathfrak{sl}(2,\mathbb{R})$. In case, $Z = X$ we have

$$(X \cdot \varphi)(a) = \frac{d}{dt}\Big|_{t=0} \varphi(a \exp(tX)) = 0 \tag{4.5.7}$$

if φ is right-K-invariant. On the other hand, if $X^a = a^{-1}Xa$, then we have

$$(X^a \cdot \varphi)(a) = \frac{d}{dt}\Big|_{t=0} \varphi(a \exp(tX^a)) = \frac{d}{dt}\Big|_{t=0} \varphi(aa^{-1} \exp(tX)a) = \frac{d}{dt}\Big|_{t=0} \varphi(\exp(tX)a) = 0 \tag{4.5.8}$$

for φ a left-K-invariant function using Proposition 2.3.4. The upshot of these two observations is that we need to decompose the Casimir element as a sum of elements of the from X^aHX (or possibly with powers of X^a, H or X). This is the so-called infinitesimal Cartan decomposition of the Casimir element. This can be done in general, even though in general the KAK-decomposition of G is harder than the Iwasawa decomposition.

Proposition 4.5.6. *Define the map $\alpha\colon A \to \mathbb{R}$ by*

$$\alpha(a(t)) = \alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t^2$$

then we have

$$\Omega = H^2 - \frac{1+\alpha(a)^2}{1-\alpha(a)^2}H + \frac{\alpha(a)^2}{(1-\alpha(a)^2)^2}\big((X^a)^2 + X^2\big) - \frac{\alpha(a)(1+\alpha(a)^2)}{(1-\alpha(a)^2)^2}X^aX$$

for any $a \in A$ satisfying $\alpha(a) \neq 1$.

The map $\alpha \colon A \to \mathbb{R}$ is known as a *root* (of the reduced root system), and in this special case we have only this root and its inverse. The abstract decomposition of Proposition 4.5.6 can be done for for general root systems.

For the proof of Proposition 4.5.6 we make use of the following lemma, cf. Proposition 3.4.32.

Lemma 4.5.7. *In the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ we have for any $a \in A$ satisfying $\alpha(a) \neq 1$*

$$\begin{aligned} Y &= \frac{\alpha(a)}{1-\alpha(a)^2}(X^a - \alpha(a)X), \\ \theta(Y) &= \frac{1}{1-\alpha(a)^2}(X - \alpha(a)X^a). \end{aligned}$$

Proof. We do the calculation for $a = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$;

$$\begin{aligned} X^a - \alpha(a)X &= \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} - t^2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & t^{-2} \\ -t^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & t^2 \\ -t^2 & 0 \end{pmatrix} = (t^{-2} - t^2)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Dividing and rewriting in terms of the root α gives the first result.

Now apply θ to the identity obtained to get

$$\theta(Y) = \frac{\alpha(a)}{1-\alpha(a)^2}(\theta(X^a) - \alpha(a)\theta(X))$$

and note that $\theta(X) = -X^t = X$. Moreover, $\theta(X^a) = \theta(a^{-1}Xa) = a^t\theta(X)(a^{-1})^t = aXa^{-1} = X^{a^{-1}}$. Changing a to a^{-1} gives

$$\theta(Y) = \frac{\alpha(a^{-1})}{1-\alpha(a^{-1})^2}(X^a - \alpha(a^{-1})X) = \frac{1}{1-\alpha(a)^2}(X - \alpha(a)X^a). \qquad \square$$

The proof of the general statement for arbitrary root systems is based on the Iwasawa decomposition on the level of the Lie algebra, see [10, Lemma 2.2].

Proof of Proposition 4.5.6. The first two terms in (4.5.6) are already of the required form, so we need to rewrite the last term, for which we use Lemma 4.5.7. So

$$\begin{aligned} Y\cdot\theta(Y) &= \frac{\alpha(a)}{(1-\alpha(a)^2)^2}(X^a-\alpha(a)X)(X-\alpha(a)X^a) \\ &= \frac{\alpha(a)}{(1-\alpha(a)^2)^2}\Big(-\alpha(a)\big((X^a)^2+X^2\big)+X^aX+\alpha(a)^2XX^a\Big) \end{aligned}$$

and we see that all terms, except the last, are of the right form. We use

$$\begin{aligned} XX^a - X^aX = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 0 & t^{-2} \\ -t^2 & 0\end{pmatrix} - \begin{pmatrix} 0 & t^{-2} \\ -t^2 & 0\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = (t^{-2}-t^2)\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} \\ \Longrightarrow \quad XX^a = X^aX + 2(\alpha(a)^{-1}-\alpha(a))H \end{aligned}$$

and we find the decomposition

$$Y\cdot\theta(Y) = \frac{-\alpha(a)^2}{(1-\alpha(a)^2)^2}\big((X^a)^2+X^2\big) + \frac{\alpha(a)(1+\alpha(a)^2)}{(1-\alpha(a)^2)^2}X^aX + \frac{2\alpha(a)^2(1-\alpha(a)^2)}{(1-\alpha(a)^2)^2}H$$

Plugging this in $\Omega = H^2 - H - Y\cdot\theta(Y)$, it remains to calculate the coefficient for H;

$$-1-\frac{2\alpha(a)^2(1-\alpha(a)^2)}{(1-\alpha(a)^2)^2} = -\frac{1-\alpha(a)^2}{1-\alpha(a)^2} - \frac{2\alpha(a)^2}{1-\alpha(a)^2} = -\frac{1+\alpha(a)^2}{1-\alpha(a)^2}. \qquad \square$$

In order to rewrite the action of the Casimir operator as a second order differential operator on bi-K-invariant functions φ, we need to pick coordinates on A. We identify $f(t)$ with $\varphi(a(t))$, so that

$$(H\cdot\varphi)(a(t)) = \frac{d}{ds}\Big|_{s=0}\varphi(a(t)\exp(sH)) = \frac{d}{ds}\Big|_{s=0}\varphi(a(te^{\frac{1}{2}s})) = \frac{d}{ds}\Big|_{s=0}f(te^{\frac{1}{2}s}) = \frac{1}{2}t\frac{df}{dt}(t)$$

and we view $f\colon (0,\infty)\to\mathbb{C}$. Then H^2 corresponds to

$$\Big(\frac{1}{2}t\frac{d}{dt}\Big)^2 = \frac{1}{4}t^2\frac{d^2}{dt^2} + \frac{1}{4}t\frac{d}{dt}$$

Because of (4.5.7) and (4.5.8) and Proposition 4.5.6 we see that $(\Omega\cdot\varphi)(a(t))$ corresponds to

$$(Df)(t) = \frac{1}{4}t^2\frac{d^2f}{dt^2}(t) + \frac{1}{4}t\frac{df}{dt}(t) - \frac{1+\alpha(a(t))^2}{1-\alpha(a(t))^2}\frac{1}{2}t\frac{df}{dt}(t). \tag{4.5.9}$$

So in order to find the spherical functions we need to determine the eigenfunctions of D as defined in (4.5.9).

Before continuing we observe that KAK-decomposition for $G=\mathrm{SL}(2,\mathbb{R})$ leads to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = \begin{pmatrix} t^{-1} & 0 \\ 0 & t\end{pmatrix}$$

and, since the left hand side is in KAK, a bi-K-invariant function $\varphi\colon \mathrm{SL}(2,\mathbb{R}) \to \mathbb{C}$ satisfies $\varphi(a(t)) = \varphi(a(t^{-1}))$. Hence, we need to find eigenfunctions f of D as in (4.5.9) satisfying $f(t) = f(t^{-1})$.

Now put $z = z(t) = -\frac{1}{4}(t-t^{-1})^2$, $f(t) = F(z(t))$, then we can rewrite $(Df)(t)$ as

$$\frac{1}{4}t^2\left(\frac{dz}{dt}(t)\right)^2\frac{d^2F}{dz^2}(z(t)) + \left(\frac{1}{4}t^2\frac{d^2z}{dt^2}(t) + \frac{1}{4}t\frac{dz}{dt}(t) - \frac{1}{2}t\frac{1+t^4}{1-t^4}\frac{dz}{dt}(t)\right)\frac{dF}{dz}(z(t))$$

and, since

$$\frac{1}{4}t^2\left(\frac{dz}{dt}(t)\right)^2 = \frac{1}{4}t^2\Big(-\frac{1}{2}(t-t^{-1})(1+t^{-2})\Big)^2 = \frac{1}{4}(t-t^{-1})^2\frac{1}{4}(t+t^{-1})^2 = z(z-1)$$

and

$$\begin{aligned}
&\frac{1}{4}t^2\frac{d^2z}{dt^2}(t) + \frac{1}{4}t\frac{dz}{dt}(t) - \frac{1}{2}t\frac{1+t^4}{1-t^4}\frac{dz}{dt}(t) = \frac{1}{4}t^2\frac{d^2z}{dt^2}(t) - \frac{1}{4}t\frac{1+3t^4}{1-t^4}\frac{dz}{dt}(t)\\
=&\frac{1}{4}t^2\big(-\frac{1}{2}(1+t^{-2})^2 + (t-t^{-1})t^{-3}\big) - \frac{1}{4}t\frac{1+3t^4}{1-t^4}\big(-\frac{1}{2}(t-t^{-1})(1+t^{-2})\big)\\
=&-\frac{1}{8}(t+t^{-1})^2 + \frac{1}{4}(t-t^{-1})t^{-1} - \frac{1}{8}(1+3t^4)t^{-2} = -\frac{1}{8}(t+t^{-1})^2 + \frac{1}{4} - \frac{3}{8}(t^2+t^{-2})
\end{aligned}$$

which equals $2z-1$, so we get D in terms of $F(z)$ as the differential operator

$$z(z-1)\frac{d^2F}{dz} + (2z-1)\frac{dF}{dz},$$

which is minus the hypergeometric differential operator (2.2.7) with $c=1$, $a+b=1$. Since the spherical functions arise as matrix entries of unitary representations, they are normalised at e by 1. Moreover, $\varphi_\lambda(a(t)) = \varphi_\lambda(a(t^{-1}))$. Since this fixes the eigenfunction to the hypergeometric differential equation, the spherical functions can be expressed as hypergeometric series;

$$\varphi_\lambda(a(t)) = {}_2F_1\left(\begin{matrix}\frac{1}{2}(1+i\lambda), \frac{1}{2}(1-i\lambda)\\ 1\end{matrix}; -\frac{1}{4}(t-t^{-1})^2\right) \tag{4.5.10}$$

and then $\Omega\cdot\varphi_\lambda = -\frac{1}{4}(1+i\lambda)(1-i\lambda)\varphi_\lambda = -\frac{1}{4}(1+\lambda^2)\varphi_\lambda$.

It remains to find the conditions on $\lambda \in \mathbb{C}$ such that there is a unitary irreducible representation π_λ of $\mathrm{SL}(2,\mathbb{R})$ with a $\mathrm{SO}(2)$-fixed vector such that $d\pi_\lambda(\Omega) = -\frac{1}{4}(1+\lambda^2)\mathbf{1}$. Since we see that Ω is $*$-invariant, we see that we require $-\frac{1}{4}(1+\lambda^2) \in \mathbb{R}$. We briefly sketch this in Section 4.5.2.

The function in (4.5.10) is a special case of what is known as a Jacobi function.

Definition 4.5.8. *The Jacobi function is defined as*

$$\varphi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1\left(\begin{matrix}\frac{1}{2}(1+\alpha+\beta+i\lambda), \frac{1}{2}(1+\alpha+\beta-i\lambda)\\ \alpha+1\end{matrix}; -\sinh^2(t)\right).$$

So we find $\varphi_\lambda(a(e^t)) = \varphi_\lambda^{(0,0)}(t)$, meaning that the spherical functions for the Gelfand pair $(\mathrm{SL}(2,\mathbb{R}), \mathrm{SO}(2))$ are given as Legendre functions, i.e. Jacobi functions for $(\alpha, \beta) = (0,0)$, just like Legendre polynomials are Jacobi polynomials for $(\alpha, \beta) = (0,0)$.

Remark 4.5.9. Put $\Delta^{(\alpha,\beta)}(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}$ for $t > 0$. Then $\varphi_\lambda^{(\alpha,\beta)}$ is an eigenfunction of the second order differential operator

$$L^{(\alpha,\beta)} = \frac{d^2}{dt^2} + \frac{\frac{d\Delta^{(\alpha,\beta)}}{dt}(t)}{\Delta^{(\alpha,\beta)}(t)} \frac{d}{dt}$$

with eigenvalue $-(\lambda^2 + (\alpha+\beta+1)^2)$. It turns out that $L^{(\alpha,\beta)}$ is symmetric and essentially self-adjoint on $(0,\infty)$ with respect to the measure $\Delta^{(\alpha,\beta)}(t)\, dt$ for suitable conditions on α, β. In particular, the spectral decomposition for the case $\alpha > -1$, $\beta \in \mathbb{R}$ with $|\beta| \le \alpha + 1$ gives the integral transform, known as the Jacobi function transform

$$\begin{aligned} \hat{f}(\lambda) &= \int_0^\infty f(t)\varphi_\lambda^{(\alpha,\beta)}(t)\Delta^{(\alpha,\beta)}(t)\, dt \\ f(t) &= \int_0^\infty \hat{f}(\lambda)\varphi_\lambda^{(\alpha,\beta)}(t)\frac{d\lambda}{|c^{(\alpha,\beta)}(\lambda)|^2} \end{aligned} \tag{4.5.11}$$

for suitable functions f. This extends to a unitary isomorphism $L^2((0,\infty), \Delta^{(\alpha,\beta)}(t)\, dt) \to L^2((0,\infty), \frac{d\lambda}{|c^{(\alpha,\beta)}(\lambda)|^2})$, where $c^{(\alpha,\beta)}$ is the so-called Harish-Chandra c-function defined by

$$c^{(\alpha,\beta)}(\lambda) = \frac{2^{-i\lambda+(\alpha+\beta+1)}\Gamma(i\lambda)\Gamma(\alpha+1)}{\Gamma(\frac{1}{2}(i\lambda+\alpha+\beta+1))\Gamma(\frac{1}{2}(i\lambda+\alpha-\beta+1))}.$$

The Jacobi function transform then gives the explicit expression for the Plancherel measure in Theorem 4.4.5 assuming the result of Section 4.5.2. See [54] for more information and references.

Exercise 4.5.10. Show that $\varphi_\lambda^{(\alpha,\beta)}$ is an eigenfunction of $L^{(\alpha,\beta)}$ with appropriate eigenvalue. Show that $\varphi_\lambda^{(\alpha,\beta)}$ is the unique eigenfunction which is even in t for this eigenvalue.

Remark 4.5.11. Since the Casimir operator is defined in $U(\mathfrak{sl}(2,\mathbb{C}))$ and $\mathfrak{sl}(2,\mathbb{C})$ is the complexification of both $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$, this then similarly gives the hypergeometric differential operator for the Legendre polynomials, i.e. the case $m = n = 0$ of Section 3.4.6. Similarly, we can deal with the differential operator arising from the Casimir element for functions that behave on the left and right via a non-trivial character of the abelian compact subgroup $K = \mathrm{SO}(2)$. It turns out that, again, we end up with the hypergeometric differential operator after conjugation with an elementary factor. This is used to determine the matrix entries as hypergeometric series. This method goes back to Bargmann [3].

Remark 4.5.12. In this setting we can see $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ as a Riemannian manifold, which is a hyperboloid. This comes with a Laplace-Beltrami operator, and it turns out that this coincides with the action of the Casimir operator. Since the Laplace-Beltrami operator is an elliptic (partial) differential operator, a classic result from the theory for partial differential equations is that eigenfunctions are real analytic. So we see that the spherical functions are automatically real analytic functions. This also applies to other matrix entries, see [39] for more information.

4.5.2 Unitary representations of $\mathrm{SL}(2,\mathbb{R})$: principal series representations

It takes us too far to classify the full unitary dual of the (non-compact) group $\mathrm{SL}(2,\mathbb{R})$. Apart from the trivial one-dimensional representation, there do not exist finite-dimensional unitary representations of $\mathrm{SL}(2,\mathbb{R})$. There are four classes of irreducible unitary representations of $\mathrm{SL}(2,\mathbb{R})$, see e.g. [39]. The discrete series fall into two classes; the negative and positive discrete series. Then there is the principal series representations and the complementary series representations. All these representations can be obtained by studying suitable induced representations from a parabolic subgroup.

We sketch some of the constructions, all details can be found in [39], [71], [16]. We let $M = Z_K(A)$ be the subgroup of K of elements that centralise A, and we let $P = MAN$. Such a subgroup P is called a minimal parabolic subgroup, and $P = MAN$ is called the Langlands decomposition of P.

Exercise 4.5.13. (i) Show that $M \cong \mathbb{Z}_2$, and that $P \subset \mathrm{SL}(2,\mathbb{R})$ is the upper triangular group.

(ii) Show that M has two irreducible representations, and let $\xi_\nu \colon A \to \mathbb{C}$ be the one-dimensional representation $a(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^\nu = e^{\nu \ln t}$ for $\nu \in \mathbb{C}$, $t > 0$. Show that for $\sigma \in \hat{M}$ we have that

$$\sigma \otimes \xi_\nu \otimes 1 \colon man \mapsto \sigma(m)\, \xi_\nu(a)$$

is a representation of $P = MAN$. Show that it is a unitary representation for $\nu \in i\mathbb{R}$.

The representations σ of M are the trivial representation, which is denoted by $+$, and the sign representation denoted by $-$. Both of them are one-dimensional. We now consider the space

$$V^{\sigma,\nu} = \{F \in C(G) \mid F(gman) = \sigma(m)^{-1}\xi_{\nu+1}(a)^{-1}F(g),\ \forall man \in P\}, \tag{4.5.12}$$

(note the shift in ν!) then this space is invariant for the left regular action λ, indeed $F \in V^{\sigma,\nu}$, then also $g \mapsto F(h^{-1}g) = (\lambda(h)F)(g)$ is $V^{\sigma,\nu}$. So we find a not necessarily unitary representation of $\mathrm{SL}(2,\mathbb{R})$ in this way, and we can wonder about completion to a Hilbert space, unitarity and the continuity requirement for representations. This way of creating representations of $G = \mathrm{SL}(2,\mathbb{R})$ from (simpler) representations of a parabolic subgroup P, is called (parabolic) induction. This procedure works for very general classes of groups G and by a general result, called Casselman's Subrepresentation Theorem, essentially all irreducible representations can be obtained as subrepresentation of a parabolically induced representation, see [10], [39].

By the Iwasawa decomposition (4.5.3) we see that $F \in V^{\sigma,\nu}$ is completely determined by its restriction to K, and actually by its restriction to K/M. We can put an inner product on $V^{\sigma,\nu}$ by using the inner product of $L^2(K)$.

Theorem 4.5.14. *For $\nu = i\lambda \in i\mathbb{R}$, the completion of $V^{\sigma,i\lambda}$ gives a unitary irreducible representation of* $\mathrm{SL}(2,\mathbb{R})$ *for $\lambda \neq 0$, which is known as the principal unitary series, and denoted $P^{\sigma,i\nu}$. Moreover, $P^{\sigma,i\lambda} \cong P^{\sigma,-i\lambda}$. For $\lambda = 0$, we have $P^{+,0}$ is an irreducible unitary representation, and $P^{-,0}$ is reducible.*

Exercise 4.5.15. We establish the unitarity of the principal series representations.

(i) Show that Haar measure on $\mathrm{SL}(2,\mathbb{R})$ in terms of the Iwasawa decomposition (4.5.3) can be written as

$$\int_{\mathrm{SL}(2,\mathbb{R})} f(g)\, dg = \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(k(\theta)a(e^t)n(x))\, e^{2t} dx\, dt\, d\theta$$

assuming that $\mathrm{SL}(2,\mathbb{R})$ is a unimodular group, since $(\mathrm{SL}(2,\mathbb{R}),\mathrm{SO}(2))$ is a Gelfand pair. (Hint: left, respectively right, invariance with respect to K, respectively N, is obvious. For right invariance with respect to A use that A normalises N, i.e. $\mathrm{Ad}(a)(N) = N$ for $a \in A$.)

(ii) For $g \in \mathrm{SL}(2,\mathbb{R})$ we define

$$g^{-1}k(\theta) = k(\psi(g,\theta))a(e^{t(g,\theta)})n(x(g,\theta))$$

with $0 \leq \psi(g,\theta) < 2\pi$, $t(g,\theta), x(g,\theta) \in \mathbb{R}$ according to the Iwasawa decomposition (4.5.3). Show that

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f\big(g^{-1}k(\theta)a(e^t)n(x)\big)e^{2t}\, dxdtd\theta \\
=&\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f\big(k(\psi(g,\theta))a(e^{t(g,\theta)+t})n(x(g,\theta)e^{-2t}+x)\big)e^{2t}\, dxdtd\theta \\
=&\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f\big(k(\psi(g,\theta))a(e^t)n(x)\big)e^{2t-2t(g,\theta)}\, dxdtd\theta
\end{aligned}$$

and using left invariance of the Haar measure and comparing with (i) derive for $f \in C(K)$ that

$$\frac{1}{2\pi}\int_0^{2\pi} f(k(\theta))\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(k(\psi(g,\theta)))\, e^{-2t(g,\theta)}\, d\theta$$

(iii) Show that the representation $P^{\sigma,i\nu}$ is unitary by observing that for $F \in V^{\sigma,i\nu}$ we have

$$F\big(g^{-1}k(\theta)\big) = \pm e^{-(1+i\nu)t(g,\theta)}F\big(k(\psi(g,\theta))\big)$$

where the sign depends on σ. Use (ii) to establish unitarity.

The proof of the unitarity of the principal unitary series in Exercise 4.5.15 does not mean that any other parabolically induced representation is not unitary. Indeed, another suitable inner product could make the representations unitary. This does indeed happen.

The principal series representations can also be realised in $L^2(\mathbb{R})$. For $P^{+,i\lambda}$ this goes as follows.

Exercise 4.5.16. (i) Check that $\pi^{+,i\lambda}\colon \mathrm{SL}(2,\mathbb{R}) \to B(L^2(\mathbb{R}))$ defined by

$$(\pi^{+,i\lambda}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f)(x) = |-bx+d|^{-1-i\lambda} f(\frac{ax-c}{-bx+d}), \qquad f \in L^2(\mathbb{R}),$$

is a homomorphism and $\pi^{+,i\lambda}(g)$ is unitary for $g \in \mathrm{SL}(2,\mathbb{R})$.

(ii) Use dominated convergence to see that $\mathrm{SL}(2,\mathbb{R}) \to L^2(\mathbb{R})$, $g \mapsto \pi^{+,i\lambda}(g)f$ for a fixed $f \in C_c^\infty(\mathbb{R})$ is continuous at $g = e \in \mathrm{SL}(2,\mathbb{R})$. Use this and the density of $C_c^\infty(\mathbb{R})$ in $L^2(\mathbb{R})$ to see that $\pi^{+,i\lambda}$ meets the continuity requirement for a unitary representation, see Definition 2.5.1 and Lemma 2.5.2.

(iii) With $\bar{N} = \{\bar{n}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in \mathbb{R}\}$ the unipotent lower triangular matrices we have that $\bar{N}MAN$ is almost all of $\mathrm{SL}(2,\mathbb{R})$, up to a set of measure zero. Define $LF\colon \mathbb{R} \to \mathbb{C}$ for $F \in V^{+,i\lambda}$ by $(LF)(x) = F(\bar{n}(x))$. It is true that $\|F\|^2 = \|LF\|^2$, and we refer to [39], so that L is an isometry and hence injective. Show that $LP^{+,i\lambda}(g) = \pi^{+,i\lambda}(g)L$ for $g \in \mathrm{SL}(2,\mathbb{R})$. Show moreover that $f \in C_c(\mathbb{R})$ are in the image of L by showing that

$$F(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = |a|^{-1-i\lambda} f(\frac{c}{a}), \qquad a \neq 0,$$

and 0 otherwise is mapped to f by L. We conclude that $\pi^{+,i\lambda} \cong P^{+,i\lambda}$.

(iv) Let $B \in B_{\mathrm{SL}(2,\mathbb{R})}(L^2(\mathbb{R}))$, show that B commutes with translations (use the action of $\bar{n}(x)$). Conclude that $\mathcal{F}(Bf)(\zeta) = m(\zeta)\mathcal{F}(f)(\zeta)$ for some bounded measurable function m, where $\mathcal{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the standard Fourier transform. Using the intertwining property for diagonal matrices, show that m is constant on the positive half line and constant on the negative half line.

Note that (iv) is not yet sufficient to conclude that B is a scalar. We can however, conclude that only spaces of functions whose Fourier transform vanishes at a half line are the only remaining candidates for invariant subspaces in $L^2(\mathbb{R})$. Such spaces are very well known in complex analysis, and they are related to Hardy spaces and this is a classical result known as a Paley-Wiener theorem. The point is now to exhibit that these spaces are not invariant under the action of $\pi^{+,i\lambda}$ by establishing an explicit function and explicit element $g \in \mathrm{SL}(2,\mathbb{R})$ and showing that the resulting function is not in this subspace, see [39] for details.

Exercise 4.5.16 then leads to the conclusion that the principal unitary series representations are irreducible unitary representations.

It turns out that only the principal unitary series representations can have non-trivial K-fixed vectors, so that only in this case we obtain spherical functions. They have to be related to the solutions of the eigenvalue problem for the Casimir operator as in Section 4.5.1. Going to the corresponding representation of the Lie algebra, the action of the Casimir element Ω on $\pi^{+,i\lambda}$ is $-\frac{1}{4}(1+\lambda^2)$.

Our next goal is to find the spherical function as discussed in the previous paragraph in a more explicit way, and in particular as a matrix entry of the principal unitary series representation $P^{+,i\lambda}$. Use the Iwasawa decomposition of Exercise 4.5.2 to define the function $H\colon \mathrm{SL}(2,\mathbb{R}) \to (0,\infty)$ by $H(g) = H(ka(t)n) = \ln t$. Then we define the function

$$\phi_\lambda(g) = \exp(-(1+i\lambda)H(g)) \tag{4.5.13}$$

so that $\phi_\lambda(kg) = \phi_\lambda(g)$, and in particular $\phi_\lambda(k) = \phi_\lambda(e) = 1$ for all $k \in K$. Moreover, $\phi_\lambda \in V^{+,i\lambda}$.

Exercise 4.5.17. Give a proof of these statements on ϕ_λ.

So we now have an expression for the corresponding bi-K-invariant function

$$\varphi_\lambda(g) = \langle P^{+,i\lambda}(g)\phi_\lambda, \phi_\lambda\rangle = \int_K \phi_\lambda(g^{-1}k)\overline{\phi_\lambda(k)}\, dk = \int_K \exp(-(1+i\lambda)H(g^{-1}k))\, dk$$

Exercise 4.5.18. (i) Show that $\varphi_\lambda(g) = \int_K \exp(-(1+i\lambda)H(g^{-1}k))\, dk$ gives a solution to the product formula of Theorem 4.1.16, and so φ_λ is a spherical function.

(ii) Show that in this explicit case

$$\varphi_\lambda(a(t)) = \frac{1}{2\pi}\int_0^{2\pi} |t^{-2}\cos^2\theta + t^2\sin^2\theta|^{-\frac{1}{2}(1+i\lambda)}\, d\theta$$

4.5.3 Product formula for spherical functions

Now that we have identified the spherical functions using Theorem 4.4.1, we apply this to find the product formula of Theorem 4.1.16 for the spherical functions explicitly. For this we need to find the KAK-decomposition for the element

$$g = \begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} s & 0 \\ 0 & s^{-1}\end{pmatrix} = \begin{pmatrix} st\cos\theta & -ts^{-1}\sin\theta \\ st^{-1}\sin\theta & s^{-1}t^{-1}\cos\theta\end{pmatrix}$$

in $\mathrm{SL}(2,\mathbb{R})$. Since the KAK-decomposition is based on the polar decomposition, and we only need the A-part, we can use this. So we take the eigenvalues of the symmetric matrix $g^t g$, since these eigenvalues λ_1, λ_2 are positive and their product is 1, we can take the diagonal part of g as the diagonal matrix with eigenvalues $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$. Let

$$T = \mathrm{Tr}\,(g^t g) = (\frac{s^2}{t^2}+\frac{t^2}{s^2})\sin^2\theta + (s^2t^2 + \frac{1}{t^2s^2})\cos^2\theta = (\frac{s^2}{t^2}+\frac{t^2}{s^2}) + (s^2-s^{-1})(t^2-t^{-2})\cos^2\theta$$

so that

$$T - 2 = (\frac{s}{t} - \frac{t}{s})^2 + (s^2 - s^{-2})(t^2 - t^{-2})\cos^2\theta$$

is positive, since we can assume without loss of generality $s \geq 1$, $t \geq 1$. If we then denote the A-part of g by

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

we have $x = \sqrt{\frac{1}{2}T + \frac{1}{2}\sqrt{T^2 - 4}}$, $x^{-1} = \sqrt{\frac{1}{2}T - \frac{1}{2}\sqrt{T^2 - 4}}$. Hence,

$$x - x^{-1} = \frac{\sqrt{T^2 - 4}}{\sqrt{\frac{1}{2}T + \frac{1}{2}\sqrt{T^2 - 4}} + \sqrt{\frac{1}{2}T - \frac{1}{2}\sqrt{T^2 - 4}}}$$

and squaring this expression, we obtain

$$\begin{aligned} (x - x^{-1})^2 &= \frac{T^2 - 4}{\frac{1}{2}T + \frac{1}{2}\sqrt{T^2 - 4} + \frac{1}{2}T - \frac{1}{2}\sqrt{T^2 - 4} + 2} \\ &= T - 2 = (\frac{s}{t} - \frac{t}{s})^2 + (s^2 - s^{-2})(t^2 - t^{-2})\cos^2\theta \end{aligned}$$

Proposition 4.5.19. *Let $\lambda \in \mathbb{R}_{>0}$, then we have*

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} \frac{1}{2}(1 + i\lambda), \frac{1}{2}(1 - i\lambda) \\ 1 \end{matrix}; -(t - t^{-1})^2\right) {}_2F_1\left(\begin{matrix} \frac{1}{2}(1 + i\lambda), \frac{1}{2}(1 - i\lambda) \\ 1 \end{matrix}; -(s - s^{-1})^2\right) = \\ &\frac{1}{2\pi}\int_0^{2\pi} {}_2F_1\left(\begin{matrix} \frac{1}{2}(1 + i\lambda), \frac{1}{2}(1 - i\lambda) \\ 1 \end{matrix}; -(\frac{s}{t} - \frac{t}{s})^2 - (s^2 - s^{-2})(t^2 - t^{-2})\cos^2\theta\right) d\theta \end{aligned}$$

Proof. By Theorem 4.1.16 we have a product formula for spherical functions. Since the spherical functions are bi-K-invariant functions, it suffices to calculate the A-part of the product of gkh for $g, h \in A$. This has been done above. Since $K = \mathrm{SO}(2) \cong \mathbb{T}$ we have the invariant integral over K as the integral over the invariant integral over the circle.

By Section 4.5.2 we have that for the parameter $\lambda \in \mathbb{R}_{>0}$ the corresponding eigenvalue for the derived representation of the Casimir operator corresponds to the principal unitary series representation, so that it is indeed a spherical function. □

Chapter 5

Matrix spherical functions

In this chapter we consider more general functions on a group G that transforms more generally under the left and right action of a compact subgroup K via another representation than the trivial one (leading to spherical functions) or characters (i.e. the one-dimensional representations of K). For the trivial representation we have studied the case of a Gelfand pair and the trivial representation in Chapter 4, leading to spherical functions. For the case $(G, K) = (\mathrm{SU}(2), \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)))$ we have studied the behaviour according to characters of K in Chapter 3. In this chapter G is an (infinite) linear group, and K is a compact subgroup. We assume throughout this chapter that G is unimodular, and K is unimodular since it is a compact group.

5.1 Matrix-valued functions and convolution

We fix a finite dimensional representation (τ, V_τ) of K, which we assume to be unitary. Then $\mathrm{End}(V_\tau)$ is a matrix algebra, and we consider functions $F\colon G \to \mathrm{End}(V_\tau)$, which are matrix valued functions. Note that $\mathrm{End}(V_\tau)$ is a Hilbert space with inner product $\langle S, T\rangle = \mathrm{Tr}\,(T^*S)$ and norm $\|T\|^2 = \mathrm{Tr}\,(T^*T)$.

Lemma 5.1.1. $\mathrm{End}(V_\tau)$ *has a* $K \times K$*-action given by*

$$(k_1, k_2) \cdot T = \tau(k_1) T \tau(k_2^{-1}), \qquad k_1, k_2 \in K, \quad T \in \mathrm{End}(V_\tau)$$

and this leads to $\mathrm{End}(V_\tau)$ *being a unitary* $K \times K$*-representation.*

Exercise 5.1.2. (i) Prove Lemma 5.1.1, where the inner product is given by $\langle T, S\rangle = \mathrm{Tr}\,(S^*T)$ for $T, S \in \mathrm{End}(V_\tau)$.

(ii) Extend the result of Lemma 5.1.1 to the case of two finite dimensional unitary representations (τ_1, V_{τ_1}) and (τ_2, V_{τ_2}) and show that $\mathrm{Hom}(V_{\tau_2}, V_{\tau_1})$ has a $K \times K$-action given by

$$(k_1, k_2) \cdot T = \tau_1(k_1) T \tau_2(k_2^{-1}), \qquad k_1, k_2 \in K, \quad T \in \mathrm{Hom}(V_{\tau_2}, V_{\tau_1}).$$

What about unitarity of this action?

Note that a suitable function space of functions on G, such as $C(G)$, $L^1(G)$, etc, also carries a $G \times G$ action by the biregular representation. Since $\mathrm{End}(V_\tau)$ is finite dimensional we can consider the space of continuous functions $F \colon G \to \mathrm{End}(V_\tau)$ as the (algebraic) tensor product $C(G) \otimes \mathrm{End}(V_\tau)$ by e.g. picking a basis for V_τ and writing F entrywise. Indeed, picking an orthonormal basis $(v_i)_{i=1}^{d_\tau}$ for V_τ we can take the corresponding matrix entries $E_{i,j} \in \mathrm{End}(V_\tau)$ defined by $E_{i,j} v_k = \delta_{j,k} v_i$ and we can write $F(g) = \sum_{i,j=1}^{d_\tau} F_{i,j}(g) E_{i,j}$ with $F_{i,j} \in C(G)$ and we identify F with $\sum_{i,j=1}^{d_\tau} F_{i,j} \otimes E_{i,j} \in C(G) \otimes \mathrm{End}(V_\tau)$. Since by restriction, $C(G)$ has a $K \times K$-action, we have a $K \times K$-action on the tensor product $C(G) \otimes \mathrm{End}(V_\tau)$ given by

$$(k_1, k_2) \cdot f(g) \otimes T = f(k_1^{-1} g k_2) \otimes \tau(k_1) T \tau(k_2^{-1}), \qquad f \in C(G),\ T \in \mathrm{End}(V_\tau).$$

Definition 5.1.3. *A continuous $F \colon G \to \mathrm{End}(V_\tau)$ is τ-invariant if $F \colon G \to \mathrm{End}(V_\tau)$ is contained in $C_\tau(G; \mathrm{End}(V_\tau)) = \big(C(G) \otimes \mathrm{End}(V_\tau)\big)^{K \times K}$, the space of invariants under the $K \times K$-action.*

Note that a continuous $F \colon G \to \mathrm{End}(V_\tau)$ is τ-invariant if and only if

$$F(k_1 g k_2) = \tau(k_1) F(g) \tau(k_2), \qquad \forall\, g \in G,\ \forall\, k_1, k_2 \in K. \tag{5.1.1}$$

Similarly, we define $L^1_\tau(G; \mathrm{End}(V_\tau))$ as the space of $K \times K$-invariants in $L^1(G) \otimes \mathrm{End}(V_\tau)$, and these are the τ-invariant integrable functions. Other spaces of τ-invariant functions can be defined similarly. Note that for $\tau = 1 \in \hat{K}$ the trivial representation, we get $C(G) = C_c(G/\!/K)$, $L^1_\tau(G; \mathrm{End}(V_\tau)) = L^1(G/\!/K)$, the bi-$K$-invariant functions. Note that $L^1(G) \otimes \mathrm{End}(V_\tau)$ and $L^1_\tau(G; \mathrm{End}(V_\tau))$ are Banach spaces for the norm

$$\|F\| = \int_G \|F(g)\|\, dg, \qquad F \in L^1_\tau(G; \mathrm{End}(V_\tau)) \text{ or } F \in L^1(G) \otimes \mathrm{End}(V_\tau).$$

The norm in the integral is the norm of $\mathrm{End}(V_\tau)$. For $F \in L^1_\tau(G; \mathrm{End}(V_\tau))$ we identify $\mathrm{End}(V_\tau)$-valued functions almost everywhere.

Define the convolution product of $F_1, F_2 \colon G \to \mathrm{End}(V_\tau)$ by

$$F_1 * F_2(x) = \int_G F_1(xg) F_2(g^{-1})\, dg = \int_G F_1(g) F_2(g^{-1}x)\, dg = \int_G F_1(xg^{-1}) F_2(g)\, dg \tag{5.1.2}$$

similar to (4.1.1) for functions F_1, F_2 for which the integral is defined, such as compactly supported continuous functions. Note that $F_1 * F_2 \colon G \to \mathrm{End}(V_\tau)$, and that this in general is non-commutative and that not even the order of F_1 and F_2 can be interchanged in the integrand since F_1 and F_2 take values in the generally non-commutative algebra $\mathrm{End}(V_\tau)$.

Proposition 5.1.4. *$L^1(G) \otimes \mathrm{End}(V_\tau)$ and $L^1_\tau(G; \mathrm{End}(V_\tau))$ are Banach algebras with respect to the convolution product* (5.1.2)*. Moreover, with*

$$F^*(x) = (F(x^{-1}))^*$$

$L^1(G) \otimes \mathrm{End}(V_\tau)$ and $L^1_\tau(G; \mathrm{End}(V_\tau))$ are Banach $*$-algebras.

Exercise 5.1.5. Note that $\mathrm{End}(V_\tau)$ is equipped with the inner product $\langle S, T\rangle = \mathrm{Tr}\,(T^*S)$ and corresponding norm $\|T\| = \sqrt{\mathrm{Tr}\,(T^*T)}$. Prove the following properties for $S, T \in \mathrm{End}(V_\tau)$.

(i) $\|T^*\| = \|T\|$,

(ii) $\|ST\| \leq \|S\|\,\|T\|$,

(iii) $\|T^*T\| \leq \|T\|^2$ and that there is $T \in \mathrm{End}(V_\tau)$ with $\|T^*T\| < \|T\|^2$.

Note that this means that $\mathrm{End}(V_\tau)$ is a Banach $*$-algebra, but not a C*-algebra.

Proof. Since $\mathrm{End}(V_\tau)$ is a Banach $*$-algebra we have

$$\begin{aligned}\int_G \|F_1 * F_2(x)\|\,dx &\leq \int_G\int_G \|F_1(xg)F_2(g^{-1})\|\,dg\,dx \leq \int_G(\int_G \|F_1(xg)\|\,dx)\|F_2(g^{-1})\|\,dg\\ &\leq \int_G \|F_1(x)\|\,dx \int_G \|F_2(g^{-1})\|\,dg = \int_G \|F_1(x)\|\,dx \int_G \|F_2(g)\|\,dg\end{aligned}$$

using the unimodularity of G. Hence, $\|F_1 * F_2\| \leq \|F_1\|\|F_2\|$, and $L^1(G)\otimes\mathrm{End}(V_\tau)$ is a Banach algebra.

To show that $L^1_\tau(G;\mathrm{End}(V_\tau))$ is a Banach algebra we need to show that $F_1 * F_2 \in L^1_\tau(G;\mathrm{End}(V_\tau))$ for $F_1, F_2 \in L^1_\tau(G;\mathrm{End}(V_\tau))$, or that convolution preservers τ-invariance. So we consider $k_1, k_2 \in K$ and

$$\begin{aligned}F_1 * F_2(k_1xk_2) &= \int_G F_1(k_1xk_2g)F_2(g^{-1})\,dg = \tau(k_1)\int_G F_1(xk_2g)F_2(g^{-1})\,dg\\ &= \tau(k_1)\int_G F_1(xg)F_2(g^{-1}k_2)\,dg = \tau(k_1)\int_G F_1(xg)F_2(g^{-1})\,dg\,\tau(k_2)\\ &= \tau(k_1)\,F_1 * F_2(x)\,\tau(k_2)\end{aligned}$$

so that $L^1_\tau(G)$ is closed under convolution.

Note that the $*$-structure is an involution, and $\|F^*\| = \|F\|$ using Exercise 5.1.5 and the unimodularity of G. Moreover, the $*$-structure preserves $L^1_\tau(G;\mathrm{End}(V_\tau))$, since for $k_1, k_2 \in K$ we have

$$\begin{aligned}F^*(k_1gk_2) &= \big(F(k_2^{-1}g^{-1}k_1^{-1}))\big)^* = \big(\tau(k_2^{-1})F(g^{-1})\tau(k_1^{-1}))\big)^*\\ &= \tau(k_1)\big(F(g^{-1})\big)^*\tau(k_2) = \tau(k_1)F^*(g)\tau(k_2)\end{aligned}$$

since τ is a unitary representation. It remains to prove that $(F_1 * F_2)^* = F_2^* * F_1^*$, which is relegated to Exercise 5.1.6. $\square$

Exercise 5.1.6. (i) Prove the last statement of Proposition 5.1.4, i.e. $(F_1 * F_2)^* = F_2^* * F_1^*$.

(ii) For different irreducible K-representations as in Exercise 5.1.2(ii), we equip $\mathrm{Hom}(V_{\tau_2}, V_{\tau_1})$ with the operator norm and we look at integrable $F\colon G \to \mathrm{Hom}(V_{\tau_2}, V_{\tau_1})$ with the transformation as in 5.1.2(ii). Then the Banach space is $L^1_{\tau_1,\tau_2}(G) = L^1_{\tau_1,\tau_2}(G; \mathrm{Hom}(V_{\tau_2}, V_{\tau_1}))$. Show that $F^* \in L^1_{\tau_2,\tau_1}(G)$ and that the convolution product gives a map

$$L^1_{\tau_1,\tau_2}(G) \times L^1_{\tau_2,\tau_3}(G) \to L^1_{\tau_1,\tau_3}(G)$$

for all $\tau_i \in \hat{K}$. What is the analogue of (i) in this setting?

(iii) Show that for $F_1, F_2 \in C_\tau(G)$ the function $g \mapsto F_1(g)(F_2(g))^*$, $G \to \mathrm{End}(V_\tau)$ is right-K-invariant. Show that $g \mapsto \mathrm{Tr}\,\big(F_1(g)(F_2(g))^*\big)$, $G \to \mathbb{C}$ is bi-K-invariant. Similarly, show that $g \mapsto (F_1(g))^* F_2(g)$, $G \to \mathrm{End}(V_\tau)$, respectively $g \mapsto \mathrm{Tr}\,\big((F_1(g))^* F_2(g)\big)$, $G \to \mathbb{C}$, is left-K-invariant, respectively bi-K-invariant. State and prove the analogues of these statement in the context of (ii).

Next we relate the space $L^1_\tau(G; \mathrm{End}(V_\tau))$ to a class of (scalar valued) functions on G. For this we need some definitions. Recall the adjoint action of G on itself by $\mathrm{Ad}(g)x = gxg^{-1}$. Then we have the contragredient action on functions on G by $\mathrm{Ad}^*(g)f(x) = f(\mathrm{Ad}(g^{-1})x) = f(g^{-1}xg)$. For a compact group G, the invariant functions are the class functions, or central functions, see Section 3.3. We call a function *K-central* if it is invariant under the action restricted to K, i.e. $f(x) = f(kxk^{-1})$ for all $k \in K$ and all $x \in G$.

Definition 5.1.7. *A function $f\colon G \to \mathbb{C}$ is of K-type $\tau \in \hat{K}$ if*

$$f(x) = \int_K f(xk) d_\tau \overline{\chi_\tau(k)}\, dk$$

for all $x \in G$. Here $\chi_\tau\colon K \to \mathbb{C}$ is the character of the representation $\tau \in \hat{K}$ and d_τ is the dimension of V_τ, i.e. $d_\tau = \dim_{\mathbb{C}} V_\tau$.

The characters of compact linear groups are discussed in Section 3.3. Recall that convolution on K with the normalised character $\xi_\tau = d_\tau \chi_\tau$ is a projection, see Exercise 4.1.3(ii). So we can rewrite the condition of Definition 5.1.7 as

$$f(x) = f *_K \xi_\tau(x) = \int_K f(xk)\xi_\tau(k^{-1})\, dk. \tag{5.1.3}$$

Note that f can be of at most one K-type τ by Exercise 4.1.3(ii) and the associativity of the convolution product on K.

Definition 5.1.8. *We denote the space of integrable functions $f\colon G \to \mathbb{C}$ which are K-central and of K-type τ by $L^1_\tau(G)^K$. Similarly, the space of continuous, K-central functions of K-type τ are denoted as $C_\tau(G)^K$.*

Note that if we take $\tau = 1 \in \hat{K}$ the trivial representation, we see that (5.1.3) is equivalent to f being right K-invariant, since $\xi_1 = 1$. Since f is K-central, it is also left K-invariant. Hence $C_1(G)^K = C_c(G/\!/K) = C_1(G; \mathrm{End}(V_1))$ and $L_1(G)^K = L^1(G/\!/K) = L^1_1(G; \mathrm{End}(V_1))$, since $V_1 = \mathbb{C}$. In this case the spaces are equal, and in Theorem 5.1.11 we show that the generalisations to $\tau \in \hat{K}$ of these spaces are isomorphic.

Proposition 5.1.9. *For $f \in C(G)$ the function*

$$Pf(x) = \int_K \int_K f(kxk^{-1}h)\xi_\tau(h^{-1})\, dhdk$$

is contained in $C_\tau(G)^K$. Moreover, $P\colon C(G) \to C_\tau(G)^K$ is a projection onto $C_\tau(G)^K$. Moreover, if f is a function of K-type σ with $\sigma \not\cong \tau$, then $Pf = 0$. Also, $P\colon L^1(G) \to L^1_\tau(G)^K$ is bounded and $L^1_\tau(G)^K$ is a Banach $*$-algebra.

Exercise 5.1.10. We sketch the proof of Proposition 5.1.9.

(i) Show that $Pf(x) = Pf(lxl^{-1})$ for $l \in K$, i.e. that Pf is K-central.

(ii) Show that $Pf(x)$ is of K-type τ and that $Pf = 0$ if f is of K-type σ with $\sigma \not\cong \tau$. (Hint: use Exercise 4.1.3.)

(iii) Show that $Pf = f$ for $f \in C_\tau(G)^K$.

(iv) Show that $\|Pf\| \leq C\|f\|$ for some constant C for all $f \in L^1(G)$.

(v) Show that $L^1_\tau(G)^K$ is closed in $L^1(G)$.

(vi) Show that the convolution product of two K-central functions of K-type τ is again a K-central function of type τ, and that $f^*(x) = \overline{f(x^{-1})}$ is also a K-central function of type τ. (Hint: use K-centrality of f and that $\overline{\xi_\tau(k)} = \xi_\tau(k^{-1})$.)

So we have associated two spaces to the irreducible unitary representation space $\tau \in \hat{K}$, namely $L^1_\tau(G)^K$ and $L^1_\tau(G; \mathrm{End}(V_\tau))$, both being Banach $*$-algebras.

Theorem 5.1.11. *The map $S = S_\tau\colon L^1_\tau(G; \mathrm{End}(V_\tau)) \to L^1_\tau(G)^K$ given by*

$$SF(x) = d_\tau \mathrm{Tr}\,(F(x))$$

is a continuous $*$-algebra isomorphism, where the inverse $S^{-1}\colon L^1_\tau(G)^K \to L^1_\tau(G; \mathrm{End}(V_\tau))$ is given by

$$S^{-1}f(x) = \int_K f(xk^{-1})\,\tau(k)\, dk$$

and S^{-1} is continuous.

We note that there is another simple relation between F and SF, stating that if we average F over the adjoint action of K, that we get SF.

Lemma 5.1.12. *With the notation of Theorem 5.1.11 we have*

$$\int_K F(kgk^{-1})\, dk = \frac{1}{d_\tau^2}(SF)(g)\mathbf{1}_{V_\tau}, \qquad \forall g \in G.$$

Proof. Note that $\tau(k_1)\int_K F(kgk^{-1})\,dk = \int_K F(kgk^{-1})\,dk\,\tau(k_1)$, so that by Schur's Lemma 2.5.7 $\int_K F(kgk^{-1})\,dk = \lambda(g)\mathbf{1}$ for some constant depending on g. Taking the trace, and using $\mathrm{Tr}\,(F(kgk^{-1}))$ is independent of $k \in K$, we find $d_\tau\lambda(g) = \mathrm{Tr}\,(F(g))$ or $\lambda(g) = d_\tau^{-1}\mathrm{Tr}\,(F(g))$. $\square$

Proof of Theorem 5.1.11. There is a lot to check for the proof. Let us first check that SF ends up in the right space. So we check SF is K-central, of K-type τ and integrable. The K-centrality follows from

$$SF(kxk^{-1}) = d_\tau\mathrm{Tr}\,(F(kxk^{-1})) = d_\tau\mathrm{Tr}\,(\tau(k)F(x)\tau(k^{-1})) = d_\tau\mathrm{Tr}\,(F(x)) = SF(x)$$

To show that $SF(x)$ is of type τ, we choose an orthonormal basis for V with corresponding matrix $\tau(k) = (\tau(k)_{j,i})_{j,i=1}^{d_\tau}$. Then $SF(x) = d_\tau\sum_{i=1}^{d_\tau} F_{i,i}(x)$ and

$$SF(xk) = d_\tau\mathrm{Tr}\,(F(xk)) = d_\tau\mathrm{Tr}\,(F(x)\tau(k)) = d_\tau\sum_{i,j=1}^{d_\tau} F_{i,j}(x)\tau_{j,i}(k)$$

so that

$$\int_K SF(xk)\xi_\tau(k^{-1})\,dk = d_\tau^2\sum_{i,j=1}^{d_\tau} F_{i,j}(x)\int_K \tau_{j,i}(k)\sum_{r=1}^{d_\tau}\overline{\tau_{r,r}(k)}\,dk = d_\tau\sum_{i=1}^{d_\tau} F_{i,i}(x) = SF(x)$$

by Schur's orthogonality relations for K, see Theorem 3.3.6. Finally, since

$$\|F\| = \int_G \sqrt{\sum_{i,j=1}^{d_\tau}|F_{i,j}(g)|^2}\,dg,$$

we see that $\int_G |F_{i,j}(g)|\,dg \leq \|F\|$, and so we see that $\int_G |SF(x)|\,dx \leq d_t\sum_{i=1}^{d_\tau}\int_G |F_{i,i}(x)|\,dx \leq d_\tau^2\|F\|$. This proves that S goes into the Banach space $L^1_\tau(G)^K$ and S is a bounded linear map.

Similarly, for $f \in L^1_\tau(G)^K$ we need to check that $S^{-1}f \in L^1_\tau(G;\mathrm{End}(V_\tau))$. The right K-behaviour follows using the right invariance of the Haar measure on K and τ being a representation. For the left K-behaviour we also need f being K-central, so

$$\begin{aligned}
S^{-1}f(k_1x) &= \int_K f(k_1xk^{-1})\,\tau(k)\,dk = \int_K f(xk^{-1}k_1)\,\tau(k)\,dk \\
&= \int_K f(xk^{-1})\,\tau(k_1k)\,dk = \tau(k_1)\int_K f(xk^{-1})\,\tau(k)\,dk = \tau(k_1)S^{-1}f(x).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|S^{-1}f\| &= \int_G \|\int_K f(xk^{-1})\,\tau(k)\,dk\|\,dx \leq \int_G\int_K \|f(xk^{-1})\,\tau(k)\|\,dkdx \\
&\leq \int_G\int_K |f(xk^{-1})|\,dkdx = \int_G |f(x)|\,dx
\end{aligned}$$

Now we need to show that S and S^{-1} are indeed each others inverse. Firstly,

$$SS^{-1}f(x) = d_\tau \mathrm{Tr}\,(S^{-1}f(x)) = d_\tau \mathrm{Tr}\,(\int_K f(xk^{-1})\tau(k)\,dk) = d_\tau \int_K f(xk^{-1})\chi(k)\,dk = f(x)$$

since f is of K-type τ. For the other way around we use again a choice of orthonormal basis, and then

$$S^{-1}SF(x) = \int_K (SF)(xk^{-1})\,\tau(k)\,dk = d_\tau \int_K \mathrm{Tr}\,(F(x)\tau(k^{-1}))\,\tau(k)\,dk$$

and taking the (m,n)-entry gives

$$\begin{aligned}\big(S^{-1}SF(x)\big)_{m,n} &= d_\tau \int_K \sum_{r,s=1}^{d_\tau} (F(x))_{r,s}(\tau(k^{-1}))_{s,r}\,(\tau(k))_{m.n}\,dk \\ &= d_\tau \sum_{r,s=1}^{d_\tau} (F(x))_{r,s} \int_K \overline{(\tau(k))_{r,s}}\,(\tau(k))_{m.n}\,dk = (F(x))_{m,n}\end{aligned}$$

by the Schur orthogonality relations for K, see Theorem 3.3.6.

We leave the last statement to Exercise 5.1.13. □

Exercise 5.1.13. For the proof of Theorem 5.1.11 we need to establish the homomorphism property. Note first that $S^{-1}f$ can be defined for any $f \in L^1(G)$. For $f \in L^1(G)$ we identify this with $f \in L^1(G; \mathrm{End}(V_\tau))$ as $f(x)\mathbf{1}$.

(i) For $f_1, f_2 \in L^1(G)$ show that $S^{-1}(f_1 * f_2) = f_1 * S^{-1}(f_2)$.

(ii) For $f_1, f_2 \in L^1(G)$ and f_2 K-central show that $f_1 * S^{-1}f_2 = S^{-1}(f_1) * S^{-1}(f_2)$ and $S^{-1}(f_1 * f_2) = S^{-1}(f_1) * f_2$.

Definition 5.1.14. *Let G be a linear group, $K \subset G$ a compact subgroup and $\tau \in \hat{K}$ an irreducible finite dimensional representation of K. The triple (G,K,τ) is a commutative triple if the convolution algebra $L^1_\tau(G; \mathrm{End}(V_\tau))$ is a commutative algebra. The pair (G,K) is a strong Gelfand pair if (G,K,τ) is a commutative triple for all $\tau \in \hat{K}$.*

Corollary 5.1.15. *(G,K,τ) is a commutative triple if and only if $L^1_\tau(G)^K$ is a commutative algebra. (G,K) is a strong Gelfand pair if and only if $L^1_\tau(G)^K$ is a commutative algebra for all $\tau \in \hat{K}$.*

Note that, trivially, a strong Gelfand pair is a Gelfand pair. This is also the case for a commutative triple up to a topological assumption, and so the unimodularity assumption on G at the beginning of Chapter 5 is actually automatically satisfied assuming we know that Gelfand pairs are unimodular, cf. Chapter 4. For the next statement we refer to [64, Prop. 3.3].

Proposition 5.1.16. *If (G, K, τ) is a commutative triple and the homogeneous space G/K is connected, then (G, K) is a Gelfand pair and in particular G is unimodular.*

The following result is the analogue of Theorem 4.4.1, see [28] and references given there.

Theorem 5.1.17. *(G, K, τ) is a commutative triple if and only if for every irreducible unitary representation of G the K-representation $\pi|_K$ contains the K-representation τ at most once, i.e. $[\pi|_K : \tau] \leq 1$ for all $\pi \in \hat{G}$.*

Note that by Frobenius reciprocity, this means that the induced representation $\mathrm{Ind}_K^G \tau$ splits multiplicity free in case G is a compact group.

Corollary 5.1.18. *(G, K) is a strong Gelfand pair if and only if every irreducible unitary representation of G splits multiplicity free when restricted to K.*

Exercise 5.1.19. Show that the examples of Gelfand pairs in Section 4.2, $(\mathrm{SU}(2), \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)))$ and $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$ are strong Gelfand pairs using (3.2.8) and (3.4.5).

We finish with a characterisation of strong Gelfand pairs. In order to describe this, note that we can associate to the pair (G, K) the pair $(G \times K, \mathrm{diag}(K))$, where $\mathrm{diag}(K) \cong K$ is embedded diagonally in the Cartesian product group $G \times K$, $K \ni k \mapsto (k, k) \in G \times K$.

Theorem 5.1.20. *(G, K) is a strong Gelfand pair if and only if $(G \times K, \mathrm{diag}(K))$ is a Gelfand pair.*

Example 5.1.21. In Exercise 5.1.19 you have shown that $\mathrm{SU}(2) \times \mathrm{SU}(2)$ with the diagonal subgroup provides us with a strong Gelfand pair. According to Theorem 5.1.20 we have that $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{diag} \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ with $\mathrm{diag}(\mathrm{diag}) \cong \mathrm{diag}$ is a Gelfand pair. So in this case this means that for any spins $\ell_1, \ell_2, \ell_3 \in \frac{1}{2}\mathbb{N}$, the trivial representation occurs at most once in the threefold tensor product $\pi_{\ell_1} \otimes \pi_{\ell_2} \otimes \pi_{\ell_3}$. By Schur's Lemma 2.5.7 the multiplicity is $[\pi_{\ell_1} \otimes \pi_{\ell_2} \otimes \pi_{\ell_3}, 1] = \dim_{\mathbb{C}} \mathrm{Hom}_K(V_{\ell_1} \otimes V_{\ell_2} \otimes V_{\ell_3}, \mathbb{C})$. Using the identification $V \otimes W^* = \mathrm{Hom}(W, V)$ we see that $\dim_{\mathbb{C}} \mathrm{Hom}_K(V_{\ell_1} \otimes V_{\ell_2} \otimes V_{\ell_3}, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Hom}_K(V_{\ell_1} \otimes V_{\ell_2}, V_{\ell_3}^*)$, which is the multiplicity $[\pi_{\ell_1} \otimes \pi_{\ell_2}, \pi_{\ell_3}^*]$. Since π_{ℓ_3} is equivalent to its contragredient representation we find

$$[\pi_{\ell_1} \otimes \pi_{\ell_2} \otimes \pi_{\ell_3}, 1] = [\pi_{\ell_1} \otimes \pi_{\ell_2}, \pi_{\ell_3}].$$

And this formula illustrates the equivalence of Theorem 5.1.20 that $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$ is a strong Gelfand pair being equivalent to $(\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$ being a Gelfand pair.

Exercise 5.1.22. We give a brief sketch of proof of Theorem 5.1.20.

(i) Let $L^1(G)^K$ be the space of integrable K-central functions, and $L^1(G \times K)^{\mathrm{diag}}$ the space of $G \times K$-integrable bi-*diag*-invariant functions of $G \times K$. Show that $L^1(G)^K$ is a Banach $*$-algebra. Show that

$$T \colon L^1(G)^K \to L^1(G \times K)^{\mathrm{diag}}, \qquad (Tf)(g, k) = f(k^{-1}g)$$

gives an isometric isomorphism of Banach $*$-algebras with inverse $(T^{-1}f)(g) = f(g, e)$.

(ii) Show that $L^1(G)^K = \bigoplus_{\tau\in\hat{K}} L^1_\tau(G)^K$ and $L^1_\tau(G)^K * L^1_\sigma(G)^K = \{0\}$ if $\tau \not\cong \sigma$.

(iii) Show that (G, K) is a strong Gelfand pair if and only if $L^1(G)^K$ is a commutative Banach $*$-algebra, and finish the proof of Theorem 5.1.20.

Exercise 5.1.23. (i) Determine the image of T restricted to $L^1_\tau(G)^K$, with T as in Exercise 5.1.22.

(ii) Let $\phi\colon G \to \mathbb{C}$ be a K-central function. Show that the following properties are equivalent:

(a) $T\phi$ satisfies the product formula for spherical functions for the Gelfand pair $(G \times K, \mathrm{diag})$, see Theorem 4.1.16, i.e.

$$\int_K (T\phi)\big((g_1,k_1)(k,k)(g_2,k_2)\big)\,dk = (T\phi)\big((g_1,k_1)\big)\,(T\phi)\big((g_2,k_2)\big),$$

for all $g_1, g_2 \in G$, and for all $k_1, k_2 \in K$.

(b) $\int_K \phi(g_1 k g_2 k^{-1})\,dk = \phi(g_1)\phi(g_2)$ for all $g_1, g_2 \in G$, cf. Proposition 4.2.9.

5.2 Matrix spherical functions

In this section we assume that (G, K, τ) is a commutative triple. So we have two commutative convolution algebras $L^1_\tau(G)^K$ and $L^1_\tau(G; \mathrm{End}(V_\tau))$, both generalising the algebra of bi-K-invariant functions in Chapter 4. We now take a look at these two commutative algebras and their characters.

Definition 5.2.1. *Let (G, K, τ) be a commutative triple. The non-zero bounded function $\Phi\colon G \to \mathrm{End}(V_\tau)$ satisfying (5.1.1) is a τ-matrix spherical function, or a matrix spherical function of type τ, if*

$$L^1_\tau(G;\mathrm{End}(V_\tau)) \to \mathbb{C}, \qquad F \mapsto \hat{F}(\Phi) = \frac{1}{d_\tau}\mathrm{Tr}\,\Big(\int_G F(g)\Phi(g^{-1})\,dg\Big) = \frac{1}{d_\tau}\mathrm{Tr}\,(F * \Phi(e))$$

is a homomorphism.

Note that the convolution product of a L^1 and L^∞-function is well-defined. Note moreover, that Φ is viewed as an element of $L^\infty_\tau(G; \mathrm{End}(V_\tau))$. A similar remark holds for the $\phi \in L^\infty_\tau(G)^K$ in Definition 5.2.2.

Definition 5.2.2. *Let (G, K, τ) be a commutative triple. The non-zero bounded function $\varphi\colon G \to \mathbb{C}$ which is K-central and of K-type τ is trace τ-spherical function if*

$$L^1_\tau(G)^K \to \mathbb{C}, \qquad f \mapsto \hat{f}(\varphi) = \int_G f(g)\varphi(g^{-1})\,dg = f * \varphi(e)$$

is a homomorphism.

Again, the convolution product is well defined. Note that the zero function would also give a homomorphism of the convolution algebras in Definitions 5.2.1 and 5.2.2, but we want to exclude the trivial case.

Lemma 5.2.3. *The following are equivalent for a commutative triple* (G, K, τ) *and* $\varphi \in L^1_\tau(G)^K$*:*

(i) φ *is a trace* τ*-spherical function;*

(ii) $f * \varphi = (f * \varphi)(e)\,\varphi \in L^\infty_\tau(G)^K$ *for all* $f \in L^1_\tau(G)^K$*.*

The statement in (ii) takes place in $L^\infty_\tau(G)^K$, since the convolution product of $L^1_\tau(G)^K$ and $L^\infty_\tau(G)^K$ ends in $L^\infty_\tau(G)^K$. Indeed, Exercise 5.1.10 shows that the convolution of K-central functions of type τ is again K-central and of type τ assuming that the integral is well-defined. So it then follows from the classical result that the convolution of a bounded function and an integrable function is a bounded function.

Since the convolution product is a smoothing operator, we see that for a continuous function $f \in L^1_\tau(G)^K$ with non-zero character, Lemma 5.2.3(ii) shows that the function φ is continuous. Next evaluating 5.2.3(ii) at e gives the following result.

Corollary 5.2.4. *The trace* τ*-spherical function is continuous and* $\varphi(e) = 1$*.*

Proof of Lemma 5.2.3. Assume (ii), then for $f_1, f_2 \in L^1_\tau(G)^K$ we have

$$(f_1 * f_2) * \varphi = f_1 * (f_2 * \varphi) = f_1 * (f_2 * \varphi(e)\varphi) = (f_2 * \varphi(e))\, f_1 * \varphi$$

and evaluating at e gives that φ is a τ-spherical function.

For the converse implication $(i) \implies (ii)$, see Exercise 5.2.5. $\square$

Exercise 5.2.5. Assume that φ is a trace τ-spherical function. Show that homomorphism property leads to

$$\int_G (f_2 * \varphi)(g^{-1}) f_1(g)\, dg = \int_G (f_2 * \varphi)(e)\varphi(g^{-1}) f_1(g)\, dg$$

for all $f_1, f_2 \in L^1_\tau(G)^K$. Conclude (ii).

For Gelfand pairs the spherical functions are characterised by the product formula, see Theorem 4.1.16. A similar, but more involved, characterisation also holds for matrix spherical functions.

Theorem 5.2.6. *Assume* $\Phi \in L^\infty_\tau(G; \mathrm{End}(V_\tau))$ *and let* $\varphi = \frac{1}{d_\tau^2} S_\tau(\Phi)$*, so* $\varphi(g) = \frac{1}{d_\tau} \mathrm{Tr}\,(\Phi(g))$*. Then the following statements are equivalent:*

(i) Φ *is a* τ*-matrix spherical function;*

(ii) φ *is a trace* τ*-spherical function;*

(iii) Φ *is non-zero and satisfies the product formula*

$$\Phi(g)\Phi(h) = \int_K \Phi(gkh)\,\overline{\xi_\tau(k)}\,dk, \qquad \forall g,h \in G;$$

(iv) φ *is non-zero and satisfies the product formula*

$$\varphi(g)\varphi(h) = \int_K \varphi(gkhk^{-1})\,dk, \qquad \forall g,h \in G;$$

(v) Φ *is non-zero and satisfies the product formula*

$$\varphi(g)\Phi(h) = \int_K \tau(k^{-1})\,\Phi(gkh)\,dk = \int_K \Phi(hkg)\,\tau(k^{-1})\,dk, \qquad \forall g,h \in G.$$

Moreover, $\Phi(e) = \mathbf{1}_{V_\tau}$.

Remark 5.2.7. Before embarking on the proof of Theorem 5.2.6, note that $\varphi \in L^\infty_\tau(G)^K$ by Theorem 5.1.11 adapted to L^∞ instead of L^1. The last statement has no meaning under the conditions of Theorem 5.2.6, since we only assume Φ to be given up to sets of measure zero. We will reduce this in a similar way to the statement in Corollary 5.2.4 for a trace τ-spherical function.

Exercise 5.2.8 can be proved directly using the definitions, and it can also be proved using the results of Theorem 5.2.6. It is instructive to understand both approaches.

Exercise 5.2.8. Recall the results of Exercises 5.1.22 and 5.1.23.

(i) Prove the following statement: assume that $\phi\colon G \to \mathbb{C}$ is a trace τ-spherical function, then $T\phi$ is a spherical function for the Gelfand pair $(G \times K, \mathrm{diag})$.

(ii) Prove the following statement: assume $\phi\colon G \to \mathbb{C}$ is a bounded K-central function such that $T\phi$ is a spherical function for the Gelfand pair $(G \times K, \mathrm{diag})$, if $\phi *_K \xi_\tau \neq 0$, then $\phi *_K \xi_\tau$ is a trace τ-spherical function.

(iii) Show that a K-central function ϕ is a positive definite function if and only if $T\phi$ is a positive definite function.

Proof of Theorem 5.2.6. We first show that (i) and (ii) are equivalent, and for this we use the fact that $S = S_\tau$ preserves convolution products for the convolution product of an L^1 and L^∞ function, cf. Theorem 5.1.11 and Exercise 5.1.13. So for any $F \in L^1_\tau(G)^K$ we get

$$S_\tau(F) * \varphi = \frac{1}{d_\tau^2} S_\tau(F) * S_\tau(\Phi) = \frac{1}{d_\tau^2} S_\tau(F * \Phi)$$

and evaluating at e we see that $S_\tau(F) * \varphi(e) = \frac{1}{d_\tau}\mathrm{Tr}\,(F * \Phi(e))$. Since $S_\tau\colon L^1_\tau(G;\mathrm{End}(V_\tau)) \to L^1_\tau(G)^K$ is an algebra isomorphism by Theorem 5.1.11, the equivalence of (i) and (ii) follows from Definitions 5.2.1 and 5.2.2.

Next we show that (iv) implies (ii). Assume (iv), and take $f \in L^1_\tau(G)^K$, then

$$\begin{aligned}(f*\varphi)(g) &= \int_G f(h)\varphi(h^{-1}g)\,dh = \int_G f(khk^{-1})\varphi(h^{-1}g)\,dh \\ &= \int_G f(h)\varphi(k^{-1}h^{-1}kg)\,dh = \int_G f(h)\varphi(h^{-1}kgk^{-1})\,dh\end{aligned}$$

since f and φ are K-central. Integrating over K gives, assuming (iv),

$$\begin{aligned}(f*\varphi)(g) &= \int_K (f*\varphi)(g)\,dk = \int_G\int_K f(h)\varphi(h^{-1}kgk^{-1})\,dh\,dk \\ &= \int_G f(h)\varphi(h^{-1})\varphi(g)\,dh = (f*\varphi)(e)\,\varphi(g)\end{aligned}$$

so that Lemma 5.2.3 proves (ii). The converse statement, i.e. (ii) implies (iv), is relegated to Exercise 5.2.8.

Assume now that (iii) holds, so using Lemma 5.1.12 we have

$$\begin{aligned}\varphi(g)\varphi(h)\mathbf{1}_{V_\tau} &= \int_K\int_K \Phi(k_1gk_1^{-1})\Phi(k_2hk_2^{-1})\,dk_1dk_2 \\ &= \int_K\int_K\int_K \Phi(k_1gk_1^{-1}k_3k_2hk_2^{-1})\,\overline{\xi_\tau(k_3)}\,dk_1dk_2dk_3.\end{aligned}$$

Now use the left invariance of the dk_2 integration, to move the k_3 in the argument of Φ to the right, where it can be taken out using that Φ is τ-invariant. So the k_3-integration is

$$\int_K \tau(k_3)\overline{\xi_\tau}(k_3)\,dk_3 = \mathbf{1}_{V_\tau}$$

by the Schur orthogonality relations of Theorem 3.3.6. So we find

$$\begin{aligned}\varphi(g)\varphi(h)\mathbf{1}_{V_\tau} &= \int_K\int_K \Phi(k_1gk_1^{-1}k_2hk_2^{-1})\,dk_1dk_2 = \int_K\int_K \Phi(k_1gk_2hk_2^{-1}k_1^{-1})\,dk_1dk_2 \\ &= \int_K \varphi(gk_2hk_2^{-1})\mathbf{1}_{V_\tau}\,dk_2\end{aligned}$$

which is (iv). The converse statement, i.e. (iv) implies (iii), is relegated to Exercise 5.2.10.

Next we prove that (iii) implies (v). Using Lemma 5.1.12 we see

$$\begin{aligned}\varphi(g)\Phi(h) &= \int_K \Phi(kgk^{-1})\Phi(h)\,dk = \int_K\int_K \Phi(kgk^{-1}k_1h)\,\overline{\xi_\tau(k_1)}dkdk_1 \\ &= \int_K\int_K \tau(k)\Phi(gk_1h)\,\overline{\xi_\tau(kk_1)}dkdk_1 = \int_K \tau(k_1^{-1})\Phi(gk_1h)\,dk_1\end{aligned}$$

where the last equality follows from the Schur orthogonality relations of Theorem 3.3.6. Since $\varphi(g)$ is a scalar, we also have

$$\varphi(g)\Phi(h) = \Phi(h)\varphi(g) = \int_K \Phi(h)\Phi(kgk^{-1})\, dk$$

and we get similarly that this equals $\int_K \Phi(hgk)\,\tau(k^{-1})\,dk$.

Next we show that (v) implies (iv). So for $g, h \in G$ we have

$$\begin{aligned}
\varphi(g)\varphi(h)\mathbf{1}_{V_\tau} &= \int_K \varphi(g)\Phi(khk^{-1})\,dk = \int_K\int_K \tau(k_1^{-1})\Phi(gk_1khk^{-1})\,dkdk_1 \\
&= \int_K\int_K \tau(k_1^{-1})\Phi(gkhk^{-1}k_1)\,dkdk_1 = \int_K\int_K \tau(k_1^{-1})\Phi(gkhk^{-1})\tau(k_1)\,dkdk_1 \\
&= \int_K\int_K \Phi(k_1^{-1}gkhk^{-1}k_1)\,dkdk_1 = \int_K \varphi(gkhk^{-1})\mathbf{1}_{V_\tau}\,dk
\end{aligned}$$

which is (iii).

Now that we have proved the equivalences of the statements (i)-(v) (up to Exercises 5.2.9 and 5.2.10), it remains to prove the last statement. First observe that φ is a trace τ-spherical function, hence continuous by Corollary 5.2.4. Then, by the equivalence of (i) and (ii) and Theorem 5.1.11, $\Phi = d_\tau^2 S_\tau^{-1}\varphi$ is also continuous. By (iii) we have

$$\Phi(g)\Phi(e) = \int_K \Phi(xk)\overline{\xi_\tau(k)}\,dk = \Phi(g)\int_K \tau(k)\overline{\xi_\tau(k)}\,dk = \Phi(g)$$

again by Schur's orthogonality relations, Theorem 3.3.6. Since $\tau(k)\Phi(e) = \Phi(k) = \Phi(e)\tau(k)$ we see that by Schur's Lemma 2.5.7, $\Phi(e)$ is a multiple of the identity. Since Φ is nonzero, $\Phi(g)\Phi(e) = \Phi(g)$ for all $g \in G$ implies $\Phi(e) = \mathbf{1}_{V_\tau}$. □

It remains to prove two of the implications of the equivalences in Theorem 5.2.6.

Exercise 5.2.9. In this exercise we prove that (ii) implies (iv). For this we need several steps, and we assume (ii) of Theorem 5.2.6.

(i) Show that for $f \in L^1_\tau(G)^K$ the function ${}_gf\colon G \to \mathbb{C}$ defined by ${}_gf(h) = \int_K f(kgk^{-1}h)\,dk$ is contained in $L^1_\tau(G)^K$. Note that ${}_ef = f$. Similarly, show that $\varphi_g(h) = \int_K \varphi(hkgk^{-1})\,dk$ is contained in $L^\infty_\tau(G)^K$. So the statement to be proved is $\varphi_g(h) = \varphi(g)\varphi(h)$.

(ii) Show that for $f \in L^1_\tau(G)^K$ we have $({}_hf * \varphi)(g) = (f * \varphi_g)(h)$.

(iii) Show that for $f \in L^1_\tau(G)^K$ we have $f * \varphi_g = (f * \varphi)(e)\,\varphi_g$ using Lemma 5.2.3.

(iv) Show that for $f \in L^1_\tau(G)^K$ we have

$$(f * \varphi)(e)\,\varphi_g(h) = ({}_hf * \varphi)(e)\,\varphi(g) = (f * \varphi)(e)\,\varphi(h)\varphi(g)$$

and prove (iv). Use Lemma 5.2.3.

Exercise 5.2.10. In this exercise we prove that (iv) implies (iii). We consider $g, h \in G$.

(i) Use Theorem 5.1.11 and (iv) to show that

$$\Phi(g)\Phi(h) = d_\tau^4 \int_K \int_K \int_K \varphi(gk_3hk_2^{-1}k_3^{-1}k_2k_1^{-1})\, \tau(k_1)\, dk_1 dk_2 dk_3$$

(ii) In (i) recognise the inner integral as $\Phi(gk_3hk_2^{-1}k_3^{-1}k_2)$ up to a factor. Use this to obtain

$$\Phi(g)\Phi(h) = d_\tau^2 \int_K \int_K \Phi(gk_3h)\tau(k_2^{-1}k_3^{-1}k_2)\, dk_2 dk_3$$

(iii) Use the Schur orthogonality relations of Theorem 3.3.6 to derive

$$d_\tau \int_K \tau(k_2^{-1}k_3^{-1}k_2)\, dk_2 = \chi_\tau(k_3^{-1})\mathbf{1}_{V_\tau}.$$

and finish the proof of the implication.

Exercise 5.2.11. Assume that $\tau \in \hat{K}$ is a one-dimensional representation, not necessarily the trivial representation. Show that $\Phi = \varphi$ and the three product formulas in Theorem 5.2.6(iii), (iv), (v) are indeed the same.

5.3 Matrix spherical functions for compact commutative triples

In this section we assume that G is a compact linear group. According to Theorem 5.1.17 we have that $[\pi|_K : \tau] \leq 1$ for all $\pi \in \hat{G}$ assuming (G, K, τ) is a compact commutative triple. We put

$$\hat{G}_\tau = \{\pi \in \hat{G} \mid [\pi|_K : \tau] = 1\} \tag{5.3.1}$$

so that the space of K-intertwiners $B_K(V_\tau, V_\pi^G)$ is one-dimensional by Schur's Lemma 2.5.7. Since we can assume that all representations are unitary, we pick a partial isometry $j_\pi \in B_K(V_\tau, V_\pi^G)$ which is then unique up to a phase factor by Schur's Lemma 2.5.7. Note that $j_\pi^* \in B_K(V_\pi^G, V_\tau)$ and $j_\pi \circ j_\pi^* \in B_K(V_\pi^G)$ is the orthogonal projection on V_τ in V_π^G and $j_\pi^* \circ j_\pi$ is the identity on V_τ. Then the function

$$\Phi_\pi : G \to \mathrm{End}(V_\tau), \qquad \Phi_\pi(g) = j_\pi^* \circ \pi(g) \circ j_\pi \tag{5.3.2}$$

is in $L_\tau^1(G, \mathrm{End}(V_\tau))$ and is independent of the choice of j_π. To see this take $k_1, k_2 \in K$ and $g \in G$ so

$$\Phi_\pi(k_1gk_2) = j_\pi^* \circ \pi(k_1)\pi(g)\pi(k_2) \circ j_\pi = \tau(k_1) \circ j_\pi^* \circ \pi(g) \circ j_\pi \circ \tau(k_2) = \tau(k_1)\Phi_\pi(g)\tau(k_2)$$

since j_π and j_π^* are K-intertwiners. Note that the function Φ_π is continuous since the representation π is continuous, and $\Phi_\pi(g)^* = \Phi_\pi(g^{-1})$ or $\Phi_\pi^* = \Phi_\pi$.

Theorem 5.3.1. *Assume (G, K, τ) is a compact commutative triple. The functions $\{\Phi_\pi \mid \pi \in \hat{G}_\tau\} \subset C_\tau(G; \mathrm{End}(V_\tau))$ are linearly independent and they span $C_\tau(G; \mathrm{End}(V_\tau))$. Moreover, we have the orthogonality relations*

$$\int_G \mathrm{Tr}\,\big((\Phi_{\pi'}(g))^* \Phi_\pi(g)\big)\, dg = \delta_{\pi,\pi'} \frac{(\dim V_\tau)^2}{\dim V_\pi^G}$$

and Φ_π is a τ-matrix spherical function.

Proof. Since the matrix entries are dense in the continuous functions by the Peter-Weyl Theorem 3.3.14, we see that functions in $C_\tau(G; \mathrm{End}(V_\tau))$ can be written as combinations of matrix entries. The only ones leading to elements in $C_\tau(G; \mathrm{End}(V_\tau))$ are then the functions Φ_π for $\pi \in \hat{G}_\tau$.

To see the orthogonality relations we pick an orthonormal basis $(v_n)_{n=1}^{d_\tau}$ of V_τ so that $(j_\pi(v_n))_{n=1}^{d_\tau}$ is part of an orthonormal basis for V_π^G, and similarly for π'. We find

$$\begin{aligned}
\mathrm{Tr}\,\big((\Phi_{\pi'}(g))^* \Phi_\pi(g)\big) &= \sum_{n=1}^{d_\tau} \langle (\Phi_{\pi'}(g))^* \Phi_\pi(g) v_n, v_n\rangle = \sum_{n=1}^{d_\tau} \langle \Phi_\pi(g) v_n, \Phi_{\pi'}(g) v_n\rangle \\
&= \sum_{m,n=1}^{d_\tau} \langle \Phi_\pi(g) v_n, v_m\rangle \langle v_m, \Phi_{\pi'}(g) v_n\rangle = \sum_{m,n=1}^{d_\tau} \langle \pi(g) j_\pi(v_n), j_\pi(v_m)\rangle \langle j_{\pi'}(v_m), \pi'(g) j_{\pi'}(v_n)\rangle \\
&= \sum_{m,n=1}^{d_\tau} \langle \pi(g) j_\pi(v_n), j_\pi(v_m)\rangle \overline{\langle \pi'(g) j_{\pi'}(v_n), j_{\pi'}(v_m)\rangle}
\end{aligned}$$

so that integrating over the compact group G then immediately gives zero for $\pi \not\cong \pi'$ by Schur's orthogonality relations of Theorem 3.3.6. In case $\pi \cong \pi'$ we may take $\pi = \pi'$, and we get

$$\begin{aligned}
\int_G \mathrm{Tr}\,\big((\Phi_\pi(g))^* \Phi_\pi(g)\big)\, dg &= \sum_{m,n=1}^{d_\tau} \int_G \langle \pi(g) j_\pi(v_n), j_\pi(v_m)\rangle \overline{\langle \pi(g) j_\pi(v_n), j_\pi(v_m)\rangle}\, dg \\
&= \sum_{m,n=1}^{d_\tau} \frac{1}{\dim V_\pi^G} \langle j_\pi(v_n), j_\pi(v_n)\rangle \langle j_\pi(v_m), j_\pi(v_m)\rangle = \frac{d_\tau^2}{\dim V_\pi^G}
\end{aligned}$$

again by Schur's orthogonality relations of Theorem 3.3.6.

In order to show that the functions Φ_π are spherical functions, we need

$$\Phi_\pi * \Phi_{\pi'} = \delta_{\pi,\pi'} \frac{\dim V_\tau}{\dim V_\pi^G} \Phi_\pi, \tag{5.3.3}$$

see Exercise 5.3.2. Since the functions Φ_π for $\pi \in \hat{G}_\tau$ span the space $L^1_\tau(G; \mathrm{End}(V_\tau))$, it suffices to check that

$$\widehat{\Phi_\sigma * \Phi_\rho}(\pi) = \hat{\Phi}_\sigma(\pi)\, \hat{\Phi}_\rho(\pi), \qquad \forall\, \sigma, \rho \in \hat{G}_\tau, \tag{5.3.4}$$

where $\hat{F}(\pi) = \frac{1}{d_\tau}\mathrm{Tr}\,(F * \Phi_\pi)(e))$ for $\pi \in \hat{G}_\tau$, cf. Definition 5.2.1. Now (5.3.3) and Theorem 5.2.6 imply that for $\Phi_\sigma \in L^1_\tau(G; \mathrm{End}(V_\tau))$ we have

$$\hat{\Phi}_\sigma(\pi) = \frac{\delta_{\sigma,\pi}}{d_\tau}\frac{d_\tau}{\dim V_\pi^G}\mathrm{Tr}\,(\Phi_\pi(e)) = \delta_{\sigma,\pi}\frac{d_\tau}{\dim V_\pi^G}.$$

So both sides of (5.3.4) vanish unless $\sigma \cong \rho \cong \pi$. In case $\sigma \cong \rho \cong \pi$, both sides are equal to $d_\tau^2(\dim V_\pi^G)^{-2}$. □

Exercise 5.3.2. Prove (5.3.3) by proving this for the matrix entries on both sides after a suitable choice of basis for V_τ. This is comparable to Exercise 4.1.3.

Exercise 5.3.3. In the setting of Theorem 5.3.1 we have a product formula for Φ_π, $\pi \in \hat{G}_\tau$, as in Theorem 5.2.6(iii). Show that this product formula follows from

$$\int_K \pi(k)\overline{\xi_\tau(k)}\,dk = P_{V_\tau}$$

where $P_{V_\tau} = j_\pi \circ j_\pi^* \colon V_\pi^G \to V_\pi^G$ is the orthogonal projection onto the (unique) copy of V_τ in V_π^G.

In this case the function $\varphi_\pi(g) = \frac{1}{d_\tau}\mathrm{Tr}\,(\Phi_\pi(g))$ is a trace with respect to the copy of V_τ embedded in V_π^G. Let $P_{V_\tau} = j_\pi \circ j_\pi^*$ be the corresponding orthogonal projection, then

$$\varphi_\pi(g) = \frac{1}{d_\tau}\mathrm{Tr}\,(\Phi_\pi(g)) = \frac{1}{d_\tau}\mathrm{Tr}_{\,V_\pi^G}(P_{V_\tau}\pi(g)P_{V_\tau}) = \frac{1}{d_\tau}\mathrm{Tr}_{\,V_\pi^G}(\pi(g)P_{V_\tau}) = \frac{1}{d_\tau}\mathrm{Tr}_{\,V_\pi^G}(P_{V_\tau}\pi(g)) \tag{5.3.5}$$

so φ_π is the trace of $\pi(g)$ restricted to the K-invariant subspace $j_\pi(V_\tau) \subset V_\pi^G$. Then the product formula of Theorem 5.2.6(iv) is similar the one for characters of compact groups viewed as spherical functions as in Proposition 4.2.9.

Exercise 5.3.4. Consider the case $G = \mathrm{SU}(2)$ and $K = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)) \cong \mathrm{U}(1)$ the diagonal subgroup. Determine all possible $\tau \in \hat{K}$ and all the corresponding spherical functions Φ (and hence φ, see Exercise 5.2.11). Extend this to the description of $L^1_{\tau_1,\tau_2}(G; \mathrm{Hom}(V_{\tau_2}, V_{\tau_1}))$ in this case, and show that all matrix entries of SU(2) are recovered by studying these spaces.

Similar to the result of Theorem 5.3.1 we have the following result.

Theorem 5.3.5. *Assume (G, K, τ) is a compact commutative triple. The functions $\{\varphi_\pi \mid \pi \in \hat{G}_\tau\} \subset C_\tau(G)^K$, $\varphi_\pi = S\Phi_\pi = \frac{1}{d_\tau}\mathrm{Tr}\,(\Phi_\pi(\cdot))$, are linearly independent and they span $C_\tau(G)^K$. Moreover, we have the orthogonality relations*

$$\int_G \varphi_\pi(g)\overline{\varphi_{\pi'}(g)}\,dg = \delta_{\pi,\pi'}\frac{1}{(\dim V_\tau)(\dim V_\pi^G)}$$

and φ_π is a trace τ spherical function.

Proof. Let $\Phi_\pi = j_\pi^* \circ \pi(g) \circ j_\pi$ and choose an orthonormal basis $(v_p)_{p=1}^{d_\tau}$ for V_τ. Then

$$\varphi_\pi(g)\overline{\varphi_{\pi'}(g)} = \frac{1}{d_\tau^2} \sum_{p,r=1}^{d_\tau} \langle \pi(g)j_\pi v_p, j_\pi v_p\rangle \overline{\langle \pi'(g)j_{\pi'} v_p, j_{\pi'} v_p\rangle}$$

Integrating over G and using the Schur orthogonality relations of Theorem 3.3.6, we obtain the orthogonality relations. The fact that these functions span $C_\tau(G)^K$ follows from the Peter Weyl Theorem 3.3.14.

Next we calculate similarly that

$$\varphi_\sigma * \varphi_\pi = \delta_{\sigma,\pi} \frac{1}{(\dim V_\tau)(\dim V_\pi^G)} \varphi_\pi$$

and this gives that φ_π is a trace τ-spherical function as in the proof of Theorem 5.3.1. $\square$

Let us now briefly consider the extension to functions $F\colon G \to \mathrm{Hom}(V_{\tau_2}, V_{\tau_1})$ for $\tau_1, \tau_2 \in \hat{K}$, see Exercises 5.1.2 and 5.1.6. As before, the representations τ_1 and τ_2 are assumed to be unitary acting in the finite-dimensional Hilbert spaces V_{τ_1} and V_{τ_2}. We assume that F satisfies

$$F(k_1 g k_2) = \tau_1(k_1) F(g) \tau_2(k_2), \qquad \forall\, k_1, k_2 \in K,\ \forall\, g \in G. \tag{5.3.6}$$

Note that we can no longer define a map as S in Theorem 5.1.11, since in general we do not have a trace on $\mathrm{Hom}(V_{\tau_2}, V_{\tau_1})$. Then $g \mapsto F(g)(F(g))^* \in \mathrm{End}(V_{\tau_1})$, respectively $g \mapsto (F(g))^* F(g) \in \mathrm{End}(V_{\tau_2})$, is a right K-invariant, respectively a left-K-invariant, function. It follows that the functions

$$g \mapsto \mathrm{Tr}_{V_{\tau_1}}\big(F(g)(F(g))^*\big), \qquad g \mapsto \mathrm{Tr}_{V_{\tau_2}}\big((F(g)^* F(g)\big) \tag{5.3.7}$$

are bi-K-invariant functions. Then we can still obtain orthogonality relations as in Proposition 5.3.6. Note that we don't have a convolution algebra of functions satisfying (5.3.6), see Exercise 5.1.6, so that we cannot require commutativity of an algebra to replace Definition 5.1.14.

Proposition 5.3.6. *Define the set*

$$\hat{G}_{\tau_1,\tau_2} = \{\pi \in \hat{G} \mid [\pi|_K : \tau_1] \geq 1 \,\wedge\, [\pi|_K : \tau_2] \geq 1\}$$

and for $\pi \in \hat{G}_{\tau_1,\tau_2}$ take $j_\pi^{\tau_i} \in \mathrm{Hom}_K(V_{\tau_i}, V_\pi^G)$, for $i = 1, 2$. Then

$$\Phi_\pi = \Phi_\pi(j_\pi^{\tau_1}, j_\pi^{\tau_2})\colon G \to \mathrm{Hom}(V_{\tau_2}, V_{\tau_1}), \qquad g \mapsto (j_\pi^{\tau_1})^* \circ \pi(g) \circ j_\pi^{\tau_2}$$

satisfies (5.3.6) *and for $\pi, \pi' \in \hat{G}_{\tau_1,\tau_2}$ we have the orthogonality relations*

$$\int_G \mathrm{Tr}_{V_{\tau_1}}\big(\Phi_\pi(j_\pi^{\tau_1}, j_\pi^{\tau_2}; g)(\Phi_{\pi'}(\tilde{j}_{\pi'}^{\tau_1}, \tilde{j}_{\pi'}^{\tau_2}; g))^*\big)\, dg = \delta_{\pi,\pi'} \frac{\mathrm{Tr}_{V_\pi^G}(j_\pi^{\tau_2}(\tilde{j}_\pi^{\tau_2})^*)\mathrm{Tr}_{V_{\tau_1}}((\tilde{j}_\pi^{\tau_1})^* j_\pi^{\tau_1})}{\dim V_\pi^G},$$

$$\int_G \mathrm{Tr}_{V_{\tau_2}}\big((\Phi_\pi(j_\pi^{\tau_1}, j_\pi^{\tau_2}; g))^* \Phi_{\pi'}(\tilde{j}_{\pi'}^{\tau_1}, \tilde{j}_{\pi'}^{\tau_2}; g)\big)\, dg = \delta_{\pi,\pi'} \frac{\mathrm{Tr}_{V_\pi^G}(j_\pi^{\tau_1}(j_\pi^{\tau_1})^*)\mathrm{Tr}_{V_{\tau_2}}((j_\pi^{\tau_2})^* \tilde{j}_\pi^{\tau_2})}{\dim V_\pi^G}$$

Moreover, the functions $\Phi_\pi(j_\pi^{\tau_1}, j_\pi^{\tau_2})$ for $\pi \in \hat{G}_{\tau_1,\tau_2}$, $j_\pi^{\tau_i}$ forming a basis of $\mathrm{Hom}(V_{\tau_i}, V_\pi^G)$, $i = 1, 2$, *span the space of continuous functions from $G \to \mathrm{Hom}(V_{\tau_2}, V_{\tau_1})$ with the transformation behaviour* (5.3.6).

Remark 5.3.7. (i) Note that the set $\hat{G}_{\tau_1,\tau_2}$ can be empty. As a simple example, consider the case $G = \mathrm{SU}(2)$ and $K = \mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(1)) \cong \mathrm{U}(1)$, then taking τ_1 an odd character of K and τ_2 an even character forces the spin ℓ of π to be of the form $\frac{1}{2}+\mathbb{N}$ and at the same time of the form $\mathbb{N}$, as follows from Chapter 3.
(ii) Since we have no analogue of a commutative triple and of Theorem 5.1.17, we do not know that $[\pi|_K\colon \tau_1] \geq 1$ implies $[\pi|_K\colon \tau_1] = \dim \mathrm{Hom}_K(V_{\tau_1}, V_\pi^G) = 1$. Since we are in the compact group situation, this number is finite. So there are possibly many choices for $j_\pi^{\tau_1}$ and $j_\pi^{\tau_2}$, which is reflected in the orthogonality relations of Proposition 5.3.6. However, in case (G,K,τ_1) and (G,K,τ_2) are both commutative triples, or in case (G,K) is a strong Gelfand pair, the orthogonality relations reduce to

$$\int_G \mathrm{Tr}_{V_{\tau_1}}\big(\Phi_\pi(g)(\Phi_{\pi'}(g))^*\big)\,dg = \delta_{\pi,\pi'}\frac{(\dim V_{\tau_1})(\dim V_{\tau_2})}{\dim V_\pi^G},$$
$$\int_G \mathrm{Tr}_{V_{\tau_2}}\big((\Phi_\pi(g))^*\Phi_{\pi'}(g)\big)\,dg = \delta_{\pi,\pi'}\frac{(\dim V_{\tau_1})(\dim V_{\tau_2})}{\dim V_\pi^G}.$$

Another interesting special case is when one of the K-representations is one-dimensional.

Exercise 5.3.8. Give a proof of Proposition 5.3.6.

In particular, the Banach $*$-algebra $L^1_\tau(G;\mathrm{End}(V_\tau))$ acts from the left on functions $F\colon G\to \mathrm{Hom}(V_\sigma,V_\tau)$ satisfying (5.3.6) whenever the convolution product is well defined for $\sigma\in\hat{K}$. We want to make this into a unitary representation of $L^1_\tau(G;\mathrm{End}(V_\tau))$, note that we do not assume (G,K,τ) a commutative triple (yet). We define $L^2_{\tau,\sigma}(G;\mathrm{Hom}(V_\sigma,V_\tau))$ as the Hilbert space completion of continuous functions $\Phi\colon G\to\mathrm{Hom}(V_\sigma,V_\tau)$ satisfying (5.3.6) with $\tau_1=\tau$, $\tau_2=\sigma$ with respect to the inner product

$$\langle\Phi,\Psi\rangle = \mathrm{Tr}_{V_\sigma}\big((\Psi^* * \Phi)(e)\big) = \int_G \mathrm{Tr}_{V_\sigma}\big(\Psi(g)^*\Phi(g)\big)\,dg.$$

Note that $\langle\Phi,\Phi\rangle = 0$ implies $\Phi(g)=0$ for all $g\in G$.

Exercise 5.3.9. Show that for $F\in L^1_\tau(G;\mathrm{End}(V_\tau))$ and $\Phi\in L^2_{\tau,\sigma}(G;\mathrm{Hom}(V_\sigma,V_\tau))$, the convolution product $F*\Phi\in L^2_{\tau,\sigma}(G;\mathrm{Hom}(V_\sigma,V_\tau))$ and that $\|F*\Phi\|\leq\|F\|\,\|\Phi\|$. (Hint: copy the idea of the proof for the convolution of a L^1 and L^2 function, which is a special case of Young's inequality.)

Proposition 5.3.10. *The map $\pi\colon L^1_\tau(G;\mathrm{End}(V_\tau))\to B(L^2_{\tau,\sigma}(G;\mathrm{Hom}(V_\sigma,V_\tau)))$ given by $\pi(F)\Phi = F*\Phi$ is a $*$-representation. Assume moreover (G,K,τ) is a commutative triple, and for every $\pi\in\hat{G}_\tau$ we have $j_\pi\in\mathrm{Hom}_K(V_\tau,V_\pi^G)$ and we pick a basis $(l_\pi^i)_{i=1}^l$ of $\mathrm{Hom}_K(V_\sigma,V_\pi^G)$, $l=\dim\mathrm{Hom}_K(V_\sigma,V_\pi^G)$, each l_π^i being an isometry, satisfying*

$$\mathrm{Tr}\left(l_\pi^i(l_\pi^k)^*\right) = \delta_{i,k}\dim V_\sigma,$$

then $g\mapsto\Phi_\pi(l_\pi^i,j_\pi;g) = j_\pi^*\circ\pi(g)\circ l_\pi^i$ forms an orthogonal basis of $L^2_{\tau,\sigma}(G;\mathrm{Hom}(V_\sigma,V_\tau))$ of eigenvectors for the commutative Banach $*$-algebra $L^1_\tau(G;\mathrm{End}(V_\tau))$.

Proof. By Exercise 5.3.9 the operator $\pi(F)$ is bounded. The homomorphism $\pi(F_1 * F_2) = \pi(F_1)\pi(F_2)$ follows from the associativity of the convolution product; $(F_1 * F_2) * \Phi = F_1 * (F_2 * \Phi)$. The unitarity follows from

$$\langle F * \Phi, \Psi \rangle = \mathrm{Tr}_{V_\sigma}((\Psi^* * F * \Phi)(e)) = \mathrm{Tr}_{V_\sigma}((F^* * \Psi)^* * \Phi)(e)) = \langle \Phi, F^* * \Psi \rangle$$

where we use the extension of the properties of the convolution products as established for $L^1_\tau(G; \mathrm{End}(V_\tau))$.

Assume now that (G, K, τ) is a commutative triple, so that we have a basis Φ_π for $\pi \in \hat{G}_\tau$ of $L^1_\tau(G; \mathrm{End}(V_\tau))$. Use Proposition 5.3.6 to conclude that

$$\langle \Phi_\pi(l^i_\pi, j_\pi; \cdot), \Phi_{\pi'}(l^k_{\pi'}, j_{\pi'}; \cdot) \rangle = \int_G \mathrm{Tr}_{V_\sigma}\left(\Phi_{\pi'}(l^k_{\pi'}, j_{\pi'}; g)^* \Phi_\pi(l^i_\pi, j_\pi; \cdot)\right) dg = \delta_{\pi,\pi'}\delta_{i,k} \frac{\dim V_\tau \dim V_\sigma}{\dim V^G_\pi}$$

so that we find an orthogonal set. The completeness follows from the Peter-Weyl Theorem 3.3.14. Now calculating the convolution product we find

$$\Phi_\pi * \Phi(j_{\pi'}, j^l_{\pi'}) = \delta_{\pi,\pi'} \frac{\dim V_\tau}{\dim V^G_\pi} \Phi(j_{\pi'}, j^l_{\pi'}). \tag{5.3.8}$$

□

Exercise 5.3.11. Prove (5.3.8).

5.4 Example: $(\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag})$

In this section we consider the example of the compact pair $(G, K) = (\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag})$. Note the case $n = 1$ gives rise to a strong Gelfand pair, see Exercise 5.1.19. However, for $n \geq 2$, this is no longer the case. But we do know that $(\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag})$ is a Gelfand pair as follows from Proposition 4.1.9 and defining $\theta \colon \mathrm{SU}(n+1) \times \mathrm{SU}(n+1) \to \mathrm{SU}(n+1) \times \mathrm{SU}(n+1)$ by $\theta(g_1, g_2) = (g_2, g_1)$, see Example 4.1.10(iii).

5.4.1 Commutative triples for $(\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag})$

We have $\mathrm{diag} \cong \mathrm{SU}(n+1)$. According to Theorem 5.1.17 we know that $(\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag}, \tau)$ with τ an irreducible $\mathrm{SU}(n+1)$-representation is a commutative triple if and only if the restriction of any representation of $\mathrm{SU}(n+1) \times \mathrm{SU}(n+1)$ restricted to the diagonal contains τ at most once. The irreducible representations of $\mathrm{SU}(n+1) \times \mathrm{SU}(n+1)$ are exterior tensor products of irreducible representations of $\mathrm{SU}(n+1)$, i.e. any irreducible representation of $\mathrm{SU}(n+1) \times \mathrm{SU}(n+1)$ is of the form $\pi_1 \times \pi_2$ acting on $V_{\pi_1} \otimes V_{\pi_2}$ by

$$(\pi_1 \times \pi_2)(g_1, g_2)\, v_1 \otimes v_2 = \pi_1(g_1)\, v_1 \otimes \pi_2(g_2)\, v_2$$

for π_i, $i = 1, 2$, irreducible representations of $\mathrm{SU}(n+1)$, see [84, Prop. 2.3.7]. Since the restriction to $K = \mathrm{diag} \cong \mathrm{SU}(n+1)$ of such an exterior tensor product is just the tensor product representation $\pi_1 \otimes \pi_2$ of $\mathrm{SU}(n+1)$, we see that $(\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag}, \tau)$ is a commutative triple if and only if for all $\pi_1, \pi_2 \in \widehat{\mathrm{SU}(n+1)}$ we have $[\pi_1 \otimes \pi_2 \colon \tau] \leq 1$ by Theorem 5.1.17. So this is gouverned by the analogue of the Clebsch-Gordan decomposition, see Corollary 3.4.3 for the case $n = 1$, for $\mathrm{SU}(n+1)$. The tensor product decomposition is in general much more complicated and not multiplicity free for $n \geq 2$.

Exercise 5.4.1. Let $(G, K) = (\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag})$, with $\theta \colon G \to G$ the flip, which has K as fixed points, $K = G^\theta$. Define $A = \{(t, t^{-1}) \mid t \in \mathrm{SU}(n+1) \text{ diagonal matrix}\}$. The goal of this exercise is to show that $G = KAK$.

(i) We denote the derivative of $\theta \colon G \to G$ as $d\theta \colon \mathfrak{g} \to \mathfrak{g}$. Identify $\mathfrak{g} = \mathfrak{su}(n+1) \oplus \mathfrak{su}(n+1)$ and show that $d\theta(X, Y) = (Y, X)$. Show that $\mathfrak{k}$, the Lie algebra of K, can be identified with the fixed points of $d\theta$. Let $\mathfrak{p}$ be the -1-eigenspace of $d\theta$. Conclude that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Determine $\mathfrak{p}$ and show that $\mathfrak{p}$ is invariant under the adjoint action of $\mathfrak{k}$, and that $\mathfrak{p}$ is not a Lie subalgebra.

(ii) Next we show that $G = K \exp(\mathfrak{p})$, which is the so-called polar decomposition. For $g \in G$ write $\theta(g)^{-1} g$, and argue that this is of the form $(\exp(X), \exp(-X))$ for $(X, -X) \in \mathfrak{p}$. Put $p = (\exp(\frac{1}{2}X), \exp(-\frac{1}{2}X)) \in \exp \mathfrak{p} \subset G$ and put $k = gp^{-1}$. Show that $k \in K$ by showing that $k\theta(k)^{-1}$ gives the identity.

(iii) Show that A is an abelian subgroup with Lie algebra $\mathfrak{a}$ having elements of the form $(T, -T)$ with T a diagonal matrix in $\mathfrak{su}(n+1)$. Note that K acts on $\mathfrak{a}$ via the adjoint action. Show that $\mathfrak{p} = \bigcup_{k \in K} \mathrm{Ad}(k)\mathfrak{a}$.

(iv) Use (ii) and (iii) to prove that $G = KAK$.

Note that the decomposition $g = k_1 a k_2$ is not unique in general.

Assume for a moment that $(\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag}, \tau)$ is commutative triple for $\tau \in \hat{K} \cong \widehat{\mathrm{SU}(n+1)}$ and let $\pi \in \hat{G}_\tau$, i.e. $\pi = \pi_1 \times \pi_2$ so that $[\pi_1 \otimes \pi_2 \colon \tau] \leq 1$ for all $\pi_1, \pi_2 \in \widehat{\mathrm{SU}(n+1)}$. Then the corresponding matrix spherical function $\Phi_\pi \colon G \to \mathrm{End}(V_\tau)$ is completely determined by its restriction to A by Exercise 5.4.1, because Φ_π is τ-invariant, see (5.1.1). Moreover, in this case

$$\Phi_\pi \colon A \to \mathrm{End}_M(V_\tau), \qquad M = Z_K(A), \tag{5.4.1}$$

since $\tau(m)\Phi_\pi(a) = \Phi_\pi(ma) = \Phi_\pi(am) = \Phi_\pi(a)\tau(m)$. In this case the centraliser of A in K, i.e. $Z_K(A) = M$, is a compact abelian goup, and

$$M = \{(t, t) \mid t \in \mathrm{SU}(n+1) \text{ diagonal element}\} \cong \mathbb{T}^n. \tag{5.4.2}$$

Exercise 5.4.2. (i) Show that $M = Z_K(A)$ is given by (5.4.2).

(ii) Determine the normaliser $N_K(A)$ of A in K, and show that the group $N_K(A)/Z_K(A)$ is a finite group which is isomorphic to $W \cong S_{n+1}$, the symmetric group on $n+1$ letters. We use W to emphasise that this group is the (reduced) Weyl group.

(iii) Put $t = \mathrm{diag}(e^{\phi_1}, \cdots, e^{i\phi_{n+1}}) \in \mathrm{SU}(n+1)$, and $a = (t, t^{-1})$. Take $w \in W$ and a representative $n_w \in N_K(A)$. Describe $w \cdot a = n_w a n_w^{-1}$ explicitly.

Because $\Phi_\pi|_A$ ends up in the M-intertwiners of $\mathrm{End}(V_\tau)$ it is useful to have the decomposition $\tau|_M$ in terms of the irreducible representations of M. Since we have $M \cong \mathbb{T}^n$, we have $\hat{M} \cong \mathbb{Z}^n$ and all its irreducible representations are one-dimensional. We can now use the following fact, specialising Kobayashi [41, Thm. 30], to this particular situation. We refer to [41] for a proof of Theorem 5.4.3.

Theorem 5.4.3. *If $\tau|_M$ splits multiplicity free, then* $(\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag}, \tau)$ *is a commutative triple.*

Since $K \cong \mathrm{SU}(n+1)$ and $M \cong \mathbb{T}^n$ can identified with the maximal torus of $\mathrm{SU}(n+1)$ we see that $\tau|_M$ splits multiplicity free if and only if the weight spaces of τ, i.e. the eigenspaces of M, are all one-dimensional. The dimensions of the weight spaces are known since the finite dimensional unitary representations of $\mathrm{SU}(n+1)$ correspond to the finite dimensional holomorphic representations of $\mathrm{SL}(n+1, \mathbb{C})$, which in turn correspond to the finite dimensional representations of the simple complex Lie algebra $\mathfrak{sl}(n+1, \mathbb{C})$ (of type A_n), see [32]. In the terminology of Humphreys [32] these are the representations for which the highest weight is a multiple of the first (or last) fundamental weight for type A_n).

Let $V = \mathbb{C}^{n+1}$ be the natural representation of $\mathrm{SU}(n+1)$, so the unitary representation $\pi\colon \mathrm{SU}(n+1) \to B(V)$ is $\pi(g) = g$ viewed as a matrix. We let $(e_i)_{i=1}^{n+1}$ be the standard orthonormal basis of V. Then by iterating the tensor product construction of Exercise 2.5.3 we get a unitary representation of $\mathrm{SU}(n+1)$ in the N-fold tensor product

$$V^{\otimes N} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{N \text{ times}}.$$

Then $V^{\otimes N}$ comes with a natural action of the symmetric group S_N by letting the permutation act on the order of tensor product, so for $s \in S_N$ we have

$$s \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_N) = v_{s(1)} \otimes v_{s(2)} \otimes \cdots \otimes v_{s(N)}.$$

Then any $\pi^{\otimes N}(g)$ commutes with any $s \in S_N$ by construction. Even more is true, the commutant of the action of $\mathrm{SU}(n+1)$ under $\pi^{\otimes N}$ is the algebra generated by the action of S_N and vice versa. This is known as Frobenius-Schur-Weyl duality. In particular, it follows that the symmetric elements in $V^{\otimes N}$ form an invariant subspace, and this space is denoted by $S^N V$.

Exercise 5.4.4. (i) Show the universal property for symmetric powers. The N-th symmetric power is the vector space $S^N V$ with a multilinear map $\iota\colon V^N \to S^N V$ so that for any multilinear symmetric map $f\colon V^N \to W$ to a vector space W, there exists a linear map $\phi\colon S^N V \to W$ so that $f = \phi \circ \iota$.

(ii) Show that $S^N V$ can be obtained by taking the quotient vector space of the tensor space $V^{\otimes N}$ by the subspace of elements of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_N - v_{s(1)} \otimes v_{s(2)} \otimes \cdots \otimes v_{s(N)}$ for all $s \in S_N$.

(iii) Show that $P = \frac{1}{N!}\sum_{s\in S_N} s\colon V^{\otimes N} \to V^{\otimes N}$ is a projection on $S^N V$.

(iv) Show that $e_1^{k_1} e_2^{k_2} \cdots e_{n+1}^{k_{n+1}}$ with $\sum_{i=1}^{n+1} k_i = N$ forms a basis for $S^N V$.

Proposition 5.4.5. *$S^N V$ is an irreducible unitary representation for $\mathrm{SU}(n+1)$, and we denote the representation $(\pi_N, S^N V)$. An orthonormal basis is given by the set $e_1^{k_1} e_2^{k_2} \cdots e_{n+1}^{k_{n+1}}$ with $\sum_{i=1}^{n+1} k_i = N$. In particular, $\dim S^N V = \binom{N+n}{N}$ and the orthonormal basis is a basis of eigenvectors for the action of M;*

$$\mathrm{diag}(t_1, \cdots, t_{N+1})\, e_1^{k_1} e_2^{k_2} \cdots e_{n+1}^{k_{n+1}} = \Big(\prod_{i=1}^{n+1} t_i^{k_i}\Big)\, e_1^{k_1} e_2^{k_2} \cdots e_{n+1}^{k_{n+1}}.$$

Exercise 5.4.6. Give a proof of Proposition 5.4.5. Hint: compare to the proof of Theorem 3.2.7.

Corollary 5.4.7. *$\pi_N|_M$ splits multiplicity free.*

Proof. Since M is abelian, all irreducible unitary representations are one-dimensional. Taking into account $\prod_{i=1}^{n+1} t_i = 1$ and $N = \sum_{i=1}^{n+1} k_i$, we eliminate t_{n+1} and k_{n+1} to find

$$\prod_{i=1}^{n+1} t_i^{k_i} = (t_1^{-1} \cdots t_n^{-1})^{N-\sum_{i=1}^n k_i} \prod_{i=1}^{n} t_i^{k_i} = \prod_{i=1}^{n} t_i^{-N+2k_i+\sum_{p=1,p\neq i}^n k_p}$$

we see that $e_1^{k_1} e_2^{k_2} \cdots e_{n+1}^{k_{n+1}}$ and $e_1^{l_1} e_2^{l_2} \cdots e_{n+1}^{l_{n+1}}$ have the same M-type if and only if $2k_i + \sum_{p=1,p\neq i}^n k_p = 2l_i + \sum_{p=1,p\neq i}^n l_p$ for all $1 \leq i \leq n$. Since the $n \times n$-matrix with all ones except for having 2's on the diagonal is invertible, we see that $k_i = l_i$ for all $1 \leq i \leq n$. $\square$

Exercise 5.4.8. Show that the $n \times n$-matrix with all ones except for 2's on the diagonal has determinant $n+1$. Hint: show that there is an eigenvalue $n+1$ and that 1 is an eigenvalue of multiplicity $n-1$ by establishing $n-1$ different eigenvectors for the eigenvalue 1.

Now Corollary 5.4.7 and Theorem 5.4.3 then show that we have a commutative triple.

Corollary 5.4.9. $(\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag}, \pi_N)$ *is a commutative triple.*

Remark 5.4.10. It is natural to ask if these are all possibilities of a commutative triple for the pair $(\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag})$. Note that Theorem 5.4.3 gives that also $(\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag}, \pi_N^*)$ is a commutative triple for the contragredient representation. It turns out that these are all commutative triples for $(\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag})$, and for this we need to understand the tensor product decomposition in more detail, see [50]. Note that for $n = 1$, we get all $\mathrm{SU}(2)$ representations in this way, so that we recover that $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$ is strong Gelfand pair.

Having the multiplicity free decomposition $\pi_N|_M = \bigoplus_{i=1}^r \sigma_i$ and $V_\tau = \bigoplus_{i=1}^r W_{\sigma_i}$ for certain $\sigma_i \in \hat{M}$ at hand, we see that $\mathrm{End}_M(V_\tau) \cong \mathbb{C}^r$ by Schur's Lemma 2.5.7, so that the matrix spherical function as in (5.4.1) can be viewed as $\Phi_\pi\colon A \to \mathbb{C}^r$. We also have the (reduced) Weyl group $W = N_K(A)/Z_K(A) = N_K(A)/M$, which is identified with S_{N+1} in Exercise 5.4.2. For $w \in W$ we take a representative $n_w \in N_K(A)$, then we have a well-defined action on A by the adjoint representation; $w \cdot a = n_w a n_w^{-1}$, for $w \in W$ and $a \in A$. Moreover, W acts on $\mathrm{End}_M(V_\tau)$ by $w \cdot T = \tau(n_w) T \tau(n_w^{-1})$. Note that $\mathrm{Tr}\,(w \cdot T) = \mathrm{Tr}\,(T)$, $w \cdot (TS) = (w \cdot T)(w \cdot S)$ and $w \cdot T^* = (w \cdot T)^*$, so that W is unitary representation on $\mathrm{End}_M(V_\tau)$. Since minimal orthogonal projections in $\mathrm{End}_M(V_\tau)$ are mapped to minimal orthogonal projections by $w \in W$, we find an action of W on the M-types $\{\sigma_1, \cdots, \sigma_r\}$ occurring in the decomposition of $V_\tau|_M$. In particular, the irreducible M-representation in $V_\tau|_M$ in the same Weyl group orbit have the same dimension.

Then we also have an action of W on $\Phi\colon A \to \mathrm{End}_M(V_\tau)$ by

$$(w \cdot \Phi)(a) = \Phi(w^{-1} \cdot a) = \Phi(n_{w^{-1}} a n_{w^{-1}}^{-1}) = \tau(n_{w^{-1}}) \Phi(a) \tau(n_{w^{-1}}^{-1}) = w^{-1} \cdot (\Phi(a))$$

so that if we know the scalar function $A \to \mathbb{C}$, $a \mapsto \Phi(a)|_{W_{\sigma_i}}$ then we also know the function $a \mapsto \Phi(a)|_{W_{\sigma_j}}$ for any σ_j in the Weyl group orbit $W\sigma_i$ of σ_i. This can be useful, in particular in case the set $\{\sigma_i \mid 1 \leq i \leq r\}$ forms a single Weyl group orbit. In this setting this only occurs if the weights of the representation π_N form a single Weyl group orbit. In the case of the symmetric power $S^N V$ this only happens for $N = 1$, i.e. for the natural representation V. The weight spaces form a single Weyl group orbit precisely for so-called minuscule highest weight representations which have all been classified, see [32, Exe. 13.13], where the minuscule weights are called non-zero dominant minimal weights.

Exercise 5.4.11. We let $(G, K) = (\mathrm{SU}(n+1) \times \mathrm{SU}(n+1), \mathrm{diag})$.

(i) Show that $\mathrm{SU}(n+1) \cong G/K$ by identifying (g_1, g_2) with $g_1 g_2^{-1}$. Show that right K-invariant functions F on G can be viewed as functions on $\mathrm{SU}(n+1)$ by identifying $F(g_1, g_2)$ with $f(g_1 g_2^{-1})$. See Section 3.4.1.

(ii) Let V_τ be a finite dimensional unitary representation of $\mathrm{SU}(n+1)$, and consider a function $f\colon \mathrm{SU}(n+1) \to V_\tau$ satisfying $f(kxk^{-1}) = \tau(k) f(x)$ for all $x, k \in \mathrm{SU}(n+1)$. Show that we can identify f with a function $F\colon G \to \mathrm{Hom}(V_\tau, \mathbb{C}) \cong V_\tau^*$ with the property (5.3.6) with $\tau_1 = \tau$ and τ_2 the trivial representation.

(iii) Let V be a finite-dimensional representation of $\mathrm{SU}(n+1)$ and assume that $T\colon V \to V \otimes V_\tau$ is a $\mathrm{SU}(n+1)$-intertwiner. Define $\chi_V\colon \mathrm{SU}(n+1) \to V_\tau$ by $\chi_T(g) = \mathrm{Tr}\,_V(T\pi_V(g))$, where $\mathrm{Tr}\,_V$ takes the trace of an operator $V \otimes V_\tau \to V \otimes V_\tau$ with respect to the finite dimensional space V, so that for orthonormal basis $(v_i)_i$ and $(u_k)_k$ of V and V_τ we have

$$\chi_T(g) = \sum_{i,k} \langle T\pi_V(g) v_i, v_i \otimes u_k \rangle u_k \in V_\tau.$$

Show that χ_T satisfies $\chi_T(kgk^{-1}) = \tau(k)\chi_T(g)$ for all $k, g \in \mathrm{SU}(n+1)$. To which function $F\colon G \to \mathrm{Hom}(V_\tau, \mathbb{C})$ does it correspond?

(iv) Argue that $f\colon \mathrm{SU}(n+1) \to V_\tau$ satisfying $f(kxk^{-1}) = \tau(k)f(x)$ for all $x, k \in \mathrm{SU}(n+1)$ is completely determined by its evaluation on a diagonal matrix, and let this abelian subgroup of $\mathrm{SU}(n+1)$ be denoted by T. Let $V_\tau|_T = \bigoplus_{\sigma\in\hat{T}} n_\sigma V_\sigma^T$ be the decomposition of V_τ with respect to the abelian subgroup T. Show that $f|_T$ takes it values in V_0^T, i.e. the subspace for the trivial T-representation. Moreover, show that $f|_T$ is invariant under S_{n+1}, which permutes the diagonal elements of T.

(v) Consider the representation $S^N V$ of $\mathrm{SU}(n+1)$. When does the trivial representation of T occur in the decomposition $S^N V$? And what is the multiplicity?

These generalised trace functions are due to Etingof and Kirillov, and have also analogues in the quantum setting as well as in the affine setting. See Kirillov [35] and references given there.

5.4.2 Matrix spherical functions for the case $n = 1$

We now focus on the case $n = 1$, since we can use the results of Chapter 3. It should be noted that most of the results in this subsection have an analogue for general n, and this is discussed briefly in Remark 5.4.36. We go back to the notation of spin ℓ-representations, so that $N = 2\ell$ in the notation of Section 5.4.1. Then we know that $(\mathrm{SU}(2)\times\mathrm{SU}(2), \mathrm{diag}, \pi_\ell)$ is a commutative triple for any $\ell \in \frac{1}{2}\mathbb{N}$. We fix ℓ and we use the notation $G = \mathrm{SU}(2)\times\mathrm{SU}(2)$, $K = \mathrm{diag} \cong \mathrm{SU}(2)$ for the remainder of this section.

Proposition 5.4.12. *For $(\ell_1, \ell_2) \in \hat{G}$ we have $(\ell_1, \ell_2) \in \hat{G}_\ell = \hat{G}_{\pi_\ell}$ if and only if $|\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2$ and $\ell_1 + \ell_2 - \ell \in \mathbb{Z}$.*

See Figure 5.4.1 for an example of $\hat{G}_2$.

Proof. Since $(\ell_1, \ell_2) \in \hat{G}_\ell$ if and only if $[\pi_{\ell_1} \otimes \pi_{\ell_2} \colon \pi_\ell] = 1$, this follows immediately from Corollary 3.4.3. □

From Proposition 5.4.12, see Figure 5.4.1, we see that we can label $\hat{G}_\ell$ with a pair (k, d) with $k \in \{-\ell, -\ell+1, \cdots, \ell\}$ and $d \in \mathbb{N}$. Then

$$(\ell_1, \ell_2) = (\frac{1}{2}(k + \ell + d), \frac{1}{2}(\ell - k + d)). \tag{5.4.3}$$

So we want to determine the matrix spherical functions,

$$\Phi_{(\ell_1,\ell_2)} = \Phi^\ell_{(\ell_1,\ell_2)} \colon G \to \mathrm{End}(V_\ell) \tag{5.4.4}$$

and it suffices to calculate its restriction to $A = \{(k(\phi), k(-\phi)) \mid 0 \leq \phi < 4\pi\}$ with the notation as in (3.2.1). In this case $M = \{(k(\phi), k(\phi)) \mid 0 \leq \phi < 4\pi\}$, and the decomposition $V_\ell|_M$ is given by (3.2.8). The one-dimensional representations of M are then precisely spanned by the basis vectors e^ℓ_n as in Section 3.2 corresponding to the character χ_{-2n}.

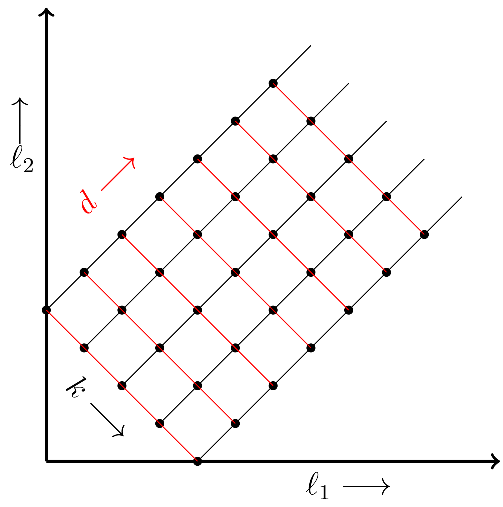

Figure 5.4.1: $\hat{G}_\ell$ for $\ell = 2$ as in Proposition 5.4.12.

Exercise 5.4.13. Show that using the Clebsch-Gordan coefficients as in Section 3.4.4 we can write for $g_1, g_2 \in \mathrm{SU}(2)$, $n, m \in \{-\ell, -\ell+1, \cdots, \ell\}$

$$\langle \Phi_{(\ell_1,\ell_2)}(g_1, g_2)e^\ell_n, e^\ell_m\rangle = \sum_{\substack{n_1,m_1=-\ell_1 \\ n_1+n_2=n,}}^{\ell_1} \sum_{\substack{n_2,m_2=-\ell_2 \\ m_1+m_2=m}}^{\ell_2} C^{\ell_1,\ell_2,\ell}_{n_1,n_2,n}\overline{C^{\ell_1,\ell_2,\ell}_{m_1,m_2,m}}\langle \pi_{\ell_1}(g_1)e^{\ell_1}_{n_1}, e^{\ell_1}_{m_1}\rangle\langle \pi_{\ell_1}(g_2)e^{\ell_2}_{n_2}, e^{\ell_2}_{m_2}\rangle$$

and in particular, when restricted to $(g_1, g_2) = (k(\phi), k(-\phi)) \in A$,

$$\langle \Phi_{(\ell_1,\ell_2)}(k(\phi), k(-\phi))e^\ell_n, e^\ell_m\rangle = \delta_{m,n} \sum_{\substack{n_1=-\ell_1 \\ n_1+n_2=n}}^{\ell_1} \sum_{n_2=-\ell_2}^{\ell_2} |C^{\ell_1,\ell_2,\ell}_{n_1,n_2,n}|^2 e^{-i\phi(n_1-n_2)}$$

Exercise 5.4.14. In this exercise we take a closer look at the product formula Theorem 5.2.6(iii) for $\Phi^\ell_{(\ell_1,\ell_2)}$ using the results of Exercise 5.4.13.

(i) Argue that it suffices to consider g, h in Theorem 5.2.6(iii) equal to $(k(\phi), k(-\phi))(h, h)$ for $h \in \mathrm{SU}(2)$, $(k(\theta), k(-\theta))$.

(ii) Let the identity act on e^ℓ_n and take inner product with e^ℓ_m to find

$$\begin{aligned}&\langle \Phi^\ell_{(\ell_1,\ell_2)}(k(\phi), k(-\phi))e^\ell_m, e^\ell_m\rangle\langle \pi_\ell(h)e^\ell_n, e^\ell_m\rangle\langle \Phi^\ell_{(\ell_1,\ell_2)}(k(\theta), k(-\theta))e^\ell_n, e^\ell_n\rangle \\ &= \int_{\mathrm{SU}(2)} \langle \Phi^\ell_{(\ell_1,\ell_2)}(k(\phi)gk(\theta), k(-\phi)gk(-\theta))e^\ell_n, e^\ell_m\rangle\overline{\xi_\ell(h^{-1}g)}\, dg\end{aligned}$$

(iii) Use Exercise 5.4.13 and the results in Chapter 3 to write both sides as Fourier series in two variables. By equating coefficients, derive an expression for the convolution product

$$\int_{\mathrm{SU}(2)} \langle \pi_{\ell_1}(g)e^{\ell_1}_{n_1}, e^{\ell_1}_{m_1}\rangle \langle \pi_{\ell_2}(g)e^{\ell_2}_{n_2}, e^{\ell_2}_{m_2}\rangle \xi_\ell(hg^{-1})\, dg$$

(iv) Show how to derive this result from the results from Chapter 3.

The Weyl group in this case is S_2, and as a representative for the non-trivial element $w \in W$ we pick $(n_1, n_1) \in N_K(A)$ with $n_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so that $w \cdot (k(\phi), k(-\phi)) = (k(-\phi), k(\phi))$. Also, $\tau(n_1)e^\ell_n = (-1)^{\ell-n}e^\ell_{-n}$, so the non-trivial Weyl group element acts on the M-types $\{-\ell, -\ell+1, \cdots, \ell\}$ by sending the M-type χ_{-2n} to χ_{2n}, i.e. $n \mapsto -n$. This gives

$$\langle \Phi_{(\ell_1,\ell_2)}(k(\phi), k(-\phi))e^\ell_n, e^\ell_n\rangle = \langle \Phi_{(\ell_1,\ell_2)}(k(-\phi), k(\phi))e^\ell_{-n}, e^\ell_{-n}\rangle \tag{5.4.5}$$

The symmetry (5.4.5) is a general feature, and also works for other situations for more general pairs (G, K). In this case we also have another obvious symmetry, namely the representation $\pi_{\ell_1} \times \pi_{\ell_2} \circ \theta$, where, as before, $\theta\colon \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SU}(2) \times \mathrm{SU}(2)$ is the flip $\theta(g_1, g_2) = (g_2, g_1)$, is equivalent to $\pi_{\ell_2} \times \pi_{\ell_1}$, where the intertwiner $P\colon V_{\ell_1} \otimes V_{\ell_2} \to V_{\ell_2} \otimes V_{\ell_1}$ is the flip; $P\colon v \otimes w \mapsto w \otimes v$. This then gives

$$\theta^*\Phi_{(\ell_1,\ell_2)} = \Phi_{(\ell_2,\ell_1)}, \quad \text{i.e. } \Phi_{(\ell_1,\ell_2)}(\theta(g_1, g_2)) = \Phi_{(\ell_2,\ell_1)}(g_1, g_2). \tag{5.4.6}$$

Let us now look closer at the case $\ell = 0$, then we are back to the case of the Gelfand pair as in Section 4.2.2. Then we see that $\hat{G}_0 = \{(\ell_1, \ell_1) \mid \ell_1 \in \frac{1}{2}\mathbb{N}\}$, cf. Figure 5.4.1. So we see that $\varphi = \Phi^0_{(\frac12,\frac12)}\colon G \to \mathbb{C}$ gives the fundamental spherical function.

Lemma 5.4.15. *The space $\mathbb{C}[\varphi]$ of polynomials in φ is the space of spherical functions on (G, K) that are polynomial in the coordinate functions.*

Sketch of proof. The proof follows the lines of Section 3.4.5. As G-representations we have

$$(\pi_{\frac12} \times \pi_{\frac12}) \otimes (\pi_{\ell_1} \times \pi_{\ell_1}) = (\pi_{\ell_1-\frac12} \times \pi_{\ell_1-\frac12}) \oplus (\pi_{\ell_1-\frac12} \times \pi_{\ell_1+\frac12}) \oplus (\pi_{\ell_1+\frac12} \times \pi_{\ell_1-\frac12}) \oplus (\pi_{\ell_1+\frac12} \times \pi_{\ell_1+\frac12})$$

and since only the first and last term on the right hand side can have a K-fixed vector we obtain

$$\varphi\, \Phi^0_{(\ell_1,\ell_1)} = C_-\Phi^0_{(\ell_1-\frac12,\ell_1-\frac12)} + C_+\Phi^0_{(\ell_1+\frac12,\ell_1+\frac12)}$$

for some constants, and it suffices to see that $C_+ \neq 0$ which can be done by expressing C_+ in terms of Clebsch-Gordan coefficients. □

Exercise 5.4.16. Fill in the details of the proof of Lemma 5.4.15.

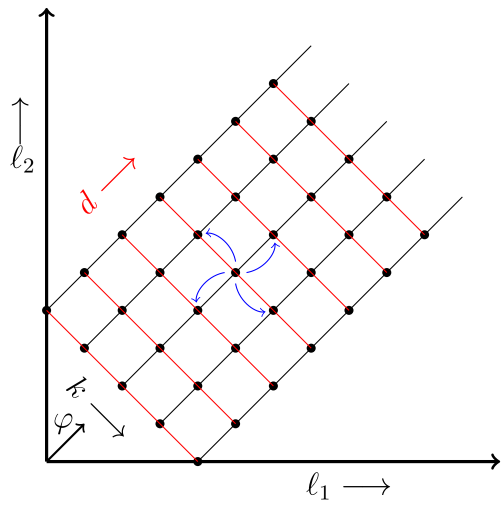

Figure 5.4.2: Figure 5.4.1 adapted for the proof of Proposition 5.4.17. The spherical direction is denoted by φ.

Having Lemma 5.4.15 we can look for an extension to the matrix spherical functions. So we go back to arbitrary ℓ and consider $\Phi^{\ell}_{(\ell_1,\ell_2)} = \Phi_{(\ell_1,\ell_2)}$. In general, if $F\colon G \to \mathrm{End}(V_\tau)$ is a τ-invariant function, i.e. satisfies (5.1.1), and $f\colon G \to \mathbb{C}$ satisfies (4.1.2), then also $fF\colon G \to \mathrm{End}(V_\tau)$ is a τ-invariant function. So we can see the space of τ-invariant functions as a module over the bi-K-invariant functions, and we now describe this in more detail following the line of thought as in Lemma 5.4.15.

Proposition 5.4.17. *Using the labeling* (5.4.3) *we have for $d \in \mathbb{N}$, $k \in \{-\ell, -\ell+1, \cdots, \ell\}$*

$$\Phi^{\ell}_{(\frac{1}{2}(k+\ell+d),\frac{1}{2}(\ell-k+d))} = \sum_{m=-\ell}^{\ell} p_{k,m;d}(\varphi)\ \Phi^{\ell}_{(\frac{1}{2}(m+\ell),\frac{1}{2}(\ell-m))}$$

where $p_{k,m;d}(\varphi)$ is a polynomial of degree at most d and $p_{k,k;d}(\varphi)$ is of degree d.

Sketch of proof. As G-representations we have

$$(\pi_{\frac{1}{2}} \times \pi_{\frac{1}{2}}) \otimes (\pi_{\ell_1} \times \pi_{\ell_2}) = (\pi_{\ell_1-\frac{1}{2}} \times \pi_{\ell_2-\frac{1}{2}}) \oplus (\pi_{\ell_1-\frac{1}{2}} \times \pi_{\ell_2+\frac{1}{2}}) \oplus (\pi_{\ell_1+\frac{1}{2}} \times \pi_{\ell_2-\frac{1}{2}}) \oplus (\pi_{\ell_1+\frac{1}{2}} \times \pi_{\ell_2+\frac{1}{2}})$$

as indicated in Figure 5.4.2. If the labels $(\ell_1 \pm \frac{1}{2}, \ell_2 \pm \frac{1}{2})$ drop out of the region, there is no associated matrix spherical function. Taking the corresponding matrix spherical functions we find this time

$$\varphi\Phi_{(\ell_1,\ell_2)} = C_+\Phi_{(\ell_1+\frac{1}{2},\ell_2+\frac{1}{2})} + C_1\Phi_{(\ell_1-\frac{1}{2},\ell_2+\frac{1}{2})} + C_2\Phi_{(\ell_1+\frac{1}{2},\ell_2-\frac{1}{2})} + C_3\Phi_{(\ell_1-\frac{1}{2},\ell_2-\frac{1}{2})}$$

and it suffices to show that $C_+ \neq 0$, rewrite in terms of the coordinates of (5.4.3) and use induction on d. $\square$

Exercise 5.4.18. Fill in the details for the proof of Proposition 5.4.17.

This gives us the opportunity to calculate

$$\mathrm{Tr}\left(\Phi^\ell_{(\frac12(k+\ell+d),\frac12(\ell-k+d))}(g)\big(\Phi^\ell_{(\frac12(k'+\ell+d'),\frac12(\ell-k'+d'))}(g)\big)^*\right) = \tag{5.4.7}$$
$$\sum_{m.m'=-\ell}^{\ell} p_{k,m;d}(\varphi(g))\mathrm{Tr}\left(\Phi^\ell_{(\frac12(m+\ell),\frac12(\ell-m))}(g)\big(\Phi^\ell_{(\frac12(m'+\ell),\frac12(\ell-m'))}(g)\big)^*\right)\overline{p_{k',m';d'}(\varphi(g))}$$

and by (5.3.7) the traces are also bi-K-invariant as well as polynomial, so that by Lemma 5.4.15 this is polynomial in φ. We put

$$W_{m,m'}(\varphi(g)) = \mathrm{Tr}\left(\Phi^\ell_{(\frac12(m+\ell),\frac12(\ell-m))}(g)\big(\Phi^\ell_{(\frac12(m'+\ell),\frac12(\ell-m'))}(g)\big)^*\right).$$

Lemma 5.4.19. *The matrix valued function $G\to M_{2\ell+1}(\mathbb{C})$, $g\mapsto (W_{m,m'}(\varphi(g)))_{m,m'=-\ell}^{\ell}$ is a positive definite matrix.*

Proof. Take a sequence $(c_n)_{n=-\ell}^{\ell}$ of complex number, then

$$\sum_{n,m=-\ell}^{\ell} c_m W_{m,n}(\varphi(g))\overline{c_n} = \mathrm{Tr}\left(T(g)(T(g))^*\right)\geq 0, \qquad T(g)=\sum_{m=-\ell}^{\ell} c_m \Phi^\ell_{(\frac12(m+\ell),\frac12(\ell-m))}(g).$$

□

Theorem 5.4.20. *The matrix valued polynomials $P_d(x) = (p_{k,m;d}(x))_{k.m=-\ell}^{\ell}$, taking values in $M_{2\ell+1}(\mathbb{C})$ satisfy the orthogonality relations*

$$\frac{2}{\pi}\int_{-1}^{1} P_d(x)W(x)P_{d'}(x)^*\sqrt{1-x^2}\,dx = \delta_{d,d'}H_d$$

where H_d is a positive definite diagonal matrix

$$(H_d)_{m,m} = \frac{(2\ell+1)^2}{(\ell+m+d+1)(\ell-m+d+1)}, \qquad m\in\{-\ell,-\ell+1,\cdots,\ell\}.$$

Note that Theorem 5.4.20 reduces to (3.4.2) in case $\ell=0$, see Section 4.2.2. So we view these matrix orthogonal polynomials as matrix analogues of Chebyshev polynomials.

Proof. Integrating (5.4.7) over G using Theorem 5.3.1 gives

$$\sum_{m.m'=-\ell}^{\ell}\int_G p_{k,m;d}(\phi(g))W_{m,m'}(\phi(g))\overline{p_{k',m';d'}(\phi(g))}\,dg = \frac{\delta_{d,d'}\delta_{k,k'}\,(2\ell+1)^2}{(\ell+k+d+1)(\ell-k+d+1)}$$

and this can be written as an integral of matrix valued functions, where we integrate entry-wise;

$$\int_G P_d(\phi(g))W(\phi(g))(P_{d'}(\phi(g)))^*\,dg = \delta_{d,d'}\,H_d.$$

Next we realise that the integral over $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ only involves the spherical function for the fundamental representation $\pi_{\frac{1}{2}} \times \pi_{\frac{1}{2}}$ of $\mathrm{SU}(2) \times \mathrm{SU}(2)$, i.e. the character of $\pi_{\frac{1}{2}}$ on $\mathrm{SU}(2)$. Using the identifications as in Section 4.2.2 and using the integral for central functions (3.4.1) we get

$$\frac{1}{\pi} \int_0^\pi P_d(\cos\theta) W(\cos\theta) (P_{d'}(\cos\theta))^* \sin^2\theta \, d\theta = \delta_{d,d'} H_d$$

and substituting $x = \cos\theta$ gives the result. $\square$

Theorem 5.4.20 shows that the matrix spherical functions gives rise to matrix orthogonal polynomials, and we recall some of the generalities in Section 5.4.3. Before doing so we first discuss some properties of the matrix weight as given in Theorem 5.4.20.

We define a matrix valued function on $A \subset G$ by

$$\Phi_0(k(\phi), k(-\phi))_{m,r} = \Big(\Phi^\ell_{\frac{1}{2}(\ell+m), \frac{1}{2}(\ell-m)}(k(\phi), k(-\phi)) \Big)_{r,r}, \qquad m, r \in \{-\ell, -\ell+1, \cdots, \ell\}. \tag{5.4.8}$$

Note that we only need to consider the (r, r)-entry, since the function takes values in $\mathrm{End}_M(V_\ell)$. By Exercise 5.4.13 we see that the expression can be made in terms of Clebsch-Gordan coefficients which occur for minimal values. Then $W = \Phi_0 \Phi_0^*$ as functions on A, and this factorisation corresponds nicely to the proof of Lemma 5.4.19. We would like to see that generically the determinant of W doesn't vanish. Unfortunately, the factorisation of $W = \Phi_0 \Phi_0^*$ on A is not helpful since it is not easy to calculate $\det(\Phi_0)$. However, it turns out that W has a very nice LDU-decomposition which allows for an explicit evaluation of $\det(W)$. The LDU-decomposition, however, is obtained by direct computation using Gegenbauer or ultraspherical polynomials, i.e. Jacobi polynomials for $\alpha = \beta$. This decomposition lacks a suitable group theoretic interpretation (yet). Then, in the notation of Theorem 5.4.20, $\det(W(x)) > 0$ for all $x \in (-1, 1)$. We refer to [49] for more information.

We now let $J \in \mathrm{End}(V_\ell)$ be defined by $J \colon e^\ell_n \mapsto e^\ell_{-n}$, so that $J = J^* = J^{-1}$. Then (5.4.5) shows that

$$\Phi_0(k(-\phi), k(\phi)) = \Phi_0(k(\phi), k(-\phi)) J$$

whereas (5.4.6) shows that

$$\Phi_0(k(-\phi), k(\phi)) = J \Phi_0(k(\phi), k(-\phi))$$

and so Φ_0, and also W, commutes with J. This proves one inclusion of the following lemma.

Lemma 5.4.21. $\{T \in \mathrm{End}(V_\ell) \mid TW(x) = W(x)T, \ \forall x \in [-1, 1]\} = \mathbb{C}[\mathbf{1}, J]$, *so the commutant algebra is two-dimensional.*

The reverse inclusion is proved by considering the commutant of the highest and one-but-highest degree polynomial of W. This is a tedious calculation, and we refer to [48], [49] for the details. Lemma 5.4.21 shows that the ± 1-eigenspaces of J are invariant subspaces for the weight W, and in particular that we can restrict the matrix orthogonal polynomials to the ± 1-subspaces of J.

Exercise 5.4.22. (i) Determine the ± 1-eigenspaces of J.

(ii) Calculate that for $\ell = \frac{1}{2}$ we get in the notation of Theorem 5.4.20

$$W(x) = \begin{pmatrix} 2 & 2x \\ 2x & 2 \end{pmatrix}$$

and that conjugation to the ± 1-eigenspaces of J gives the Jacobi weights for $(\alpha, \beta) = (\frac{1}{2}, \frac{3}{2})$ and $(\alpha, \beta) = (\frac{3}{2}, \frac{1}{2})$.

5.4.3 Matrix orthogonal polynomials of a single variable

There are several references for matrix orthogonal polynomials, see e.g. [15], [65] and references given there. In particular, the matrix orthogonality goes back to Krein in the 1940s. For information on matrix measures, see [6], [45] and references given there. For information on the decomposition of matrix measures we refer to [51], [75].

Matrix orthogonal polynomials are functions of a real variable taking values in a matrix algebra such that the function can be written as

$$p(x) = x^n A_n + x^{n-1} A_{n-1} + \cdots + x A_1 + A_0 = \sum_{k=0}^{n} x^k A_k, \qquad A_k \in M_N(\mathbb{C}) \tag{5.4.9}$$

or $p \in M_N(\mathbb{C})[x]$, and here $M_N(\mathbb{C})$ is the space of all $N \times N$-matrices over $\mathbb{C}$ as in Section 2.3. The polynomial in (5.4.9) has degree n if $A_n \neq 0$ (which denotes the matrix with all zeroes).

We consider measures on the real line $\mathbb{R}$ equipped with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ taking values in $M_N(\mathbb{C})$, but we want to have a positive measure. So we consider $\mu\colon \mathcal{B}(\mathbb{R}) \to M_N(\mathbb{C})$ so that for all $X \in \mathcal{B}(\mathbb{R})$ the matrix $\mu(X)$ is positive definite and

$$\mu(\bigcup_{i\in\mathbb{N}} X_i) = \sum_{i\in\mathbb{N}} \mu(X_i)$$

for any sequence $(X_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{B}(\mathbb{R})$. The convergence is unconditional, so that in particular, after choosing coordinates, we have N^2 complex measures $\mu_{i,j}$. Then the trace, $\tau_\mu = \sum_{i=1}^{N} \mu_{i,i}$ is a (positive) Borel measure called the trace-measure, and each of the entries is absolutely continuous with respect to the trace measure. In particular, denoting the Radon-Nikodym derivative of $\mu_{i,j}$ with respect to the trace measure, any matrix measure can be written as the product of matrix valued weight function W taking its values in the positive definite matrices and a (positive) Borel measure τ_μ. For matrix valued functions F and G we define a sesquilinear form by

$$\langle F, G\rangle = \int_{\mathbb{R}} F(x) W(x) (G(x))^* \, d\tau_\mu(x) \in M_N(\mathbb{C})$$

whenever the integral is well-defined, i.e. when each of the entries is well-defined. We now assume that all matrix polynomials are integrable, i.e. each polynomial is absolutely integrable with respect to the variation of each of the complex measures $\mu_{i,j}$. This is equivalent to the trace measure having finite moments, see [6, § 2]. Assuming that the only matrix valued functions F with $\langle F,F\rangle = 0$ are the functions $F=0$ a.e., we have a corresponding Hilbert space $L^2(\mu)$ with a $M_N(\mathbb{C})$-valued inner product.

Exercise 5.4.23. Give an example of a matrix valued weight so that all moments exist, but that there exist functions F not equal to zero a.e. with $\langle F,F\rangle = 0$.

Now we get matrix orthonormal polynomials $(\tilde{P}_n)_{n=0}^{\infty}$, $\tilde{P}_n$ of degree n, with invertible leading coefficient, so that

$$\langle \tilde{P}_n, \tilde{P}_m\rangle = \int_{\mathbb{R}} \tilde{P}_n(x)W(x)(\tilde{P}_m(x))^*\, d\tau_\mu(x) = \delta_{m,n}\mathbf{1} \in M_N(\mathbb{C}).$$

Then the polynomials are orthonormal matrix polynomial. Note that also $Q_n(x) = U_n\tilde{P}_n(x)$ for U_n a constant unitary matrix also leads to orthonormal matrix polynomials. For this reason, the polynomials are often normalised to be monic, i.e. the leading coefficient matrix is taken to be the identity, then $(P_n)_{n\in\mathbb{N}}$, $P_n(x) = x^n\mathbf{1} + \sum_{k=0}^{n-1} A_k x^k$, satisfies

$$\langle P_n, P_m\rangle = \int_{\mathbb{R}} P_n(x)W(x)(P_m(x))^*\, d\tau_\mu(x) = \delta_{m,n}H_n \in M_N(\mathbb{C}).$$

for an invertible positive definite matrix H_n.

Lemma 5.4.24. *The monic orthogonal polynomials satisfy a three-term recurrence of the form*

$$x\, P_n(x) = P_{n+1}(x) + B_nP_n(x) + C_nP_{n-1}(x)$$

for $B_n, C_n \in M_N(\mathbb{C})$. Here we take the convention $P_{-1}(x) = 0$,

Exercise 5.4.25. (i) Give a proof of Lemma 5.4.24. Express B_n and C_n in terms of matrix valued integrals.

(ii) Give the corresponding statement for the orthonormal matrix polynomials $\tilde{P}_n$, and give the proof, cf. Theorem 2.1.2.

(iii) Let M be an invertible matrix in $M_N(\mathbb{C})$, show that $\mu_M\colon \mathcal{B}(\mathbb{R}) \to M_N(\mathbb{C})$, $\mu_M(A) = M\mu(A)M^*$ is a measure as well. Relate the orthonormal and monic matrix polynomials for μ and μ_M to each other, as well as the matrices B_n and C_n of Lemma 5.4.24 for these cases.

Remark 5.4.26. One could also define the orthogonal matrix polynomials with respect to the matrix sesquilinear form

$$\langle F,G\rangle = \int_{\mathbb{R}} (G(x))^*W(x)F(x)\, d\tau_\mu(x).$$

Then the corresponding monic orthogonal polynomials satisfy a relation of the form as in Lemma 5.4.24, but with multiplication of the constant matrices from the right. Note that these choices can be transformed into each other by taking the adjoints of the orthogonal polynomials.

Exercise 5.4.25(iii) suggests that we can reduce matrix measures and corresponding matrix orthogonal polynomials to smaller sizes if there exists an invertible M so that $M\mu(A)M^* = \mathrm{diag}(\mu_1(A), \mu_2(A))$ where μ_1 and μ_2 are matrix measures for smaller size N_1 and N_2. Conversely, we can construct from e.g. N scalar valued measures $\mu_1, \cdots, \mu_N$ on, say $[-1, 1]$, a matrix measure by putting $\mu(A) = M\mathrm{diag}(\mu_1(A), \cdots, \mu_N(A))M^*$ for an invertible matrix M. In order to understand a possible decomposition of a matrix measure μ, we state the following proposition.

Proposition 5.4.27. *Let $\mu\colon \mathcal{B}(\mathbb{R}) \to M_N(\mathbb{C})$ be matrix valued measure taking values in the positive definite matrices of size N. Define the spaces*

$$\mathcal{A}(\mu) = \{T \in M_N(\mathbb{C}) \mid T\mu(X) = \mu(X)T^*, \ \forall X \in \mathcal{B}(\mathbb{R})\},$$
$$A(\mu) = \{T \in M_N(\mathbb{C}) \mid T\mu(X) = \mu(X)T, \ \forall X \in \mathcal{B}(\mathbb{R})\},$$

where $A(\mu)$ is the commutant algebra of the image of μ. Then

(i) $\mathcal{A}(\mu)$ is a real vector space containing the identity, and for $T \in \mathcal{A}(\mu)$ and polynomial p with real coefficients, $p(T) \in \mathcal{A}(\mu)$;

(ii) $A(\mu)$ is a complex $*$-algebra, and its set of self-adjoint elements $A_h(\mu)$ is equal to $A_h(\mu) = \mathcal{A}(\mu) \cap (\mathcal{A}(\mu))^*$;

(iii) $\mathcal{A}(\mu)$ is $*$-invariant if and only if $A_h(\mu) = \mathcal{A}(\mu)$;

(iv) the map $A(\mu) \times \mathcal{A}(\mu) \to \mathcal{A}(\mu)$ $(S, T) \mapsto STS^*$ gives an action of the algebra $A(\mu)$ on the vector space $\mathcal{A}(\mu)$;

(v) μ can be reduced to $M\mathrm{diag}(\mu_1, \mu_2)M^*$, M invertible, if and only if $\mathcal{A}(\mu)$ contains an element which is not a multiple of the identity.

Exercise 5.4.28. The purpose of this exercise is to sketch a proof of the statements of Proposition 5.4.27.

(i) Prove Proposition 5.4.27(i) and the first statement of (ii).

(ii) Prove the rest of Proposition 5.4.27(ii) by showing first that $\mathcal{A}(\mu)$ cannot contain skew-adjoint elements.

(iii) Prove Proposition 5.4.27(iii) using (ii), and that $T = T^*$ for all $T \in \mathcal{A}(\mu) \cap (\mathcal{A}(\mu))^*$.

(iv) Prove Proposition 5.4.27(iv).

(v) Prove Proposition 5.4.27(v). Show first that if μ is reducible then $MPM^{-1} \in \mathcal{A}(\mu)$ for a suitable projection P. Show conversely that for a $T \in \mathcal{A}(\mu)$ we can reduce to T in Jordan normal form by changing μ to $M\mu M^*$, and then use the (generalised) eigenspace of T.

Exercise 5.4.29. Let $T \in \mathcal{A}(\mu)$, and assume that $(P_n)_{n\in\mathbb{N}}$ are the corresponding monic polynomials satisfying the three-term recurrence of Lemma 5.4.24. Show that $TP_n(x) = P_n(x)T$, $TB_n = B_nT$ and $TC_n = C_nT$ for all $n \in \mathbb{N}$.

5.4.4 Matrix spherical functions for the case $n = 1$: continued

From Theorem 5.4.20 and Lemma 5.4.21 and Section 5.4.3 we see that the matrix valued spherical functions for the pair $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$ lead to matrix orthogonal polynomials, which are reducible to a setting of a 2×2-block matrix orthogonal polynomials, cf. Exercise 5.4.22. Now, the matrix spherical functions are eigenfunctions of a matrix valued differential operator similar to the situation for the matrix entries as in Section 3.4.6. In fact, since $U(\mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}) = U(\mathfrak{su}(2)_{\mathbb{C}}) \otimes U(\mathfrak{su}(2)_{\mathbb{C}})$ we have two natural operators for which the matrix spherical functions are eigenfunctions, namely from a Casimir element in the first leg and a Casimir element in the second leg of the tensor product.

Recall $k(t) = \begin{pmatrix} e^{i\frac{1}{2}t} & 0 \\ 0 & e^{i\frac{1}{2}t} \end{pmatrix}$ from (3.2.1), and put $a(t) = (k(t), k(-t)) \in A \subset \mathrm{SU}(2) \times \mathrm{SU}(2)$ as in Exercise 5.4.1 for the case $n = 1$. In order to calculate the action of the Casimir elements as differential operators for the matrix spherical functions, we proceed in a similar way as in Section 3.4.6. We use the notation J_0, J_+, J_- for the basis of the Lie algebra $\mathfrak{su}(2)$ and its complexification $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$. Then the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$.

Lemma 5.4.30. *Let $\alpha \colon A \to \mathbb{C}$, $a(t) = (k(t), k(-t)) \mapsto e^{it}$ be a character of the abelian group A, then the adjoint action of A on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$ decomposes as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha^{-1}}$. Here*

(i) the space $\mathfrak{g}_0 = \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}}$, where $\mathfrak{m}_{\mathbb{C}} = \mathbb{C}(J_0, J_0)$ and $\mathfrak{a}_{\mathbb{C}} = \mathbb{C}(J_0, -J_0)$, is eigenspace for the trivial character of A;

(ii) the space $\mathfrak{g}_\alpha = \mathbb{C}(J_+ + iJ_-, 0) \oplus \mathbb{C}(0, J_+ - iJ_-)$ is the eigenspace for the character α of A;

(iii) the space $\mathfrak{g}_{\alpha^{-1}} = \mathbb{C}(J_+ - iJ_-, 0) \oplus \mathbb{C}(0, J_+ + iJ_-)$ is the eigenspace for the character α^{-1} of A.

Exercise 5.4.31. Give a proof of Lemma 5.4.30.

With the notation of Exercise 5.4.1 for θ, we also have the derived map, also denoted $\theta \colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$, where $\mathfrak{k}_{\mathbb{C}}$, respectively $\mathfrak{p}_{\mathbb{C}}$, is the $+1$-eigenspace, respectively the -1-eigenspace, for the complexified Lie algebra setting. We now proceed along the lines of Section 3.4.6 to obtain an expression for suitable quadratic Casimir elements. Recall the notation $X^a = \mathrm{Ad}(a^{-1})X$ for $a \in A$, $X \in \mathfrak{g}_{\mathbb{C}}$.

Lemma 5.4.32. *Let $U = (J_+ + iJ_-, J_+ + iJ_-) \in \mathfrak{k}_{\mathbb{C}}$, $V = (J_+ - iJ_-, J_+ - iJ_-) \in \mathfrak{k}_{\mathbb{C}}$, then*

$$
(J_+ + iJ_-, 0) = \frac{e^{it}}{1 - e^{2it}}(U^{a(t)} - e^{it}U), \qquad (J_+ - iJ_-, 0) = \frac{e^{-it}}{1 - e^{-2it}}(V^{a(t)} - e^{-it}V),
$$
$$
(0, J_+ - iJ_-) = \frac{e^{it}}{1 - e^{2it}}(V^{a(t)} - e^{it}V), \qquad (0, J_+ + iJ_-) = \frac{e^{-it}}{1 - e^{-2it}}(U^{a(t)} - e^{-it}U).
$$

Exercise 5.4.33. Prove Lemma 5.4.32. Hint: compare to Proposition 3.4.32.

Now we rewrite the Casimir operator in the first component as

$$\begin{aligned}\Omega_1 &= (\Omega, 0) = (-J_0^2 - J_-^2 - J_+^2, 0) = (-J_0^2 + iJ_0 - (J_+ + iJ_-)(J_+ - iJ_-), 0) \\ &= (-J_0^2 + iJ_0, 0) - \frac{1}{(1-e^{2it})(1-e^{-2it})}(U^{a(t)} - e^{it}U)(V^{a(t)} - e^{-it}V). \end{aligned} \tag{5.4.10}$$

This then gives a first step in the infinitesimal Cartan decomposition of the Casimir operator Ω_1. The result is given in Theorem 5.4.34, and $\Omega_2 = (0, \Omega)$.

Theorem 5.4.34. *We have*

$$\begin{aligned}\Omega_1 = &-\big(\frac{1}{2}(J_0, J_0) + \frac{1}{2}(J_0, -J_0)\big)^2 - \frac{2\cos t}{\sin t}(J_0, J_0) - \frac{\cos t}{2\sin t}(J_0, -J_0) \\ &- \frac{1}{4\sin^2 t}\big(U^{a(t)}V^{a(t)} - e^{it}V^{a(t)}U - e^{-it}U^{a(t)}V + UV\big), \\ \Omega_2 = &-\big(\frac{1}{2}(J_0, J_0) - \frac{1}{2}(J_0, -J_0)\big)^2 + \frac{2\cos t}{\sin t}(J_0, J_0) - \frac{\cos t}{2\sin t}(J_0, -J_0) \\ &- \frac{1}{4\sin^2 t}\big(U^{a(t)}V^{a(t)} - e^{-it}V^{a(t)}U - e^{it}U^{a(t)}V + UV\big).\end{aligned}$$

Note that the expressions in Theorem 5.4.34 are the infinitesimal Cartan or KAK-decomposition for the two Casimir elements. So one can write the action of one of the Casimir elements on a function F satisfying (5.1.1) as a differential operator for F restricted to A. Indeed, a smooth function $F\colon G \to \mathrm{End}(V_\tau)$ is determined by its restriction to A by Exercise 5.4.1 and then the action of the Casimir elements gives rise to a second-order differential operator. In particular, we find

$$\Omega_1 \cdot \Phi^\ell_{(\ell_1,\ell_2)}|_A = \ell_1(\ell_1+1)\Phi^\ell_{(\ell_1,\ell_2)}|_A, \qquad \Omega_2 \cdot \Phi^\ell_{(\ell_1,\ell_2)}|_A = \ell_2(\ell_2+1)\Phi^\ell_{(\ell_1,\ell_2)}|_A.$$

Note that in cas $\ell = 0$, i.e. the trivial representation of $K = \mathrm{diag} \cong \mathrm{SU}(2)$, all terms U, V and (J_0, J_0) act trivially, and the actions of Ω_1 and Ω_2 coincide. In this case we know that the spherical functions are the characters of SU(2), i.e. expressible in terms of Chebyshev polynomials, see Section 3.4.1. In that case we obtain second-order differential operator for the Chebyshev polynomials of the second kind, which is a special case of the second-order hypergeometric differential operator.

Proof. Using the expression (5.4.10) for Ω_1 we obtain

$$\begin{aligned}\Omega_1 &= (-J_0^2 + iJ_0, 0) + \frac{1}{(1-e^{2it})(1-e^{-2it})}(U^{a(t)} - e^{it}U)(V^{a(t)} - e^{-it}V) \\ &= (-J_0^2 + iJ_0, 0) - \frac{1}{4\sin^2 t}\big(U^{a(t)}V^{a(t)} - e^{it}UV^{a(t)} - e^{-it}U^{a(t)}V + UV\big)\end{aligned}$$

and since we need to get the elements of $\mathfrak{k}_\mathbb{C}$ in the conjugated version to the left we need the commutation

$$[U, V^{(a(t))}] = (2ie^{it}J_0, 2ie^{-it}J_0). \tag{5.4.11}$$

So we finally need to rewrite the remaining term

$$(-J_0^2 + iJ_0, 0) + \frac{e^{it}}{4\sin^2 t}(2ie^{it}J_0, 2ie^{-it}J_0)$$

in terms of the generator $(J_0, -J_0)$ of $\mathfrak{a}_\mathbb{C}$ and the generator (J_0, J_0) of $\mathfrak{m}_\mathbb{C}$. Now write $(J_0, 0) = \frac{1}{2}(J_0, -J_0) + \frac{1}{2}(J_0, J_0)$ and $(2ie^{it}J_0, 2ie^{-it}J_0) = 2i\cos t(J_0, J_0) - 2\sin t(J_0, -J_0)$. Collecting the terms and simplifying the trigonometric terms gives the result for Ω_1. A similar calculation gives Ω_2. □

Exercise 5.4.35. (i) Prove (5.4.11), and check that the final expression for Ω_1 follows.

(ii) Prove the expression for Ω_2. You can either do this by mimicking the calculation of the proof of Theorem 5.4.34 or you can use the automorphism θ.

Note that we have obtained a matrix differential operator for the matrix spherical functions. Theorem 5.4.34 gives $\Phi^\ell_{(\ell_1,\ell_2)}$ as eigenfunction of two commuting matrix differential operators. Writing τ also for the corresponding representation of the complexified Lie algebra $\mathfrak{k}_\mathbb{C} = \mathfrak{su}(2)_\mathbb{C}$ we find that the matrix spherical function $\Phi^\ell_{(\ell_1,\ell_2)}$ restricted to A satisfies a second order differential operator arising from the expression for Ω_1 in Theorem 5.4.34;

$$\begin{aligned}
&-\frac{1}{4}\frac{d^2F}{dt^2}(t) - \frac{1}{2}\tau(J_0)\frac{dF}{dt}(t) - \frac{1}{4}\tau(J_0)^2F(t) - 2\frac{\cos t}{\sin t}\tau(J_0)F(t) - \frac{\cos t}{2\sin t}\frac{dF}{dt}(t) \\
&-\frac{1}{4\sin^2 t}\Big(\tau(J_+ + iJ_-)\tau(J_+ - iJ_-)F(t) - e^{it}\tau(J_+ - iJ_-)F(t)\tau(J_+ + iJ_-) \\
&- e^{-it}\tau(J_+ + iJ_-)F(t)\tau(J_+ - iJ_-) + F(t)\tau(J_+ + iJ_-)\tau(J_+ - iJ_-)\Big) = \ell_1(\ell_1+1)\,F(t),
\end{aligned} \tag{5.4.12}$$

where $F(t) = \Phi^\ell_{(\ell_1,\ell_2)}(a(t))$ and $\tau = \pi_\ell$, the spin ℓ-representation of $\mathrm{SU}(2)$. Note that in (5.4.12) the elements $\tau(J_0)$ can be put on either side of F and its derivatives, since J_0 corresponds to to the action of M and $F(t) \in \mathrm{End}_M(V_\tau)$. For the other elements in (5.4.12) the order is indeed relevant. Note that (5.4.12) has a companion second order differential operator for Ω_2. Since Ω_1 and Ω_2 commute, the corresponding differential operators also commute. There is in general a rather large subalgebra of elements of the universal algebra that gives rise to an algebra of differential operators having the matrix spherical functions as eigenfunctions, see Dixmier [19, Ch. 9].

It is indeed possible to rewrite the second order differential equation (5.4.12) and its companion to a second order differential equation for the matrix orthogonal polynomials in Theorem 5.4.20. For this one needs to find a first order differential equation for the minimal $\Phi^\ell_{(\ell_1,\ell_2)}$, $\ell_1 + \ell_2 = \ell$, and this can always be done. Note that by Theorem 5.4.34 we have

$$\Omega_1 - \Omega_2 = -(J_0, J_0)(J_0, -J_0) - \frac{4\cos t}{\sin t}(J_0, J_0) + \frac{i}{2\sin t}\big(V^{a(t)}U - U^{a(t)}V\big)$$

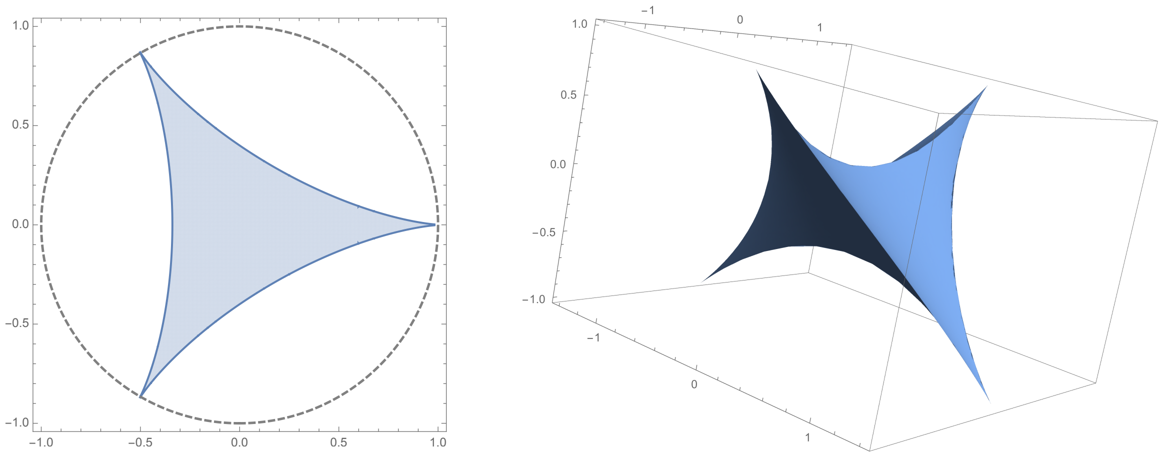

Figure 5.4.3: The figure on the left corresponds to the orthogonality region for the case $n = 2$. This is the area enclosed by Steiner's hypocycloid. The figure on the right is the three-dimensional region of orthogonality for $n = 3$.

which gives the first order differential equation

$$\begin{aligned}
&-\tau(J_0)\frac{dF}{dt}(t) - \frac{4\cos t}{\sin t}\tau(J_0)F(t) \\
&+\frac{i}{2\sin t}\Big(\tau(J_+ - iJ_-)F(t)\tau(J_+ + iJ_-) - \tau(J_+ + iJ_-)F(t)\tau(J_+ - iJ_-)\Big) \\
&= \big(\ell_1(\ell_1+1) - \ell_2(\ell_2+1)\big)\, F(t),
\end{aligned} \tag{5.4.13}$$

where $F(t) = \Phi^{\ell}_{(\ell_1,\ell_2)}(a(t))$ and $\tau = \pi_\ell$ as before. Note that both sides of (5.4.13) reduce to 0 in case $\ell = 0$, i.e. τ is trivial, since then $\ell_1 = \ell_2$. Note that is a phenomenon which doesn't arise for scalar orthogonal polynomials, since in that case the order of the relevant differential operator is at least two.

The second order differential operators can be used to explicitly determine the matrix orthogonal polynomials of Theorem 5.4.20. However, an essential step in this determination is a very convenient explicit LDU-decomposition of the weight function W. The LDU-decomposition then also gives rise to yet another second order differential operator. Unfortunately, there is no (or better 'not yet') group theoretic interpretation of the LDU-decomposition of the weight. We refer to the papers by Koornwinder [55] and the papers [48], [49]. The explicit LDU-decomposition has the advantage that one can use analytic methods to extend in the parameters. For the special case of the matrix orthogonal polynomials arising in this situation the analytic methods are clarified in [47].

Remark 5.4.36. Most of the programme sketched in this Section 5.4 for $n = 1$ has its analogue for $n > 1$ as well. It is indeed possible to describe $\hat{G}_\tau$ for $G = \mathrm{SU}(n+1)\times\mathrm{SU}(n+1)$ and τ being the representation in terms of homogeneous polynomials as presented in Section 5.4.1. This requires some additional background in the representation theory of compact Lie groups. In the case $n > 1$ one can also introduce matrix orthogonal polynomials, and in general the polynomials are orthogonal polynomials of n variables on more complicated areas in $\mathbb{R}^n$, see Figure 5.4.3 for the areas in the case $n = 2$ and $n = 3$. In this setting the polynomials are eigenfunctions of second order partial differential operators and of a first order partial differential operator. In this case the polynomials are irreducible, at least for low sizes. We refer to [50] for more information.

Appendices

Appendix A

Lie algebras

This Appendix contains some results on Lie algebras, and all statements can be found in Humphreys [32], see also [7], [8]

A.1 Lie algebras

We consider Lie algebras over $\mathbb{R}$ or $\mathbb{C}$.

Definition A.1.1. *A* Lie algebra $\mathfrak{g}$ *is a finite-dimensional complex or real vector space equipped with a bilinear map* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$*, called the* Lie bracket*, satisfying*

(i) (skew-symmetry) $[X, Y] = -[Y, X]$*,* $\forall\, X, Y \in \mathfrak{g}$

(ii) (Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$*,* $\forall\, X, Y, Z \in \mathfrak{g}$

The linear subspace $\mathfrak{k} \subset \mathfrak{g}$ is a *Lie subalgebra* if $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, and it is a *Lie ideal* if $[\mathfrak{k}, \mathfrak{g}] \subset \mathfrak{k}$. Note that, by skew symmetry, there is no difference between a left- and right ideal. A Lie subalgebra is a Lie algebra in itself.

The Lie algebra is *abelian* if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.

A linear map $\phi\colon \mathfrak{g} \to \mathfrak{g}'$ between two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ is a *Lie algebra homomorphism* if it preserves the Lie bracket; $\phi([X, Y]) = [\phi(X), \phi(Y)]$. And $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic if there exists a Lie algebra homomorphism which is an isomorphism. Note that the kernel of ϕ is Lie ideal of $\mathfrak{g}$ and that the image of ϕ is a Lie subalgebra of $\mathfrak{g}'$.

For a complex vector space V the space $\mathrm{End}(V)$ of linear maps of V to V has a bracket given by $[T, S] = TS - ST$. Skew-symmetry is obvious, and the Jacobi identity follows from the associativity. So in case V is finite-dimensional, $\mathrm{End}(V)$ is a complex Lie algebra. Any associative algebra (with or without unit) gives a (possibly infinite-dimensional) Lie algebra for the commutator bracket. A *representation* of a Lie algebra is a vector space V (not necessarily finite-dimensional) with a Lie algebra homomorphism $\phi\colon \mathfrak{g} \to \mathrm{End}(V)$.

An important example of a representation is the *adjoint representation*, which is

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad \mathrm{ad}\, X\colon Y \mapsto \mathrm{ad}\, X(Y) = [X, Y] \tag{A.1.1}$$

and $\mathrm{ad}\, X\colon \mathfrak{g} \to \mathfrak{g}$ is a linear map, and moreover $X \mapsto \mathrm{ad}\, X$ is linear, both by the bilinearity of the Lie bracket. To show that ad preserves the bracket we need to show that $\mathrm{ad}\, [X,Y] = \mathrm{ad}\, X \circ \mathrm{ad}\, Y - \mathrm{ad}\, Y \circ \mathrm{ad}\, X$, which is equivalent to

$$[[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]], \qquad \forall Z \in \mathfrak{g}$$

and this is the Jacobi identity (using the skew-symmetry). Using the adjoint representation we define the *Killing form* $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, which is a bilinear form defined by $B(X,Y) = \mathrm{Tr}\,(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$. Note that the Killing form is symmetric; $B(X,Y) = B(Y,X)$ for all $X, Y \in \mathfrak{g}$ and that

$$\begin{gathered} B(\mathrm{ad}\, X(Y), Z) = B([X,Y],Z) = \mathrm{Tr}\,(\mathrm{ad}\,[X,Y] \circ \mathrm{ad}\, Z) = \\ \mathrm{Tr}\,(\mathrm{ad}\, X \circ \mathrm{ad}\, Y \circ \mathrm{ad}\, Z) - \mathrm{Tr}\,(\mathrm{ad}\, Y \circ \mathrm{ad}\, X \circ \mathrm{ad}\, Z) = \\ \mathrm{Tr}\,(\mathrm{ad}\, Y \circ \mathrm{ad}\, Z \circ \mathrm{ad}\, X) - \mathrm{Tr}\,(\mathrm{ad}\, Y \circ \mathrm{ad}\, X \circ \mathrm{ad}\, Z) = \\ \mathrm{Tr}\,(\mathrm{ad}\, Y \circ [\mathrm{ad}\, Z, \mathrm{ad}\, X]) = -\mathrm{Tr}\,(\mathrm{ad}\, Y \circ \mathrm{ad}\,[X,Z]) = -B(Y, \mathrm{ad}\, X(Z)) \end{gathered}$$

or $\mathrm{ad}\, X$ is antisymmetric with respect to the Killing form.

Example A.1.2. The first example is $\mathfrak{gl}(n,\mathbb{C}) = \mathrm{End}(\mathbb{C}^n)$, which is a Lie algebra since $\mathrm{End}(\mathbb{C}^n)$ is an associative (unital) algebra. Other examples can be obtained as Lie subalgebras of $\mathfrak{gl}(n,\mathbb{C})$. First, after choosing a basis of the vector space $\mathbb{C}^n$, we can take $\mathfrak{d}(n,\mathbb{C})$ of diagonal matrices, and since diagonal matrices commute, this is an example of an abelian Lie algebra. An important example is

$$\mathfrak{sl}(n,\mathbb{C}) = \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid \mathrm{Tr}\, X = 0\}$$

which is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ since the trace of a commutator is zero. The Killing form on $\mathfrak{sl}(n,\mathbb{C})$ is $B(X,Y) = 2n\mathrm{Tr}\,(XY)$, where the right hand side is the trace of the product of two $n \times n$-matrices X and Y. For $n = 2$ we take the basis H, E, F with

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and commutation relations

$$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H. \tag{A.1.2}$$

Then $\mathrm{ad}\, H$, $\mathrm{ad}\, E$ and $\mathrm{ad}\, F$ correspond to the following 3×3-matrices with respect to the basis $\{H, E, F\}$;

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

So $B(H,H) = 8$, $B(H,E) = 0$, $B(H,F) = 0$, $B(E,E) = 0$, $B(E,F) = 4$, $B(F,F) = 0$. Indeed, we see $B(X,Y) = 4\mathrm{Tr}\,(XY)$.

Exercise A.1.3. Show that $\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C}\mathbf{1}$ is a direct sum of Lie ideals.

Now a *semisimple Lie algebra* is a Lie algebra for which the Killing form B is non-degenerate, i.e. $B(X,Y)=0$ for all $Y \in \mathfrak{g}$ implies $X=0$. This is equivalent to $\mathfrak{g}$ being the direct sum of simple Lie algebras; $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ with all $\mathfrak{g}_i$'s simple Lie algebras, i.e. $\mathfrak{g}_i$ does not contain a non-trivial Lie ideal.

Definition A.1.4. *A $*$-structure on a Lie algebra $\mathfrak{g}$ is an antilinear involution $*\colon \mathfrak{g} \to \mathfrak{g}$ satisfying* $[X^*,Y^*] = [Y,X]^*$.

Note that -1-eigenspace of $*$ gives $\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid X^* = -X\}$. Now $\mathfrak{g}_0$ is a Lie algebra, since for $X,Y \in \mathfrak{g}_0$ we have

$$[X,Y]^* = [Y^*,X^*] = [Y,X] = -[X,Y] \quad \Longrightarrow \quad [X,Y] \in \mathfrak{g}_0.$$

Note that $\mathfrak{g}_0$ is a Lie algebra considered as a real vector space, since $*$ is antilinear, so we consider the $\mathfrak{g}_0$ as a *real form* of $\mathfrak{g}$. Then $i\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid X^* = X\}$ is the $+1$-eigenspace of $*$, and the decomposition of $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ into eigenspaces is to be considered as the complexification of the real Lie algebra $\mathfrak{g}_0$. Any real Lie algebra $\mathfrak{g}$ can be complexified $\mathfrak{g}_\mathbb{C} = \mathbb{C} \otimes \mathbb{R}\mathfrak{g}$, and the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ has a Lie bracket that comes from extending to complex scalars.

Example A.1.5. Take $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, and let the $*$-operator be the standard matrix adjoint. Then $\mathfrak{g}_0 = \{X \in \mathfrak{sl}(n,\mathbb{C}) \mid X^* = -X\}$ is the real Lie algebra of traceless anti-selfadjoint matrices, which is the real Lie algebra $\mathfrak{su}(n)$ corresponding to the special unitary Lie algebra. Note that for $X \in \mathfrak{g}_0$, we have $B(X,X) = 2n\mathrm{Tr}\,(XX) = -2n\mathrm{Tr}\,(X^*X) < 0$ for $X \neq 0$, so that the Killing form is negative definite on $\mathfrak{g}_0$.

Exercise A.1.6. Take $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, and let

$$J = \begin{pmatrix} \mathbf{1}_r & 0 \\ 0 & -\mathbf{1}_s \end{pmatrix}, \qquad r+s=n,$$

be the $n \times n$ matrix with r 1's and s -1's on the diagonal. So $J^2 = \mathbf{1}$, $J = J^*$. We reserve the $*$-notation for the standard adjoint of a matrix, and we define $X^\dagger = JX^*J$. Show that $\dagger$ is a $*$-operation on $\mathfrak{g}$, and show that the real Lie algebra is

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid JX^* + XJ = 0\}$$

which is the real Lie algebra $\mathfrak{su}(r,s)$. In particular, for the case $n=2$, $r=s=1$, we get the real Lie algebra $\mathfrak{su}(1,1)$. Show that $\mathfrak{su}(1,1) = C\mathfrak{sl}(2,\mathbb{R})C^{-1}$ with $C = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$.

Next assume that we have a linear group G such that $\mathfrak{g}$ is the corresponding Lie algebra, i.e. $\exp(tX) \in G$ for all t in some interval $(-r,r)$, $r>0$ if and only if $X \in \mathfrak{g}$, cf. Proposition 2.3.7 for the case of $\mathrm{SU}(n)$ with Lie algebra $\mathfrak{su}(n)$. Then for $g \in G$ and $X \in \mathfrak{g}$ we have

$gXg^{-1} \in \mathfrak{g}$, since $\exp(tgXg^{-1}) = g\exp(tX)g^{-1} \in G$ for $t \in (-r,r)$. This gives the adjoint action of G on $\mathfrak{g}$

$$\mathrm{Ad}\colon G \to \mathrm{End}(\mathfrak{g}), \qquad \mathrm{Ad}(g)\colon X \mapsto gXg^{-1}. \tag{A.1.3}$$

Note that this gives a finite dimensional representation of G on $\mathfrak{g}$.

More general, for a representation $\pi\colon G \to B(V)$ in the space V, we assume that the map $t \mapsto \pi(\exp(tX))v$, $v \in V$, is differentiable at $t = 0$, then we define

$$d\pi(X)v = \frac{d}{dt}\Big|_{t=0} \pi(\exp(tX))v. \tag{A.1.4}$$

In particular, for the adjoint representation of G on $\mathfrak{g}$ we get

$$\begin{aligned} d\mathrm{Ad}(X)Y =& \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\exp(tX))Y = \frac{d}{dt}\Big|_{t=0} \exp(tX)Y\exp(-tX) \\ =& XY - YX = [X,Y] = \mathrm{ad}\,(X)(Y) \end{aligned}$$

so that $d\mathrm{Ad} = \mathrm{ad}$, which is a Lie algebra representation. Assuming that the conditions of differentiability are satisfied, we get similarly

$$\begin{aligned} d\pi(gXg^{-1})v &= \frac{d}{dt}\Big|_{t=0} \pi(\exp(tgXg^{-1}))v = \frac{d}{dt}\Big|_{t=0} \pi(g\exp(tX)g^{-1}))v \\ &= \frac{d}{dt}\Big|_{t=0} \pi(g)\pi(\exp(tX))\pi(g^{-1})v = \pi(g)\Big(\frac{d}{dt}\Big|_{t=0} \pi(\exp(tX))\Big)\pi(g^{-1})v \end{aligned}$$

so $d\pi(gXg^{-1}) = \pi(g)d\pi(X)\pi(g^{-1})$ whenever the action is well-defined on $v \in V$. Replacing $g = \exp(tY)$ and taking derivative with respect to t and setting $t = 0$, we get (after flipping the role of X and Y) that the derived representation satisfies

$$d\pi([X,Y]) = [d\pi(X), d\pi(Y)] \tag{A.1.5}$$

so that $d\pi$ gives a representation of $\mathfrak{g}$ in V. This always works well in case V is a finite-dimensional representation, but in other cases the operators $d\pi(X)$ can be (and generally are) unbounded operators and we have to be careful with the domains of the operators.

So we see that in general a representation of the linear G leads to a representation of the Lie algebra $\mathfrak{g}$. It is not true that in general representations of $\mathfrak{g}$ can be integrated to representations of G, even for finite dimensional representations. This depends on the topology of G.

Exercise A.1.7. Let G be a linear group with Lie algebra $\mathfrak{g}$. Let $\pi\colon G \to \mathrm{End}(V)$ be a finite-dimensional representation with derivate $d\pi\colon \mathfrak{g} \to \mathrm{End}(V)$. Prove the following statements:

(i) Assume W is an invariant subspace for π, then W is an invariant subspace for $d\pi$.

(ii) Assume that $d\pi$ is irreducible, then π is irreducible.

(iii) Assume that π is a unitary representation of G. Show that $d\pi(X)$ is skew-adjoint for all $X \in \mathfrak{g}$.

(iv) Assume that $\pi_1 \cong \pi_2$ as G-representations. Show that $d\pi_1 \cong d\pi_2$ as representations of the Lie algebra $\mathfrak{g}$.

Noet that

Remark A.1.8. Note that any $X \in \mathfrak{g}$ gives a first order differential operator on smooth functions on G by

$$(X \cdot f)(g) = \frac{d}{dt}\Big|_{t=0} f(g \exp(tX))$$

and this gives X as a left-invariant vectorfield. Indeed, $X \cdot (\lambda(h)f) = \lambda(h)(X \cdot f)$ for any $h \in G$, where λ is the left regular action as in Example 2.5.4. Note that this action is the derivative of the right regular representation.

A.2 The universal enveloping algebra

In this section we consider a complex Lie algebra $\mathfrak{g}$. The definition of the universal enveloping via its universal property is as follows.

Definition A.2.1. *For a Lie algebra* $\mathfrak{g}$ *the* universal enveloping algebra *is a pair* (U, i)*, where* $U = U(\mathfrak{g})$ *is an associative (unital) algebra and* $i\colon \mathfrak{g} \to U$ *is a linear map satisfying*

$$i([X, Y]) = i(X)i(Y) - i(X)i(Y)$$

such that for any linear map $j\colon \mathfrak{g} \to U'$ *with* U' *an associative (unital) algebra and* j *satisfying* $j([X, Y]) = j(X)j(Y) - j(X)j(Y)$ *there exists a unique algebra homomorphism* $\phi\colon U \to U'$ *such that* $\phi \circ i = j$*.*

Note that we require i and j to be a Lie algebra homomorphisms. The universality proves directly that $U(\mathfrak{g})$ is unique up to isomorphism of algebras. The existence follows by showing that $T(\mathfrak{g})/\mathcal{I}$ has the required property. Here $T(\mathfrak{g})$ is the tensor algebra for the vector space $\mathfrak{g}$ and $\mathcal{I}$ is the two-sided ideal generated by $X \otimes Y - Y \otimes X - [X, Y]$ for all $X, Y \in \mathfrak{g}$. For $X_{i_1}, \cdots, X_{i_r} \in \mathfrak{g}$ we let $X_{i_1} \cdots X_{i_r}$ be the corresponding image of $X_{i_1} \otimes \cdots \otimes X_{i_r} \in T(\mathfrak{g})$.

The *Poincaré-Birkhoff-Witt theorem* or PBW-theorem states that a basis for $\mathfrak{g}$ induces a basis for $U(\mathfrak{g})$ as vector spaces.

Theorem A.2.2 (PBW)**.** *Let* $\{X_1, \cdots, X_n\}$ *be a basis for the vector space* $\mathfrak{g}$*, then*

$$\{X_1^{m_1} \cdots X_n^{m_n} \mid m_i \in \mathbb{N}\}$$

is a basis for the vector space $U(\mathfrak{g})$*.*

Note that $U(\mathfrak{g})$ is an infinite dimensional associative unital algebra. The unit 1 corresponds to $m_i = 0$ for all i.

Corollary A.2.3. $\mathfrak{g} \to U(\mathfrak{g})$ *is injective, so* $\mathfrak{g} \subset U(\mathfrak{g})$*.*

Corollary A.2.4. *If $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$, then $U(\mathfrak{g})$ is a free $U(\mathfrak{k})$-module.*

Here a free module means that $U(\mathfrak{g})$ can be written uniquely as a sum of basis elements times $U(\mathfrak{k})$. To see how this follows from the PBW-theorem, take a basis $\{X_1, \cdots, X_k\}$ of $\mathfrak{k}$ and pick $\{Y_1, \cdots, Y_n\}$ such that $\{X_1, \cdots, X_k, Y_1, \cdots, Y_n\}$ forms a basis of $\mathfrak{g}$, then $U(\mathfrak{g})$ has basis $Y_1^{m_1} \cdots Y_n^{m_n}$, $m_i \in \mathbb{N}$, as $U(\mathfrak{k})$-module.

Corollary A.2.5. *If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, with $\mathfrak{k}$ and $\mathfrak{s}$ Lie subalgebras, then $U(\mathfrak{g}) = U(\mathfrak{k}) \otimes U(\mathfrak{s})$.*

By the universality property of Definition A.2.1 we see that there is a $1-1$-correspondence between finite-dimensional Lie algebra representations and finite dimensional representations of the universal enveloping algebras. Indeed, starting from a Lie algebra representation $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ we know that it extends to an algebra homomorphism $\phi\colon U(\mathfrak{g}) \to \mathrm{End}(V)$. Conversely, we can restrict a representation of $U(\mathfrak{g})$ from $U(\mathfrak{g})$ to $\mathfrak{g}$ by Corollary A.2.3 in order to obtain a representation of $\mathfrak{g}$.

In case the Lie algebra is abelian, the universal enveloping algebra is the symmetric algebra of $\mathfrak{g}$; $U(\mathfrak{g}) = S(\mathfrak{g})$, which we can view as the polynomial algebra $\mathbb{C}[X_1, \cdots, X_r]$ for a basis $\{X_i\}_{i=1}^r$ of the abelian Lie algebra $\mathfrak{g}$.

Remark A.2.6. We can extend Remark A.1.8 to the universal enveloping algebra so that we can view elements of $U(\mathfrak{g})$ as higher order left-invariant differential operators on smooth functions of G. Indeed, for $X_1, \cdots, X_m$ we define the m-th order differential operator on a smooth function f by

$$(X_1 \cdots X_m) \cdot f(g) = \frac{d}{dt_1}\Big|_{t_1=0} \cdots \frac{d}{dt_m}\Big|_{t_m=0} f(g \exp(t_1 X_1) \cdots \exp(t_m X_m))$$

and the corresponding differential operator is left invariant as for the case of a first order differential operator as in Remark A.1.8. This indeed gives an action of $U(\mathfrak{g})$, since it is the extension of the derivative of the right regular representation. Note that the action satisfies $X \cdot (Y \cdot f) = (XY) \cdot f$.

For a semisimple Lie algebra $\mathfrak{g}$ we can take a basis $\{X_i\}_i$ of the Lie algebra $\mathfrak{g}$, and we take the dual basis $\{X^i\}_i$ with respect to the Killing form, $B(X_i, X^j) = \delta_{i,j}$. This is well-defined since the Killing form is non-degenerate for a semisimple Lie algebra. Then the *Casimir element*

$$\Omega = \sum_i X_i X^i \in Z(U(\mathfrak{g}))$$

is central. Moreover, it is independent of the choice of basis. In particular, in any finite dimensional irreducible representation π of G with Lie algebra $\mathfrak{g}$, the element $d\pi(\Omega)$, the extension of the Lie algebra representation to $U(\mathfrak{g})$, acts as a scalar, say c_π, by Schur's Lemma 2.5.7. This means that any matrix coefficient of the representation π is an eigenfunction to the second order partial differential operator corresponding to Ω for the eigenvalue c_π.

Finally, note that if $\mathfrak{g}$ has a $*$-structure, then $U(\mathfrak{g})$ is a $*$-algebra, where $*$ is an antilinear antimultiplicative involution on $U(\mathfrak{g})$, so that $U(\mathfrak{g})$ is a $*$-algebra.

A.3 Representations of $\mathfrak{sl}(2,\mathbb{C})$

The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of 2×2-complex matrices of trace zero is a three-dimensional complex vector space with a basis $\{H,E,F\}$ as in Example A.1.2. Assume the representation $\sigma\colon \mathfrak{sl}(2,\mathbb{C})\to \mathrm{End}(V)$ is an irreducible finite dimensional representation of the Lie algebra. Taking an eigenvector $v\in V$ for an eigenvalue $\lambda\in\mathbb{C}$ for $\sigma(H)$, then $\sigma(E)v$ satisfies

$$\sigma(H)\sigma(E)v=\sigma(E)\sigma(H)v+2\sigma(E)v=(\lambda+2)\sigma(E)v$$

since $[H,E]=2E$. Similarly, $\sigma(H)\sigma(F)v=(\lambda-2)\sigma(F)v$. Since eigenvectors for different eigenvalues are linearly independent and the representation is finite-dimensional, we may assume that there exists $v_0\in V$ with $\sigma(H)v=\lambda_0 v_0$ and $\sigma(E)v_0=0$. Define inductively $v_{k+1}=\sigma(F)v_k$, so that $\sigma(H)v_k=(\lambda_0-2k)v_k$. Then

$$\sigma(E)v_k=k(\lambda_0-k+1)v_{k-1} \tag{A.3.1}$$

by induction on k. The case $k=0$ follows by $\sigma(E)v_0=0$, and for $k=1$ we can check

$$\sigma(E)v_1=\sigma(E)\sigma(F)v_0=\sigma(F)\sigma(E)v_0+\sigma(H)v_0=0+\lambda_0v_0=\lambda_0v_0$$

and the induction step goes similarly;

$$\begin{aligned}\sigma(E)v_{k+1}=&\sigma(E)\sigma(F)v_k=\sigma(F)\sigma(E)v_k+\sigma(H)v_k=k(\lambda_0-k+1)\sigma(F)v_{k-1}+(\lambda_0-2k)v_k\\ =&(\big(k(\lambda_0-k+1)+(\lambda_0-2k)\big)v_k=(k+1)(\lambda_0-k)v_k\end{aligned}$$

So if $v_k\neq 0$, $k\in\{0,\cdots,n\}$, and $v_{n+1}=0$, then $\sigma(E)v_{k+1}=0$ implying $\lambda_0=n$.

Lemma A.3.1. *For each $n\in\mathbb{N}$ there is an $n+1$-dimensional representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ acting on V with basis $\{v_0,\cdots,v_n\}$ by*

$$H\cdot v_k=(n-2k)v_k,\quad E\cdot v_k=\begin{cases}0 & k=0\\ k(n-k+1)v_{k-1} & k\geq 1\end{cases},\quad F\cdot v_k=\begin{cases}0 & k=n\\ v_{k+1} & k\leq n-1\end{cases}.$$

Now we take the standard adjoint on $\mathfrak{sl}(2,\mathbb{C})$ and $U(\mathfrak{sl}(2,\mathbb{C}))$. Then the corresponding real form $\mathfrak{g}_0=\mathfrak{sl}(2,\mathbb{C})_0=\mathfrak{su}(2)$. So we ask if the representations of Lemma A.3.1 are unitarisable, i.e. if we can impose an inner product on V such that $H^*=H$ and $E^*=F$. Now $H^*=H$ requires $\langle v_k,v_l\rangle=0$ for $k\neq l$. And $E^*=F$ gives

$$\|v_{k+1}\|^2=\langle F\cdot v_k,F\cdot v_k\rangle=\langle EF\cdot v_k,v_k\rangle=(k+1)(n-k)\|v_k\|^2$$

so that normalising $\|v_0\|^2=1$ we get $\|v_k\|^2=(-1)^k k!\,(-n)_k$, which is indeed positive. Normalising we get that $\mathfrak{sl}(2,\mathbb{C})$ has a finite dimensional unitary representation for the $*$-operator in the $n+1$-dimensional Euclidean vector space with orthonormal basis $\{f_0,\cdots,f_n\}$ given by

$$H\cdot f_k=(n-2k)f_k,\ E\cdot f_k=\sqrt{k(n-k+1)}f_{k-1},\ F\cdot f_k=\sqrt{(k+1)(n-k)}f_{k+1}, \tag{A.3.2}$$

which has the advantage that notation is uniform. Indeed, the vectors killed by E and F follow from the coefficients in (A.3.2). In order to match with the SU(2)-representations of Chapter 3, we put $\ell = \frac{1}{2}n$, and relabel $e^\ell_n = f_{\ell+n}$, $n \in \{-\ell, -\ell+1, \cdots, \ell\}$, so that

$$H \cdot e^\ell_n = -2n e^\ell_n, \quad E \cdot e^\ell_n = \sqrt{(\ell+n)(\ell-n+1)}e^\ell_{n-1}, \quad F \cdot e^\ell_n = \sqrt{(\ell+n+1)(\ell-n)}e^\ell_{n+1}.$$

Next we calculate the Casimir element of $U(\mathfrak{sl}(2,\mathbb{C}))$. Taking the form in Example A.1.2 we see that the basis dual to $\{H, E, F\}$ with respect to the Killing form B is given by $\{\frac{1}{8}H, \frac{1}{4}F, \frac{1}{4}E\}$. So the Casimir element is (after multiplying by 2)

$$\Omega = \frac{1}{4}H^2 + \frac{1}{2}(EF + FE) = \frac{1}{4}H^2 + \frac{1}{2}H + FE = \frac{1}{4}H^2 - \frac{1}{2}H + EF, \tag{A.3.3}$$

so

$$\Omega \cdot e^\ell_n = \big(\frac{1}{4}(-2n)^2 - \frac{1}{2}(-2n) + (\ell+n+1)(\ell-n)\big)e^\ell_n = \ell(\ell+1)e^\ell_n. \tag{A.3.4}$$

Indeed, the Casimir operator acts by a constant as indicated by Schur's Lemma.

Exercise A.3.2. Show by direct calculation the Casimir element Ω in (A.3.3) is central, i.e. show that $H\Omega = \Omega H$, $E\Omega = \Omega E$ and $F\Omega = \Omega F$ in $U(\mathfrak{sl}(2,\mathbb{C}))$ using (A.1.2).

Note that we can take a basis of the Lie algebra $\mathfrak{su}(2)$ of traceless skew-adjoint 2×2-matrices$\mathfrak{sl}(2,\mathbb{C})$ by putting

$$J_0 = \frac{1}{2}iH, \qquad J_+ = \frac{1}{2}(F - E), \qquad J_- = \frac{i}{2}(E + F)$$

or explicitly

$$J_0 = \begin{pmatrix} \frac{1}{2}i & 0 \\ 0 & -\frac{1}{2}i \end{pmatrix}, \quad J_+ = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & \frac{1}{2}i \\ \frac{1}{2}i & 0 \end{pmatrix}.$$

The commutation relations become

$$[J_0, J_+] = -J_-, \qquad [J_0, J_-] = J_+, \qquad [J_-, J_+] = J_0.$$

Then $\Omega = -J_0^2 - J_-^2 - J_+^2$

Appendix B

Banach algebras

This Appendix contains some results on Banach algebras, and all statements can be found in [14], [23], [68], [34], [86].

B.1 Banach algebras

Recall that a Banach space is a vector space X equipped with a norm $\|\cdot\|\colon X \to [0,\infty)$ such that the space is topologically complete. With an algebra we always assume an algebra over $\mathbb{C}$ in which associativity and the distributive laws hold. In general an algebra in this appendix is neither commutative nor unital, i.e. having a unit element.

Definition B.1.1. *A Banach space X for which X is also an algebra is a Banach algebra if*

$$\|xy\| \leq \|x\|\,\|y\|, \qquad \forall x, y \in X.$$

In case X is a $*$-algebra, i.e. $*\colon X \to X$ is an antilinear antimultiplicative involution, we say that X is a Banach $*$-algebra if $\|x^*\| = \|x\|$ for all $x \in X$.

Example B.1.2. The convolution algebra $L^1(G)$ for G a linear group is a Banach algebra. This is a commutative algebra if G is an abelian group (and the converse also holds). In general, $L^1(G)$ does not contain a unit. The unit should be the Dirac delta at the identity of g, which is only in $L^1(G)$ if G is a discrete group and the Haar measure is the counting measure. For unimodular G the $*$-structure is given by $f^*(g) = \overline{f(g^{-1})}$.

Another example, the space of bounded continuous functions $C_b(\mathbb{R})$ on $\mathbb{R}$ is a Banach algebra with respect to the supremum norm $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$ and the algebra structure is given by pointwise multiplication. The unit is the constant function 1. It becomes a Banach $*$-algebra for $f^*(x) = \overline{f(x)}$.

In the second example we have $\|f^*f\|_\infty = \|f\|_\infty^2$, i.e. $C_b(\mathbb{R})$ is a C*-algebra, whereas $L^1(G)$ is not a C*-algebra (unless G trivial).

In case X is a Banach algebra without unit, we can equip $X \times \mathbb{C}$ with a Banach algebra structure with unit as follows: $(x,\lambda)(y,\mu) = (xy+\lambda y+\mu x, \lambda\mu)$ and $(x,\lambda)+(y,\mu) = (x+y, \lambda+\mu)$

and norm $\|(x,\lambda)\| = \|x\| + |\lambda|$. Then $X_1 = X \times \mathbb{C}$ is a unital Banach algebra with unit $(0,1)$ such that $X \to X_1$, $x \mapsto (x,0)$ is an isometric isomorphism.

Lemma B.1.3. *For X a unital abelian Banach algebra the correspondence between homomorphisms $h\colon X \to \mathbb{C}$ and maximal ideals in X is a bijection. The space of maximal ideals is denoted by Σ.*

One can show that for a (not necessarily abelian nor unital) Banach algebra X any (algebra) homomorphism $X \to \mathbb{C}$ is continuous with norm at most 1. If X is unital, then the norm is 1.

The space Σ of maximal ideals can be viewed as a subspace of the dual Banach space X^*. The space X^* comes equipped with the weak $*$-topology, which is the topology induced by the seminorms $\{p_x \mid x \in X\}$ given by $p_x(x^*) = |x^*(x)|$. Then the Banach-Alaoglu theorem states that the unit ball in X^* is compact with respect to the weak $*$-topology. Then this makes Σ a topological space, and it is a compact Hausdorff space.

Theorem B.1.4. *Let X be a unital commutative Banach algebra, then the map $X \to C(\Sigma)$ given by $x \mapsto \hat{x}$, $\hat{x}(h) = h(x)$, is the Gelfand transform of x. The Gelfand transform induces a continuous homomorphism $X \to C(\Sigma)$ with kernel the intersection of all maximal ideals.*

In case of a non-unital commutative Banach algebra, one needs to restrict to so-called modular ideals, and Σ is no longer compact. In case of $L^1(\mathbb{R})$ the Gelfand transform can be identified with the Fourier transform, see [14], [34].

We also use representations of Banach $*$-algebras. Recall that a unitary representation of a Banach $*$-algebra X is a homomorphism $\pi\colon X \to B(H)$ for some Hilbert space H such that $\pi(x^*) = \pi(x)^*$. We always assume that π is not the zero map. The representation is called irreducible if the only closed invariant subspaces are the trivial subspaces $\{0\}$ and H.

Proposition B.1.5. *Let X be a commutative Banach $*$-algebra and $\pi\colon X \to B(H)$ an irreducible representation. Then H is one-dimensional.*

Proof. Pick a non-zero self-adjoint element $x \in X$, then $\pi(x)$ is a self-adjoint operator on the Hilbert space H. Apply the spectral theorem to $\pi(x)$, so that we find a projection valued measure E on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$. Then for any $A \in \mathcal{B}(\mathbb{R})$, $E(A)$ is a projection that commutes with $\pi(x)$ and with all operators that commute with $\pi(x)$. Since X is commutative, it means that $E(A)$ commutes with $\pi(y)$ for all $y \in X$. By irreducibility, $E(A)$ projects on $\{0\}$ or on H, so $E(A) = 0$ or $E(A) = \mathbf{1}$. So $\pi(x)$ is a non-zero multiple of the identity. If $x \in X$ is not self-adjoint, write $x = y + iz$, $y = \frac{1}{2}(x + x^*)$, $z = \frac{1}{2i}(x - x^*)$ to conclude that any element acts as a multiple of the identity. Hence, H is one-dimensional. $\square$

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