Analysis 1

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Chapter 1

Introduction

This is the set of lecture notes for the course Analysis 1. The notes are deliberately written in English in order to have the notes accessible to a large audience. For several mathematical terms we have introduced the Dutch translation directly after the introduction of the English mathematical term, so e.g. after defining a bounded set we add “begrensde verzameling” as the Dutch translation. In these lecture notes we have added to each chapter a set of exercises, but there are also exercises throughout the text. You should try to do the exercises in order to increase your understanding. For some exercises there are hints in Appendix B. Moreover, in Appendix A there are additional exercises which discuss extensions of results discussed in these lecture notes. The exercises of Appendix A are a bit more challenging, but they do come with hints and intermediate steps.

There are many names throughout the text; Abel, d'Alemenbert, Bolzano, Borel, Cauchy, Dirichlet, Heine, Newton, Riemann, Taylor, Weierstrass, etc. The names of these mathematicians are often attached to classical results and form part of the mathematical culture, and you can find more information on their life and work at the website

`https://mathshistory.st-andrews.ac.uk/`.

This set of lecture notes is a very concise introduction to the basics of real analysis, and there are many more elaborate sources available. We mention previously used books by Tao (Fields Medal in 2006) [6] and Garling [3]. A very classic book is Rudin's book [5]. These and other books can be consulted for more information. We need several notions from the course *Inleiding in de wiskunde*, see [4], and this includes the real numbers and some of its properties and the notion of a function. This is recalled in Chapter 2.

The notes have been used in the academic year 2020-2021, 2021-2022 and the feedback by students and colleagues has been very useful. Thanks to all who have pointed out errors and typos, and have come up with suggestions.

1.1 Why analysis?

Historically, functions were considered as explicit expression, which is more of a calculus approach. Once the idea of limits arose in relation to differentation, integral, series, etc., they naturally lead to various problems of which we sketch a few here very briefly. This was resolved by the rigorous development of analysis, where we study the abstract development of these notions involving limits. The key person in this development is Weierstrass. A nice discussion of this history can be found in Bressoud's book [2].

Let us discuss a few simple examples which give rise to these paradoxes. The first one goes back to the Middle Ages, and proves that $0 = 1$ via a sequence of equalities as follows:

$$\begin{aligned} 0 &= (1-1) + (1-1) + (1-1) + (1-1) + (1-1) + (1-1) + \cdots \\ &= 1 + (-1+1) + (-1+1) + (-1+1) + (-1+1) + (-1+1) + \cdots \\ &= 1 + 0 + 0 + 0 + 0 + \cdots = 1. \end{aligned}$$

On the other hand we could also try to find the value of

$$\begin{aligned} S &= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\ &= 1 - S \qquad \Longrightarrow \qquad S = \frac{1}{2}, \end{aligned}$$

so what is it? Should we attach the value 0, $\frac{1}{2}$ or 1 to $1-1+1-1+1-1+1-1+1-1+1-1+\cdots$? They all seem to be reasonable values for this infinite sum.

We get in the same vein the equality $0 = 1$ by summing all the entries of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 & 0 & \\ \vdots & & & \ddots & \ddots & & \end{pmatrix}$$

Summing over the rows first, we get 1, and summing over the columns first we obtain 0. Why do these two ways of summing the same numbers lead to different outcomes?

An example from Calculus is the following; take $f_n \colon \mathbb{R} \to \mathbb{R}$, defined by $f_n(x) = ((x+n)^2 + 1)^{-1}$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{1}{(x+n)^2 + 1}\, dx = \pi$$

and on the other hand $\int_{\mathbb{R}} \lim_{n\to\infty} f_n(x)\, dx = 0$. So we cannot interchange limit and integration without additional conditions. Another example for an integral over a finite interval is the following. Put $f_0(x) = 64(x - \frac{1}{4})$ for $\frac{1}{4} \le x \le \frac{3}{8}$ and $f_0(x) = 64(\frac{1}{2} - x)$ for $\frac{3}{8} \le x \le \frac{1}{2}$ and $f_0(x) = 0$ elsewhere. Put $f_n(x) = 2^n f_0(2^n x)$. Then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 1 \ne 0 = \int_0^1 (\lim_{n\to\infty} f_n(x))\, dx$$

since $\lim_{n\to\infty} f_n(x) = 0$ for $x \in [0,1]$.

The common phenomenon in these cases is that they deal with limit processes, and it shows that we cannot work carelessly with limit processes. So how do we need to treat limit processes rigorously in such a way that we can ensure that interchanging limits is indeed valid? This is the main topic of Analysis 1.

After recalling some notions in Chapter 2 and fixing notation, we start by studying sequences and limits of sequences in Chapter 3. This is used to describe the topological nature (what is an 'open' or 'closed' set?) of the real numbers in Chapter 4. Next we discuss real valued functions on $\mathbb{R}$ and the important notion of continuity in Chapter 5. In Chapter 6 we discuss differentiability of functions, and in Chapter 7 we introduce the Riemann integral and derive some of its main properties. In Chapter 8 we study series of numbers, i.e. infinite sums of numbers, and series of functions.

This course is not about calculating limits, derivatives, integrals, series, etc., but it is about what are precisely the conditions that make the Fundamental Theorem of Calculus valid, when can limit processes be interchanged, etc. In *Calculus A*, see [1], you have learned how to do the explicit calculations, and in this course you will learn to understand why this is true. So the emphasis is on general structures and on proofs, not on specific functions. Although sometimes an example will be discussed and we define the logarithm function, the exponential function and the Γ-function in the exercises, see Exercises 7.6.9, 7.6.14.

Chapter 2

Prerequisites

The purpose of Chapter 2 is to recall some of the notions that play an important part in this course. The most important notions we need are the notion of *supremum* of a subset of the real numbers and the notion of a *function*.

2.1 Notation and numbers

We use the standard notation $\mathbb{N}$ for the set of natural numbers “natuurlijke getallen”. Here the number zero, 0, is also considered as a natural number. So

$$\mathbb{N} = \{0, 1, 2, 3, 4, \cdots\}$$

and the inclusion of 0 is a matter of convention. Moreover we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

Here $\mathbb{Z}$ denotes the set of integers “gehele getallen”:

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

$\mathbb{Q}$ denotes the set of rational numbers “rationale getallen”, i.e. the fractions obtained from $\mathbb{Z}$:

$$\mathbb{Q} = \{\frac{n}{m} \mid n, m \in \mathbb{Z}, m \neq 0\}.$$

The real numbers “reële getallen” $\mathbb{R}$ have been introduced in Inleiding Wiskunde [4], and we discuss the important properties of $\mathbb{R}$ for this course in Section 2.2. The natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ all come with a natural ordering: $x \leq y$.

Finally, $\mathbb{C}$ denotes the complex numbers “complexe getallen”. Recall that a complex number $z \in \mathbb{C}$ can be written as

$$z = a + ib = r(\cos\phi + i\sin\phi) = re^{i\phi}$$

where $a = \Re z \in \mathbb{R}$ is the real part of z, $b = \Im z \in \mathbb{R}$ is the imaginary part of z and $r = |z| = \sqrt{z\bar{z}} \geq 0$ is the modulus of z and ϕ is the argument of z. Recall that $\bar{z} = a - ib$ is the complex conjugate. The element i is the imaginary unit satisfying $i^2 = -1$, and this defines the multiplication and addition in $\mathbb{C}$. Explicitly, recall that

$$z + w = (a + ib) + (c + id) = (a + c) + i(b + d),$$
$$z \cdot w = (a + ib) \cdot (c + id) = (ac - bd) + i(ad + bc).$$

For subsets A and B of the real numbers we use the standard notation $A \subset B$ to indicate that all elements of A are contained in B. In particular, this can also mean that $A = B$, i.e. $A = B \Rightarrow A \subset B$. Similarly, $A \cup B$ denotes the union of A and B, and $A \cap B$ denotes the intersection of A and B;

$$A \cup B = \{x \in \mathbb{R} \mid x \in A \text{ or } x \in B\}, \quad A \cap B = \{x \in \mathbb{R} \mid x \in A \text{ and } x \in B\}.$$

The complement of the set $A \subset \mathbb{R}$ is the set $A^c = \{x \in \mathbb{R} \mid x \notin A\}$. We also use the notation $A \setminus B = \{x \in A \mid x \notin B\} = A \cap B^c$.

Exercise 2.1.1. Assume that we have sets B_α, $\alpha \in I$, indexed by an index set I. Then

$$\bigcap_{\alpha \in I} B_\alpha = \{x \in \mathbb{R} \mid \forall \alpha \in I \colon x \in B_\alpha\}, \qquad \bigcup_{\alpha \in I} B_\alpha = \{x \in \mathbb{R} \mid \exists \alpha \in I \colon x \in B_\alpha\}.$$

Show the de Morgan's laws

$$\left(\bigcap_{\alpha \in I} B_\alpha\right)^c = \bigcup_{\alpha \in I} B_\alpha^c, \qquad \left(\bigcup_{\alpha \in I} B_\alpha\right)^c = \bigcap_{\alpha \in I} B_\alpha^c.$$

We use the following notation for intervals "intervallen" as subsets of $\mathbb{R}$, where in general we assume $a \leq b$;

$$\begin{aligned} (a,b) &= \{x \in \mathbb{R} \mid a < x < b\}, & (a,b] &= \{x \in \mathbb{R} \mid a < x \leq b\}, \\ [a,b) &= \{x \in \mathbb{R} \mid a \leq x < b\}, & [a,b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\}, \\ (a,\infty) &= \{x \in \mathbb{R} \mid a < x\}, & [a,\infty) &= \{x \in \mathbb{R} \mid a \leq x\}, \\ (-\infty,b) &= \{x \in \mathbb{R} \mid x < b\}, & (-\infty,b] &= \{x \in \mathbb{R} \mid x \leq b\}. \end{aligned}$$

So in particular, $[a,a] = \{a\}$ and $(a,a] = [a,a) = (a,a) = \emptyset$ is the empty set, and any interval with $b < a$ is also defined as the empty set $\emptyset$.

The absolute value (or in case of $\mathbb{C}$, the modulus) satisfies the following properties. First, $|x| = 0$ if and only if $x = 0$ and the triangle inequality "driehoeksongelijkheid"

$$\forall x, y \qquad |x + y| \leq |x| + |y| \tag{2.1.1}$$

holds for real numbers as well as complex numbers. Sometimes it is convenient to have the reverse triangle inequality "omgekeerde driehoeksongelijkheid";

$$\forall x, y \qquad \big||x| - |y|\big| \leq |x - y|. \tag{2.1.2}$$

Exercise 2.1.2. (i) Show that $|x| \leq |x-y| + |y|$ and conclude that $|x| - |y| \leq |x-y|$.

(ii) Show that $|y| \leq |x-y| + |x|$ and conclude that $|x| - |y| \geq -|x-y|$.

(iii) Finish the proof of (2.1.2).

We recall the notation for sums and products;

$$\sum_{i=n}^{m} a_i = a_n + a_{n+1} + \cdots + a_m, \qquad \prod_{i=n}^{m} a_i = a_n \cdot a_{n+1} \cdots a_m$$

here n and m are integers and a_i, $i \in \{n, n+1, \cdots, m\}$, are numbers. In case $m < n$ we define the empty sum as 0 (being the identity for addition) and the empty product as 1 (being the identity for multiplication).

Finally, we recall the factorial and binomial coefficients;

$$n! = \prod_{i=1}^{n} i, \qquad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

for which we assume that $n, k \in \mathbb{N}$ with $k \leq n$. Following the convention above we set $0! = 1$. The binomial coefficients $\binom{n}{k}$ for which $k \in \mathbb{Z} \setminus \{0, 1, \cdots, n\}$ are set to 0. Then Newton's binomial formula states that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \tag{2.1.3}$$

which generalises the familiar $(x+y)^2 = x^2 + 2xy + y^2$ to arbitrary positive integer powers. A generalisation of Newton's binomial formula (2.1.3) is presented in Chapter 8.

Exercise 2.1.3. (i) Show Pascal's identity;

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

(ii) Prove Newton's binomial formula by induction on n.

2.2 The real numbers

The most important property of the real numbers $\mathbb{R}$ is the supremum or least upper bound "kleinste bovengrens".

Definition 2.2.1. *A subset $A \subset \mathbb{R}$ is* bounded from above *"van boven begrensd" if there exists $M \in \mathbb{R}$ so that each $a \in A$ is smaller than or equal to M. Or*

$$\exists\, M \in \mathbb{R} \quad \forall\, a \in A \qquad a \leq M.$$

A subset $A \subset \mathbb{R}$ *is* bounded from below *"van onderen begrensd" if*

$$\exists\, L \in \mathbb{R} \quad \forall\, a \in A \qquad a \geq L.$$

A subset $A \subset \mathbb{R}$ *is* bounded *"begrensd" if the set* A *is bounded from above and bounded from below.*
A subset $A \subset \mathbb{R}$ *is* unbounded *"onbegrensd" if the set* A *is not bounded.*

Exercise 2.2.2. Show that the set A is bounded if and only if

$$\exists\, M > 0 \quad \forall\, a \in A \qquad |a| \leq M.$$

Exercise 2.2.3. (i) Give an example of a non-empty bounded set A.

(ii) Give an example of a non-empty unbounded set A.

(iii) Give an example of a non-empty unbounded set A, which is bounded from above.

(iv) Give an example of a non-empty unbounded set A, which is bounded from below.

In Inleiding Wiskunde [4] the real numbers have been constructed and Theorem 2.2.4 has been proved.

Theorem 2.2.4 (Completeness of $\mathbb{R}$)**.** *Let* $A \subset \mathbb{R}$ *be a non-empty subset, which is bounded from above. Then* A *has a least upper bound, and this least upper bound is denoted* $\sup(A)$*.*

The notation $\sup(A)$ stands for *supremum* "supremum" of A. The fact that the supremum is the *least* upper bound means the following: for any $L < \sup(A)$ there exists $a \in A$ with $L < a \leq \sup(A)$;

$$L < \sup(A) \quad \Longrightarrow \quad \exists\, a \in A \quad L < a$$

Exercise 2.2.5. Show that the least upper bound is unique.

Exercise 2.2.6. Prove the following statement: let $A \subset \mathbb{R}$ be a non-empty subset, which is bounded from below. Then A has a greatest lower bound, and this greatest lower bound is denoted $\inf(A)$. This is called the *infimum* "infimum".

Exercise 2.2.7. Let $A \subset \mathbb{R}$ be a non-empty set, which is bounded from below. Show that

$$L > \inf(A) \quad \Longrightarrow \quad \exists\, a \in A \quad a < L.$$

Show that the infimum is unique.

Remark 2.2.8. It is sometimes convenient to extend the real numbers $\mathbb{R}$ with the elements $+\infty$ and $-\infty$, and we denote the extended real numbers by $\mathbb{R}^*$. The standard ordering can then be extended by saying that $x \leq +\infty$ and $x \geq -\infty$ for all elements x in the extended number field. Then we can define the supremum of a non-empty set $A \subset \mathbb{R}$ which is not bounded from above as $\sup(A) = +\infty$. Similarly, we can define infimum of a non-empty set $A \subset \mathbb{R}$ which is not bounded from below as $\inf(A) = -\infty$. Note that one has to be very careful with extending the standard arithmetic from $\mathbb{R}$ to the extended real numbers, and we will avoid this. The convention can also be extended to the empty set $\emptyset$, and then we put $\sup(\emptyset) = -\infty$ and $\inf(\emptyset) = \infty$.

2.3 Functions

Recall that functions can be defined in terms of the graph. We think of functions as maps

$$f\colon A \to B$$

from the domain "domein" A to its codomain "codomein" B with the property that for all $a \in A$ there exists a unique $b \in B$ with (a, b) element of the corresponding graph. Then we write $f(a) = b$. Recall that the range "bereik" or image "beeld" is given by

$$\operatorname{Ran}(f) = f(A) = \{f(a) \mid a \in A\} \subset B.$$

More generally we write for $C \subset A$

$$f(C) = \{f(a) \mid a \in C\} \subset B.$$

The function f is surjective "surjectief" (also known as onto) if its range equals its codomain, i.e. $f(A) = B$. The function f is injective "injectief" (also known as one-to-one) if

$$f(a_1) = f(a_2) \quad \Longrightarrow \quad a_1 = a_2.$$

The function f is bijective "bijectief" if f is injective and surjective. In case f is a bijection (i.e. a bijective function), there exists an inverse function $f^{-1}\colon B \to A$ defined by $f^{-1}(b) = a$ if and only if $f(a) = b$.

Remark 2.3.1. It is important to realise that the domain and codomain of the function play a role in injectivity and surjectivity issues. E.g. consider the function f defined by $f(x) = x^2$. Then f is an injective function when considered as $f\colon [0,\infty) \to \mathbb{R}$, but $f\colon \mathbb{R} \to \mathbb{R}$ is not injective. Similarly, $f\colon [0,\infty) \to [0,\infty)$ is surjective, whereas $f\colon [0,\infty) \to \mathbb{R}$ is not surjective.

Exercise 2.3.2. Assume we have functions $f\colon A \to B$ and $g\colon B \to C$, and the composition $g \circ f\colon A \to C$.

(i) Assume that f and g are surjective, show that $g \circ f$ is surjective.

(ii) Assume that f and g are injective, show that $g \circ f$ is injective.

(iii) Assume that f and g are bijective, show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

For a subset $U \subset B$ of the codomain, we define the inverse image "volledig origineel" or "inverse beeld" as

$$f^{-1}(U) = \{a \in A \mid f(a) \in U\}.$$

Note that we use the notation f^{-1} even though f is not necessarily a bijection.

Exercise 2.3.3. Let $f\colon A \to B$ be a function, and let U and V be subsets of the codomain B. Show that

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V), \qquad f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V), \qquad f^{-1}(U^c) = (f^{-1}(U))^c$$

where $U^c = B \setminus U$ and $(f^{-1}(U))^c = A \setminus f^{-1}(U)$ are complements with respect to the codomain and domain, respectively.

For a function $f\colon A \to B$ we can define for a subset $Y \subset A$ its restriction, or restricted function, $f|_Y\colon Y \to B$ by

$$\forall y \in Y \qquad f|_Y(y) = f(y).$$

Note that this is a different function, since the domains differ.

Note that for real valued functions $f\colon A \to \mathbb{R}$ and $g\colon A \to \mathbb{R}$ for some set $A \subset \mathbb{R}$ we can define the linear combination $cf + dg\colon A \to \mathbb{R}$ for $c, d \in \mathbb{R}$ by

$$(cf + dg)(a) = c \cdot f(a) + d \cdot g(a)$$

and also the product fg and quotient $\frac{f}{g}$ (assuming that $\forall a \in A$ we have $g(a) \neq 0$) by

$$(fg)(a) = f(a) \cdot g(a), \qquad \frac{f}{g}(a) = \frac{f(a)}{g(a)},$$

where we have used the dot $\cdot$ to emphasise the product. This is not used in general.

We say that for a function $f\colon A \to \mathbb{R}$ with domain $A \subset \mathbb{R}$ that f is *strictly increasing* “strikt stijgend” if $\forall\, x, y \in A$ we have that $x < y \;\Rightarrow\; f(x) < f(y)$. Analogously, f is *strictly decreasing* “strikt dalend” if $\forall\, x, y \in A$ we have that $x < y \;\Rightarrow\; f(x) > f(y)$. We also use the definition f is *increasing* “stijgend” if $\forall\, x, y \in A$ we have that $x < y \;\Rightarrow\; f(x) \leq f(y)$. Analogously, f is *decreasing* “dalend” if $\forall\, x, y \in A$ we have that $x < y \;\Rightarrow\; f(x) \geq f(y)$.

Chapter 3

Sequences

In Chapter 3 we start with the important notion of convergence of sequences, and this is the first instance where we rigorously define what a limit is. We then derive various properties of sequences, one of the most important ones being the Bolzano-Weierstrass Theorem 3.2.12.

3.1 Sequences and subsequences

Definition 3.1.1. *A* sequence *"rij" is a function $a\colon \mathbb{N} \to \mathbb{R}$, denoted as $(a_n)_{n\in\mathbb{N}}$.*

Other notations in use are

$$(a_0, a_1, a_2, a_3, a_4, \cdots), \qquad (a_n)_{n=0}^{\infty}$$

where the first is considered as an ordered set. It is not essential to start at $n = 0$, and sometimes it is more convenient to consider the sequence from another starting point, e.g. $(\frac{1}{n})_{n=1}^{\infty}$. We recall from Section 2.3 the definition of increasing functions specialised to functions on $\mathbb{N}$.

Definition 3.1.2. *A sequence $(a_n)_{n=0}^{\infty}$ is called*

- increasing *"stijgend" if $\forall n \in \mathbb{N}\ a_n \leq a_{n+1}$;*
- strictly increasing *"strikt stijgend" if $\forall n \in \mathbb{N}\ a_n < a_{n+1}$;*
- decreasing *"dalend" if $\forall n \in \mathbb{N}\ a_n \geq a_{n+1}$;*
- strictly decreasing *"strikt dalend" if $\forall n \in \mathbb{N}\ a_n > a_{n+1}$;*
- constant *if $\forall n \in \mathbb{N}\ a_n = a_{n+1}$.*

A sequence is called monotonous *"monotoon" if the sequence is an increasing sequence or a decreasing sequence.*

Exercise 3.1.3. Show that Definition 3.1.2 is the same as the definitions for (strictly) increasing and (strictly) decreasing functions as in Section 2.3 when considering a sequence $a\colon \mathbb{N} \to \mathbb{R}$ as a function on the domain $\mathbb{N} \subset \mathbb{R}$. Hint: see Lemma 3.2.10.

Remark 3.1.4. Note that the notions of Definition 3.1.2 overlap. For example, a constant sequence is also a decreasing (or increasing) sequence. In particular, a constant sequence is also a monotonous sequence. Note as well that there are sequences that do not not have any of the properties in Definition 3.1.2, e.g. $(\frac{(-1)^n}{n+1})_{n\in\mathbb{N}}$.

Definition 3.1.5. *Let $a\colon \mathbb{N} \to \mathbb{R}$ be a sequence. For a strictly increasing function $f\colon \mathbb{N} \to \mathbb{N}$ the sequence obtained by composition, i.e. $a \circ f\colon \mathbb{N} \to \mathbb{R}$, is called a* subsequence *"deelrij" of the sequence $(a_n)_{n=0}^{\infty}$. Using the notation $f(j) = n_j$ a subsequence is denoted as $(a_{n_j})_{j=0}^{\infty}$.*

Looking at the sequence $(a_n)_{n\in\mathbb{N}} = (\frac{(-1)^n}{n+1})_{n\in\mathbb{N}}$, we can obtain the sequence $(\frac{1}{2n+1})_{n\in\mathbb{N}}$ as a subsequence by taking $f(n) = 2n$. Note that the subsequence is monotonous, since it is decreasing.

Exercise 3.1.6. Consider the sequence $(a_n)_{n=0}^{\infty}$ and define sequences $(b_n)_{n=0}^{\infty}$, $(c_n)_{n=0}^{\infty}$ and $(d_n)_{n=0}^{\infty}$ by $b_n = a_{2n}$, $c_n = a_{3n}$, $d_n = a_{4n}$. Indicate whether or not the following are true:

(a) $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$;

(b) $(c_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$;

(c) $(d_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$;

(d) $(c_n)_{n=0}^{\infty}$ is a subsequence of $(b_n)_{n=0}^{\infty}$;

(e) $(d_n)_{n=0}^{\infty}$ is a subsequence of $(b_n)_{n=0}^{\infty}$;

(f) $(d_n)_{n=0}^{\infty}$ is a subsequence of $(c_n)_{n=0}^{\infty}$.

Proposition 3.1.7. *Any sequence has a monotonous subsequence.*

Proof. Let us denote the sequence $(a_n)_{n\in\mathbb{N}}$, and we define the set

$$\mathscr{H} = \{n \in \mathbb{N} \mid \forall m > n \quad a_n > a_m\}.$$

So $n \in \mathscr{H}$ if a_n dominates the remaining terms in the sequence. The subset $\mathscr{H} \subset \mathbb{N}$ is either a finite or an infinite subset of $\mathbb{N}$.

Assume first that $\mathscr{H}$ is an infinite subset of $\mathbb{N}$, then we construct a sequence $(n_i)_{i\in\mathbb{N}}$, $n_i \in \mathscr{H}$ for all $i \in \mathbb{N}$, with $n_i < n_{i+1}$ for all i. The subsequence $(a_{n_i})_{i\in\mathbb{N}}$ is a decreasing subsequence, which is even strictly decreasing.

Next we assume that $\mathscr{H}$ is a finite subset of $\mathbb{N}$. So there exists $N \in \mathbb{N}$ so that $n < N$ for all $n \in \mathscr{H}$. We define an increasing subsequence inductively. We take $n_0 = N$, and since $n_0 \notin \mathscr{H}$ there exists $n_1 \in \mathbb{N}$ with $n_1 > n_0$ and $a_{n_0} \leq a_{n_1}$. In general, having defined $a_{n_0}, a_{n_1}, \cdots, a_{n_k}$, $k \in \mathbb{N}$, with $n_i < n_{i+1}$ and $a_{n_i} \leq a_{n_{i+1}}$ for all $i \in \{0, 1, \cdots, k-1\}$, we define n_{k+1} as follows. Since $n_k > N$ we have $n_k \notin \mathscr{H}$, so that there exists $n_{k+1} \in \mathbb{N}$ with $n_{k+1} > n_k$ and $a_{n_k} \leq a_{n_{k+1}}$. It follows that $(a_{n_k})_{k\in\mathbb{N}}$ is an increasing subsequence. $\square$

In Exercise 3.4.1 you are asked to prove a refinement of Proposition 3.1.7.

The boundedness of sequences is defined in terms of the boundedness of the image of $a\colon \mathbb{N} \to \mathbb{R}$, see Definition 2.2.1.

Definition 3.1.8. *The sequence* $(a_n)_{n\in\mathbb{N}}$ *is* bounded from above *if the set* $\{a_n \mid n \in \mathbb{N}\} \subset \mathbb{R}$ *is bounded from above.*
The sequence $(a_n)_{n\in\mathbb{N}}$ *is* bounded from below *if the set* $\{a_n \mid n \in \mathbb{N}\} \subset \mathbb{R}$ *is bounded from below.*
The sequence $(a_n)_{n\in\mathbb{N}}$ *is* bounded *if the set* $\{a_n \mid n \in \mathbb{N}\} \subset \mathbb{R}$ *is bounded.*

Exercise 3.1.9. Consider the sequence $(a_n)_{n=0}^{\infty}$, and assume that for each $j \in \mathbb{N}$ there is a subsequence $(a_{n_k^{(j)}})_{k=0}^{\infty}$. We moreover assume that each of these subsequences is a subsequence of the previous one, so for all $j \in \mathbb{N}$ the sequence $(a_{n_k^{(j+1)}})_{k=0}^{\infty}$ is a subsequence of $(a_{n_k^{(j)}})_{k=0}^{\infty}$. Show that $(a_{n_k^{(k)}})_{k=0}^{\infty}$ is a subsequence of the the sequence $(a_n)_{n=0}^{\infty}$. This subsequence is the so-called diagonal subsequence.

3.2 Convergent sequences

Definition 3.2.1. *The sequence* $(a_n)_{n=0}^{\infty}$ *is* convergent *"convergent" if there exists* $L \in \mathbb{R}$ *so that for all* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *so that for all* $n \in \mathbb{N}$ *with* $n \geq N$ *we have* $|a_n - L| < \varepsilon$. *Or,*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \qquad |a_n - L| < \varepsilon.$$

Then we say that $(a_n)_{n=0}^{\infty}$ *is convergent to* L *and we denote this by*

$$\lim_{n\to\infty} a_n = L.$$

Example 3.2.2. Let $a_n = c$ for all $n \in \mathbb{N}$, so we consider the constant sequence. Then $(a_n)_{n\in\mathbb{N}}$ is convergent, and $L = c$. Indeed, for any $\varepsilon > 0$ we can take $N = 0$, since for any $n \in \mathbb{N}$ we have $|a_n - L| = |c - c| = 0 < \varepsilon$.

Exercise 3.2.3. Assume that $(a_n)_{n\in\mathbb{N}}$ is convergent with $\lim_{n\to\infty} a_n = 0$. Show that $(|a_n|)_{n\in\mathbb{N}}$ is convergent with $\lim_{n\to\infty} |a_n| = 0$. What happens if the sequence converges to a non-zero limit? Hint: use the reversed triangle inequality (2.1.2).

Note that we can check Definition 3.2.1 by taking an arbitrary $\varepsilon > 0$ and proving that there exists $N \in \mathbb{N}$, which in general depends on ε, so that for all $n \geq N$ we have $|a_n - L| < \varepsilon$.

Note also that convergence of a sequence expresses information on the tail of the sequence. If we would change a finite numbers of terms in a convergent sequence, then the series remains convergent with the same limit.

Exercise 3.2.4. Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence and $\lim_{n\to\infty} a_n = L$. Assume that $(b_n)_{n\in\mathbb{N}}$ is a sequence such that there exists $M \in \mathbb{N}$ so that for all $n \geq M$ we have $b_n = a_n$. Show that $(b_n)_{n\in\mathbb{N}}$ is a convergent sequence and $\lim_{n\to\infty} b_n = L$.

Exercise 3.2.5. (i) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence and $L \in \mathbb{R}$. Assume that all subsequences are convergent with limit L. Show that $(a_n)_{n\in\mathbb{N}}$ is convergent with $\lim_{n\to\infty} a_n = L$. Hint: argue by contradiction.

(ii) Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence with $\lim_{n\to\infty} a_n = L$. Let $(a_{n_j})_{j\in\mathbb{N}}$ be an arbitrary subsequence of $(a_n)_{n\in\mathbb{N}}$. Show that $(a_{n_j})_{j\in\mathbb{N}}$ is a convergent sequence and that $\lim_{j\to\infty} a_{n_j} = L$.

Remark 3.2.6. We use the adjective *divergent* "divergent" for a sequence which is not convergent. A special class of divergent sequences are the sequences that diverge to infinity. Even though such sequences are not convergent, we sometimes use the notation

$$\lim_{n\to\infty} a_n = +\infty$$

for such a divergent sequence. We will not use this kind of divergence, but we can define $\lim_{n\to\infty} a_n = +\infty$ as follows

$$\forall\, M \in \mathbb{R} \quad \exists\, N \in \mathbb{N} \quad \forall\, n \geq N \qquad a_n > M$$

Exercise 3.2.7. Give a suitable definition for $\lim_{n\to\infty} a_n = -\infty$.

Before we continue, we discuss a relation between convergence and boundedness of a sequence.

Proposition 3.2.8. *A convergent sequence is bounded.*

Proof. With the notation as in Definition 3.2.1, we take $\varepsilon = 1$ and the corresponding N. Then for all $n \geq N$ we have $|a_n - L| < 1$, so that $L - 1 < a_n < L + 1$ and hence $|a_n| < 1 + |L|$. If we now put

$$M = \max(|a_0|, |a_1|, \cdots, |a_{N-1}|, 1 + |L|)$$

then $|a_n| \leq M$ for all $n \in \mathbb{N}$. So $(a_n)_{n\in\mathbb{N}}$ is bounded. □

The importance of Proposition 3.2.8 is that an unbounded sequence is divergent, so it can be used as a criterion for divergence. For example, the sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = n$ or $a_n = \sqrt{n}$ is divergent.

We can now formulate a basic and important result, which gives a large class of convergent sequences.

Theorem 3.2.9. *An increasing sequence $(a_n)_{n\in\mathbb{N}}$ which is bounded from above is convergent. Moreover, in this case*

$$\lim_{n\to\infty} a_n = \sup\{a_n \mid n \in \mathbb{N}\}.$$

Note that the set $\{a_n \mid n \in \mathbb{N}\}$ is non-empty (it contains a_0), and bounded from above by Definition 3.1.8. So by Theorem 2.2.4, the supremum exists. We often use the abbreviation

$$\sup\{a_n \mid n \in \mathbb{N}\} = \sup_{n\in\mathbb{N}} a_n.$$

Before starting the proof of Theorem 3.2.9, we need a result on increasing sequences.

Lemma 3.2.10. *Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence. Then for any $n \in \mathbb{N}$ we have that for all $m \geq n$ the inequality $a_m \geq a_n$ holds.*

Proof. We can prove this by induction on m, the initial case $m = n$ obviously being true. The induction step follows from $a_{m+1} \geq a_m \geq a_n$, where the last inequality is the induction hypothesis. □

Proof of Theorem 3.2.9. Put $L = \sup\{a_n \mid n \in \mathbb{N}\} = \sup_{n\in\mathbb{N}} a_n$. In order to verify the conditions of Definition 3.2.1 we pick an arbitrary $\varepsilon > 0$. We need to find $N \in \mathbb{N}$, which may depend on ε, as in Definition 3.2.1. In order to do so, we observe that $L - \varepsilon$ is not an upper bound for $\{a_n \mid n \in \mathbb{N}\}$, since L is the *least* upper bound. So there exists $a_N \in \{a_n \mid n \in \mathbb{N}\}$ with $L - \varepsilon < a_N \leq L$. Now this gives the required N.

To see that N meets the condition of Definition 3.2.1 we take $n \geq N$ arbitrary. Then, by Lemma 3.2.10, we have $a_n \geq a_N$, so that

$$L - \varepsilon < a_N \leq a_n.$$

On the other hand, L is an upper bound, so that $a_n \leq L$ as well. So in total, we get for any $n \geq N$ that

$$L - \varepsilon < a_N \leq a_n \leq L \quad \Longrightarrow \quad L - \varepsilon < a_n < L + \varepsilon \quad \Longrightarrow$$
$$-\varepsilon < a_n - L < \varepsilon \quad \Longrightarrow \quad |a_n - L| < \varepsilon$$

so that we have established Definition 3.2.1. □

Exercise 3.2.11. (i) Consider the statement: a decreasing sequence which is bounded from below is convergent, and it converges to its infimum.

(a) Give a direct proof of (i) mimicking the proof of Theorem 3.2.9.

(b) Give another proof of (i) using the result of Theorem 3.2.9.

(ii) Prove the following statement: a bounded monotonous sequence is convergent.

Now that we have Theorem 3.2.9 and its variants in Exercise 3.2.11 available we can establish some explicit limits rigorously. But first we establish an important theoretical consequence, the Bolzano-Weierstrass Theorem 3.2.12.

Theorem 3.2.12 (Bolzano-Weierstrass)**.** *A bounded sequence has a convergent subsequence.*

Proof. Take a monotonous subsequence of the sequence, which we can do by Proposition 3.1.7. Since the sequence is bounded, the monotonous subsequence is bounded. By Theorem 3.2.9 and Exercise 3.2.11, the monotonous subsequence is convergent. □

This proof is a simple combination of the established results, and we discuss another proof of the Bolzano-Weierstrass Theorem 3.2.12 in Exercise 3.4.9.

Exercise 3.2.13. The boundedness condition is essential in the Bolzano-Weierstrass Theorem 3.2.12. Show that the unbounded sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = n$ has no convergent subsequence. Hint: show that $|a_n - a_m| \geq 1$ for all $n \neq m$.

Next we use Theorem 3.2.9 and Exercise 3.2.11 to establish a well-known limit rigorously.

Proposition 3.2.14. *The sequence* $(a_n)_{n=1}^{\infty}$ *with* $a_n = \frac{1}{n}$ *is convergent, and*

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

By Exercise 3.2.11(i) we can expect that $\inf\limits_{n\geq 1} \dfrac{1}{n}$ plays a role. So we establish this first.

Lemma 3.2.15. *Let* $A = \{\frac{1}{n} \mid n \in \mathbb{N}, n \geq 1\} \subset \mathbb{R}$, *then* $\inf(A) = 0$.

Proof. Firstly, A is not empty, since it contains 1. Moreover, it is bounded from below by 0, since $\frac{1}{n} > 0$ for all $n \geq 1$. Hence, by Exercise 2.2.6, the infimum exists, and $\inf(A) \geq 0$ since 0 is a lower bound. In order to show that $\inf(A) = 0$, we show that $\inf(A) > 0$ is not possible.

So assume that $L = \inf(A) > 0$, then $2L > L > 0$. In particular, $2L$ is not a lower bound for the set A. So by Exercise 2.2.7, there exists $\frac{1}{N} \in A$ with

$$L \leq \frac{1}{N} < 2L \quad \Longrightarrow \quad \frac{1}{2N} < L.$$

Since $\frac{1}{2N} \in A$, we see that L is not a lower bound for A. This gives the required contradiction. □

Proof of Proposition 3.2.14. First we establish that $a_n = \frac{1}{n}$ is a decreasing sequence for $n \geq 1$. Now for any natural number $n \geq 1$ we have

$$\frac{1}{n+1} \leq \frac{1}{n} \quad \Longleftrightarrow \quad n+1 \geq n \quad \Longleftrightarrow \quad 1 \geq 0.$$

By Lemma 3.2.15 we know that the sequence $(a_n)_{n=1}^{\infty}$ is bounded from below, so that by Exercise 3.2.11(i) and again Lemma 3.2.15, we have

$$\lim_{n\to\infty} \frac{1}{n} = \inf\{\frac{1}{n} \mid n \in \mathbb{N}, n \geq 1\} = 0.$$ □

Corollary 3.2.16. *We have the following results on real, rational and natural numbers:*

(i) $\forall\, x \in \mathbb{R}\ \exists\, n \in \mathbb{N}$ *with* $x \leq n$*;*

(ii) $\forall\, x, y \in \mathbb{R}$ *with* $x < y\ \exists\, q \in \mathbb{Q}$ *with* $x < q < y$*;*

(iii) (Archimedean property) $\forall\, x > 0$, $\forall\, \varepsilon > 0$, $\exists N \in \mathbb{N}$ *with* $N\varepsilon > x$*;*

(iv) if $x \leq y + \varepsilon\ \forall\, \varepsilon > 0$, *then* $x \leq y$.

Proof. For (i) we can take $n = 0$ if $x \leq 0$. For $x > 0$ we have $0 < \frac{1}{x}$ so that $\frac{1}{x}$ is not a lower bound for A as in Lemma 3.2.15. So there exists $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{1}{x}$ or $x < n$. Now (iii) follows from (i) by replacing x by $\frac{x}{\varepsilon}$. Finally, (ii) and (iv) are proved in Exercise 3.2.17. $\square$

Exercise 3.2.17. We first prove Corollary 3.2.16(ii).

(i) Show that it suffices to consider the case $0 \leq x < y$. Hint: the cases $x < y \leq 0$ and $x < 0 < y$ are simple or can be reduced to this case.

(ii) Show that there exists $N \in \mathbb{N}$ with $0 < \frac{1}{N} < y - x$.

(iii) Show that there exists $k \in \mathbb{N}$ with $\frac{k}{N} \leq x < \frac{k+1}{N}$. Hint: consider $A = \{m \in \mathbb{N} \mid m \leq Nx\}$ and take k the greatest element of A after showing that A is non-empty and finite (use (i)).

(iv) Finish the proof of Corollary 3.2.16(ii).

For the proof of Corollary 3.2.16(iv), reduce first to the case $y = 0$. Then argue by contradiction; so assume that $0 < x$, and use (ii) to find $\varepsilon \in \mathbb{Q}$ with $\varepsilon > 0$ and $0 < \varepsilon < x$ giving a contradiction.

Exercise 3.2.18. In order to check that a sequence is convergent, see Definition 3.2.1, we need to find the value L. The goal of this exercise is to prove that for a convergent sequence the L in Definition 3.2.1 is uniquely determined. So let $M \in \mathbb{R}$ satisfy the condition in Definition 3.2.1.

(i) Show that for any $n \in \mathbb{N}$

$$0 \leq |L - M| \leq |L - a_n| + |M - a_n|$$

(ii) Show that for any $\varepsilon > 0$ we have $|L - M| < \varepsilon$.

(iii) Conclude that $L = M$ using Corollary 3.2.16.

Now that we have worked with convergent sequences, we can also consider how sequences relate to the arithmetic of the real numbers. This is given in Theorem 3.2.19

Theorem 3.2.19. *Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be convergent sequences with $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$.*

(i) For any $c \in \mathbb{R}$ the sequence $(ca_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} ca_n = cL$.

(ii) The sequence $(a_n + b_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty}(a_n + b_n) = L + M$.

(iii) For $c, d \in \mathbb{R}$ the sequence $(ca_n + db_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty}(ca_n + db_n) = cL + dM$.

(iv) The sequence $(a_n b_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} a_n b_n = LM$.

(v) *If* $b_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $M \neq 0$*, then the sequence* $(\frac{a_n}{b_n})_{n\in\mathbb{N}}$ *is convergent and* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$.

(vi) *Assume that there exists* $N \in \mathbb{N}$ *so that for all* $n \geq N$ *we have* $a_n \leq b_n$*, then* $L \leq M$.

(vii) (Sandwich principle) *Let* $(c_n)_{n\in\mathbb{N}}$ *be a sequence and assume that there exists* $N \in \mathbb{N}$ *so that for all* $n \geq N$ *we have* $a_n \leq c_n \leq b_n$*. If* $L = M$*, then* $(c_n)_{n\in\mathbb{N}}$ *is a convergent sequence and* $\lim_{n\to\infty} c_n = L$.

Exercise 3.2.20. Prove Theorem 3.2.19(i).

(i) First consider the case $c = 0$.

(ii) Consider $c \neq 0$, and use $|ca_n - cL| = |c|\,|a_n - L|$.

Proof of Theorem 3.2.19(ii). We need to show

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \qquad |(a_n + b_n) - (L + M)| < \varepsilon.$$

So we pick $\varepsilon > 0$ arbitrarily and we need to etablish the existence of $N \in \mathbb{N}$ with the above property.

Now observe that

$$|(a_n + b_n) - (L + M)| = |(a_n - L) + (b_n - M)| \leq |a_n - L| + |b_n - M|$$

by the triangle inequality, and we can make each of the terms as small as we want for sufficiently large n. So we show that we can makes each of these terms smaller than $\frac{1}{2}\varepsilon$. To make this precise, we use the convergence of the sequence $(a_n)_{n\in\mathbb{N}}$ to L as in Definition 3.2.1 with $\frac{1}{2}\varepsilon$ instead of ε. So we know that there exists $N_1 \in \mathbb{N}$ so that $\forall n \geq N_1$ we have $|a_n - L| < \frac{1}{2}\varepsilon$. Similarly, we know that there exists $N_2 \in \mathbb{N}$ so that $\forall n \geq N_2$ we have $|b_n - M| < \frac{1}{2}\varepsilon$. Now we put $N = \max(N_1, N_2)$, so that for any $n \geq N$ we have

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Since we have taken $\varepsilon > 0$ arbitrarily, we have established the result. □

Exercise 3.2.21. (i) Prove Theorem 3.2.19(iii) by combining (i) and (ii).

(ii) Prove Theorem 3.2.19(iv) by using

$$|a_n b_n - LM| \leq |a_n|\,|b_n - M| + |M|\,|a_n - L| \leq C\,|b_n - M| + |M|\,|a_n - L|$$

for some constant C, since the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded by Proposition 3.2.8.

Proof of Theorem 3.2.19(v). We first consider the special case that $(a_n)_{n\in\mathbb{N}}$ is the constant sequence $a_n = 1$ for all n. Observe that

$$|\frac{1}{b_n} - \frac{1}{M}| = \frac{1}{|b_n||M|}|b_n - M|$$

and we want the fraction $\frac{1}{|b_n||M|}$ on the right hand side to be bounded, at least for sufficiently large n. For this we use the convergence of the sequence $(b_n)_{n\in\mathbb{N}}$, and we take $\varepsilon = \frac{1}{2}|M| > 0$ since $M \neq 0$, so there exists $N_1 \in \mathbb{N}$ so that for all $n \geq N_1$ we have

$$|b_n - M| \leq \frac{1}{2}|M| \quad \Longrightarrow \quad -\frac{1}{2}|M| < b_n - M < \frac{1}{2}|M| \quad \Longrightarrow \quad M - \frac{1}{2}|M| < b_n < M + \frac{1}{2}|M|$$

In case $M > 0$, we find $b_n > \frac{1}{2}M$, and in case $M < 0$ we find $b_n < \frac{1}{2}M$, so that in both case $|b_n| > \frac{1}{2}|M|$. So this means that for all $n \geq N_1$ we have the estimate

$$|\frac{1}{b_n} - \frac{1}{M}| \leq \frac{2}{|M|^2}|b_n - M|$$

Now take an arbitrary $\varepsilon > 0$, and we use the convergence of the sequence $(b_n)_{n\in\mathbb{N}}$ to find a $N_2 \in \mathbb{N}$ so that for all $n \geq N_2$ we have

$$|b_n - M| < \frac{1}{2}|M|^2\varepsilon.$$

Put $N = \max(N_1, N_2)$ so that both estimates hold for all $n \geq N$, and this gives

$$\forall n \geq N \quad |\frac{1}{b_n} - \frac{1}{M}| \leq \frac{2}{|M|^2}|b_n - M| < \frac{2}{|M|^2}\frac{1}{2}|M|^2\varepsilon = \varepsilon.$$

Now that we have proved this special case, the general case follows from part (iv). □

Exercise 3.2.22. (i) Prove Theorem 3.2.19(vi) by showing that $L > M$ leads to a contradiction. Assuming $L > M$ apply the convergence criteria for the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ with $\varepsilon = \frac{1}{3}(L-M)$ and show that for all $N \in \mathbb{N}$ there exists $n \geq N$ with $a_n > b_n$.

(ii) Discuss whether or not the following refinement of Theorem 3.2.19(vi) is true: assume that there exists $N \in \mathbb{N}$ so that for all $n \geq N$ we have $a_n < b_n$, then $L < M$? Give a proof or a counterexample.

Proof of Theorem 3.2.19(vii). Pick $\varepsilon > 0$, and choose $N_1 \in \mathbb{N}$ so that for all $n \geq N_1$ we have $|a_n - L| < \varepsilon$, or $L - \varepsilon < a_n < L + \varepsilon$, and similarly choose $N_2 \in \mathbb{N}$ so that for all $n \geq N_2$ we have $|b_n - L| < \varepsilon$, since $M = L$. Then for all $n \geq N = \max(N_1, N_2)$ we have

$$L - \varepsilon < a_n \leq c_n \leq b_n < L + \varepsilon \quad \Longrightarrow \quad |c_n - L| < \varepsilon.$$

□

Exercise 3.2.23. Assume $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be convergent sequences with $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence and assume that there exists $N \in \mathbb{N}$ so that for all $n \geq N$ we have $a_n \leq c_n \leq b_n$. Can we conclude that $(c_n)_{n\in\mathbb{N}}$ is a convergent sequence and $L \leq \lim_{n\to\infty} c_n \leq M$? Give a proof or a counterexample.

3.3 Cauchy sequences, limsup and liminf

Definition 3.3.1. *A sequence* $(a_n)_{n\in\mathbb{N}}$ *is a* Cauchy sequence *"Cauchyrij" if for any* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *so that for all integers* $n \geq N$ *and all integers* $m \geq N$ *we have* $|a_n - a_m| < \varepsilon$, *i.e.*

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m, n \geq N \qquad |a_n - a_m| < \varepsilon.$$

Upon comparing with Definition 3.2.1 we don't need to know the additional information on the limit L. Cauchy sequences and convergent sequences will turn out to be equivalent, see Theorem 3.3.11, and we first prove the easy implication in Proposition 3.3.2.

Note that it is important that the estimates holds for all n and m larger than N. For instance, the sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = \sqrt{n}$ satisfies that for any $\varepsilon > 0$ we have $|a_{n+1} - a_n| < \varepsilon$ for n sufficiently large, but it is not a Cauchy sequence. This can be seen from e.g. Exercise 3.3.3.

Proposition 3.3.2. *A convergent sequence is a Cauchy sequence.*

Proof. Take $\varepsilon > 0$ arbitrarily, and since $(a_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} a_n = L$, we can find $N \in \mathbb{N}$ so that for all $n \geq N$ we have $|a_n - L| < \frac{1}{2}\varepsilon$. Now, we take $n \geq N$ and $m \geq N$ arbitrarily, and we get

$$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon. \qquad \square$$

So we expect that we can prove properties of Cauchy sequences analogously to corresponding statements for convergent sequences. A first example is in Exercise 3.3.7.

Exercise 3.3.3. Show that a Cauchy sequence is bounded. Hint: mimick the proof of Proposition 3.2.8 by replacing L by a suitable element of the Cauchy sequence $(a_n)_{n\in\mathbb{N}}$.

In order to prove the converse of Proposition 3.3.2 we introduce the limsup (or limes superior or limit superior or upper limit) and liminf (or limes inferior or limit inferior or lower limit).

Definition 3.3.4. *For a bounded sequence* $(a_n)_{n\in\mathbb{N}}$, *we define the sequence* $(A_k)_{k\in\mathbb{N}}$ *by*

$$A_k = \sup_{n\geq k} a_n = \sup\{a_n \mid n \geq k\}.$$

The sequence $(A_k)_{k\in\mathbb{N}}$ *is a decreasing bounded sequence, and we define the* limsup *"limsup"*

$$\limsup_{n\to\infty} a_n = \lim_{k\to\infty} A_k = \inf_{k\in\mathbb{N}} A_k.$$

Similarly, we define the increasing bounded sequence $(B_k)_{k\in\mathbb{N}}$ *by*

$$B_k = \inf_{n\geq k} a_n = \inf\{a_n \mid n \geq k\}$$

and we define the liminf *"liminf"*

$$\liminf_{n\to\infty} a_n = \lim_{k\to\infty} B_k = \sup_{k\in\mathbb{N}} B_k.$$

Remark 3.3.5. (i) Note that the sequences $(A_k)_{k\in\mathbb{N}}$ and $(B_k)_{k\in\mathbb{N}}$ are bounded, since, if $|a_n| \le M$ for all $n \in \mathbb{N}$, then $|A_k| \le M$ for all $k \in \mathbb{N}$ and $|B_k| \le M$ for all $k \in \mathbb{N}$. To see that $(B_k)_{k\in\mathbb{N}}$ is increasing, note that

$$\inf_{n\ge k+1} a_n \ge \inf_{n\ge k} a_n$$

since $\inf_{n\ge k} a_n = \min(a_k, \inf_{n\ge k+1} a_n)$. By Theorem 3.2.9, the bounded increasing $(B_k)_{k\in\mathbb{N}}$ converges to its supremum.
(ii) Another notation in use for the limsup and liminf is

$$\limsup_{n\to\infty} a_n = \overline{\lim_{n\to\infty}}\, a_n, \qquad \liminf_{n\to\infty} a_n = \underline{\lim_{n\to\infty}}\, a_n,$$

and note that we can write

$$\limsup_{n\to\infty} a_n = \inf_{k\in\mathbb{N}} \sup_{n\ge k} a_n, \qquad \liminf_{n\to\infty} a_n = \sup_{k\in\mathbb{N}} \inf_{n\ge k} a_n.$$

Exercise 3.3.6. Show that $(A_k)_{k\in\mathbb{N}}$ of Definition 3.3.4 is a decreasing sequence.

As an example, we take the sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = (-1)^n$, so the sequence alternates between $+1$ and -1. Then $A_k = 1$ for all $k \in \mathbb{N}$, and $B_k = -1$ for all $k \in \mathbb{N}$, so the sequences in this example are constant sequences, so we see

$$\limsup_{n\to\infty}(-1)^n = 1, \qquad \liminf_{n\to\infty}(-1)^n = -1.$$

This example shows that in general

$$\inf_{k\in\mathbb{N}} \sup_{n\ge k} a_n \ne \sup_{k\in\mathbb{N}} \inf_{n\ge k} a_n$$

and that one has to be careful with the order in which one takes suprema and infima! In order to get acquainted with limsup and liminf, Exercise 3.3.7 is very useful.

Exercise 3.3.7. Assume $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence.

(i) Show that for $x > \limsup_{n\to\infty} a_n$ we have

$$\exists\, N \in \mathbb{N} \quad \forall\, n \ge N \qquad a_n < x$$

or, for any element to the right of the limsup there are only finitely many elements of the sequence greater or equal than this element.

(ii) Show that for $x < \limsup_{n\to\infty} a_n$ we have

$$\forall\, N \in \mathbb{N} \quad \exists\, n \ge N \qquad a_n > x$$

or, at any small distance to the left of the limsup there there infinitely many elements of the sequence.

(iii) Formulate and prove the corresponding statements for the liminf.

See also the proof of Proposition 3.3.9, where the properties of Exercise 3.3.6 are used.

Proposition 3.3.8. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a bounded sequence, then*

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

and for a convergent subsequence $(a_{n_j})_{j\in\mathbb{N}}$ *of* $(a_n)_{n\in\mathbb{N}}$ *we have*

$$\liminf_{n\to\infty} a_n \leq \lim_{j\to\infty} a_{n_j} \leq \limsup_{n\to\infty} a_n.$$

Proof. First, observe that, using the notation as in Definition 3.3.4,

$$B_k = \inf_{n\geq k} a_n \leq \sup_{n\geq k} a_n = A_k$$

so that by Theorem 3.2.19(vi) the first statement follows.

Similarly, for the subsequence $(a_{n_j})_{j\in\mathbb{N}}$ we get

$$B_{n_j} = \inf_{n\geq n_j} a_n \leq a_{n_j} \leq \sup_{n\geq n_j} a_{n_j} = A_{n_j}.$$

By Exercise 3.2.5, the corresponding subsequences $(A_{n_j})_{j\in\mathbb{N}}$ and $(B_{n_j})_{j\in\mathbb{N}}$ are convergent with the same limits. Applying twice Theorem 3.2.19(vi) we get the result. □

The convergence of the sequence can be characterised in terms of the liminf and the limsup of the sequence.

Proposition 3.3.9. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a bounded sequence. The sequence* $(a_n)_{n\in\mathbb{N}}$ *is convergent to the value* L *if and only if* $L = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

Remark 3.3.10. In Proposition 3.3.9 we have assumed that the sequence is bounded in order to have $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ to be well-defined. If we allow for the liminf and limsup to take values in the extended real line, i.e. $\mathbb{R}$ extended with $+\infty$ and $-\infty$, cf. Remark 2.2.8, then we can drop the boundedness assumption in Proposition 3.3.9.

Proof. We first assume that the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent with limit L. So for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that for all $n \geq N$ we have

$$|a_n - L| < \varepsilon \quad \Longleftrightarrow \quad L - \varepsilon < a_n < L + \varepsilon.$$

So in particular, we see that for any $k \geq N$ we have

$$L - \varepsilon \leq B_k = \inf_{n\geq k} a_n \leq \sup_{n\geq k} a_n = A_k \leq L + \varepsilon,$$

using the notation as in Definition 3.3.4. Since we know that the limits of $(A_k)_{k\in\mathbb{N}}$ and $(B_k)_{k\in\mathbb{N}}$ exist, we find

$$L-\varepsilon \le \liminf_{n\to\infty} a_n = \lim_{k\to\infty} B_k \le \limsup_{n\to\infty} a_n = \lim_{k\to\infty} A_k \le L+\varepsilon,$$

and hence

$$|L-\limsup_{n\to\infty} a_n| \le \varepsilon \quad \text{and} \quad |L-\liminf_{n\to\infty} a_n| \le \varepsilon.$$

Since $\varepsilon > 0$ is choosen arbitrarily, we see that $L = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

Conversely, we assume that the limsup and liminf are equal (and equal to L). We need to show that the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent to L, i.e. for $\varepsilon > 0$ arbitrary we need to prove the existence of $N \in \mathbb{N}$ so that for all $n \ge N$ we have $|a_n - L| < \varepsilon$.

Now, $L+\varepsilon > \limsup_{n\to\infty} a_n$ and by Exercise 3.3.7(i) we find $N_1 \in \mathbb{N}$ so that for all $n \ge N_1$ we have $a_n < \limsup_{n\to\infty} a_n + \varepsilon = L+\varepsilon$. Similarly, by Exercise 3.3.7(iii), we find $N_2 \in \mathbb{N}$ so that for all $n \ge N_2$ we have $L-\varepsilon = \liminf_{n\to\infty} a_n - \varepsilon < a_n$. Put $N = \max(N_1, N_2)$, then we have for all $n \ge N$

$$L-\varepsilon < a_n < L+\varepsilon \quad \Longrightarrow \quad |a_n - L| < \varepsilon. \qquad \square$$

We can now formulate the equivalence of convergent sequences, see Definition 3.2.1, and Cauchy sequences, see Definition 3.3.1. Theorem 3.3.11 is also stated as the completeness of $\mathbb{R}$, and it is possible to show that it is equivalent to Theorem 2.2.4, but this will not be done in this course.

Theorem 3.3.11. *A sequence is convergent if and only if it is a Cauchy sequence.*

Proof. In Proposition 3.3.2 we have shown that a convergent sequence is a Cauchy sequence, so it suffices to show that a Cauchy sequence is convergent. So let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence, then it is bounded by Exercise 3.3.3. So $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ exist. By Proposition 3.3.9 it suffices to show that they are equal. We will prove that for any $\varepsilon > 0$ we have

$$0 \le \limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n \le \varepsilon. \tag{3.3.1}$$

Since $\varepsilon > 0$ is arbitrary, (3.3.1) implies $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$ by Corollary 3.2.16(iv). Recall that the first inequality is from Proposition 3.3.8.

So take $\varepsilon > 0$, then $(a_n)_{n\in\mathbb{N}}$ being a Cauchy sequence, we find $N \in \mathbb{N}$ so that for all $n, m \ge N$ we have $|a_n - a_m| < \frac{1}{2}\varepsilon$ (and we take $\frac{1}{2}\varepsilon$ to get to ε in (3.3.1)). Fix $m = N$ then we have for all $n \ge N$

$$a_N - \frac{1}{2}\varepsilon < a_n < a_N + \frac{1}{2}\varepsilon \qquad \Longrightarrow$$
$$\forall k \ge N \quad a_N - \frac{1}{2}\varepsilon \le B_k = \inf_{n\ge k} a_n \le A_k = \sup_{n\ge k} a_n \le a_N + \frac{1}{2}\varepsilon$$

and taking the limit $k \to \infty$ gives

$$a_N - \frac{1}{2}\varepsilon \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq a_N + \frac{1}{2}\varepsilon$$

which yields (3.3.1). □

Exercise 3.3.12. An alternative proof of the statement that a Cauchy sequence is a convergent sequence in Theorem 3.3.11 is the following.

(i) Use Exercise 3.3.3 and the Bolzano-Weierstrass Theorem 3.2.12 to show that a Cauchy sequence has a convergent subsequence.

(ii) Show that a Cauchy sequence with a convergent subsequence is a convergent sequence.

3.4 Exercises

Exercise 3.4.1. Show that the following refinement of Proposition 3.1.7 is valid: any sequence has a strictly decreasing subsequence or a strictly increasing subsequence or a constant subsequence. Hint: modify the second part of the proof.

Exercise 3.4.2. Let $r \geq 0$, and consider the sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = r^n$.

(i) Show that for $0 \leq r \leq 1$ the sequence $(r^n)_{n\in\mathbb{N}}$ is convergent and determine its limit. Hint: mimick the proof of Proposition 3.2.14.

(ii) Show that for $-1 < r \leq 1$ the sequence $(r^n)_{n\in\mathbb{N}}$ is convergent and determine its limit.

(iii) Show that for $|r| > 1$ and $r = -1$ the sequence $(r^n)_{n\in\mathbb{N}}$ is divergent.

Exercise 3.4.3. Prove that for $M > 0$ we have $\lim_{n\to\infty} \sqrt[n]{M} = 1$.

Exercise 3.4.4. Let $|r| < 1$. Show that the sequence $(nr^n)_{n\in\mathbb{N}}$ for $|r| < 1$ is convergent and $\lim_{n\to\infty} nr^n = 0$. Hint: assume first that $0 < r < 1$ and show that the sequence is decreasing for n sufficiently large, say for $n \geq N$. Next set $L = \inf_{n\geq N} nr^n \geq 0$, and assume that $L > 0$. Then $r^{-1}L$ is not an infimum, and derive a contradiction.

Exercise 3.4.5. (i) Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be bounded sequences. Show that for any $c, d \in \mathbb{R}$ the sequences $(ca_n + db_n)_{n\in\mathbb{N}}$ is bounded as well. Show that $(a_nb_n)_{n\in\mathbb{N}}$ is bounded as well.

(ii) Assume that $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence and that $(b_n)_{n\in\mathbb{N}}$ is a bounded sequence. Is the sequence $(a_nb_n)_{n\in\mathbb{N}}$ bounded? Give a proof or a counterexample.

(iii) Assume that $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence with $\lim_{n\to\infty} a_n = 0$, and let $(b_n)_{n\in\mathbb{N}}$ be an arbitrary sequence. Is the sequence $(a_nb_n)_{n\in\mathbb{N}}$ bounded?

Exercise 3.4.6. (i) Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences, and we assume that $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence and $(b_n)_{n\in\mathbb{N}}$ is a divergent sequence. What can you say about the convergence or divergence of the sequence $(a_n+b_n)_{n\in\mathbb{N}}$? Give a proof of your statement.

(ii) Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be divergent sequences. What can you say about the convergence or divergence of the sequence $(a_n+b_n)_{n\in\mathbb{N}}$? Give a proof of your statement.

Exercise 3.4.7. Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence. Show that there exists a convergent subsequence $(a_{n_j})_{j\in\mathbb{N}}$ with

$$\lim_{j\to\infty} a_{n_j} = \limsup_{n\to\infty} a_n.$$

Prove that similarly there also exists a convergent subsequence converging to $\liminf_{n\to\infty} a_n$. Hint: use Exercise 3.3.7 and Exercise 3.1.9.

Exercise 3.4.8. Prove or disprove the following statements:

(i) Assume $(a_n)_{n=0}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty} a_n = 0$ and let $(b_n)_{n=0}^{\infty}$ be a bounded sequence. Then $(a_nb_n)_{n=0}^{\infty}$ is convergent and $\lim_{n\to\infty} a_nb_n = 0$.

(ii) Assume $(a_n)_{n=0}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty} a_n = L$ with $L\neq 0$ and let $(b_n)_{n=0}^{\infty}$ be a bounded sequence. Then $(a_nb_n)_{n=0}^{\infty}$ has a convergent subsequence.

(iii) Assume $(a_n)_{n=0}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty} a_n = L$ with $L\neq 0$ and let $(b_n)_{n=0}^{\infty}$ be a bounded sequence. Then $(a_nb_n)_{n=0}^{\infty}$ is convergent and $\lim_{n\to\infty} a_nb_n = L$.

(iv) Assume $(a_n)_{n=0}^{\infty}$ is a convergent sequence with $\lim_{n\to\infty} a_n = 0$ and let $(b_n)_{n=0}^{\infty}$ be a sequence. Then $(a_nb_n)_{n=0}^{\infty}$ is convergent and $\lim_{n\to\infty} a_nb_n = 0$.

Exercise 3.4.9. We develop an alternative proof of the Bolzano-Weierstrass Theorem 3.2.12. So we assume that we have a bounded sequence $(a_n)_{n\in\mathbb{N}}$, and we assume the sequence is contained in the bounded interval $[b,c]$.

(i) Assume that $(a_n)_{n\in\mathbb{N}}$ is a finite set, i.e. the image of the function $a\colon \mathbb{N}\to\mathbb{R}$ is finite. Show that $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence, which can be taken to be constant.

(ii) Define $d=\frac{1}{2}(b+c)$ to be the middle of the interval $[b,c]$. Argue that at least one of the intervals $[b,d]$ or $[d,c]$ contains an infinite number of the elements of the sequence $(a_n)_{n\in\mathbb{N}}$, i.e. at least one of $\{n\in\mathbb{N}\mid a_n\in[b,d]\}$ or $\{n\in\mathbb{N}\mid a_n\in[d,c]\}$ is infinite.

(iii) Set $b_0=b$, $c_0=c$, and we put $b_1=b$, $c_1=d$ in case $[b,d]$ contains an infinite number of the elements of the sequence $(a_n)_{n\in\mathbb{N}}$, and otherwise we put $b_1=d$, $c_1=c_0$.

(iv) Iterate the above construction to find intervals $[b_k,c_k]$ with $[b_{k+1},c_{k+1}]\subset[b_k,c_k]$ and such that $I_k=\{n\in\mathbb{N}\mid a_n\in[b_k,c_k]\}$ is infinite.

(v) Show that the length of $[b_k,c_k]$ tends to zero. Hint: Exercise 3.4.2. Conclude that $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ are convergent sequences with $b_k<c_k$ for all $k\in\mathbb{N}$ and that $\lim_{k\to\infty} b_k = \lim_{k\to\infty} c_k$.

(vi) Label the infinite sets I_k by an increasing function $\mathbb{N} \to I_k$, say $(n_j^k)_{j\in\mathbb{N}}$. Show that Exercise 3.1.9 is applicable and show that the diagonal subsequence $(a_{n_k^{(k)}})_{k\in\mathbb{N}}$ is convergent. Hint: use Theorem 3.2.19(vii).

Exercise 3.4.10. Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence, and assume that $(a_{n_j})_{j\in\mathbb{N}}$ is a subsequence. Show that $(a_{n_j})_{j\in\mathbb{N}}$ is bounded and that

$$\liminf_{n\to\infty} a_n \leq \liminf_{j\to\infty} a_{n_j} \leq \limsup_{j\to\infty} a_{n_j} \leq \limsup_{n\to\infty} a_n.$$

Exercise 3.4.11. Suppose that $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are bounded sequences. Show that

$$\begin{aligned}\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \leq \liminf_{n\to\infty}(a_n+b_n) \leq \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \\ \leq \limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n\end{aligned}$$

Exercise 3.4.12. Suppose that $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are bounded sequences. Assume that $(a_n)_{n=0}^{\infty}$ is convergent and $\lim_{n\to\infty} a_n = L$ and that $L > 0$. Show that

$$\limsup_{n\to\infty} a_n b_n = L \limsup_{n\to\infty} b_n.$$

Exercise 3.4.13. Compare this exercise to Exercise 3.3.7. Assume $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence, and assume that $L \in \mathbb{R}$ has the following properties:

(i) for all $x > L$ we have

$$\exists\, N \in \mathbb{N} \quad \forall\, n \geq N \qquad a_n < x$$

(ii) for all $x < L$ we have

$$\forall\, N \in \mathbb{N} \quad \exists\, n \geq N \qquad a_n > x$$

Prove that $L = \limsup_{n\to\infty} a_n$. This means that the properties of Exercise 3.3.7 characterise limsup. Naturally, a similar characterisation is valid for liminf. State and prove the corresponding result. Hint: by Exercise 3.3.7 we know that $\limsup_{n\to\infty} a_n$ satisfies these conditions. Show that these conditions fix L uniquely.

Exercise 3.4.14. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence with $c_n > 0$ for all $n \in \mathbb{N}$. Assume moreover that the sequences $(\sqrt[n]{c_n})_{n=1}^{\infty}$ and $(\frac{c_{n+1}}{c_n})_{n=0}^{\infty}$ are bounded. We want to prove:

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

One inequality follows from Proposition 3.3.8.

(i) Prove the first inequality. Hint: Let $L = \liminf_{n\to\infty} \frac{c_{n+1}}{c_n}$ and show that for any $\varepsilon > 0$ we have $L - \varepsilon \leq \liminf_{n\to\infty} \sqrt[n]{c_n}$. The result of Exercise 3.4.3 can be handy.

(ii) Prove the last inequality.

(iii) Let $\alpha > 0$, show that $\lim_{n\to\infty} \sqrt[n]{n^\alpha} = 1$.

(iv) Extend the statement to the case that the sequences $(\sqrt[n]{c_n})_{n=1}^\infty$ and $(\frac{c_{n+1}}{c_n})_{n=0}^\infty$ are not necessarily bounded, see Remark 3.2.6.

Exercise 3.4.15. Assume $(a_n)_{n=0}^\infty$ is a bounded sequence. Define $s_N = \sum_{n=0}^{N} a_n$.

(i) Show that

$$\liminf_{n\to\infty} a_n \leq \liminf_{N\to\infty} \frac{s_N}{N+1} \leq \limsup_{N\to\infty} \frac{s_N}{N+1} \leq \limsup_{n\to\infty} a_n$$

Hint: use Exercise 3.4.13.

(ii) Assume moreover that the sequence $(a_n)_{n=0}^\infty$ is convergent to its limit L. Show that the sequence $(\frac{s_N}{N+1})_{N=0}^\infty$ is convergent with limit L.

(iii) Is the following converse valid? If $\lim_{N\to\infty} \dfrac{s_N}{N+1} = L$, then $\lim_{N\to\infty} a_N = L$.

Exercise 3.4.16. Using the conventions as in Remark 2.2.8, give a definition of the limsup and liminf for a general sequence $(a_n)_{n\in\mathbb{N}}$, so a not necessarily bounded sequence, see also Remark 3.3.10.

Exercise 3.4.17. We define a complex sequence $(z_n)_{n\in\mathbb{N}}$ as a function $z\colon \mathbb{N} \to \mathbb{C}$. We say that the complex sequence is convergent if there exists $z \in \mathbb{C}$ so that

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \qquad |z_n - z| < \varepsilon$$

where we use the modulus of $\mathbb{C}$, see Section 2.1. We denote this by $\lim_{n\to\infty} z_n = z$.

(i) Upon writing $z_n = a_n + ib_n$ in Cartesian coordinates, i.e. $a_n = \Re z_n$, $b_n = \Im z_n$, show that $(z_n)_{n\in\mathbb{N}}$ is a convergent complex sequence with $\lim_{n\to\infty} z_n = z$ if and only if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent (real) sequences with $\lim_{n\to\infty} a_n = \Re z$ and $\lim_{n\to\infty} b_n = \Im z$.

(ii) Let $(z_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ be convergent complex sequences with $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$. Prove the following statements:

(a) for $\alpha, \beta \in \mathbb{C}$, the complex sequence $(\alpha z_n + \beta w_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} \alpha z_n + \beta w_n = \alpha z + \beta w$;

(b) the complex sequence $(z_n w_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} z_n w_n = zw$;

(c) assuming additionally that $w_n \neq 0$ for all $n \in \mathbb{N}$ and $w \neq 0$, then the complex sequence $(\frac{z_n}{w_n})_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} \frac{z_n}{w_n} = \frac{z}{w}$;

(d) the complex sequence $(\overline{z_n})_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} \overline{z_n} = \overline{z}$;

(e) the real sequence $(|z_n|)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty}|z_n| = |z|$.

(iii) Formulate and prove a complex analogue of the Bolzano-Weierstrass Theorem 3.2.12.

Exercise 3.4.18. Equipping the d-dimensional real vector space $\mathbb{R}^d$ with standard length, i.e.

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_d \end{pmatrix}, \qquad \|\mathbf{v}\| = \sqrt{\sum_{i=1}^{d} \mathbf{v}_i^2}$$

we say that a sequence of vectors in $\mathbb{R}^d$ is a function $\mathbb{N} \to \mathbb{R}^d$. The sequence $(\mathbf{v}_n)_{n\in\mathbb{N}}$ converges to $\mathbf{v} \in \mathbb{R}^d$ if

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \qquad \|\mathbf{v}_n - \mathbf{v}\| < \varepsilon.$$

Show that $(\mathbf{v}_n)_{n\in\mathbb{N}}$ converges to $\mathbf{v} \in \mathbb{R}^d$ if and only if it converges in each coordinate, i.e. for all $i \in \{1, \cdots, d\}$ the real sequence $((\mathbf{v}_n)_i)_{n\in\mathbb{N}}$ converges to $\mathbf{v}_i$.

State and prove a similar statement for sequences in the d-dimensional complex vector space $\mathbb{C}^d$.

Chapter 4

Topology of $\mathbb{R}$

In this chapter an introduction to the topology of the real numbers is given. So we describe what open and closed sets are, and Proposition 4.1.4 shows that indeed the open sets form a topology as will be discussed in the course Topology (2nd year). The Heine-Borel Theorem 4.3.3 is an important result as it describes the sequentially compact sets of $\mathbb{R}$.

4.1 Open and closed sets

Definition 4.1.1. *For $a \in \mathbb{R}$ and $\varepsilon > 0$ we define the* ε-neighbourhood *"ε-omgeving" $N_\varepsilon(a)$ of a by*

$$N_\varepsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \varepsilon\} = (a - \varepsilon, a + \varepsilon).$$

For a subset $A \subset \mathbb{R}$ we say that $a \in A$ is an interior point *"inwendig punt" if there exists $\varepsilon > 0$ so that $N_\varepsilon(a) \subset A$.*

So $\frac{1}{2}$ is an interior point of $A = (0, 1)$, and also of $A = [0, 1]$ and of $A = (0, 1]$. However, $\frac{1}{2}$ is not an interior point of $A = [\frac{1}{2}, 1]$ nor of $A = \{\frac{1}{n+1} \mid n \in \mathbb{N}\}$.

Definition 4.1.2. *For $A \subset \mathbb{R}$ we define its* interior set *"inwendige verzameling" as*

$$A^\circ = \{a \in A \mid \exists\, \varepsilon > 0 \ N_\varepsilon(a) \subset A\} = \{a \in A \mid a \text{ is interior point of } A\} \subset A.$$

The set A is an open set *"open verzameling" if $A^\circ = A$.*

Remark 4.1.3. Note that in general $A^\circ \subset A$, so that it suffices to prove $A \subset A^\circ$ to conclude that A is an open set.

We start by describing some properties of open sets in Proposition 4.1.4.

Proposition 4.1.4. *Let A and B be subsets of $\mathbb{R}$.*

(i) If $A \subset B$, then $A^\circ \subset B^\circ$.

(ii) A° is an open set, i.e. $(A^\circ)^\circ = A^\circ$.

(iii) A° *is the largest open set in* A*, i.e. if* $B \subset A$ *and* B *is open, then* $B \subset A^\circ$.

(iv) Let A_α *be an open set for all* $\alpha \in I$*, then* $\bigcup_{\alpha \in I} A_\alpha$ *is an open set, i.e. the union of arbitrarily many open sets is open.*

(v) Let $A_1, A_2, \cdots, A_N$ *be open sets, then* $\bigcap_{i=1}^{N} A_i$ *is an open set, i.e. the intersection of finitely many open sets is open.*

(vi) $\emptyset$ *and* $\mathbb{R}$ *are open sets.*

Proof. To prove (i), take $a \in A^\circ$, so that there exists $\varepsilon > 0$ and $N_\varepsilon(a) \subset A \subset B$. Hence, $a \in B^\circ$.

To prove (ii) it suffices to show that $A^\circ \subset (A^\circ)^\circ$ by Remark 4.1.3. Take $a \in A^\circ$, and we need to prove that there is an ε-neighbourhood of a contained in A°. Since $a \in A^\circ$, we have some $\varepsilon > 0$ so that $N_\varepsilon(a) \subset A$. Observe that any $b \in N_{\frac{1}{2}\varepsilon}(a)$ is also an interior point of A, since $N_{\frac{1}{2}\varepsilon}(b) \subset N_\varepsilon(a) \subset A$. In particular, $N_{\frac{1}{2}\varepsilon}(a) \subset A^\circ$, and hence a is an interior point of A°, or $a \in (A^\circ)^\circ$.

To prove (iii), take an open set $B \subset A$, so that by (i) we have $B^\circ \subset A^\circ$, and A° is an open set by (ii). Since B is open, we have $B = B^\circ \subset A^\circ$.

To prove (iv), take $a \in \bigcup_{\alpha \in I} A_\alpha$ arbitrarily. So there exists $\alpha_0 \in I$ with $a \in A_{\alpha_0}$. Since A_{α_0} is open, there exists $\varepsilon > 0$ with $N_\varepsilon(a) \subset A_{\alpha_0} \subset \bigcup_{\alpha \in I} A_\alpha$, so $a \in (\bigcup_{\alpha \in I} A_\alpha)^\circ$. Thus $\bigcup_{\alpha \in I} A_\alpha$ is open.

To prove (v), take $a \in \bigcap_{i=1}^{N} A_i$ arbitrarily. So $a \in A_i$ for all $i \in \{1, \cdots, N\}$. Since A_i is open, there exists $\varepsilon_i > 0$ so that $N_{\varepsilon_i}(a) \subset A_i$. Now put $\varepsilon = \min_{1 \le i \le N} \varepsilon_i > 0$, then $N_\varepsilon(a) \subset A_i$ for all $i \in \{1, \cdots, N\}$. Hence, $N_\varepsilon(a) \subset \bigcap_{i=1}^{N} A_i$ and $a \subset (\bigcap_{i=1}^{N} A_i)^\circ$ and $\bigcap_{i=1}^{N} A_i$ is an open set. □

Note that Proposition 4.1.4(ii) shows that being an interior point is an 'open property', in the sense that if a is an interior point, then there is a ε-neighbourhood of a consisting of interior points.

Exercise 4.1.5. (i) Show the last statement, i.e. prove Proposition 4.1.4(vi). Hint: for the empty set $\emptyset$ there is nothing to prove, and for $\mathbb{R}$ all points are interior.

(ii) Assume A_i is an open set for all $i \in \mathbb{N}$. Can we strenghten Proposition 4.1.4(v) to conclude that the intersection $\bigcap_{i \in \mathbb{N}} A_i$ is an open set? Provide a proof or a counterexample.

Definition 4.1.6. *Let* $A \subset \mathbb{R}$ *and* $x \in \mathbb{R}$*, then we call* x *a* closure point *"afsluitingspunt" of the set* A *if*

$$\forall \varepsilon > 0 \quad \exists a \in A \qquad |x - a| < \varepsilon.$$

We call x *a* limit point *"limietpunt" of the set* A *if*

$$\forall \varepsilon > 0 \quad \exists a \in A \qquad 0 < |x - a| < \varepsilon.$$

So the condition of being a closure point means

$$\forall\, \varepsilon > 0 \qquad N_\varepsilon(x) \cap A \neq \emptyset$$

and the condition of a limit point means

$$\forall\, \varepsilon > 0 \qquad N_\varepsilon(x) \setminus \{x\} \cap A \neq \emptyset$$

Note that a limit point is in particular a closure point. Note that $x \in A$ is always a closure point for A, but not necessarily a limit point of A. Indeed, take $A = (0,1) \cup \{2\}$, then 2 is a closure point, but it is not a limit point. A point of A which is a closure point, but not a limit point, is occasionally called an *isolated point* "geïsoleerd punt".

Proposition 4.1.7. *Let $A \subset \mathbb{R}$ and let $x \in \mathbb{R}$. The following statements are equivalent:*

(i) x is a closure point of A;

(ii) there exists a sequence $(a_n)_{n\in\mathbb{N}}$ with $\forall\, n \in \mathbb{N}$ $a_n \in A$ and $\lim_{n\to\infty} a_n = x$.

Exercise 4.1.8. We describe the proof of Proposition 4.1.7.

(i) For (ii) implies (i): use Definition 3.2.1.

(ii) For (i) implies (ii): in Definition 4.1.6 choose $\varepsilon = \frac{1}{n+1}$ for $n \in \mathbb{N}$ and put

$$A_n = \{a \in A \mid |x - a| < \frac{1}{n+1}\}.$$

Show that $A_n \neq \emptyset$ and construct the convergent sequence.[1]

Definition 4.1.9. *For $A \subset \mathbb{R}$ we define its* closure *"afsluiting"*

$$\overline{A} = \{x \in \mathbb{R} \mid x \text{ is a closure point of } A\}.$$

The set $A \subset \mathbb{R}$ is a closed set *"gesloten verzameling" if $A = \overline{A}$.*

Remark 4.1.10. Note that $A \subset \overline{A}$ is valid for any set, so that it suffices to prove $\overline{A} \subset A$ to conclude that A is a closed set. Note also that we have for any set

$$A^\circ \subset A \subset \overline{A}$$

and that an open set is defined by the first inclusion being an equality and that a closed set is defined by the second inclusion being an equality.

Next we establish the analogue of Proposition 4.1.4 for closed sets.

Proposition 4.1.11. *Let A and B be subsets of* $\mathbb{R}$.

[1] Here we use the *axiom of choice* "keuzeaxioma" for a countable number of sets.

(i) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(ii) $\overline{A}$ is a closed set, i.e. $\overline{\overline{A}} = \overline{A}$.

(iii) $\overline{A}$ is the smallest closed set containing A, i.e. if $A \subset B$ and B is closed, then $\overline{A} \subset B$.

(iv) Let A_α be a closed set for all $\alpha \in I$, then $\bigcap_{\alpha \in I} A_\alpha$ is a closed set, i.e. the intersection of arbitrarily many closed sets is closed.

(v) Let $A_1, A_2, \cdots, A_N$ be closed sets, then $\bigcup_{i=1}^{N} A_i$ is a closed set, i.e. the union of finitely many closed sets is closed.

(vi) $\emptyset$ and $\mathbb{R}$ are closed sets.

Let us emphasise that in Proposition 4.1.4(iv) and in Proposition 4.1.11(iv) the index set I can be arbitrary. In particular, I can be an uncountable infinite set.

Exercise 4.1.12. Prove Proposition 4.1.11, cf. the proof of Proposition 4.1.4. Hints for the proofs of the parts of Proposition 4.1.11 are given below.

(i) Observe that if x is a closure point of A and $A \subset B$, then x is a closure point of B.

(ii) Let $b \in \overline{\overline{A}}$ and let $\varepsilon > 0$ be arbitrary. Find $c \in \overline{A}$ with $|c - b| < \frac{1}{2}\varepsilon$. Since $c \in \overline{A}$, find $a \in A$ with $|a - c| < \frac{1}{2}\varepsilon$. Conclude that $b \in \overline{A}$. Show that this suffices to finish the proof.

(iii) Compare the proof of Proposition 4.1.4(iii).

(iv) Let b be a closure point of $\bigcap_{\alpha \in I} A_\alpha$, and pick $\varepsilon > 0$, and let $a \in \bigcap_{\alpha \in I} A_\alpha$ with $|b - a| < \varepsilon$. Argue that b is a closure point of A_α for all $\alpha \in I$. Use that A_α is closed, and argue that $b \in \bigcap_{\alpha \in I} A_\alpha$.

(v) Assume $b \notin \bigcup_{i=1}^{N} A_i$, then for all i we have $b_i \notin \overline{A_i}$. So there exists $\varepsilon_i > 0$ so that $|b - a_i| \geq \varepsilon_i$ for all $a_i \in A_i$. Argue that there exists $\varepsilon > 0$ so that $|b - a| \geq \varepsilon$ for all $a \in \bigcup_{i=1}^{N} A_i$. Conclude that $b \notin \overline{\bigcup_{i=1}^{N} A_i}$ and finish the proof.

(vi) Use the definition.

Exercise 4.1.13. Show that $\overline{A} = A \cup \{x \in \mathbb{R} \mid x \text{ limit point of } A\}$.

Lemma 4.1.14. *Let $A \subset \mathbb{R}$. Then A is an open set if and only if its complement $A^c = \mathbb{R} \setminus A$ is a closed set.*

So we also have that a set is closed if and only its complement is open. Note that by Proposition 4.1.11(vi) and Proposition 4.1.4(vi) we have $\emptyset$ and $\mathbb{R}$ as sets which are both open and closed. Note that there exist sets A which are neither closed nor open. An example is $A = (0,1]$, since

$$A^\circ = (0,1) \subset A = (0,1] \subset \overline{A} = [0,1]$$

and neither of the inclusions is an equality.

Proof of Lemma 4.1.14. Observe that

$$\begin{aligned} x \in (A^\circ)^c &\iff x \notin A^\circ \\ &\iff \forall\, \varepsilon > 0 \quad N_\varepsilon(x) \not\subset A \\ &\iff \forall\, \varepsilon > 0 \quad N_\varepsilon(x) \cap A^c \neq \emptyset \\ &\iff x \in \overline{A^c} \end{aligned}$$

so that the complement of the interior of a set equals the closure of the complement set, i.e. $(A^\circ)^c = \overline{A^c}$. So if A is an open set, then $A^c = (A^\circ)^c = \overline{A^c}$ and A^c is a closed set. If A^c is a closed set, then $A^c = \overline{A^c} = (A^\circ)^c$ and taking complements gives $A = A^\circ$, hence A is an open set. □

Exercise 4.1.15. Prove Proposition 4.1.11 from Proposition 4.1.4 using Lemma 4.1.14.

The following characterisation in Lemma 4.1.16 of a point in the closure of a set A is often useful.

Lemma 4.1.16. *Let $A \subset \mathbb{R}$ be a set. Then $x \in \overline{A}$ if and only if there exists a convergent sequence $(a_n)_{n\in\mathbb{N}}$ with $\forall n \in \mathbb{N}\ a_n \in A$ and $\lim_{n\to\infty} a_n = x$.*

Exercise 4.1.17. Prove Lemma 4.1.16 using Proposition 4.1.7.

Definition 4.1.18. *The* boundary *"rand" of a set $A \subset \mathbb{R}$ is $\partial A = \overline{A} \setminus A^\circ$.*

So the boundary of a set is its closure minus its interior, and it means that the boundary of $(0,1)$, $(0,1]$, $[0,1)$ and $[0,1]$ are all the same, namely $\{0,1\}$.

Proposition 4.1.19. *(i) The boundary ∂A of the set A is a closed set.*

(ii) $a \in \partial A$ if and only if $\forall\, \varepsilon > 0$ we have $N_\varepsilon(a) \cap A \neq \emptyset$ and $N_\varepsilon(a) \cap A^c \neq \emptyset$.

Proof. Using Lemma 4.1.14 we see that

$$\partial A = \overline{A} \setminus A^\circ = \overline{A} \cap (A^\circ)^c = \overline{A} \cap \overline{(A^c)},$$

which is a closed since it is the intersection of two closed sets, see Proposition 4.1.11(iv). This proves (i).

The proof of (ii) is Exercise 4.1.20. □

Exercise 4.1.20. We show part (ii) of Proposition 4.1.19.

(i) Show that $a \in \overline{A}$ implies $N_\varepsilon(a) \cap A \neq \emptyset$ for any $\varepsilon > 0$.

(ii) Show that $a \notin A^\circ$ implies $N_\varepsilon(a) \cap A^c \neq \emptyset$ for any $\varepsilon > 0$.

(iii) Show that $a \in \partial A$ implies that $\forall\, \varepsilon > 0$ we have $N_\varepsilon(a) \cap A \neq \emptyset$ and $N_\varepsilon(a) \cap A^c \neq \emptyset$.

(iv) Prove the converse of this statement, and finish the proof of Proposition 4.1.19(ii).

Exercise 4.1.21. Let $A \subset \mathbb{R}$ be a set.

(i) Show that $(\partial A)^c = A^\circ \cup (A^c)^\circ$.

(ii) Show that $\partial(A^c) = \partial A$.

4.2 Relatively open sets and relatively closed sets

In Section 4.1 open and closed sets have been defined with respect to $\mathbb{R}$ as the ambient space. We also need the notion of open and closed sets with respect to a fixed subset $X \subset \mathbb{R}$.

Definition 4.2.1. *For $X \subset \mathbb{R}$ a fixed subset, the* relative closure with respect to X *"relatieve afsluiting met betrekking tot X" of the subset $A \subset X$ is $\overline{A} \cap X \subset X$. The set $A \subset X$ is* relatively closed with respect to X *"relatief gesloten met betrekking tot X" if $A = \overline{A} \cap X$.*

In case the ambient space X is clear from the context, we leave out the "with respect to X" from the terminology. As an example, take $X = (0, 1)$, and $A = (0, \frac{1}{2}]$. Note that A is not closed in $\mathbb{R}$, but that $A = \overline{A} \cap X$ and hence A is a closed set relative to X. However, relatively closed sets can be related to closed sets in $\mathbb{R}$ as follows.

Proposition 4.2.2. *Let $A \subset X \subset \mathbb{R}$. The following statements are equivalent:*

(i) A is relatively closed with respect to X;

(ii) there exists a closed set $F \subset \mathbb{R}$ so that $A = F \cap X$;

(iii) for any sequence $(a_n)_{n\in\mathbb{N}}$ with the properties $\forall n \in \mathbb{N}\ a_n \in A$ and $\lim_{n\to\infty} a_n = b \in X$ we have $b \in A$.

Proof. (i) $\Longrightarrow$ (ii): take $F = \overline{A}$, and it follows from Definition 4.2.1.

(ii) $\Longrightarrow$ (i): note that $A = F \cap X \subset F$, so that $\overline{A} \subset \overline{F} = F$ by Proposition 4.1.11(i). So $\overline{A} \cap X \subset F \cap X = A$, and since trivially $A \subset \overline{A} \cap X$ we have $A = \overline{A} \cap X$ and A is relatively closed with respect to X.

(i) $\Longrightarrow$ (iii): assume that A is relatively closed, and that we a have a convergent sequence in A with limit b in X. In order to show that the limit is in A, observe that by Proposition 4.1.7 we have $b \in \overline{A}$. Hence, $b \in \overline{A} \cap X = A$.

(iii) $\Longrightarrow$ (i): since $A \subset X$ we have $A \subset \overline{A} \cap X$, and we have to show the reversed inclusion. So pick $b \in \overline{A} \cap X$, then by Proposition 4.1.7 we have a sequence $(a_n)_{n\in\mathbb{N}}$ in A with $\lim_{n\to\infty} a_n = b \in X$. Hence, by the assumption (iii), we have $b \in A$. So $\overline{A} \cap X \subset A$, and the reversed inclusion is established. □

Exercise 4.2.3. Let $X \subset \mathbb{R}$. Show that the following properties, cf. Proposition 4.1.11, are valid:

(i) Let A_α be a closed set relative to X for all $\alpha \in I$, then $\bigcap_{\alpha\in I} A_\alpha$ is a closed set relative to X.

(ii) Let $A_1, A_2, \cdots, A_N$ be closed sets relative to X, then $\bigcup_{i=1}^{N} A_i$ is a closed set relative to X.

(iii) $\emptyset$ and X are closed sets relative to X.

Hint: use Proposition 4.1.11 and Proposition 4.2.2. What can you say about possible analogues of the other statements of Proposition 4.1.11? You should first think about what the closure with respect to X is.

Definition 4.2.4. *Assume $X \subset \mathbb{R}$ a fixed subset, and let $A \subset X$. Then $a \in A$ is called an* interior point of A relative to X *"inwendig punt van A met betrekking tot X" if $\exists\, \varepsilon > 0$ so that $N_\varepsilon(a) \cap X \subset A$. The set $A \subset X$ is* relatively open with respect to X *"relatief open met betrekking tot X" if $\forall\, a \in A$ the point a is an interior point of A relative to X.*

Proposition 4.2.5. *Let $A \subset X \subset \mathbb{R}$. The following statements are equivalent:*

(i) A is relatively open with respect to X;

(ii) there exists a open set $U \subset \mathbb{R}$ so that $A = U \cap X$;

(iii) $X \setminus A$ is closed relative to X.

Proof. (i) $\Longrightarrow$ (ii): since for each $a \in A$ there exists $\varepsilon = \varepsilon_a > 0$ so that $N_\varepsilon(a) \cap X \subset A$. Now put $U = \cup_{a\in A} N_{\varepsilon_a}(a)$, which is open by Proposition 4.1.4(iv). Then $A \subset U \cap X$, and conversely $U \cap X = \cup_{a\in A}(N_{\varepsilon_a}(a) \cap X) \subset A$, since $N_{\varepsilon_a}(a) \cap X \subset A$ by Definition 4.2.4.

(ii) $\Longrightarrow$ (i): take $a \in A = U \cap X$ arbitrarily. Then, since $a \in U$ there exists $\varepsilon > 0$ with $N_\varepsilon(a) \subset U$ as U is an open set. So then $N_\varepsilon(a) \cap X \subset U \cap X = A$. Hence, a is an interior point of A relative to X. Since a was arbitrary, A is open relative to X.

(ii) $\Longrightarrow$ (iii): since $A = U \cap X$, we have $X \setminus A = U^c \cap X$ and since U^c is closed by Lemma 4.1.14 we have $X \setminus A$ is relatively closed with respect to X by Proposition 4.2.2.

(iii) $\Longrightarrow$ (ii): using Proposition 4.2.2 there exists a closed set $F \subset \mathbb{R}$ with $X \setminus A = F \cap X$. So $A = F^c \cap X$, and F^c is open by Lemma 4.1.14. □

Exercise 4.2.6. Let $X \subset \mathbb{R}$. Show that the following properties, cf. Proposition 4.1.11, are valid:

(i) Let A_α be an open set relative to X for all $\alpha \in I$, then $\bigcup_{\alpha\in I} A_\alpha$ is an open set relative to X

(ii) Let $A_1, A_2, \cdots, A_N$ be open sets relative to X, then $\bigcap_{i=1}^{N} A_i$ is an open set relative to X.

(iii) $\emptyset$ and X are open sets.

Hint: use Proposition 4.2.5 and Proposition 4.2.2. What can you say about possible analogues of the other statements of Proposition 4.1.4? First you need to define the interior with respect to X.

4.3 Sequentially compact sets

Definition 4.3.1. *The set $A \subset \mathbb{R}$ is* sequentially compact *"rijcompact" if for any sequence $(a_n)_{n\in\mathbb{N}}$ with $\forall\, n \in \mathbb{N}$ $a_n \in A$, there exists a convergent subsequence $(a_{n_j})_{j\in\mathbb{N}}$ with its limit contained in A; $\lim_{j\to\infty} a_{n_j} \in A$.*

Exercise 4.3.2. (i) Show that a sequentially compact set A is a closed set. Hint: use Lemma 4.1.16.

(ii) Assume that A is a sequentially compact set, and let $B \subset A$ be a closed set. Show that B is a sequentially compact set.

(iii) Assume that $A \subset B$ and that A is not sequentially compact. Can we conclude that B is not sequentially compact?

We see that $\mathbb{N}$ is not sequentially compact, since the sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = n$ has no convergent subsequence, see Exercise 3.2.13. Similarly, we see that $\mathbb{Z}$ and $\mathbb{Q}$ are also not sequentially compact subsets of $\mathbb{R}$.

The Heine-Borel Theorem 4.3.3 characterises sequentially compact sets.

Theorem 4.3.3 (Heine-Borel)**.** *Let $A \subset \mathbb{R}$. The set A is sequentially compact if and only if A is a closed and bounded set.*

Proof. We first assume that A is sequentially compact. In Exercise 4.3.2(i) we have observed that A is a closed set. In order to prove that A is also bounded, we argue by contradiction. So assume that A is unbounded, then

$$\forall n \in \mathbb{N} \qquad A_n = \{a \in A \mid |a| \geq n\} \neq \emptyset.$$

We consider a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A_n$ for all $n \in \mathbb{N}$. Since A is sequentially compact, there exists a convergent subsequence $(a_{n_j})_{j\in\mathbb{N}}$. By Proposition 3.2.8 we see that $(a_{n_j})_{j\in\mathbb{N}}$ is bounded, but $a_{n_j} \in A_{n_j}$ so that $|a_{n_j}| \geq n_j \geq j$ which tends to infinity as j increases, since

$n_j = f(j)$ for a strictly increasing function $f\colon \mathbb{N} \to \mathbb{N}$, see Definition 3.1.5. This gives the required contradiction, and A is bounded.

Conversely, assume that A is a bounded and closed set. We take an arbitrary sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A$ for all $n \in \mathbb{N}$ and we have to show that the sequence has a convergent subsequence $(a_{n_j})_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} a_{n_j} \in A$. Now, since A is bounded, the sequence $(a_n)_{n\in\mathbb{N}}$ is bounded. Hence, by the Bolzano-Weierstrass Theorem 3.2.12 the sequence $(a_n)_{n\in\mathbb{N}}$ has convergent subsequence. Since the set A is closed, the limit of the subsequence is in A by Lemma 4.1.16. So A is sequentially compact. □

Exercise 4.3.4. (i) Show that the union of two sequentially compact sets is a sequentially compact set.

(ii) Show that the intersection of two sequentially compact sets is a sequentially compact set.

(iii) Is the union of arbitrarily many sequentially compact sets a sequentially compact set?

(iv) Is the intersection of arbitrarily many sequentially compact sets a sequentially compact set?

4.4 Exercises

Exercise 4.4.1. (i) Let A be a non-empty bounded closed set. Show that $\sup A \in \overline{A}$ and $\inf A \in \overline{A}$.

(ii) Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence, and define the set $A = \{a_n \mid n \in \mathbb{N}\}$. Show that $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are elements of $\overline{A}$. Hint: use Exercise 3.3.7.

Exercise 4.4.2. (i) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(ii) Show that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$, and give an example where the inclusion is a strict inclusion.

Exercise 4.4.3. (i) Show that a set consisting of a finite number of elements is closed.

(ii) Is $\mathbb{N}$ a closed or open set with respect to $\mathbb{Z}$? Or is it neither open nor closed?

Exercise 4.4.4. Let $A \subset \mathbb{R}$ be a non-empty set. Assume that $x \in \mathbb{R}$ and that A is a sequentially compact set. Show that there exists $c \in A$ with $|c - x| = \inf\{|x - a| \mid a \in A\}$.

Exercise 4.4.5. For $A \subset \mathbb{R}$, $B \subset \mathbb{R}$ define $A + B = \{a + b \mid a \in A, b \in B\}$.

(i) Let A, B be open sets. Show that $A + B$ is an open set.

(ii) Let A, B be closed sets. Is $A + B$ a closed set?

(iii) Let A, B be closed sets and assume A is sequentially compact. Show that $A + B$ is a closed set.

Exercise 4.4.6. We call a set $A \subset \mathbb{R}$ *pathwise connected* "padsgewijs samenhangend" if for any two elements x and y we have $tx + (1-t)y \in A$ for all $t \in [0,1]$. Otherwise stated, the image of the path $[0,1] \ni t \mapsto tx + (1-t)y$ is contained in the set A.

Show that an interval is pathwise connected. Conversely, assume that $A \subset \mathbb{R}$ is pathwise connected, show that A is an interval.

Exercise 4.4.7. Define the *derived set* "afgeleide verzameling" of the set $A \subset \mathbb{R}$ as

$$A' = \{x \in \mathbb{R} \mid x \text{ limit point of } A\}.$$

(i) Show that $x \in A'$ if and only if there exists a convergent sequence $(a_n)_{n\in\mathbb{N}}$ with $\forall\, n \in \mathbb{N}$ we have $a_n \in A \setminus \{x\}$ and $\lim_{n\to\infty} a_n = x$.

(ii) Show that $A' \subset \overline{A}$ and that A' is a closed set.

(iii) Let A be an open set. Show that $A' = \overline{A}$.

(iv) Show that in general $\overline{A} = A' \cup \{x \in A \mid x \text{ isolated point of } A\}$ with a disjoint union.

(v) Show that in general $(A')' \neq A'$.

Exercise 4.4.8. We say that a set $A \subset \mathbb{R}$ is a *compact set* "compacte verzameling" if for any covering of A by open sets there exists a finite subcover. Or, for any collection $\{B_\alpha \mid \alpha \in I\}$ with $\forall \alpha \in I$ the set B_α is an open set and

$$A \subset \bigcup_{\alpha\in I} B_\alpha$$

(which is the covering of A by open sets) there exists $N \in \mathbb{N}$ and $\alpha_1 \in I$, $\alpha_2 \in I$, $\cdots$, $\alpha_N \in I$ so that

$$A \subset \bigcup_{i=1}^{N} B_{\alpha_i}$$

(which is the finite subcover).

The notions of sequentially compactness and compactness are equivalent for subsets of $\mathbb{R}$. The purpose of Exercise 4.4.8 is to prove one of the implications, namely compactness implies sequentially compactness. So assume $A \subset \mathbb{R}$ to be compact, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in A. We want to show that there exists a convergent subsequence $(a_{n_j})_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} a_{n_j} = x \in A$. We proceed as follows.

(i) Assume $A \subset \mathbb{R}$ to be compact, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in A. Then $\exists\, x \in A$ so that for all $\varepsilon > 0$ the set $\{n \in \mathbb{N} \mid a_n \in N_\varepsilon(x)\}$ is infinite. We prove this in the following steps:

(a) Arguing by contradiction, we assume

$$\forall\, x \in A \quad \exists\, \varepsilon = \varepsilon(x) > 0 \qquad |\{n \in \mathbb{N} \mid a_n \in N_\varepsilon(x)\}| < \infty$$

Show that $\bigcup_{x\in A} N_\varepsilon(x)$ is a covering of A by open sets.

(b) Conclude that there exist $N \in \mathbb{N}$ and $x_1 \in A, \cdots, x_N \in A$ so that $A \subset \bigcup_{i=1}^{N} N_{\varepsilon(x_i)}(x_i)$. Derive the contradiction.

(ii) Using (i), construct a convergent subsequence of $(a_{n_j})_{j \in \mathbb{N}}$, convergent to x, by choosing $\varepsilon = \frac{1}{j+1}$.

Chapter 5

Real functions

In this chapter we study real valued functions. In particular, we study the properties of continuous functions. Continuity at a point is rephrased in terms of sequences and continuity is rephrased in terms of open sets. We study properties of continuous functions, and we consider also uniform continuous functions. Sequences of functions are considered in the pointwise and uniform setting.

5.1 Continuous functions

Definition 5.1.1. *Let $f\colon A \to \mathbb{R}$ be a real valued function with domain $A \subset \mathbb{R}$. Let $E \subset A$ and assume $x_0 \in \overline{E}$. Then we say that f* converges to $L \in \mathbb{R}$ at x_0 through E *"f convergeert naar $L \in \mathbb{R}$ in x_0 door E" if*

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in E \qquad |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

We use the notation

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = L \quad \text{or} \quad \lim_{x \to x_0; x \in E} f(x) = L$$

In case $E = A$, we leave the dependence on E from the notation. The additional usage of E in Definition 5.1.1 can be used to define left and right limits, see Exercise 5.5.2. In case $E = \{x_0\}$, we trivially have $\lim_{x \to x_0; x \in \{x_0\}} f(x) = f(x_0)$.

The dependence on E is not that relevant, since the limit of a function only depends on the local behaviour of the function f. We formulate this in Lemma 5.1.2, and the proof is sketched in Exercise 5.1.3.

Lemma 5.1.2. *For any $\delta_0 > 0$ we have*

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = L \qquad \Longleftrightarrow \qquad \lim_{\substack{x \to x_0 \\ x \in E \cap (x_0 - \delta_0, x_0 + \delta_0)}} f(x) = L.$$

Exercise 5.1.3. We indicate how to prove Lemma 5.1.2.

(i) Show that for $x_0 \in \overline{E}$ we also have that $x_0 \in \overline{E \cap (x_0 - \delta_0, x_0 + \delta_0)}$

(ii) Argue that the implication from left to right is trivial.

(iii) Show that for $x_0 \in \overline{E \cap (x_0 - \delta_0, x_0 + \delta_0)}$ we also have that $x_0 \in \overline{E}$.

(iv) Prove the implication from right to left. Hint: replace δ in Definition 5.1.1 by $\delta' = \min(\delta, \delta_0)$.

Proposition 5.1.4. *Using the notation of Definition 5.1.1,*

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = L.$$

if and only if for any sequence $(a_n)_{n \in \mathbb{N}}$ with $a_n \in E$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = x_0$ we have that the sequence $(f(a_n))_{n \in \mathbb{N}}$ is convergent and $\lim_{n\to\infty} f(a_n) = L$.

Proof. Assume first that $\lim_{x \to x_0; x \in E} f(x) = L$ and that $(a_n)_{n \in \mathbb{N}}$ is a sequence with $a_n \in E$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = x_0$. We need to prove that $(f(a_n))_{n \in \mathbb{N}}$ is a convergent sequence with $\lim_{n\to\infty} f(a_n) = L$. So take $\varepsilon > 0$ arbitrary, then by Definition 5.1.1 there exists a $\delta > 0$ so that for $x \in E$ and $|x_0 - x| < \delta$ we have $|f(x) - L| < \varepsilon$. Since $(a_n)_{n \in \mathbb{N}}$ converges to x_0, there exists $N \in \mathbb{N}$ so that for all $n \geq N$ we have $|x_0 - a_n| < \delta$. Hence, for this $N \in \mathbb{N}$ we have for all $n \geq N$ that $|L - f(a_n)| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $\lim_{n\to\infty} f(a_n) = L$.

Conversely, assume that for any sequence $(a_n)_{n \in \mathbb{N}}$ with $a_n \in E$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = x_0$ we have that the sequence $(f(a_n))_{n \in \mathbb{N}}$ is convergent and $\lim_{n\to\infty} f(a_n) = L$. In order to prove that

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = L.$$

we argue by contradiction. Taking the negation of Definition 5.1.1 we see

$$\exists\, \varepsilon_0 > 0 \quad \forall\, \delta > 0 \quad \exists\, x \in E \qquad |x_0 - x| < \delta \ \wedge \ |L - f(x)| \geq \varepsilon_0.$$

In order to create a sequence that contradicts our assumption, we take consecutively $\delta = \frac{1}{n+1}$ for $n \in \mathbb{N}$. The corresponding $x \in E$ with $|x_0 - x| < \frac{1}{n+1}$ and $|L - f(x)| \geq \varepsilon_0$, we call a_n. From the property $|x_0 - a_n| < \frac{1}{n+1}$ we see that $\lim_{n\to\infty} a_n = x_0$. From the $|L - f(a_n)| \geq \varepsilon_0$, we see that the sequence $(f(a_n))_{n \in \mathbb{N}}$ cannot converge to L. This is the required contradiction. □

Proposition 5.1.4 gives the opportunity to connect limits of functions to limits of sequences, so that we can apply the machinery of Chapter 3.

Corollary 5.1.5. *Let $f\colon A \to \mathbb{R}$ and $g\colon A \to \mathbb{R}$ be real-valued functions with domain $A \subset \mathbb{R}$. Let $E \subset A$ and assume $x_0 \in \overline{E}$ and that*

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x) = L, \qquad \lim_{\substack{x \to x_0 \\ x \in E}} g(x) = M.$$

Then the following properties hold:

(i) *the limit L is uniquely determined;*

(ii) *for real values $c, d \in \mathbb{R}$ the function $cf + dg$ converges to $cL + dM$ at x_0 through E*

$$\lim_{\substack{x \to x_0 \\ x \in E}} cf(x) + dg(x) = cL + dM$$

(iii) *the function fg converges to LM at x_0 through E*

$$\lim_{\substack{x \to x_0 \\ x \in E}} f(x)g(x) = LM$$

(iv) *assuming that $\forall\, x \in E$ we have $g(x) \neq 0$ and moreover that $M \neq 0$, then the function $\frac{f}{g}$ converges to $\frac{L}{M}$ at x_0 through E*

$$\lim_{\substack{x \to x_0 \\ x \in E}} \frac{f(x)}{g(x)} = \frac{L}{M}$$

Exercise 5.1.6. (i) Give a proof of Corollary 5.1.5 using Proposition 5.1.4 and the results of Chapter 3, in particular Theorem 3.2.19 and Exercise 3.2.18.

(ii) Formulate and prove the statements for limits of functions that correspond to Theorem 3.2.19(vi) and the sandwich principle Theorem 3.2.19(vii).

Exercise 5.1.7. Give another proof of Lemma 5.1.2 using Proposition 5.1.4.

Definition 5.1.8. *Let $f \colon A \to \mathbb{R}$ be a real-valued function, $x_0 \in A$. Then f* is continuous at *$x_0 \in A$ "f is continu in $x_0 \in A$" if*

$$\lim_{x \to x_0; x \in A} f(x) = f(x_0)$$

and $f \colon A \to \mathbb{R}$ is a continuous function *"f is een continue functie" if $\forall\, x_0 \in A$ the function f is continuous at x_0.*

Exercise 5.1.9. Assume that $f \colon A \to \mathbb{R}$ is continuous, and that $Y \subset A$. Show that the restricted function $f|_Y \colon Y \to A$ is continuous.

Proposition 5.1.10. *Let $f \colon A \to \mathbb{R}$ be a function, and $x_0 \in A$. The following statements are equivalent:*

(i) *f is continuous at x_0;*

(ii) *for any sequence $(a_n)_{n \in \mathbb{N}}$ with $\forall\, n \in \mathbb{N}\ a_n \in A$ and $\lim_{n \to \infty} a_n = x_0$ the image sequence $(f(a_n))_{n \in \mathbb{N}}$ is convergent and $\lim_{n \to \infty} f(a_n) = f(x_0)$.*

(iii) *$\forall\, \varepsilon > 0\ \exists\, \delta > 0$ so that for all $a \in A$ with $|x_0 - a| < \delta$ we have $|f(x_0) - f(a)| < \varepsilon$.*

Proof. The equivalence of (i) and (ii) follows from Proposition 5.1.4. The equivalence of (i) and (iii) follow from Definition 5.1.1 and Definition 5.1.8. □

As before, Proposition 5.1.10 enables us to use the machinery of Chapter 3 to obtain results on continuous functions.

Corollary 5.1.11. *Let $f\colon A \to \mathbb{R}$ and $g\colon A \to \mathbb{R}$ be real-valued functions with domain $A \subset \mathbb{R}$. Assume that f and g are continuous in $x_0 \in A$. Then the following properties hold:*

(i) for real values $c, d \in \mathbb{R}$ the function $cf + dg$ is continuous in x_0;

(ii) the function fg is continuous in x_0;

(iii) assuming that $\forall\, x \in A$ we have $g(x) \neq 0$, then the function $\frac{f}{g}$ is continuous in x_0;

Exercise 5.1.12. (i) Give a proof of Corollary 5.1.11.

(ii) Assuming the conditions of Corollary 5.1.11. Show that the functions $\max(f, g)$ and $\min(f, g)$ are continuous at x_0. Here $\max(f, g)\colon A \to \mathbb{R}$ is the function defined by $\max(f, g)(a) = \max(f(a), g(a))$ for $a \in A$, and similarly for $\min(f, g)$.

Exercise 5.1.13. Assume that $f\colon A \to \mathbb{R}$ with domain $A \subset \mathbb{R}$ is continuous at $x_0 \in A$, and assume that $g\colon B \to \mathbb{R}$ and that $f(A) \subset B$. Moreover, assume that f is continuous at $x_0 \in A$ and that g is continuous in $f(x_0) \in B$. Show that the composition $g \circ f\colon A \to \mathbb{R}$ is continuous in x_0. Hint: use Proposition 5.1.10(ii).

The following result of Theorem 5.1.14 is important as it will give the definition of continuous functions in a more general context in the later course on Topology.

Theorem 5.1.14. *The function $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}$, is continuous if and only if for any open set $U \subset \mathbb{R}$ its inverse image $f^{-1}(U)$ is an open set relative to A.*

Note that $f^{-1}(U)$ is an open set relative to A means that $f^{-1}(U) = V \cap A$ for an open set $V \subset \mathbb{R}$, see Section 4.2, in particular Proposition 4.2.5. Since complements of open sets are closed sets, and taking inverse image and complements commute, see Section 2.3, the following corollary follows.

Corollary 5.1.15. *The function $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}$, is continuous if and only if for any closed set $U \subset \mathbb{R}$ its inverse image $f^{-1}(U)$ is a closed set relative to A.*

Proof of Theorem 5.1.14. Assume that f is continuous, and take $U \subset \mathbb{R}$ an open set. Take $x \in f^{-1}(U)$, or $f(x) \in U$. Since U is open, $\exists\, \varepsilon > 0$ with $N_\varepsilon(f(x)) \subset U$. Using that f is continuous at $x \in A$ we find $\delta > 0$ so that

$$|x - a| < \delta \text{ and } a \in A \qquad \Longrightarrow \qquad |f(x) - f(a)| < \varepsilon$$

which we can rephrase as

$$a \in N_\delta(x) \cap A \qquad \Longrightarrow \qquad f(a) \in N_\varepsilon(f(x)) \subset U$$

which gives $N_\delta(x) \cap A \subset f^{-1}(U)$. So $f^{-1}(U)$ is open relative to the domain A.

Conversely, assume that the inverse image of any open set U is open. Take $x \in A$ and $\varepsilon > 0$ arbitrary, and consider the open set $U = N_\varepsilon(f(x))$. Then $x_0 \in f^{-1}(U)$, and since $f^{-1}(U)$ is open with respect to A, there exists $\delta > 0$ so that $N_\delta(x) \cap A \subset f^{-1}(N_\varepsilon(f(x))$. This means that for $a \in N_\delta(x_0) \cap A$, i.e. any $a \in A$ with $|x - a| < \delta$, we have $f(a) \in N_\varepsilon(f(x))$, i.e. that $|f(a) - f(x)| < \varepsilon$. So f is continuous in x, and since $x \in A$ was arbitrary, f is continuous. □

Exercise 5.1.16. It is possible to restricted the open sets U in Theorem 5.1.14 to half intervals.

Show that the equivalence of Theorem 5.1.14 and Corrollary 5.1.15 can be extended to be equivalent to the following conditions:

(i) for all $c \in \mathbb{R}$ the sets

$$U_c = f^{-1}((c,\infty)) = \{a \in A \mid f(a) > c\}, \qquad L_c = f^{-1}((-\infty,c)) = \{a \in A \mid f(a) < c\}$$

are relatively open with respect to A,

(ii) for all $c \in \mathbb{R}$ the sets

$$V_c = f^{-1}([c,\infty)) = \{a \in A \mid f(a) \geq c\}, \qquad M_c = f^{-1}((-\infty,c]) = \{a \in A \mid f(a) \leq c\}$$

are relatively closed with respect to A.

So, show that $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}$, is continuous if and only if (i) holds and if and only if (ii) holds.

5.2 Properties of continuous functions

Continuous functions on a bounded closed interval have a series of additional interesting properties. By the Heine-Borel Theorem 4.3.3 we know that a bounded and closed interval is a sequentially compact set, and a number of results in this section can be generalised to continuous functions on sequentially compact sets, see Section 5.5.

Definition 5.2.1. *The function $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}$, is called* bounded from above *if its range or image $f(A) \subset \mathbb{R}$ is bounded from above as in Definition 2.2.1. Similarly, $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}$, is called* bounded from below *if its range or image $f(A) \subset \mathbb{R}$ is bounded from below. And $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}$, is called* bounded *if its range or image $f(A) \subset \mathbb{R}$ is bounded.*

Using Definition 2.2.1 we see that $f\colon A \to \mathbb{R}$ is bounded from above if

$$\exists\, M \in \mathbb{R} \quad \forall\, a \in A \qquad f(a) \leq M$$

and $f\colon A \to \mathbb{R}$ is bounded from below if

$$\exists\, M \in \mathbb{R} \quad \forall\, a \in A \qquad f(a) \geq M$$

and $f\colon A \to \mathbb{R}$ is bounded if

$$\exists\, M \in \mathbb{R} \quad \forall\, a \in A \qquad |f(a)| \leq M.$$

Lemma 5.2.2. *Let $f\colon [a,b] \to \mathbb{R}$, $a < b$, be a continuous function, then f is bounded.*

Remark 5.2.3. It is important to notice that it is essential in Lemma 5.2.2 that the interval $[a,b]$ is sequentially compact. Please try to find explicit examples of unbounded continuous functions on domains such as e.g. $(0,1]$ and on $[0,\infty)$.

The proof of Lemma 5.2.2 is very similar to part of the proof of the Heine-Borel Theorem 4.3.3.

Proof of Lemma 5.2.2. Assume f is not bounded, then for all $n \in \mathbb{N}$ we have

$$A_n = \{x \in [a,b] \mid |f(x)| > n\} \neq \emptyset$$

Now we pick $a_n \in A_n$ for each $n \in \mathbb{N}$, and we obtain a sequence $(a_n)_{n\in\mathbb{N}}$ in the domain $[a,b]$. Since $[a,b]$ is a bounded set, the Bolzano-Weierstrass Theorem 3.2.12 gives a convergent subsequence $(a_{n_j})_{n\in\mathbb{N}}$ with $\lim_{j\to\infty} a_{n_j} = x_0 \in [a,b]$, since $[a,b]$ is a closed set.

Because f is continuous, it is continuous at x_0. By Proposition 5.1.10 we know that the image sequence $(f(a_{n_j}))_{j\in\mathbb{N}}$ is a convergent sequence. In particular, $(f(a_{n_j}))_{j\in\mathbb{N}}$ is a bounded sequence by Proposition 3.2.8. But, by construction, we have

$$\forall j \in \mathbb{N} \qquad |f(a_{n_j})| > n_j \geq j$$

and this shows that the sequence $(f(a_{n_j}))_{j\in\mathbb{N}}$ is unbounded. This is the required contradiction. □

Exercise 5.2.4. Let $f\colon A \to \mathbb{R}$, with domain $A \subset \mathbb{R}$ a sequentially compact set, be a continuous function. Show that f is bounded. Hint: mimick the proof of Lemma 5.2.2 using the Heine-Borel Theorem 4.3.3.

Definition 5.2.5. *Let $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}$, be a function. We say that f attains its* maximum *"maximum" in $x_0 \in A$ if*

$$\forall\, a \in A \qquad f(a) \leq f(x_0).$$

Similarly, f attains its minimum *"minimum" in $x_0 \in A$ if*

$$\forall\, a \in A \qquad f(a) \geq f(x_0).$$

A maximum or minimum as in Definition 5.2.5 is also called a *global maximum* or a *global minimum.*

Proposition 5.2.6. *Let $f\colon [a,b] \to \mathbb{R}$, $a < b$, be a continuous function, then there exists $x_0 \in [a,b]$ such that f attains its maximum at x_0.*

Proof. Consider the range $f([a,b])$, then by Lemma 5.2.2 this set is bounded. Since it is also not empty –it contains $f(a) \in f([a,b])$– the supremum exists by Theorem 2.2.4, so that we can define $m = \sup(f([a,b]))$. We have to prove the existence of $x_0 \in [a,b]$ with $f(x_0) = m$.

We do this by establishing x_0 as a limit of a suitable convergent sequence. In order to construct the sequence, we consider for any $n \in \mathbb{N}$ the set

$$A_n = \{x \in [a,b] \mid m - \frac{1}{n+1} < f(x) \leq m\}.$$

Then $A_n \neq \emptyset$ for all $n \in \mathbb{N}$, since if it would be empty the element $m - \frac{1}{n+1}$ would be an upper bound for $f(A)$ smaller than its supremum. Pick $a_n \in A_n$ for each $n \in \mathbb{N}$, and consider the sequence $(a_n)_{n\in\mathbb{N}}$. Since $[a,b]$ is a sequentially compact set, by the Heine-Borel Theorem 4.3.3, this sequence has a convergent subsequence with limit in $[a,b]$, i.e. there exists a subsequence $(a_{n_j})_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} a_{n_j} = x_0 \in [a,b]$. Then, f being continuous at x_0, we have by Proposition 5.1.10 $\lim_{j\to\infty} f(a_{n_j}) = f(x_0) \in f(A)$. By construction we have

$$\forall j \in \mathbb{N} \qquad m - \frac{1}{n_j+1} < f(a_{n_j}) \leq m$$

so that $\lim_{j\to\infty} m - \frac{1}{n_j+1} = m$ and Theorem 3.2.19(vii) show that $\lim_{j\to\infty} f(a_{n_j}) = m$. By uniqueness of the limit of a sequence, see Exercise 3.2.18, we have $f(x_0) = m$. □

Remark 5.2.7. Note that in the proof of Proposition 5.2.6 we have that $(f(a_n))_{n\in\mathbb{N}}$ is a convergent sequence with $\lim_{n\to\infty} f(a_n) = m$. Is it also true that $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence with $\lim_{n\to\infty} a_n = x_0$?

Corollary 5.2.8. *Let $f\colon [a,b] \to \mathbb{R}$, $a < b$, be a continuous function, then there exists $x_0 \in [a,b]$ such that f attains its minimum at x_0.*

Exercise 5.2.9. Give a proof of Corollary 5.2.8. For this you can mimick the proof of Proposition 5.2.6 using the $\inf(f([a,b]))$, or you can set up a proof reducing to the statement of Proposition 5.2.6 by switching to $-f$.

Exercise 5.2.10. Let $f\colon A \to \mathbb{R}$ be a continuous function, and let A be a sequentially compact set. Show that f attains its maximum and its minimum.

Theorem 5.2.11 (Intermediate Value Theorem "Tussenwaardestelling")**.** *Let $a < b$ and assume that $f\colon [a,b] \to \mathbb{R}$ is continuous. Assume $y \in \mathbb{R}$ is between $f(a)$ and $f(b)$, i.e. $y \in [f(a), f(b)]$ or $y \in [f(b), f(a)]$, then there exists $x_0 \in [a,b]$ with $f(x_0) = y$.*

Proof. We assume that $f(a) \leq f(b)$. In case $f(a) = f(b)$, we take $x_0 = a$. So we assume that $f(a) < f(b)$, and we also assume $f(a) < y < f(b)$, since otherwise we can take $x_0 = a$ (in case $f(a) = y$) or $x_0 = b$ (in case $f(b) = y$). We now define the set

$$A = \{x \in [a,b] \mid f(x) < y\} \subset [a,b]$$

then A is a bounded set, and A is non-empty, since $a \in A$. We claim that $x_0 = \sup(A)$ meets the criteria. This will follow from the following two observations: $f(x_0) \leq y$ and $f(x_0) \geq y$. Observe that $x_0 < b$, or see Exercise 5.2.12.

Firstly, we show that $f(x_0) \leq y$. Observe that for $n \in \mathbb{N}$, $x_0 - \frac{1}{n+1}$ is not an upper bound, so there exists $a_n \in A$ with

$$x_0 - \frac{1}{n+1} < a_n \leq x_0.$$

In particular, $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence and $\lim_{n\to\infty} a_n = x_0$ by Theorem 3.2.19(vii). Since f is continuous, $\lim_{n\to\infty} f(a_n) = f(x_0)$, and since for all n we have $a_n \in A$ it follows that $f(a_n) < y$, so that by Theorem 3.2.19(vi) we have that $f(x_0) \leq y$.

Secondly, we show that $f(x_0) \geq y$. Since $x_0 < b$ we consider the sequence $(x_0 + \frac{1}{n})_{n=n_0}^{\infty}$ for $n_0 \in \mathbb{N}$ so that $x_0 + \frac{1}{n_0} \leq b$. Then $x_0 + \frac{1}{n} \notin A$ for all $n \geq n_0$, so that $f(x_0 + \frac{1}{n}) \geq y$. Since $\lim_{n\to\infty} x_0 + \frac{1}{n} = x_0$, the continuity of f gives $\lim_{n\to\infty} f(x_0 + \frac{1}{n}) = f(x_0)$, and by Theorem 3.2.19(vi), we find $f(x_0) \geq y$. □

Exercise 5.2.12. In the proof of Theorem 5.2.11 we need to show that $x_0 = \sup(A) < b$ in case $f(a) < y < f(b)$. By construction $\sup(A) \leq b$. We show that $\sup(A) = b$ leads to a contradiction.

(i) Show that there is a sequence $(a_n)_{n\in\mathbb{N}}$ with $\forall n \in \mathbb{N}\ a_n \in A$ and $\lim_{n\to\infty} a_n = b$.

(ii) Use the continuity of f at b and $y < f(b)$ to obtain a contradiction.

Corollary 5.2.13. *Let $f\colon [a,b] \to \mathbb{R}$, $a < b$, be a continuous function, and let m and l be the supremum and infimum of the image of f, i.e. $m = \sup f([a,b])$, $l = \inf f([a,b])$. Then $f([a,b]) = [l,m]$.*

Exercise 5.2.14. Combine Proposition 5.2.6, Corollary 5.2.8 and Theorem 5.2.11 to prove Corollary 5.2.13. Hint: find $c, d \in [a,b]$ with $f(c) = l$ and $f(d) = m$, and restrict f to the interval $[c,d]$ if $c < d$ and apply Theorem 5.2.11 to f restricted to $[c,d]$.

In the case of Corollary 5.2.13 we cannot generalise to a continuous function $f\colon A \to \mathbb{R}$ on a sequentially compact set A. In this case the important property is the connectedness of the interval instead of the sequentially compactness of the domain, see Exercise 4.4.6.

5.3 Uniformly continuous functions

Definition 5.3.1. *The function $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}$, is called* uniformly continuous *"uniform continu" if*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, a \in A \qquad |x - a| < \delta \quad \Longrightarrow \quad |f(x) - f(a)| < \varepsilon.$$

Comparing Definition 5.1.8 with Definition 5.3.1 we see that a uniform continuous function is continuous. The term uniformly can be explained as follows: in Definition 5.1.8 the $\delta > 0$ depends on both $x \in A$ and on $\varepsilon > 0$, whereas Definition 5.3.1 the $\delta > 0$ only depends on $\varepsilon > 0$, so this is uniform in $x \in A$.

Lemma 5.3.2. *Assume that $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}$, is a uniformly continuous function. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in A, then $(f(a_n))_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Note that Cauchy sequences in $\mathbb{R}$ are convergent sequences by Theorem 3.3.11. Note that in Lemma 5.3.2 it is not required that the limit of the Cauchy seqeuence in the domain A is contained in the domain A. The limit is contained in the closure $\overline{A}$ of the domain.

Proof. We need to prove that

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n, m \geq N \qquad |f(a_n) - f(a_m)| < \varepsilon.$$

Pick $\varepsilon > 0$ arbitrarily, then, by Definition 5.3.1, $\exists \delta > 0$ so that for all $x, a \in A$ with $|x-a| < \delta$ we have $|f(x) - f(a)| < \varepsilon$. Since $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists $N \in \mathbb{N}$ so that for all $n, m \geq N$ we have $|a_n - a_m| < \delta$. So for this N we have for all $n, m \geq N$ that $|f(a_n) - f(a_m)| < \varepsilon$. □

Exercise 5.3.3. Show that $f\colon (0,1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous.

Corollary 5.3.4. *Assume that $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}$, is a uniformly continuous function, and that A is bounded, then $f(A)$ is bounded.*

Exercise 5.3.5. Give a proof of Corollary 5.3.4 using Lemma 5.3.2 and the boundedness of Cauchy sequences, see Exercise 3.3.3. Hint: argue by contradiction as in the proof of Lemma 5.2.2.

The analogue of Proposition 5.1.10 is Proposition 5.3.8. For this characterisation we need the notion of equivalent sequences.

Definition 5.3.6. *The sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are* equivalent sequences *"equivalente rijen" if the sequence $(a_n - b_n)_{n\in\mathbb{N}}$ is convergent and* $\lim_{n\to\infty} a_n - b_n = 0$.

So Definition 5.3.6 means

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \qquad |a_n - b_n| < \varepsilon.$$

Exercise 5.3.7. (i) Give an example of equivalent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ which are both divergent.

(ii) Show that if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are equivalent sequences, and $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence with $\lim_{n\to\infty} a_n = L$, then $(b_n)_{n\in\mathbb{N}}$ is a convergent sequence with $\lim_{n\to\infty} b_n = L$.

(iii) Conclude that equivalent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are either both convergent (and with the same limit) or both divergent.

(iv) Assume that the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are equivalent sequences. Consider the subsequences $(a_{n_j})_{j\in\mathbb{N}}$ and $(b_{n_j})_{j\in\mathbb{N}}$ which are subsequences for the same increasing function $f\colon \mathbb{N} \to \mathbb{N}$, $n_j = f(j)$. Show that $(a_{n_j})_{j\in\mathbb{N}}$ and $(b_{n_j})_{j\in\mathbb{N}}$ are equivalent series. Can we adapt this statement to allow for different functions labelling the subsequence?

Proposition 5.3.8. *Let $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}$, be a function. Then the following are equivalent:*

(i) f is uniformly continuous,

(ii) for each pair of equivalent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in A, the sequences $(f(a_n))_{n\in\mathbb{N}}$ and $(f(b_n))_{n\in\mathbb{N}}$ are equivalent sequences.

The proof of Proposition 5.3.8 is in certain respects similar to the proof of Proposition 5.1.10. In Exercise 5.3.9 you are asked to fill in the details of the proof of Proposition 5.3.8.

Exercise 5.3.9. We indicate a proof of Proposition 5.3.8.

(i) Show that (i) implies (ii) by writing out the definitions mimicking the proof of Lemma 5.3.2.

(ii) To show that (ii) implies (i) argue by contradiction. Use that f not uniformly continuous means that

$$\exists\,\varepsilon_0 > 0 \quad \forall\,\delta > 0 \quad \exists\, a_\delta, b_\delta \in A \qquad |a_\delta - b_\delta| < \delta \text{ and } |f(a_\delta) - f(b_\delta)| \geq \varepsilon_0.$$

As in the proof of Proposition 5.1.10 take $\delta = \frac{1}{n+1}$, $n \in \mathbb{N}$, and construct sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ establishing a contradiction with (ii).

Theorem 5.3.10. *Assume that $f\colon [a,b] \to \mathbb{R}$, $a < b$, is continuous, then f is uniformly continuous.*

Theorem 5.3.10 can be extended to a continuous function $f\colon A \to \mathbb{R}$ on a sequentially compact set. Theorem 5.3.10 has an easier proof using the definition of compactness as in Exercise 4.4.8, see Exercise 5.5.15. For completeness we give the proof using the Heine-Borel Theorem 4.3.3 for sequentially compact sets.

Proof. We use a contradiction, so we assume that f is not uniformly continuous. By Proposition 5.3.8 there exists two equivalent sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in the domain $[a,b]$ so that the images $(f(x_n))_{n\in\mathbb{N}}$, $(f(y_n))_{n\in\mathbb{N}}$ are not equivalent, i.e. the sequence $(f(x_n) - f(y_n))_{n\in\mathbb{N}}$ does not converge to 0. This means that

$$\exists\,\varepsilon_0 > 0 \quad \forall N \geq 0 \quad \exists\, n \geq N \qquad |f(x_n) - f(y_n)| \geq \varepsilon_0.$$

So we have a subsequence $(f(x_{n_j}) - f(y_{n_j}))_{j\in\mathbb{N}}$ with

$$\forall\, j \in \mathbb{N} \qquad |f(x_{n_j}) - f(y_{n_j})| \geq \varepsilon_0. \tag{5.3.1}$$

(Explain how to construct this subsequence.)

Since $[a,b]$ is closed and bounded, it is sequentially compact by the Heine-Borel Theorem 4.3.3, so that we know that the sequence $(x_{n_j})_{j\in\mathbb{N}}$ has a convergent subsequence, which we denote by $(x_{n_{j_k}})_{k\in\mathbb{N}}$;

$$\lim_{k\to\infty} x_{n_{j_k}} = L \in [a,b].$$

But then we also have

$$\lim_{k\to\infty} y_{n_{j_k}} = L \in [a,b]$$

by Exercise 5.3.7(ii) and (iv), since the subsequences $(x_{n_{j_k}})_{k\in\mathbb{N}}$ and $(y_{n_{j_k}})_{k\in\mathbb{N}}$ are equivalent sequences. By the continuity of f we find that

$$\lim_{k\to\infty} f(x_{n_{j_k}}) = f(L) = \lim_{k\to\infty} f(y_{n_{j_k}})$$

and thus

$$\lim_{k\to\infty} f(x_{n_{j_k}}) - f(y_{n_{j_k}}) = 0,$$

which contradicts (5.3.1). □

5.4 Convergence of sequences of functions

We now consider sequences of functions defined on a fixed domain A.

Definition 5.4.1. *Assume that for each $n \in \mathbb{N}$ we have a function $f_n\colon A \to \mathbb{R}$ and that we have a function $f\colon A \to \mathbb{R}$. Then the sequence $(f_n)_{n\in\mathbb{N}}$ of functions* converges pointwise *"convergeert puntsgewijs" to f if for all $x \in A$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ is convergent and*

$$\forall\, x \in A \qquad \lim_{n\to\infty} f_n(x) = f(x).$$

We say $\lim_{n\to\infty} f_n = f$ with pointwise convergence.

Example 5.4.2. Take the function $f_n\colon [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$. Then we see that, use Exercise 3.4.2,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

so that, even though all the f_n's are continuous functions, the limit function is not continuous. So pointwise convergence is too weak to preserve the notion of continuity.

Definition 5.4.3. *Assume that for each $n \in \mathbb{N}$ we have a function $f_n\colon A \to \mathbb{R}$ and that we have a function $f\colon A \to \mathbb{R}$. Then the sequence $(f_n)_{n\in\mathbb{N}}$ of functions* converges uniformly *"convergeert uniform" to f if*

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n \ge N \quad \forall\, x \in A \qquad |f_n(x) - f(x)| < \varepsilon.$$

We say $\lim_{n\to\infty} f_n = f$ uniformly, or with uniform convergence.

If we reformulate Definition 5.4.1 as in Definition 5.4.3 we get

$$\forall\, x \in A \quad \forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n \ge N \qquad |f_n(x) - f(x)| < \varepsilon,$$

and we see that in Definition 5.4.1 the N will in general depend on both $x \in A$ and on $\varepsilon > 0$, whereas in Definition 5.4.3 the N will only depend on $\varepsilon > 0$ and works for arbitrary $x \in A$. You should check that the sequence in Example 5.4.2 does not converge uniformly.

The importance of uniform convergence of functions, is that it does preserve continuity in the limit. This is the content of Theorem 5.4.4.

Theorem 5.4.4. *Assume that* $\lim_{n\to\infty} f_n = f$ *uniformly and that for all* $n \in \mathbb{N}$ *the function* $f_n : A \to \mathbb{R}$ *is continuous. Then* $f : A \to \mathbb{R}$ *is continuous.*

Note that Theorem 5.4.4 shows that the convergence in Example 5.4.2 is not uniform.

Proof. The proof depends on the inequality

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \tag{5.4.1}$$

which holds for all $x, y \in A$ and for all $n \in \mathbb{N}$. Now take $\varepsilon > 0$ arbitrarily, then, since the convergence is uniform, we have $N \in \mathbb{N}$ so that for all $n \geq N$ and for all $x \in A$ we have

$$|f(x) - f_n(x)| < \frac{1}{3}\varepsilon.$$

We take $n = N$ in (5.3.1), so that

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{2}{3}\varepsilon + |f_N(x) - f_N(y)|$$

and next we use the continuity of f at x to find $\delta > 0$ so that

$$\forall\, y \in A \quad |x - y| < \delta \implies |f_N(x) - f_N(y)| < \frac{1}{3}\varepsilon.$$

Plugging this in, we see that $\exists\, \delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have shown that f is continuous at $x \in A$. Since $x \in A$ is arbitrary, the limit function $f : A \to \mathbb{R}$ is continuous. □

For sequences, convergence is equivalent to being a Cauchy sequence, see Theorem 3.3.11. Theorem 5.4.5 is the analogue of this statement for a sequence of functions converging uniformly.

Theorem 5.4.5. *Let* $A \subset \mathbb{R}$ *and for all* $n \in \mathbb{N}$ *we have a function* $f_n : A \to \mathbb{R}$*. Then the sequence* $(f_n)_{n\in\mathbb{N}}$ *converges uniformly if and only if*

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, m, n \geq N \quad \forall\, x \in A \qquad |f_n(x) - f_m(x)| < \varepsilon.$$

Exercise 5.4.6. The proof of Theorem 5.4.5 can be modelled on the proof of Theorem 3.3.11.

(i) First assume that $\lim_{n\to\infty} f_n = f$ uniformly. Apply Definition 5.4.3 with $\frac{1}{2}\varepsilon$ and use the triangle inequality

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)|$$

to prove the statement.

(ii) Conversely, assume the condition of Theorem 5.4.5. Conclude that for $x \in A$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence, and, by Theorem 3.3.11, a convergent sequence. So $\lim_{n\to\infty} f_n(x) = f(x)$, and we have pointwise convergence $\lim_{n\to\infty} f_n = f$. It remains to show that the convergence is uniform. Show that this is true by showing that one can take the limit $m \to \infty$ in the condition of Theorem 5.4.5.

5.5 Exercises

Exercise 5.5.1. Assume that $A \subset \mathbb{R}$ and that a is a limit point of A. Assume that $f\colon A \setminus \{a\} \to \mathbb{R}$ satisfies $\lim_{x\to a; x\in A\setminus\{a\}} f(x) = L$. Define $g\colon A \to \mathbb{R}$ by $g(a) = L$ and $g(x) = f(x)$ for all $x \in A \setminus \{a\}$. Show that g is continuous in a.

Exercise 5.5.2. Let $a < b$ and let $f\colon (a, b) \to \mathbb{R}$ be a function and let $c \in [a, b]$. We define the left limit as

$$\lim_{x\nearrow c} f(x) = \lim_{x\to c; x\in(a,c)} f(x)$$

and similarly we define the right limit

$$\lim_{x\searrow c} f(x) = \lim_{x\to c; x\in(c,b)} f(x).$$

Show that for $c \in (a, b)$ the limit $\lim_{x\to c; x\in(a,b)\setminus\{c\}} f(x)$ exists if and only if the left and right limit exist and are equal. In that case

$$\lim_{x\to c; x\in(a,b)\setminus\{c\}} f(x) = \lim_{x\searrow c} f(x) = \lim_{x\nearrow c} f(x).$$

Exercise 5.5.3. Let $a < b$ and let $f\colon (a, b) \to \mathbb{R}$ be a function. We say that f is *Lipschitz continuous* “Lipschitzcontinu” if $\exists\, M \in \mathbb{R}$ we have $|f(x) - f(y)| \leq M|x - y|$ for $\forall\, x, y \in (a, b)$.

(i) Show that a Lipschitz continuous function is uniformly continuous.

(ii) Show that $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is Lipschitz continuous.

Exercise 5.5.4. For a function $f\colon A \to \mathbb{R}$ on a domain $A \subset \mathbb{R}$ an element $x \in A$ is a *fixed point* “vast punt” or “dekpunt” if $f(x) = x$.

(i) Let $f\colon [0, 1] \to \mathbb{R}$ be a continuous function with $f(0) = 1$ and $f(1) = 0$. Show that f has a fixed point in $(0, 1)$.

(ii) Let $f\colon [0, 1] \to [0, 1]$ be a continuous function. Show that f has a fixed point.

Exercise 5.5.5. Let $f\colon [0, 1] \to \mathbb{R}$ be a continuous function with $f(0) = f(1)$. Show that there exists $x \in [0, \frac{1}{2}]$ with $f(x) = f(x + \frac{1}{2})$.

Exercise 5.5.6. A *polynomial* “polynoom” or “veelterm” p is a function of the form $p(x) = \sum_{n=0}^{N} a_n x^n$, where we assume $a_N \neq 0$. Then we say that the polynomial p is of *degree* “graad” N. Show that a polynomial is continuous. Hint: e.g. by induction on the degree N using Corollary 5.1.11.

Exercise 5.5.7. Assume that $f\colon (0, \infty) \to \mathbb{R}$ and $L \in \mathbb{R}$.

(i) Give a definition of $\lim\limits_{x\to\infty} f(x) = L$.

(ii) Show that if f is an increasing function, then $\lim_{x\to\infty} f(x) = L$ if and only f is bounded, and in that case $L = \sup_{x\in[0,\infty)} f(x)$. Hint: compare Theorem 3.2.9.

(iii) Assume that $\lim_{x\to\infty} f(x)$ exists, show that $\lim_{x\searrow 0} f(\frac{1}{x})$ exists and

$$\lim_{x\to\infty} f(x) = \lim_{x\searrow 0} f(\frac{1}{x})$$

using the notation of Exercise 5.5.2.

Exercise 5.5.8. We assume that $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the addition formula $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Assume that f is continuous and show that there exists $c \in \mathbb{R}$ with $f(x) = cx$.

Exercise 5.5.9. Give an alternative proof for the Intermediate Value Theorem 5.2.11 using subdivision of the interval $[a, b]$ following the proof of the Bolzano-Weierstrass Theorem 3.2.12 as in Exercise 3.4.9.

Exercise 5.5.10. Assume that $f_n\colon A \to \mathbb{R}$ are bounded functions labelled by $n \in \mathbb{N}$. Assume moreover, that $\lim_{n\to\infty} f_n = f$ uniformly. Show that $f\colon A \to \mathbb{R}$ is a bounded function.

Exercise 5.5.11. Assume that $f_n\colon A \to \mathbb{R}$ are uniformly continuous functions labelled by $n \in \mathbb{N}$. Assume moreover, that $\lim_{n\to\infty} f_n = f$ uniformly. Show that $f\colon A \to \mathbb{R}$ is a uniformly continuous function. Hint: analyse the proof of Theorem 5.4.4.

Exercise 5.5.12. Assume that $f\colon A \to \mathbb{R}$ is uniformly continuous.

(i) Assume that $x_0 \in \overline{A} \setminus A$, show that $\lim_{a\to x_0, a\in A} f(x)$ exists. Hint: use Lemma 5.3.2 and Theorem 3.3.11 to prove this, and make sure that the limit is independent of the choice of the Cauchy sequence.

(ii) Show that there exists a continuous function $g\colon \overline{A} \to \mathbb{R}$ extending f, i.e. $g|_A = f$.

(iii) Show that the continuous function $g\colon \overline{A} \to \mathbb{R}$ extending f as in (ii) is uniquely determined.

Exercise 5.5.13. Prove the following generalisation of Lemma 5.2.2, see also Exercise 5.2.4: let $f\colon A \to \mathbb{R}$ be a continuous function with $A \subset \mathbb{R}$ a compact set. Show that f is bounded. Hint: use Exercise 4.4.8 for the definition of compact set.

Exercise 5.5.14. Corollary 5.2.13 can be stated more generally as follows. Let $A \subset \mathbb{R}$ be a compact set. Let $f\colon A \to \mathbb{R}$ be continuous, then $f(A)$ is a compact set. Give a proof of this statement using the definition of compact set as in Exercise 4.4.8. Hint: use Theorem 5.1.14.

Exercise 5.5.15. Theorem 5.3.10 can be generalised to the following statement. Let $A \subset \mathbb{R}$ be a compact set, and let $f\colon A \to \mathbb{R}$ be a continuous function. Show that f is a uniformly continuous function. Hint: use the definition of Exercise 4.4.8, and proceed as in Exercise 5.5.14.

Exercise 5.5.16. Assume $a < b$ and that $f\colon [a,b] \to \mathbb{R}$ is a continuous strictly increasing function.

(i) Show that $f\colon [a,b] \to [f(a), f(b)]$ is a bijection.

(ii) Show that $f^{-1}\colon [f(a), f(b)] \to [a,b]$ is a strictly increasing function.

(iii) Show that $f^{-1}\colon [f(a), f(b)] \to [a,b]$ is a continuous function.

(iv) Show that $f\colon [0,\infty) \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{N}$, is a continuous strictly increasing function on each interval $[0,R]$, $R > 0$. Hint: see Exercise 5.5.6 for continuity and you can prove that f is strictly increasing by induction on n.

(v) Conclude that for $x \geq 0$ the root $\sqrt[n]{x}$ is uniquely defined.

(vi) Show that $f\colon [0,\infty) \to \mathbb{R}$, $f(x) = \sqrt[n]{x}$ is a continuous strictly increasing function.

Exercise 5.5.17. Let $A \subset \mathbb{R}$, and consider a *complex valued function* "complexwaardige functie" $f\colon A \to \mathbb{C}$. Then f is continuous at $x \in A$ if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \qquad |x-y| < \delta \wedge y \in A \implies |f(x) - f(y)| < \varepsilon,$$

where we use the modulus for complex numbers as in Section 2.1. And $f\colon A \to \mathbb{C}$ is continuous if f is continuous at all points of the domain A.

(i) Show that the complex function $f\colon A \to \mathbb{C}$ is continuous if and only if its imaginary and real parts are continuous functions on A, i.e. $\Im f\colon A \to \mathbb{R}$, $\Re f\colon A \to \mathbb{R}$ defined by $\Im f(x) = \Im(f(x))$, $\Re f(x) = \Re(f(x))$ are continuous (real valued) functions.

(ii) We say that $B \subset \mathbb{C}$ is a *bounded complex set* "begrensde complexe verzameling" if there exists $R \in \mathbb{R}$ so that $B \subset \{z \in \mathbb{C} \mid |z| \leq R\}$. Show the following analogue of Lemma 5.2.2: let $f\colon A \to \mathbb{C}$ be a continuous function and assume that the domain A is sequentially compact, then f is bounded, i.e. $f(A) \subset \mathbb{C}$ is a bounded set.

(iii) Note that we can define a uniformly continuous function $f\colon A \to \mathbb{C}$ as in Definition 5.3.1 using the modulus. Prove the analogue of Theorem 5.3.10: assume $f\colon A \to \mathbb{C}$ be a continuous function on a sequentially compact set, then f is uniformly continuous.

(iv) Having a sequence of functions $f_n\colon A \to \mathbb{C}$, $n \in \mathbb{N}$, on the same domain $A \subset \mathbb{R}$, we can copy the Definition 5.4.3 to the situation of complex valued functions. State and prove the analogues of Theorem 5.4.4 and Theorem 5.4.5.

(v) Are there theorems in Chapter 5 which have no analogue for complex valued functions?

Chapter 6

Differentiable functions

In this chapter we discuss the notion of the derivative of a function. Differentiability is related to having a good linear approximation, and we extend this to higher order differentiability and Taylor polynomial approximation. There is an intimate connection between derivatives and extremal values of functions, which is expressed in Rolle's Theorem 6.2.4 and its important consequence, the Mean Value Theorem 6.2.5. We discuss the relation between differentiability and inverse functions in the Inverse Function Theorem 6.3.1.

6.1 Differentiable functions

Definition 6.1.1. *Let $f\colon A \to \mathbb{R}$ be a function, and assume $x_0 \in A$ is a limit point of A. Then f is* differentiable at x_0 *"differentieerbaar in x_0" if the limit*

$$\lim_{\substack{x \to x_0 \\ x \in A\setminus\{x_0\}}} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. In that case the limit is denoted by $f'(x_0)$, and this is called the derivative of f at x_0 *"afgeleide van f in x_0".*

Remark 6.1.2. (i) Requiring that x_0 is a limit point of the domain A of f, means that we cannot define the derivative of a function in a point of the domain which is not a limit point, i.e. the derivative is not defined in isolated points of the domain.
(ii) By Corollary 5.1.5(i) the derivative of f at x_0 is uniquely defined.
(iii) Differentiability of a function f at a point x_0 is a local property, which follows from the fact that limits of functions only depend on local behaviour, see Lemma 5.1.2. This is formalised in Exercise 6.1.3

Exercise 6.1.3. Assume that $f\colon A \to \mathbb{R}$ is a function, $x_0 \in A$ is a limit point of A and that f is differentiable at x_0. Let $Y \subset A$ and assume that $x_0 \in Y$ is a limit point of Y. Show that the restriction $f|_Y\colon Y \to \mathbb{R}$ is differentiable at x_0 and that $(f|_Y)'(x_0) = f'(x_0)$. Is the converse of this statement valid?

We reformulate the differentiablity in terms of approximation by a linear function. This is known as Newton's approximation, and corresponds to the first order Taylor approximation, see Section 6.4.

Proposition 6.1.4 (Newton approximation)**.** *Let $f\colon A \to \mathbb{R}$ be a function and assume that $x_0 \in A$ is a limit point of A. Then the following statements are equivalent:*

(i) f is differentiable at x_0 with derivative L;

(ii) we have $\forall\, \varepsilon > 0 \quad \exists\, \delta > 0$

$$|x - x_0| < \delta \ \wedge \ x \in A \quad \Longrightarrow \quad |f(x) - \big(f(x_0) + L(x - x_0)\big)| \leq \varepsilon |x - x_0|.$$

Note that we use $\leq$ in (ii) instead of $<$ in order to make sure that the estimate is also valid for $x = x_0$, which is the point excluded in the limit in Definition 6.1.1.

Proof. We just write out the definitions. So (i) is equivalent to

$$\lim_{\substack{x \to x_0 \\ x \in A \setminus \{x_0\}}} \frac{f(x) - f(x_0)}{x - x_0} = L \quad \Longleftrightarrow$$

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad 0 < |x - x_0| < \delta \ \wedge \ x \in A \quad \Longrightarrow \quad \Big|\frac{f(x) - f(x_0)}{x - x_0} - L\Big| < \varepsilon$$

Now we can replace $<$ by $\leq$ in the last estimate. Multiplying the inequality by $(x - x_0)$ we see that the inequality is equivalent to

$$|f(x) - f(x_0) - L(x - x_0)| \leq \varepsilon |x - x_0|$$

for $x \in A$ with $0 < |x - x_0| < \delta$. Since it is trivially valid for $x = x_0$, we have obtained the result. □

Exercise 6.1.5. Use Proposition 6.1.4 to show that the functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = c$ and $g(x) = x$ are differentiable in $x_0 \in \mathbb{R}$. Determine $f'(x_0)$ and $g'(x_0)$.

Corollary 6.1.6. *Let $f\colon A \to \mathbb{R}$ be a function, and $x_0 \in A$ a limit point of A. Assume that f is differentiable in x_0. Then f is continuous in x_0.*

Proof. Using the Newton approximation of Proposition 6.1.4 we see that for all $\varepsilon' > 0$ we can find $\delta' > 0$ so that for all $x \in A$ with $|x - x_0| < \delta'$ we have

$$|f(x) - f(x_0) - L(x - x_0)| \leq \varepsilon' |x - x_0| \quad \Longleftrightarrow$$
$$-\varepsilon' |x - x_0| + L(x - x_0) \leq f(x) - f(x_0) \leq \varepsilon' |x - x_0| + L(x - x_0)$$

so that $|f(x) - f(x_0)| \leq (\varepsilon' + |L|)|x - x_0|$.

To show that f is continuous at x_0, we use Proposition 5.1.10(iii). So pick $\varepsilon > 0$ arbitrary, then we set $\varepsilon' = 1$ and we take the corresponding $\delta' > 0$. We define $\delta = \min(\delta', \frac{\varepsilon}{2(1+|L|)})$, then

$$|f(x) - f(x_0)| \leq (1 + |L|)|x - x_0| \leq (1 + |L|)\delta \leq \frac{1}{2}\varepsilon < \varepsilon. \qquad \square$$

Exercise 6.1.7. Give a proof of Corollary 6.1.6 using Definition 6.1.1 directly. Hint: show that

$$\lim_{\substack{x\to x_0\\ x\in A\setminus\{x_0\}}} f(x)-f(x_0) = \Big(\lim_{\substack{x\to x_0\\ x\in A\setminus\{x_0\}}} \frac{f(x)-f(x_0)}{x-x_0}\Big)\Big(\lim_{\substack{x\to x_0\\ x\in A\setminus\{x_0\}}} x-x_0\Big)$$

Exercise 6.1.8. Show that the function $f\colon \mathbb{R}\to\mathbb{R}$, $f(x)=|x|$ is continuous at 0, and that f is not differentiable at 0. Hint: use Exercise 5.1.12 for the first statement.

There exist continuous functions for which in each point of the domain the derivative does not exist. The first example was constructed by Weierstrass. The standard construction of such a function requires series, as discussed in Chapter 8, see Exercise 4 in Appendix A for an explicit construction.

Next we prove the standard rules for differentation in Theorem 6.1.9.

Theorem 6.1.9. *Let $f\colon A\to\mathbb{R}$, $g\colon A\to\mathbb{R}$ be functions, let $x_0\in A$ be a limit point of A, and assume that f and g are differentiable at x_0. Then*

(i) $af+bg\colon A\to\mathbb{R}$ for $a,b\in\mathbb{R}$ is differentiable in x_0 and $(af+bg)'(x_0)=af'(x_0)+bg'(x_0)$ (sum rule);

(ii) $fg\colon A\to\mathbb{R}$ is differentiable in x_0 and $(fg)'(x_0)=f'(x_0)g(x_0)+f(x_0)g'(x_0)$ (product rule);

(iii) additionally assume $\forall\, x\in A\ g(x)\neq 0$, then $\frac{f}{g}\colon A\to\mathbb{R}$ is differentiable in x_0 and

$$\Big(\frac{f}{g}\Big)'(x_0)=\frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g(x_0)^2}$$

(quotient rule).

The proof of Theorem 6.1.9 is sketched in Exercise 6.1.10.

Exercise 6.1.10. The proof of Theorem 6.1.9 is based on Definition 6.1.1.

(i) Argue first that in order to prove the sum rule, i.e. Theorem 6.1.9(i), it suffices to prove the case $a\in\mathbb{R}$ and $b=0$ and the case $a=b=1$. Show the first case using

$$\frac{(af)(x)-(af)(x_0)}{x-x_0}=a\frac{f(x)-f(x_0)}{x-x_0}$$

and the second case using

$$\frac{(f+g)(x)-(f+g)(x_0)}{x-x_0}=\frac{f(x)-f(x_0)}{x-x_0}+\frac{g(x)-g(x_0)}{x-x_0}$$

using Corollary 5.1.5.

(ii) To prove the product rule, i.e. Theorem 6.1.9(ii), use

$$\frac{(fg)(x)-(fg)(x_0)}{x-x_0} = \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} = \frac{f(x)-f(x_0)}{x-x_0}g(x) + f(x_0)\frac{g(x)-g(x_0)}{x-x_0}$$

and use Corollary 5.1.5 and Corollary 6.1.6.

(iii) To prove the quotient rule, i.e. Theorem 6.1.9(iii), first argue that the general case follows from the case $f(x) = 1$ and the product rule and Exercise 6.1.5. For the remaining special case use

$$\frac{\left(\frac{1}{g}\right)(x)-\left(\frac{1}{g}\right)(x_0)}{x-x_0} = \frac{-1}{g(x)g(x_0)}\frac{g(x)-g(x_0)}{x-x_0}$$

and use Corollary 5.1.5 and Corollary 6.1.6.

The chain rule of Theorem 6.1.11 is a very important result, but is a bit harder to prove and we rely on a reformulation of the Newton approximation of Proposition 6.1.4.

Theorem 6.1.11. *Let $f\colon A \to \mathbb{R}$ be a function, $x_0 \in A$ a limit point of A and assume that f is differentiable at x_0. Moreover, we assume that $f(A) \subset B$, and that $g\colon B \to \mathbb{R}$ is a function and that $y_0 = f(x_0) \in B$ is a limit point of B and that g is differentiable at y_0. Then the composition $g \circ f\colon A \to \mathbb{R}$ is differentiable at x_0 and*

$$(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0).$$

Before starting the proof of the chain rule of Theorem 6.1.11, we rewrite the Newton approximation of Proposition 6.1.4 as follows. Defining

$$u\colon A \setminus \{x_0\} \to \mathbb{R}, \qquad u(x) = \frac{f(x)-f(x_0)}{x-x_0} - f'(x_0) \text{ or } f(x)-f(x_0) = (x-x_0)(f'(x_0)+u(x))$$

we see that Proposition 6.1.4 is equivalent to $\lim_{x\to x_0;x\in A\setminus\{x_0\}} u(x) = 0$. We can extend $u\colon A \to \mathbb{R}$ by defining $u(x_0) = 0$, so that $f(x)-f(x_0) = (x-x_0)(f'(x_0)+u(x))$ remains valid, and we have $\lim_{x\to x_0;x\in A} u(x) = 0$, i.e. u is continuous at x_0.

Proof of Theorem 6.1.11. We define v for g around $y_0 = f(x_0)$ similarly. So we have

$$f(x)-f(x_0) = (x-x_0)(f'(x_0)+u(x)), \qquad g(y)-g(y_0) = (y-y_0)(g'(y_0)+v(y))$$

and $\lim_{x\to x_0;x\in A} u(x) = 0$, $\lim_{y\to y_0;y\in B} v(y) = 0$, where $v\colon B \to \mathbb{R}$. Now we get

$$\begin{aligned} g(f(x)) - g(f(x_0)) &= (f(x)-f(x_0))(g'(y_0)+v(f(x))) \\ = (x-x_0)(f'(x_0)+u(x))(g'(y_0)+v(f(x))) &= (x-x_0)(g'(f(x_0))\, f'(x_0)+U(x)) \end{aligned}$$

with $U(x) = u(x)g'(f(x_0)) + v(f(x))f'(x_0) + u(x)v(f(x))$. Then $U(x_0) = 0$, since $u(x_0) = 0$ and $v(f(x_0)) = v(y_0) = 0$.

It remains to show that U is continuous at x_0. By assumption u is continuous at x_0 and v is continuous at y_0. Since f is differentiable at x_0, f is continuous at x_0 by Corollary 6.1.6. By Exercise 5.1.13, the composition $A \ni x \mapsto v(f(x)) \in \mathbb{R}$ is continuous at x_0. Corollary 5.1.11 shows that U is continuous at x_0. By the remark before starting the proof, this is equivalent to the Newton approximation, so Proposition 6.1.4 shows that $g \circ f$ is differentiable in x_0 and $(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0)$. □

The proof of Theorem 6.1.11 obscures the roles of ε and δ in the Newton approximation of Proposition 6.1.4 a bit. It is an instructive, but tedious, exercise to work out all the details involving the various ε's and δ's.

6.2 Extremal values and the derivative

Recall that in Definition 5.2.5 we have defined the notion of maximum and minimum of a function. We now refine this to a local maximum and a local minimum of a function. Sometimes we refer to the maximum and minimum of Definition 5.2.5 as a a global maximum and a global minimum.

Definition 6.2.1. *Let $f\colon A \to \mathbb{R}$ be a function, then f* has a local maximum in $x_0 \in A$ *"f heeft een lokaal maximum in $x_0 \in A$" if there exists $\delta > 0$ so that*

$$\forall\, x \in A \cap (x_0 - \delta, x_0 + \delta) \qquad f(x) \le f(x_0)$$

and f has a local minimum in x_0 *"f heeft een lokaal minimum in $x_0 \in A$" if there exists $\delta > 0$ so that*

$$\forall\, x \in A \cap (x_0 - \delta, x_0 + \delta) \qquad f(x) \ge f(x_0).$$

Note that we can rephrase Definition 6.2.1 as follows: the function $f\colon A \to \mathbb{R}$ has a local maximum at x_0 if and only if $\exists\, \delta > 0$ so that $f|_{A\cap(x_0-\delta,x_0+\delta)}$ has a (global) maximum in the sense of Definition 5.2.5. A similar statement holds for the local minimum.

Proposition 6.2.2. *Assume that the function $f\colon A \to \mathbb{R}$ is differentiable at $x_0 \in A$, and that f attains a local maximum or a local minimum at x_0. If $x_0 \in A^\circ$, i.e. x_0 is an interior point of the domain, then $f'(x_0) = 0$.*

Proof. We assume that f attains a local maximum. The case of a local minimum can be proved similarly or we can reduce to this case by considering $-f$.

Since x_0 is an interior point of A, we can assume that there exists $\delta > 0$ so that $(x_0 - \delta, x_0 + \delta) \subset A$, and then we see that numerator of

$$\frac{f(x) - f(x_0)}{x - x_0}$$

is always non-positive, $f(x) - f(x_0) \le 0$ for all $|x - x_0| < \delta$. Since the limit

$$\lim_{x \to x_0, x \in (x_0 - \delta, x_0 + \delta)\setminus\{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} = L$$

exists, we can take any sequence $(x_n)_{n\in\mathbb{N}}$ of elements in $(x_0-\delta, x_0+\delta)\setminus\{x_0\}$ converging to x_0 in order to calculate the limit, see Proposition 5.1.4. First take a sequence converging to x_0 for which $x_n < x_0$ for all $n\in\mathbb{N}$, so that the denominator is negative. It follows that $\frac{f(x_n)-f(x_0)}{x_n-x_0}\geq 0$ for all $n\in\mathbb{N}$, so that taking the limit shows $L\geq 0$ by Theorem 3.2.19(vi). Similarly, taking a sequence converging to x_0 with $x_n > x_0$ for all $n\in\mathbb{N}$ shows similarly that $L\leq 0$. Combining the statements gives $L=0$, or $f'(x_0)=0$. □

Note that the proof makes essential use of the fact that x_0 is an interior point of the domain, and the statement of Proposition 6.2.2 is not valid if we drop this assumption. An easy example is $f\colon[0,1]\to\mathbb{R}$, $f(x)=x$, which has a local maximum at $x_0=1$ and the derivative at 1 is not equal to 0.

Definition 6.2.3. *Let $f\colon A\to\mathbb{R}$ be a function, then f is called* differentiable *"differentieerbaar" if for all $x_0\in A$ the function f is differentiable at x_0. Then the derivative is defined as a function $f'\colon A\to\mathbb{R}$.*

In general, for a function $f\colon A\to\mathbb{R}$ we can define the derivative as a function $f'\colon B\to\mathbb{R}$ where B is the subset of A defined by $B=\{x_0\in A\mid f \text{ differentiable at } x_0\}$.

Theorem 6.2.4 (Rolle)**.** *Let $a<b$ and $g\colon[a,b]\to\mathbb{R}$ be a continuous function. Assume moreover that g is differentiable on (a,b). If $g(a)=g(b)$, then there exists $c\in(a,b)$ with $g'(c)=0$.*

Proof. In case $g\colon[a,b]\to\mathbb{R}$ is constant, i.e. $g(x)=g(a)=g(b)$ for all $x\in[a,b]$, we can take any $c\in(a,b)$, cf. Exercise 6.1.5.

In case g is not constant, it attains a maximum and a minimum by Proposition 5.2.6, and least one of them is not attained in a or b. So there exists $c\in(a,b)$ where g attains a maximum or a minimum, so that by Proposition 6.2.2 we have $g'(c)=0$. □

Rolle's Theorem 6.2.4 has important consequences, the most important one being the Mean Value Theorem.

Theorem 6.2.5 (Mean Value Theorem "Middelwaardestelling")**.** *Let $a<b$ and assume $f\colon[a,b]\to\mathbb{R}$ is a continuous function, which is differentiable on (a,b). Then there exists $c\in(a,b)$ so that*

$$\frac{f(b)-f(a)}{b-a}=f'(c).$$

Proof. We reduce to Rolle's Theorem 6.2.4 by considering

$$g\colon[a,b]\to\mathbb{R},\qquad g(x)=f(x)-\Big(\frac{f(b)-f(a)}{b-a}(x-a)+f(a)\Big).$$

Since we modify by a linear function, which is differentiable by Exercise 6.5.1, we see that g is continuous on $[a,b]$ and differentiable on (a,b) using Theorem 6.1.9. Now $g(a)=0$ and $g(b)=0$, so that all conditions of Rolle's Theorem 6.2.4 are met, and so we find $c\in(a,b)$ with $g'(c)=0$. Rewriting in terms of f gives the result. □

The Mean Value Theorem 6.2.5 has a generalisation, which we customarily also call a theorem. Its importance is not as high as the Mean Value Theorem. The generalisation is also known as Cauchy's Mean Value Theorem.

Theorem 6.2.6 (Generalised Mean Value Theorem "Gegeneraliseerde Middelwaardestelling")**.** *Let $a < b$ and assume $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ are continuous functions, so that f and g are differentiable on (a,b). Assume $g(a) \neq g(b)$, and that $g'(x) \neq 0$ for all $x \in (a,b)$. Then there exists $c \in (a,b)$ so that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Exercise 6.2.7. Before checking the proof of Theorem 6.2.6, you should check the mistake in the following 'proof'. Apply the Mean Value Theorem 6.2.5 to f and to g and conclude

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f(b)-f(a)}{b-a}\frac{b-a}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof of Theorem 6.2.6. We reduce to Rolle's Theorem 6.2.4 by introducing an auxiliary function $h\colon [a,b] \to \mathbb{R}$ defined by

$$h(x) = (g(x)-g(b))(f(a)-f(b)) - (f(x)-f(b))(g(a)-g(b)).$$

Since h is a linear combination of f and g it is continuous by Corollary 5.1.11 and differentiable on (a,b) by Theorem 6.1.9. Since $h(a) = 0$ and $h(b) = 0$, we find by Rolle's Theorem 6.2.4 a $c \in (a,b)$ with $h'(c) = 0$ or

$$0 = h'(c) = g'(c)(f(a)-f(b)) - f'(c)(g(a)-g(b)).$$

Since $g(a) \neq g(b)$ and $g'(c) \neq 0$, the result follows. □

Note that if we drop the conditions $g(a) \neq g(b)$ or that g' is non-zero in Theorem 6.2.6, we can still conclude the existence of $c \in (a,b)$ for which we have

$$g'(c)(f(a)-f(b)) = f'(c)(g(a)-g(b)). \tag{6.2.1}$$

As a corollary we can formulate one of the many versions of l'Hôpital's rule for limits.

Corollary 6.2.8. *Assume that $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$ are continuous and differentiable functions. Let $x_0 \in (a,b)$ and assume $f(x_0) = 0 = g(x_0)$ and $g'(x) \neq 0$ for all $x \in (a,b)$. Assume that*

$$\lim_{x\to x_0, x\in(a,b)\setminus\{x_0\}} \frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to x_0, x\in(a,b)\setminus\{x_0\}} \frac{f(x)}{g(x)} = L.$$

Proof. First observe that $g(x) \neq 0$ for $x \in (a,b) \setminus \{x_0\}$. Indeed, if $g(x) = 0$, then $g|_{[x,x_0]}$ (in case $x < x_0$) would satisfy the conditions of Rolle's Theorem 6.2.4, so that there would exist $c \in (x, x_0)$ with $g'(c) = 0$, contradicting the assumptions on g.

Then we can apply the Generalised Mean Value Theorem 6.2.6 to get

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c)}{g'(c)}$$

for some c strictly between x and x_0. Since $\lim_{x \to x_0, x \in (a,b)} \frac{f'(x)}{g'(x)} = L$ we know that for any $\varepsilon > 0$ there exists $\delta > 0$ so that $|c - x_0| < \delta$ and $x \in (a,b)$ implies

$$|\frac{f'(c)}{g'(c)} - L| < \varepsilon.$$

So for $0 < |x - x_0| < \delta$ and $x \in (a,b)$ we have

$$|\frac{f(x)}{g(x)} - L| = |\frac{f'(c)}{g'(c)} - L| < \varepsilon$$

since $|c - x_0| < \delta$ as c is between x and x_0. This proves $\lim_{x \to x_0, x \in (a,b) \setminus \{x_0\}} \frac{f(x)}{g(x)} = L$. □

6.3 The inverse function theorem

Theorem 6.3.1 (Inverse Function Theorem "Inversefunctiestelling")**.** *Let $f \colon A \to B$, A and B subsets of $\mathbb{R}$, be a bijection. Let $f^{-1} \colon B \to A$ be the inverse function. Assume that x_0 is a limit point of A and that f is differentiable at x_0 and $f'(x_0) \neq 0$. Let $y_0 = f(x_0)$ and assume that f^{-1} is continuous at y_0. Then f^{-1} is differentiable at y_0 and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

In case we already know that f^{-1} is differentiable at y_0, then the result follows from the chain rule, Theorem 6.1.11. Indeed, differentiating the composition $f^{-1} \circ f$ at x_0 gives

$$f^{-1}(f(x)) = x \quad \Longrightarrow \quad (f^{-1})'(f(x_0))\, f'(x_0) = 1$$

implying the result.

Proof. Firstly, taking a sequence $(x_n)_{n=1}^{\infty}$ in $A \setminus \{x_0\}$ converging to x_0, it follows by continuity of f at x_0 –Corollary 6.1.6– that $\lim_{n \to \infty} f(x_n) = f(x_0) = y_0$. Since $f(x_n) \in B$ for all n, and by bijectivity $f(x_n) \neq f(x_0)$, we see that y_0 is a limit point of B.

It remains to show that

$$\lim_{\substack{y \to y_0 \\ y \in B \setminus \{y_0\}}} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}.$$

By Proposition 5.1.4 it suffices to show that for any sequence $(y_n)_{n=1}^\infty$ with $y_n \in B \setminus \{y_0\}$ for all $n \geq 1$ and $\lim_{n\to\infty} y_n = y_0$ that

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \frac{1}{f'(x_0)}.$$

So we pick any such sequence, and we find $x_n \in A \setminus \{x_0\}$ with $f(x_n) = y_n$. This gives a sequence $(x_n)_{n=1}^\infty$ in $A \setminus \{x_0\}$ by bijectivity. Moreover, since f^{-1} is continuous at y_0 we have by Proposition 5.1.10

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f^{-1}(y_n) = f^{-1}(y_0) = x_0.$$

So now we can rewrite

$$\frac{f^{-1}(y_n) - f(y_0)}{y_n - y_0} = \frac{x_n - x_0}{y_n - y_0} = \frac{1}{\frac{y_n - y_0}{x_n - x_0}} = \frac{1}{\frac{f(x_n) - f(x_0)}{x_n - x_0}}$$

Note that for any n the numerator and denominator of all these fractions are non-zero. Since the $\lim_{n\to\infty} \frac{f(x_n)-f(x_0)}{x_n - x_0}$ exists and is $f'(x_0) \neq 0$, we can apply Theorem 3.2.19(v). This gives

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f(y_0)}{y_n - y_0} = \frac{1}{\lim_{n\to\infty} \frac{f(x_n)-f(x_0)}{x_n - x_0}} = \frac{1}{f'(x_0)}.$$ □

6.4 Higher order differentiability and Taylor approximation

We can reformulate the Mean Value Theorem 6.2.5 as

$$f(x) = f(c) + f'(t)(x - c),$$

i.e. we approximate the value of f at x with the constant function $f(c)$ and then we make an error which is expressed as $(x - c)$ times the derivative at some intermediate point t. We study this for higher degree polynomials, which we define now.

Definition 6.4.1. *Let $f\colon (a, b) \to \mathbb{R}$ be a function for which the derivatives f', f'', f''', until $f^{(n)}$, for some $n \in \mathbb{N}$ exist as functions on (a, b). Let $c \in (a, b)$, then the* Taylor polynomial of degree n for f at c *"Taylorpolynonoom of Taylorveelterm van graad n voor f rond c" is*

$$T_n(x) = T_n(x; f, c) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k$$

$$= f(c) + f'(c)(x - c) + \frac{1}{2}f''(c)(x - c)^2 + \frac{1}{6}f'''(c)(x - c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

The case $n = 1$ for differentiable f gives $T_1(x; f, c) = f(c) + f'(c)(x - c)$ whose graph is the tangent line to the graph of f in $(c, f(c))$. Then Proposition 6.1.4 shows how well the linear Taylor polynomial, i.e. of degree 1, approximates f.

The question to be answered is how well the Taylor polynomial approximates the original function f.

Theorem 6.4.2 (Taylor approximation "Taylorbenadering")**.** *Assume that $f\colon (a, b) \to \mathbb{R}$ has derivatives up to order $n + 1$. Let $c \in (a, b)$ and put*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k + E_n(x) = T_n(x; f, c) + E_n(x; f, c)$$

then have the following expressions for the remainder $E_n(x) = E_n(x; f, c)$. The Cauchy form of the remainder says that for any $p \in \mathbb{N}$, $p \geq 1$, there exists t between c and x so that

$$E_n(x; f, c) = \frac{f^{(n+1)}(t)}{n!\, p}(x - c)^p (x - t)^{n+1-p}.$$

The case $p = n + 1$ gives the Lagrange form of the remainder; there exists t between c and x so that

$$E_n(x; f, c) = \frac{f^{(n+1)}(t)}{(n + 1)!}(x - c)^{n+1}$$

Note that the Lagrange form of the remainder is almost the same as the next term in the Taylor polynomial except that the $n + 1$-th derivative has to be evaluated at an intermediate point t instead of c. Another form for the remainder in terms of an integral is given in Corollary 7.4.7.

Corollary 6.4.3. *Assume that $f\colon (a, b) \to \mathbb{R}$ has derivatives up to order $n + 1$ and that $f^{(n+1)}\colon (a, b) \to \mathbb{R}$ is bounded by M, then for all $x \in (a, b)$*

$$|f(x) - T_n(x; f, c)| \leq \frac{M}{(n + 1)!}|x - c|^{n+1}.$$

Remark 6.4.4. Assume that we have two functions defined on some interval (a, b) containing $c \in (a, n)$, then we say that $f(x) = \mathcal{O}(g(x))$ as $x \to c$ if there exists a constant M so that

$$|f(x)| \leq M|g(x)|, \qquad \forall\, x \in (a, b),\ 0 < |x - c| < \delta$$

for some $\delta > 0$. The $\mathcal{O}$ stands for "order", and the notation was introduced by Landau. It is customary to drop x, when it is clear from the context, so then one has $f = \mathcal{O}(g)$ as $x \to c$. So we can rephase Corollary 6.4.3 as

$$f(x) = T_n(x; f, c) + \mathcal{O}((x - c)^{n+1}).$$

Exercise 6.4.5. (i) Assume that $f_1(x) = \mathcal{O}(g_1(x))$ and $f_2(x) = \mathcal{O}(g_2(x))$ as $x \to c$. Show that $\mathcal{O}(f_1 f_2) = g_1 g_2$ as $x \to c$.

(ii) Assume that $f_1(x) = \mathcal{O}(g(x))$ and $f_2(x) = \mathcal{O}(g(x))$ as $x \to c$. Show that $f_1 + f_2 = \mathcal{O}(g)$. What can you say in case about the order of $f_1 + f_2$ in case $f_1(x) = \mathcal{O}(g_1(x))$ and $f_2(x) = \mathcal{O}(g_2(x))$ as $x \to c$?

(iii) Assume that f and g are functions on an interval (a, b) containing $c \in (a, b)$. Assume that f and g are functions satisfying the conditions of Corollary 6.4.3, and that for $k \in \mathbb{N}$ with $k < n - 1$ we have $g^{(r)}(c) = 0$ for $r \in \mathbb{N}$ and $0 \leq r \leq k$ and $g^{(k+1)}(c) \neq 0$. Assume that $f^{(r)}(c) = 0$ for $r \in \{0, \cdots, k\}$, show that

$$\lim_{x \to c; x \in (a,b) \setminus \{c\}} \frac{f(x)}{g(x)} = \frac{f^{(k+1)}(c)}{g^{(k+1)}(c)}.$$

You should compare this to l'Hôpital's rule of Corollary 6.2.8.

Example 6.4.6. Consider $f(x) = (1 - x)^\alpha$, then we prove by induction on n that

$$f^{(n)}(x) = \Big(\prod_{i=0}^{n-1} (-\alpha + i)\Big)(1 - x)^{\alpha - n} \quad \Longrightarrow \quad T_n(x; f, 0) = \sum_{k=0}^{n} \frac{1}{k!}\Big(\prod_{i=0}^{k-1} (-\alpha + i)\Big)x^k$$

and for $|x| < \frac{1}{2}$ and $n > \alpha$ we have $|f^{(n+1)}(x)| \leq \big|\prod_{i=0}^{n} (-\alpha + i)\big| 2^{n+1-\alpha}$. So Corollary 6.4.3 gives for $|x| < \frac{1}{2}$ and $n > \alpha$

$$|(1 - x)^\alpha - \sum_{k=0}^{n} \frac{1}{k!}\Big(\prod_{i=0}^{k-1} (-\alpha + i)\Big)x^k| \leq \Big|\prod_{i=0}^{n} (-\alpha + i)\Big| \frac{2^{n+1-\alpha}}{n!} |x|^{n+1}.$$

Proof of Theorem 6.4.2. Since we take $c \in (a, b)$ arbitrary, we can also view the Taylor polynomial (in x) as a function of c;

$$F \colon (a, b) \to \mathbb{R}, \qquad F(c) = f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k$$

so that $F(x) = f(x)$ and $F(c) = T_n(x; f, c)$ so that for the remainder $E_n(x; f, c)$ we find

$$F(x) - F(c) = E_n(x; f, c).$$

Since f has derivatives up to order $n+1$, and the other terms involving c are polynomials, we see that F is differentiable. Using Exercise 6.5.1 and Theorem 6.1.9 we calculate the derivative of F by

$$\begin{aligned} F'(c) &= f'(c) + \sum_{k=1}^{n} \Big(\frac{f^{(k+1)}(c)}{k!}(x - c)^k - k\frac{f^{(k)}(c)}{k!}(x - c)^{k-1} \Big) \\ &= f'(c) + \sum_{k=2}^{n+1} \frac{f^{(k)}(c)}{(k-1)!}(x - c)^{k-1} - \sum_{k=1}^{n} \frac{f^{(k)}(c)}{(k-1)!}(x - c)^{k-1} = \frac{f^{(n+1)}(c)}{n!}(x - c)^n. \end{aligned}$$

Now we take a function $G\colon (a,b) \to \mathbb{R}$, which is continuous and differentiable. Then we can apply the Generalised Mean Value Theorem 6.2.6 in the form of (6.2.1) to find a t between c and x such that

$$G'(t)(F(x) - F(c)) = F'(t)(G(x) - G(c)).$$

In particular, if we additionally assume that $G'(t) \neq 0$ for t between c and x we have an expression

$$E_n(x; f, c) = F(x) - F(c) = \big(G(x) - G(c)\big)\frac{F'(t)}{G'(t)}$$

for the remainder $E_n(x; f, c)$ and it remains to choose G carefully. Take $G(y) = (x-y)^p$, for $p \in \mathbb{N}$, $p \geq 1$, so that

$$E_n(x; f, c) = -(x-c)^p \frac{f^{(n+1)}(t)}{n!}(x-t)^n \frac{1}{-p(x-t)^{p-1}} = \frac{f^{(n+1)}(t)}{n!\, p}(x-c)^p(x-t)^{n+1-p}. \quad \square$$

6.5 Exercises

Exercise 6.5.1. We show that polynomials, see Exercise 5.5.6 are differentiable.

(i) Show that $f_n(x) = x^n$, $n \in \mathbb{N}$, $f_n\colon \mathbb{R} \to \mathbb{R}$, is differentiable and $f_n'(x) = nx^{n-1}$. Give a proof using induction on n and Theorem 6.1.9 or give a proof using Newton's binomial summation (2.1.3), cf proof of Theorem 8.4.1.

(ii) Let P_N be the space of polynomials on $\mathbb{R}$ of degree at most N. Show that P_N is a finite-dimensional real vector space.

(iii) Show that taking derivatives gives a linear map $\frac{d}{dx}\colon P_N \to P_N$.

(iv) Show that the linear map $\frac{d}{dx}\colon P_N \to P_N$ is nilpotent. (Recall that a linear map $T\colon V \to V$ on a vector space is nilpotent if there exists $k \in \mathbb{N}$ so that $T^k = 0$.)

Exercise 6.5.2. Let $a < b$ and $f\colon [a,b] \to \mathbb{R}$ be a function. Give a suitable definition of the left and right derivative of f at a and b. Explain why in Proposition 6.2.2 the requirement that x_0 is an interior point is necessary. Establish an example in which a maximum is obtained in the endpoint a.

Exercise 6.5.3. Define $f_n\colon [-1,1] \to \mathbb{R}$ by $f_n(x) = \sqrt{x^2 + \frac{1}{(n+1)^2}}$ for $n \in \mathbb{N}$.

(i) Show that f_n is a differentiable function.

(ii) Show that $\lim_{n\to\infty} f_n$ converges uniformly, and determine the limit.

(iii) Show that the uniform limit of differentiable functions is not necessarily a differentiable function, cf. Theorem 5.4.4 which states that continuity is preserved under uniform convergence.

See Theorem 7.5.2 for a partial answer.

Exercise 6.5.4. Let $f\colon A \to \mathbb{R}$ be a function on an open set A. Then f is a continuously differentiable "continu differenteerbare" function if f is differentable and $f'\colon A \to \mathbb{R}$ is continuous. The class of continuously differentiable functions is denoted by $C^1(A)$. Inductively, $f\colon A \to \mathbb{R}$ is k-times continuously differentiable ($k \in \mathbb{N}$, $k \geq 2$) if f is differentiable and $f'\colon A \to \mathbb{R}$ is $k-1$-times continuously differentiable, where a 1-times continuously differentiable function is a continuously differentiable function. The class of k-times continuously differentiable functions is denoted by $C^k(A)$.

We conventially define $C^0(A)$ as the space of continuous functions on A. So $f \in C^k(A)$ indicates how smooth a function is. These kind of spaces can also be defined for more general sets.

(i) Show that $C^k(A)$ is a real vector space.

(ii) Show that $C^{k+1}(A) \subset C^k(A)$.

(iii) Show that $f(x) = |x|$ defines a function in $C^0(\mathbb{R}) \setminus C^1(\mathbb{R})$.

(iv) More generally, set $f(x) = |x|^{k+1}$, $k \in \mathbb{N}$. In which space is this function contained?

(v) Show that for each $N \in \mathbb{N}$ we have $P_N \subset C^\infty(A) = \bigcap_{n\in\mathbb{N}} C^k(A)$ with P_N as in Exercise 6.5.1.

Exercise 6.5.5. Assume that $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$ are functions having derivatives up to order n. Show Leibniz's formula

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)$$

generalising the product formula, see Theorem 6.1.9. Hint: use induction on n.

Exercise 6.5.6. Give a proof of Proposition 6.2.2 using Proposition 6.1.4.

Exercise 6.5.7. Let $a < b$ and $f\colon [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on (a,b). Assume that $f'(c) = 0$ for all $c \in (a,b)$. Show that f is a constant function.

Exercise 6.5.8. Let $f, g, h\colon [a,b] \to \mathbb{R}$ be continuous functions and assume that f, g, h are differentiable on (a,b). Consider

$$d(x) = \det \begin{pmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{pmatrix}.$$

Show that there exists $c \in (a,b)$ with

$$f'(c)\big(g(a)h(b) - h(a)g(b)\big) - g'(c)\big(f(a)h(b) - h(a)f(b)\big) + h'(c)\big(f(a)g(b) - g(a)f(b)\big) = 0.$$

Exercise 6.5.9. Assume that $f\colon [a,b] \to \mathbb{R}$ is continuous and that f is differentiable on (a,b).

(i) Assume that $f'(c) > 0$ for all $c \in (a,b)$. Show that f is a strictly increasing function.

(ii) Assume that $f'(c) \geq 0$ for all $c \in (a,b)$. Show that f is an increasing function.

(iii) Assume that f is a strictly increasing function. Can we conclude that $f'(c) > 0$ for all $c \in (a,b)$?

Exercise 6.5.10. Building on Exercise 5.5.16 show that the function $f\colon [0,\infty) \to \mathbb{R}$ defined as $f(x) = \sqrt[n]{x}$ is differentiable and calculate its derivative. Hint: use Theorem 6.3.1.

Exercise 6.5.11. We sketch another proof of Theorem 6.4.2 with the Lagrange form of the remainder using the Mean Value Theorem 6.2.5.

(i) We assume $c < x$, the other case can be done similarly. Define an auxiliary function

$$h(x) = f(x) - T_n(x; f, c) - C\frac{(x-c)^{n+1}}{(n+1)!}$$

for a constant C to be determined later. Show that h has derivatives up to order $n+1$. Show that $h(c) = 0$, $h'(c) = 0$, $\cdots$, $h^{(n)}(c) = 0$.

(ii) Pick C so that $h(x) = 0$. Show that there exists $t_1 \in (c,x)$ with $h'(t_1) = 0$. Hint: Mean Value Theorem 6.2.5.

(iii) Show that for $r \in \{1, \cdots, n\}$ there exists $t_{r+1} \in (c, t_r)$ with $h^{(r+1)}(t_{r+1}) = 0$. Hint: Mean Value Theorem 6.2.5 on $h^{(r)}$ and induction on r.

(iv) Show that $t = t_{n+1}$ gives the Lagrange form of the remainder of Theorem 6.4.2.

Exercise 6.5.12. Let $f\colon I \to \mathbb{R}$, I an open interval, $a \in I$. Assume that f is differentiable and that $f'\colon I \to \mathbb{R}$ is differentiable at a. Show that

$$\lim_{h\to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = f''(a)$$

Hint: Show first $\lim_{h\to 0} \frac{f'(a+h)-f'(a-h)}{h} = 2f''(a)$ and use l'Hôpital's rule, see Corollary 6.2.8.

Exercise 6.5.13. Assume we have functions $f, g\colon \mathbb{R} \to \mathbb{R}$. We say $f = \mathcal{O}(g)$ as $x \to \infty$ if there exists $M > 0$ and $x_0 \in \mathbb{R}$ so that for $x \geq x_0$ we have $|f(x)| \leq M|g(x)|$. And we say $f = o(g)$ as $x \to \infty$ if for all $\varepsilon > 0$ there exists $x_0 \in \mathbb{R}$ so that for $x \geq x_0$ we have $|f(x)| \leq \varepsilon |g(x)|$.

(i) Show that if $f = o(g)$ as $x \to \infty$, then $f = \mathcal{O}(g)$ as $x \to \infty$.

(ii) Give an example that the converse of (i) is not true.

Chapter 7

The Riemann integral

In this chapter we define integration in the sense of the Riemann integral. The idea is to define the integral of a piecewise constant function on a bounded interval first. We define what Riemann integrability is in terms of the integral of piecewise constant functions. We show that continuous functions are Riemann integrable. The Fundamental Theorem of Calculus is proved, and we discuss various consequences such as some of the classical integration rules. We show that the uniform limit of Riemann integrable functions is again Riemann integrable, and that limit and integral can be interchanged in this case. As an application we prove a result on the interplay between differentiation and uniform convergene of a sequence of functions.

7.1 Piecewise constant functions

Definition 7.1.1. *The* length *"lengte" $|I|$ of a bounded interval I is defined as $b-a$ whenever $b \geq a$ and $I = (a,b)$, $I = [a,b)$, $I = (a,b]$ or $I = [a,b]$.*

So the length of an interval is not influenced by whether the endpoints are contained in the interval or not. Note that for a non-empty bounded interval we have $|I| = \sup_{x\in I} x - \inf_{x\in I} x$. We follow the convention that $|I| = 0$ if $b < a$, i.e. $|\emptyset| = 0$.

Definition 7.1.2. *Let $I \subset \mathbb{R}$ be a bounded interval. A* partition *"partitie" is a finite collection $\mathscr{P}$ of intervals $J \subset I$ so that*

$$\forall\, x \in I \quad \exists!\, J \in \mathscr{P} \qquad x \in J.$$

Partitions of intervals behave well with respect to the length.

Proposition 7.1.3. *Let $I \subset \mathbb{R}$ be a bounded interval, and let $\mathscr{P}$ be a partition of I. Then*

$$|I| = \sum_{J\in\mathscr{P}} |J|$$

Exercise 7.1.4. We sketch a proof of Proposition 7.1.3. We introduce notation for a non-empty bounded interval J, namely $b_J = \inf_{x \in J} x$ is left endpoint of J and $e_J = \sup_{x \in J} x$ is the right endpoint. Since $|\emptyset| = 0$ by convention, we reformulate Proposition 7.1.3 as

$$|I| = \sum_{J \in \mathscr{P}} |J| = \sum_{J \in \mathscr{P}; J \neq \emptyset} |J| = \sum_{J \in \mathscr{P}; J \neq \emptyset} (e_J - b_J).$$

Now prove that $\sum_{J \in \mathscr{P}; J \neq \emptyset} (e_J - b_J) = e_I - b_I$ by induction on the number of elements of the partition $\mathscr{P}$.

(i) Prove the statement in case that $\mathscr{P}$ has 1 element.

(ii) Prove the induction step in case $b_I \in I$ (or $e_I \in I$).

(iii) Prove the induction step in case $b_I \notin I$ (or $e_I \notin I$).

We can order the endpoints and beginpoints of the intervals contained in $\mathscr{P}$ in an increasing sequence $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_N$ so that $x_0 = b_I$, $x_N = e_I$, and for all $I \in \mathscr{P}$, $I \neq \emptyset$, there exists $i \in \{1, \cdots, N\}$ with $b_I = x_{i-1}$, $e_I = x_i$ and for any $i \in \{0, 1, \cdots, N\}$ there exists $I \in \mathscr{P}$ so that $x_i = e_I$ or $x_i = b_I$.

Definition 7.1.5. *Let $I \subset \mathbb{R}$ be a bounded interval, and let $\mathscr{P}$ and $\mathscr{Q}$ be partitions of I. The partition $\mathscr{Q}$ is* finer *"fijner" than the partition $\mathscr{P}$, or equivalently the partition $\mathscr{P}$ is* coarser *"grover" than the partition $\mathscr{Q}$, if*

$$\forall J \in \mathscr{Q} \quad \exists K \in \mathscr{P} \qquad J \subset K.$$

Note that $\mathscr{P}$ being a partition implies that the $K \in \mathscr{P}$ with $J \subset K$ is uniquely determined for non-empty $J \in \mathscr{Q}$.

Proposition 7.1.6. *Let I be a bounded interval with partitions $\mathscr{P}$ and $\mathscr{Q}$, then*

$$\mathscr{P} \# \mathscr{Q} = \{K \cap J \mid K \in \mathscr{P}, J \in \mathscr{Q}\}$$

is a partition of I which is finer than $\mathscr{P}$ and finer than $\mathscr{Q}$.

The partition $\mathscr{P} \# \mathscr{Q}$ is the *common refinement* "gemeenschappelijke verfijning" of the partitions $\mathscr{P}$ and $\mathscr{Q}$.

Proof. Note that for two intervals the intersection is again an interval, where we consider the empty set as an interval as well. Moreover, $\mathscr{P} \# \mathscr{Q}$ has a finite number of elements. Take $x \in I$, then there exists a unique interval $K \in \mathscr{P}$ and a unique interval $J \in \mathscr{Q}$ with $x \in K$ and $x \in J$. So $K \cap J$ is the unique interval in $\mathscr{P} \# \mathscr{Q}$ containing x.

Since $K \cap J \subset K$ and $K \cap J \subset J$, we have that $\mathscr{P} \# \mathscr{Q}$ is a refinement of both $\mathscr{P}$ and $\mathscr{Q}$. $\square$

Definition 7.1.7. *Let $I \subset \mathbb{R}$ be a bounded interval. The function $f\colon I \to \mathbb{R}$ is a* piecewise constant *"stukgewijs constante" function if there exists a partition $\mathcal{P}$ of I and a function $c\colon \mathcal{P} \to \mathbb{R}$ so that for all $x \in J$ we have $f(x) = c_J$. We put*

$$\mathrm{PC}(I) = \{f\colon I \to \mathbb{R} \mid f \text{ piecewise constant}\}.$$

Note that this is well-defined, since each $x \in I$ is contained in exactly one $J \in \mathcal{P}$. This means that $f|_J(x) = c_J$ for all $x \in J$. Furthermore, a piecewise constant function can have various partitions for which it is constant, e.g. any finer partition will also do. The range of a piecewise constant function is a finite set; $f(I) = \{c_J \mid J \in \mathcal{P}, J \neq \emptyset\}$. In particular, any $f \in \mathrm{PC}(I)$ is bounded.

Exercise 7.1.8. Assume that $f \in \mathrm{PC}(I)$ and assume that $f\colon I \to \mathbb{R}$ is piecewise constant with respect to the partition $\mathcal{P}$ of I. Let $\mathcal{Q}$ be a partition of I which is finer than $\mathcal{P}$. Show that f is piecewise constant with respect to the partition $\mathcal{Q}$. Hint: show that for a non-empty interval $J \in \mathcal{P}$ there exists a partition $\mathcal{Q}_J$ of J of intervals contained in $\mathcal{Q}$ and $\mathcal{Q} = \bigcup_{J\in\mathcal{P}} \mathcal{Q}_J$ as a disjoint union. Now define $d\colon \mathcal{Q} \to \mathbb{R}$ as $d_K = c_J$ if $K \in \mathcal{Q}_J$.

Exercise 7.1.9. Show that $\mathrm{PC}(I)$, for $I \subset \mathbb{R}$ a bounded interval, is a vector space, which is closed under multiplication. Hint: take the common refinement of the partition $\mathcal{P}$ for $f \in \mathrm{PC}(I)$ and the partition $\mathcal{Q}$ for $g \in \mathrm{PC}(I)$ to have a partition for $f+g$ and fg.

Definition 7.1.10. *Let $I \subset \mathbb{R}$ be a bounded interval, and $f\colon I \to \mathbb{R}$ a piecewise constant function for the partition $\mathcal{P}$, then we define the* integral of f over I *"de integraal van f over het interval I" by*

$$\mathrm{pc}\!\int_{I;[\mathcal{P}]} f(x)\,dx = \sum_{J\in\mathcal{P}} c_J |J|$$

using the notation as in Definition 7.1.7.

A first step is to show that the definition of the integral of a piecewise continuous function is independent of the choice of partition.

Lemma 7.1.11. *Let $I \subset \mathbb{R}$ be a bounded interval, and $f\colon I \to \mathbb{R}$ a piecewise constant function for the partition $\mathcal{P}$ and for the partition $\mathcal{Q}$, then*

$$\mathrm{pc}\!\int_{I;[\mathcal{P}]} f(x)\,dx = \mathrm{pc}\!\int_{I;[\mathcal{Q}]} f(x)\,dx$$

Lemma 7.1.11 shows that we can remove the dependence $\mathcal{P}$ from the notation, and we put

$$\mathrm{pc}\!\int_I f(x)\,dx = \mathrm{pc}\!\int_{I;[\mathcal{P}]} f(x)\,dx$$

for any partition $\mathcal{P}$ for which f is piecewise constant.

Proof of Lemma 7.1.11. We first assume that the partition $\mathcal{Q}$ a refinement is of the partition $\mathcal{P}$. Using Exercise 7.1.8 we have for every $J \in \mathcal{P}$ a partition $\mathcal{Q}_J$ of J, and we have the disjoint union $\mathcal{Q} = \bigcup_{J\in\mathcal{P}} \mathcal{Q}_J$. Then for $K \in \mathcal{Q}_J$ we have $d_K = c_J$.

Now

$$\mathrm{pc}\int_{I;[\mathcal{Q}]} f(x)\,dx = \sum_{K\in\mathcal{Q}} d_K|K| = \sum_{J\in\mathcal{P}}\sum_{K\in\mathcal{Q}_J} d_K|K| = \sum_{J\in\mathcal{P}} c_J \sum_{K\in\mathcal{Q}_J} |K| =$$
$$\sum_{J\in\mathcal{P}} c_J|J| = \mathrm{pc}\int_{I;[\mathcal{P}]} f(x)\,dx$$

using Proposition 7.1.3. And this proves the statement in case $\mathcal{Q}$ is a refinement of $\mathcal{P}$.

In the case of general partitions $\mathcal{P}$ and $\mathcal{Q}$, we use this result and the common refinement of Proposition 7.1.6 to have

$$\mathrm{pc}\int_{I;[\mathcal{Q}]} f(x)\,dx = \mathrm{pc}\int_{I;[\mathcal{P}\#\mathcal{Q}]} f(x)\,dx = \mathrm{pc}\int_{I;[\mathcal{P}]} f(x)\,dx. \qquad \square$$

Theorem 7.1.12. *Let $I \subset \mathbb{R}$ be a bounded interval, and let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be piecewise constant functions.*

(i) For $a, b \in \mathbb{R}$ we have $a\,f + b\,g \in PC(I)$ and

$$\mathrm{pc}\int_I a\,f(x) + b\,g(x)\,dx = a\,\mathrm{pc}\int_I f(x)\,dx + b\,\mathrm{pc}\int_I g(x)\,dx.$$

(ii) If $f \geq g$, i.e. $\forall\, x \in I$ we have $f(x) \geq g(x)$, then

$$\mathrm{pc}\int_I f(x)\,dx \geq \mathrm{pc}\int_I g(x)\,dx.$$

(iii) In case $f(x) = c$ for all $x \in I$, i.e. f is piecewise constant with respect to $\mathcal{P} = \{I\}$, then

$$\mathrm{pc}\int_I f(x)\,dx = c|I|.$$

(iv) Assume that $\{J, K\}$ is a partition of I, then $f|_J$ and $f|_K$ are piecewise constant functions on J and K and

$$\mathrm{pc}\int_I f(x)\,dx = \mathrm{pc}\int_J f|_J(x)\,dx + \mathrm{pc}\int_K f|_K(x)\,dx.$$

(v) Assume that $I \subset J$, with J a bounded interval. Extend $f\colon I \to \mathbb{R}$ to

$$F\colon J \to \mathbb{R}, \qquad F(x) = \begin{cases} f(x), & x \in I,\\ 0 & x \in J \setminus I \end{cases}$$

then $F \in \mathrm{PC}(J)$ and

$$\mathrm{pc}\int_I f(x)\,dx = \mathrm{pc}\int_J F(x)\,dx.$$

Exercise 7.1.13. Give a proof of Theorem 7.1.12. Hint: use Exercise 7.1.9. Take refinements of the partitions of f and g in case (i) and (ii), and take a refinement of the partition for f and $\{J, K\}$ for (iv), and find a suitable refinement for that partition of F in case (v). Then use Lemma 7.1.11.

7.2 The Riemann integral

Definition 7.2.1. *Let $A \subset \mathbb{R}$ and let $f \colon A \to \mathbb{R}$ and $g \colon A \to \mathbb{R}$ be functions on the same domain A. Then we say that f* majorises *"majoreert" g, notation $f \geq g$, if $\forall\, x \in A$ we have $f(x) \geq g(x)$. Equivalently, g* minorises *"minoreert" f and we write $g \leq f$.*

In general it is not true that any two functions can be compared, e.g. for $f \colon [0,1] \to \mathbb{R}$ and $g \colon [0,1] \to \mathbb{R}$ defined by $f(x) = x$ and $g(x) = 1 - x$, we have neither $f \geq g$ nor $f \leq g$. This is a partial ordening.

Definition 7.2.2. *Let $I \subset \mathbb{R}$ be a bounded interval, and $f \colon I \to \mathbb{R}$ a bounded function. We define the* upper Riemann integral *"Riemannbovenintegraal"*

$$\overline{\int_I} f(x)\,dx = \inf\{\mathrm{pc}\!\int_I g(x)\,dx \mid g \in \mathrm{PC}(I), g \geq f\}$$

We define the lower Riemann integral *"Riemannonderintegraal"*

$$\underline{\int_I} f(x)\,dx = \sup\{\mathrm{pc}\!\int_I g(x)\,dx \mid g \in \mathrm{PC}(I), g \leq f\}$$

Remark 7.2.3. We check that infimum and supremum in Definition 7.2.2 are indeed well-defined. Since, we assume that f is bounded, we have $M > 0$ so that $\forall\, x \in I$ we have $|f(x)| \leq M$. It means that the constant function $g(x) = M$, respectively $h(x) = -M$, majorises, respectively minorises, f, i.e. $f \leq g$, respectively $f \geq h$. In particular, both sets are not empty, the first containing $M|I|$ and the second containing $-M|I|$. By the same observation and Theorem 7.1.12(ii), we see that $-M|I|$ is a lower bound for the first set and $M|I|$ is an upper bound for the second set in Definition 7.2.2. So the lower and upper Riemann integral are well-defined, and the key to this is the boundedness of f.

Lemma 7.2.4. *Let $I \subset \mathbb{R}$ be a bounded interval, and let $f \colon I \to \mathbb{R}$ be a bounded function, i.e. $\exists\, M > 0\ \forall\, x \in I\ |f(x)| \leq M$. Then*

$$-M|I| \leq \underline{\int_I} f(x)\,dx \leq \overline{\int_I} f(x)\,dx \leq M|I|.$$

Proof. The first and last inequality have been observed in Remark 7.2.3. To prove the remaining inequality we choose arbitrary $h, g \in \mathrm{PC}(I)$ with $h \leq f \leq g$. By Theorem 7.1.12(ii) we have that

$$\mathrm{pc}\!\int_I h(x)\,dx \leq \mathrm{pc}\!\int_I g(x)\,dx.$$

Since this inequality holds for all $h \in \mathrm{PC}(I)$, we can take the supremum over all such $h \in \mathrm{PC}(I)$ with $h \leq f$. This then gives

$$\underline{\int}_I f(x)\,dx \leq \mathrm{pc}\!\int_I g(x)\,dx.$$

Since this holds for all $g \in \mathrm{PC}(I)$ with $f \leq g$, we can take the infimum over all such g and retain the inequality. This gives the required inequality. □

With all these preparations we can finally say when a function is Riemann integrable, and define its Riemann integral.

Definition 7.2.5. *Let $I \subset \mathbb{R}$ be a bounded interval, and $f\colon I \to \mathbb{R}$ is a bounded function. The function f is* Riemann integrable *"Riemannintegreerbaar" if*

$$\underline{\int}_I f(x)\,dx = \overline{\int}_I f(x)\,dx.$$

In that case we define the Riemann integral of f over I *"Riemannintegraal van f over het interval I" as*

$$\int_I f(x)\,dx = \underline{\int}_I f(x)\,dx = \overline{\int}_I f(x)\,dx.$$

A classical example is the function $f\colon [0,1] \to \mathbb{R}$ defined as follows

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

This is a bounded function, which is not Riemann integrable. You should check that in this case $\underline{\int}_I f(x)\,dx = 0$ and $\overline{\int}_I f(x)\,dx = 1$, see Exercise 7.6.2.

We use the notation $\int_I f(x)\,dx$ to stick to the notation of calculus, but the x in the notation is a dummy variable, and we could as well have used s, t, y, u, etc. We will do so occasionally in order to avoid confusion.

Before discussing larger classes of integrable functions, we should at least check that for piecewise constant functions we get the same result.

Lemma 7.2.6. *Let $I \subset \mathbb{R}$ be a bounded interval, and $f \in \mathrm{PC}(I)$. Then f is a Riemann integrable function and*

$$\int_I f(x)\,dx = \mathrm{pc}\!\int_I f(x)\,dx.$$

Proof. We have already observed that any $f \in \mathrm{PC}(I)$ is bounded. Since $f \in \mathrm{PC}(I)$ and $f \leq f$ trivially, we find

$$\mathrm{pc}\!\int_I f(x)\,dx \leq \sup\{\mathrm{pc}\!\int_I g(x)\,dx \mid g \in \mathrm{PC}(I), g \leq f\} = \underline{\int}_I f(x)\,dx$$
$$\mathrm{pc}\!\int_I f(x)\,dx \geq \inf\{\mathrm{pc}\!\int_I g(x)\,dx \mid g \in \mathrm{PC}(I), g \geq f\} = \overline{\int}_I f(x)\,dx$$

which gives

$$\overline{\int}_I f(x)\,dx \le \text{pc}\!\!\int_I f(x)\,dx \le \underline{\int}_I f(x)\,dx \le \overline{\int}_I f(x)\,dx,$$

where the last inequality follows from Lemma 7.2.4. Since this means that all inequalities have to be equalities, we find that

$$\underline{\int}_I f(x)\,dx = \overline{\int}_I f(x)\,dx = \text{pc}\!\!\int_I f(x)\,dx$$

which means that f is Riemann integrable and that its Riemann integral equals the integral for piecewise constant functions. □

7.3 Riemann integrable functions

We have defined what a Riemann integrable function is in Definition 7.2.5 and we have seen that a piecewise constant function is Riemann integrable and the notion of both integrals coincide. So we can ask how Theorem 7.1.12 generalises to Riemann integrable functions. We list such properties in Theorem 7.3.1.

Theorem 7.3.1. *Let $I \subset \mathbb{R}$ be a bounded interval, and let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be bounded functions, which are Riemann integrable functions.*

(i) For $a, b \in \mathbb{R}$, the function $a\,f + b\,g\colon I \to \mathbb{R}$ is Riemann integrable and

$$\int_I a\,f(x) + b\,g(x)\,dx = a\int_I f(x)\,dx + b\int_I g(x)\,dx.$$

(ii) If $f \ge g$, then

$$\int_I f(x)\,dx \ge \int_I g(x)\,dx.$$

(iii) Assume that $\{J, K\}$ is a partition of I, then $f|_J$ and $f|_K$ are Riemann integrable functions on J and K and

$$\int_I f(x)\,dx = \int_J f|_J(x)\,dx + \int_K f|_K(x)\,dx.$$

(iv) Assume that $I \subset J$, with J a bounded interval. Extend $f\colon I \to \mathbb{R}$ to

$$F\colon J \to \mathbb{R}, \qquad F(x) = \begin{cases} f(x), & x \in I, \\ 0 & x \in J \setminus I \end{cases}$$

then F is Riemann integrable on J and

$$\int_I f(x)\,dx = \int_J F(x)\,dx.$$

(v) *The functions* $\max(f, g)$ *and* $\min(f, g)$ *(pointwise defined) are Riemann integrable. In particular,* $|f|$ *is Riemann integrable.*

Corollary 7.3.2. *Assume* $I \subset \mathbb{R}$ *to be a finite interval, and that* $f: I \to \mathbb{R}$ *is a Riemann integrable function. Then*

$$\left| \int_I f(x)\, dx \right| \le \int_I |f(x)|\, dx.$$

Proof. By Theorem 7.3.1(v) we know that $|f|: I \to \mathbb{R}$ and $-|f|: I \to \mathbb{R}$ are Riemann integrable. Since $-|f| \le f \le |f|$, we find by Theorem 7.3.1(i)

$$-\int_I |f(x)|\, dx \le \int_I f(x)\, dx \le \int_I |f(x)|\, dx$$

which is the required estimate. □

There is quite a bit to prove for Theorem 7.3.1. We prove parts of it directly, and other parts are relegated to the exercises.

Proof of Theorem 7.3.1(i). We split this into two cases; the case $a \in \mathbb{R}$ and $b = 0$, and the case $a = b = 1$, since these imply the result.

In the case $b = 0$, we have to distinguish between the cases $a > 0$, $a = 0$ and $a < 0$. In case $a = 0$ we are back to a constant, hence piecewise constant, function and Lemma 7.2.6 gives the result. We leave the (easier) case as Exercise 7.3.3, and we assume $a < 0$. We will prove that for all $\eta > 0$ we have

$$0 \le \overline{\int}_I a\, f(x)\, dx - \underline{\int}_I a\, f(x)\, dx < \eta, \tag{7.3.1}$$

which proves that lower Riemann integral of af and the upper Riemann integral of af are equal, and so $af: I \to \mathbb{R}$ is Riemann integrable.

Pick $\varepsilon > 0$, and then there exist $f_u, f_l \in \mathrm{PC}(I)$ with $f_l \le f \le f_u$ so that

$$\mathrm{pc}\!\int_I f_u(x)\, dx - \varepsilon < \overline{\int}_I f(x)\, dx = \int_I f(x)\, dx = \underline{\int}_I f(x)\, dx < \mathrm{pc}\!\int_I f_l(x)\, dx + \varepsilon$$

using the definition of lower and upper Riemann integral as supremum and infimum in Definition 7.2.2 and the fact that f is Riemann integrable, so that the lower and upper Riemann integral are equal, see Definition 7.2.5. Observe that $af_u, af_l \in \mathrm{PC}(I)$ and $af_u \le af \le af_l$ since $a < 0$, so that

$$\begin{aligned}
\overline{\int}_I a\, f(x)\, dx &\le \mathrm{pc}\!\int_I a\, f_l(x)\, dx = a\, \mathrm{pc}\!\int_I f_l(x)\, dx < a \int_I f(x)\, dx - a\varepsilon, \\
\underline{\int}_I a\, f(x)\, dx &\ge \mathrm{pc}\!\int_I a\, f_u(x)\, dx = a\, \mathrm{pc}\!\int_I f_u(x)\, dx > a \int_I f(x)\, dx + a\varepsilon.
\end{aligned}$$

Note that we use Theorem 7.1.12(i). Subtracting gives (7.3.1) for $\eta = -2a\varepsilon$. Since $a < 0$ and ε can be taken arbitrarily positive it follows that $af\colon I \to \mathbb{R}$ is a Riemann integrable function. Moreover, the estimates also yield that $\int_I a\,f(x)\,dx = a\int_I f(x)\,dx$.

Next we consider the case $a = b = 1$, i.e. we show that the sum of two Riemann integrable functions is Riemann integrable, and the integral of the sum is the sum of the integrals. Take $f_u, g_u \in \mathrm{PC}(I)$ so that $f \leq f_u$, $g \leq g_u$, then $f + g \leq f_u + g_u$ and

$$\overline{\int}_I f(x) + g(x)\,dx \leq \mathrm{pc}\!\int_I f_u(x) + g_u(x)\,dx = \mathrm{pc}\!\int_I f_u(x)\,dx + \mathrm{pc}\!\int_I g_u(x)\,dx$$

using Theorem 7.1.12. Taking the infimum over all $g_u \in \mathrm{PC}(I)$ majorising g, we get

$$\overline{\int}_I f(x) + g(x)\,dx \leq \mathrm{pc}\!\int_I f_u(x)\,dx + \overline{\int}_I g(x)\,dx$$

and next taking the infimum over all $f_u \in \mathrm{PC}(I)$ majorising f we get

$$\overline{\int}_I f(x) + g(x)\,dx \leq \overline{\int}_I f(x)\,dx + \overline{\int}_I g(x)\,dx.$$

Analogously, we get

$$\underline{\int}_I f(x) + g(x)\,dx \geq \underline{\int}_I f(x)\,dx + \underline{\int}_I g(x)\,dx$$

and the combination gives

$$\begin{aligned}\int_I f(x)\,dx + \int_I g(x)\,dx &= \underline{\int}_I f(x)\,dx + \underline{\int}_I g(x)\,dx \leq \underline{\int}_I f(x) + g(x)\,dx \\ &\leq \overline{\int}_I f(x) + g(x)\,dx \leq \overline{\int}_I f(x)\,dx + \overline{\int}_I g(x)\,dx = \int_I f(x)\,dx + \int_I g(x)\,dx\end{aligned}$$

using the Riemann integrability of f and g for the first and last equality and Lemma 7.2.4 in the middle inequality. So the inequalities are equalities, proving that $f+g\colon I \to \mathbb{R}$ is Riemann integrable, and that the Riemann integral of $f + g$ is the sum of the Riemann integrals of f and g. □

Exercise 7.3.3. Prove the case $a > 0$, $b = 0$ of Theorem 7.3.1(i). Note that it would suffice to treat the case $a = -1$ in the proof given above.

Exercise 7.3.4. (i) Prove Theorem 7.3.1(ii). Reduce to the case $g = 0$ by replacing f by $f - g$ and using Theorem 7.3.1(i).

(ii) Prove Theorem 7.3.1(iii). Hint: show that

$$\overline{\int}_J f|_J(x)\,dx + \overline{\int}_K f|_K(x)\,dx \leq \overline{\int}_I f(x)\,dx$$

and a similar expression for the lower Riemann integral.

(iii) Prove Theorem 7.3.1(iv). Hint: show that

$$\overline{\int}_J F(x)\,dx \le \overline{\int}_I f(x)\,dx$$

and a similar expression for the lower Riemann integral.

Proof of Theorem 7.3.1(v). Pick $\varepsilon > 0$ arbitrarily. Choose $f_u, g_u, f_l, g_l \in \mathrm{PC}(I)$ satisfy $f_l \le f \le f_u$, $g_l \le g \le g_u$ with

$$\begin{aligned} \mathrm{pc}\!\int_I f_u(x)\,dx - \varepsilon < \int_I f(x)\,dx < \mathrm{pc}\!\int_I f_l(x)\,dx + \varepsilon, \\ \mathrm{pc}\!\int_I g_u(x)\,dx - \varepsilon < \int_I g(x)\,dx < \mathrm{pc}\!\int_I g_l(x)\,dx + \varepsilon. \end{aligned} \tag{7.3.2}$$

Then $\max(f_l, g_l) \le \max(f, g) \le \max(f_u, g_u)$ and $\max(f_l, g_l), \max(f_u, g_u) \in \mathrm{PC}(I)$ (Why?), and we find

$$0 \le \overline{\int}_I \max(f,g)(x)\,dx - \underline{\int}_I \max(f,g)(x)\,dx \le \mathrm{pc}\!\int_I \max(f_u,g_u)(x) - \max(f_l,g_l)(x)\,dx.$$

Define the nonnegative function $h = f_u - f_l + g_u - g_l \in \mathrm{PC}(I)$, then we have $\mathrm{pc}\!\int_I h(x)\,dx < 4\varepsilon$ by (7.3.2) and

$$\begin{aligned} f_u = f_l + (f_u - f_l) \le f_l + h, \qquad g_u = g_l + (g_u - g_l) \le g_l + h, \\ \Longrightarrow \quad \max(f_u, g_u) \le \max(f_l + h, g_l + h) = \max(f_l, g_l) + h. \end{aligned}$$

This shows

$$0 \le \overline{\int}_I \max(f,g)(x)\,dx - \underline{\int}_I \max(f,g)(x)\,dx \le \mathrm{pc}\!\int_I h(x)\,dx < 4\varepsilon$$

and so the upper and lower Riemann integral are equal, and $\max(f, g)$ is Riemann integrable. The proof for $\min(f, g)$ is analogous. □

Exercise 7.3.5. Prove Theorem 7.3.1(v) for $\min(f, g)$ by either redoing the above proof for the minimum or by relating $\min(f, g)$ to a suitable maximum and using earlier parts of Theorem 7.3.1.

A slightly more complicated proof is required to show that the product of Riemann integrable functions is Riemann integrable.

Theorem 7.3.6. *Let $I \subset \mathbb{R}$ be a bounded interval, and let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$ be bounded functions. Assume that f and g are Riemann integrable, then the product $fg\colon I \to \mathbb{R}$ is a Riemann integrable function.*

Proof. First of all, note that fg is a bounded function. We reduce to the case that we can assume both f and g to be non-negative functions.

In order to see this, we write $f = f^+ - f^-$ with $f^+ = \max(f, 0)$, $f^- = -\min(f, 0)$, so that f^+ and f^- are non-negative functions. Since f is integrable, we know from Theorem 7.3.1(v) and (ii) that both f^+ and f^- are Riemann integrable. Doing the same for g we see that we can write the product of f and g as a combination of four products of non-negative functions;

$$fg = f^+g^+ - f^-g^+ - f^+g^- + f^-g^-.$$

So proving the statement for non-negative functions, shows that each term on the right hand side is Riemann integrable, and, using Theorem 7.3.1 again, shows that fg is Riemann integrable.

So we assume $f \geq 0$ and $g \geq 0$. Since the functions are bounded, we find that there exist constants $M_f \geq 0$ and $M_g \geq 0$ so that

$$\forall\, x \in I \qquad 0 \leq f(x) \leq M_f \text{ and } 0 \leq g(x) \leq M_g.$$

We will show that $\overline{\int}_I f(x)g(x)\,dx - \underline{\int}_I f(x)g(x)\,dx$ is smaller than any positive number. Since it is non-negative by Lemma 7.2.4, we see that this implies $\overline{\int}_I f(x)g(x)\,dx = \underline{\int}_I f(x)g(x)\,dx$, proving that the product is Riemann integrable.

To prove the statement, we pick $\varepsilon > 0$ arbitrary. Since the lower Riemann integral of f is a supremum, we see that we have a $f_l \in \mathrm{PC}(I)$ with $f_l \leq f$ and

$$\underline{\int}_I f(x)\,dx - \varepsilon < \mathrm{pc}\!\int_I f_l(x)\,dx.$$

Since $f \geq 0$ we can assume that $f_l \geq 0$ as well. Similarly, since the upper Riemann integral of f is an infimum, we see that we have a $f_u \in \mathrm{PC}(I)$ with $f_u \geq f$ and

$$\overline{\int}_I f(x)\,dx + \varepsilon > \mathrm{pc}\!\int_I f_u(x)\,dx.$$

Since f is Riemann integrable, the lower Riemann integral equals the upper Riemann integral, so that we find

$$\mathrm{pc}\!\int_I f_u(x)\,dx - \varepsilon < \int_I f(x)\,dx < \mathrm{pc}\!\int_I f_l(x)\,dx + \varepsilon \quad \Longrightarrow \quad 0 \leq \mathrm{pc}\!\int_I f_u(x) - f_l(x)\,dx < 2\varepsilon$$

We find similarly functions $g_l, g_u \in \mathrm{PC}(I)$ with the analogous properties for g.

Having these four piecewise constant functions at hand, we have

$$0 \leq fg \leq f_u g_u, \qquad 0 \leq f_l g_l \leq fg$$

and here we use the positivity of f and g. Since $f_u g_u$ and $f_l g_l$ are elements of $\mathrm{PC}(I)$ we find

$$\overline{\int}_I f(x)g(x)\,dx \leq \mathrm{pc}\!\int_I f_u(x)g_u(x)\,dx, \qquad \underline{\int}_I f(x)g(x)\,dx \geq \mathrm{pc}\!\int_I f_l(x)g_l(x)\,dx,$$

so that

$$0 \leq \overline{\int_I} f(x)g(x)\,dx - \underline{\int_I} f(x)g(x)\,dx \leq \mathrm{pc}\int_I f_u(x)g_u(x) - f_l(x)g_l(x)\,dx.$$

Estimating the integrand

$$\begin{aligned} 0 \leq f_u(x)g_u(x) - f_l(x)g_l(x) &= (f_u(x) - f_l(x))g_u(x) + f_l(x)(g_u(x) - g_l(x)) \\ &\leq M_g(f_u(x) - f_l(x)) + M_f(g_u(x) - g_l(x)) \end{aligned}$$

we get

$$0 \leq \overline{\int_I} f(x)g(x)\,dx - \underline{\int_I} f(x)g(x)\,dx$$
$$\leq M_g \mathrm{pc}\int_I f_u(x) - f_l(x)\,dx + M_f \mathrm{pc}\int_I g_u(x) - g_l(x)\,dx \leq 2\varepsilon(M_f + M_g).$$

Since $\varepsilon > 0$ is arbitrary, so that $\overline{\int}_I f(x)g(x)\,dx - \underline{\int}_I f(x)g(x)\,dx$ is smaller than any positive number, as claimed. □

We have seen that piecewise constant functions are Riemann integrable, and Theorem 7.3.1 and Theorem 7.3.6 show how to create Riemann integrable functions from other functions. There exist several sufficient conditions for Riemann integrability, and we discuss a few of this results. Some statements and proofs are relegated to the exercises.

Theorem 7.3.7. *Let $I \subset \mathbb{R}$ be a bounded interval, and assume that $f\colon I \to \mathbb{R}$ is uniformly continuous. Then f is a Riemann integrable function.*

Corollary 7.3.8. *Let $a < b$ and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function, then f is a Riemann integrable function.*

Proof. By Theorem 5.3.10 it follows that f is uniformly continuous, so that the corollary follows from Theorem 7.3.7. □

Proof of Theorem 7.3.7. First recall that $f(I)$ is bounded by Corollary 5.3.4, since I is a bounded set.

The strategy of the proof is similar to the proof of Theorem 7.3.6, and we will show that $\overline{\int}_I f(x)\,dx - \underline{\int}_I f(x)\,dx$ is smaller than any positive number. So we pick $\varepsilon > 0$ arbitrary. Since f is uniformly continuous, there exists $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Since I is a bounded interval, there exists $a < b$ with $I = [a,b]$, $I = (a,b]$, $I = [a,b)$, or $I = (a,b)$, i.e. $a = \inf_{x\in I} x$ and $b = \sup_{x\in I} x$. Take $N \in \mathbb{N}$ so that $\frac{b-a}{N} < \delta$, and we define the intervals

$$J_i = (a + \frac{b-a}{N}(i-1), a + \frac{b-a}{N} i]$$

for $i \in \{1, \cdots, N\}$. We take J_1 to be open, respectively closed, at the left endpoint a if I is open, respectively closed, at the left endpoint. Similarly, we take J_N to be open, respectively closed, at the right endpoint b if I is open, respectively closed, at the right endpoint. Then we have described a partition $\mathscr{P} = \{J_1, \cdots, J_N\}$. With this partition we associate two functions $f_u, f_l \in \mathrm{PC}(I)$ by

$$\forall\, x \in J_i \qquad f_u(x) = \sup_{y \in J_i} f(y) \text{ and } f_l(x) = \inf_{y \in J_i} f(y).$$

By construction $f_l \le f \le f_u$ and so

$$\overline{\int}_I f(x)\,dx \le \mathrm{pc}\!\int_I f_u(x)\,dx = \sum_{i=1}^{N} |J_i| \sup_{y \in J_i} f(y),$$
$$\underline{\int}_I f(x)\,dx \ge \mathrm{pc}\!\int_I f_l(x)\,dx = \sum_{i=1}^{N} |J_i| \inf_{y \in J_i} f(y)$$

implying that

$$0 \le \overline{\int}_I f(x)\,dx - \underline{\int}_I f(x)\,dx \le \sum_{i=1}^{N} \frac{b-a}{N} \big(\sup_{y \in J_i} f(y) - \inf_{y \in J_i} f(y)\big).$$

Since for all $x, y \in J_i$ we know that $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$, we have

$$f(y) - \varepsilon < f(x) < f(y) + \varepsilon \quad \Longrightarrow \quad \sup_{x \in J_i} f(x) \le f(y) + \varepsilon \quad \Longrightarrow \quad \sup_{x \in J_i} f(x) \le \inf_{y \in J_i} f(y) + \varepsilon.$$

Plugging this in the previous estimate, and we find

$$0 \le \overline{\int}_I f(x)\,dx - \underline{\int}_I f(x)\,dx \le \sum_{i=1}^{N} \frac{b-a}{N} \varepsilon = \varepsilon(b-a)$$

proving the required estimate. □

Remark 7.3.9. Note that in general we cannot replace the supremum and infimum over J_i in the proof of Theorem 7.3.7 by maximum and minimum even though the function is uniformly continuous since the interval J_i is not closed in general.

7.4 The fundamental theorem of calculus

The fundamental theorem of calculus relates integrals, i.e. areas under a graph, to derivatives, i.e. direction of a tangential line to a graph. Naturally, the graphs are not the same. We discuss the fundamental theorem of calculus in two parts.

Theorem 7.4.1 (Fundamental Theorem of Calculus "Hoofdstelling van de Integraalrekening", Part I)**.** *Let $a < b$ and assume that $f\colon [a,b] \to \mathbb{R}$ is a Riemann integrable function. Define*

$$F\colon [a,b] \to \mathbb{R}, \qquad F(x) = \int_{[a,x]} f(s)\, ds$$

then we have

(i) F is uniformly continuous;

(ii) if f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof. First note that f restricted to a subinterval $[a,x]$ of $[a,b]$ is Riemann integrable by Theorem 7.3.1(iii), so that F is well-defined. Let $x, y \in [a,b]$ with $y \geq x$, then

$$F(y) - F(x) = \int_{[a,y]} f(s)\, ds - \int_{[a,x]} f(s)\, ds = \int_{(x,y]} f(s)\, ds$$

using Theorem 7.3.1(iii). Since f is assumed to be Riemann integrable, it is a bounded function, say $|f(x)| \leq M$. Then by Lemma 7.2.4 we have

$$-M|(x,y]| \leq F(y) - F(x) \leq M|(x,y]| \quad \Longrightarrow \quad |F(y) - F(x)| \leq M|x-y|.$$

The same inequality holds for $y \leq x$ by interchanging the roles of x and y. It follows that F is Lipschitz continuous, see Exercise 5.5.3, and thus uniformly continuous, which proves part (i).

For (ii) we employ the characterisation of differentiability using Newton's approximation of Proposition 6.1.4. So we need to prove that $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ so that

$$|x - x_0| < \delta \quad \Longrightarrow \quad |F(x) - (F(x_0) + f(x_0)(x - x_0))| \leq \varepsilon |x - x_0|.$$

So take $\varepsilon > 0$ arbitrary. Then by Proposition 5.1.10 we have $\delta > 0$ so that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$ and thus

$$f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon.$$

First assume $x > x_0$ and $|x - x_0| < \delta$, then we find

$$\begin{gathered}
F(x) - F(x_0) = \int_{(x_0,x]} f(s)\, ds \qquad \Longrightarrow \\
(x - x_0)\big(f(x_0) - \varepsilon\big) \leq F(x) - F(x_0) \leq (x - x_0)\big(f(x_0) + \varepsilon\big) \qquad \Longrightarrow \\
-\varepsilon(x - x_0) \leq F(x) - F(x_0) - f(x_0)(x - x_0) \leq \varepsilon(x - x_0) \qquad \Longrightarrow \\
|F(x) - (F(x_0) + f(x_0)(x - x_0))| \leq \varepsilon |x - x_0|.
\end{gathered}$$

The case $x < x_0$ and $|x - x_0| < \delta$ is treated analogously and gives the same result. □

The first part of the Fundamental Theorem of Calculus, Theorem 7.4.1, suggests that we can evaluate the integral by finding F, since $\int_{[a,b]} f(x)\,dx = F(b)$ in the notation of Theorem 7.4.1.

Definition 7.4.2. *Let $I \subset \mathbb{R}$ be bounded interval, and assume that $f\colon I \to \mathbb{R}$ is a function. Then f has an* antiderivative *or* primitive *"primitieve" function if there exists $F\colon I \to \mathbb{R}$, which is a differentiable function and for all $x \in I$ we have $F'(x) = f(x)$.*

By Exercise 6.5.7 we see that if f has an antiderivative (or primitive) function F, then F is uniquely determined up to a constant. Indeed, if f would have two different antiderivative functions F and G, then $(F-G)' = F' - G' = f - f = 0$, and Exercise 6.5.7 shows that $F - G$ is a constant, since the domain is an interval.

Lemma 7.4.3. *Let $a < b$ and assume that $f\colon [a,b] \to \mathbb{R}$ is continuous. Then f has an antiderivative function.*

Proof. By Corollary 7.3.8 we know that f is Riemann integrable. By Theorem 7.4.1 we see that $F(x) = \int_{[a,x]} f(s)\,ds$ is an antiderivative function for f. □

Theorem 7.4.4 (Fundamental Theorem of Calculus "Hoofdstelling van de Integraalrekening", Part II)**.** *Let $a < b$ and $f\colon [a,b] \to \mathbb{R}$ is a Riemann integrable function and assume that $F\colon [a,b] \to \mathbb{R}$ is an antiderivative function for f. Then*

$$\int_{[a,b]} f(x)\,dx = F(b) - F(a) = F\Big|_a^b$$

Proof. The proof follows by showing that for each $g, h \in \mathrm{PC}([a,b])$ with $g \le f \le h$ we have

$$\mathrm{pc}\!\int_{[a,b]} g(x)\,dx \le F(b) - F(a) \le \mathrm{pc}\!\int_{[a,b]} h(x)\,dx \tag{7.4.1}$$

so that by taking the supremum over the left hand side over all $g \in \mathrm{PC}([a,b])$, and next taking the infimum over the right hand side over all $h \in \mathrm{PC}([a,b])$ we obtain

$$\overline{\int}_{[a,b]} f(x)\,dx \le F(b) - F(a) \le \underline{\int}_{[a,b]} f(x)\,dx,$$

which proves the result since the left hand side and the right hand side are equal to $\int_{[a,b]} f(x)\,dx$, since f is a Riemann integrable function.

In order to prove (7.4.1) we take $g, h \in \mathrm{PC}([a,b])$ with respect to the same partition $\mathcal{P}$, which we can assume by Proposition 7.1.6. Furthermore, we assume that all intervals $J \in \mathcal{P}$ are non-empty sets, i.e. $\forall\, J \in \mathcal{P}$ we have $J \neq \emptyset$. Take $J \in \mathcal{P}$, then J is of the form (b_J, e_J), $(b_J, e_J]$, $[b_J, e_J)$, or $[b_J, e_J]$ and we define $F[J] = F(e_J) - F(b_J)$. We assume $b_J < e_J$. Since F

is differentiable on $[a,b]$, we can apply the Mean Value Theorem 6.2.5 to obtain the existence of a point $t_J \in (b_J, e_J)$ with

$$\frac{F(e_J)-F(b_J)}{e_J-b_J} = F'(t_J) = f(t_J) \quad\Longrightarrow\quad |J| \inf_{x\in J} f(x) \leq f(t_J)|J| = F[J] \leq |J| \sup_{x\in J} f(x)$$

and in this form it also holds in case J consists of one point, $e_J = b_J$. Since $g \leq f \leq h$ we have $g(y) \leq \inf_{x\in J} f(x)$ for all $y \in J$ since g is constant on J and similarly $\sup_{x\in J} f(x) \leq h(y)$ for all $y \in J$. So summing over all $J \in \mathscr{P}$ gives

$$\mathrm{pc}\!\int_{[a,b]} g(x)\,dx \leq \sum_{J\in\mathscr{P}} |J| \inf_{x\in J} f(x) \leq \sum_{J\in\mathscr{P}} F[J] \leq \sum_{J\in\mathscr{P}} |J| \sup_{x\in J} f(x) \leq \mathrm{pc}\!\int_{[a,b]} h(x)\,dx.$$

Since $\sum_{J\in\mathscr{P}} F[J] = F(b)-F(a)$, compare with Proposition 7.1.3, we have obtained (7.4.1). □

As a first application of the Fundamental Theorem of Calculus we obtain the substitution rule.

Corollary 7.4.5 (Substitution "substitutieregel")**.** *Assume $g\colon [a,b] \to \mathbb{R}$ is an increasing function, which is differentiable and such that $g'\colon [a,b] \to \mathbb{R}$ is a Riemann integrable function. Assume that $[g(a), g(b)]$ is a finite interval and that $f\colon [g(a), g(b)] \to \mathbb{R}$ is a continuous function. Then*

$$\int_{[g(a),g(b)]} f(s)\,ds = \int_{[a,b]} f(g(x))\,g'(x)\,dx.$$

The conditions on f and g in the substitution rule of Corollary 7.4.5 can be considerably relaxed, but this would take a much longer proof. You should check the appropriate analogue for a decreasing function g.

Proof. Let F be any antiderivative function for f, see Lemma 7.4.3. Then F is a differentiable function, by Theorem 7.4.1. By the chain rule of Theorem 6.1.11 we have that $F\circ g\colon [a,b] \to \mathbb{R}$ is a differentiable function, and $(F\circ g)'(x) = F'(g(x))\,g'(x) = f(g(x))\,g'(x)$. Since $f\circ g\colon [a,b] \to \mathbb{R}$ is continuous, it is a Riemann integrable function. By Theorem 7.3.6 it follows that $[a,b] \ni x \mapsto f(g(x))\,g'(x)$ is a Riemann integrable function with antiderivative $F \circ g$. By Theorem 7.4.4 we have

$$F(g(b)) - F(g(a)) = \int_{[a,b]} f(g(x))\,g'(x)\,dx.$$

On the other hand, since F is an antiderivative function for f, we also have by Theorem 7.4.4 that

$$F(g(b)) - F(g(a)) = \int_{[g(a),g(b)]} f(s)\,ds.$$

Comparing the expressions proves the substitution rule. □

As a second application of the Fundamental Theorem of Calculus we obtain the for integration by parts.

Corollary 7.4.6 (Integration by parts "partieel integreren")**.** *Assume $F\colon [a,b] \to \mathbb{R}$ and $G\colon [a,b] \to \mathbb{R}$ be differentiable functions. Moreover, assume that the derivatives $F'\colon [a,b] \to \mathbb{R}$ and $G'\colon [a,b] \to \mathbb{R}$ are Riemann integrable functions. Then the functions $F'G\colon [a,b] \to \mathbb{R}$ and $FG'\colon [a,b] \to \mathbb{R}$ are Riemann integrable functions and*

$$\int_{[a,b]} F(x)G'(x)\,dx = F(b)G(b) - F(a)G(a) - \int_{[a,b]} F'(x)G(x)\,dx.$$

Proof. Since F and G are differentiable, the functions F and G are continuous by Corollary 6.1.6, and hence by Corollary 7.3.8, the functions F and G are Riemann integrable. By Theorem 7.3.6, we have that $F'G$ and FG' are Riemann integrable. By Theorem 7.3.1 we have that $F'G + FG' = (FG)'$ is Riemann integrable, and by Theorem 7.4.4 we get

$$FG\Big|_a^b = \int_{[a,b]} (FG)'(x)\,dx = \int_{[a,b]} F'(x)G(x)\,dx + \int_{[a,b]} F(x)G'(x)\,dx. \qquad \square$$

So integration by parts and the substitution rule can be viewed as the integrated versions of the product rule Theorem 6.1.9 and the chain rule Theorem 6.1.11 using the fundamental theorem of calculus Theorem 7.4.4.

Integration by parts can be used to obtain an expression for the remainder in the Taylor polynomial in terms of an integral, and this should be compared to Theorem 6.4.2.

Corollary 7.4.7. *Assume that $f\colon (a,b) \to \mathbb{R}$ has derivatives up to order $n+1$, and that $f^{(n+1)}\colon (a,b) \to \mathbb{R}$ is continuous. Let $c \in (a,b)$ and put*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_n(x) = T_n(x; f, c) + E_n(x; f, c)$$

then we have for $b > x \geq c$

$$E_n(x; f, c) = \frac{1}{n!} \int_{[c,x]} (x-s)^n\, f^{(n+1)}(s)\,ds.$$

Proof. The proof follows by induction on n and integration by parts. We assume that $b > x \geq c$, and we leave the case $a < x \leq c$ as Exercise 7.4.8. The case $n = 0$ is

$$f(x) - f(c) = \int_{[c,x]} f'(s)\,ds$$

which follows from Theorem 7.4.4, since f' is continuous on $[c,x]$ and thus Riemann integrable by Lemma 7.4.3.

In order to do the inductive step we assume that all derivatives up to order $n+2$ exist and that $f^{(n+2)}$ is continuous. Subtracting

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_n(x)$$

$$f(x) = \sum_{k=0}^{n+1} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1}(x)$$

gives

$$0 = E_n(x) - \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} - E_{n+1}(x).$$

Using the induction hypothesis and the elementary integral $\int_{[c,x]}(x-s)^n\,ds = \frac{1}{n+1}(x-c)^{n+1}$ gives

$$\begin{aligned} E_{n+1}(x) &= E_n(x) - \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} \\ &= \frac{1}{n!}\int_{[c,x]}(x-s)^n f^{(n+1)}(s)\,ds - \frac{f^{(n+1)}(c)}{n!}\int_{[c,x]}(x-s)^n\,ds \\ &= \frac{1}{n!}\int_{[c,x]}(x-s)^n\big(f^{(n+1)}(s) - f^{(n+1)}(c)\big)\,ds \end{aligned}$$

Now apply the integration by parts of Corollary 7.4.6 with $F(s) = f^{(n+1)}(s) - f^{(n+1)}(c)$, $G'(s) = (x-s)^n$ and note that the boundary terms vanish. This gives the result. □

Exercise 7.4.8. Formulate and show the statement of Corollary 7.4.7 for the case $x \leq c$.

7.5 Limit of functions and the Riemann integral

From Chapter 1 we have seen that we cannot interchange limit and integration in general, but Theorem 7.5.1 states that this can be done in case the convergence is uniform, see Definition 5.4.3.

Theorem 7.5.1. *Let $I \subset \mathbb{R}$ be a finite interval. Assume that $\forall\, n \in \mathbb{N}$ the function $f_n\colon I \to \mathbb{R}$ is Riemann integrable function, and assume that $\lim_{n\to\infty} f_n = f$ in the uniform convergence to a function $f\colon I \to \mathbb{R}$. Then f is a Riemann integrable function and*

$$\int_I f(x)\,dx = \lim_{n\to\infty}\int_I f_n(x)\,dx.$$

Note that, since the f_n's are Riemann integrable, they are bounded functions. Since uniform convergence of bounded functions gives a bounded function, see Exercise 5.5.10, the function f is bounded. We prove this (and do Exercise 5.5.10) in the proof of Theorem 7.5.1.

Proof. So pick $\varepsilon > 0$ arbitrary. Since the convergence is uniform, we know that there exists $N \in \mathbb{N}$ so that for all $n \geq N$ and for all $x \in I$ we have $|f_n(x) - f(x)| < \varepsilon$. In particular, we see that for all $n \geq N$

$$\forall\, x \in I \qquad f_n(x) - \varepsilon \leq f(x) \leq f_n(x) + \varepsilon.$$

Taking $\varepsilon = 1$, and using that the corresponding f_N is bounded, say by M, i.e. $|f(x)| \leq M$ for all $x \in I$, we have $|f(x)| \leq M + 1$ for all $x \in I$. So the upper and lower integral for f exist.

We will show that they differ by an arbitrarily small positive number, and hence are equal. This implies the Riemann integrability of the limit function f.

We use Exercise 7.6.7, to get for all $n \geq N$

$$\begin{aligned}\int_I f_n(x) - \varepsilon\, dx = \underline{\int}_I f_n(x) - \varepsilon\, dx \leq \underline{\int}_I f(x)\, dx \leq \overline{\int}_I f(x)\, dx \\ \leq \overline{\int}_I f_n(x) + \varepsilon\, dx = \int_I f_n(x) + \varepsilon\, dx\end{aligned}$$

where the first and last equality follow from the fact that f_n is Riemann integrable, and, of course, that the constant function $\pm\varepsilon$ is Riemann integrable as well.

It follows that

$$0 \leq \overline{\int}_I f(x)\, dx - \underline{\int}_I f(x)\, dx \leq 2\varepsilon\, |I|.$$

Since $\varepsilon > 0$ is arbitrary and I is finite, this shows that f is Riemann integrable. Moreover, plugging this back it into the inequality, we see that for all $n \geq N$ we have

$$\Big| \int_I f(x)\, dx - \int_I f_n(x)\, dx \Big| \leq \varepsilon\, |I|$$

showing that $\int_I f(x)\, dx = \lim_{n\to\infty} \int_I f_n(x)\, dx$. □

Recall that in Chapter 6 we have not discussed the relation between uniform convergence of a sequence of functions and differentiability. In general, this is a bit more complicated, but using Theorem 7.5.1 and the Fundamental Theorem of Calculus Theorems 7.4.1, 7.4.4 we can prove the following result.

Proposition 7.5.2. *For all $n \in \mathbb{N}$, the function $f_n\colon [a,b] \to \mathbb{R}$ is assumed to be differentiable with continuous derivative $f_n'\colon [a,b] \to \mathbb{R}$. Assume that $(f_n')_{n\in\mathbb{N}}$ converges uniformly to g. If there exists $x_0 \in [a,b]$ for which the series $(f_n(x_0))_{n\in\mathbb{N}}$ converges, say $\lim_{n\to\infty} f_n(x_0) = L$, then the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $f\colon [a,b] \to \mathbb{R}$. Moreover, f is a differentiable function with derivative $f' = g$.*

Proof. Assume for convenience that $x > x_0$, then we can use the Fundamental Theorem of Calculus, see Theorems 7.4.1, 7.4.4, to write

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(x_0) + \int_{[x_0,x]} f_n'(s)\, ds = L + \int_{[x_0,x]} g(s)\, ds$$

using Theorem 7.5.1. So we have established that the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to $f(x) = L + \int_{[x_0,x]} g(s)\, ds$, which is a differentiable function with derivative g, again by the Fundamental Theorem of Calculus.

It remains to show that the convergence is uniform. For this note that

$$|f(x) - f_n(x)| \leq |L - f_n(x_0)| + \int_{[x_0,x]} |f_n'(s) - g(s)|\, ds \leq |L - f_n(x_0)| + |b-a| \sup_{s\in[a,b]} |f_n'(s) - g(s)|$$

and note that the right hand side is independent of $x \in [a,b]$. Pick $\varepsilon > 0$, then there exists $N_1 \in \mathbb{N}$ so that for all $n \geq N_1$ we have $|L - f_n(x_0)| < \frac{1}{2}\varepsilon$ since $\lim_{n\to\infty} f_n(x_0) = L$. Similarly, there exists $N_2 \in \mathbb{N}$ so that for all $n \geq N_2$ we have

$$\forall x \in [a,b] \qquad |f_n'(s) - g(s)| < \frac{1}{2|b-a|}\varepsilon.$$

So taking $N = \max(N_1, N_2)$ we have that for all $n \geq N$ that

$$\forall x \in [a,b] \qquad |f(x) - f_n(x)| \leq |L - f_n(x_0)| + |b-a| \sup_{s\in[a,b]} |f_n'(s) - g(s)| < \varepsilon. \qquad \square$$

7.6 Exercises

Exercise 7.6.1. Let $I \subset \mathbb{R}$ be a bounded interval, and let $f\colon I \to \mathbb{R}$ be a bounded function. Let $\mathscr{P}$ be a partition of I and define the upper Riemann sum

$$U(f,\mathscr{P}) = \sum_{J\in\mathscr{P}; J\neq\emptyset} \big(\sup_{x\in J} f(x)\big)|J|$$

and the lower Riemann sum

$$L(f,\mathscr{P}) = \sum_{J\in\mathscr{P}; J\neq\emptyset} \big(\inf_{x\in J} f(x)\big)|J|.$$

(i) Explain why we need to exclude the empty set in the sums on the right hand side.

(ii) Assume that $g \in \mathrm{PC}(I)$ majorises f, and that g is piecewise constant with respect to the partition $\mathscr{P}$. Show that $\mathrm{pc}\!\int_I g(x)\,dx \geq U(f,\mathscr{P})$. State and prove a similar statement for a $h \in \mathrm{PC}(I)$ minorising f.

(iii) Show that the upper, respectively lower, Riemann integral are equal to the infimum, respectively supremum, of the upper, respectively lower, Riemann sum over all partitions of I;

$$\overline{\int_I} f(x)\,dx = \inf_{\mathscr{P}} U(f,\mathscr{P}), \qquad \underline{\int_I} f(x)\,dx = \sup_{\mathscr{P}} L(f,\mathscr{P}).$$

The sums $U(f,\mathscr{P})$, $L(f,\mathscr{P})$ are also named Darboux sums.

Exercise 7.6.2. Define the function $\chi\colon \mathbb{R} \to \mathbb{R}$ by $\chi(x) = 1$ if $x \in \mathbb{Q}$ and $\chi(x) = 0$ if $x \in \mathbb{R}\setminus\mathbb{Q}$. This means that χ is the indicator function of $\mathbb{Q}$. Show that χ is not a Riemann integrable function on the bounded interval $[0,a]$. This example is due to Dirichlet. Hint: use Corollary 3.2.16(ii) and Exercise 7.6.1.

Exercise 7.6.3. Define a function $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{2n+2}, & x \in (\frac{1}{2n+3}, \frac{1}{2n+2}], \\ \frac{-1}{2n+1}, & x \in (\frac{1}{2n+2}, \frac{1}{2n+1}], \\ 0, & x = 0. \end{cases}$$

for $n \in \mathbb{N}$. Is $f \in \mathrm{PC}([0,1])$? If not, is f Riemann integrable on $[0,1]$?

Exercise 7.6.4. Consider the functions $f_n\colon [0,1] \to \mathbb{R}$ defined by $f_n(x) = (nx+1)^{-1}$. Show that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwise, but not uniformly. Show that in this case

$$\lim_{n\to\infty} \int_{[0,1]} f_n(x)\,dx = \int_{[0,1]} \lim_{n\to\infty} f_n(x)\,dx.$$

Exercise 7.6.5. Assume that we have proved the case $f = g$ of Theorem 7.3.6, i.e. that for $f\colon I \to \mathbb{R}$ a Riemann integrable we have that $f^2\colon I \to \mathbb{R}$ is a Riemann integrable function. Prove Theorem 7.3.6 in generality. Hint: write fg in terms of squares of functions.

Exercise 7.6.6. Let $a < b$ and assume $f\colon [a,b] \to \mathbb{R}$ is a continuous function. Show that there exists $c \in (a,b)$ with

$$\int_{[a,b]} f(x)\,dx = (b-a)f(c).$$

Hint: use Theorem 6.2.5.

Exercise 7.6.7. Let $I \subset \mathbb{R}$ be a finite interval, and let $f\colon I \to \mathbb{R}$ and $g\colon I \to \mathbb{R}$.

(i) Show that

$$\underline{\int}_I f(x) + g(x)\,dx \geq \underline{\int}_I f(x)\,dx + \underline{\int}_I g(x)\,dx.$$

Hint: see the proof of Theorem 7.3.1(i).

(ii) Can we conclude that

$$\underline{\int}_I f(x) + g(x)\,dx = \underline{\int}_I f(x)\,dx + \underline{\int}_I g(x)\,dx?$$

If not, can you formulate additional conditions on f and g making the equalities valid?

(iii) Show that, using $-f\colon I \to \mathbb{R}$ defined by $(-f)(x) = -f(x)$,

$$\overline{\int}_I (-f)(x)\,dx = -\underline{\int}_I f(x)\,dx.$$

Exercise 7.6.8. Let $a < b$. Show that a monotonous function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable. Here a function $f\colon A \to \mathbb{R}$, $A \subset \mathbb{R}$, is a *monotonous function* "monotone functie" if f is either an increasing function on A or a decreasing function on A. Hint: take an equidistant partition of $[a,b]$, i.e. $|P| = N$ and for all $J \in \mathscr{P}$ we have $|J| = \frac{b-a}{N}$. Show that in case of an increasing function

$$0 \leq \overline{\int} f(x)\,dx - \underline{\int} f(x)\,dx \leq \frac{b-a}{N}\big(f(b) - f(a)\big).$$

This can be done directly or using Exercise 7.6.1.

Exercise 7.6.9. Note that $f(x) = \frac{1}{x}$ gives a continuous function $f\colon (0,\infty) \to \mathbb{R}$, so that by Corollary 7.3.8 f is integrable on intervals of the form $[1,x]$, $x > 1$ and $[x,1]$, $x > 0$. We define, see Lemma 7.4.3, its primitive as

$$\ln\colon (0,\infty) \to \mathbb{R}, \qquad \ln(x) = \int_{[1,x]} \frac{1}{t}\,dt \ (x \geq 1), \quad \ln(x) = -\int_{[x,1]} \frac{1}{t}\,dt \ (0 < x < 1).$$

Note that the calculus convention $\int_{[a,b]} f(x)\,dx = -\int_{[b,a]} f(x)\,dx$ for $a > b$ would come in handy.

(i) Show that $\ln(xy) = \ln(x) + \ln(y)$ for $x > 1$, $y > 1$. Hint: use Theorem 7.3.1(iii) to split $[1,xy]$ in $[1,x]$ and $(x,xy]$ and use Corollary 7.4.5.

(ii) Adapt Corollary 7.4.5 to show that $\ln(xy) = \ln(x) + \ln(y)$ for $x > 0$, $y > 0$.

(iii) Show that $\ln(x^r) = r\ln(x)$. Hint: first show this for $r \in \mathbb{N}$, next $r \in \mathbb{Z}$ and $r \in \mathbb{Q}$. Then use continuity of the function $x \mapsto a^x$ for $a > 0$ (which you may assume).

(iv) Show that ln is a differentiable function, which is a strictly increasing function. Show that $\lim_{x\to\infty} \ln(x) = \infty$ in the folllowing sense

$$\forall\, M \in \mathbb{R} \quad \exists\, K \in (0,\infty) \quad \forall\, x \geq K \qquad \ln(x) > M.$$

Similarly, show that $\lim_{x\searrow 0} \ln(x) = -\infty$, i.e.

$$\forall\, M \in \mathbb{R} \quad \exists\, \delta > 0 \quad \forall\, x \in (0,\delta) \qquad \ln(x) < M.$$

(v) Show that $\lim_{x\searrow 0} x\ln(x) = 0$. Hint: show that $f(x) = x\ln(x)$, $f\colon (0,a) \to \mathbb{R}$ is negative, differentiable, decreasing (use Exercise 6.5.9) for $a > 0$ sufficiently small. Taking any sequence $(x_n)_{n\in\mathbb{N}}$ in $(0,a)$ converging to 0, show that $\lim_{n\to\infty} f(x_n) = 0$ using Exercise 3.4.4.

(vi) Show that for any $\alpha > 0$ we have $\lim_{x\to\infty} \frac{\ln(x)}{x^\alpha} = 0$ and $\lim_{x\searrow 0} x^\alpha \ln(x) = 0$.

(vii) Argue that $\ln\colon (0,\infty) \to \mathbb{R}$ is a bijection. Let $\exp\colon \mathbb{R} \to (0,\infty)$ be its inverse. Show that exp is a differentiable function, and show that the derivative of exp equals exp and that $\exp(0) = 1$. Hint: use Theorem 6.3.1.

(viii) Show that $\exp(x+y) = \exp(x)\exp(y)$ for all $x, y \in \mathbb{R}$.

(ix) Put $e = \exp(1)$, i.e. $e \in (0,\infty)$, is the unique number satisfying $\int_{[1,e]} \frac{1}{t}\,dt = 1$. Show that $\exp(x) = e^x$. Hint: first show this for $x \in \mathbb{N}$, next $x \in \mathbb{Z}$ and $x \in \mathbb{Q}$. Then use continuity of the function $x \mapsto a^x$ for $a > 0$ (which you may assume). We have $e = 2.718281828459045\cdots$, and in Exercise 8.7.9 you can show that $e \notin \mathbb{Q}$.

(x) Let $\alpha > 0$. Show that $\lim_{x\to\infty} \frac{\exp(x)}{x^\alpha} = \infty$ and $\lim_{x\to\infty} x^\alpha \exp(-x) = 0$.

Exercise 7.6.10. Let $f\colon [a,\infty) \to \mathbb{R}$ a function so that $f|_{[0,R]} \to \mathbb{R}$ is a Riemann integrable function for each $R \geq a$. Then the *improper integral* "oneigenlijke integraal" $\int_{[a,\infty)} f(x)\,dx$ exists if

$$\lim_{R\to\infty} \int_{[a,R]} f(x)\,dx$$

exists, so we assume the limit to be finite. Recall that this means that there exists $L \in \mathbb{R}$, the limit value, so that

$$\forall \varepsilon > 0 \quad \exists M \geq a \quad \forall R \geq M \qquad |L - \int_{[a,R]} f(x)\,dx| < \varepsilon.$$

Then $\int_{[a,\infty)} f(x)\,dx = \lim_{R\to\infty} \int_{[a,R]} f(x)\,dx$ and we say the improper integral is convergent. Otherwise the improper integral is divergent.

(i) Assume that $f\colon [a,\infty) \to \mathbb{R}$ and $g\colon [a,\infty) \to \mathbb{R}$ satisfy $0 \leq f \leq g$. Assume moreover that f and g are Riemann integrable functions on the bounded interval $[a,R]$ for all $R > a$.

  (a) Assume that the improper integral $\int_{[a,\infty)} g(x)\,dx$ is convergent. Show that the improper integral $\int_{[a,\infty)} f(x)\,dx$ is convergent.

  (b) Assume that the improper integral $\int_{[a,\infty)} f(x)\,dx$ is divergent. Show that the improper integral $\int_{[a,\infty)} g(x)\,dx$ is divergent.

(ii) Let $\alpha > 0$. Show that $\int_{[1,\infty)} \frac{1}{x^\alpha}\,dx$ is convergent if and only if $\alpha > 1$.

Exercise 7.6.11. Let $I \subset \mathbb{R}$ be a bounded interval, and $f\colon I \to \mathbb{R}$. Then f is called *piecewise continuous* "stuksgewijs continu" if there exists a partition $\mathcal{P}$ of I so that for all $J \in \mathcal{P}$ the function $f|_J\colon J \to \mathbb{R}$ is continuous. Show that if $f\colon I \to \mathbb{R}$ is piecewise continuous and bounded, then f is a Riemann integrable function on I. Hint: use Theorem 7.3.1.

Exercise 7.6.12. In this exercise we show that not all differentiable functions have Riemann integrable derivatives.

(i) Show that $F(x) = x^2 \sin(x^{-3})$ for $x \neq 0$ and $F(0) = 0$ defines a differentiable function $F\colon [-1,1] \to \mathbb{R}$.

(ii) Show that F' is not Riemann integrable on $[-1,1]$.

Exercise 7.6.13. Let $f\colon (a,b] \to \mathbb{R}$ be a function so that $f|_{[a+\varepsilon,b]}\colon [a+\varepsilon,b] \to \mathbb{R}$ is a Riemann integrable function for each $b-a > \varepsilon > 0$. Then the *improper integral* "oneigenlijke integraal" $\int_{(a,b]} f(x)\,dx$ exists if

$$\lim_{\varepsilon \searrow 0} \int_{[a+\varepsilon,b]} f(x)\,dx$$

exists. Then $\int_{(a,b]} f(x)\,dx = \lim_{\varepsilon \searrow 0} \int_{[a+\varepsilon,b]} f(x)\,dx$ and we say the improper integral is convergent. Otherwise the improper integral is divergent.

(i) State and prove the analogue of Exercise 7.6.10(i) for this improper integral.

(ii) Let $\alpha > 0$. Show that the improper integral $\int_{(0,1]} \frac{1}{x^\alpha}\,dx$ is convergent if and only if $0 < \alpha < 1$. Note that the integrand $\frac{1}{x^\alpha}$ is unbounded on $(0,1]$, and so we cannot discuss its Riemann integrability.

(iii) Let us consider the case $\alpha = 1$. The function $f(x) = \frac{1}{x}$ is well defined on the domain $[-1,0) \cup (0,1]$. Show that for all ε with $1 > \varepsilon > 0$

$$\int_{[-1,-\varepsilon]} \frac{1}{x}\,dx + \int_{[\varepsilon,1]} \frac{1}{x}\,dx = 0$$

even though the limit $\varepsilon \searrow 0$ of each term separately is not finite.

(iv) Prove that for a function $f\colon [-1,1] \to \mathbb{R}$ which is continuously differentiable the limit

$$\lim_{\varepsilon \searrow 0} \int_{[-1,-\varepsilon] \cup [\varepsilon,1]} \frac{f(x)}{x}\,dx$$

exists. This is known as a *principal value integral* "hoofdwaarde-integraal".

Exercise 7.6.14. (i) Show that the improper integral $\int_{[1,\infty)} e^{-t} t^{x-1}\,dt$ is absolutely convergent for $x > 0$.

(ii) Show that the function $t \mapsto e^{-t} t^{x-1}$ is Riemann integrable on $[0,1)$ for $x \geq 1$. And show that for $0 < x < 1$ the improper integral $\int_{(0,1)} e^{-t} t^{x-1}\,dt$ is convergent.

(iii) Conclude that the improper integral $\int_{(0,\infty)} e^{-t} t^{x-1}\,dt$ exists for all $x > 0$, and we define the function $\Gamma\colon (0,\infty) \to \mathbb{R}$ by

$$\Gamma(x) = \int_{(0,\infty)} e^{-t} t^{x-1}\,dt$$

which was introduced by Euler in 1729.

(iv) Show that $\Gamma(x+1) = x\,\Gamma(x)$. Hint: use Corollary 7.4.6 on a bounded interval and take suitable limits to incorporate the improper integrals.

(v) Calculate $\Gamma(1) = 1$. Hint: use Theorem 7.4.4 and Exercise 7.6.9.

(vi) Show that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

Exercise 7.6.15. Let $I \subset \mathbb{R}$ be a bounded interval, and assume $f\colon I \to \mathbb{C}$ is a complex valued function as in Exercise 5.5.17. We define the complex valued function $f\colon I \to \mathbb{C}$ to be Riemann integrable if both real valued functions $\Re f\colon I \to \mathbb{R}$ and $\Im f\colon I \to \mathbb{R}$ are Riemann integrable, and in that case we define

$$\int_I f(x)\,dx = \int_I (\Re f)(x)\,dx + i\int_I (\Im f)(x)\,dx.$$

Discuss and prove the analogues of Theorem 7.3.1 and Corollary 7.3.2, and the Fundamental Theorem of Calculus, Theorems 7.4.1, 7.4.4. Can you also obtain an analogue of Corollary 7.3.2?

Chapter 8

Series

In this chapter we deal with infinite summations, known as series. After introducing convergent series, we discuss several criteria for convergent series, mainly for absolutely convergent series. Then we consider series of functions, especially power series, and we study their convergence properties. For power series we show how such series can be integrated and differentiated. We also discuss conditions which allow to interchange summation in iterated series. Finally, we discuss power series in the complex setting.

8.1 Convergent series

We know how to sum a finite number of terms, and then we can use all the rules for elementary arithmetic. But how do we give meaning to infinite sums? See Chapter 1 for an indication that care is needed. Such an infinite sum is called a *series* “reeks”.

Definition 8.1.1. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Define the* partial sum *“partiële som” $S_N = \sum_{n=0}^{N} a_n$ for $N \in \mathbb{N}$. If the sequence $(s_N)_{N=0}^{\infty}$ is a convergent sequence, then we say that the* series *“reeks” $\sum_{n=0}^{\infty} a_n$ is* convergent *“convergent” and the result is*

$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N.$$

If the sequence $(s_N)_{N=0}^{\infty}$ is not convergent, we say that the series $\sum_{n=0}^{\infty} a_n$ is divergent *“divergent”.*

Note that we label the series starting from 0, and this is not essential as one can see by shifting or relabelling the sequence to be summed. So we can also say that the sequence $\sum_{k=m}^{\infty} a_k$ is convergent if the sequence $(S_N)_{N=m}^{\infty}$ of partial sums $S_N = \sum_{k=m}^{N} a_k$ is convergent.

Exercise 8.1.2. Show that convergence only depends on the tail of the sequence. Or, given the sequence $(a_n)_{n\in\mathbb{N}}$, the series $\sum_{n=0}^{\infty} a_n$ is convergent if and only if there exists $m \in \mathbb{N}$ such that the series $\sum_{n=m}^{\infty} a_n$ is convergent. How are the outcomes of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=m}^{\infty} a_n$ related assuming they are convergent series?

Example 8.1.3. We know the geometric sum

$$\sum_{n=0}^{N} r^n = \begin{cases} \dfrac{1-r^{N+1}}{1-r}, & r \neq 1. \\ N+1, & r = 1. \end{cases}$$

(If you don't know this sum, then you should prove it by induction on N or by proving the identity you get by multiplying both sides by $1-r$ and do the case $r=1$ separately.) Observe that the case $r=1$ can also be obtained by a limit $r \to 1$ using Corollary 6.2.8. Use Exercise 3.4.2 to conclude that the series $\sum_{n=0}^{\infty} r^n$ is convergent if and only if $|r| < 1$ and in that case

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \qquad |r| < 1.$$

This is a very important example, and it is called the *geometric series* "meetkundige reeks". It is used in various results in this chapter.

Since we have defined convergent series in terms of convergent sequences, we can transpose the statements of Theorem 3.2.19 to the case of series.

Theorem 8.1.4. *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be convergent series with $\sum_{n=0}^{\infty} a_n = L$ and $\sum_{n=0}^{\infty} b_n = M$.*

(i) For any $c \in \mathbb{R}$ the series $\sum_{n=0}^{\infty} ca_n$ is convergent and $\sum_{n=0}^{\infty} ca_n = cL$.

(ii) The series $\sum_{n=0}^{\infty}(a_n + b_n)$ is convergent and $\sum_{n=0}^{\infty}(a_n + b_n) = L + M$.

(iii) For $c, d \in \mathbb{R}$ the series $\sum_{n=0}^{\infty}(ca_n + db_n)$ is convergent and $\sum_{n=0}^{\infty}(ca_n + db_n) = cL + dM$.

(iv) Assume that for all $n \in \mathbb{N}$ we have $a_n \leq b_n$, then $L \leq M$, or $\sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{\infty} b_n$.

Exercise 8.1.5. Give a proof of Theorem 8.1.4 using Definition 8.1.1 and Theorem 3.2.19.

As an exercise we discuss the situation of a *telescoping series* "telescopende reeks". This is a series of the form

$$\sum_{n=0}^{\infty}(a_n - a_{n+1}) = (a_0 - a_1) + (a_1 - a_2) + (a_2 - a_3) + \cdots$$

where terms in the n-the summand of the series cancel with terms from the $n+1$-th summand.

Proposition 8.1.6 (Telescoping series)**.** *Let $(a_n)_{n=0}^{\infty}$ be a sequence.*
The series $\sum_{n=0}^{\infty}(a_n - a_{n+1})$ is convergent if and only if the sequence $(a_n)_{n=0}^{\infty}$ is convergent. In that case we have

$$\sum_{n=0}^{\infty}(a_n - a_{n+1}) = a_0 - \lim_{n\to\infty} a_n.$$

Proof. We have to consider the partial sum

$$S_N = \sum_{n=0}^{N}(a_n - a_{n+1}) = \sum_{n=0}^{N} a_n - \sum_{n=1}^{N+1} a_n = a_0 - a_{N+1}$$

by Definition 8.1.1. So we see that the limit of the partial sums exist if and only if $\lim_{N\to\infty} a_{N+1}$ exists. Since this is equivalent with $\lim_{n\to\infty} a_n$ exists, we have proved the first statement.

In this case we have

$$\sum_{n=0}^{\infty}(a_n - a_{n+1}) = \lim_{N\to\infty}(a_0 - a_{N+1}) = a_0 - \lim_{n\to\infty} a_n. \qquad \square$$

See Exercise 8.7.1 for some explicit examples of telescoping series.

The following result leads to a criterium for divergence of a series. It is a simple, but important result.

Theorem 8.1.7. *Assume that the series* $\sum_{n=0}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Corollary 8.1.8. *If* $(a_n)_{n\in\mathbb{N}}$ *is a sequence for which* $\lim_{n\to\infty} a_n$ *does not exist or for which* $\lim_{n\to\infty} a_n = L$ *exists, but* $L \neq 0$*, then the series* $\sum_{n=0}^{\infty} a_n$ *is divergent.*

At a later stage we will see many examples of divergent series $\sum_{n=0}^{\infty} a_n$ for which $a_n \to 0$, the most prominent being the harmonic series, see Example 8.2.5. So the converse of Theorem 8.1.7 is false!

Proof of Theorem 8.1.7. Let $S_N = \sum_{n=0}^{N} a_n$ be the partial sum, then we know that $(S_N)_{N=0}^{\infty}$ is convergent, say $\lim_{N\to\infty} S_N = L$. Then also $\lim_{N\to\infty} S_{N+1} = L$ and we have

$$a_{N+1} = S_{N+1} - S_N \implies \lim_{N\to\infty} a_{N+1} = \lim_{N\to\infty} S_{N+1} - S_N = \lim_{N\to\infty} S_{N+1} - \lim_{N\to\infty} S_N = L - L = 0,$$

using Theorem 3.2.19. $\square$

In the following convergence criterium we use Theorem 3.3.11.

Theorem 8.1.9. *The series* $\sum_{n=0}^{\infty} a_n$ *is convergent if and only if*

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > m \geq N \;:\; \Big|\sum_{j=m+1}^{n} a_j\Big| < \varepsilon.$$

Proof. By definition $\sum_{n=0}^{\infty} a_n$ is convergent if and only if the sequence $(S_N)_{N=0}^{\infty}$ of partial sums is convergent. By Theorem 3.3.11 this is equivalent to $(S_N)_{N=0}^{\infty}$ being a Cauchy sequence, which by Definition 3.3.1 means

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n, m \geq N \;:\; |S_n - S_m| < \varepsilon.$$

We can assume that $n > m$ without loss of generality, and we observe that

$$S_n - S_m = \sum_{j=0}^{n} a_j - \sum_{j=0}^{m} a_j = \sum_{j=m+1}^{n} a_j$$

proving the result. $\square$

Definition 8.1.10. *The series* $\sum_{n=0}^{\infty} a_n$ *is* absolutely convergent *"absoluut convergent" if the series* $\sum_{n=0}^{\infty} |a_n|$ *is a convergent series.*

Example 8.1.11. The geometric series of Example 8.1.3 is absolutely convergent for $|r| < 1$.

The notion of absolute convergence is stronger than the notion of convergence.

Proposition 8.1.12. *If the series* $\sum_{n=0}^{\infty} a_n$ *is absolutely convergent, then the series* $\sum_{n=0}^{\infty} a_n$ *is convergent. Moreover, in that case*

$$\Big|\sum_{n=0}^{\infty} a_n\Big| \leq \sum_{n=0}^{\infty} |a_n|.$$

Proof. Note that for finite sums we have

$$\Big|\sum_{i=n}^{m} a_i\Big| \leq \sum_{i=n}^{m} |a_i|, \tag{8.1.1}$$

which follows from the triangle inequality (2.1.1) and induction on the number of terms in the sum. Since the series $\sum_{n=0}^{\infty} |a_n|$ converges, we can reformulate by Theorem 8.1.9 that

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n > m \geq N \,:\, \Big|\sum_{j=m+1}^{n} a_j\Big| \leq \sum_{j=m+1}^{n} |a_j| < \varepsilon.$$

using (8.1.1). By Theorem 8.1.9 it follows that $\sum_{n=0}^{\infty} a_n$ is convergent.

With $S_N = \sum_{n=0}^{N} a_n$ and $T_N = \sum_{n=0}^{N} |a_n|$, we have $|S_N| \leq T_N$ for all $N \in \mathbb{N}$. Taking the limit and using that the absolute value is a continuous function and Proposition 5.1.10 we see that

$$\Big|\sum_{n=0}^{\infty} a_n\Big| = |\lim_{N\to\infty} S_N| = \lim_{N\to\infty} |S_N| \leq \lim_{N\to\infty} T_N = \sum_{n=0}^{\infty} |a_n|.$$ □

Definition 8.1.13. *The series* $\sum_{n=0}^{\infty} a_n$ *is called* relatively convergent *"relatief convergent" or "voorwaardelijk convergent" if* $\sum_{n=0}^{\infty} a_n$ *is convergent and* $\sum_{n=0}^{\infty} |a_n|$ *is divergent.*

From this discussion it is clear that we have to pay attention to series with positive terms. We first reformulate Theorem 3.2.9.

Theorem 8.1.14. *Let* $(a_n)_{n\in\mathbb{N}}$ *satisfy* $\forall\, n \in \mathbb{N}\ a_n \geq 0$*, and let* $(S_N)_{N\in\mathbb{N}}$ *be sequence of partial sums;* $S_N = \sum_{n=0}^{N} a_n$*. Then* $\sum_{n=0}^{\infty} a_n$ *is convergent if and only if the sequence* $(S_N)_{N\in\mathbb{N}}$ *is bounded. In that case*

$$\sum_{n=0}^{\infty} a_n = \sup_{N\in\mathbb{N}} S_N.$$

Proof. First observe that if $\sum_{n=0}^{\infty} a_n$ is convergent, the sequence $(S_N)_{N\in\mathbb{N}}$ is convergent, and hence, by Proposition 3.2.8, bounded. Conversely, note that

$$S_{N+1} = \sum_{n=0}^{N+1} a_n = S_N + a_{N+1} \geq S_N$$

since $a_{N+1} \geq 0$. So $(S_N)_{N\in\mathbb{N}}$ is an increasing sequence, and it is bounded by assumption. So, by Theorem 3.2.9, $(S_N)_{N\in\mathbb{N}}$ is convergent, meaning that $\sum_{n=0}^{\infty} a_n$ is convergent.

In case this condition is satisfied, Theorem 3.2.9 also implies the value of the series as a limit of the sequence of partial sums. □

Theorem 8.1.14 allows to find a characterisation of relatively convergent series. This characterisation can be used to show that we can attach any value to a relatively convergent sequence by changing the order of summation, see Exercise 8.7.6 for the details in a specific example.

Proposition 8.1.15. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence, and put for all $n \in \mathbb{N}$ $a_n^+ = \max(a_n, 0)$, $a_n^- = -\min(a_n, 0)$, so that $a_n = a_n^+ - a_n^-$, $|a_n| = a_n^+ + a_n^-$ and $(a_n^+)_{n\in\mathbb{N}}$ and $(a_n^-)_{n\in\mathbb{N}}$ are non-negative sequences. If $\sum_{n=0}^{\infty} a_n$ is relatively convergent, then both $\sum_{n=0}^{\infty} a_n^+$ and $\sum_{n=0}^{\infty} a_n^-$ are divergent.*

Remark 8.1.16. Note that a convergent series $\sum_{n=0}^{\infty} a_n$ for which at least one of the series $\sum_{n=0}^{\infty} a_n^+$ or $\sum_{n=0}^{\infty} a_n^-$ diverges, cannot be absolutely convergent since $|a_n| \geq a_n^+$ and $|a_n| \geq a_n^-$.

Proof of Proposition 8.1.15. Let $S_N = \sum_{n=0}^{N} a_n$, $S_N^+ = \sum_{n=0}^{N} a_n^+$, $S_N^- = \sum_{n=0}^{N} a_n^-$, be the corresponding partial sums. Then $(S_N^+)_{N\in\mathbb{N}}$ and $(S_N^-)_{N\in\mathbb{N}}$ are increasing sequences, so that by Theorem 8.1.14 the boundedness of $(S_N^+)_{N\in\mathbb{N}}$, respectively $(S_N^-)_{N\in\mathbb{N}}$, determines the convergence $\sum_{n=0}^{\infty} a_n^+$, respectively $\sum_{n=0}^{\infty} a_n^-$.

We prove Proposition 8.1.15 by excluding all the other possibilities. First, assume that $(S_N^+)_{N\in\mathbb{N}}$ and $(S_N^-)_{N\in\mathbb{N}}$ are bounded. Since $|a_n| = a_n^+ + a_n^-$, it follows by Theorem 3.2.19 that $\sum_{n=0}^{\infty} |a_n|$ is convergent as sum of two convergent series. This contradicts that the series $\sum_{n=0}^{\infty} a_n$ is not absolutely convergent.

Next we assume that one sequence of partial sums is bounded, and that the other one is unbounded. We assume that $(S_N^+)_{N\in\mathbb{N}}$ is bounded and that $(S_N^-)_{N\in\mathbb{N}}$ is unbounded. The other case proceeds similarly. This means that $\sum_{n=0}^{\infty} a_n^+$ is convergent and $\sum_{n=0}^{\infty} a_n^-$ is divergent. Then we write $a_n^- = a_n^+ - a_n$, and since the last two terms correspond to a convergent series, we have that $\sum_{n=0}^{\infty} a_n^-$ is convergent by Theorem 8.1.4, contradicting the divergence.

So the only possibility remaining is that $\sum_{n=0}^{\infty} a_n^+$ and $\sum_{n=0}^{\infty} a_n^-$ both diverge, and the sequences of the partial sums diverge to ∞. □

We have indicated that relatively convergent series behave badly under rearrangement of the series, see Exercise 8.7.6. The absolutely convergent series do not suffer from this drawback.

Theorem 8.1.17. *Assume that $\sum_{n=0}^{\infty} a_n$ is an absolutely convergent series. Let $\tau\colon \mathbb{N} \to \mathbb{N}$ be a bijection, then $\sum_{n=0}^{\infty} a_{\tau(n)}$ is absolutely convergent and*

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_{\tau(n)}.$$

Proof. We first prove the statement under the additional assumption that $\forall\, n \in \mathbb{N}$ we have $a_n \geq 0$. Let $S_N = \sum_{n=0}^{N} a_n$, then, by Theorem 8.1.14, there exists $M \in \mathbb{R}$ so that $\forall\, N \in \mathbb{N}$ we have $S_N \leq M$. We take $M = \sup_{N\in\mathbb{N}} S_N$. Now, using that $a_n \geq 0$ for all n,

$$\sum_{k=0}^{K} a_{\tau(k)} \leq \sum_{n=0}^{N} a_n \leq M, \qquad N = \max_{k\in\{0,\cdots,K\}} \tau(k).$$

It follows that the sequence of partial sums for $\sum_{n=0}^{\infty} a_{\tau(n)}$ is bounded by M, so that by Theorem 8.1.14 $\sum_{n=0}^{\infty} a_{\tau(n)}$ is convergent. Moreover, taking the limit $K \to \infty$ in this inequality and using Theorem 8.1.14 again, we have

$$\sum_{k=0}^{\infty} a_{\tau(k)} \leq M = \sup_{N\in\mathbb{N}} S_N = \sum_{n=0}^{\infty} a_n.$$

Now apply this to the series $\sum_{n=0}^{\infty} b_n$ with $b_n = a_{\tau(n)}$ and replace τ by τ^{-1}, to see that $\sum_{k=0}^{\infty} a_{\tau(k)} = \sum_{n=0}^{\infty} a_n$.

In the general case we write, cf. Proposition 8.1.15, $a_n = a_n^+ - a_n^-$, with $a_n^\pm \geq 0$ and $|a_n| = a_n^+ + a_n^-$. It follows that

$$\sum_{n=0}^{N} a_n^+ \leq \sum_{n=0}^{N} |a_n|$$

and this is bounded independent of N, since $\sum_{n=0}^{\infty} a_n$ is absolutely convergent. So we conclude that $\sum_{n=0}^{\infty} a_n^+$, and similarly $\sum_{n=0}^{\infty} a_n^-$, is convergent. Since $a_{\tau(n)} = a_{\tau(n)}^+ - a_{\tau(n)}^-$, we have by the proof for the case of a series with positive terms that

$$\sum_{n=0}^{\infty} a_{\tau(n)}^+ = \sum_{n=0}^{\infty} a_n^+, \qquad \sum_{n=0}^{\infty} a_{\tau(n)}^- = \sum_{n=0}^{\infty} a_n^-.$$

Now use Theorem 8.1.4 to see that

$$\sum_{n=0}^{\infty} a_{\tau(n)} = \sum_{n=0}^{\infty} a_{\tau(n)}^+ - \sum_{n=0}^{\infty} a_{\tau(n)}^-, \qquad \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n^+ - \sum_{n=0}^{\infty} a_n^-,$$

and this finishes the proof in the general case. □

8.2 Convergence criteria for series

In this section we discuss some more general convergence criteria. Note that we have already described several convergence criteria in Proposition 8.1.6, Theorem 8.1.9, Theorem 8.1.14 and a divergence criterium in Corollary 8.1.8. Most of the criteria deal with series of positive terms, and hence deal with criteria for absolute convergence. The exception to this is the first criterion.

Theorem 8.2.1 (Alternating series or Leibniz criterion "alternerende reekscriterium")**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence of non-negative terms, i.e. $a_n \geq 0$ and $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$. Then the alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ is convergent if and only if* $\lim_{n\to\infty} a_n = 0$.

Proof. If $\sum_{n=0}^{\infty}(-1)^n a_n$ is convergent, then, by Theorem 8.1.7, we have $\lim_{n\to\infty}(-1)^n a_n = 0$. Using Exercise 3.2.3 we also have $\lim_{n\to\infty} a_n = 0$.

Conversely, let $S_N = \sum_{n=0}^{N}(-1)^n a_n$ be the partial sum. Then we have

$$S_{2N+2} = S_{2N} + \underbrace{a_{2N+2} - a_{2N+1}}_{\leq 0} \leq S_{2N} \qquad S_{2N+1} = S_{2N-1} + \underbrace{a_{2N} - a_{2N+1}}_{\geq 0} \geq S_{2N-1}$$

so we find that the subsequence $(S_{2N})_{N\in\mathbb{N}}$ is decreasing and the subsequence $(S_{2N+1})_{N\in\mathbb{N}}$ is increasing. Moreover, we can compare the subsequences, since for any N we have

$$S_1 \leq S_3 \leq \cdots \leq S_{2N+1} = S_{2N} - a_{2N+1} \leq S_{2N} \leq S_{2N-2} \leq \cdots \leq S_2 \leq S_0.$$

So $(S_{2N})_{N\in\mathbb{N}}$ is a decreasing sequence, which is bounded from below, so that by Theorem 3.2.9 the sequence is convergent. Similarly, $(S_{2N+1})_{N\in\mathbb{N}}$ is an increasing sequence, which is bounded from above and hence convergent. So the limits exist and

$$\lim_{N\to\infty} S_{2N} = L, \qquad \lim_{N\to\infty} S_{2N+1} = M,$$

and it remains to prove that $\lim_{N\to\infty} S_N$ exists if and only if $L = M$ if and only if $\lim_{n\to\infty} a_n = 0$. This is referred to Exercise 8.2.2. □

Exercise 8.2.2. We finish the proof of Theorem 8.2.1.

(i) Assume that subsequences of even and odd parts of $(a_n)_{n=0}^{\infty}$ are convergent; i.e. $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = M$. Prove that $(a_n)_{n=0}^{\infty}$ is convergent if and only if $L = M$.

(ii) In the context of the proof of Theorem 8.2.1 show that with $\lim_{N\to\infty} S_{2N} = L$, $\lim_{N\to\infty} S_{2N+1} = M$ we have $L = M$ if and only if $\lim_{n\to\infty} a_n = 0$.

All the other convergence criteria are focused on series with positive terms, so essentially for establishing absolute converence.

Theorem 8.2.3 (Majorising criteron)**.** *Let $(a)_{n\in\mathbb{N}}$, $(b)_{n\in\mathbb{N}}$ be sequences satisfying $\forall\, n \in \mathbb{N}$ $0 \leq a_n \leq b_n$. Then we have the following statements:*

(i) *if* $\sum_{n=0}^{\infty} b_n$ *is a convergent series, then* $\sum_{n=0}^{\infty} a_n$ *is a convergent series;*

(ii) *if* $\sum_{n=0}^{\infty} a_n$ *is a divergent series, then* $\sum_{n=0}^{\infty} b_n$ *is a divergent series.*

We say the series $\sum_{n=0}^{\infty} b_n$ *majorises* "majoriseert" the series $\sum_{n=0}^{\infty} b_n$, or equivalently that $\sum_{n=0}^{\infty} a_n$ *minorises* "minoriseert" the series $\sum_{n=0}^{\infty} a_n$. A related, but more refined convergence criterion is in Exercise 8.7.5.

Proof. Let $S_N = \sum_{n=0}^{N} a_n$, $T_N = \sum_{n=0}^{N} b_n$ be the partial sums for the series. Then $(S_N)_{N=0}^{\infty}$ and $(T_N)_{N=0}^{\infty}$ are increasing sequences, see proof of Theorem 8.1.14. Then for all $N \in \mathbb{N}$ we have $S_N \leq T_N$. In particular, if $\sum_{n=0}^{\infty} b_n$ is a convergent series, then $(T_N)_{N=0}^{\infty}$ is a bounded sequence by Theorem 8.1.14, so that $(S_N)_{N=0}^{\infty}$ is a bounded sequence. Again by Theorem 8.1.14 this means that $\sum_{n=0}^{\infty} a_n$ is a convergent series.

Similarly, in case $\sum_{n=0}^{\infty} a_n$ is a divergent series, the sequence $(S_N)_{N=0}^{\infty}$ is an unbounded sequence by Theorem 8.1.14. Hence, $(T_N)_{N=0}^{\infty}$ is an unbounded sequence, and, again by Theorem 8.1.14, $\sum_{n=0}^{\infty} b_n$ is a divergent series. □

Theorem 8.2.4 (Cauchy)**.** *Assume that* $(a_n)_{n\in\mathbb{N}}$ *is a decreasing non-negative sequence. Then* $\sum_{n=0}^{\infty} a_n$ *is a convergent series if and only if* $\sum_{k=0}^{\infty} 2^k a_{2^k}$ *is a convergent series.*

Example 8.2.5. An important application of Cauchy's Theorem 8.2.4 is that we can consider the sequence

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^\alpha} = \sum_{m=1}^{\infty} \frac{1}{m^\alpha}.$$

We see that this series is convergent if and only if

$$\sum_{k=0}^{\infty} \frac{2^k}{(2^k+1)^\alpha}$$

is a convergent series. For $\alpha > 1$, we see that we can majorise this series by $\sum_{k=0}^{\infty} \frac{2^k}{(2^k)^\alpha} = \sum_{k=0}^{\infty} (2^{1-\alpha})^k$, which is a convergent geometric series, see Example 8.1.3. If $0 < \alpha \leq 1$, we see that $\lim_{k\to\infty} \frac{2^k}{(2^k+1)^\alpha} \neq 0$, so that by Corollary 8.1.8 the series is divergent. In conclusion we have for $\alpha > 0$ that

$$\sum_{m=1}^{\infty} \frac{1}{m^\alpha}$$

is a convergent series if and only if $\alpha > 1$. The case $\alpha = 1$ is known as the *harmonic series* "harmonische reeks", and in particular the harmonic series $\sum_{m=1}^{\infty} \frac{1}{m}$ is a divergent series.

Exercise 8.2.6. We indicate a proof of Theorem 8.2.4. Denote by $S_N = \sum_{n=0}^{N} a_n$ and $T_K = \sum_{k=0}^{K} 2^k a_{2^k}$ the corresponding partial sums.

(i) Show that for $k \in \mathbb{N}$ and $k \geq 1$ we have

$$2^{k-1} a_{2^{k-1}} \geq a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k} \geq 2^{k-1} a_{2^k}$$

using that $(a_n)_{n\in\mathbb{N}}$ is a decreasing sequence.

(ii) Sum the inequalities over $k \in \{1, \cdots, K\}$ to get

$$T_{K-1} \geq S_{2^K} - (a_0 + a_1) \geq \frac{1}{2}(T_K - a_1)$$

(iii) Using that $(a_n)_{n\in\mathbb{N}}$ is a nonnegative sequence, derive from the inequalities from (ii) that $(S_N)_{N=0}^{\infty}$ is a bounded sequence if and only if $(T_K)_{K=0}^{\infty}$ is bounded.

(iv) Finish the proof of Theorem 8.2.4.

The following criteria are based on comparison with the geometric series, see Example 8.1.3 and Example 8.1.11.

Theorem 8.2.7 (Cauchy)**.** *Consider the series $\sum_{n=0}^{\infty} a_n$, and let*

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \in [0, \infty]$$

where we put $\alpha = \infty$ if the sequence diverges to infinity, see Remark 3.2.6. Then we have the following statements:

(i) if $\alpha < 1$, then $\sum_{n=0}^{\infty} a_n$ is an absolutely convergent series;

(ii) if $\alpha > 1$, then $\sum_{n=0}^{\infty} a_n$ is a divergent series.

The criterion of Theorem 8.2.7 is also known as the *root criterion* “wortelkenmerk”.

Proof. Assume $1 < \alpha < \infty$, then we pick $\varepsilon > 0$ so that $\alpha - \varepsilon > 1$. Then there exists $N \in \mathbb{N}$ so that for all $K \geq N$ we have

$$\sup_{n\geq K} \sqrt[n]{|a_n|} > \alpha - \varepsilon \quad \Longrightarrow \quad \forall\, N \in \mathbb{N} \quad \exists\, k \geq N \qquad |a_k| > (\alpha - \varepsilon)^k.$$

Since $\alpha - \varepsilon > 1$, we see that $\lim_{n\to\infty} a_n$ is not equal to 0, so that by Corollary 8.1.8 the series is divergent. The case $\alpha = +\infty$ is in Exercise 8.2.8. This proves the second statement.

Next we assume $0 \leq \alpha < 1$, recall that from Definition 3.3.4

$$\alpha = \lim_{N\to\infty} \sup_{n\geq N} \sqrt[n]{|a_n|} = \inf_{N\in\mathbb{N}} \sup_{n\geq N} \sqrt[n]{|a_n|}.$$

Choose $\varepsilon > 0$ so that $\alpha + \varepsilon < 1$, then there exists $N \in \mathbb{N}$ so that for all $K \geq N$ we have $\sup_{n\geq K} \sqrt[n]{|a_n|} < \alpha + \varepsilon$. This implies that for all $n \geq N$ we have

$$\sqrt[n]{|a_n|} \leq \alpha + \varepsilon \quad \Longrightarrow \quad |a_n| \leq (\alpha + \varepsilon)^n$$

so that we can majorise the series $\sum_{n=N}^{\infty} |a_n|$ by the the tail of the geometric series $\sum_{n=N}^{\infty}(\alpha + \varepsilon)^n$. Since $0 < \alpha + \varepsilon < 1$, this series is absolutely convergent, and so by Theorem 8.2.3 the series $\sum_{n=N}^{\infty} |a_n|$ is convergent. By Exercise 8.1.2, we see that the tail of the series determines the convergence properties, so $\sum_{n=0}^{\infty} |a_n|$ is convergent. $\square$

Exercise 8.2.8. Give a proof of the case $\alpha = +\infty$ of Theorem 8.2.7. Mimick the case of $1 < \alpha < \infty$ by showing that $\lim_{n\to\infty} a_n$ is not equal to 0 using Remark 3.2.6.

Theorem 8.2.9 (d'Alembert)**.** *Consider the series $\sum_{n=0}^{\infty} a_n$, and assume that for all $n \in \mathbb{N}$ we have $a_n \neq 0$. Then we have the following statements:*

(i) if $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then $\sum_{n=0}^{\infty} a_n$ is an absolutely convergent series;

(ii) if $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then $\sum_{n=0}^{\infty} a_n$ is a divergent series.

This criterion is also known as the *quotient criterion* "quotiëntenkenmerk". Theorem 8.2.9 follows directly from Theorem 8.2.7 and Exercise 3.4.14. See Exercise 8.2.11.

Remark 8.2.10. Note that for the series $\sum_{m=1}^{\infty} \frac{1}{m^\alpha}$ we have

$$\lim_{m\to\infty} \frac{(m+1)^\alpha}{m^\alpha} = 1, \qquad \lim_{m\to\infty} \frac{1}{\sqrt[m]{m^\alpha}} = 1$$

for all $\alpha > 0$ and by Example 8.2.5 we see that for $0 < \alpha \leq 1$ the series is divergent and for $\alpha > 1$ the series is absolutely convergent. So we conclude that in case $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1$ in Theorem 8.2.7 and that in case $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1$ in Theorem 8.2.9 we cannot come to a conclusion on the convergence properties of the involved series $\sum_{n=0}^{\infty} a_n$.

Exercise 8.2.11. Here we indicate a direct proof of Theorem 8.2.9 by comparing the series with the geometric series as in the proof of Theorem 8.2.7.

(i) Assume $L = \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, show that there exists $0 < r < 1$ and $N \in \mathbb{N}$ so that for all $n \geq N$ we have $\frac{|a_{n+1}|}{|a_n|} \leq r$. Hint: choose $\varepsilon > 0$ so that $r = L + \varepsilon < 1$, and use Definition 3.3.4.

(ii) Continuing (i), show that $\sum_{n=N}^{\infty} a_n$ is absolutely convergent by comparing it to the geometric series, see Example 8.1.3, Example 8.1.11. Hint: show that $|a_{N+k}| \leq |a_N| r^k$ for $k \in \mathbb{N}$.

(iii) Continuing (i) and (ii), show that $\sum_{n=0}^{\infty} a_n$ is absolutely convergent. Hint: use Exercise 8.1.2.

(iv) Assume $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$. Show that $\lim_{n\to\infty} a_n$ is not zero. Hint: compare the proof of Theorem 8.2.7(i).

8.3 Power series

In this section we consider series where each of the term is a particular kind of function, namely a power function (i.e. of the form x^n). It is related to extending the Taylor polynomials as in Section 6.4 to a series. We start more generally with a series of functions.

Definition 8.3.1. *Assume $A \subset \mathbb{R}$ and that for all $n \in \mathbb{N}$ we have a function $f_n\colon A \to \mathbb{R}$, then we say that the* series $\sum_{n=0}^{\infty} f_n$ converges pointwise *"reeks $\sum_{n=0}^{\infty} f_n$ convergeert puntsgewijs" if the sequence $(S_N)_{N=0}^{\infty}$ of functions defined by $S_N\colon A \to \mathbb{R}$ with $S_N(x) = \sum_{n=0}^{N} f_n(x)$ converges pointwise to a function $f\colon A \to \mathbb{R}$, and then $\sum_{n=0}^{\infty} f_n = f$ pointwise. Similarly, the* series $\sum_{n=0}^{\infty} f_n$ converges uniformly *"reeks $\sum_{n=0}^{\infty} f_n$ convergeert uniform" if the sequence $(S_N)_{N=0}^{\infty}$ of functions converges uniformly to a function $f\colon A \to \mathbb{R}$, and then $\sum_{n=0}^{\infty} f_n = f$ uniformly.*

Theorem 8.3.2 gives a criterion for uniform convergence, which is known as the *Weierstrass M-test.*

Theorem 8.3.2 (Weierstrass)**.** *Let $A \subset \mathbb{R}$ and for all $n \in \mathbb{N}$ we have a function $f_n\colon A \to \mathbb{R}$. Assume that there exists a sequence $(M_n)_{n\in\mathbb{N}}$ with*

$$\forall\, n \in \mathbb{N} \quad \forall\, x \in A \qquad |f_n(x)| \leq M_n.$$

If the series $\sum_{n=0}^{\infty} M_n$ converges, then $\sum_{n=0}^{\infty} f_n$ converges uniformly.

Proof. Let $S_N(x) = \sum_{n=0}^{N} f_n(x)$ be the partial sum, and assume $\sum_{n=0}^{\infty} M_n$ converges. For $\varepsilon > 0$ there exists $K \geq 0$ so that for all $N > M \geq K$ we have

$$|S_N(x) - S_M(x)| \leq \sum_{n=M+1}^{N} |f_n(x)| \leq \sum_{n=M+1}^{N} M_n < \varepsilon$$

using Theorem 8.1.9. Now Theorem 5.4.5 shows that the convergence is uniform. □

In general, the convergence properties of series of functions depend on the structure of the functions f_n. We consider the special case that the f_n's are powers of x.

Definition 8.3.3. *A* power series *"machtreeks" is a series of the form*

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

for a sequence $(a_n)_{n=0}^{\infty}$, which are the coefficients of the power series *"coëfficiënten van de machtreeks", and $c \in \mathbb{R}$ is the* centre *"centrum" or "middelpunt" of the power series. Finally, $x \in \mathbb{R}$ is considered to be a variable.*

A power series is a rather formal object, since we do not know what the convergence properties are of the series. But it is clear that it converges for $x = c$, since then all terms except the $n = 0$ term (recall that $0^0 = 1$) vanish. Note that the centre is not that important, since we can always shift the centre to 0 by putting $y = x - c$. The region of convergence of a power series is particularly nice, and this is the content of the next result.

Theorem 8.3.4. *For the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ we have one of the following options:*

(i) the power series converges for all $x \in \mathbb{R}$;

(ii) the power series converges only for $x = c$;

(iii) there exists $R \in (0, \infty)$ so that for all $x \in (c - R, c + R)$ the power series is absolutely convergent and for all $x \in (-\infty, c - R) \cup (c + R, \infty)$ the power series is divergent.

Then $R = \frac{1}{\alpha}$ with $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ with the convention that $\alpha = 0$ corresponds to (i) and $\alpha = \infty$ corresponds to (ii). Moreover, the power series converges uniformly on $[c - R + \varepsilon, c + R - \varepsilon]$ for any $\varepsilon > 0$.

Definition 8.3.5. *For a power series $\sum_{n=0}^{\infty} a_n(x - c)^n$ we define the* radius of convergence *"convergentiestraal" $R \in [0, \infty)$ as in Theorem 8.3.4(ii) and we set the radius of convergence $R = \infty$ in case of Theorem 8.3.4(i). The radius of convergence is in $[0, \infty] = [0, \infty) \cup \{\infty\}$.*

Since the f_n's are continuous functions, we see that $f\colon (c - R, c + R) \to \mathbb{R}$, $f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$ is a continuous function by Theorem 5.4.4.

Lemma 8.3.6 indicates why there is an interval of convergence as in Theorem 8.3.4, again by comparing to the geometric series.

Lemma 8.3.6. *Assume the power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $x \in \mathbb{R}$ with $|x| = r$, then it also converges absolutely for all x with $|x| < r$.*

Proof. The assumption shows that $\sum_{n=0}^{\infty} |a_n| r^n$ is a convergent series with positive summands. Pick $x \in \mathbb{R}$ with $|x| = s < r$, then we write

$$|a_n x^n| = |a_n| s^n = |a_n| r^n \left(\frac{s}{r}\right)^n.$$

Since $\sum_{n=0}^{\infty} |a_n| r^n$ is a convergent series, we have that $(|a_n| r^n)_{n=0}^{\infty}$ is bounded, since it is a convergent sequence by Theorem 8.1.7 and using Proposition 3.2.8. So $\exists\, M$ with $|a_n| r^n \leq M$ for all $n \in \mathbb{N}$. Put $b_n = M t^n$, $t = \frac{s}{r} \in [0, 1)$, then $0 \leq |a_n x^n| \leq b_n$ and $\sum_{n=0}^{\infty} b_n$ is convergent, being a multiple of the geometric series of Example 8.1.3. So Theorem 8.2.3 shows that $\sum_{n=0}^{\infty} |a_n x^n|$ is convergent. □

For the proof of Theorem 8.3.4 we don't need Lemma 8.3.6.

Proof of Theorem 8.3.4. Without loss of generality we can take the centre $c = 0$. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = |x| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = |x|\, \alpha.$$

By Theorem 8.2.7 we see that $|x|\,\alpha < 1$ implies that the power series $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent, and that $|x|\,\alpha > 1$ implies that the power series $\sum_{n=0}^{\infty} a_n x^n$ is divergent.

Pick $\varepsilon > 0$ and $\varepsilon < R$, then for all $|x - c| \leq R - \varepsilon$ we have

$$|a_n(x - c)^n| \leq |a_n||R - \varepsilon|^n, \qquad \sum_{n=0}^{\infty} |a_n||R - \varepsilon|^n < \infty$$

so that by the Weierstrass M-test Theorem 8.3.2 with $f_n(x) = a_n(x - c)^n$, $M_n = |a_n||R - \varepsilon|^n$ the uniform convergence follows. □

Exercise 8.3.7. Show that we can also characterise the radius of convergence in the following two ways

$$R = \sup\{r \geq 0 \mid (|a_n| r^n)_{n=0}^\infty \text{ is a bounded sequence}\}$$
$$R = \sup\{r \geq 0 \mid \sum_{n=0}^\infty |a_n| r^n \text{ is a convergent series}\}$$

with the appropriate adaptation in case $R = \infty$.

Example 8.3.8. Writing the geometric series as

$$\sum_{n=0}^\infty x^n = \frac{1}{1-x}, \qquad |x| < 1$$

we see that $R = 1$. The series $\sum_{n=0}^\infty \frac{x^n}{n!}$ has radius of convergence $R = \infty$, and we see from Exercise 8.7.8 that this is the exponential function $\exp(x) = e^x$, see Exercise 7.6.9. An example of a power series with radius of convergence $R = 0$ is $\sum_{n=0}^\infty n^n x^n$.

8.4 Functions represented by power series

We assume that we have a power series $\sum_{n=0}^\infty a_n (x-c)^n$ with radius of convergence R. Then

$$f\colon (c-R, c+R) \to \mathbb{R}, \qquad f(x) = \sum_{n=0}^\infty a_n (x-c)^n \tag{8.4.1}$$

defines a continuous function by Theorem 8.3.4 and Theorem 5.4.4.

Theorem 8.4.1. *For any $\varepsilon > 0$ the function f of (8.4.1) is Riemann integrable on the interval $[c-R+\varepsilon, c+R-\varepsilon]$ and its antiderivative function F normalised by $F(c) = 0$ has a power series expansion*

$$F(x) = \sum_{n=0}^\infty \frac{a_n}{n+1}(x-c)^{n+1} = \sum_{n=1}^\infty \frac{a_{n-1}}{n}(x-c)^n$$

with the same radius of convergence. The function f of (8.4.1) is differentiable and its derivative f' has a power series expansion

$$f'(x) = \sum_{n=1}^\infty n\, a_n\, (x-c)^{n-1} = \sum_{n=0}^\infty (n+1) a_{n+1} (x-c)^n$$

with the same radius of convergence.

Iterating the last statement we get that functions defined by power series have derivatives of all order, and by evaluating at c we obtain the relation between f and the coefficients a_n.

Corollary 8.4.2. *The function f of* (8.4.1) *has derivatives of all order, and*

$$\forall\, n \in \mathbb{N} \qquad a_n = \frac{f^{(n)}(c)}{n!}.$$

In particular, if $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ and $f(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$ are power series for the function f on $(c-R, c+R)$ for some $R > 0$, then $b_n = c_n$ for all $n \in \mathbb{N}$.

Corollary 8.4.2 motivates that a power series for the function f is also called the Taylor series.

Proof of Theorem 8.4.1. First observe that

$$\limsup_{n\to\infty} \sqrt[n]{|\frac{a_{n-1}}{n}|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} \limsup_{n\to\infty} \sqrt[n]{|a_{n-1}|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

by Exercise 3.4.14 and Theorem 3.2.19. By Theorem 8.3.4 we see that the radius of convergence for the power series for F is equal to R.

Note that for any $\varepsilon > 0$ the function $f\colon [c - R + \varepsilon, c + R - \varepsilon] \to \mathbb{R}$ is continuous, and thus Riemann integrable by Corollary 7.3.8. With $S_N(x) = \sum_{n=0}^{N} a_n (x-c)^n$ we have $\lim_{N\to\infty} S_N = f$ uniformly by Theorem 8.3.4. By Theorem 7.5.1 we have

$$F(x) = \int_{[c,x]} f(y)\, dy = \lim_{N\to\infty} \int_{[c,x]} S_N(y)\, dy = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{a_n}{n+1} (x-c)^{n+1}$$

which gives the result.

Considering the derivative, we first check that the radius of convergence is the same;

$$\limsup_{n\to\infty} \sqrt[n]{|(n+1)a_{n+1}|} = \lim_{n\to\infty} \sqrt[n]{n+1} \limsup_{n\to\infty} \sqrt[n]{|a_{n+1}|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

by Exercise 3.4.14 and Theorem 3.2.19. By Theorem 8.3.4 we see that the radius of convergence for the power series for f' is equal to R.

We now assume that $c = 0$. We give a direct proof for the derivative. Using Newton's binomial formula (2.1.3) we obtain for $h \neq 0$

$$\big((x+h)^n - x^n\big) = hnx^{n-1} + \sum_{k=2}^{n} \binom{n}{k} h^k x^{n-k} \qquad \Longrightarrow$$

$$|(x+h)^n - x^n - hnx^{n-1}| \leq \frac{|h|^2}{\delta^2} \sum_{k=2}^{n} \binom{n}{k} \delta^k |x|^{n-k} \leq \frac{|h|^2}{\delta^2} (\delta + |x|)^n$$

assuming that $|h| < \delta$. Now assume that $|x| < R$ and $\delta > 0$ so that $|x| + \delta < R$, then the series

$$\sum_{n=0}^{\infty} a_n x^n, \quad \sum_{n=0}^{\infty} a_n (x+h)^n, \quad g(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

are absolutely convergent power series for $|h| < \delta$. Then we have

$$\Big|\frac{f(x+h)-f(x)}{h} - g(x)\Big| = \Big|\sum_{n=2}^{\infty} a_n\Big(\frac{(x+h)^n - x^n}{h} - nx^{n-1}\Big)\Big| \le \frac{|h|}{\delta^2}\sum_{n=2}^{\infty} |a_n|(\delta+|x|)^n.$$

Since the series in the right hand side is convergent and independent of h, we find that

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = g(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

proving the required result. □

Exercise 8.4.3. The proof of the statement of Theorem 8.4.1 for the derivative of a power series has been proved directly using Definition 6.1.1 of the derivative. Show that this statement can also be derived from Proposition 7.5.2. Hint: let f_n correspond to the partial sum of the power series.

Remark 8.4.4. Theorem 8.4.1 shows that the class of functions $f\colon (c-R, c+R) \to \mathbb{R}$, $R > 0$, having a representation as a power series is quite limited. In particular, such a function has to be C^∞, see Exercise 6.5.4. However, this is not sufficient. There exist functions in C^∞, which are non-zero in any interval $(-r, r)$ but for which the power series centered at 0 is identically equal to 0, see Exercise 8.7.14.

Example 8.4.5. Recalling the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad |x| < R = 1$$

we can use Theorem 8.4.1 to obtain more explicit series. Integrating, and changing x to $-x$ gives

$$\ln(1+x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}, \qquad |x| < 1.$$

By induction on $k \in \mathbb{N}$, or repeatedly applying Theorem 8.4.1, we obtain

$$(1-x)^{-k} = \sum_{n=0}^{\infty}\frac{k(k+1)\cdots(k+n-1)}{n!}x^n, \qquad |x| < 1. \tag{8.4.2}$$

Note that this result can also be obtained using convolution products, see Exercise 8.7.10.

The result of Example 8.4.5 can be generalised to more general values of the parameter.

Theorem 8.4.6 (Binomial sum “Binomiaalreeks”)**.** *Assume $\alpha \in \mathbb{R}$ then we have*

$$(1-x)^{\alpha} = \sum_{n=0}^{\infty}\frac{(-\alpha)(-\alpha+1)\cdots(-\alpha+n-1)}{n!}x^n, \qquad |x| < 1.$$

A proof of Theorem 8.4.6 for $|x| < \frac{1}{2}$ is discussed in Exercise 8.7.15. Note that Theorem 8.4.6 contains the cases of Newton's binomial formula (2.1.3) as a special case for $\alpha \in \mathbb{N}$. Example 8.4.5 is contained for $\alpha \in \mathbb{Z}_{<0}$. The standard proof of Theorem 8.4.6 is by showing that both functions satisfy the same first order differential equation and initial condition, so that the result follows from existence and uniqueness of initial value problems, but this is outside the scope of the course.

Now that we have obtained the power series expansion, or Taylor series expansion, of a number of functions we can consider what happens at the endpoints of the interval of convergence. First of all, we note that we can have all kinds of convergence, i.e. absolute convergence, relative convergence and divergence, at the endpoints. Take the power series $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^\alpha}$, $\alpha > 0$, then its radius of convergence $R = 1$. For $\alpha > 1$ the convergence at both endpoints is absolute. For $0 < \alpha \leq 1$ the series diverges for $x = 1$ and converges relatively for $x = -1$. So in general we cannot say much about the behaviour and value at the endpoints, but in special cases Abel's Theorem 8.4.7 can be used.

Theorem 8.4.7 (Abel)**.** *Assume that the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has radius of convergence R. Let $f\colon (c-R, c+R) \to \mathbb{R}$ be defined by $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$. Assume that $\sum_{n=0}^{\infty} a_n R^n$ converges, then*

$$\lim_{x\to c+R; x\in(c-R,c+R)} f(x) = \sum_{n=0}^{\infty} a_n R^n.$$

So Abel's Theorem 8.4.7 states that if a power series converges in an endpoint, then the corresponding function f extends continuously to this endpoint. Naturally, there is a similar statement for the other endpoint, which one can obtain by replacing a_n by $(-1)^n a_n$. Note that we do not require that $\sum_{n=0}^{\infty} a_n R^n$ converges absolutely, it also works for relative convergence.

Proof. By scaling and translating the variable x we can reduce to the case $c = 0$ and $R = 1$. Let $S_N = \sum_{n=0}^{N} a_n$ be the partial sum of the convergent series, and then we have $\lim_{N\to\infty} S_N = S$. We need to show that $\lim_{x\to 1; x\in(-1,1)} f(x) = S$.

In order to do this we rewrite the expression for f, cf. Exercise 8.7.12. Let

$$\sum_{n=0}^{N} a_n x^n = \sum_{n=0}^{N} (S_n - S_{n-1}) x^n = \sum_{n=0}^{N} S_n x^n - \sum_{n=0}^{N-1} S_n x^{n+1} = (1-x)\sum_{n=0}^{N-1} S_n x^n + S_N x^N$$

where we set $S_{-1} = 0$. Since $(S_N)_{N\in\mathbb{N}}$ is a convergent sequence, it is bounded by Proposition 3.2.8. Using Exercise 3.4.2, we see that $\lim_{N\to\infty} S_N x^N = 0$ for $|x| < 1$. Since the limit of the right hand side exists, we see by Theorem 3.2.19 that the limit of $\sum_{n=0}^{N-1} S_n x^n$ exists, and we find

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = (1-x)\sum_{n=0}^{\infty} S_n x^n, \qquad |x| < 1.$$

In order to see that $\lim_{x\to 1; x\in(-1,1)} f(x) = S$ we need to show that for all $\varepsilon > 0$ there exists $\delta > 0$ so that for all $x \in (1-\delta, 1)$ we have $|f(x) - S| < \varepsilon$. So we pick $\varepsilon > 0$ arbitrary, and since $\lim_{N\to\infty} S_N = S$ we find $N \in \mathbb{N}$ so that for all $n \geq N$ we have $|S_n - S| < \frac{1}{2}\varepsilon$. Write

$$f(x) - S = (1-x)\sum_{n=0}^{\infty}(S_n - S)x^n = (1-x)\sum_{n=0}^{N-1}(S_n - S)x^n + (1-x)\sum_{n=N}^{\infty}(S_n - S)x^n$$

using $(1-x)\sum_{n=0}^{\infty} x^n = 1$. This leads to

$$|f(x) - S| \leq (1-x)\sum_{n=0}^{N-1}|S_n - S|x^n + (1-x)\frac{1}{2}\varepsilon\sum_{n=N}^{\infty}x^n = (1-x)\sum_{n=0}^{N-1}|S_n - S|x^n + x^N\frac{1}{2}\varepsilon$$

where we assume $x \in (0,1)$. Estimating $x^N < 1$ and $\sum_{n=0}^{N-1}|S_n - S|x^n \leq \sum_{n=0}^{N-1}|S_n - S| = M$ we find

$$|f(x) - S| \leq (1-x)M + \frac{1}{2}\varepsilon < \varepsilon$$

for $(1-x)M < \frac{1}{2}\varepsilon$ or take $\delta = \min(\frac{1}{2M}\varepsilon, 1)$. □

Example 8.4.8. The classical example of Abel's Theorem 8.4.7 is that the alternating harmonic series equals

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = \ln(2).$$

Apply Theorem 8.4.7 to Example 8.4.5 using Theorem 8.2.1.

8.5 Iterated series

Given a function $a\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}$, we can consider the series

$$\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{m,n}, \quad \sum_{m=0}^{\infty}\Big(\sum_{n=0}^{\infty} a_{m,n}\Big), \quad \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty} a_{m,n}\Big)$$

and ask whether these are equal. In Chapter 1 we have seen that we can not take for granted that these series are equal. Although we have defined series, and so we have a meaning for the last two expressions, we don't have a meaningful definition for the first expression. Recall from Inleiding Wiskunde [4] that a countable infinite set X is a set for which a bijection $\phi\colon \mathbb{N} \to X$ exists.

Definition 8.5.1. *Let X be a countable infinite set and let $a\colon X \to \mathbb{R}$ be a function. The series $\sum_{x\in X} a(x)$ is called an* absolute convergent series on X *"absoluut convergente reeks op X" if there exists a bijection $\phi\colon \mathbb{N} \to X$ so that the series*

$$\sum_{n=0}^{\infty} a(\phi(n))$$

is absolutely convergent in the sense of Definition 8.1.10.

Note that if there are two bijections $\phi\colon \mathbb{N} \to X$, $\psi\colon \mathbb{N} \to X$, then $\tau = \psi^{-1} \circ \phi\colon \mathbb{N} \to \mathbb{N}$ is a bijection. By Theorem 8.1.17, we see that $\sum_{n=0}^{\infty} a(\phi(n))$ and $\sum_{n=0}^{\infty} a(\psi(n))$ are both absolutely convergent. So for an absolutely convergent series on X we can define its value

$$\sum_{x\in X} a(x) = \sum_{n=0}^{\infty} a(\phi(n)), \tag{8.5.1}$$

which is independent of the choice of bijection according to Theorem 8.1.17. Note that dropping the requirement of absolute convergence in favour of convergence gives the problem that the outcome depends on the choice of bijection $\phi\colon X \to \mathbb{N}$, cf. Exercise 8.7.6.

Now we can take $X = \mathbb{N} \times \mathbb{N}$, and then the absolute convergence of $\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{m,n}$ is defined.

Theorem 8.5.2 (Fubini)**.** *Let $a\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a function so that the series*

$$\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{m,n}$$

converges absolutely. Then for all $n \in \mathbb{N}$ the series $\sum_{m=0}^{\infty} a_{m,n}$ converges absolutely, and for all $m \in \mathbb{N}$ the series $\sum_{n=0}^{\infty} a_{m,n}$ converges absolutely. Moreover,

$$\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{m,n} = \sum_{m=0}^{\infty}\Big(\sum_{n=0}^{\infty} a_{m,n}\Big) = \sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty} a_{m,n}\Big).$$

Note that the example of Chapter 1 is not an absolutely convergent series on $\mathbb{N} \times \mathbb{N}$.

The proof is reminiscent of the proof of Theorem 8.1.17.

Proof. We take up the proof in case $a_{m,n} \geq 0$ for all $(m,n) \in \mathbb{N} \times \mathbb{N}$. We put $L = \sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{m,n}$. Since $a_{m,n} \geq 0$ we see that for any finite subset $Y \subset \mathbb{N} \times \mathbb{N}$ we have

$$\sum_{(m,n)\in Y\subset\mathbb{N}\times\mathbb{N}} a_{m,n} \leq L,$$

since under a bijection $\phi\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ we have that $\phi^{-1}(Y)$ will correspond to a finite set of $\mathbb{N}$. Then for $K = \max(\phi^{-1}(Y))$ we have

$$\sum_{(m,n)\in Y\subset\mathbb{N}\times\mathbb{N}} a_{m,n} \leq \sum_{n=0}^{K} a_{\phi(n)} = S_K \leq L = \sup_{K\in\mathbb{N}} S_K,$$

since $a_{m,n} \geq 0$.

Now take $n \in \mathbb{N}$ arbitrarily and put $Y = \{0, 1, \cdots, M\} \times \{n\}$, then we have that

$$\sum_{m=0}^{M} a_{m,n} \leq L.$$

Since this holds for all $M \in \mathbb{N}$, the partial sums are bounded and by Theorem 8.1.14 $\sum_{m=0}^{\infty} a_{m,n}$ is convergent. Similarly, for all $m \in \mathbb{N}$ the series $\sum_{n=0}^{\infty} a_{m,n}$ converges absolutely.

Next we take $Y = \{0, 1, \cdots, M\} \times \{0, 1, \cdots, N\}$, so that

$$\sum_{(m,n)\in Y \subset \mathbb{N}\times\mathbb{N}} a_{m,n} = \sum_{m=0}^{M} \Big(\sum_{n=0}^{N} a_{m,n} \Big) \leq L.$$

Since this holds for all N we can take the supremum over $N \in \mathbb{N}$, which by Theorem 8.1.14 gives, with $b_m = \sum_{n=0}^{\infty} a_{m,n}$,

$$\sum_{m=0}^{M} b_m = \sum_{m=0}^{M} \Big(\sum_{n=0}^{\infty} a_{m,n} \Big) \leq L.$$

Since this is valid for all $M \in \mathbb{N}$, again Theorem 8.1.14 gives that $\sum_{m=0}^{\infty} b_m$ converges absolutely. So we conclude that

$$\sum_{m=0}^{\infty} \Big(\sum_{n=0}^{\infty} a_{m,n} \Big) \leq L.$$

Similarly, $\sum_{n=0}^{\infty} \Big(\sum_{m=0}^{\infty} a_{m,n} \Big)$ converges absolutely and

$$\sum_{n=0}^{\infty} \Big(\sum_{m=0}^{\infty} a_{m,n} \Big) \leq L.$$

Now, replacing in the above $a_{m,n}$ by $|a_{m,n}|$ we have established the absolute convergence of all series involved. It remains to prove the equality of the series involved.

We return to the case $a_{m,n} \geq 0$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$. It suffices to prove that for all $\varepsilon > 0$ we have

$$\sum_{m=0}^{\infty} \Big(\sum_{n=0}^{\infty} a_{m,n} \Big) \geq L - \varepsilon.$$

For $\varepsilon > 0$, $L - \varepsilon$ is not an upper bound for the increasing sequence $(S_K)_{K\in\mathbb{N}}$, so that there exists $P \in \mathbb{N}$ so that for all $p \geq P$ we have

$$L - \varepsilon < S_p = \sum_{\phi(\{0,\cdots,p\})\subset\mathbb{N}\times\mathbb{N}} a_{m,n} \leq L$$

Take $p = P$ and use that $\phi(\{0, \cdots, p\}) \subset \mathbb{N} \times \mathbb{N}$ is a finite subset, hence there exist $M \in \mathbb{N}$ and $N \in \mathbb{N}$ so that $\phi(\{0, \cdots, p\}) \subset \{0, \cdots, M\} \times \{0, \cdots, N\}$, and so

$$L - \varepsilon < S_p = \sum_{\phi(\{0,\cdots,p\})\subset\mathbb{N}\times\mathbb{N}} a_{m,n} \leq \sum_{m=0}^{M} \Big(\sum_{n=0}^{N} a_{m,n} \Big) \leq L$$

since $a_{m,n} \geq 0$. Taking the supremum over N and next the supremum over M gives the required estimate. Since $\varepsilon > 0$ is arbitrary, we obtain $\sum_{m=0}^{\infty}\Big(\sum_{n=0}^{\infty} a_{m,n}\Big) = L$. Similarly, we obtain $\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty} a_{m,n}\Big) = L$. This proves Fubini's Theorem 8.5.2 in case of nonnegative summands. The general case follows from this, and this is in Exercise 8.5.3. □

Exercise 8.5.3. In order to prove the general case of Theorem 8.5.2 from the special case of nonnegative summands, we proceed as in Theorem 8.1.17. Put $a_{m,n} = a^+_{m,n} - a^-_{m,n}$, and finish the proof of Theorem 8.5.2 as in the proof of Theorem 8.1.17.

Lemma 8.5.4. *Let X be a countable infinite set and let $a\colon X \to \mathbb{R}$ be a function. The series $\sum_{x\in X} a(x)$ is an absolute convergent series on X if and only if*

$$\sup\{\sum_{x\in Y} |a(x)| \mid Y \subset X, |Y| < \infty\} < \infty.$$

Exercise 8.5.5. Prove Lemma 8.5.4. Hint: consider the proof of Theorem 8.5.2.

Example 8.5.6. Note that in case $a_{m,n} = c_m b_n$ we have the trivial case that the double series is absolutely convergent if and only if the series $\sum_{m=0}^{\infty} c_m$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent, and then

$$\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{m,n} = \Big(\sum_{m=0}^{\infty} c_m\Big)\Big(\sum_{n=0}^{\infty} b_n\Big)$$

8.6 Complex power series

In Exercise 3.4.17 we have introduced complex sequences and convergent complex sequences. So we can immediately copy Definition 8.1.1 to the complex setting.

Definition 8.6.1. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. Define the* partial sum *"partiële som" $S_N = \sum_{n=0}^{N} a_n$ for $N \in \mathbb{N}$. If the sequence $(s_N)_{N=0}^{\infty}$ is a convergent complex sequence, then we say that the* complex series *"complexe reeks" $\sum_{n=0}^{\infty} a_n$ is* convergent *"convergent" and the result is*

$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N.$$

If the sequence $(s_N)_{N=0}^{\infty}$ is not convergent, we say that the series $\sum_{n=0}^{\infty} a_n$ is divergent *"divergent".*

Now the definition of an absolutely convergent series for complex series is exactly the same as the Definition 8.1.10.

Exercise 8.6.2. Check the results of Section 8.1 and Section 8.2 and determine which results go through for complex series. Note that in particular the convergence criteria for complex series are the same, and proofs are the same.

This then gives the option to define complex power series.

Definition 8.6.3. *A* complex power series *"complexe machtreeks" is a complex series of the form*

$$\sum_{n=0}^{\infty} a_n (z-c)^n$$

for a complex sequence $(a_n)_{n=0}^{\infty}$*, which are the* coefficients of the power series *"coëfficiënten van de complexe machtreeks", and* $c \in \mathbb{C}$ *is the* centre *"centrum" or "middelpunt" of the complex power series. Finally,* $z \in \mathbb{C}$ *is considered to be a complex variable.*

Then we can prove the same result on the radius of the convergence.

Theorem 8.6.4. *For the complex power series* $\sum_{n=0}^{\infty} a_n(z-c)^n$ *we have one of the following options:*

(i) the power series converges for all $z \in \mathbb{C}$*;*

(ii) the power series converges only for $z = c$*;*

(iii) there exists $R \in [0, \infty)$ *so that for all* $z \in \mathbb{C}$ *with* $|z-c| < R$ *the complex power series is absolutely convergent and for all* $z \in \mathbb{C}$ *with* $|z-c| > R$ *the complex power series is divergent.*

Then $R = \frac{1}{\alpha}$ *with* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ *with the convention that* $\alpha = 0$ *corresponds to (i) and* $\alpha = \infty$ *corresponds to (ii).*

Note that $\{z \in \mathbb{C} \mid |z-c| < R\}$ is an open disc of radius R centered at c in the complex plane $\mathbb{C}$. This explains why R is called the radius of convergence.

Exercise 8.6.5. Give a proof of Theorem 8.6.4 along the lines of the proof of Theorem 8.3.4.

Exercise 8.6.6. Prove the complex analogue of the geometric series: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for complex z with $|z| < 1$.

In Exercise 8.7.8 we derive an explicit power series for the exponential function $\exp\colon \mathbb{R} \to (0, \infty)$ introduced in Exercise 7.6.9. We can then extend this to a complex power series to define the complex exponential function by

$$\exp\colon \mathbb{C} \to \mathbb{C}, \qquad \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

which converges for all $z \in \mathbb{C}$, i.e. the radius of convergence is $R = \infty$.

Proposition 8.6.7. *For all* $z, w \in \mathbb{C}$ *we have* $\exp(z+w) = \exp(z)\exp(w)$*. Moreover, for* $t \in \mathbb{R}$*, we have Euler's formula*

$$\exp(it) = \cos(t) + i\sin(t).$$

Proof. Note that we can interchange summations for absolutely convergent complex series, so

$$\exp(z+w) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{n!} \frac{n!}{k!\,(n-k)!} z^k w^{n-k}$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} z^k \sum_{n=k}^{\infty} \frac{1}{(n-k)!} w^{n-k} = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \sum_{p=0}^{\infty} \frac{1}{p!} w^p = \exp(z)\exp(w)$$

using the complex analogue of Newton's binomial sum (2.1.3) and putting $n = k + p$.

Restricting $z = it$ to the imaginary axis, and splitting the terms in the real and imaginary part gives

$$\exp(it) = \sum_{n=0}^{\infty} i^n \frac{t^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!}$$

using the absolute convergence. The result of Exercise 8.7.13 then gives the result. □

8.7 Exercises

Exercise 8.7.1. Consider the following series. Analyse the convergence properties and evaluate the series when convergent.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - \frac{1}{4}}$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+k)}$, $k \in \mathbb{N}$, $k \geq 2$.

Is the iterated series $\sum_{k=2}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n(n+k)}$ convergent? Hint: telescoping series as in Proposition 8.1.6.

Exercise 8.7.2. Show that for $\alpha > 0$ the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^\alpha}$$

is convergent. For which $\alpha > 0$ is the series absolutely convergent, relatively convergent, divergent? What happens with the convergence properties (i.e. absolute convergence, relative convergence or divergence) in case $\alpha = 0$ or $\alpha < 0$?

Exercise 8.7.3. Let $(a_n)_{n=0}^{\infty}$ be a decreasing sequence of nonnegative numbers. Assume $\lim_{n\to\infty} a_n = 0$. By Theorem 8.2.1 the series $\sum_{n=0}^{\infty}(-1)^n a_n$ is convergent.
Let $S_N = \sum_{n=0}^{N}(-1)^n a_n$ be the partial sum, and let $L = \sum_{n=0}^{\infty}(-1)^n a_n$. Show the error estimate

$$|L - S_N| \leq a_N.$$

Exercise 8.7.4. We define the sequence $(a_n)_{n=1}^{\infty}$ as follows:

$$a_{2k-1} = \frac{1}{k}, \qquad a_{2k} = \int_{[k,k+1]} \frac{1}{t}\, dt = \ln(k+1) - \ln(k) \tag{8.7.1}$$

(i) Show that $(a_n)_{n=1}^{\infty}$ is a nonnegative decreasing sequence and that $\lim_{n\to\infty} a_n = 0$.

(ii) Conclude that the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent, and put

$$\gamma = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

(iii) Show that

$$S_{2N-1} = \sum_{n=1}^{2N-1}(-1)^{n-1}a_n = \sum_{k=1}^{N}\frac{1}{k} - \int_{[1,N]}\frac{1}{t}\, dt$$

and conclude that

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{k} - \ln(N) = \gamma.$$

Estimate $|\gamma - \sum_{k=1}^{N}\frac{1}{k} + \ln(N)|$.

The constant $\gamma = 0.5772156649\cdots$ is known as Euler's constant or as the Euler-Mascheroni constant. (It is still open what kind of number γ is: irrational, algebraic, transcendental.)

Exercise 8.7.5. Theorem 8.2.3 can be refined. Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ be sequences of nonnegative numbers and assume that $b_n > 0$ for all $n \in \mathbb{N}$. Assume $\lim_{n\to\infty}\frac{a_n}{b_n} = L$, and we allow the case $L = \infty$.

(i) Assume $0 < L < \infty$. Show the following statement:
the series $\displaystyle\sum_{n=0}^{\infty} a_n$ is convergent if and only if the series $\displaystyle\sum_{n=0}^{\infty} b_n$ is convergent.

(ii) Derive a statement in case $L = 0$, and prove the statement.

(iii) Derive a statement in case $L = \infty$, i.e. the sequence $(\frac{a_n}{b_n})_{n=0}^{\infty}$ diverges to ∞. Prove your statement.

(iv) Use the result to discuss the convergence properties of:

$$\sum_{n=1}^{\infty} \frac{n^2 + 3\sqrt{n} + 5}{n^3 + 6n\sqrt[3]{5n+3} + 9}, \qquad \sum_{n=1}^{\infty} \frac{n - \sqrt[4]{n} + \pi}{6n^2\sqrt{n} + 6n\sqrt{9n + \pi^2} + 137}.$$

Exercise 8.7.6. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$, which is the alternating harmonic series. This is a relatively convergent series using Theorem 8.2.1 and Example 8.2.5. By Example 8.4.8 the value is $\ln(2)$. We show that it can converge to any value L by reordering the terms. For convenience we take $L > 1$, but this is not essential.

(i) First take positive terms such

$$1 + \frac{1}{3} + \cdots + \frac{1}{2k_1 - 1} < L \leq 1 + \frac{1}{3} + \cdots + \frac{1}{2k_1 + 1}.$$

Explain why this can be done. Define $\tau(n) = 2n - 1$ for $n \in \{1, \cdots, k_1\}$.

(ii) Next take negative terms so that

$$\sum_{n=0}^{k_1} \frac{1}{2n+1} - \frac{1}{2} - \frac{1}{4} - \frac{1}{2k_2} < L \leq \sum_{n=0}^{k_1} \frac{1}{2n+1} - \frac{1}{2} - \frac{1}{4} - \frac{1}{2k_2 - 2}.$$

Explain why this can be done. Define $\tau(k_1 + n) = 2n$ for $n \in \{1, \cdots, k_2\}$.

(iii) Now add positive terms to overshoot L, and then add negative terms until you undershoot L. Construct τ, and show that $\tau\colon \mathbb{N} \setminus \{0\} \to \mathbb{N} \setminus \{0\}$ is a bijection, and that

$$\sum_{n=1}^{\infty} \frac{(-1)^{\tau(n)-1}}{\tau(n)} \text{ is convergent with value } L.$$

Note that the proof can be adapted to any value L, and that it can be adapted to any relatively convergent series using Proposition 8.1.15.

Exercise 8.7.7. (i) Show that

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n},$$

and determine the radius of convergence. Hint: use the geometric series.

(ii) Integrate the result to find a power series expansion for $\arctan(x)$ around 0.

(iii) Show that

$$\frac{1}{4}\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

Is this series absolutely or relatively convergent?

Exercise 8.7.8. Consider the exponential function as defined in Exercise 7.6.9.

(i) Let $T_N(x)$ be the Taylor polynomial for exp at the centre 0. Show that

$$T_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}.$$

(ii) Show that for all $x \in [-r, r]$, $r > 0$, the estimate

$$|\exp(x) - T_N(x)| \leq \exp(r)\frac{r^{N+1}}{(N+1)!}$$

holds. Hint: Corollary 6.4.3.

(iii) Show that the sequence $(T_N)_{N=0}^{\infty}$ converges uniformly to $\exp(x)$ on the interval $[-r, r]$. Conclude that

$$\forall\, x \in \mathbb{R} \qquad \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Exercise 8.7.9. Now $e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$.

(i) Show that $2 \leq e \leq 3$.

(ii) Assume that $e \in \mathbb{Q}$, and write $e = s/t$ with $s, t \in \mathbb{Z}$. Show that

$$t!e = \sum_{n=0}^{t} \frac{t!}{n!} + \sum_{n=t+1}^{\infty} \frac{t!}{n!}, \qquad S_1 = \sum_{n=0}^{t} \frac{t!}{n!}, \quad S_2 = \sum_{n=t+1}^{\infty} \frac{t!}{n!}.$$

And prove that $S_1 \in \mathbb{N}$ and $0 \leq S_2 \leq 1$. (Hint: show that $(t+k)! \geq 2^k t!$ for $k \in \mathbb{N}$ and $t \geq 1$.)

(iii) Assume $e \in \mathbb{Q}$, improve the previous estimate slightly to obtain to obtain a contradiction. Conclude that $e \notin \mathbb{Q}$, i.e. e is irrational.

Exercise 8.7.10. Given two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ centered at 0, we define the product by collecting the powers of x in the product;

$$\Big(\sum_{m=0}^{\infty} a_m x^m\Big)\Big(\sum_{n=0}^{\infty} b_n x^n\Big) = \sum_{p=0}^{\infty} c_p x^p, \qquad c_p = \sum_{m+n=p} a_m b_n$$

which is the *convolution product* “convolutieproduct” of the power series.

(i) Assume that for all $n \in \mathbb{N}$, $a_n \geq 0$ and $b_n \geq 0$. Assume that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent. Show that $\sum_{p=0}^{\infty} c_p$ is convergent and that

$$\Big(\sum_{m=0}^{\infty} a_m\Big)\Big(\sum_{n=0}^{\infty} b_n\Big) = \sum_{p=0}^{\infty} c_p, \qquad c_p = \sum_{m+n=p} a_m b_n = \sum_{n=0}^{p} a_{p-n} b_n$$

Hint: prove an estimate of the form

$$\Big(\sum_{n=0}^{\lfloor K/2 \rfloor} a_n\Big)\Big(\sum_{n=0}^{\lfloor K/2 \rfloor} b_n\Big) \leq \sum_{p=0}^{K} c_p \leq \Big(\sum_{n=0}^{K} a_n\Big)\Big(\sum_{n=0}^{K} b_n\Big)$$

where $\lfloor K/2 \rfloor$ is the largest integer smaller than or equal to $K/2$.

(ii) Let R_a, respectively R_b, be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$, respectively $\sum_{n=0}^{\infty} b_n x^n$. Show that the radius of convergence R of the power series $\sum_{p=0}^{\infty} c_p x^p$ satisfies $R \geq \min(R_a, R_b)$, and for $|x| < \min(R_a, R_b)$ we have

$$\Big(\sum_{m=0}^{\infty} a_m x^m\Big)\Big(\sum_{n=0}^{\infty} b_n x^n\Big) = \sum_{p=0}^{\infty} c_p x^p$$

(iii) Prove (8.4.2) with induction on k using the geometric series and taking convolution products.

(iv) Show that for $k, l \in \mathbb{N}$ we have

$$\frac{(k+l)(k+l+1)\cdots(k+l+p-1)}{p!} = \sum_{n=0}^{p} \frac{k(k+1)\cdots(k+n-1)}{n!} \frac{l(l+1)\cdots(l+p-n-1)}{(p-n)!}$$

by taking the convolution products of two series of the form (8.4.2).

(v) Generalise this to the situation where k and l can be arbitrary real numbers α and β. This identity is known as the Chu-Vandermonde sum. Hint: use Theorem 8.4.6 and convolutions.

Exercise 8.7.11. Assume that we have a function $f\colon [1,\infty) \to \mathbb{R}$ so that f is a nonnegative decreasing function, which is Riemann integrable on $[1, R]$ for any $R \geq 1$. Show that the series $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if the improper integral $\int_{[1,\infty)} f(x)\,dx$ is convergent (as in Exercise 7.6.10). Hint: show that

$$\sum_{n=2}^{N} f(n) \leq \int_{[1,N]} f(x)\,dx \leq \sum_{n=1}^{N-1} f(n)$$

and use Exercise 5.5.7(ii). Use this to give another proof of, see Example 8.2.5,

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \text{ is convergent} \iff \alpha > 1.$$

Exercise 8.7.12. Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be sequences, and put $S_N = \sum_{n=0}^{N} a_n$.

(i) Show that for $r, s \in \mathbb{N}$, $r \leq s$, we have

$$\sum_{n=r}^{s} a_n b_n = S_s b_s - S_{r-1} b_r + \sum_{n=r}^{s-1} S_n (b_n - b_{n+1})$$

where empty sums are set to 0. This is *summation by parts* "partiële sommatie", and compare this with Corollary 7.4.6. Hint: $a_n = S_n - S_{n-1}$.

(ii) Assume that $(S_N)_{N=0}^{\infty}$ is a bounded sequence, and that $(b_n)_{n\in\mathbb{N}}$ is a decreasing sequence of nonnegative numbers satisfying $\lim_{n\to\infty} b_n = 0$. Show that $\sum_{n=0}^{\infty} a_n b_n$ is convergent. Hint: use Theorem 8.1.9 and (i).

(iii) Show that the non-trival implication of the alternating series test of Theorem 8.2.1 follows from (ii).

Exercise 8.7.13. Determine the derivatives of the sine and cosine functions sin and cos, and show that the sequence of Taylor polynomials centered at 0 converge to sin and cos. Prove that

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \qquad \sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

and that the radius of convergence is ∞ for both series.

Exercise 8.7.14. Define the function $f\colon \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0, & x \leq 0, \\ \exp(-\frac{1}{x}), & x > 0. \end{cases}$$

(i) Show inductively that f has derivatives of any order and that $f^{(k)}(0) = 0$. Hint: use Exercise 7.6.9.

(ii) Conclude that the power series expansion for f around 0 is identically equal to 0. And conclude that for all $r > 0$ the power series expansion does not converge to f on $(-r, r)$.

Exercise 8.7.15. Use Example 6.4.6 to show that Theorem 8.4.6 is valid for $|x| < \frac{1}{2}$, i.e. the series of Taylor polynomials converge uniformly to $(1-x)^\alpha$ on an interval $[-\frac{1}{2}+\varepsilon, \frac{1}{2}+\varepsilon]$.

Exercise 8.7.16. (i) Put $a\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ defined by $a_{m,n} = \frac{1}{m^2-n^2}$ for $m \neq n$ and $a_{m,m} = 0$. Is the series $\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{m,n}$ absolutely convergent?

(ii) Put $b\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ defined by $b_{m,n} = \frac{1}{m^2+n^2}$ for $(m,n) \neq (0,0)$ and $b_{0,0} = 0$. Is the series $\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} b_{m,n}$ absolutely convergent?

Appendices

Appendix A

Additional exercises

A.1 We prove $\sqrt{2} \notin \mathbb{Q}$. Consider the equation $p^2 - 2q^2 = \pm 1$ (allowing both signs on the right hand side).

(i) Show that $p_1 = 1$, $q_1 = 1$ satisfy the equation. Define p_n, q_n for $n \in \mathbb{N}$, $n \geq 1$ by

$$(p_1 - \sqrt{2}q_1)^n = p_n - \sqrt{2}q_n, \qquad (p_1 + \sqrt{2}q_1)^n = p_n + \sqrt{2}q_n,$$

(a) Show that $p_n, q_n \in \mathbb{Z}$. (Hint: Newton's binomial formula.)

(b) Show that $p_n^2 - 2q_n^2 = (-1)^n$, so that p_n, q_n satisfy the equation.

(ii) Prove that the sequence $(p_n+\sqrt{2}q_n)_{n=1}^{\infty}$ is unbounded. Conclude that $(p_n-\sqrt{2}q_n)_{n=1}^{\infty}$ converges to 0, but that all terms are different from 0, $p_n-\sqrt{2}q_n \neq 0\ \forall\, n \in \mathbb{N}$. (Hint: use $p_n^2 - 2q_n^2 = (-1)^n$.)

(iii) Prove that $\sqrt{2} \notin \mathbb{Q}$. (Hint: argue by contradiction. Put $\sqrt{2} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, and show that the sequence $(bp_n - aq_n)_{n=1}^{\infty}$ is contained in $\mathbb{Z}$. But it converges to 0, and alle elements are different from 0.)

A.2 Let $f\colon A \to \mathbb{R}$ be a function with domain $A \subset \mathbb{R}$. We assume A has no isolated points, and additionally that f is a bounded function,

(i) Assume $a \in \overline{A}$ and define $m\colon (0,\infty) \to \mathbb{R}$ by

$$m(\delta) = \sup_{x \in N_\delta^*(a)\cap A} f(x), \qquad N_\delta^*(a) = \{x \in \mathbb{R} \mid 0 < |x-a| < \delta\}$$

Show that $\lim\limits_{\delta\to 0,\delta>0} m(\delta)$ existst. Then we can define

$$\limsup_{x\to a, x\in A} f(x) = \lim_{\delta\to 0,\delta>0} m(\delta)$$

(ii) Prove that (a) and (b) are equivalent:

(a) $L = \limsup_{x\to a, x\in A} f(x)$

(b) the following two properties hold:
- there exists a sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} f(a_n) = L$
- for all $r > L$ there exists $\delta > 0$ so that $f(x) < r$ for all $x \in N^*_\delta(a) \cap A$

(iii) Assume A is a closed interval. Show that the statements (a) and (b) are equivalent:

(a) for all $a \in A$ we have $\limsup_{x\to a, x\in A} f(x) \leq f(a)$.

(b) for all $r \in \mathbb{R}$ the set $\{x \in A \mid f(x) < r\}$ is open (relatively with respect to A)

(iv) A function $f\colon A \to \mathbb{R}$ satisfying the conditions in (iii) is called an *upper semicontinuous function*. Define an appropriate notion of a lower semicontinuous function and show that $f\colon A \to \mathbb{R}$ is a continuous function if and only if the function $f\colon A \to \mathbb{R}$ is both lower and upper semicontinuous. Hint: use Theorem 5.1.14.

A.3 The purpose of this exercise is to show that a continuous function $f\colon [a,b] \to \mathbb{R}$ can be approximated by polynomials on $[a,b]$ using uniform convergence, i.e. we show that there exists a sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials with $\lim_{n\to\infty} p_n = f$ uniformly. This is known as the Weierstrass Theorem (and it is generalisation is known as the Stone-Weierstrass Theorem). We follow a proof due to Bernstein.

(i) Show that we can restrict without loss of generalisation to the interval $[0,1]$. Hint: consider the function $g\colon [0,1] \to \mathbb{R}$ defined by $g(x) = f(a + x(b-a))$ and observe this preserves polynomials.

(ii) Show that it suffices to prove

$$\forall \varepsilon > 0 \quad \exists p \quad \forall x \in [0,1] \qquad |f(x) - p(x)| < \varepsilon$$

where $p\colon [0,1] \to \mathbb{R}$ is a polynomial.

(iii) For a continuous function $f\colon [0,1] \to \mathbb{R}$ we define the Bernstein polynomial on $[0,1]$

$$B_n(x; f) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} f\left(\frac{i}{n}\right).$$

Show the following properties

(a) if $f(x) \geq g(x)$ for all $x \in [0,1]$, then $B_n(x; f) \geq B_n(x; g)$

(b) $B_n(x; -f) = -B_n(x; f)$ and $B_n(x; f+g) = B_n(x; f) + B_n(x; g)$ for continuous functions f and g.

(iv) We calculate the Bernstein polynomial for some simple functions. Show that

(a) for constant function $f(x) = C$, we have $B_n(x; f) = C$;

(b) assume $f(x) = x$, then $B_n(x; f) = x$;

(c) assume $f(x) = x^2$, then $B_n(x; f) = x^2 + \frac{x-x^2}{n}$.

Hint: use Newton's binomium for (a), and use this also for (b) and (c) upon rewriting binomial coefficients.

(v) Let $f\colon [0,1] \to \mathbb{R}$ be a fixed continuous function. Pick $\varepsilon > 0$ arbitrary. Show that there exists a constant $\alpha > 0$ so that for all $x, y \in [0,1]$ we have

$$|f(x) - f(y)| < \frac{1}{2}\varepsilon + \alpha(x-y)^2.$$

Hint: use Theorem 5.3.10 to conclude that f is uniformly continuous, so that there exists $\delta > 0$ so that for all $x, y \in [0,1]$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < \frac{1}{2}\varepsilon$. For $|x - y| \geq \delta$ use that f is bounded, say by M and that we can take $\alpha = \frac{2M}{\delta^2}$.

(vi) Put $F(x) = \frac{1}{2}\varepsilon + \alpha(x-y)^2$, so that by (iii), (iv) we have

$$B(x, F) = \frac{1}{2}\varepsilon + \alpha y^2 - 2\alpha y x + \alpha x^2 + \alpha\frac{x - x^2}{n}.$$

Show that

$$|B_n(x; f) - f(y)| \leq \frac{1}{2}\varepsilon + \alpha y^2 - 2\alpha y x + \alpha x^2 + \frac{\alpha}{4n}$$

Hint: conclude from (v) that $-B(x, F) < B(x, f) - f(y) < B(x, F)$.

(vii) Take $x = y$ in (vi) to show that for all $y \in [0,1]$

$$|B_n(x; f) - f(x)| \leq \frac{1}{2}\varepsilon + \frac{\alpha}{4n}$$

and conclude that there exists a polynomial p so that $|p(x) - f(x)| < \varepsilon$ for all $y \in [0,1]$.

A.4 As we have seen, $f\colon [-1,1] \to \mathbb{R}$, $f(x) = |x|$, is a continuous function which is not diffentiable at $x = 0$. In this exercise we construct a continuous function, which is not differentiable at any point of its domain. The first example of such a function was studied by Weierstrass in 1872.

(i) We extend the absolute value as a function $f\colon [-1,1] \to \mathbb{R}$, $f(x) = |x|$ to a function $f\colon \mathbb{R} \to \mathbb{R}$ by requiring $f(x+2) = f(x)$. Sketch the graph of f, and show that for $x, y \in \mathbb{R}$ we have $|f(x) - f(y)| \leq |x - y|$, i.e. f is Lipschitz continuous, see Exercise 5.5.3.

(ii) Define

$$g(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n f(4^n x).$$

Show that the series converges uniformly in x, and conclude that g is a continuous function. Hint: use the Weierstrass M-test of Theorem 8.3.2 and that $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}$.

(iii) Take $x \in \mathbb{R}$ arbitrarily. We want to show that there exists a sequence $(\delta_m)_{m\in\mathbb{N}}$ with $\lim_{m\to\infty} \delta_m = 0$ and

$$\left|\frac{g(x+\delta_m)-g(x)}{\delta_m}\right| \geq \frac{1}{2}(3^m+1)$$

(a) Put $\delta_m = \frac{1}{2}4^{-m}$ if $(4^m x, 4^m(x+\frac{1}{2}4^{-m}))$ contains no integer, and put $\delta_m = -\frac{1}{2}4^{-m}$ if $(4^m(x-\frac{1}{2}4^{-m}), 4^m x)$ contains no integer. Explain why this can be done.

(b) Put

$$\gamma_n = \frac{f(4^n(x+\delta_m)) - f(4^n x)}{\delta_m}.$$

Show that $\gamma_n = 0$ for $m > n$ and that $|\gamma_n| \leq 4^n$ for $0 \leq n \leq m$.

(c) Conclude that

$$\frac{g(x+\delta_m)-g(x)}{\delta_m} = \sum_{n=0}^{m}\left(\frac{3}{4}\right)^n \gamma_n$$

and give the required inequality by

$$\left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n\right| \geq 3^m - \sum_{n=0}^{m-1} 3^n$$

and sum this using the geometric sum.

(iv) Use (iii) to prove that g is not diffentiable at x.

A.5 In Exercise 4.4.8 compactness of a subset of $\mathbb{R}$ is defined as every open cover has a finite subcover. In Exercise 4.4.8 it is shown that compactness implies sequentially compact. The purpose of this exercise is to prove the converse under an additional condition, showing that in $\mathbb{R}$ the notions of compactness and sequentially compactness are equivalent.

So we assume $A \subset \mathbb{R}$ to be a sequentially compact set, and we assume that $A \subset \bigcup_{i\in\mathbb{N}} B_i$, with B_i an open set for all $i \in \mathbb{N}$. So we assume that A has a countable cover by open sets.

(i) Assume that there exists no finite subcover of this countable cover. Construct a sequence $(a_n)_{n\in\mathbb{N}}$ as follows: $a_0 \in A$, $a_k \in A \setminus (B_0 \cup \cdots B_{k-1})$ for $k \geq 1$. Show that this is a well-defined sequence in A.

(ii) Let $x \in A$ be the limit of a convergent subsequence of $(a_n)_{n\in\mathbb{N}}$. Show that $x \in A \setminus \bigcup_{i=0}^{N} B_i$ for all $N \in \mathbb{N}$. Hint: use that $A \setminus \bigcup_{i=0}^{N} B_i$ is a closed set (in the relative topology of A).

(iii) Show that $x \in A \setminus \bigcup_{i=0}^{\infty} B_i$, and arrive at a contradiction with $\bigcup_{i\in\mathbb{N}} B_i$ covering A.

(iv) Assuming Lindelöf's theorem, stating that in $\mathbb{R}$ any open cover of A has a countable subcover, show that a sequentially compact set A is a compact set.

A.6 **Claim** *A non-empty open set in* $\mathbb{R}$ *is a countable union of disjoint open intervals.* The purpose of this exercise is to prove this claim. Note that by Proposition 4.1.4(iv) such a countable union of disjoint open intervals is indeed an open set in $\mathbb{R}$.

(i) Let $\emptyset \neq A \subset \mathbb{R}$ be an open set. Define the collection $\mathscr{D}$ of intervals

$$\mathscr{D} = \{I = (a, b) \subset A \mid I \subset (c, d) \subset A \;\Rightarrow\; a = c \text{ en } b = d\}$$

and show that $\cup_{I \in \mathscr{D}} I \subset A$.

(ii) Show that $A \subset \cup_{I \in \mathscr{D}} I$.

(iii) Show that $I, J \in \mathscr{D}$ with $I \neq J$ implies $I \cap J = \emptyset$.

(iv) Show that the collection $\mathscr{D}$ is countable. (Hint: establish an injection $\mathscr{D} \to \mathbb{Q}$.)

(v) Prove the claim.

A.7 **Claim** *Assume that* $f_n \colon A \to \mathbb{R}$ *for all* $n \in \mathbb{N}$ *and that* $\lim_{n\to\infty} f_n = f$ *uniformly. Assume that* x_0 *is a limit point of* A *and that for all* $n \in \mathbb{N}$ *we have*

$$\lim_{x \to x_0; x \in A} f_n(x) = L_n$$

Then the sequence $(L_n)_{n\in\mathbb{N}}$ *converges and*

$$\lim_{x \to x_0; x \in A} f(x) = \lim_{n\to\infty} L_n.$$

The purpose of this exercise is to prove this statement, which says that in this case limits can be interchanged:

$$\lim_{x \to x_0; x \in A} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{x \to x_0; x \in A} f_n(x).$$

(i) Pick $\varepsilon > 0$ and determine $N \in \mathbb{N}$ so that for all $m, n \geq N$ we have $|f_n(x) - f_m(x)| < \varepsilon$. Show that for such $m, n \geq N$ we have $|L_n - L_m| \leq \varepsilon$.

(ii) Conclude that $(L_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and that $L = \lim_{n\to\infty} L_n$ is defined.

(iii) Write

$$|f(x) - L| \leq |f(x) - f_n(x)| + |f_n(x) - L_n| + |L_n - L|$$

and show that for any $\varepsilon > 0$ there exists a $\delta > 0$ so that $|x - x_0| < \delta$ and $x \in A$. Hint: compare with the proof of Theorem 5.4.4.

A.8 Assume $(a_k)_{k\in\mathbb{N}}$ is a sequence with $a_k \neq -1 \;\forall\, k \in \mathbb{N}$. Define the sequence $(p_n)_{n\in\mathbb{N}}$ by

$$p_n = \prod_{k=0}^{n} (1 + a_k) = (1 + a_0)(1 + a_1) \cdots (1 + a_n).$$

We think of $(p_n)_{n\in\mathbb{N}}$ as the sequence of partial products in analogy with the sequence of partial sum for a series. Then we say that the infinite product $\prod_{k=0}^{\infty}(1 + a_k)$ is *convergent* if the squence $(p_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} p_n = p \neq 0$. In case $p = 0$ or if the sequence $(p_n)_{n\in\mathbb{N}}$ is divergent, we call the infinite product $\prod_{k=0}^{\infty}(1 + a_k)$ *divergent.*

(i) Assume that the infinite product $\prod_{k=0}^{\infty}(1+a_k)$ is convergent. Show $\lim_{k\to\infty} a_k = 0$.

(ii) Assume now that $a_k > -1\ \forall\, k \in \mathbb{N}$. Prove that $\prod_{k=0}^{\infty}(1+a_k)$ is convergent if and only if $\sum_{k=0}^{\infty} \ln(1+a_k)$ is convergent.

(iii) Assume now that $a_k \geq 0\ \forall\, k \in \mathbb{N}$. Prove that $\prod_{k=0}^{\infty}(1+a_k)$ is convergent if and only if $\sum_{k=0}^{\infty} a_k$ is convergent. (Hint: use $1 + x \leq \exp(x)$.)

A.9 In Section 5.4 we have seen that uniform convergence of a sequence of functions implies pointwise convergence. Under suitable additional conditions an instance of pointwise convergence implies uniform convergence. This particular instance is known as Dini's Theorem.

Assume that for all $n \in \mathbb{N}$ the function $f_n \colon A \to \mathbb{R}$ is continuous, and we assume that sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to a function $f \colon A \to \mathbb{R}$; $\lim_{n\to\infty} f_n(x) = f(x)$. We assume that $f \colon A \to \mathbb{R}$ is continuous. Assume moreover that for each $x \in A$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ is an increasing sequence and that A is compact as in Exercise 4.4.8.

(i) Pick $\varepsilon > 0$ arbitrarily, and let

$$A_n = \{x \in A \mid f(x) - \varepsilon < f_n(x)\}, \qquad n \in \mathbb{N}.$$

Show that A_n is open (in the relative topology with respect A) and that $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$.

(ii) Show that $A \subset \bigcup_{n\in\mathbb{N}} A_n$ and use Exercise 4.4.8 to conclude that $\exists\, N \in \mathbb{N}$ with $A \subset A_N$.

(iii) Show that the convergence $\lim_{n\to\infty} f_n = f$ is uniform.

A.10 In Proposition 7.5.2 a statement about the interaction of uniform convergence of derivatives has been given. In this exercise we prove a similar statement under weaker conditions. We prove the following statement:

Claim *For all $n \in \mathbb{N}$, the function $f_n \colon [a,b] \to \mathbb{R}$ is assumed to be differentiable with derivative $f_n' \colon [a,b] \to \mathbb{R}$. Assume that $(f_n')_{n\in\mathbb{N}}$ converges uniformly to g. If there exists $x_0 \in [a,b]$ for which the series $(f_n(x_0))_{n\in\mathbb{N}}$ converges, say $\lim_{n\to\infty} f_n(x_0) = L$, then the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $f \colon [a,b] \to \mathbb{R}$. Moreover, f is a differentiable function with derivative $f' = g$.*

Note that the difference with Proposition 7.5.2 is that we do not assume the derivative f_n' to be Riemann integrable.

(i) Show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $f \colon [a,b] \to \mathbb{R}$. Proceed as follows.

(a) Pick $\varepsilon > 0$ arbitrarily, and determine $N \in \mathbb{N}$ so that for all $n, m \geq N$ we have

$$|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)}.$$

Show that for all $x, t \in [a, b]$ we have for all $n, m \geq N$

$$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| < \frac{1}{2}\varepsilon.$$

Hint: use the Mean Value Theorem 6.2.5.

(b) Use (a) with $t = x_0$ and $\lim_{n\to\infty} f_n(x_0) = L$ to find

$$\forall\, n, m \geq N \quad \forall\, x \in [a, b] \qquad |f_n(x) - f_m(x)| < \varepsilon.$$

(c) Use Theorem 5.4.5 to conclude that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $f\colon [a, b] \to \mathbb{R}$.

(ii) Fix $x_0 \in [a, b]$ and consider the corresponding differential quotients:

$$\phi_n(x) = \frac{f_n(x) - f_n(x_0)}{x - x_0}, \qquad \phi(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

defined for $x \neq x_0$. Then from (i) we have $\lim_{n\to\infty} \phi_n = \phi$ uniformly for $x \in A\backslash\{x_0\}$. Use Exercise A.7 to show

$$\lim_{x\to x_0; x\in A} \phi(x) = \lim_{n\to\infty} f_n'(x_0)$$

and finish the proof of the claim.

A.11 Let $\phi\colon [0, \infty) \to [0, \infty)$ be a continuous, strictly increasing function with $\phi(0) = 0$.

(i) Show that ϕ has an inverse function $\psi\colon [0, \infty) \to [0, \infty)$, which is continuous and strictly increasing.

(ii) Conclude that ϕ and ψ are Riemann integrable functions on bounded intervals and show that Young's inequality

$$ab \leq \int_{[0,a]} \phi(x)\, dx + \int_{[0,b]} \psi(x)\, dx$$

holds for $a, b \geq 0$. Show that equality holds if and only if $b = \phi(a)$. Hint: interpret the inequality in geometric terms.

(iii) Conclude that for $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad a, b \geq 0.$$

A.12 Fix a bounded interval $I \subset \mathbb{R}$, and let $f, g\colon I \to \mathbb{R}$ be bounded functions, which are Riemann integrable. Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and assume that $|f|^p\colon I \to \mathbb{R}$ and $|g|^q\colon I \to \mathbb{R}$ are Riemann integrable, where $|f|^p(x) = |f(x)|^p$ and similarly for $|g|^q$.

(i) Assume that $\int_I |f|^p(x)\,dx = 1$ and that $\int_I |g|^q(x)\,dx = 1$, show that

$$\int_I |f(x)g(x)|\,dx \leq 1$$

Hint: use Young's inequality from Exercise A.11(iii).

(ii) Show Hölder's inequality

$$\int_I |f(x)g(x)|\,dx \leq \left(\int_I |f|^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_I |g|^q(x)\,dx\right)^{\frac{1}{q}}$$

Hint: first do the special case $\int_I |f|^p(x)\,dx = 0$ or $\int_I |g|^q(x)\,dx = 0$, and in case $\int_I |f|^p(x)\,dx > 0$ and $\int_I |g|^q(x)\,dx > 0$ normalise f and g to reduce to the case (i).

(iii) Show the Cauchy-Schwarz inequality for integrals;

$$\int_I |f(x)g(x)|\,dx \leq \sqrt{\int_I |f|^2(x)\,dx}\sqrt{\int_I |g|^2(x)\,dx}$$

assuming that all integrals exist as Riemann integrals.

A.13 Fix a bounded interval $I \subset \mathbb{R}$, and let $f, g \colon I \to \mathbb{R}$ be bounded functions, which are Riemann integrable. Let $p \in [1, \infty)$ and assume that $|f|^p \colon I \to \mathbb{R}$, $|g|^p \colon I \to \mathbb{R}$ and $|f+g|^p \colon I \to \mathbb{R}$ are Riemann integrable for all $p \geq 1$.

(i) Show that

$$\int_I |f(x) + g(x)|\,dx \leq \int_I |f(x)|\,dx \leq + \int_I |g(x)|\,dx.$$

(ii) Let $p > 1$. Show that

$$\int_I |f(x) + g(x)|^p\,dx \leq \int_I |f(x) + g(x)|^{p-1}\,|f(x)|dx + \int_I |f(x) + g(x)|^{p-1}\,|g(x)|dx.$$

and

$$\int_I |f(x) + g(x)|^{p-1}\,|g(x)|dx \leq \left(\int_I |f(x) + g(x)|^p\,dx\right)^{\frac{p-1}{p}} \left(\int_I |g(x)|^p\,dx\right)^{\frac{1}{p}}.$$

Hint: use Hölder's inequality of Exercise A.12(ii).

(iii) Use the previous estimate to show that

$$\int_I |f(x) + g(x)|^p\,dx \leq \left(\int_I |f(x) + g(x)|^p\,dx\right)^{\frac{p-1}{p}} \times \left(\left(\int_I |f|^p(x)\,dx\right)^{\frac{1}{p}} + \left(\int_I |g|^p(x)\,dx\right)^{\frac{1}{p}}\right).$$

(iv) Prove Minkowski's inequality

$$\left(\int_I |f(x)+g(x)|^p\,dx\right)^{\frac{1}{p}} \le \left(\int_I |f|^p(x)\,dx\right)^{\frac{1}{p}} + \left(\int_I |g|^p(x)\,dx\right)^{\frac{1}{p}}.$$

A.14 Assume we have real or complex sequences $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$.

(i) Let $p, q \in (1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$. Show that

$$\sum_{k=0}^{\infty} |a_k b_k| \le \left(\sum_{k=0}^{\infty} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=0}^{\infty} |b_k|^q\right)^{\frac{1}{q}}$$

which is Hölder's inequality for series. (Hint: mimick the proof of Exercise A.12.) Explain that the result is valid in the real line extended with ∞.

(ii) Show that

$$\sum_{k=0}^{\infty} |a_k b_k| \le \left(\sup_{k\in\mathbb{N}} |a_k|\right) \sum_{k=0}^{\infty} |b_k|,$$

which we view as corresponding to the case $q = 1$, $p = \infty$ of (i). Explain that the result is valid in the real line extended with ∞.

(iii) Show that for $p \ge 1$ we have

$$\left(\sum_{k=0}^{\infty} |a_k+b_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=0}^{\infty} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=0}^{\infty} |b_k|^p\right)^{\frac{1}{p}},$$

which is Minkowski's inequality for series. (Hint: mimick the proof of Exercise A.13.) Explain that the result is valid in the real line extended with ∞.

A.15 Assume $(a_n)_{n\in\mathbb{N}}$ is a sequence with $a_n > 0$ for all $n \in \mathbb{N}$.

(i) Assume the limit L in

$$L = \lim_{n\to\infty} n\big(1 - \frac{a_{n+1}}{a_n}\big)$$

exists. Show that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$.

(ii) Assume additionally $L > 1$. Show that there exists $N \in \mathbb{N}$ so that $0 \le (n-1)a_n - na_{n+1}$ for all $n \ge N$. (Hint: pick $\varepsilon > 0$ so that $L - \varepsilon > 1$ and the corresponding N, and argue that $(L-\varepsilon-1)a_n < (n-1)a_n - na_{n+1}$ for $n \ge N$.)

(iii) We assume $L > 1$. Show that the series $\sum_{n=0}^{\infty}(n-1)a_n - na_{n+1}$ is convergent. (Hint: Show that $\lim_{n\to\infty} na_{n+1}$ exists.)

(iv) We assume $L > 1$. Show that the series $\sum_{n=0}^{\infty} a_n$ is convergent.

(v) Use the result in this exercise to show that $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ is convergent for $\alpha > 1$. (Hint: also use the Mean Value Theorem.)

Appendix B

Hints for selected exercises

B.1 Chapter 2

2.2.5 Suppose that L and M are the values. Note that there are three cases: $L = M$, $L < M$, $M < L$, and show that the last two lead to a contradiction.

2.2.6 Consider $-A = \{x \in \mathbb{R} \mid -x \in A\}$ and apply Theorem 2.2.4. Argue that $\sup(-A) = -\inf(A)$.

2.3.2 Use the definitions.

B.2 Chapter 3

3.1.6 true, true, true, false, true, false.

3.1.9 Label the subsequence

$$\begin{array}{l}
\mathbf{n_0^{(0)}} < n_1^{(0)} < n_2^{(0)} < n_3^{(0)} < n_4^{(0)} < n_5^{(0)} < n_6^{(0)} < n_7^{(0)} < n_8^{(0)} < n_9^{(0)} < \cdots \\
n_0^{(1)} < \mathbf{n_1^{(1)}} < n_2^{(1)} < n_3^{(1)} < n_4^{(1)} < n_5^{(1)} < n_6^{(1)} < n_7^{(1)} < n_8^{(1)} < n_9^{(1)} < \cdots \\
n_0^{(2)} < n_1^{(2)} < \mathbf{n_2^{(2)}} < n_3^{(2)} < n_4^{(2)} < n_5^{(2)} < n_6^{(2)} < n_7^{(2)} < n_8^{(2)} < n_9^{(2)} < \cdots \\
n_0^{(3)} < n_1^{(3)} < n_2^{(3)} < \mathbf{n_3^{(3)}} < n_4^{(3)} < n_5^{(3)} < n_6^{(3)} < n_7^{(3)} < n_8^{(3)} < n_9^{(3)} < \cdots \\
n_0^{(4)} < n_1^{(4)} < n_2^{(4)} < n_3^{(4)} < \mathbf{n_4^{(4)}} < n_5^{(4)} < n_6^{(4)} < n_7^{(4)} < n_8^{(4)} < n_9^{(4)} < \cdots \\
n_0^{(5)} < n_1^{(5)} < n_2^{(5)} < n_3^{(5)} < n_4^{(5)} < \mathbf{n_5^{(5)}} < n_6^{(5)} < n_7^{(5)} < n_8^{(5)} < n_9^{(5)} < \cdots \\
n_0^{(6)} < n_1^{(6)} < n_2^{(6)} < n_3^{(6)} < n_4^{(6)} < n_5^{(6)} < \mathbf{n_6^{(6)}} < n_7^{(6)} < n_8^{(6)} < n_9^{(6)} < \cdots \\
\vdots
\end{array}$$

an show that $n_{k+1}^{(k+1)} > n_k^{(k)}$ for all $k \in \mathbb{N}$.

3.2.7 $\forall M \in \mathbb{R}\, \exists N \in \mathbb{N}\, \forall n \geq N\ a_n < M$.

3.2.22 (ii) false.

3.2.23 $(c_n)_{n=0}^{\infty}$ is not necessarily convergent, but if it is convergent then its limit is contained in $[L, M]$.

3.3.7 For (i) and (ii), write out the definition and use the properties of sup and inf. For (iii) For $x < \liminf_{n\to\infty} a_n$ we have

$$\exists N \in \mathbb{N} \quad \forall n \geq N \qquad a_n > x$$

or, to the left of the liminf there are only finitely many elements of the sequence. For $x > \liminf_{n\to\infty} a_n$ we have

$$\forall N \in \mathbb{N} \quad \exists n \geq N \qquad a_n < x$$

or, at any small distance to the right of the liminf there there infinitely many elements of the sequence.

3.4.2 The case $r = 1$ and $r = -1$ should be done separately. Follow the lines of the proof of Proposition 3.2.14 for the case $0 < r < 1$ and show that $\inf\{r^n \mid n \in \mathbb{N}\} = 0$. For $|r| < 1$ use the sandwich principle of Theorem 3.2.19(vii). For $|r| > 1$ show that the sequence is unbounded.

3.4.3 Note that this is more of a calculus exercise. First consider $0 < M <$ and show that $(\sqrt[n]{M})_{n=1}^{\infty}$ is an increasing subsequence bounded above by 1. Next show that its supremum is at least 1 using Exercise 3.4.2. Use Theorem 3.2.19 to reduce the case $M > 1$ to the case $0 < M < 1$.

An alternative proof goes as follows: take $M > 1$ and put $x_n = \sqrt[n]{M} - 1$, then $x_n > 0$ and, using the binomial formula (2.1.3),

$$1 + nx_n \leq \sum_{k=0}^{n} \binom{n}{k} x_n^k = (1 + x_n)^n = M \quad \Longrightarrow \quad 0 < x_n < \frac{M-1}{n}$$

and use Theorem 3.2.19(vii) and Proposition 3.2.14. The case $0 < M < 1$ follows from the case $M > 1$ using Theorem 3.2.19.

3.4.6 (i) divergent, (ii) no conclusion, can be convergent (taken $b = -a_n$ for all n) or divergent (take $b_n = a_n$).

3.4.8 true, true, false, false.

3.4.11 For the case

$$\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \leq \limsup_{n\to\infty} a_n + b_n.$$

Let $L = \liminf_{n\to\infty} a_n = \sup_k \inf_{n\geq k} a_n$, so for arbitrary $\varepsilon > 0$ there exists $K \in \mathbb{N}$ with for all $k \geq K$ we have $L - \varepsilon < \inf_{n\geq k} a_n \leq L$. Hence, for all $n \geq K$ we have $L - \varepsilon < a_n$, cf. Exercise 3.3.7. Conclude that for all $n \geq K$ we have $a_n + b_n > b_n + L - \varepsilon$, and taking suprema over $n \geq k$ for $k \geq K$ gives $\sup_{n\geq k}(a_n + b_n) \geq L - \varepsilon + \sup_{n\geq k} b_n$. Taking the infimum over $k \geq K$ gives $\inf_{k\geq K} \sup_{n\geq k}(a_n + b_n) \geq L - \varepsilon + \inf_{k\geq K} \sup_{n\geq k} b_n$, and since we deal with decreasing sequences, this is sufficient to conclude

$$\limsup_{n\to\infty} a_n + b_n \geq \liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n - \varepsilon.$$

Since $\varepsilon > 0$ arbitrary, we get the required inequality using Corollary 3.2.16(iv).

3.4.14 (i) First show that $L = 0$ is trivial. Next assume $L > 0$, and take $\varepsilon > 0$ arbitrarily assuming $L - \varepsilon > 0$. $L - \varepsilon$ is not an upper bound for the increasing subsequence $\inf_{j\geq n} \frac{c_{j+1}}{c_j}$, so $\exists\, N \in \mathbb{N}$ with

$$L - \varepsilon < \inf_{j\geq N} \frac{c_{j+1}}{c_j} \leq L$$

In particular, $\forall\, j \geq N$ we have $c_{j+1} > (L - \varepsilon)c_j$, and thus $c_{N+j} > (L - \varepsilon)^j c_N$ for all $j \in \mathbb{N}$ (by induction). Then for $n \geq N$ we have $c_n \geq A(L - \varepsilon)^n$ with $A = c_N(L - \varepsilon)^{-N}$. So $\sqrt[n]{c_n} \geq (L - \varepsilon)\sqrt[n]{A}$ for $n \geq N$. By Exercise 3.4.3 we get $\liminf_{n\to\infty} \sqrt[n]{c_n} \geq (L - \varepsilon)$.

B.3 Chapter 4

4.1.5 (ii) Take $A_i = (-\frac{1}{i+1}, 1)$, then $\bigcap_{i\in\mathbb{N}} A_i = [0, 1)$ (prove this). This is not an open set.

4.1.21 Use $\partial A = \overline{A} \cap (A^\circ)^c$ and the de Morgan rules, see Section 2.1.

4.4.4 Observe that $c + \frac{1}{n}$ is not a lower bound. Pick $a_n \in A$ with $L \leq |c - a_n| < L + \frac{1}{n}$, where $L = \inf_{a\in A} |c - a|$. Consider the sequence $(a_n)_{n=1}^{\infty}$.

4.4.5 (ii) No, take e.g. $A = \{n \in \mathbb{N} \mid n \geq 1\}$ and $B = \{-n + \frac{1}{n} \mid n \in \mathbb{N}, n \geq 2\}$. Then $0 \notin A + B$, but since the sequence $(\frac{1}{n})_{n=2}^{\infty}$ is contained in $A + B$, we have $0 \in \overline{A + B}$. For (iii) pick a sequence $(a_n + b_n)_{n\in\mathbb{N}}$ converging to $x \in \overline{A + B}$ and pick a convergent subsequence of $(a_{n_j})_{j\in\mathbb{N}}$. Show that $(b_{n_j})_{j\in\mathbb{N}}$ is convergent, and that it converges in B. Finish the proof by showing that $x = a + b$ using the limits of the convergent subsequences.

4.4.7 For (i) consider Lemma 4.1.16. For (ii) the inclusion is trivial, and use (i) for the other statement. For (iii) $A \subset A'$ by definition of an open set, and for $x \in \partial A$ it follows from Proposition 4.1.19. For (iv) write out the definitions, and for (v) consider $A = \{\frac{1}{n} \mid n \in \mathbb{N}, n \geq 1\}$.

B.4 Chapter 5

5.1.6 (ii) Let $f\colon A \to \mathbb{R}$ and $g\colon A \to \mathbb{R}$ be real-valued functions with domain $A \subset \mathbb{R}$. Let $E \subset A$ and assume $x_0 \in \overline{E}$ and that

$$\lim_{\substack{x\to x_0\\ x\in E}} f(x) = L, \qquad \lim_{\substack{x\to x_0\\ x\in E}} g(x) = M.$$

(vi) assume that there exists a neighbourhood $N_\delta(x_0)$ so that for all $x \in A \cap N_\delta(x_0)$ we have that $f(x) \le g(x)$, then $L \le M$;

(vii) assume that there exists a function $h\colon A \to \mathbb{R}$ so that for all $x \in A$ one has $f(x) \le h(x) \le g(x)$ and $L = M$. Then $\lim_{\substack{x\to x_0\\ x\in E}} h(x)$ exists and $\lim_{\substack{x\to x_0\\ x\in E}} h(x) = L = M$.

5.1.12 (i). Use Theorem 3.2.19 and Proposition 5.1.10.

(ii) By Proposition 5.1.10 it suffices to show the following addition to Theorem 3.2.19: if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are convergent sequences with $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then the sequences $(\max(a_n, b_n))_{n\in\mathbb{N}}$ and $(\min(a_n, b_n))_{n\in\mathbb{N}}$ are convergent with $\lim_{n\to\infty} \max(a_n, b_n) = \max(L, M)$ and $\lim_{n\to\infty} \min(a_n, b_n) = \min(L, M)$. To prove the first statement, choose $\varepsilon > 0$, then there exists $N_a \in \mathbb{N}$ so that for all $n \ge N_a$ we have $|a_n - L| < \varepsilon$ or $L - \varepsilon < a_n < L + \varepsilon$. Similarly, there exists $N_b \in \mathbb{N}$ so that for all $n \ge N_b$ we have $|b_n - M| < \varepsilon$ or $M - \varepsilon < b_n < M + \varepsilon$. Thus, for all $n \ge N = \max(N_a, N_b)$ we have

$$\max(L, M) - \varepsilon = \max(L - \varepsilon, M - \varepsilon) < \max(a_n, b_n) < max(L + \varepsilon . M + \varepsilon) = \max(L, M) + \varepsilon$$

or $|\max(a_n, b_n) - \max(L, M)| < \varepsilon$ for all $n \ge N$. So the limit for the maximum is proved, and for the minimum it is the same proof.

5.1.16 For any $c \in \mathbb{R}$ we have that

$$U_c = f^{-1}((c, \infty)) = \{a \in A \mid f(a) > c\}, \qquad L_c = f^{-1}((-\infty, c)) = \{a \in A \mid f(a) < c\}$$

are open relative to the domain A. Now take $x \in A$ arbitrarily, and we show that f is continuous in $x \in A$. Take $\varepsilon > 0$, then the set

$$U_{f(x)-\varepsilon} \cap L_{f(x)+\varepsilon} = \{a \in A \mid f(x) - \varepsilon < f(a) < f(x) + \varepsilon\}$$

is relatively open and contains x. So $\exists\, \delta > 0$ with $N_\delta(x) \cap A \subset U_{f(x)-\varepsilon} \cap L_{f(x)+\varepsilon}$, and this precisely means

$$|x - a| < \delta \text{ and } a \in A \qquad \Longrightarrow \qquad |f(x) - f(a)| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the function f is continuous at x. And since $x \in A$ is arbitrary, $f\colon A \to \mathbb{R}$ is continuous.

5.2.10 First observe that $f(A) \neq \emptyset$ and $f(A)$ is bounded, by Exercise 5.2.4. Put $m = \sup(f(A))$ and take a sequence in $f(A)$ converging to m. Take a corresponding sequence in A under f and use that A is sequentially compact to find a convergent subsequence. Show that f attains the maximum in the limit of the convergent subsequence.

5.3.3 Take $a_n = \frac{1}{n}$, $n \geq 1$.

5.2.14 Take an open cover of $f(A)$, by taking inverse images there is an open cover of A using Theorem 5.1.14. Now use the definition, and take images.

5.4.6 (ii) Fix $x \in A$, so $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence, and so convergent by Theorem 3.3.11. The limit is defined as $f(x) = \lim_{n\to\infty} f_n(x)$, and $f\colon A \to \mathbb{R}$ is a the limit function. So this gives pointwise convergence $\lim_{n\to\infty} f_n = f$.

Observe that $|f_n(x) - f_m(x)| < \varepsilon$ is the same as $f_n(x) - \varepsilon < f_m(x) < f_n(x) + \varepsilon$. Since $\lim_{m\to\infty} f_m(x) = f(x)$ exists, Theorem 3.2.19 gives $f_n(x) - \varepsilon \leq f(x) \leq f_n(x) + \varepsilon$, i.e. $|f(x) - f_n(x)| \leq \varepsilon$. Or, by taking the limit $m \to \infty$ in the condition of Theorem 5.4.5 gives

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall\, n \geq N \quad \forall\, x \in A \qquad |f(x) - f_n(x)| \leq \varepsilon$$

which gives uniform convergence.

5.5.3 (i) Take $\delta = \frac{\varepsilon}{M}$ if $M > 0$, the case $M = 0$ being trivial.
(ii) Use the reverse triangle inequality (2.1.2).

5.5.4 In both cases show that $g(x) = f(x) - x$ is a continuous function with $g(0)g(1) \leq 0$. Apply Theorem 5.2.11 to g.

5.5.5 Consider $g\colon [0, \frac{1}{2}] \to \mathbb{R}$ defined by $g(x) = f(x + \frac{1}{2}) - f(x)$ and consider $g(0)$ and $g(\frac{1}{2})$. Use Theorem 5.2.11 for g.

5.5.7 (i) Assume that $f\colon (0, \infty) \to \mathbb{R}$ and $L \in \mathbb{R}$ then a definition of $\lim_{x\to\infty} f(x) = L$ is

$$\forall\, \varepsilon > 0 \quad \exists\, M \in \mathbb{R} \quad \forall\, x \geq M \qquad |L - f(x)| < \varepsilon.$$

5.5.8 Take $c = f(1)$, first prove the statement for $x \in \mathbb{N}$, next for $x \in \mathbb{Z}$ and for $x \in \mathbb{Q}$. Then use continuity to get it for $x \in \mathbb{R}$.

5.5.9 Assume for convenience that $f(a) < y < f(b)$. Put $a_0 = a$, $b_0 = b$, and let $d = \frac{1}{2}(a + b)$. If $f(d) = y$ we are done, otherwise put $a_1 = d$, $b_1 = b$ in case $f(d) < y$ or $a_1 = a$, $b_1 = d$ in case $f(d) > y$. Then we have $f(a_1) < y < f(b_1)$. Construct two sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ inductively with $f(a_n) \leq y \leq f(b_n)$. Show that the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ are convergent using Theorem 3.2.9 and its consequences, and that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$. Use the continuity of f to establish $f(c) = y$.

5.5.14 Let $f(A) \subset \bigcup_{\alpha\in I} B_\alpha$ be an arbitrary open cover. Then $A \subset \bigcup_{\alpha\in I} f^{-1}(B_\alpha)$, and by Theorem 5.1.14 the sets $f^{-1}(B_\alpha)$ are open for all $\alpha \in I$. By definition of compactness of A, there exists a finite subcover $A \subset \bigcup_{i=1}^{N} f^{-1}(B_{\alpha_i})$, and then $f(A) \subset \bigcup_{i=1}^{N} B_{\alpha_i}$ is a finite subcover of the open cover.

B.5 Chapter 6

6.2.7 Note that the 'c' in Theorem 6.2.5 depends on f, so the c in the numerator depends on f and the one in the denominator depends on g. They will in general not be the same.

6.5.6 Proposition 6.1.4 gives that for all $\varepsilon > 0$ there exists $\delta > 0$ so that $|x - x_0| < \delta$ implies (after rewriting)

$$-\varepsilon|x - x_0| + f'(x_0)(x - x_0) \le f(x) - f(x_0) \le \varepsilon|x - x_0| + f'(x_0)(x - x_0).$$

Since f has a maximum in x_0, the middle term is nonpositive. If $f'(x_0) > 0$ then we take $\varepsilon = \frac{1}{2}f'(x_0)$ and we see that the left hand is strictly positive for $x > x_0$, which gives a contradiction. Similarly, $f'(x_0) < 0$ leads to a contradiction, so $f'(x_0) = 0$.

6.5.7 Use the Mean Value Theorem 6.2.5.

6.5.8 Use Rolle's Theorem 6.2.4.

6.5.9 Use the Mean Value Theorem 6.2.5. (iii): no.

B.6 Chapter 7

7.6.1 Let $g \in \mathrm{PC}(I)$ with partition $\mathscr{P}$ and $g \ge f$. Pick $J \in \mathscr{P}$, $J \ne \emptyset$, so for $x \in J$ we have $c_J = g(x) \ge f(x)$, so that $c_J \ge \sup_{x\in J} f(x)$. Conclude that $c_J\,|J| \ge (\sup_{x\in J} f(x))\,|J|$ implying

$$\mathrm{pc}\!\int_I g(x)\,dx \ge \sum_{J\in\mathscr{P};J\ne\emptyset} \Big(\sup_{x\in J} f(x)\Big)\,|J| \;=\; U(f,\mathscr{P}).$$

Conclude for $g \in \mathrm{PC}(I)$ with $g \ge f$ that

$$\inf\{U(f,\mathscr{P})\colon \mathscr{P} \text{ partition of } I\} \;\le\; \mathrm{pc}\!\int_I g(x)\,dx.$$

Take the infimum over $g \in \mathrm{PC}(I)$, $g \ge f$, gives

$$\inf\{U(f,\mathscr{P})\colon \mathscr{P} \text{ partition of } I\} \;\le\; \overline{\int_I} f(x)\,dx.$$

For the reverse inequality, realise $U(,\mathscr{P}) = \mathrm{pc}\!\int_I g(x)\,dx$ for $g \in \mathrm{PC}(I)$, $g \geq f$. Choose $\varepsilon > 0$ arbitray, find partition $\mathscr{P}_0$ so that

$$\inf\{U(f,\mathscr{P})\colon \mathscr{P} \text{ partition of } I\} \geq U(f,\mathscr{P}_0) - \varepsilon = \mathrm{pc}\!\int_I g(x)\,dx - \varepsilon \geq \overline{\int}_I f(x)\,dx - \varepsilon.$$

Conclude

$$\inf\{U(f,\mathscr{P})\colon \mathscr{P} \text{ partition of } I\} \geq \overline{\int}_I f(x)\,dx.$$

So $\inf\{U(f,\mathscr{P})\colon \mathscr{P} \text{ partition of } I\} = \overline{\int}_I f(x)\,dx$. Prove the other statement analogously.

7.6.2 Show that for each bounded interval with non-empty interior J we have $\sup_{x\in J} \chi(x) = 1$ and $\inf_{x\in J} \chi(x) = 0$.

7.6.3 No, yes.

7.6.7 (ii) No, take $f = \chi$, $g = -\chi$ as in Exercise 7.6.2.

7.6.8 Assume f to be increasing. Choose $\mathscr{P}$ an equidistant partition, i.e. $J_i = (a + \frac{b-a}{N}(i-1), a + \frac{b-a}{N}i]$ for $i \in \{1, \cdots, N\}$ (and where we adapt J_1 to be closed at the left endpoint a). Then pick $g \in \mathrm{PC}(I)$ with $g \geq f$ defined by $g|_{J_i} = f(a + \frac{b-a}{N}i)$ and pick $h \in \mathrm{PC}(I)$ with $h \leq f$ defined by $h|_{J_i} = f(a + \frac{b-a}{N}(i-1))$. Note that we use f increasing to see that $h \leq f \leq g$. Then

$$0 \leq \overline{\int}_{[a,b]} f(x)\,dx - \underline{\int}_{[a,b]} f(x)\,dx \leq \mathrm{pc}\!\int_{[a,b]} g(x)\,dx - \mathrm{pc}\!\int_{[a,b]} h(x)\,dx$$
$$= \sum_{i=1}^{N} \frac{b-a}{N}\big(f(a + \frac{b-a}{N}i) - f(a + \frac{b-a}{N}(i-1))\big) = \frac{b-a}{N}(f(b) - f(a)).$$

So by choosing N big, we can make this as small as an arbitrarily choosen ε. So $\overline{\int}_{[a,b]} f(x)\,dx = \underline{\int}_{[a,b]} f(x)\,dx$ and f is Riemann integrable.

7.6.10 (i) Observe that the functions $F\colon [a,\infty) \to \mathbb{R}$, $F(R) = \int_{[a,R]} f(x)\,dx$ and $G\colon [a,\infty) \to \mathbb{R}$ $G(R) = \int_{[a,R]} g(x)\,dx$ are increasing and $0 \leq F \leq G$. Since increasing functions F, respectively G, have limits $R \to \infty$ if and only if F, respectively G, is bounded, see Exercise 5.5.7, the results of (i) follow.

7.6.13 (iv) Write

$$\int_{[-1,-\varepsilon]\cup[\varepsilon,1]} \frac{f(x)}{x}\,dx = \int_{[\varepsilon,1]} \frac{f(x) - f(-x)}{x}\,dx.$$

using the substitution rule of Corollary 7.4.5 for a decreasing function. Now check that $u\colon [0,1] \to \mathbb{R}$ defined by

$$u(x) = \begin{cases} \frac{f(x)-f(-x)}{x}, & x > 0 \\ 2f'(0) & x = 0 \end{cases}$$

is a continuous function. Only a check for $x = 0$ is required, and this can be obtained using l'Hôpital's rule of Corollary 6.2.8. So we find that the limit is $\int_{[0,1]} u(x)\,dx$. (Show this using the definition and the fact that u is bounded on $[0, 1]$ by Lemma 5.2.2.

B.7 Chapter 8

8.4.3 In Proposition 7.5.2, let f_n correspond to the partial sum of the power series and take $x_0 = c$. Use that the radius of convergence is unaltered and Theorem 8.3.4.

8.7.1 All series are absolutely convergent, majorise by a series of the form $\sum_{k=1}^{\infty} \frac{1}{n^2}$ (or use Exercise 8.7.5). Use

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \qquad \frac{1}{n^2 - \frac{1}{4}} = \frac{1}{(n-\frac{1}{2})(n+\frac{1}{2})} = \frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}}$$

For the last observe that we need to extend the idea of telescoping series

$$\frac{1}{n(n+k)} = \frac{1}{k}\Big(\frac{1}{n} - \frac{1}{n+k}\Big)$$

so that the cancelling of terms occurs between the n-th and $(n+k)$-th term. Adapt the proof Proposition 8.1.6 accordingly. The iterated series is not convergent (after summing over n), the remaining series over k has terms larger than $\frac{1}{k}$ so that the harmonic series and Theorem 8.2.3 show divergence.

8.7.2 Absolutely convergent if and only if $\alpha > 1$. Relatively convergent for all $\alpha > 0$. For $\alpha \leq 0$ all divergent, by Corollary 8.1.8.

8.7.4 (i) $a_n \geq 0$ is clear, and also that $\lim_{k\to\infty} a_{2k-1} = 0$ and

$$\lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} \ln((1 + \frac{1}{k}) = 0$$

using Exercise 7.6.9. Now use Exercise 8.2.2. To see it is a decreasing sequence observe that

$$\frac{1}{k} \geq \int_{[k,k+1]} \frac{1}{t}\,dt \geq \frac{1}{k+1}$$

since $f(t) = \frac{1}{t}$ is decreasing.

(ii) Apply Theorem 8.2.1.

(iii) Apply Exercise 7.6.9 and Exercise 8.7.3.

8.7.5 (i) Observe that for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that for all $n \geq N$

$$(L - \varepsilon)b_n < a_n < (L + \varepsilon)b_n.$$

Choose $\varepsilon > 0$ wisely, and proceed as in the proof of Theorem 8.2.3.

(ii) Assume $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then we have

- if $\sum_{n=0}^{\infty} b_n$ convergent, then $\sum_{n=0}^{\infty} a_n$ convergent;
- if $\sum_{n=0}^{\infty} a_n$ divergent, then $\sum_{n=0}^{\infty} b_n$ divergent.

The proof follows from $0 \leq a_n < \varepsilon b_n$ for n sufficiently large.

(iii) Assume $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then we have

- if $\sum_{n=0}^{\infty} a_n$ convergent, then $\sum_{n=0}^{\infty} b_n$ convergent;
- if $\sum_{n=0}^{\infty} b_n$ divergent, then $\sum_{n=0}^{\infty} a_n$ divergent.

The proof follows from $a_n > M b_n \geq 0$ for n sufficiently large for $M > 0$.

(iv) divergent, absolutely convergent.

8.7.12 The result of Exercise 8.7.12 is known as Dirichlet's test. (i)

$$\sum_{n=r}^{s} a_n b_n = \sum_{n=r}^{s} (S_n - S_{n-1}) b_n = \sum_{n=r}^{s} S_n b_n - \sum_{n=r-1}^{s-1} S_n b_{n+1}$$
$$= \sum_{n=r}^{s-1} S_n (b_n - b_{n+1}) + S_s b_s - S_{r-1} b_r$$

(ii) Use $|S_n| \leq M$, and that $b_n \geq b_{n+1} \geq 0$ to find

$$\Big|\sum_{n=r}^{s} a_n b_n\Big| \leq M(b_s + b_r) + M \sum_{n=r}^{s-1} (b_n - b_{n+1}) = 2M(b_s + b_r)$$

Since $\lim_{n\to\infty} b_n = 0$ we can estimate the sum, and now use Theorem 8.1.9.

(iii) Take $a_n = (-1)^n$, so that $|S_n| \leq 1$.

8.7.13 Observe

$$\frac{d^{2n}}{dx^{2n}} \sin(x) = (-1)^n \sin(x), \qquad \frac{d^{2n+1}}{dx^{2n+1}} \sin(x) = (-1)^n \cos(x)$$

are bounded, so by Corollary 6.4.3 the Taylor polynomials converge uniformly to $\sin(x)$. Similarly for $\cos(x)$.

8.7.14 (i) Only at 0 we need to consider the continuity of the derivatives. Use the result of Exercise 7.6.9 to see that

$$\lim_{x\searrow 0} p(\frac{1}{x}) \exp(-1/x) = \lim_{y\to\infty} p(y) \exp(-y) = 0$$

for any polynomial.

8.7.16 no, yes.

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