q-SPECIAL FUNCTIONS, BASIC HYPERGEOMETRIC SERIES AND OPERATORS

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ABSTRACT. In the lecture notes we start off with an introduction to the q-hypergeometric series, or basic hypergeometric series, and we derive some elementary summation and transformation results. Then the q-hypergeometric difference equation is studied, and in particular we study solutions given in terms of power series at 0 and at ∞. Factorisations of the corresponding operator are considered in terms of a lowering operator, which is the q-derivative, and the related raising operator. Next we consider the q-hypergeometric operator in a special case, and we show that there is a natural Hilbert space –a weighted sequence space– on which this operator is symmetric. Then the corresponding eigenfunctions are polynomials, which are the little q-Jacobi polynomials. These polynomials form a family in the q-Askey scheme, and so many important properties are well known. In particular, we show how the orthogonality relations and the three-term recurrence for the little q-Jacobi polynomials can be obtained using only the factorisation of the corresponding operator. As a next step we consider the q-hypergeometric operator in general, which leads to the little q-Jacobi functions. We sketch the derivation of the corresponding orthogonality using the connection between various eigenfunctions. The link between the q-hypergeometric operators with different parameters is studied in general using q-analogues of fractional derivatives, and this gives transmutation properties for this operator. In the final parts of these notes we consider partial extensions of this approach to orthogonal polynomials and special functions. The first extension is a brief introduction to the Askey-Wilson functions and the corresponding integral transform. The second extension is concerned with a matrix-valued extension of the q-hypergeometric difference equation and its solutions.

CONTENTS

1. INTRODUCTION

Basic hypergeometric series have been introduced a long time ago, and important contributions go back to Euler, Heine, Rogers, Ramanujan, etc. The importance and the history of the basic hypergeometric series is clearly indicated in Askey's foreword to the book on basic hypergeometric series

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by Gasper and Rahman [18]. Since the work of Askey, Andrews, Ismail, and coworkers many new results on classes of special functions and orthogonal polynomials in terms of basic hypergeometric series have been obtained. The relation to representation theory of quantum groups and related structures in e.g. mathematical physics and combinatorics has given the topic a new boost in the recent decades.

In these lecture notes we focus on the basic hypergeometric series of type ${}_2\varphi_1$ by studying the corresponding q-difference operator to which these series are eigenfunctions. The study of general q-difference operators go back to Birkhoff and Trjitzinsky in the 1930s. In §2 we first introduce the basic hypergeometric series, and we derive some elementary summation and transformation formulas needed in the sequel. However, we will not prove all the necessary transformation formulas, but refer to Gasper and Rahman [18] when necessary. Section 2 is based on the book [18] by Gasper and Rahman, which is the basic reference for basic hypergeometric series. In §3 we then discuss the corresponding q-difference operator in more detail, by studying the solutions obtained by Frobenius's method. We also look at the decomposition of the operator using the standard q-difference operator. Next, in §4 we consider a special case of the q-difference operator, namely the one which can be related to polynomial eigenfunctions for functions supported on $q^{\mathbb{N}}$. This essentially leads to the little q-Jacobi polynomials, for which we derive the natural orthogonality measure, the corresponding orthogonality relations, and the three-term recurrence relation by using the shift operators. These shift operators are the operators in factorisations of the difference operator.

In §5 we study a more general case. This leads to general orthogonality for ${}_2\varphi_1$-series, which we derive by calculating the spectral measure of the corresponding measure. In §6 we study the transmutation properties of the basic hypergeometric q-difference operator. In §7 we then lift this to the level of Askey-Wilson polynomials and the Askey-Wilson functions.

In §8 we make a first start in order to lift the results on little q-Jacobi polynomials of §4 and little q-Jacobi functions of §5 to the matrix-valued extensions.

There are many related results available in the literature, and we indicate several developments in the notes to each section. In particular, it is not clear if the results of §4 and §5 can be extended to the level of the Askey-Wilson functions as in §7 or the matrix-valued analogues of §8.

By $\mathbb{N}$ we denote the natural numbers starting at 0. The standing assumption on q is $0 < q < 1$.

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2. Basic hypergeometric series

The basic hypergeometric series are analogues of the much better known hypergeometric series and hypergeometric functions. The hypergeometric series ${}_2F_1(a,b;c;z)$ as well as the analogous Thomae series ${}_{r+1}F_r$ and the more general hypergeometric ${}_rF_s$-series are discussed in detail in e.g. [4], [7], [25], [31], [32], [55], [61], [65] and many other standard textbooks. Recall the notation for that standard hypergeometric function

$$ {}_2F_1(a,b;c;z) \;=\; {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right) \;=\; \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n\, n!} z^n, \tag{2.1} $$

for this series (and for its sum when it converges) assuming $c \neq 0, -1, -2, \cdots$. This is the *(ordinary) hypergeometric series* or the *Gauss hypergeometric series*. The series converges absolutely for $|z| < 1$, and for $|z| = 1$ when $\Re(c-a-b) > 0$, see Exercise 1. Many important functions, such as the logarithm, arcsin, exponential, classical orthogonal polynomials can be expressed in terms of Gauss hypergeometric series. $(a)_n$ denotes the *shifted factorial* or *Pochhammer symbol* or *raising factorial* defined by

$$(a)_0 = 1, \qquad (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n = 1, 2, \dots . \tag{2.2}$$

More generally, one can define hypergeometric series with more parameters.

Around the mid 19th-century Heine introduced the series

$$1 + \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)} z + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})} z^2 + \cdots, \tag{2.3}$$

where it is assumed that $q \neq 1$, $c \neq 0, -1, -2, \dots$ and the principal value of each power of q is taken. This series converges absolutely for $|z| < 1$ when $|q| < 1$ and it tends termwise to Gauss' series as $q \to 1$, because

$$\lim_{q\to 1} \frac{1-q^a}{1-q} = a. \tag{2.4}$$

The ratio $(1-q^a)/(1-q)$ considered in (2.4) is called a *q-number* (or *basic number*) and it is denoted by $[a]_q$. One should realise that other notations for q-numbers, such as $\frac{q^a - q^{-a}}{q-q^{-1}}$, are also in use. It is also called a q-analogue, q-deformation, q-extension, or a q-generalization of the complex number a. In terms of q-numbers the *q-number factorial* $[n]_q!$ is defined for a nonnegative integer n by $[n]_q! = \prod_{k=1}^n [k]_q$, and the corresponding *q-number shifted factorial* is defined by $[a]_{q;n} = \prod_{k=0}^{n-1} [a+k]_q$. Clearly, $[a]_{q;n} = (1-q)^{-n}(q^a;q)_n$, with the notation (2.6) and $\lim_{q\to 1}[a]_{q;n} = (a)_n$. The series in (2.3) is usually called *Heine's series* or, in view of the base q, the *basic hypergeometric series* or *q-hypergeometric series*, or simply a *q-series*.

2.1. **Notation for basic hypergeometric series.** Analogous to Gauss's notation for the hypergeometric function, Heine used the notation $\varphi(a, b, c, q, z)$ for his series. However, since one would like to also be able to consider the case when q to the power a, b, or c is replaced by zero, it is now customary to define the *basic hypergeometric series*

$$\varphi(a, b; c; q, z) \equiv {}_2\varphi_1(a, b; c; q, z) \equiv {}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n, \tag{2.5}$$

where

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n = 1, 2, \dots, \end{cases} \tag{2.6}$$

is the *q-shifted factorial* and for general a and b it is assumed that $c \neq q^{-m}$ for $m = 0, 1, \dots$. Some other notations that have been used in the literature for the product $(a;q)_n$ are $(a)_{q,n}$, $[a]_n$ (not to be confused with $[a]_q$).

Unless stated otherwise, when dealing with nonterminating basic hypergeometric series we shall assume that $|q| < 1$ and that the parameters and variables are such that the series converges absolutely. Note that if $|q| > 1$, then we can perform an inversion with respect to the base by setting $p = q^{-1}$ and using the identity

$$(a;q)_n = (a^{-1};p)_n (-a)^n p^{-\binom{n}{2}} \tag{2.7}$$

to convert the series (2.9) to a similar series in base p with $|p|<1$, see (2.11). The inverted series will have a finite radius of convergence if the original series does.

More generally, we call the series $\sum_{n=0}^{\infty} u_n$ a (unilateral) *hypergeometric series* if the quotient u_{n+1}/u_n is a rational function of n. Similarly, a series $\sum_{n=0}^{\infty} v_n$ a *basic hypergeometric series* (with base q) if the quotient v_{n+1}/v_n is a rational function of q^n for a fixed base q. The most general form of the quotient is

$$\frac{v_{n+1}}{v_n} = \frac{(1-a_1q^n)(1-a_2q^n)\cdots(1-a_rq^n)}{(1-q^{n+1})(1-b_1q^n)\cdots(1-b_sq^n)}\,(-q^n)^{1+s-r}\,z. \tag{2.8}$$

normalising $v_0=1$. Generalising Heine's series, we define an ${}_r\varphi_s$ *basic hypergeometric series* by

$$\begin{aligned} {}_r\varphi_s(a_1,a_2,\ldots,a_r;b_1,\ldots,b_s;q,z) \equiv {}_r\varphi_s\left(\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix};q,z\right) \\ = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n \end{aligned} \tag{2.9}$$

with $\binom{n}{2}=n(n-1)/2$, where $q\neq 0$ when $r>s+1$.

Remark 2.1. If $0<|q|<1$, the ${}_r\varphi_s$ series converges absolutely for all z if $r\leq s$ and for $|z|<1$ if $r=s+1$. This series also converges absolutely if $|q|>1$ and $|z|<|b_1b_2\cdots b_sq|/|a_1a_2\cdots a_r|$. It diverges for $z\neq 0$ if $0<|q|<1$ and $r>s+1$, and if $|q|>1$ and $|z|>|b_1b_2\cdots b_sq|/|a_1a_2\cdots a_r|$, unless it terminates.

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notations

$$(a_1,a_2,\ldots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, \qquad n\in\mathbf{N}. \tag{2.10}$$

As is customary, the ${}_r\varphi_s$ notation is also used for the sums of these series inside the circle of convergence and for their analytic continuations (called *basic hypergeometric functions*) outside the circle of convergence. To switch from base q to base q^{-1} we note

$${}_r\varphi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix};q,z\right) = \sum_{n=0}^{\infty} \frac{(a_1^{-1},\ldots,a_r^{-1};q^{-1})_n}{(q^{-1},b_1^{-1},\ldots,b_s^{-1};q^{-1})_n}\left(\frac{a_1\cdots a_r z}{b_1\cdots b_s q}\right)^n \tag{2.11}$$

assuming the upper and lower parameters are non-zero.

It is important to note that in case one of the upper parameters is of the form q^{-n} for $n\in\mathbf{N}$ the series in (2.9) terminates. From now on, unless stated otherwise, whenever $q^{-j},q^{-k},q^{-m},q^{-n}$ appear as numerator parameters in basic series it will be assumed that j, k, m, n, respectively, are nonnegative integers. For terminating series it is sometimes useful to switch the order of summation, which is given by

$$\begin{aligned} {}_{r+1}\varphi_s\left(\begin{matrix} a_1,\ldots,a_r,q^{-n} \\ b_1,\ldots,b_s\end{matrix};q,z\right) = \frac{(a_1,\ldots,a_r;q)_n}{(b_1\ldots,b_s;q)_n}\left(\frac{z}{q}\right)^n\left((-1)^n q^{\binom{n}{2}}\right)^{s-r-1} \\ \times \sum_{k=0}^{n} \frac{(q^{1-n}/b_1,\ldots,q^{1-n}/b_s,q^{-n};q)_k}{(q,q^{1-n}/a_1,\ldots,q^{1-n}/a_r;q)_k}\left(\frac{b_1\cdots b_s}{a_1\cdots a_r}\,\frac{q^{n+1}}{z}\right)^k \end{aligned} \tag{2.12}$$

for non-zero parameters.

Observe that the series (2.9) has the property that if we replace z by z/a_r and let $a_r\to\infty$, then the resulting series is again of the form (2.9) with r replaced by $r-1$. Because this is not the case for the ${}_r\varphi_s$ series defined without the factors $\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}$ in the books of Bailey [7] and Slater

[61] and we wish to be able to handle such limit cases, we have chosen to define the series ${}_r\varphi_s$ as in (2.9). There is no loss in generality since the Bailey and Slater series can be obtained from the $r = s+1$ case of (2.9) by choosing s sufficiently large and setting some of the parameters equal to zero.

For negative subscripts, the *q-shifted factorials* as defined in (2.6) are defined by

$$(a;q)_{-n} = \frac{1}{(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{-n})} = \frac{1}{(aq^{-n};q)_n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a;q)_n}, \tag{2.13}$$

where $n = 0, 1, \ldots$. We also define

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \tag{2.14}$$

for $|q| < 1$. Since the infinite product in (2.14) diverges when $a \neq 0$ and $|q| > 1$, whenever $(a;q)_\infty$ appears in a formula, we shall assume that $|q| < 1$. In particular, for $|q| < 1$ and z an integer

$$(a;q)_z = \frac{(a;q)_\infty}{(aq^z;q)_\infty}, \tag{2.15}$$

which is a notation that we also employ for complex z, where we take standard branch cut for the complex power.

The basic hypergeometric series

$${}_{r+1}\varphi_r\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, \ldots\ , b_r \end{matrix}; q, z\right)$$

is called *k-balanced* if $b_1 b_2 \cdots b_r = q^k a_1 a_2 \cdots a_{r+1}$ and $z = q$, and a 1-balanced basic hypergeometric series is called *balanced* (or *Saalschützian*). The basic hypergeometric series ${}_{r+1}\varphi_r$ is *well-poised* if the parameters satisfy the relations

$$qa_1 = a_2 b_1 = a_3 b_2 = \cdots = a_{r+1} b_r;$$

very-well-poised if, in addition, $a_2 = qa_1^{\frac12}, a_3 = -qa_1^{\frac12}$.

For very-well-poised series the following notation is in use:

$$\begin{aligned}{}_{r+1}W_r\left(a_1; a_4, a_5, \ldots, a_{r+1}; q, z\right) &= {}_{r+1}\varphi_r\left(\begin{matrix} a_1, qa_1^{\frac12}, -qa_1^{\frac12}, a_4, \ldots, a_{r+1} \\ a_1^{\frac12}, -a_1^{\frac12}, qa_1/a_4, \ldots, qa_1/a_{r+1} \end{matrix}; q, z\right) \\ &= \sum_{k=0}^{\infty} \frac{1-a_1 q^{2k}}{1-a_1} \frac{(a_1, a_4, \cdots, a_{r+1}; q)_k}{(q, qa_1/a_4, \cdots, qa_{r+1}/a_1} z^k.\end{aligned} \tag{2.16}$$

The q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}} \tag{2.17}$$

and satisfies the following recurrences

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q. \tag{2.18}$$

The generalized q-binomial coefficient is defined for complex α, β by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_q = \frac{(q^{\beta+1}, q^{\alpha-\beta+1}; q)_\infty}{(q, q^{\alpha+1}; q)_\infty} \tag{2.19}$$

and then (2.18) remains valid for complex α.

We end Section 2.1 with some useful identities for q-shifted factorials;

$$
\begin{gathered}
(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} \\
(a^{-1}q^{1-n};q)_n = (a;q)_n(-a^{-1})^n q^{-\binom{n}{2}}, (a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}\,(-qa^{-1})^k q^{\binom{k}{2}-nk}, \\
(a;q)_{n+k} = (a;q)_n(aq^n;q)_k, \\
(aq^n;q)_k = \frac{(a;q)_k(aq^k;q)_n}{(a;q)_n}, \\
(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}, \\
(aq^{2k};q)_{n-k} = \frac{(a;q)_n(aq^n;q)_k}{(a;q)_{2k}}, \\
(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}, \\
(aq^{-n};q)_k = \frac{(a;q)_k(qa^{-1};q)_n}{(a^{-1}q^{1-k};q)_n} q^{-nk}, \\
(a;q)_{2n} = (a;q^2)_n(aq;q^2)_n, \\
(a;q)_{3n} = (a;q^3)_n(aq;q^3)_n(aq^2;q^3)_n, \\
(a^2;q^2)_n = (a;q)_n(-a;q)_n, \\
(a^3;q^3)_n = (a;q)_n(\omega a;q)_n(\omega^2 a;q)_n, \qquad \omega = e^{2\pi i/3}
\end{gathered}
$$

and similar expressions for $(a;q)_{kn}$ and $(a^k;q^k)_n$, $k = 4, 5, \cdots$.

2.2. **Some summation and transformation formulae.** There are many summation and transformation results for basic hypergeometric series available, and we only give a few basic results. We give precise references in case we need more advanced summation or transformation formulae.

The most fundamental result is the q-binomial theorem, stating

$$
{}_1\varphi_0\left(\begin{matrix} a \\ - \end{matrix}\,;q,z\right) = \frac{(az;q)_\infty}{(z;q)_\infty}, \qquad |z|<1. \tag{2.20}
$$

Its terminating version reads

$$
{}_1\varphi_0\left(\begin{matrix} q^{-n} \\ - \end{matrix}\,;q,z\right) = (q^{-n}z;q)_n, \qquad n \in \mathbb{N}. \tag{2.21}
$$

The proof is sketched in Exercise 6. We discuss a few consequences of the q-binomial theorem.

First, we write

$$
\frac{(az;q)_\infty}{(z;q)_\infty}\frac{(z;q)_\infty}{(bz;q)_\infty} = \frac{(az;q)_\infty}{(bz;q)_\infty} \quad\Longrightarrow\quad {}_1\varphi_0\left(\begin{matrix} a \\ - \end{matrix}\,;q,z\right)\,{}_1\varphi_0\left(\begin{matrix} 1/b \\ - \end{matrix}\,;q,bz\right) = {}_1\varphi_0\left(\begin{matrix} a/b \\ - \end{matrix}\,;q,bz\right)
$$

and this is a product of analytic functions, so that the coefficients of the power series have to be equal. This gives

$$\sum_{k+p=n} \frac{(a;q)_k}{(q;q)_k} \frac{(1/b;q)_p}{(q;q)_p} b^p = \frac{(a/b;q)_n}{(q;q)_n} b^n \quad \Longrightarrow \quad {}_2\varphi_1 \left(\begin{matrix} q^{-n}, a \\ c \end{matrix} ; q, q \right) = \frac{(c/a;q)_n}{(c;q)_n} a^n \tag{2.22}$$

after relabeling. This is the q-Chu-Vandermonde formula.

Another application of the q-binomial formula (2.20) is Heine's transformation formula. Heine showed

$$ {}_2\varphi_1(a,b;c;q,z) = \frac{(b,az;q)_\infty}{(c,z;q)_\infty} \, {}_2\varphi_1(c/b,z;az;q,b), \tag{2.23}$$

where $|z| < 1$ and $|b| < 1$. By iterating the result

$$\begin{aligned} {}_2\varphi_1(a,b;c;q,z) &= \frac{(c/b,bz;q)_\infty}{(c,z;q)_\infty} \, {}_2\varphi_1(abz/c,b;bz;q,c/b) \\ &= \frac{(abz/c;q)_\infty}{(z;q)_\infty} \, {}_2\varphi_1(c/a,c/b;c;q,abz/c). \end{aligned} \tag{2.24}$$

with appropriate conditions on the parameters for the last two series to be convergent. Heine's formula (2.23) can directly be obtained from the q-binomial theorem (2.20);

$$\begin{aligned} {}_2\varphi_1(a,b;c;q,z) &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^\infty \frac{(a;q)_n (cq^n;q)_\infty}{(q;q)_n (bq^n;q)_\infty} z^n \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^\infty \frac{(a;q)_n}{(q;q)_n} z^n \sum_{m=0}^\infty \frac{(c/b;q)_m}{(q;q)_m} (bq^n)^m \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^\infty \frac{(c/b;q)_m}{(q;q)_m} b^m \sum_{n=0}^\infty \frac{(a;q)_n}{(q;q)_n} (zq^m)^n \\ &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^\infty \frac{(c/b;q)_m}{(q;q)_m} b^m \frac{(azq^m;q)_\infty}{(zq^m;q)_\infty} \\ &= \frac{(b,az;q)_\infty}{(c,z;q)_\infty} \, {}_2\varphi_1 \left(\begin{matrix} c/b, z \\ az \end{matrix} ; q, b \right), \end{aligned} \tag{2.25}$$

which gives (2.23). The implied convergence of the series above is assumed to hold. Limit cases of Heine's transformation formulas (2.23), (2.24) are

$$\begin{aligned} (c;q)_\infty \, {}_1\varphi_1 \left(\begin{matrix} a \\ c \end{matrix} ; q, z \right) &= (a,z;q)_\infty \, {}_2\varphi_1 \left(\begin{matrix} c/a, 0 \\ z \end{matrix} ; q, a \right) \\ &= (c/a;q)_\infty \, {}_2\varphi_1 \left(\begin{matrix} az/c, a \\ 0 \end{matrix} ; q, \frac{c}{a} \right) \\ &= (az/c,c;q)_\infty \, {}_2\varphi_1 \left(\begin{matrix} c/a, 0 \\ c \end{matrix} ; q, \frac{az}{c} \right) \\ &= (z;q)_\infty \, {}_1\varphi_1 \left(\begin{matrix} az/c \\ z \end{matrix} ; q, c \right), \end{aligned} \tag{2.26}$$

so that in particular $(c;q)_\infty \, {}_1\varphi_1(0;c;q,z) = (z;q)_\infty \, {}_1\varphi_1(0;z;q,c)$ is symmetric in c and z. This symmetry is observed by Koornwinder and Swarttouw [51] in their study of the q-Hankel transform for the ${}_1\varphi_1$-q-Bessel functions. Taking a limit in (2.26) we obtain

$$(c;q)_\infty\, {}_0\varphi_1\left(\begin{matrix} - \\ c \end{matrix};q,z\right) = (z/c,c;q)_\infty\, {}_2\varphi_1\left(\begin{matrix} 0,0 \\ c \end{matrix};q,\frac{z}{c}\right) = {}_1\varphi_1\left(\begin{matrix} z/c \\ 0 \end{matrix};q,c\right). \tag{2.27}$$

Consider the q-integral on an interval $[0,a]$, defined by

$$\int_0^a f(t)\,d_q(t) = (1-q)a\sum_{k=0}^\infty q^k f(aq^k) \tag{2.28}$$

whenever the function f is such that the series in (2.28) converges. Note that we can view (2.28) as a Riemann sum for $\int_0^a f(t)\,dt$ on a non-equidistant partition of the interval $[0,a]$. Using the notation (2.28) we can rewrite (2.23) as

$$\begin{aligned} {}_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,z\right) &= \frac{(b,c/b,z;q)_\infty}{(q,c,z;q)_\infty}\sum_{k=0}^\infty \frac{(q^{k+1},azq^k;q)_\infty}{(q^kc/b,zq^k;q)_\infty} b^k \\ &= \frac{(b,c/b,z;q)_\infty}{(q,c,z;q)_\infty}\frac{1}{1-q}\int_0^1 \frac{(qt,azt;q)_\infty}{(tc/b,tz;q)_\infty} t^{-1+\log_q b}\,d_qt \end{aligned}$$

which can be considered as a q-analogue of Euler's integral representation

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt, \qquad \Re c > \Re b > 0.$$

for the hypergeometric series, see e.g. [3], [65].

Another integral representation is the Watson integral representation. In Watson's formula we assume $0<q<1$, and then

$${}_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,z\right) = \frac{-1}{2\pi}\frac{(a,b;q)_\infty}{(q,c;q)_\infty}\int_{-i\infty}^{i\infty}\frac{(q^{1+s},cq^s;q)_\infty}{(aq^s,bq^s;q)_\infty}\frac{\pi(-z)^s}{\sin(\pi s)}\,ds \tag{2.29}$$

for $|z|<1$ and $|\arg(-z)|<\pi$. The contour runs from $-i\infty$ to $i\infty$ via the imaginary axis with indentations such that the poles of $1/\sin(\pi s)$ lie to the right of the contour and the poles of $1/(aq^s,bq^s;q)_\infty$ lie to the left of the contour. Then Watson's formula (2.29) follows by a residue calculation and estimates on the behaviour of the integrand, see [18, §4.2]. By then flipping the contour, and evaluating the integral using the residues at the poles of $1/(aq^s,bq^s;q)_\infty$ and performing the right estimates shows the connection formula, see [18, §4.3];

$$\begin{aligned} {}_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,z\right) &= \frac{(b,c/a;q)_\infty(az,q/az;q)_\infty}{(c,b/a;q)_\infty(z,q/z;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix} a,aq/c \\ aq/b \end{matrix};q,cq/abz\right) \\ &+ \frac{(a,c/b;q)_\infty}{(c,a/b;q)_\infty}\frac{(bz,q/bz;q)_\infty}{(z,q/z;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix} b,bq/c \\ bq/a \end{matrix};q,cq/abz\right). \end{aligned} \tag{2.30}$$

which gives the analytic continuation to the region $|\arg(-z)|<\pi$, with c and a/b not integer powers of q, and $a,b,z\neq 0$. Note that the coefficients in (2.30) are related to theta functions,. Indeed, because of Jacobi's triple product identity, see [18, §1.6],

$$\theta(z) = (z,q/z;q)_\infty \qquad \Longrightarrow \qquad \theta(q^kz) = (-z)^k q^{-\frac{1}{2}k(k-1)}\theta(z) \tag{2.31}$$

θ is a renormalized Jacobi theta function.

2.3. **Exercises.**

1. Use Raabe's test to show that ${}_2F_1(a,b;c,z)$, $|z|=1$, converges absolutely for $\Re(c-a-b)>0$.
2. Prove the statements on convergence of the basic hypergeometric series as in Remark 2.1.
3. Prove (2.12).
4. Prove (2.18).
5. Prove the useful identities for q-shifted factorials.
6. Askey's proof of the q-binomial theorem (2.20) goes as follows. Denote the ${}_1\varphi_0$-series by $h_a(z)$ and show that

$$h_a(z) - h_{aq}(z) = -az\, h_{aq}(z), \qquad h_a(z) - h_a(qz) = (1-a)z\, h_{aq}(z) \quad \Longrightarrow$$
$$h_a(z) = \frac{1-az}{1-z}\, h_a(qz).$$

Iterate and use the analyticity and the value at $z=0$ to finish the proof.

Notes. The basic reference for basic hypergeometric series is the standard book [18] by Gasper and Rahman, or the first edition of [18]. The book by Gasper and Rahman contains a wealth of information on basic hypergeometric series. There are older books containing chapters on basic hypergeometric series, e.g. Bailey [7], Slater [61], as well as the Heine's book –the second edition of *Handbuch der Kugelfunktionen* of 1878, see references in [18]. More modern books on special functions having chapters on basic hypergeometric series are e.g. [4], [25]. Another useful reference is the lecture notes by Ismail [26].

3. Basic hypergeometric q-difference equation

An important aspect of the hypergeometric series ${}_2F_1$ is that it can be used to describe the solutions to the *hypergeometric differential equation*

$$z(1-z)\frac{d^2 f}{dz^2}(z) + (c-(a+b+1)z)\frac{df}{dz}(z) - abf(z) = 0, \tag{3.1}$$

see e.g. [4], [25], [55], [65]. In particular,

$$u_1(z) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; z\right), \quad c \neq 0, -1, -2, \cdots$$

solves the hypergeometric differential equation (3.1) as can be checked directly by plugging the power series expansion. Other solutions expressible in terms of hypergeometric series are e.g.

$$u_2(z) = z^{1-c}\, {}_2F_1\left(\begin{matrix} a-c+1, b-c+1 \\ 2-c \end{matrix} ; z\right), \quad c \neq 2, 3, \cdots$$
$$u_3(z) = z^{-a}\, {}_2F_1\left(\begin{matrix} a, a-c+1 \\ a-b+1 \end{matrix} ; \frac{1}{z}\right), \quad a-b \neq -1, -2, \cdots$$
$$u_4(z) = z^{-b}\, {}_2F_1\left(\begin{matrix} b, b-c+1 \\ b-a+1 \end{matrix} ; \frac{1}{z}\right), \quad b-a \neq -1, -2, \cdots.$$

The differential equation (3.1) is a Fuchsian differential equation with three regular singular points at 0, 1 and ∞. So one usually also considers the similar solutions in terms of power series around $z=1$, but these solutions do not have appropriate q-analogues.

So in general we have two linearly independent solutions in terms of power series around 0 and two linearly independent solutions in terms of power series around ∞. Since the solution space of

(3.1) is 2-dimensional, there are all kinds of relations between these solutions. One of the classical relations between hypergeometric series is given by

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\,{}_2F_1\left(\begin{matrix} a,a-c+1 \\ a-b+1 \end{matrix};\frac{1}{z}\right) \\
+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\,{}_2F_1\left(\begin{matrix} b,b-c+1 \\ b-a+1 \end{matrix};\frac{1}{z}\right)
\end{aligned}
\tag{3.2}
$$

for $|\arg(-z)| < \pi$.

The Jacobi polynomials are special cases of the hypergeometric series ${}_2F_1$; explicitly

$$
P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix};\frac{1}{2}(1-x)\right). \tag{3.3}
$$

So in particular, the Jacobi polynomials are eigenfunctions to a second-order differential operator. This differential operator can then be studied on the weighted L^2 spaces with respect to the beta-weight $(1-x)^\alpha(1-x)^\beta$ on $[-1,1]$. The differential operator is a self-adjoint operator on a suitable domain with compact resolvent. The orthogonality of the Jacobi polynomials is related to the orthogonality of the eigenvectors of the corresponding differential operator.

The Jacobi functions are

$$
\phi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1\left(\begin{matrix} \frac{1}{2}(\alpha+\beta+1+i\lambda), \frac{1}{2}(\alpha+\beta+1-i\lambda) \\ \alpha+1 \end{matrix};-\sinh^2 t\right) \tag{3.4}
$$

and these are eigenfunctions of a related second order differential operator, after a change of variables. The corresponding Jacobi function transform arises from the spectral decomposition of the differential operator, see [48] for more information as well as the link to representation theory of non-compact symmetric spaces of rank one.

3.1. **Basic hypergeometric q-difference equation.** For fixed $q \neq 1$, the q-derivative operator D_q is defined by

$$
D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}, \qquad x \neq 0, \tag{3.5}
$$

and $D_q f(0) = f'(0)$ assuming the derivative exists. Then $D_q f(x)$ tends to $f'(x)$ as $q \to 1$ for differentiable f. We can iterate; $D_q^n f = D_q(D_q^{n-1} f)$, $n = 1,2,\cdots$. The q-difference operator D_q applied to the ${}_2\varphi_1$-series:

$$
D_q^n\,{}_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,z\right) = \frac{(a,b;q)_n}{(c;q)_n(1-q)^n}\,{}_2\varphi_1\left(\begin{matrix} aq^n, bq^n \\ cq^n \end{matrix};q,z\right) \tag{3.6}
$$

which can be checked directly. Moreover, $u(z) = {}_2\varphi_1(a,b;c;q,z)$ satisfies (for $|z| < 1$ and in the formal power series sense) the second order q-difference equation

$$
z(c-abqz)D_q^2 u + \left[\frac{1-c}{1-q} + \frac{(1-a)(1-b)-(1-abq)}{1-q}z\right] D_q u - \frac{(1-a)(1-b)}{(1-q)^2}u = 0, \tag{3.7}
$$

which is a q-analogue of the hypergeometric differential equation (3.1). Indeed, replacing a, b, c with q^a, q^b, q^c and taking formal limits, shows that (3.7) tends to (3.1) as $q \to 1$. Explicitly, (3.7) is

$$
(c-abz)\,u(qz) + (-(c+q)+(a+b)z)\,u(z) + (q-z)\,u(z/q) = 0 \tag{3.8}
$$

for a,b,c non-zero complex numbers. We consider (3.8) as the *basic hypergeometric q-difference equation*. Note that if u is a solution to (3.8), and C is a q-periodic function, i.e. $C(qz) = C(z)$, then Cu is also a solution to (3.8).

Proposition 3.1. *The functions*

$$
\begin{aligned}
u_1(z) &= {}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix} ; q, z\right), \quad c \neq q^{-n},\ n = 0, 1, 2 \cdots \\
u_2(z) &= z^{1-\log_q(c)}\ {}_2\varphi_1\left(\begin{matrix} qa/c, qb/c \\ q^2/c \end{matrix} ; q, z\right), \quad c \neq q^{n+2},\ n = 0, 1, 2 \cdots,
\end{aligned} \tag{3.9}
$$

and the functions

$$
\begin{aligned}
u_3(z) &= z^{-\log_q(a)}\ {}_2\varphi_1\left(\begin{matrix} a, qa/c \\ qa/b \end{matrix} ; q, \frac{qc}{abz}\right), \quad a \neq bq^{-n-1},\ n = 0, 1, 2 \cdots, \\
u_4(z) &= z^{-\log_q(b)}\ {}_2\varphi_1\left(\begin{matrix} b, qb/c \\ qb/a \end{matrix} ; q, \frac{qc}{abz}\right), \quad b \neq aq^{-n-1},\ n = 0, 1, 2 \cdots.
\end{aligned} \tag{3.10}
$$

are solutions of the basic hypergeometric q-difference equation (3.8).

Since the map $z \mapsto qz$ has two fixed points on the Riemann sphere, namely $z = 0$ and $z = \infty$, it is natural to consider power series expansion solutions of (3.8) at $z = 0$ and $z = \infty$ using the Frobenius method.

Proof. We make the Ansatz

$$
u(z) = \sum_{n=0}^{\infty} a_n z^{n+\mu}, \qquad a_n \in \mathbb{C},\ a_0 \neq 0,\ \mu \in \mathbb{C}
$$

Plugging such a solution into (3.8) and collecting the coefficients of $z^{n+\mu}$, we require

$$
0 = a_0 z^{\mu}\big(cq^{\mu} - (c+q) + q^{1-\mu}\big) +
$$
$$
\sum_{n=1}^{\infty} z^{\mu+n}\Big(a_n\big(cq^{\mu+n} - (c+q) + q^{1-\mu-n}\big) + a_{n-1}\big(-abq^{\mu+n-1} + a + b - q^{1-\mu-n}\big)\Big)
$$

So the coefficient of z^{μ} has to be zero, and this gives the *indicial equation*

$$
0 = cq^{\mu} - (c+q) + q^{1-\mu} = (q^{\mu} - 1)(c - q^{1-\mu}).
$$

So we find $q^{\mu} = 1$ or $q^{\mu} = q/c$.

In case $q^{\mu} = 1$ we find the recurrence relation

$$
a_n\big(cq^n - (c+q) + q^{1-n}\big) = a_{n-1}\big(abq^{n-1} - (a+b) + q^{1-n}\big) \quad \Longrightarrow
$$
$$
a_n = a_{n-1}\frac{(aq^{n-1} - 1)(b - q^{1-n})}{(cq^n - q)(1 - q^{-n})} = a_{n-1}\frac{(1 - aq^{n-1})(1 - bq^{n-1})}{(1 - cq^n)(1 - q^{n+1})} = a_0\frac{(a, b; q)_n}{(c, q; q)_n},
$$

so we find the solution u_1 for $\mu = 0$. If we take more generally $\mu = \frac{2\pi i}{\log q}k$, $k \in \mathbb{Z}$, then we multiply u_1 by the q-periodic function $z \mapsto z^{\frac{2\pi i}{\log q}k}$.

In case $q^{\mu} = q/c$ we find the recurrence relation

$$
a_n\big(q^{1+n} - (c+q) + cq^{-n}\big) = a_{n-1}\big(\frac{ab}{c}q^n - (a+b) + cq^{-n}\big) \quad \Longrightarrow
$$
$$
a_n = a_{n-1}\frac{(q^n\frac{a}{c} - 1)(b - cq^{-n})}{(q^n - 1)(q - cq^{-n})} = a_{n-1}\frac{(1 - q^n a/c)(1 - bq^n/c)}{(1 - q^n)(1 - q^{n+1}/c)} = a_0\frac{(qa/c, qb/c; q)_n}{(q, q^2/c; q)_n},
$$

so we find the solution u_2 for $\mu = 1 - \log_q(c)$, and again if we add an integer multiple of $\frac{2\pi i}{\log q}$, we multiply by a q-periodic function.

Similarly we obtain the solutions u_3, u_4 by replacing the Ansatz by $u(z) = \sum_{n=0}^{\infty} a_n z^{-n-\mu}$, $a_n \in \mathbb{C}$, $a_0 \neq 0$, $\mu \in \mathbb{C}$. We leave this as Exercise 2. $\square$

Remark 3.2. Note that for generic parameters the solutions u_1 and u_2, respectively u_3 and u_4, are linearly independent (over q-periodic functions). So we expect that these solutions satisfy relations amongst each other. In particular, (2.30) gives

$$u_1(z) = C_3(z)u_3(z) + C_4(z)u_4(z)$$
$$C_3(z) = C_3(z;a,b;c) = \frac{(b, c/a;q)_\infty (az, q/az;q)_\infty}{(c, b/a;q)_\infty (z, q/z;q)_\infty} z^{\log_q(a)}, \qquad C_4(z) = C_3(z;b,a;c)$$

Note that indeed, $C_3(qz) = C_3(z)$ using (2.31) and so C_3 and C_4 are q-periodic functions.

We rewrite the basic hypergeometric equation (3.8) as

$$(c - abz)\,\frac{u(qz) - u(z)}{z} + (1 - z/q)\,\frac{u(z/q) - u(z)}{z/q} = (1-a)(1-b)u(z). \tag{3.11}$$

We consider the left hand side as an operator acting on functions u, so we put

$$\big(Lu\big)(z) = \big(L^{a,b,c}u\big)(z) = (c - abz)\,\frac{u(qz) - u(z)}{z} + (1 - z/q)\,\frac{u(z/q) - u(z)}{z/q}, \tag{3.12}$$

so that upon using the normalized q-difference operator, cf. (3.5),

$$\big(\tilde{D}_q u\big)(z) = \frac{u(qz) - u(z)}{z}$$

we can rewrite (3.11) as

$$\big(Lu\big)(z) = (c - abz)\,\big(\tilde{D}_q u\big)(z) - (1 - z/q)\,\big(\tilde{D}_q u\big)(z/q) = (1-a)(1-b)u(z), \tag{3.13}$$

so that the operator L in left hand side of (3.11), (3.13) can be written as the composition $L = S \circ \tilde{D}_q$, where

$$S = S^{a,b,c}, \qquad \big(Sf\big)(z) = (c - abz)f(z) - (1 - z/q)f(z/q). \tag{3.14}$$

Note that when acting on polynomials, $\tilde{D}_q$ maps polynomials of degree n to polynomials of degree $n-1$ and S maps polynomials of degree n to polynomials of degree $n+1$. So we see that $L = S \circ \tilde{D}_q$ gives a factorisation in terms of a lowering operator and a raising operator. It is then common to consider the reversed composition and consider it as the Darboux transform of L. Lemma 3.3 shows that the Darboux transform is again of the same class.

Lemma 3.3. *We have*

$$\big(\tilde{D}_q \circ S^{a,b,c} f\big)(z) = \frac{1}{q}\big(L^{aq,bq,cq} f\big)(z) + (1-q)(ab - q^{-1})f(z),$$

and if $L^{a,b,c}u = (1-a)(1-b)u$, then $f = \tilde{D}_q u$ satisfies $L^{aq,bq,cq} f = (1-aq)(1-bq)f$.

Proof. A straightforward calculation gives

$$\begin{aligned}
\big(\tilde{D}_q \circ S^{a,b,c} f\big)(z) &= \frac{1}{z}\Big((c - abqz)f(qz) - (1-z)f(z) - (c - abz)f(z) + (1 - z/q)f(z/q)\Big) \\
&= (c - abqz)\frac{f(qz) - f(z)}{z} + (1 - z/q)\frac{f(z/q) - f(z)}{z/q}\frac{1}{q} + ab(1-q)f(z) + (1 - q^{-1})f(z) \\
&= \frac{1}{q}\left((cq - abq^2 z)\frac{f(qz) - f(z)}{z} + (1 - z/q)\frac{f(z/q) - f(z)}{z/q}\right) + (1-q)(ab - q^{-1})f(z)
\end{aligned}$$

and the term in brackets is precisely the operator $L^{aq,bq,cq}$ acting on f by (3.12).

Apply the first result to $f = \tilde{D}_q u$, so that

$$\frac{1}{q}\big(L^{aq,bq,cq}f\big)(z) + (1-q)(ab-q^{-1})f(z) = \big(\tilde{D}_q \circ S^{a,b,c} \circ \tilde{D}_q u\big)(z)$$
$$= \big(\tilde{D}_q \circ L^{a,b,c}u\big)(z) = (1-a)(1-b)\big(\tilde{D}_q u\big)(z) = (1-a)(1-b)f(z)$$

so that

$$\big(L^{aq,bq,cq}f\big)(z) = \big(q(1-a)(1-b) - q(1-q)(ab-q^{-1})\big)f(z)$$
$$= \big(q - aq - bq + abq - abq + abq^2 + 1 - q)\big)f(z) = (1-aq)(1-bq)f(z).$$

This proves the second statement, and it is in line with (3.6). □

3.2. **Exercises.**

1. Prove (3.6).
2. Show the second part of Proposition 3.1, see also Theorem 8.3.

Notes. The solutions follow unpublished notes by Koornwinder, and the factorisation and the Darboux transform seems to be well known.

4. Basic hypergeometric q-difference equation: polynomial case

We first consider the basic hypergeometric q-difference operator on a space which we can identify with a sequence space on $\mathbb{N}$. This is closely connected to the little q-Jacobi polynomials, and we derive its orthogonality and recurrence properties from properties of this operator.

4.1. **The difference equation in a special case.** Replace $z = z_0 q^{k+1}$ in (3.8) and put $u_k = u(z_0 q^{k+1})$ then we get

$$(c - abz_0q^{k+1})\,u_{k+1} + \Big(-(c+q) + (a+b)z_0q^{k+1}\Big)\,u_k + (q - z_0q^{k+1})\,u_{k-1} = 0 \quad \Longrightarrow$$
$$(cq^{-k-1} - abz_0)\,u_{k+1} - (c+q)q^{-k-1}\,u_k + (q^{-k} - z_0)\,u_{k-1} = -(a+b)z_0\,u_k$$

Note that in the special case $z_0 = 1$ the coefficient of u_{-1} is zero. So we consider the operator L on sequences $u = (u_k)_{k\in\mathbb{N}}$ by

$$\begin{aligned}(Lu)_k &= (cq^{-k-1} - ab)\,u_{k+1} - \big((cq^{-k-1} - ab) + (q^{-k} - 1)\big)\,u_k + (q^{-k} - 1)\,u_{k-1}\\ &= (cq^{-k-1} - ab)\,(u_{k+1} - u_k) + (q^{-k} - 1)\,(u_{k-1} - u_k), \qquad k \geq 1,\\ (Lu)_0 &= (cq^{-1} - ab)\,(u_1 - u_0)\end{aligned} \tag{4.1}$$

Note that putting $\varphi(q^k) = u_k$, we can view L as

$$(L\varphi)(x) = (cq^{-1} - abx)\,\frac{\varphi(xq) - \varphi(x)}{x} + (1-x)\frac{\varphi(x/q) - \varphi(x)}{x}, \quad x = q^k, k \geq 1$$
$$(L\varphi)(1) = (cq^{-1} - ab)\,\big(\varphi(q) - \varphi(1)\big)$$

so $L\varphi$ is expressible in terms of $D_q\varphi$ and $D_{q^{-1}}\varphi$. Considering L as an operator acting on functions by

$$(L\varphi)(x) = (cq^{-1} - abx)\,\frac{\varphi(xq) - \varphi(x)}{x} + (1-x)\frac{\varphi(x/q) - \varphi(x)}{x}, \tag{4.2}$$

we see that L preserves the polynomials and moreover that L also preserves the degree. Note that L of (4.2) is slightly different from the L in (3.12) since there is a q-shift in the argument.

Letting $\mathbb{C}_N[x]$ be the polynomials of degree less than or equal to N, we see that $L\colon \mathbb{C}_N[x] \to \mathbb{C}_N[x]$ and moreover that L is lower-triangular with respect to the basis $\{x^k \mid 0 \leq k \leq N\}$. Then

$$Lx^n = \big(-ab(q^n-1) + (1-q^{-n})\big)x^n + \text{l.o.t.} = (1-abq^n)(1-q^{-n})x^n + \text{l.o.t.}$$

where l.o.t. means 'lower order terms'.

Lemma 4.1. *For each degree n, the operator L as in (4.2) has a polynomial eigenfunction of degree n with eigenvalue $(1-abq^n)(1-q^{-n})$.*

Proof. A lower triangular operator has its eigenvalues on the diagonal. Since the eigenvalues are different for different n, all eigenvalues have algebraic and geometric multiplicity equal to 1.

From the basic hypergeometric difference equation and Proposition 3.1 we see that the polynomial eigenfunction is

$${}_2\varphi_1\left(\begin{matrix} q^{-n}, abq^n \\ c \end{matrix}; q, qx\right).$$

$\square$

Consider the Hilbert space $\ell^2(\mathbb{N}, w)$ of weighted ℓ^2-sequences with inner product

$$\langle u, v\rangle = \sum_{k=0}^{\infty} u_k \overline{v_k} w_k$$

for some positive sequence $w = (w_k)_{k\in\mathbb{N}}$.

Proposition 4.2. *Take $0<q<1$, $ab, c \in \mathbb{R}$, then with*

$$w_k = c^k \frac{(abq/c;q)_k}{(q;q)_k}, \qquad c>0, abq<c,$$

the operator L with $D(L) = \{u = (u_k)_{k\in\mathbb{N}} \mid u_k \neq 0$ for at most finitely many $k \in \mathbb{N}\}$ is symmetric;

$$\langle Lu, v\rangle = \langle u, Lv\rangle, \qquad \forall\, u, v \in D(L).$$

Proof. We consider for finite sequences $(u)_k$, $(v)_k$ the difference of the inner products. Note that in particular all sums are finite, so that absolute convergence of all series involved is automatic.

We do the calculation slightly more general by not making any assumptions on the sequences $(u)_k$, $(v)_k$, but by chopping off the inner product. Denote $\langle u, v\rangle_N = \sum_{k=0}^N u_k\overline{v_k}w_k$ then obviously $\lim_{N\to\infty}\langle u, v\rangle_N = \langle u, v\rangle$ for any two sequences $(u)_k$, $(v)_k \in \ell^2(\mathbb{N}, w)$. Now

$$\begin{gathered}
\langle Lu, v\rangle_N - \langle u, Lv\rangle_N = (cq^{-1}-ab)(u_1-u_0)\overline{v_0}w_0 - u_0\overline{(cq^{-1}-ab)(v_1-v_0)}w_0 + \\
\sum_{k=1}^N \Big((cq^{-k-1}-ab)\,(u_{k+1}-u_k) + \,(q^{-k}-1)\,(u_{k-1}-u_k)\Big)\overline{v_k}w_k - \\
\sum_{k=1}^N u_k \overline{\Big((cq^{-k-1}-ab)\,(v_{k+1}-v_k) + \,(q^{-k}-1)\,(v_{k-1}-v_k)\Big)} w_k = \\
(cq^{-1}-ab)w_0\big(u_1\overline{v_0} - u_0\overline{v_1}\big) + \sum_{k=1}^N \Big((cq^{-k-1}-ab)\,u_{k+1} + \,(q^{-k}-1)\,u_{k-1}\Big)\overline{v_k}w_k - \\
\sum_{k=1}^N u_k \overline{\Big((cq^{-k-1}-ab)\,v_{k+1} + (q^{-k}-1)\,v_{k-1}\Big)} w_k
\end{gathered}$$

since the coefficients are real. Relabeling gives

$$\begin{aligned}\langle Lu,v\rangle_N-\langle u,Lv\rangle_N=(cq^{-1}-ab)w_0\big(u_1\overline{v_0}-u_0\overline{v_1}\big)+\\ \sum_{k=2}^{N+1}u_k(cq^{-k}-ab)\overline{v_{k-1}}\frac{w_{k-1}}{w_k}w_k-\sum_{k=1}^{N}u_k(q^{-k}-1)\,\overline{v_{k-1}}w_k\\ +\sum_{k=0}^{N-1}u_k(q^{-1-k}-1)\,\overline{v_{k+1}}\frac{w_{k+1}}{w_k}w_k-\sum_{k=1}^{N}u_k(cq^{-k-1}-ab)\,\overline{v_{k+1}}w_k.\end{aligned}$$

Since we want L to be symmetric, most of the terms have to cancel for all sequences $(u)_k$, $(v)_k$. So we need to impose

$$(cq^{-k}-ab)\frac{w_{k-1}}{w_k}=(q^{-k}-1),\qquad (q^{-1-k}-1)\frac{w_{k+1}}{w_k}=(cq^{-k-1}-ab) \tag{4.3}$$

which give the same recurrence relation for w_k;

$$w_k=\frac{(cq^{-k}-ab)}{(q^{-k}-1)}w_{k-1}=c\frac{(1-abq^k/c)}{(1-q^k)}w_{k-1}=c^k\frac{(abq/c;q)_k}{(q;q)_k}w_0.$$

Since we need $w_k>0$, we require $c>0$ and $abq/c<1$.

Taking this value for w_k we see that most of the terms in the sum cancels, and we get

$$\begin{aligned}\langle Lu,v\rangle_N-\langle u,Lv\rangle_N=(cq^{-1}-ab)w_0\big(u_1\overline{v_0}-u_0\overline{v_1}\big)+\\ u_{N+1}(q^{-N-1}-1)\overline{v_N}w_{N+1}-u_1(q^{-1}-1)\overline{v_0}w_1\\ +u_0(cq^{-1}-ab)\overline{v_1}w_0-u_N(cq^{-N-1}-ab)\,\overline{v_{N+1}}w_N\\ =u_{N+1}(q^{-N-1}-1)\overline{v_N}w_{N+1}-u_N(cq^{-N-1}-ab)\,\overline{v_{N+1}}w_N.\end{aligned}$$

Using (4.3) one obtains

$$\langle Lu,v\rangle_N-\langle u,Lv\rangle_N=\sqrt{(q^{-N-1}-1)(cq^{-N-1}-ab)}\sqrt{w_Nw_{N+1}}\big(u_{N+1}\overline{v_N}-u_N\,\overline{v_{N+1}}\big) \tag{4.4}$$

In particular, for finitely supported sequences $(u)_k$, $(v)_k$ we have $\langle Lu,v\rangle=\langle u,Lv\rangle$ by taking $N\gg 0$, so that L is symmetric with respect to the domain of the finitely supported sequences in $\ell^2(\mathbb{N},w)$. $\square$

Remark 4.3. We do not study the spectral analysis of L, but see §4.4. In the analysis the form in (4.4) is closely related to the study of self-adjoint extensions of L.

Let us now assume that u and v correspond to the polynomial eigenvalues of L as in Lemma 4.1. So this means $u_k=A+Bq^k+\mathcal{O}(q^{2k})$, $v_k=C+Dq^k+\mathcal{O}(q^{2k})$ as $k\to\infty$. For these values we see that (4.4) gives

$$\begin{aligned}\langle Lu,v\rangle_N-\langle u,Lv\rangle_N=\\ \sqrt{(q^{-N-1}-1)(cq^{-N-1}-ab)}\sqrt{w_Nw_{N+1}}\Big(\big(A+Bq^{N+1}+\mathcal{O}(q^{2N})\big)\big(\bar C+\bar Dq^N+\mathcal{O}(q^{2N})\big)-\\ \big(A+Bq^N+\mathcal{O}(q^{2N})\big)\big(\bar C+\bar Dq^{N+1}+\mathcal{O}(q^{2N})\big)\Big)=\\ \sqrt{(q^{-N-1}-1)(cq^{-N-1}-ab)}\sqrt{w_Nw_{N+1}}\Big((B\bar Cq+A\bar D)q^N-(B\bar C+A\bar Dq)q^N+\mathcal{O}(q^{2N})\Big)=\\ \sqrt{(q^{-1}-q^N)(cq^{-1}-abq^N)}\sqrt{w_Nw_{N+1}}\Big((B\bar C-A\bar D)(q-1)+\mathcal{O}(q^N)\Big).\end{aligned}$$

In particular, since we want to $\lim_{N\to\infty}\langle Lu,v\rangle_N - \langle u,Lv\rangle_N = 0$ for such sequences (u_k) and (u_k) and since $w_N = \mathcal{O}(c^N)$, we need $c<1$. Assuming $0<c<1$ and $abq<c$ we see that any polynomial p gives a sequence $(p(q^k))_k \in \ell^2(\mathbb{N},w)$.

4.2. **The little q-Jacobi polynomials.** In order to get to the little q-Jacobi polynomials we relabel. We (re-)define the polynomials from Lemma 4.1 as the little q-Jacobi polynomials;

$$p_n(x;\alpha,\beta;q) = {}_2\varphi_1\left(\begin{matrix} q^{-n},\alpha\beta q^{n+1} \\ \alpha q\end{matrix}; q, qx\right). \tag{4.5}$$

So we have specialized (a,b,c) to $(q^{-n},\alpha\beta q^{n+1},\alpha q)$ and the conditions $0<c<1$, $abq<c$ translate into $0<\alpha<q^{-1}$, $\beta<q^{-1}$. By (3.6) we have

$$\begin{aligned}\big(D_q p_n(\cdot;\alpha,\beta;q)\big)(x) &= \frac{(1-q^{-n})(1-\alpha\beta q^{n+1})}{(1-c)(1-q)}\,{}_2\varphi_1\left(\begin{matrix} q^{1-n},\alpha\beta q^{n+2} \\ \alpha q^2\end{matrix}; q, qx\right) \\ &= \frac{(1-q^{-n})(1-\alpha\beta q^{n+1})}{(1-c)(1-q)} p_{n-1}(x;\alpha q,\beta q;q)\end{aligned} \tag{4.6}$$

Let us also rename the weight of Proposition 4.2 to the new labeling. We get

$$w(\alpha,\beta;q)_k = (\alpha q)^k \frac{(\beta q;q)_k}{(q;q)_k}$$

and we denote the corresponding Hilbert space $\ell^2(\mathbb{N},w)$ by $\ell^2(\mathbb{N};\alpha,\beta;q)$. A polynomial sequences in $\ell^2(\mathbb{N};\alpha,\beta;q)$ is a sequence of the form $u_k = p(q^k)$ for some polynomial p. Note that these sequences are indeed in $\ell^2(\mathbb{N};\alpha,\beta;q)$, and we denote them by $\mathcal{P}$.

Lemma 4.4. *$\tilde{D}_q$ is an unbounded map from $\ell^2(\mathbb{N};\alpha,\beta;q)$ to $\ell^2(\mathbb{N};\alpha q,\beta q;q)$. As its domain we take the polynomial sequences $\mathcal{P}$. Then we have*

$$\langle \tilde{D}_q p, r\rangle_{\ell^2(\mathbb{N};\alpha q,\beta q;q)} = \langle p, S^{\alpha,\beta} r\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)}, \qquad \forall\, p,r\in\mathcal{P}$$

where

$$S^{\alpha,\beta}\colon \mathcal{P}\to\mathcal{P}, \qquad \big(S^{\alpha,\beta} r\big)_k = (\alpha q)^{-1}\frac{(1-q^k)}{(1-\beta q)} r(q^{k-1}) - \frac{(1-\beta q^{k+1})}{(1-\beta q)} r(q^k)$$

and $-\alpha q(1-\beta q)S^{\alpha,\beta}$ corresponds to $S^{a,b,c}$ as in (3.14) with $a=q^{-n}$, $b=\alpha\beta q^{n+1}$, $c=\alpha q$ and $z=q^{k+1}$.

Proof. Note that

$$\begin{aligned}\langle \tilde{D}_q p, r\rangle_{\ell^2(\mathbb{N};\alpha q,\beta q;q)} &= \sum_{k=0}^\infty \frac{p(x^{k+1})-p(q^k)}{q^k}\overline{r(q^k)}(\alpha q^2)^k\frac{(\beta q^2;q)_k}{(q;q)_k} \\ = \sum_{k=1}^\infty p(x^k)\overline{r(q^{k-1})}(\alpha q)^{k-1}\frac{(\beta q^2;q)_{k-1}}{(q;q)_{k-1}} &- \sum_{k=0}^\infty p(x^k)\overline{r(q^k)}(\alpha q)^k\frac{(\beta q^2;q)_k}{(q;q)_k} \\ = -p(1)\overline{r(1)} + \sum_{k=1}^\infty p(x^k)&\left(\overline{r(q^{k-1})}(\alpha q)^{k-1}\frac{(\beta q^2;q)_{k-1}}{(q;q)_{k-1}} - \overline{r(q^k)}(\alpha q)^k\frac{(\beta q^2;q)_k}{(q;q)_k}\right) \\ = -p(1)\overline{r(1)} + \sum_{k=1}^\infty p(x^k)&\left(\overline{r(q^{k-1})}(\alpha q)^{-1}\frac{(1-q^k)}{(1-\beta q)} - \overline{r(q^k)}\frac{(1-\beta q^{k+1})}{(1-\beta q)}\right)(\alpha q)^k\frac{(\beta q;q)_k}{(q;q)_k}.\end{aligned}$$

In this derivation we use that all sums converge absolutely, so that we can split the series and rearrange them. This calculation gives, using that $\alpha, \beta, q \in \mathbb{R}$,

$$\big(S^{\alpha,\beta}r\big)_k = (\alpha q)^{-1}\frac{(1-q^k)}{(1-\beta q)}r(q^{k-1}) - \frac{(1-\beta q^{k+1})}{(1-\beta q)}r(q^k)$$

which is again in $\mathcal{P}$ since $r \in \mathcal{P}$. Note that the formula is also valid for $k=0$.

We leave the fact that $\tilde{D}_q$ is unbounded to the reader. □

Note that $S^{\alpha,\beta}$ raises the degree of the polynomial by 1. We have already observed how $\tilde{D}_q$ acts on a little q-Jacobi polynomial, but we also want to find out for $S^{\alpha,\beta}$.

Proposition 4.5. *The little q-Jacobi polynomials are orthogonal in $\ell^2(\mathbb{N};\alpha,\beta;q)$. Moreover,*

$$\big(\tilde{D}_q p_n(\cdot;\alpha,\beta;q)\big)(x) = \frac{(1-q^{-n})(1-\alpha\beta q^{n+1})}{(1-c)}p_{n-1}(x;\alpha q,\beta q;q),$$
$$\big(S^{\alpha,\beta}p_{n-1}(\cdot;\alpha q,\beta q;q)\big)(x) = \frac{1}{\alpha q}\frac{1-\alpha q}{1-\beta q}p_n(x;\alpha,\beta;q)$$

Proof. Note that the second order difference operator has the little q-Jacobi polynomial as eigenfunction, see Lemma 4.1. In the relabeling the operator L is given by

$$\big(L^{\alpha,\beta}f\big)(x) = \big(Lf\big)(x) = \alpha(1-\beta qx)\frac{f(qx)-f(x)}{x} + (1-x)\frac{f(x/q)-f(x)}{x}$$

see (4.2) with (a,b,c,x) replaced by $(q^{-n}, \alpha\beta q^{n+1}, \alpha q, qx)$. Since the decomposition of $L^{\alpha,\beta}$ of (3.14) corresponds, up to a scalar, with the operator $S^{\alpha,\beta}$ we see that $\big(S^{\alpha,\beta}p_{n-1}(\cdot;\alpha q,\beta q;q)\big)(x)$ has to be a multiple of $p_n(x;\alpha,\beta;q)$. By considering the evaluation at 0, and using $p_n(0;\alpha,\beta;q)=1$, we see that the multiple is

$$\frac{(1/\alpha q - 1)}{1-\beta q} = \frac{1}{\alpha q}\frac{1-\alpha q}{1-\beta q}$$

since we can write for a polynomial r

$$\big(S^{\alpha,\beta}r\big)(x) = \frac{1}{\alpha q}\frac{(1-x)}{(1-\beta q)}r(x/q) - \frac{(1-\beta qx)}{(1-\beta q)}r(x).$$

In order to show the orthogonality, we consider the following inner product for $k \le n$, using the raising and lowering operators $S^{\alpha,\beta}$ and $\tilde{D}_q$;

$$\begin{aligned}\langle x^k, p_n(\cdot;\alpha,\beta;q)\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)} &= \alpha q\frac{(1-\beta q)}{(1-\alpha q)}\langle x^k, S^{\alpha,\beta}p_{n-1}(\cdot;\alpha q,\beta q;q)\rangle_{\ell^2(\mathbb{N};\alpha q,\beta q;q)}\\ &= (q^k-1)\alpha q\frac{(1-\beta q)}{(1-\alpha q)}\langle x^{k-1}, p_{n-1}(\cdot;\alpha q,\beta q;q)\rangle_{\ell^2(\mathbb{N};\alpha q,\beta q;q)}\end{aligned}$$

since $\tilde{D}_q x^k = (q^k-1)x^{k-1}$. In particular, we get 0 for the inner product in case $k=0$. By iterating the procedure we get

$$\begin{aligned}\langle x^k, p_n(\cdot;\alpha,\beta;q)\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)} &= \alpha q\frac{(1-\beta q)}{(1-\alpha q)}\langle x^k, S^{\alpha,\beta}p_{n-1}(\cdot;\alpha q,\beta q;q)\rangle_{\ell^2(\mathbb{N};\alpha q,\beta q;q)}\\ &= (-1)^p(q^{k-p+1};q)_p\alpha^p q^{\frac{1}{2}p(p+1)}\frac{(\beta q;q)_p}{(\alpha q;q)_p}\langle x^{k-p}, p_{n-p}(\cdot;\alpha q^p,\beta q^p;q)\rangle_{\ell^2(\mathbb{N};\alpha q^p,\beta q^p;q)}\end{aligned}$$

and for $k<n$ this gives zero for $p=k+1\le n$. Hence, the little q-Jacobi polynomial $p_n(\cdot;\alpha,\beta;q)$ of degree n is orthogonal to all monomials of degree $<n$. So the the little q-Jacobi polynomials are orthogonal. □

Note that as an immediate corollary to the proof, we also obtain

$$\begin{aligned}\langle p_n(\cdot;\alpha,\beta;q),p_n(\cdot;\alpha,\beta;q)\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)} &= \mathrm{lc}\big(p_n(\cdot;\alpha,\beta;q)\big)\langle x^n,p_n(\cdot;\alpha,\beta;q)\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)}\\ =&\mathrm{lc}\big(p_n(\cdot;\alpha,\beta;q)\big)(-1)^n(q;q)_n\alpha^n q^{\frac12 n(n+1)}\frac{(\beta q;q)_n}{(\alpha q;q)_n}\langle 1,1\rangle_{\ell^2(\mathbb{N};\alpha q^n,\beta q^n;q)},\end{aligned} \tag{4.7}$$

where $\mathrm{lc}(p)$ denotes the leading coefficient of the polynomial p. So the shift operators can be used to find the squared norm of the orthogonal polynomials in terms of the squared norm of the constant function 1.

The leading coefficient can be calculated directly from the definition (4.5);

$$\mathrm{lc}\big(p_n(\cdot;\alpha,\beta;q)\big)=\frac{(q^{-n},\alpha\beta q^{n+1};q)_n}{(q,\alpha q;q)_n}q^n=(-1)^n q^{-\frac12 n(n-1)}\frac{(\alpha\beta q^{n+1};q)_n}{(\alpha q;q)_n}.$$

The squared norm of the constant function 1 follows from the q-binomial sum (2.20);

$$\langle 1,1\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)}=\sum_{k=0}^\infty(\alpha q)^k\frac{(\beta q;q)_k}{(q;q)_k}={}_1\varphi_0\left(\begin{matrix}\beta q\\ -\end{matrix};q,\alpha q\right)=\frac{(\alpha\beta q^2;q)_\infty}{(\alpha q;q)_\infty}.$$

This gives the following orthogonality relations for the little q-Jacobi polynomials from the analysis of the hypergeometric q-difference operator on a very specific set of points.

Theorem 4.6. *Let $0<\alpha<q^{-1}$, $\beta<q^{-1}$ and consider the little q-Jacobi polynomials*

$$p_n(x;\alpha,\beta;q)={}_2\varphi_1\left(\begin{matrix}q^{-n},\alpha\beta q^{n+1}\\ \alpha q\end{matrix};q,qx\right)$$

and the inner product space

$$\langle f,g\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)}=\sum_{k=0}^\infty f(q^k)\overline{g(q^k)}(\alpha q)^k\frac{(\beta q;q)_k}{(q;q)_k}$$

then the little q-Jacobi polynomials satisfy

$$\begin{aligned}\langle p_n(\cdot;\alpha,\beta;q),p_m(\cdot;\alpha,\beta;q)\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)}&=\delta_{m,n}h_n(\alpha,\beta;q)\\ h_n(\alpha,\beta;q)=(\alpha q)^n\frac{(q,\beta q;q)_n}{(\alpha q,\alpha\beta q;q)_n}&\frac{1-\alpha\beta q}{1-\alpha\beta q^{2n+1}}\frac{(\alpha\beta q^2;q)_\infty}{(\alpha q;q)_\infty}.\end{aligned}$$

Proof. This is a combination of the results in this section, and we are left with calculating the squared norm. Now

$$\begin{aligned}h_n(\alpha,\beta;q)&=\mathrm{lc}\big(p_n(\cdot;\alpha,\beta;q)\big)(-1)^n(q;q)_n\alpha^n q^{\frac12 n(n+1)}\frac{(\beta q;q)_n}{(\alpha q;q)_n}\langle 1,1\rangle_{\ell^2(\mathbb{N};\alpha q^n,\beta q^n;q)}\\ &=(-1)^n q^{-\frac12 n(n-1)}\frac{(\alpha\beta q^{n+1};q)_n}{(\alpha q;q)_n}(-1)^n(q;q)_n\alpha^n q^{\frac12 n(n+1)}\frac{(\beta q;q)_n}{(\alpha q;q)_n}\frac{(\alpha\beta q^{2n+2};q)_\infty}{(\alpha q^{n+1};q)_\infty}\\ &=\frac{(q,\beta q;q)_n}{(\alpha q,\alpha\beta q;q)_n}\frac{(\alpha q)^n}{1-\alpha\beta q^{2n+1}}\frac{(\alpha\beta q;q)_\infty}{(\alpha q;q)_\infty}\end{aligned}$$

which is equal to the stated value. ∎

4.3. **The three-term recurrence relation for the little q-Jacobi polynomials.** The shift operators $\tilde{D}_q$ and $S^{\alpha,\beta}$ play an essential role in the derivation of the orthogonality relations for the little q-Jacobi polynomials in Theorem 4.6. As a final application we show how one can obtain the coefficients in the three-term recurrence relation for the monic little q-Jacobi polynomials. Let $\tilde{p}_n(x;\alpha,\beta;q)$ be the monic little q-Jacobi polynomials. Because of the monicity we have a three-term recurrence of the form

$$x\,\tilde{p}_n(x;\alpha,\beta;q) = \tilde{p}_{n+1}(x;\alpha,\beta;q) + b_n\tilde{p}_n(x;\alpha,\beta;q) + c_n\tilde{p}_{n-1}(x;\alpha,\beta;q).$$

The value of c_n can then be calculated from the knowledge we already have obtained;

$$\begin{aligned} c_n\langle \tilde{p}_{n-1}(x;\alpha,\beta;q), \tilde{p}_{n-1}(x;\alpha,\beta;q)\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)} &= \langle x\,\tilde{p}_n(x;\alpha,\beta;q), \tilde{p}_{n-1}(x;\alpha,\beta;q)\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)} \\ = \langle \tilde{p}_n(x;\alpha,\beta;q), x\tilde{p}_{n-1}(x;\alpha,\beta;q)\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)} &= \langle \tilde{p}_n(x;\alpha,\beta;q), \tilde{p}_n(x;\alpha,\beta;q)\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)} \end{aligned}$$

using that multiplication by x is self-adjoint. So we find

$$\begin{aligned} c_n &= \frac{\langle \tilde{p}_n(x;\alpha,\beta;q), \tilde{p}_n(x;\alpha,\beta;q)\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)}}{\langle \tilde{p}_{n-1}(x;\alpha,\beta;q), \tilde{p}_{n-1}(x;\alpha,\beta;q)\rangle_{\ell^2(\mathbb{N};\alpha,\beta;q)}} = \left(\frac{\mathrm{lc}\big(p_{n-1}(\cdot;\alpha,\beta;q)\big)}{\mathrm{lc}\big(p_n(\cdot;\alpha,\beta;q)\big)}\right)^2 \frac{h_n(\alpha,\beta;q)}{h_{n-1}(\alpha,\beta;q)} \\ &= \left(-q^{n-1}\frac{(1-\alpha\beta q^n)(1-\alpha q^n)}{(1-\alpha\beta q^{2n-1})(1-\alpha\beta q^{2n})}\right)^2 \alpha q \frac{(1-q^n)(1-\beta q^n)(1-\alpha\beta q^{2n-1})}{(1-\alpha q^n)(1-\alpha\beta q^n)(1-\alpha\beta q^{2n+1})} \\ &= \alpha q^{2n-1}\frac{(1-q^n)(1-\alpha q^n)(1-\beta q^n)(1-\alpha\beta q^n)}{(1-\alpha\beta q^{2n-1})(1-\alpha\beta q^{2n})^2(1-\alpha\beta q^{2n+1})}. \end{aligned}$$

Writing $\tilde{p}_n(x;\alpha,\beta;q) = x^n + r_n(\alpha,\beta)x^{n-1} + \text{l.o.t.}$, it follows that upon comparing coefficients of x^n in the three-term recurrence relation that

$$r_n(\alpha,\beta) = r_{n+1}(\alpha,\beta) + b_n \qquad \Longrightarrow \qquad b_n = r_n(\alpha,\beta) - r_{n+1}(\alpha,\beta).$$

We could read it off from the explicit expression (4.5), but we use the shift operators to find the values. Indeed, since, by Proposition 4.5,

$$\begin{gathered} \big(\tilde{D}_q\tilde{p}_n(\cdot;\alpha,\beta;q)\big)(x) = (q^n-1)\,\tilde{p}_{n-1}(x;\alpha q,\beta q;q) \quad \Longrightarrow \\ (q^{n-1}-1)r_n(\alpha,\beta) = (q^n-1)r_{n-1}(\alpha q,\beta q) \quad \Longrightarrow \quad r_n(\alpha,\beta) = \frac{(1-q^n)}{(1-q^{n-1})}r_{n-1}(\alpha q,\beta q) \quad \Longrightarrow \\ r_n(\alpha,\beta) = \frac{(1-q^n)}{(1-q^{n-p})}r_{n-p}(\alpha q^p,\beta q^p) = \frac{(1-q^n)}{(1-q)}r_1(\alpha q^{n-1},\beta q^{n-1}). \end{gathered}$$

In order to determine $r_1(\alpha,\beta)$ we use the shift operator as well. By Proposition 4.5 we have that $p_1(x;\alpha,\beta;q)$ is a multiple of

$$\begin{gathered} \big(S^{\alpha,\beta}1\big)(x) = \frac{1}{\alpha q}\frac{(1-x)}{(1-\beta q)} - \frac{(1-\beta q x)}{(1-\beta q)} = \frac{\beta q - 1/\alpha q}{(1-\beta q)}x + \frac{1/\alpha q - 1}{(1-\beta q)} = \frac{\beta q - 1/\alpha q}{(1-\beta q)}\left(x + \frac{1/\alpha q - 1}{\beta q - 1/\alpha q}\right) \\ \Longrightarrow \qquad r_1(\alpha,\beta) = \frac{1/\alpha q - 1}{\beta q - 1/\alpha q} = -\frac{1-\alpha q}{1-\alpha\beta q^2} \end{gathered}$$

Proposition 4.7. *The monic little q-Jacobi polynomials $\tilde p_n(x;\alpha,\beta;q)$ satisfy the three-term recurrence relation*

$$
\begin{aligned}
x\,\tilde p_n(x;\alpha,\beta;q) &= \tilde p_{n+1}(x;\alpha,\beta;q) + b_n\tilde p_n(x;\alpha,\beta;q) + c_n\tilde p_{n-1}(x;\alpha,\beta;q),\\
b_n &= \frac{q^n\left((1+\alpha)-\alpha(1+\beta)(1+q)q^n+\alpha\beta q(1+\alpha)q^{2n}\right)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+2})}\\
c_n &= \alpha q^{2n-1}\frac{(1-q^n)(1-\alpha q^n)(1-\beta q^n)(1-\alpha\beta q^n)}{(1-\alpha\beta q^{2n-1})(1-\alpha\beta q^{2n})^2(1-\alpha\beta q^{2n+1})}.
\end{aligned}
$$

The value for b_n is seemingly different from the classical value given in e.g. [32, (3.12.4)]. We leave this to Exercise 2.

Proof. We have already established the value for c_n. It remains to finish the calculation of b_n. This is done as follows;

$$
\begin{aligned}
b_n = r_n(\alpha,\beta) - r_{n+1}(\alpha,\beta) &= \frac{(1-q^n)}{(1-q)}r_1(\alpha q^{n-1},\beta q^{n-1}) - \frac{(1-q^{n+1})}{(1-q)}r_1(\alpha q^n,\beta q^n)\\
&= -\frac{(1-q^n)(1-\alpha q^n)}{(1-q)(1-\alpha\beta q^{2n})} + \frac{(1-q^{n+1})(1-\alpha q^{n+1})}{(1-q)(1-\alpha\beta q^{2n+2})}
\end{aligned}
$$

By working this out we get

$$
b_n = \frac{q^n}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+2})}\left((1+\alpha)-\alpha(1+\beta)(1+q)q^n+\alpha\beta q(1+\alpha)q^{2n}\right)
$$

□

Note that the value

$$
r_n(\alpha,\beta) = -\frac{(1-q^n)(1-\alpha q^n)}{(1-q)(1-\alpha\beta q^{2n})}
$$

also corresponds with (4.5) taking into account division by the leading coefficient.

Remark 4.8. Let us view the operator $L = L^{\alpha,\beta}$ as an operator acting on polynomials as well as the operator M which is acting by multiplication. They can be viewed as generators (up to an affine scaling) of a limit of the Zhedanov algebra, also known as the Askey-Wilson algebra. We refer to [50] for the precise formulation and related references. Moreover, the Zhedanov algebra as well as its degenerations in [50] have relations that can be interpreted as non-homogeneous Serre relations in quantum algebras, and this type of relations hold for generators of quantum symmetric pairs as studied by Gail Letzter and coworkers, see e.g. [46, §5.3]. It is not clear what the connection entails.

4.4. **Relation to Al-Salam–Chihara polynomials.** We have circumvented the precise analytic study of the basic q-difference equation for the little q-Jacobi polynomials or the basic hypergeometric series. The reason is that this analytic study is somewhat complicated since the self-adjoint extension of the symmetric operator as in Lemma 4.4 depend in general on parameters. Indeed, $(L, D((L))$ as in Lemma 4.4 is not essentially self-adjoint in general. We explain this in this section by relating to a non-determinate moment problem.

We can also relate the eigenvalue equation to orthogonal polynomials. Indeed, rewriting $Lu=\lambda u$ gives

$$
\lambda u_k(\lambda) = (cq^{-k-1}-ab)\left(u_{k+1}(\lambda)-u_k(\lambda)\right) + \left(q^{-k}-1\right)\left(u_{k-1}(\lambda)-u_k(\lambda)\right)
$$

which we can consider as a three-term recurrence for orthogonal polynomials with initial values $u_0(\lambda) = 0$ (and $u_{-1}(\lambda) = 0$). In order to determine these polynomials, we first look at the monic version. So we put $u_k(\lambda) = \alpha_k r_k(\lambda)$ with

$$\frac{\alpha_{k+1}}{\alpha_k}(cq^{-k-1} - ab) = 1$$

so that the recurrence relation becomes

$$\lambda r_k(\lambda) = r_{k+1}(\lambda) - \big((cq^{-k-1} - ab) + (q^{-k} - 1)\big) r_k(\lambda) + (q^{-k} - 1)(cq^{-k} - ab)\, r_{k-1}(\lambda)$$

and putting $2\mu = \alpha(\lambda - ab - 1)$, $p_k(\mu) = \alpha^{-k} r_k(\alpha(2\mu - ab - 1))$ we get

$$2\mu p_k(\mu) = p_{k+1}(\mu) - \frac{1}{\alpha}(cq^{-k-1} + q^{-k}) p_k(\mu) + \frac{ab}{\alpha^2}(1 - q^{-k})(1 - cq^{-k}/ab)\, p_{k-1}(\mu)$$

and finally taking $\alpha = -\sqrt{ab}$ we find

$$2\mu p_k(\mu) = p_{k+1}(\mu) + q^{-k}(c/q\sqrt{ab} + 1/\sqrt{ab}) p_k(\mu) + (1 - q^{-k})(1 - cq^{-k}/ab)\, p_{k-1}(\mu). \tag{4.8}$$

Now (4.8) can be matched to [32, §3.8], so that $p_k(\mu)$ can be identified with the Al-Salam–Chihara polynomials $Q_k(\mu; c/q\sqrt{ab}, 1/\sqrt{ab}|q^{-1})$ in base $q^{-1} > 1$.

Theorem 4.9 gives a characterization of the (in-)determinacy of the Al-Salam–Chihara polynomials in case the base is bigger than 1, and it is due to Askey and Ismail [5, Thm. 3.2, p. 36].

Theorem 4.9. *Consider the sequence of monic polynomials*

$$xv_n(x) = v_{n+1}(x) + Aq^{-n} + (1 - q^{-n})(C - Bq^{-n}) v_{n-1}(x)$$

with $0 < q < 1$, $B \geq 0$, $B > C$ and initial conditions $v_{-1}(x) = 0$, $v_0(x) = 1$. Then the corresponding moment problem is indeterminate if and only if

$$A^2 > 4B, \quad \textit{and} \quad q \geq |\beta^2 B|$$

where $(1 - At + Bt^2) = (1 - t/\alpha)(1 - t/\beta)$ with $|\alpha| \geq |\beta|$.

The conditions $0 < q < 1$, $B \geq 0$, $B > C$ in Theorem 4.9 ensure that the conditions of Favard's theorem, see e.g. [10], [13], [14], [25], [35], are met. So there is an orthogonality measure for which the polynomials $v_n(x)$ are orthogonal. In the determinate case this measure is uniquely determined by the polynomials, whereas in the indeterminate case there are infinitely many orthogonality measures for these polynomials. In the indeterminate case this means that the operator L with domain $D(L)$ the finite linear combinations as in Proposition 4.2 is not essentially self-adjoint, see e.g. [14], [35], [60].

The proof of Theorem 4.9 follows by observing that a moment problem is indeterminate if and only if $\sum_{n=0}^{\infty} |p_n(i)|^2 < \infty$ for the corresponding orthonormal polynomials $p_n(x)$, see e.g. [1]. Askey and Ismail then determine the asymptotic behaviour of the Al-Salam–Chihara polynomials by applying Darboux's method, see e.g. [54], to the generating function for the Al-Salam–Chihara polynomials.

Comparing Theorem 4.9 with (4.8) we see that we can apply Theorem 4.9 with $(A, B, C) = (c/q\sqrt{ab} + 1/\sqrt{ab}, c/abq, 1)$ and the same base q. So the requirement for Favard's theorem translates to $c/abq > 1$ or $c > abq$, which we now assume. Then the first condition $A^2 > 4B$ translates to

$$\left(\frac{c}{q\sqrt{ab}} + \frac{1}{\sqrt{ab}}\right)^2 > \frac{4c}{abq} \iff \left(\frac{c}{q\sqrt{ab}} - \frac{1}{\sqrt{ab}}\right)^2 > 0,$$

which is always true unless $c = q$. For the second condition we factorise

$$1 - At + Bt^2 = 1 - (c/q\sqrt{ab} + 1/\sqrt{ab})t + \frac{c}{abq}t^2 = (1 - \frac{ct}{q\sqrt{ab}})(1 - \frac{t}{\sqrt{ab}})$$

so that $\{\alpha,\beta\} = \{\frac{q\sqrt{ab}}{c}, \sqrt{ab}\}$. So $\beta = \sqrt{ab}$ if $c \leq q$, and then $q \geq |\beta^2 B|$ is equivalent to $c \leq q^2$. Next $\beta = \frac{q\sqrt{ab}}{c}$ if $c \geq q$ and then $q \geq |\beta^2 B|$ is equivalent to $q \geq 1$. We conclude that L is not essentially self-adjoint if $0 < c < q$.

4.5. **Exercises.**

1. Show that $\tilde{D}_q$ as in Lemma 4.4 is unbounded.
2. Look up the standard value for b_n as in Proposition 4.7 and establish the equality with the value as given in Proposition 4.7.
3. Show that the three-term recurrence relation for the litle q-Jacobi polynomials as in Proposition 4.7 gives a relation for the Al-Salam–Chihara polynomials which is related to the q-difference operator for the Al-Salam–Chihara polynomials.

Notes. The little q-Jacobi polynomials were introduced by Andrews and Askey [3] in 1977. The link to the quantum $SU(2)$-group as matrix elements of unitary representations by Vaksman & Soibelman, Koornwinder and Masuda, Mimachi, Nakagami, Noumi, and Ueno at the end of the 1980s has led to many results on (subclasses of) little q-Jacobi polynomials, see the references in the lecture notes [49] by Koornwinder. The usage of the shift operators to obtain the explicit results is a technique that can be useful in other applications, such as multivariable setting or in the matrix-valued case, see also [49] for this approach for little and big q-Jacobi polynomials. The duality between little q-Jacobi polynomials and Al-Salam–Chihara polynomials is observed by Rosengren [57] and it is also observed by Groenevelt [20]. This duality –but in a dual way– also plays an important role in the study of the quantum analogue of the Laplace-Beltrami operator on bounded quantum symmetric domain, see Vaksman [67]. For the corresponding Zhedanov algebra, the duality is described in [50]. The duality can also be extended to big q-Jacobi polynomials and continuous dual q^{-1}-Hahn polynomials, see [41]. In general, this duality is related to explicit solutions of explicit indeterminate moment problems, and several examples are known. A vector-valued analogue of [41] is given by Groenevelt [21].

5. Basic hypergeometric q-difference equation: non-polynomial case

We now consider the basic hypergeometric q-difference equation in a more general version. In the general version we cannot restrict naturally to a simple domain. We have to take all the general q-line $zq^{\mathbb{Z}}$ into account.

5.1. **Doubly infinite Jacobi operators.** In this section we briefly review the spectral analysis of a doubly infinite Jacobi operator, i.e. a three-term recurrence on the Hilbert space $\ell^2(\mathbb{Z})$. This section requires some knowledge from functional analysis, in particular of symmetric, unbounded, and self-adjoint operators and the spectral theorem.

We consider an operator on the Hilbert space $\ell^2(\mathbb{Z})$ of the form

$$L\, e_k = a_k\, e_{k+1} + b_k\, e_k + a_{k-1}\, e_{k-1}, \qquad a_k > 0,\ b_k \in \mathbb{R}, \tag{5.1}$$

where $\{e_k\}_{k\in\mathbb{Z}}$ is the standard orthonormal basis of $\ell^2(\mathbb{Z})$. If $a_i = 0$ for some $i \in \mathbb{Z}$, then L splits as the direct sum of two Jacobi operators, so that we are essentially back to two three-term recurrence operators related to two sets of orthogonal polynomials. Recall that a three-term recurrence operator on $\ell^2(\mathbb{N})$ is a Jacobi operator. The spectral analysis is closely related to the orthogonality of the corresponding orthogonal polynomials, and is essentially a proof of Favard's theorem, see [14], [35], [60]. So we will assume that $a_i \neq 0$ for all $i \in \mathbb{Z}$. We call L a Jacobi operator on $\ell^2(\mathbb{Z})$ or a doubly infinite Jacobi operator.

The domain $\mathcal{D}(L)$ of L is the dense subspace $\mathcal{D}(\mathbb{Z})$ of finite linear combinations of the basis elements e_k, $k \in \mathbb{Z}$. This makes L a densely defined symmetric operator.

We extend the action of L to an arbitrary vector $v = \sum_{k=-\infty}^{\infty} v_k e_k \in \ell^2(\mathbb{Z})$ by

$$L^* v = \sum_{k=-\infty}^{\infty} (a_k v_{k+1} + b_k v_k + a_{k-1} v_{k-1}) e_k,$$

which is not an element of $\ell^2(\mathbb{Z})$ in general. Define

$$\mathcal{D}^* = \{v \in \ell^2(\mathbb{Z}) \mid L^* v \in \ell^2(\mathbb{Z})\}.$$

Lemma 5.1. *$(L^*, \mathcal{D}^*)$ is the adjoint of $(L, \mathcal{D}(\mathbb{Z}))$.*

The proof of Lemma 5.1 requires a bit of Hilbert space theory, and we leave it to the Exercise 1.

In particular, L^* commutes with complex conjugation, so its deficiency indices are equal. Here the deficiency indices (n_+, n_-) are defined as

$$\begin{aligned} n_+ &= \dim\ker(L^* - z) = \dim\ker(L^* - i), && \Im z > 0 \\ n_- &= \dim\ker(L^* - z) = \dim\ker(L^* + i), && \Im z < 0 \end{aligned}$$

since the dimension is constant in the upper and lower half plane. The solution space of $L^* v = z\, v$ is two-dimensional, since v is completely determined by any initial data (v_{n-1}, v_n) for any fixed $n \in \mathbb{Z}$. So the deficiency indices are equal to (i, i) with $i \in \{0, 1, 2\}$. From the general theory of self-adjoint operators, see [14], we have that $(L, \mathcal{D}(L))$ has self-adjoint extensions since the deficiency indices $n_- = n_+$. In case $n_- = n_+ = 0$, the operator $(L^*, \mathcal{D}^*)$ is self-adjoint, and this case will be generally assumed in this section.

5.1.1. *Relation to Jacobi operators.* To the operator L we associate two Jacobi operators J^+ and J^- acting on $\ell^2(\mathbb{N})$ with orthonormal basis denoted by $\{f_k\}_{k\in\mathbb{N}}$ in order to avoid confusion. Define

$$J^+ f_k = \begin{cases} a_k f_{k+1} + b_k f_k + a_{k-1} f_{k-1}, & \text{for } k \geq 1, \\ a_0 f_1 + b_0 f_0, & \text{for } k = 0, \end{cases}$$

$$J^- f_k = \begin{cases} a_{-k-2} f_{k+1} + b_{-k-1} f_k + a_{-k-1} f_{k-1}, & \text{for } k \geq 1, \\ a_{-2} f_1 + b_{-1} f_0, & \text{for } k = 0, \end{cases}$$

and extend by linearity to $\mathcal{D}(\mathbb{N})$, the space of finite linear combinations of the basis vectors $\{f_k\}_{k=0}^{\infty}$ of $\ell^2(\mathbb{N})$. Then $J^\pm$ are densely defined symmetric operators with deficiency indices $(0,0)$ or $(1,1)$ corresponding to whether the associated Hamburger moment problems is determinate or indeterminate, see [1], [9], [14], [35], [60]. The following theorem, due to Masson and Repka [52], relates the deficiency indices of L and $J^\pm$.

Theorem 5.2 (Masson and Repka)**.** *The deficiency indices of L are obtained by summing the deficiency indices of J^+ and the deficiency indices of J^-.*

For the proof of Theorem 5.2 we refer to [52], [35].

We define the Wronskian $[u, v]_k = a_k(u_{k+1} v_k - u_k v_{k+1})$. The Wronskian is also known as the Casorati determinant.

Lemma 5.3. *The Wronskian $[u, v] = [u, v]_k$ is independent of k for $L^* u = z\, u$, $L^* v = z\, v$. Moreover, $[u, v] \neq 0$ if and only if u and v are linearly independent solutions.*

The proof is straightforward, see Exercise 2.

Now using Lemma 5.3

$$\begin{aligned}
&\sum_{k=M}^{N} (L^*u)_k \bar{v}_k - u_k \overline{(L^*v)_k} \\
&= \sum_{k=M}^{N} (a_k u_{k+1} + b_k u_k + a_{k-1} u_{k-1}) \bar{v}_k - u_k (a_k \bar{v}_{k+1} + b_k \bar{v}_k + a_{k-1} \bar{v}_{k-1}) \\
&= \sum_{k=M}^{N} [u, \bar{v}]_k - [u, \bar{v}]_{k-1} = [u, \bar{v}]_N - [u, \bar{v}]_{M-1},
\end{aligned}$$

so that, cf. Proposition 4.2,

$$B(u, v) = \lim_{N\to\infty} [u, \bar{v}]_N - \lim_{M\to-\infty} [u, \bar{v}]_M, \qquad u, v \in \mathcal{D}^*.$$

In particular, if J^- and J^+ are essentially self-adjoint, $(L^*, \mathcal{D}^*)$ is self-adjoint, and then

$$\lim_{M\to-\infty} [u, \bar{v}]_M = 0 \quad \text{and} \quad \lim_{N\to\infty} [u, \bar{v}]_N = 0.$$

5.1.2. *The Green kernel and the resolvent operator.* From on we assume that J^- and J^- have deficiency indices $(0, 0)$, so that J^- and J^+ are essentially self-adjoint and by Theorem 5.2 the deficiency indices of L are $(0, 0)$. We refer to e.g. [35], [52], for the case that one of the operators has deficiency indices $(0, 0)$ and the other on $(1, 1)$. This can also be analysed in this framework. In case L has deficiency indices $(2, 2)$ the restriction of the domain of a self-adjoint extension of L to the Jacobi operator $J^\pm$ does not in general correspond to a self-adjoint extension of $J^\pm$, cf. [14, Thm. XII.4.31], so that this is the most difficult situation. We restrict ourselves to the case of essentially self-adjoint L or equivalently that $J^\pm$ have both deficiency indices $(0, 0)$, i.e. the adjoint of L is self-adjoint.

Let $z \in \mathbb{C}\backslash\mathbb{R}$, so that we know that $L^* - z\mathrm{Id}$ has an inverse in $B(\ell^2(\mathbb{Z}))$, the bounded linear operators on $\ell^2(\mathbb{Z})$. The inverse is denoted by $R(z)$, and is called the *resolvent operator*. Introduce the spaces

$$\begin{aligned}
S_z^- &= \{\{f_k\}_{k=-\infty}^{\infty} \mid L^* f = z\, f \text{ and } \sum_{k=-\infty}^{-1} |f_k|^2 < \infty\}, \\
S_z^+ &= \{\{f_k\}_{k=-\infty}^{\infty} \mid L^* f = z\, f \text{ and } \sum_{k=0}^{\infty} |f_k|^2 < \infty\}.
\end{aligned} \tag{5.2}$$

Since the solution of a three-term recurrence operator is completely determined by two starting values v_0, v_1, we find $\dim S_z^\pm \leq 2$. The deficiency index n_+, respectively n_-, for L^* is precisely $\dim(S_z^+ \cap S_z^-)$ for $\Im z > 0$, respectively $\Im z < 0$. From the general theory of orthogonal polynomials we know that $\dim(S_z^\pm) \geq 1$, and in case of deficiency indices $(0, 0)$ of $J^\pm$ we actually have $\dim(S_z^\pm) = 1$. Consequently, in the case of a self-adjoint $(L^*, \mathcal{D}^*)$ we have $\dim(S_z^\pm) = 1$ and $\dim(S_z^+ \cap S_z^-) = 0$.

Choose $\Phi_z \in S_z^-$, so that Φ_z is determined up to a constant. We assume $\overline{(\Phi_z)_k} = (\Phi_{\bar{z}})_k$, which we can do since L^* commutes with complex conjugation. Let $\varphi_z \in S_z^+$, such that $\overline{(\varphi_z)_k} = (\varphi_{\bar{z}})_k$. We may assume

1. $[\varphi_z, \Phi_z] \neq 0$,
2. $\tilde{\varphi}_z$, defined by $(\tilde{\varphi}_z)_k = 0$ for $k < 0$ and $(\tilde{\varphi}_z)_k = (\varphi_z)_k$ for $k \geq 0$, is contained in the domain $\mathcal{D}^*$ of the self-adjoint L^*.

Let $(L^*, \mathcal{D}^*)$ be the self-adjoint extension of L, assuming, as before, that $J^\pm$ have deficiency indices $(0,0)$. Let $\varphi_z \in S_z^+$, $\Phi_z \in S_z^-$ as before. We define the Green kernel for $z \in \mathbb{C}\backslash\mathbb{R}$ by

$$G_{k,l}(z) = \frac{1}{[\varphi_z, \Phi_z]} \begin{cases} (\Phi_z)_k\, (\varphi_z)_l, & k \leq l, \\ (\Phi_z)_l\, (\varphi_z)_k, & k > l. \end{cases}$$

So $\{G_{k,l}(z)\}_{k=-\infty}^\infty, \{G_{k,l}(z)\}_{l=-\infty}^\infty \in \ell^2(\mathbb{Z})$ and $\ell^2(\mathbb{Z}) \ni v \mapsto G(z)v$ given by

$$(G(z)v)_k = \sum_{l=-\infty}^{\infty} v_l G(z)_{k,l} = \langle v, \overline{G_{k,\cdot}(z)}\rangle$$

is well-defined. For $v \in \mathcal{D}(\mathbb{Z})$ we have $G(z)v \in \mathcal{D}^*$.

Proposition 5.4. *The resolvent of $(L^*, \mathcal{D}^*)$ is given by $R(z) = G(z)$ for $z \in \mathbb{C}\backslash\mathbb{R}$.*

For the proof of Proposition 5.4 we refer to [35], and we give here the basic calculation. For $v \in \mathcal{D}(\mathbb{Z})$

$$\begin{aligned}
[\varphi_z, \Phi_z]\big((L^* - z)G(z)v\big)_k = &\sum_{l=-\infty}^{k-1} v_l\big(a_k(\varphi_z)_{k+1} + (b_k - z)(\varphi_z)_k + a_{k-1}(\varphi_z)_{k-1}\big)(\Phi_z)_l \\
&+ \sum_{l=k+1}^{\infty} v_l\big(a_k(\Phi_z)_{k+1} + (b_k - z)(\Phi_z)_k + a_{k-1}(\Phi_z)_{k-1}\big)(\varphi_z)_l \\
&+ v_k\big(a_k(\Phi_z)_k(\varphi_z)_{k+1} + (b_k - z)(\Phi_z)_k(\varphi_z)_k + a_{k-1}(\Phi_z)_{k-1}(\varphi_z)_k\big) \\
= &v_k a_k\big((\Phi_z)_k(\varphi_z)_{k+1} - (\Phi_z)_{k+1}(\varphi_z)_k\big) = v_k[\varphi_z, \varphi_z]
\end{aligned}$$

and canceling the Wronskian gives the result.

With Proposition 5.4 we can calculate

$$\langle G(z)u, v\rangle = \sum_{k,l=-\infty}^{\infty} G_{k,l}(z) u_l \bar{v}_k = \frac{1}{[\varphi_z, \Phi_z]} \sum_{k \leq l} (\Phi_z)_k(\varphi_z)_l\big(u_l\bar{v}_k + u_k\bar{v}_l\big)(1 - \frac{1}{2}\delta_{k,l}), \tag{5.3}$$

Now the spectral theorem, see [14, §XII.4], [58, Ch. 13], can be stated as follows. In particular, one sees that the resolvent in terms of the Green kernel gives the spectral decomposition by the Stieltjes-Perron inversion formula.

Theorem 5.5 (Spectral theorem)**.** *Let $T\colon \mathcal{D}(T) \to \mathcal{H}$ be an unbounded self-adjoint linear map with dense domain $\mathcal{D}(T)$ in the Hilbert space $\mathcal{H}$, then there exists a unique spectral measure E such that $T = \int_\mathbb{R} t\, dE(t)$, i.e. $\langle Tu, v\rangle = \int_\mathbb{R} t\, dE_{u,v}(t)$ for $u \in \mathcal{D}(T)$, $v \in \mathcal{H}$. Moreover, E is supported on the spectrum $\sigma(T)$, which is contained in $\mathbb{R}$. Moreover, the Stieltjes-Perron inversion formula is valid;*

$$E_{u,v}\big((a,b)\big) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i}\int_{a+\delta}^{b-\delta} \langle R(x + i\varepsilon)u, v\rangle - \langle R(x - i\varepsilon)u, v\rangle\, dx.$$

Recall that a spectral measure E is a self-adjoint orthogonal projection-valued measure on the Borel sets of $\mathbb{R}$ such that $E(\mathbb{R}) = \mathrm{Id}$, $E(\emptyset) = 0$, $E(A \cap B) = E(A)E(B)$ for Borel sets A and B and that σ-finite additivity with respect to the strong operator topology holds, i.e. for all $x \in \mathcal{H}$ and any sequence $(A_i)_{i\in\mathbb{N}}$ of mutually disjoint Borel sets we have

$$E\Big(\bigcup_{i\in\mathbb{N}} A_i\Big)x = \sum_{i\in\mathbb{N}} E(A_i)x.$$

In particular $E_{x,y}(B) = \langle E(B)x, y\rangle$ for $x, y \in \mathcal{H}$ and B a Borel set gives a complex Borel measure on $\mathbb{R}$, which is positive in case $x = y$.

5.2. **The basic hypergeometric difference equation.** This example is based on Appendix A in [42], which was greatly motivated by Kakehi [30] and unpublished notes by Koornwinder. On a formal level the result can be obtained as a limit case of the orthogonality of the Askey-Wilson polynomials, see [43] for a precise formulation.

We take the coefficients as

$$a_k = \frac{1}{2}\sqrt{(1-\frac{q^{-k}}{r})(1-\frac{cq^{-k}}{d^2r})}, \qquad b_k = \frac{q^{-k}(c+q)}{2dr},$$

where we assume $0 < q < 1$, and $r < 0$, $c > 0$, $d \in \mathbb{R}$. This assumption is made in order to get the expression under the square root sign positive. There are more possible choices in order to achieve this, see [42, App. A]. Note that a_k and b_k are bounded for $k < 0$, so that J^- is self-adjoint. Hence, the deficiency indices of L are $(0,0)$ or $(1,1)$ by Theorem 5.2.

Lemma 5.6. *Put*

$$\begin{aligned}
w_k^2 &= d^{2k}\frac{(cq^{1-k}/d^2r;q)_\infty}{(q^{1-k}/r;q)_\infty},\\
f_k(\mu(y)) &= {}_2\varphi_1\left(\begin{matrix} dy, d/y \\ c\end{matrix}\,;q, rq^k\right), \qquad c\notin q^{-\mathbb{N}}, \quad \mu(y)=\frac{1}{2}(y+y^{-1}),\\
g_k(\mu(y)) &= q^kc^{-k}\,{}_2\varphi_1\left(\begin{matrix} qdy/c, qd/cy \\ q^2/c\end{matrix}\,;q, rq^k\right) \qquad \mu(y)=\frac{1}{2}(y+y^{-1}),\ c\notin q^{2+\mathbb{Z}}\\
F_k(y) &= (dy)^{-k}\,{}_2\varphi_1\left(\begin{matrix} dy, qdy/c \\ qy^2\end{matrix}\,;q, \frac{q^{1-k}c}{d^2r}\right) \qquad y^2\notin q^{-\mathbb{N}},
\end{aligned}$$

then, with $z = \mu(y)$, we have that $u_k(z) = w_kf_k(\mu(y))$, $u_k(z) = w_kg_k(\mu(y))$, $u_k(z) = w_kF_k(y)$ and $u_k(z) = w_kF_k(y^{-1})$ define solutions to

$$z\,u_k(z) = a_k\,u_{k+1}(z) + b_k\,u_k(z) + a_{k-1}\,u_{k-1}(z).$$

Proof. Put $u_k(z) = w_kv_k(z)$, then $v_k(z)$ satisfies

$$2z\,v_k(z) = (d-\frac{cq^{-k}}{dr})\,v_{k+1}(z) + q^{-k}\frac{c+q}{dr}\,v_k(z) + (d^{-1}-\frac{q^{1-k}}{dr})\,v_{k-1}(z)$$

and this is precisely the second order q-difference equation that has the solutions given, see Proposition 3.1 and Section 3. □

The asymptotics of the solutions of Lemma 5.6 can be given as follows. First observe that $w_{-k} = \mathcal{O}(d^{-k})$ as $k\to\infty$, and using

$$w_k^2 = c^k\frac{(rq^k, d^2r/c, cq/d^2r;q)_\infty}{(d^2rq^k/c, r, q/r;q)_\infty} \Rightarrow w_k = \mathcal{O}(c^{\frac{1}{2}k}),\ k\to\infty.$$

Now $f_k(\mu(y)) = \mathcal{O}(1)$ as $k\to\infty$, and $g_k(\mu(y)) = \mathcal{O}((q/c)^k)$ as $k\to\infty$. Similarly, $F_{-k}(y) = \mathcal{O}((dy)^k)$ as $k\to\infty$.

Proposition 5.7. *The operator L is essentially self-adjoint for $0 < c \le q^2$, and L has deficiency indices $(1,1)$ for $q^2 < c < 1$. Moreover, for $z\in\mathbb{C}\backslash\mathbb{R}$ the one-dimensional space S_z^- is spanned by $wF(y)$ with $\mu(y) = z$ and $|y| < 1$. For $0 < c \le q^2$ the one-dimensional space S_z^+ is spanned by $wf(z)$, and for $q^2 < c < 1$ the two-dimensional space S_z^+ is spanned by $wf(z)$ and $wg(z)$.*

The proof of Proposition 5.7 relies on criteria establishing the defect indices of Jacobi operators. We refer to [35, p. 79] for the application of these criteria leading to Proposition 5.7. Since we restrict ourselves to the self-adjoint setting, we assume from now on that $0 < c \leq q^2$.

The Wronskian

$$[wF(y), wF(y^{-1})] = \lim_{k\to-\infty} a_k w_{k+1} w_k \bigl(F_{k+1}(y)F_k(y^{-1}) - F_k(y)F_{k+1}(y^{-1})\bigr) = \frac{1}{2}(y^{-1} - y)$$

using $a_k \to \frac{1}{2}$ as $k \to -\infty$ and the asymptotics of F_k and w_k as $k \to -\infty$. Note that the Wronskian is non-zero for $y \neq \pm 1$ or $z \neq \pm 1$. Since $wF(y)$ and $wF(y^{-1})$ are linearly independent solutions to the recurrence fo $L^* f = zf$ for $z \in \mathbb{C}\backslash\mathbb{R}$, we see that we can express $f_k(\mu(y))$ in terms of $F_k(y)$ and $F_k(y^{-1})$. These solutions are related by the expansion

$$\begin{aligned} f_k(\mu(y)) &= c(y)F_k(y) + c(y^{-1})F_k(y^{-1}), \\ c(y) &= \frac{(c/dy, d/y, dry, q/dry; q)_\infty}{(y^{-2}, c, r, q/r; q)_\infty}, \end{aligned} \tag{5.4}$$

for $c \notin q^{-\mathbb{N}}$, $y^2 \notin q^{\mathbb{Z}}$, which is a reformulation of (2.30). This shows that we have

$$[wf(\mu(y)), wF(y)] = \frac{1}{2} c(y^{-1})(y - y^{-1}). \tag{5.5}$$

Since we assume that $0 < c \leq q^2$, L is essentially self-adjoint, or L^* is self-adjoint. Then for $z \in \mathbb{C}\backslash\mathbb{R}$ we have $\varphi_z = wf(z)$ and $\Phi_z = wF(y)$, where $z = \mu(y)$ and $|y| < 1$. In particular, it follows that $\varphi_{x\pm i\varepsilon} \to \varphi_x$ as $\varepsilon \downarrow 0$. For the asymptotic solution Φ_z we have to be more careful in computing the limit of z to the real axis. For $x \in \mathbb{R}$ satisfying $|x| > 1$ we have $\varphi_{x\pm i\varepsilon} \to wF_y$ as $\varepsilon \downarrow 0$, where $y \in (-1, 1)\backslash\{0\}$ is such that $\mu(y) = x$. If $x \in [-1, 1]$, then we put $x = \cos\chi = \mu(e^{i\chi})$ with $\chi \in [0, \pi]$, and then $\Phi_{x-i\varepsilon} \to wF_{e^{i\chi}}$ and $\Phi_{x+i\varepsilon} \to wF_{e^{-i\chi}}$ as $\varepsilon \downarrow 0$.

We calculate the integrand in the Stieltjes-Perron inversion formula of Theorem 5.5 using Lemma 5.6 and Proposition 5.7 in the case $|x| < 1$, where $x = \cos\chi = \mu(e^{i\chi})$. For $u, v \in \mathcal{D}(\mathbb{Z})$ we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \langle G(x + i\varepsilon)u, v\rangle - \langle G(x - i\varepsilon)u, v\rangle = \\
&\lim_{\varepsilon\downarrow 0} \sum_{k\leq l} \Bigl(\frac{(\Phi_{x+i\varepsilon})_k(\varphi_{x+i\varepsilon})_l}{[\varphi_{x+i\varepsilon}, \Phi_{x+i\varepsilon}]} - \frac{(\Phi_{x-i\varepsilon})_k(\varphi_{x-i\varepsilon})_l}{[\varphi_{x-i\varepsilon}, \Phi_{x-i\varepsilon}]}\Bigr)\bigl(u_l\bar{v}_k + u_k\bar{v}_l\bigr)(1 - \frac{1}{2}\delta_{k,l}) = \\
&2\sum_{k\leq l} \Bigl(\frac{w_kF_k(e^{-i\chi})w_lf_l(\cos\chi)}{c(e^{i\chi})(e^{-i\chi} - e^{i\chi})} - \frac{w_kF_k(e^{i\chi})w_lf_l(\cos\chi)}{c(e^{-i\chi})(e^{i\chi} - e^{-i\chi})}\Bigr)\bigl(u_l\bar{v}_k + u_k\bar{v}_l\bigr)(1 - \frac{1}{2}\delta_{k,l}) = \\
&2\sum_{k\leq l} \Bigl(w_kw_lf_l(\cos\chi)\frac{c(e^{-i\chi})F_k(e^{-i\chi}) + c(e^{i\chi})F_k(e^{i\chi})}{c(e^{i\chi})c(e^{-i\chi})(e^{-i\chi} - e^{i\chi})}\bigl(u_l\bar{v}_k + u_k\bar{v}_l\bigr)(1 - \frac{1}{2}\delta_{k,l}) = \\
&2\sum_{k\leq l} \Bigl(\frac{w_kw_lf_l(\cos\chi)f_k(\cos\chi)}{c(e^{i\chi})c(e^{-i\chi})(e^{-i\chi} - e^{i\chi})}\bigl(u_l\bar{v}_k + u_k\bar{v}_l\bigr)(1 - \frac{1}{2}\delta_{k,l}) = \\
&\frac{2}{c(e^{i\chi})c(e^{-i\chi})(e^{-i\chi} - e^{i\chi})} \sum_{l=-\infty}^{\infty} w_lf_l(\cos\chi)u_l \sum_{k=-\infty}^{\infty} w_kf_k(\cos\chi)\bar{v}_k
\end{aligned}$$

using the expansion (5.4) and the Wronskian in (5.5). Now integrate over the interval (a, b) with $-1 \leq a < b \leq 1$ and replacing x by $\cos\chi$, so that $\frac{1}{2\pi i}dx = (e^{i\chi} - e^{-i\chi})d\chi/4\pi$. We obtain, with

$a = \cos\chi_a$, $b = \cos\chi_b$, and $0 \le \chi_b < \chi_a \le \pi$,

$$E_{u,v}\big((a,b)\big) = \frac{1}{2\pi}\int_{\chi_b}^{\chi_a} \big(\mathcal{F}u\big)(\cos\chi)\overline{\big(\mathcal{F}v\big)(\cos\chi)}\frac{d\chi}{|c(e^{i\chi})|^2},$$

$$\big(\mathcal{F}u\big)(x) = \langle u, \varphi_x\rangle = \sum_{l=-\infty}^{\infty} w_l f_l(\cos\chi)u_l.$$

This shows that $[-1,1]$ is contained in the continuous spectrum of L.

For $|x| > 1$ we can calculate as above the integrand in the Stieltjes-Perron inversion formula, but now we have to use that $x = \mu(y)$ with $|y| < 1$. This gives

$$\lim_{\varepsilon\downarrow 0}\langle G(x+i\varepsilon)u,v\rangle = 2\sum_{k\le l}\frac{w_k F_k(y) w_l f_l(y)}{c(y^{-1})(y-y^{-1})}\big(u_l\bar v_k + u_k \bar v_l\big)(1-\frac{1}{2}\delta_{k,l}).$$

The limit $\lim_{\varepsilon\downarrow 0}\langle G(x+i\varepsilon)u,v\rangle$ gives the same result, so we can only have discrete mass points for $|x|>1$ in the spectral measure at the zeroes of the Wronskian, i.e. at the zeroes of $y \mapsto c(y^{-1})$ with $|y| < 1$ or at $y = \pm 1$. Let us assume that all zeroes of the c-function are simple, so that the spectral measure at these points can be easily calculated.

The zeroes of the c-function can be read off from the expressions in (5.4), and they are

$$\{cq^k/d \mid k\in\mathbb{N}\}, \quad \{dq^k \mid k \in \mathbb{N}\}, \quad \{q^k/dr \mid k\in\mathbb{Z}\}.$$

Assuming that $|c/d| < 1$ and $|d| < 1$, we see that the first two sets do not contribute. (In the more general case we have that the product is less than 1, since the product equals $c < 1$. We leave this extra case to the reader.) The last set, labeled by $\mathbb{Z}$ always contributes to the spectral measure. Now for $u, v \in \mathcal{D}(\mathbb{Z})$ we let $x_p = \mu(y_p)$, $y_p = q^p/dr$, $p \in \mathbb{Z}$, with $|q^p/dr| > 1$, so that by the Stieltjes-Perron inversion formula and Cauchy's residue theorem we find

$$E_{u,v}(\{x_p\}) = \mathrm{Res}_{y=y_p^{-1}}\Big(\frac{-1}{c(y^{-1})y}\Big) w_k F_k(y_p^{-1}) w_l f_l(x_p)\big(u_l\bar v_k + u_k\bar v_l\big)(1-\frac{1}{2}\delta_{k,l})$$

after substituting $x = \mu(y)$. Now from (5.4) we find $f_k(x_p) = c(y_p)F_k(y_p^{-1})$, since $c(y_p^{-1}) = 0$ and we assume here that $c(y_p) \ne 0$. Hence, we can symmetrise the sum again and find

$$E_{u,v}(\{x_p\}) = \Big(\mathrm{Res}_{y=y_p}\frac{1}{c(y^{-1})c(y)y}\Big)\big(\mathcal{F}u\big)(x_p)\overline{\big(\mathcal{F}v\big)(x_p)}$$

switching to the residue at y_p.

We can combine the calculations in the following theorem. Note that most of the regularity conditions can be removed by continuity after calculating explicitly all the residues. The case of an extra set of finite mass points is left to the reader, as stated above. Of course, there are also other possibilities for choices of the parameters c, d and r for which the expression under the square root sign in a_k in (5.1) is positive. See [42, App. A] for details.

Theorem 5.8. *Assume $r < 0$, $0 < c \le q^2$, $d \in \mathbb{R}$ with $|d| < 1$ and $|c/d| < 1$ such that the zeroes of $y \mapsto c(y)$ are simple and $c(y) = 0$ implies $c(y^{-1}) \ne 0$. Then the spectral measure for the Jacobi operator on $\ell^2(\mathbb{Z})$ defined by is given by, $A \subset \mathbb{R}$ a Borel set,*

$$\langle E(A)u,v\rangle = \int_{\cos\chi\in[-1,1]\cap A}\big(\mathcal{F}u\big)(\cos\chi)\overline{\big(\mathcal{F}v\big)(\cos\chi)}\frac{d\chi}{|c(e^{i\chi})|^2}$$
$$+ \sum_{p\in\mathbb{Z}, |q^p/dr|>1, \mu(q^p/dr)\in A}\Big(\mathrm{Res}_{y=q^p/dr}\frac{1}{c(y^{-1})c(y)y}\Big)\big(\mathcal{F}u\big)(\mu(q^p/dr))\overline{\big(\mathcal{F}v\big)(\mu(q^p/dr))}.$$

Proof. It only remains to prove that ± 1 is not contained in the point spectrum. These are precisely the points for which $F(y)$ and $F(y^{-1})$ are not linearly independent solutions. We have to show that $\varphi_{\pm 1} \notin \ell^2(\mathbb{Z})$, and this can be done by determining its asymptotic behaviour as $k \to -\infty$, see [30], [41] for more information. □

Take $A = \mathbb{R}$ and $u = e_k$ and $v = e_l$, then we find the following orthogonality relations for the ${}_2\varphi_1$-series as in Lemma 5.6.

Corollary 5.9. *With the notation and assumptions as in Theorem 5.8 we have*

$$\int_0^\pi f_k(\cos\chi) f_l(\cos\chi) \frac{d\chi}{|c(e^{i\chi})|^2} + \sum_{p\in\mathbb{Z}, |q^p/dr|>1} \Big(\mathrm{Res}_{y=q^p/dr} \frac{1}{c(y^{-1})c(y)y}\Big) f_k(\mu(\frac{q^p}{dr})) f_l(\mu(\frac{q^p}{dr})) = \frac{\delta_{k,l}}{w_k^2}.$$

Remark 5.10. Theorem 5.8 and Corollary 5.9 have been obtained under the condition that the operator $L^*, \mathcal{D}^*)$ is self-adjoint, or that $0 < c \le q^2$. In [35] it is shown that in case $q^2 < c < 1$, there exists a self-adjoint extension of $(L, \mathcal{D}(\mathbb{Z}))$ such that the same decomposition in Theorem 5.8 and Corollary 5.9 remain valis. The case $c = q$ is a bit more intricate and requires a limiting process, since, see Proposition 3.1, u_1 and u_2 are the same, see [22, App. C] for the details.

Remark 5.11. The result in Corollary 5.9 can be viewed as q-analogue of the integral transform pair of the Jacobi functions, see (3.4) for the definition. The Jacobi function transform is an integral transform pair with a ${}_2F_1$-series, the Jacobi function, as integral kernel, see [48] for details.

Another possible option is to obtain the result of Corollary 5.9 as a limiting case of the orthogonality relation of the Askey-Wilson polynomials, which is comparable to the limit transition of the Jacobi polynomials to the Bessel functions, see [43]. q-Analogues of the Bessel functions in terms of ${}_2\varphi_1$-series have also been studied in [33]. Taking a similar limit in the little q-Jacobi polynomials leads to the little q-Bessel functions (or ${}_1\varphi_1$-q-Bessel function or as the Hahn-Exton q-Bessel function) studied by Koornwinder and Swarttouw [51]. These q-Bessel functions have been studied intensively, see e.g. [15], [16], [45] as well as other references.

5.3. **Exercises.**

1. Prove Lemma 5.1.
2. Prove Lemma 5.3.

Notes. The results of this section have been motivated by the paper by Kakehi [30] and unpublished notes by Koornwinder. The results and techniques have been very useful in the study of various problems related to harmonic analysis on the non-compact quantum group analogue of $SU(1,1)$. In particular, we have used [42, App. A], where more general sets of parameters have been studied, see also [35]. A special case is studied in [22, App. C]. In [34] another case related to a non-selfadjoint operator is studied in detail. There is also an approach to doubly infinite Jacobi operators as in §5.1, due to Krein, and this is to relate it to a 2×2-matrix valued three term recurrence on $\mathbb{N}$, see e.g. Berezanskiĭ [8, Ch. VII]. This leads to the theory of matrix-valued orthogonal polynomials.

See [35] for the solution to Exercise 1 and Exercise 2.

6. Transmutation properties for the basic q-difference equation

In §3 we have discussed the factorisation of the basic q-difference operator. The Darboux factorisation in §3 is related to a q-shift in both parameters. Here we discuss a related shift operator, but we use a relabeling of the parameters. Moreover, the shift is more general and leads to a q-analogue of fractional integral operators and other type of factorisations of the basic q-difference operator.

We rewrite the second order hypergeometric q-difference operator as studied in §3 as

$$L = L^{(a,b)} = a^2(1+\frac{1}{x})\big(T_q - \mathrm{Id}\big) + (1+\frac{aq}{bx})\big(T_q^{-1} - \mathrm{Id}\big), \tag{6.1}$$

where $T_q f(x) = f(qx)$ for suitable functions f in a suitable Hilbert space. So we have eigenfunctions to L in terms of basic hypergeometric series, see Proposition 3.1. The little q-Jacobi function is defined as

$$\varphi_\lambda(x;a,b;q) = {}_2\varphi_1\left(\begin{matrix} a\sigma, a/\sigma \\ ab \end{matrix}; q, -\frac{bx}{a}\right), \quad \lambda = \frac{1}{2}(\sigma+\sigma^{-1}) = \mu(\sigma) \tag{6.2}$$

The little q-Jacobi function satisfies

$$L\varphi_\lambda(\cdot;a,b;q) = (-1-a^2+2a\lambda)\varphi_\lambda(\cdot;a,b;q).$$

We note that the little q-Jacobi functions are eigenfunctions for the eigenvalue λ of

$$\mathcal{L}^{(a,b)} = \frac{1}{2a}L^{(a,b)} + \frac{1}{2}(a+a^{-1}) = \frac{a}{2}(1+\frac{1}{x})T_q - (\frac{a}{2x}+\frac{q}{2bx})\mathrm{Id} + \frac{1}{2a}(1+\frac{aq}{bx})T_q^{-1}. \tag{6.3}$$

For simplicity we assume that $a,b>0$, $ab<1$ and $y>0$, but the results hold, mutatis mutandis, for the more general range of the parameters as discussed in [42, App. A]. Then the operator L is an unbounded symmetric operator on the Hilbert space $\mathcal{H}(a,b;y)$ of square integrable sequences $u=(u_k)_{k\in\mathbb{Z}}$ with respect to the weights

$$\sum_{k=-\infty}^{\infty} |u_k|^2 (ab)^k \frac{(-byq^k/a;q)_\infty}{(-yq^k;q)_\infty}, \tag{6.4}$$

where the operator L is initially defined on the sequences with finitely many non-zero entries, see §5, and where $x = yq^k$.

The goal is to give a general factorisation property in Theorem 6.4 and Theorem 6.6. As a motivation we start by giving a Darboux factorisation of the second order q-difference operator $L^{(a,b)}$ or $\mathcal{L}^{(a,b)}$, related to the one in §3.

The backward q-derivative operator is $B_q = M_{1/x}(1-T_q^{-1})$, where M_g is the operator of multiplication by g; $\big(M_g f\big)(x) = g(x)f(x)$, and $T_q f(x) = f(qx)$. Then B_q is closely related to $\tilde{D}_q$ of §3 with inverted base $q \leftrightarrow q^{-1}$. Now we check that

$$\big(B_q\varphi_\lambda(\cdot;a,b;q)\big)(x) = \frac{b(1-a\sigma)(1-a/\sigma)}{qa(1-ab)}\,\varphi_\lambda(x;aq,b;q). \tag{6.5}$$

Considering $\mathcal{H}(a,b;y)$ as an L^2-space with discrete weights $(ab)^k(-byq^k/a;q)_\infty/(-yq^k;q)_\infty$ at the point yq^k, $k\in\mathbb{Z}$, we look at B_q as a (densely defined unbounded) operator from $\mathcal{H}(a,b;y)$ to $\mathcal{H}(aq,b;y)$. Its adjoint, up to a constant depending only on y, is given by

$$A(a,b) = M_{1+bx/aq} - abM_{1+x}T_q, \tag{6.6}$$

and it is a straightforward calculation to show that

$$\big(A(a,b)\varphi_\lambda(\cdot;aq,b;q)\big)(x) = (1-ab)\,\varphi_\lambda(x;a,b;q) \tag{6.7}$$

and that $-bL^{(a,b)} = aqA(a,b)\circ B_q$, with the notation as in (6.1). This calculation is essentially the same as done in §3.

Since B_q and $A(a,b)$ are triangular with respect to the standard orthogonal basis of Dirac delta's at yq^k of $\mathcal{H}(a,b;y)$, this means that we have a Darboux factorisation of $L^{(a,b)}$. Also,

$$-b(L^{(aq,b)}+(1-q)(1-qa^2)) = aq^2 B_q \circ A(a,b),$$

from which we deduce

$$B_q \circ L^{(a,b)} = L^{(aq,b)} \circ B_q \quad \text{and} \quad L^{(a,b)} \circ A(a,b) = A(a,b) \circ L^{(aq,b)}.$$

It is the purpose of this section to generalize these intertwining properties to arbitrary powers of B_q.

Introduce the operator W_ν, $\nu \in \mathbb{C}$, acting on functions defined on $[0,\infty)$ by

$$\big(W_\nu f\big)(x) = x^\nu \sum_{l=0}^\infty f(xq^{-l}) q^{-l\nu} \frac{(q^\nu;q)_l}{(q;q)_l}, \qquad x \in [0,\infty), \tag{6.8}$$

assuming that the infinite sum is absolutely convergent if $\nu \notin -\mathbb{N}$. So we want f sufficiently decreasing on a q-grid tending to infinity, e.g. $f(xq^{-l}) = \mathcal{O}(q^{l(\nu+\varepsilon)})$ for some $\varepsilon > 0$. Note that for $\nu \in -\mathbb{N}$ the sum in (6.8) is finite and $W_0 = \mathrm{Id}$ and $W_{-1} = B_q$.

This operator is a q-analogue of the Weyl fractional integral operator as used in [47, §3], [48, §5.3] for the Abel transform. With the notation

$$\int_a^\infty f(t)\, d_q t = a \sum_{k=0}^\infty f(xq^{-k}) q^{-k}$$

for the q-integral we see that for $n \in \mathbb{N}$ the operator W_n is an iterated q-integral;

$$\big(W_n f\big)(x) = \int_x^\infty \int_{x_1}^\infty \cdots \int_{x_{n-1}}^\infty f(x_n)\, d_q x_n d_q x_{n-1} \ldots d_q x_1. \tag{6.9}$$

In the following lemma we collect some results on W_ν, where we use the function space

$$\mathcal{F}_\rho = \{f\colon [0,\infty) \to \mathbb{C} \mid |f(xq^{-l})| = \mathcal{O}(q^{l\rho}),\ l \to \infty,\ \forall x \in (q,1]\}, \qquad \rho > 0. \tag{6.10}$$

Recall that $\mathcal{L}^{(a,b)}$ is defined in (6.3).

Lemma 6.1. *Let $\nu, \mu \in \mathbb{C}\backslash(-\mathbb{N})$.*

(i) *W_ν preserves the space of compactly supported functions,*
(ii) *$W_\nu\colon \mathcal{F}_\rho \to \mathcal{F}_{\rho-\Re\nu}$ for $\rho > \Re\nu > 0$,*
(iii) *$W_\nu \circ W_\mu = W_{\nu+\mu}$ on $\mathcal{F}_\rho$ for $\rho > \Re(\mu+\nu) > 0$,*
(iv) *$W_\nu \circ B_q = B_q \circ W_\nu = W_{\nu-1}$ on $\mathcal{F}_\rho$ for $\rho > \Re\nu - 1 > 0$, and $B_q^n \circ W_n = \mathrm{Id}$ for $n \in \mathbb{N}$ on $\mathcal{F}_\rho$ for $\rho > n$,*
(v) *$\mathcal{L}^{(aq^{-\nu},b)} \circ W_\nu = W_\nu \circ \mathcal{L}^{(a,b)}$, valid for compactly supported functions.*

Remark 6.2. It follows from (iii) that $W_{-n} = B_q^n$, $n \in \mathbb{N}$, and $W_0 = \mathrm{Id}$.

Proof. The first statement is immediate from (6.8). For (ii) we use that for $f \in \mathcal{F}_\rho$ and $x \in (q,1]$ we have

$$|W_\nu f(xq^{-k})| \le M \sum_{l=0}^\infty q^{(k+l)\rho} q^{-(k+l)\Re\nu} \frac{(q^{\Re\nu};q)_l}{(q;q)_l} = M q^{k(\rho-\Re\nu)} \frac{(q^\rho;q)_\infty}{(q^{\rho-\Re\nu};q)_\infty}$$

by the q-binomial theorem for $\rho > \Re\nu$. The third statement is a consequence of interchanging summations, valid for $f \in \mathcal{F}_\rho$, $\rho > \Re(\mu+\nu)$, and

$$\sum_{k+l=p} \frac{(q^\mu;q)_k (q^\nu;q)_l}{(q;q)_k (q;q)_l} q^{-(l+k)\mu - l\nu} = q^{-p(\mu+\nu)} \frac{(q^{\mu+\nu};q)_p}{(q;q)_p},$$

which is the q-Chu-Vandermonde summation formula (2.22). For (iv) we note that $B_q\colon \mathcal{F}_\rho \to \mathcal{F}_{\rho+1}$, then the first statement of (iv) is a simple calculation involving q-shifted factorials, which reduces the second statement of (iv) to verifying the easy case $n=1$. For (v) recall (6.3), so that $\mathcal{L}^{(aq^{-\nu},b)}(W_\nu f)(x)$ and $W_\nu(\mathcal{L}^{(a,b)}f)(x)$ involve the values $f(xq^{-k})$, $k+1\in\mathbb{N}$. A straightforward calculation using q-shifted factorials shows that the coefficients of $f(xq^{-k})$ in $\mathcal{L}^{(aq^{-\nu},b)}(W_\nu f)(x)$ and $W_\nu(\mathcal{L}^{(a,b)}f)(x)$ are equal. □

The asymptotically free solution $\Phi_\sigma(yq^k;a,b;q)$ is defined by

$$\Phi_\sigma(yq^k;a,b;q) = (a\sigma)^{-k}\,{}_2\varphi_1\left(\begin{matrix} a\sigma, q\sigma/b \\ q\sigma^2 \end{matrix}; q, -\frac{q^{1-k}}{y}\right). \tag{6.11}$$

so that, see (2.30),

$$\begin{aligned} \varphi_\lambda(yq^k;a,b;q) &= c(\sigma;a,b;q)\Phi_\sigma(yq^k;a,b;q) + c(\sigma^{-1};a,b;q)\Phi_{\sigma^{-1}}(yq^k;a,b;q), \\ c(\sigma;a,b,y;q) &= \frac{(b/\sigma, a/\sigma;q)_\infty}{(\sigma^{-2},ab;q)_\infty}\frac{(-by\sigma, -q/by\sigma;q)_\infty}{(-by/a,-qa/by;q)_\infty}, \end{aligned} \tag{6.12}$$

valid for $\sigma^2\notin q^{\mathbb{Z}}$. Then Φ_σ is the asymptotically free solution;

$$L\Phi_\sigma(\cdot;a,b;q) = (-1-a^2+2a\lambda)\Phi_\sigma(\cdot;a,b;q)$$

on $yq^{\mathbb{Z}}$ with, as before, $\lambda=\mu(\sigma)=\frac12(\sigma+\sigma^{-1})$.

The asymptotically free solution $\Phi_\sigma(yq^k;a,b;q)\in\mathcal{F}_\rho$ for $q^\rho > |a\sigma|$ as follows from (6.11). A calculation using the q-binomial formula gives, cf. (2.20),

$$\big(W_\nu\Phi_\sigma(\cdot;a,b;q)\big)(yq^k) = y^\nu\frac{(a\sigma;q)_\infty}{(aq^{-\nu}\sigma;q)_\infty}\Phi_\sigma(yq^k;aq^{-\nu},b;q), \tag{6.13}$$

for $|a\sigma|<q^\nu$ in accordance with Lemma 5.1(v). Note that (6.13) is a q-analogue of Bateman's formula, cf. [17], [47].

Lemma 6.3. *Define the operator*

$$S(a,b) = M_{\frac{(-x;q)_\infty}{(-bx/a;q)_\infty}}\circ T_{b/a}, \qquad T_{b/a}f(x) = f(\frac{b}{a}x),$$

then $S(a,b)^{-1}\circ\mathcal{L}^{(a,b)}\circ S(a,b) = \mathcal{L}^{(b,a)}$. *In particular,* $\tilde W_\nu^{(a,b)} = S(a,bq^{-\nu})\circ W_\nu\circ S(a,b)^{-1}$ *satisfies the intertwining property* $\mathcal{L}^{(a,bq^{-\nu})}\circ\tilde W_\nu^{(a,b)} = \tilde W_\nu^{(a,b)}\circ\mathcal{L}^{(a,b)}$.

Note that $S(a,b)^{-1}=S(b,a)$ and that $S(a,b)\colon\mathcal{H}(b,a;yb/a)\to\mathcal{H}(a,b;y)$ is an isometric isomorphism. For $f\in\mathcal{F}_\rho$ we see that $\big(S(a,b)f\big)(xq^{-l}) = \mathcal{O}(|a/b|^lq^{l\rho})$, so that $S(a,b)f\in\mathcal{F}_{\rho+\ln(|a/b|)/\ln q}$.

Proof. It follows from (6.3) that

$$\begin{aligned} &\mathcal{L}^{(a,b)}\big(x\mapsto\frac{(-x;q)_\infty}{(-bx/a;q)_\infty}f(x)\big)(x) = \\ &\qquad \frac{(-x;q)_\infty}{(-bx/a;q)_\infty}\Big(\frac{b}{2}(1+\frac{a}{bx})f(qx) + \frac{1}{2b}(1+\frac{q}{x})f(xq^{-1}) - \frac12(\frac{a}{x}+\frac{q}{bx})f(x)\Big) \end{aligned}$$

and the term in parentheses can be written as $T_{b/a}\circ\mathcal{L}^{(b,a)}\circ T_{a/b}$ applied to f. The second statement then follows from Lemma 5.1(v). □

It follows directly from (6.2), (6.11) and the last equation of (2.24),

$$\begin{aligned}\big(S(a,b)\varphi_\lambda(\cdot;b,a;q)\big)(x) &= \varphi_\lambda(x;a,b;q),\\ \big(S(a,b)\Phi_\sigma(\cdot;b,a;q)\big)(x) &= \Phi_\sigma(x;a,b;q).\end{aligned} \tag{6.14}$$

Theorem 6.4. *Let $a,b\in\mathbb{C}\backslash\{0\}$, $\nu,\mu\in\mathbb{C}$ with $|q^{\nu-\mu}b/a|<1$. Define the operator*

$$\begin{aligned}\big(W_{\nu,\mu}(a,b)f\big)(x) &= \frac{(-x;q)_\infty}{(-xq^{-\mu};q)_\infty} q^{-\mu^2}\big(\frac{b}{a}\big)^\mu x^{\mu+\nu}\\ &\times\sum_{p=0}^\infty f(xq^{-\mu-p})\,q^{-p\nu}\frac{(q^\nu;q)_p}{(q;q)_p}\,{}_3\varphi_2\left(\begin{matrix}q^{-p},q^{-\mu},-q^{1+\mu-\nu}a/bx\\ q^{1-p-\nu},-q^{\mu+1}/x\end{matrix};q,q^{1-\mu}\frac{b}{a}\right)\end{aligned}$$

for any function f with $|f(xq^{-p})|=\mathcal{O}(q^{p(\varepsilon+\nu)})$ for some $\varepsilon>0$. Then

$$W_{\nu,\mu}(a,b)\circ\mathcal{L}^{(a,b)}=\mathcal{L}^{(aq^{-\nu},bq^{-\mu})}\circ W_{\nu,\mu}(a,b)$$

on the space of compactly supported functions and for $|a\sigma|<q^\nu$

$$\big(W_{\nu,\mu}(a,b)\Phi_\sigma(\cdot;a,b;q)\big)(yq^k)=y^{\mu+\nu}\frac{(a\sigma,b\sigma;q)_\infty}{(aq^{-\nu}\sigma,bq^{-\mu}\sigma;q)_\infty}\Phi_\sigma(yq^k;aq^{-\nu},bq^{-\mu};q).$$

Proof. It follows from Lemma 6.1(v) and Lemma 6.3 that the operator

$$W_{\nu,\mu}(a,b)=\tilde{W}_\mu^{(aq^{-\nu},b)}\circ W_\nu=S(aq^{-\nu},bq^{-\mu})\circ W_\mu\circ S(b,aq^{-\nu})\circ W_\nu$$

satisfies the required interwining property. For $f\in\mathcal{F}_\rho$ with $\rho>\Re\nu$ we can interchange summations, which leads to the sum with a terminating ${}_3\varphi_2$ as kernel. Note that the ${}_3\varphi_2$-series in the kernel of $W_{\nu,\mu}(a,b)$ behaves as

$${}_2\varphi_1\left(\begin{matrix}q^{-\mu},-q^{1+\mu-\nu}a/bx\\ -q^{1+\mu}/x\end{matrix};q,q^{\nu-\mu}\frac{b}{a}\right)$$

as $p\to\infty$.

The statement for the action on $\Phi_\sigma(\cdot;a,b;q)$ follows immediately from (6.13) and (6.14). □

The results in Theorem 6.4 deal with the fractional q-derivative W_ν related to the point at ∞, and these operators act nicely on the eigenfunctions Φ_σ at ∞ of the operator $\mathcal{L}^{(a,b)}$. We want to have similar statements on a suitable intertwining operator that acts nicely on the eigenfunctions φ_λ at 0 of the operator $\mathcal{L}^{(a,b)}$. In order to get results in this direction, see Theorem 6.6, we take appropriate adjoints of the previous construction. Consider W_ν, $\nu\in\mathbb{C}\setminus(-\mathbb{N})$, as a densely defined unbounded operator from $\mathcal{H}(aq^\nu,b;y)$ to $\mathcal{H}(a,b;y)$ and define $R_\nu^{(a,b)}$ as its adjoint, so

$$\langle R_\nu^{(a,b)}f,g\rangle_{\mathcal{H}(aq^\nu,b;y)}=\langle f,W_\nu g\rangle_{\mathcal{H}(a,b;y)} \tag{6.15}$$

for all compactly supported functions g, cf. Lemma 5.1(i). Here we use the identification of $\mathcal{H}(a,b;y)$ as a weighted L^2-space on a discrete set, see §1. A q-integration by parts shows

$$\big(R_\nu^{(a,b)}f\big)(yq^p)=y^\nu\frac{(-byq^p/a;q)_\infty}{(-byq^{p-\nu}/a;q)_\infty}\sum_{l=0}^\infty f(yq^{p+l})(ab)^l\frac{(q^\nu,-yq^p;q)_l}{(q,-byq^p/a;q)_l}. \tag{6.16}$$

Now define, for functions f, the operator

$$\big(A_\nu^{(a,b)}f\big)(x)=\frac{(-bx/a;q)_\infty}{(-bxq^{-\nu}/a;q)_\infty}\sum_{l=0}^\infty f(xq^l)(ab)^l\frac{(q^\nu,-x;q)_l}{(q,-bx/a;q)_l}, \tag{6.17}$$

so that $A_\nu^{(a,b)}\Big|_{\mathcal{H}(a,b;y)} = y^{-\nu}R_\nu^{(a,b)}$. Note that $A_\nu^{(a,b)}$ is well-defined for bounded functions assuming $|ab|<1$. Recall that the dense domain of finite linear combinations of the basis vectors for $\mathcal{L}^{(a,b)}$ corresponds to the functions compactly supported in $(0,\infty)$.

Lemma 6.5. *$\mathcal{L}^{(aq^\nu,b)}\circ A_\nu^{(a,b)} = A_\nu^{(a,b)}\circ\mathcal{L}^{(a,b)}$ on the space of functions compactly supported in $(0,\infty)$. Moreover,*

$$\big(A_\nu^{(a,b)}\varphi_\lambda(\cdot;a,b;q)\big)(x) = \frac{(abq^\nu;q)_\infty}{(ab;q)_\infty}\varphi_\lambda(x;aq^\nu,b;q).$$

Defining $\tilde A_\nu^{(a,b)} = S(a,bq^\nu)\circ A_\nu^{(b,a)}\circ S(b,a)$ we have $\mathcal{L}^{(a,bq^\nu)}\circ\tilde A_\nu^{(a,b)} = \tilde A_\nu^{(a,b)}\circ\mathcal{L}^{(a,b)}$, and

$$\big(\tilde A_\nu^{(a,b)}\varphi_\lambda(\cdot;a,b;q)\big)(x) = \frac{(abq^\nu;q)_\infty}{(ab;q)_\infty}\varphi_\lambda(x;a,bq^\nu;q).$$

Proof. Note that (6.16) and (6.17) show that the operators $R_\nu^{(a,b)}$ and $A_\nu^{(a,b)}$ preserve the space of functions compactly supported in $(0,\infty)$. The intertwining property for $R_\nu^{(a,b)}$ follows from (6.15) and Lemma 5.1, and hence for $A_\nu^{(a,b)}$.

To calculate the action of $A_\nu^{(a,b)}$ on the little q-Jacobi function we use the last equation of (2.24) to write

$$\varphi_\lambda(x;a,b;q) = \frac{(-x;q)_\infty}{(-bx/a;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix} b\sigma, b/\sigma \\ ab\end{matrix};q,-x\right) \tag{6.18}$$

Using this in (6.17), interchanging summations, which is easily justified for $|x|<1$, and using the q-binomial theorem gives

$$\big(A_\nu^{(a,b)}\varphi_\lambda(\cdot;a,b;q)\big)(x) = \frac{(abq^\nu,-x;q)_\infty}{(ab,-bxq^{-\nu}/a;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix} b\sigma, b/\sigma \\ abq^\nu\end{matrix};q,-x\right)$$

and using (6.18) again gives the result for $|x|<1$. The general case follows by analytic continuation in x, see (6.11), since the convergence in (6.17) for f the little q-Jacobi function is uniform on compact sets for x.

The statements for $\tilde A_\nu^{(a,b)}$ follow from the corresponding statements for $A_\nu^{(a,b)}$ and Lemma 5.2 and (6.14). $\square$

Theorem 6.6. *Let $a,b>0$, $ab<1$, $\nu>0$ and $\mu\in\mathbb{C}\setminus\mathbb{Z}_{\le 0}$. Define the operator*

$$\begin{aligned}\big(A_{\nu,\mu}(a,b)f\big)(x) &= \frac{(-bxq^\mu/a;q)_\infty}{(-bxq^{\mu-\nu}/a;q)_\infty} \\ &\qquad\times\sum_{k=0}^\infty f(xq^{\mu+k})\,(ab)^k\frac{(q^\nu,-xq^\mu;q)_k}{(q,-bxq^\mu/;q)_k}\,{}_3\varphi_2\left(\begin{matrix} q^{-k}, q^\mu, -bxq^{\mu-\nu}/a \\ q^{1-\nu-k}, -xq^\mu\end{matrix};q,q\right)\end{aligned}$$

for any bounded function. Then $\mathcal{L}^{(aq^\nu,bq^\mu)}\circ A_{\nu,\mu}(a,b) = A_{\nu,\mu}(a,b)\circ\mathcal{L}^{(a,b)}$ on the space of functions compactly supported in $(0,\infty)$. Moreover,

$$\big(A_{\nu,\mu}(a,b)\varphi_\lambda(\cdot;a,b;q)\big)(x) = \frac{(abq^{\nu+\mu};q)_\infty}{(ab;q)_\infty}\,\varphi_\lambda(x;aq^\nu,bq^\mu;q).$$

Proof. Define

$$\begin{aligned}A_{\nu,\mu}(a,b) &= \tilde A_\mu^{(aq^\nu,b)}\circ A_\nu^{(a,b)} \\ &= S(aq^\nu,bq^\mu)\circ A_\mu^{(b,aq^\nu)}\circ S(b,aq^\nu)\circ A_\nu^{(a,b)}\end{aligned}$$

then it follows from Lemma 5.3 that the intertwining property is valid. The action on a function f can be calculated and for f compactly suppported in $(0,\infty)$ we find the explicit result with the ${}_3\varphi_2$-series as kernel. We can extend the result to bounded f if we require $\nu > 0$.

The action of $A_{\nu,\mu}(a,b)$ on the little q-Jacobi function follows from Lemma 5.3. □

These results can be used to obtain several identities involving the kernels of the transforms $A_\nu^{(a,b)}$ and $W_{\nu,\mu}(a,b)$, involving the transform of §5. We refer to [40] for examples.

6.1. Exercises.

1. Gasper's q-analogue [17, (1.8)] of Erdélyi's fractional integral is

$$
{}_2\varphi_1\left(\begin{matrix} ar\sigma, ar/\sigma \\ abrs \end{matrix}; q, -\frac{byq^l}{ar}\right) = \frac{(ab, rs; q)_\infty}{(q, abrs; q)_\infty} \sum_{k=0}^\infty (ab)^k \frac{(q^{k+1}, -byq^{k+l}/a; q)_\infty}{(rsq^k, -byq^{k+l}/ar; q)_\infty}
\times {}_3\varphi_2\left(\begin{matrix} q^{-k}, r, ar/b \\ rs, -arq^{1-l-k}/by \end{matrix}; q, q\right) {}_2\varphi_1\left(\begin{matrix} a\sigma, a/\sigma \\ ab \end{matrix}; q, -\frac{byq^{l+k}}{a}\right)
$$

for $|rs| < 1$, $|ab| < 1$. Derive this from Theorem 6.6.

Notes. The result of this section are based on [40], and they focus on the fractional analogues of the q-derivative for the asymptotically free solution of the second order q-difference equation. Several other results related to this factorisation of the q-difference equation are presented in [40]. For the classical situation this is related to factoring the Jacobi function transform as a product of the Abel transform followed by the (standard) Fourier transform, see [48] for details.

7. Askey-Wilson level

At the level of the Askey-Wilson polynomials and Askey-Wilson functions one considers the second-order q-difference operator

7.1. Askey-Wilson polynomials.

In this section we briefly recall the basic properties of the Askey-Wilson polynomials. We formulate these properties using the concept of duality.

The Askey-Wilson polynomials $p_n(x) = p_n(x; a, b, c, d; q)$, $n \in \mathbb{N}$, are defined by

$$
p_n(x) = p_n(x; a, b, c, d \mid q) = {}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, ax, ax^{-1} \\ ab, ac, ad \end{matrix}; q, q\right) \tag{7.1}
$$

see [6]. Note that p_n is a polynomial in $z = \mu(x) = \frac{1}{2}(x + x^{-1})$, but we consider it as Laurent polynomial in x. Usually, see [6], [31], [32], the normalization is chosen differently in order to make the Askey-Wilson polynomials symmetric in a, b, c and d. The Askey-Wilson polynomials $\{p_n\}_{n\in\mathbb{N}}$ form a basis of the polynomial algebra $\mathbb{C}[z] = \mathbb{C}[x + x^{-1}]$ consisting of eigenfunctions of the Askey-Wilson second order q-difference operator

$$
L = \alpha(x)(T_q - 1) + \alpha(x^{-1})(T_q^{-1} - 1), \qquad \alpha(x) = \frac{(1-ax)(1-bx)(1-cx)(1-dx)}{(1-x^2)(1-qx^2)}, \tag{7.2}
$$

where $(T_q^{\pm 1} f)(x) = f(q^{\pm 1}x)$.

The eigenvalue of L corresponding to the Askey-Wilson polynomial p_n is $\mu(\gamma_n)$, where $\gamma_n = \tilde{a}q^n$, $\tilde{a} = \sqrt{q^{-1}abcd}$, and

$$
\big(Lp_n\big)(x) = \mu(\gamma_n)p_n(x), \qquad \mu(\gamma) = -1 - \tilde{a}^2 + \tilde{a}(\gamma + \gamma^{-1}). \tag{7.3}
$$

In order to describe the orthogonality relations concisely we recall the dual parameters to (a,b,c,d). We extend the definition of $\tilde a$ to

$$\tilde a = \sqrt{q^{-1}abcd}, \qquad \tilde b = ab/\tilde a = q\tilde a/cd, \qquad \tilde c = ac/\tilde a = q\tilde a/bd, \qquad \tilde d = ad/\tilde a = q\tilde a/bc. \tag{7.4}$$

Lemma 7.1. *The assignment $(a,b,c,d,t) \mapsto (\tilde a, \tilde b, \tilde c, \tilde d, \tilde t)$ defined by (7.4) is an involution.*

Lemma 7.1 follows by calculation.

The orthogonality relations for the Askey-Wilson polynomials hold quite generally, see [6], [18], [32], but we assume $0 < q < 1$ as usual and moreover that a, b, c and d are positive and less than one. Then Askey and Wilson [6] proved the orthogonality relations

$$\frac{1}{2\pi i C_0}\int_{x\in\mathbb{T}} p_n(x)p_m(x)\Delta(x)\frac{dx}{x} = \delta_{m,n}\frac{\operatorname*{Res}_{x=\gamma_0}\left(\frac{\tilde\Delta(x)}{x}\right)}{\operatorname*{Res}_{x=\gamma_n}\left(\frac{\tilde\Delta(x)}{x}\right)} \tag{7.5}$$

where $\delta_{m,n}$ is the Kronecker delta and $\mathbb{T}$ is the counterclockwise oriented unit circle in the complex plane, with the weight function given by

$$\Delta(x) = \frac{\big(x^2, 1/x^2; q\big)_\infty}{\big(ax, a/x, bx, b/x, cx, c/x, dx, d/x; q\big)_\infty}, \tag{7.6}$$

and with $\widetilde\Delta(x)$ the weight function $\Delta(x)$ with respect to dual parameters. Here the positive normalization constant C_0 is given by the Askey-Wilson integral

$$C_0 = \frac{1}{2\pi i}\int_{x\in\mathbb{T}} \Delta(x)\frac{dx}{x} = \frac{2\big(abcd;q\big)_\infty}{\big(q, ab, ac, ad, bc, bd, cd; q\big)_\infty}.$$

Various different proofs of the Askey-Wilson integral exist, see e.g. references in [26]. The original proof follows by an elaborate residue calculus, and now there are many different approaches to the Askey-Wilson integral as well as to its various extensions, see [18] for references.

Having the dual parameters (7.4), the explicit expression (7.1) for the Askey-Wilson polynomials show that the duality relation

$$p_n(aq^m; a,b,c,d;q) = p_m(\tilde a q^n; \tilde a, \tilde b, \tilde c, \tilde d; q), \qquad m,n\in\mathbb{N} \tag{7.7}$$

holds. The deeper understanding of the duality (7.4) stems from affine Hecke algebraic considerations, see [53]. This duality takes an even nicer form in the case of the Askey-Wilson functions, see 7.2. The duality (7.7) also shows that the three-term recurrence relations for the Askey-Wilson polynomials $p_m(\cdot;\tilde a,\tilde b,\tilde c,\tilde d;q)$, $m\in\mathbb{N}$, follows from the eigenvalue equations $Lp_n = \mu(\gamma_n)p_n$, $n\in\mathbb{N}$, by applying the duality (7.7), see e.g. [6] and [53].

The orthogonality relations written in the form (7.5) exhibit the duality (7.7) of the Askey-Wilson polynomials on the level of the orthogonality relations, since it expresses the quadratic norms explicitly in terms of the dual weight function $\widetilde\Delta$. This description of the quadratic norms was proved in [53].

7.2. **Askey-Wilson function transform.** We define the Askey-Wilson function transform and we state the main result concerning the Askey-Wilson function transform. For this we more generally need to consider general, i.e. non-polynomial, eigenfunctions to

$$\big(Lf\big)(x) = \mu(\gamma)f(x), \tag{7.8}$$

which reduces to the Askey-Wilson polynomial for $\gamma=\gamma_n$, $n\in\mathbb{N}$, and enjoys the same duality properties. The solutions of (7.8) have been studied by Ismail and Rahman [27].

Two linearly independent solutions of the eigenvalue equation (7.8) can be derived from Ismail's and Rahman's [27, (1.11)–(1.16)] solutions for the three term recurrence relation of the associated Askey-Wilson polynomials. The solutions are given in terms of very well poised ${}_8\varphi_7$ series, in particular

$$\begin{aligned}\phi_\gamma(x) =&\phi_\gamma(x;a,b,c;d\mid q) = \frac{\left(qax\gamma/\tilde{d}, qa\gamma/\tilde{d}x;q\right)_\infty}{\left(\tilde{a}\tilde{b}\tilde{c}\gamma, q\gamma/\tilde{d}, q\tilde{a}/\tilde{d}, qx/d, q/dx;q\right)_\infty} \\ &\times {}_8W_7\left(\tilde{a}\tilde{b}\tilde{c}\gamma/q; ax, a/x, \tilde{a}\gamma, \tilde{b}\gamma, \tilde{c}\gamma; q, q/\tilde{d}\gamma\right), \qquad |q/\tilde{d}\gamma|<1\end{aligned} \tag{7.9}$$

is a solution to (7.8). This solution is called the *Askey-Wilson function*. Lemma 7.2 shows that the Askey-Wilson function satisfies the same duality, and moreover extends the Askey-Wilson polynomials of (7.1).

Lemma 7.2. *The Askey-Wilson function satisfies the duality and reduction formulas*

$$\phi_\gamma(x;a;b,c;d\mid q) = \phi_x(\gamma;\tilde{a};\tilde{b},\tilde{c};\tilde{d}\mid q)$$
$$\phi_{\gamma_n}(x) = \frac{1}{\left(bc, qa/d, q/ad;q\right)_\infty} p_n(x), \qquad n\in\mathbb{N}.$$

and $\phi_{\gamma^{\pm1}}(x^{\pm1}) = \phi_\gamma(x)$ *for all possible choices.*

Note that duality (7.7) for the Askey-Wilson polynomials is a special case of the duality of ϕ_γ in Lemma 7.2.

Proof. The proof rests on a formula expressing a very-well poised ${}_8\varphi_7$-series as a sum of two balanced ${}_4\varphi_3$-series given by Bailey's formula [18, (III.36)]. This gives

$$\begin{aligned}\phi_\gamma(x) =& \frac{1}{\left(bc, qa/d, q/ad;q\right)_\infty} {}_4\phi_3\left(\begin{matrix} ax, a/x, \tilde{a}\gamma, \tilde{a}/\gamma \\ ab, ac, ad\end{matrix}; q, q\right) \\ &+ \frac{\left(ax, a/x, \tilde{a}\gamma, \tilde{a}/\gamma, qb/d, qc/d;q\right)_\infty}{\left(qx/d, q/dx, q\gamma/\tilde{d}, q/\tilde{d}\gamma, ab, ac, bc, qa/d, ad/q;q\right)_\infty} \\ &\qquad \times {}_4\phi_3\left(\begin{matrix} qx/d, q/dx, q\gamma/\tilde{d}, q/\tilde{d}\gamma \\ qb/d, qc/d, q^2/ad\end{matrix}; q, q\right),\end{aligned} \tag{7.10}$$

hence $\phi_\gamma(x)$ extends to a meromorphic function in x and γ for generic parameters a,b,c and d, with possible poles at $x^{\pm1} = q^{1+k}/d$, $k\in\mathbb{N}$, and $\gamma^{\pm1} = q^{1+k}/\tilde{d}$, $k\in\mathbb{N}$. It follows from (7.10) that $\phi_{\gamma^{\pm1}}(x^{\pm1}) = \phi_\gamma(x)$ (all sign combinations possible), and that ϕ_γ satisfies the duality relation by inspection.

Finaly, observe that the meromorphic continuation (7.10) of $\phi_\gamma(x)$ implies that

$$\phi_{\gamma_n}(x) = \frac{1}{\left(bc, qa/d, q/ad;q\right)_\infty} p_n(x), \qquad n\in\mathbb{N}. \tag{7.11}$$

Indeed, the factor $\left(\tilde{a}/\gamma;q\right)_\infty$ in front of the second ${}_4\varphi_3$ in (7.10) vanishes for $\gamma=\gamma_n=\tilde{a}q^n$ for $n\in\mathbb{N}$. □

At this stage we need to specify a particular parameter domain for the five parameters (a,b,c,d,t) in order to ensure positivity of measures.

Definition 7.3. *Let V be the set of parameters $(a,b,c,d,t)\in\mathbb{R}^5$ satisfying*

$$t<0, \qquad 0<b,c\le a<d/q, \qquad bd, cd\ge q, \qquad ab, ac<1.$$

Observe that $b,c<1$ and $d>q$ for all $(a,b,c,d,t)\in V$. We extend the duality of (7.4) to

$$\tilde t = 1/qadt. \tag{7.12}$$

The domain V is self-dual extending Lemma 7.1.

Lemma 7.4. *The assignment $(a,b,c,d,t)\mapsto(\tilde a,\tilde b,\tilde c,\tilde d,\tilde t)$ defined by (7.4) and (7.12), is an involution on V.*

Proof. Again by direct verification. □

From now on we consider $(a,b,c,d,t)\in V$ fixed.

In order to motivate the measure which we will introduce we look at other solutions for the eigenvalue equation for L. Observe that the eigenvalue equation (7.8) is asymptotically of the form

$$\tilde a^2\bigl(f(qx)-f(x)\bigr)+\bigl(f(q^{-1}x)-f(x)\bigr)=\mu(\gamma)f(x) \tag{7.13}$$

when $|x|\to\infty$. For generic γ, the asymptotic eigenvalue equation (7.13) has a basis $\{\Phi^{\mathrm{free}}_\gamma,\Phi^{\mathrm{free}}_{\gamma^{-1}}\}$ of solutions on the q-line $I=\{dtq^k\}_{k\in\mathbb{Z}}$, where

$$\Phi^{\mathrm{free}}_\gamma(dtq^k)=(\tilde a\gamma)^{-k},\qquad k\in\mathbb{Z}.$$

Furthermore, for generic γ there exists a unique solution $\Phi_\gamma(x)$ of the eigenvalue equation (7.8) on I of the form $\Phi_\gamma(x)=\Phi^{\mathrm{free}}_\gamma(x)g(x)$, where g has a convergent power series expansion around ∞ with constant coefficient equal to one. The solution Φ_γ is the *asymptotically free solution* of the eigenvalue equation (7.8).

Actually, an explicit expression for Φ_γ can be obtained from the study of Ismail and Rahman on the associated Askey-Wilson polynomials, in which they study solutions of the eigenvalue equation (7.8). Starting with [27, (1.13)] and applying the transformation formula [18, (III.23)] for very well poised ${}_8\varphi_7$'s we obtain

$$\begin{aligned}\Phi_\gamma(x)=&\frac{\bigl(qa\gamma/\tilde a x,qb\gamma/\tilde a x,qc\gamma/\tilde a x,q\tilde a\gamma/dx,d/x;q\bigr)_\infty}{\bigl(q/ax,q/bx,q/cx,q/dx,q^2\gamma^2/dx;q\bigr)_\infty}\\&\times{}_8W_7\bigl(q\gamma^2/dx;q\gamma/\tilde a,q\gamma/\tilde d,\tilde b\gamma,\tilde c\gamma,q/dx;q,d/x\bigr)\Phi^{free}_\gamma(x)\end{aligned}\tag{7.14}$$

for $x\in I$ with $|x|\gg0$. We now expand the Askey-Wilson function $\phi_\gamma(x)$ as a linear combination of the asymptotically free solutions $\Phi_\gamma(x)$ and $\Phi_{\gamma^{-1}}(x)$ for $x\in I$ with $|x|\gg0$. Since these are all solutions to the same eigenvalue equation, we can expect a relation with coefficients being constants or q-periodic functions.

Proposition 7.5. *Let $x\in I$ with $|x|\gg0$. Then we have the c-function expansion*

$$\phi_\gamma(x)=\widetilde c(\gamma)\Phi_\gamma(x)+\widetilde c(\gamma^{-1})\Phi_{\gamma^{-1}}(x)$$

for generic γ, where the c-function is given by

$$c(\gamma)=c(\gamma;a;b,c;d;q;t)=\frac{1}{\bigl(ab,ac,bc,qa/d;q\bigr)_\infty\theta(qadt)}\frac{\bigl(a/\gamma,b/\gamma,c/\gamma;q\bigr)_\infty\theta(\gamma/dt)}{\bigl(q\gamma/d,1/\gamma^2;q\bigr)_\infty}.$$

using the notation (2.31) and $\widetilde c(\gamma)=c(\gamma;\tilde a;\tilde b,\tilde c;\tilde d;q;\tilde t)$.

We call $\widetilde c(\gamma)=c(\gamma;\tilde a;\tilde b,\tilde c;\tilde d;q;\tilde t)$ the dual c-function, with the dual parameter $\tilde t$ defined by (7.12).

Proof. The proof requires some calculation. The essential ingredients are as follows. First apply Bailey's three term recurrence relation [18, (III.37)] with its parameters specialized as

$$a \to q\gamma^2/dx, \quad b \to q/dx, \quad c \to q\gamma/\tilde{a}, \quad d \to q\gamma/\tilde{d}, \quad e \to \tilde{b}\gamma, \quad f \to \tilde{c}\gamma.$$

This gives an expansion of the required form with explicit coefficients $\widetilde{c}(\gamma)$ and $\widetilde{c}(\gamma^{-1})$, which at a first glance still depend on $x \in I$. Using the theta function (2.31) and its functional equation (2.31) we see that the coefficients are independent of x. □

For the moment we furthermore assume that $x \mapsto 1/c(x)c(x^{-1})$ only has simple poles. This imposes certain generic conditions on the parameters (a, b, c, d, t), which can be removed at a later stage by a continuity argument.

It is convenient to renormalize the function $1/c(x)c(x^{-1})$ as follows,

$$W(x) = \frac{1}{c(x)c(x^{-1})c_0} = \frac{\big(qx/d, q/dx, x^2, 1/x^2; q\big)_\infty}{\big(ax, a/x, bx, b/x, cx, c/x; q\big)_\infty \theta(dtx)\theta(dt/x)}, \tag{7.15}$$

where c_0 is the positive constant

$$c_0 = \frac{\big(ab, ac, bc, qa/d; q\big)_\infty^2 \theta(adt)^2}{a^2}.$$

It follows from (7.15) and (7.6) that

$$W(x) = \frac{\theta(dx)\theta(d/x)}{\theta(dtx)\theta(dt/x)}\Delta(x). \tag{7.16}$$

By (2.31), the quotient of theta functions in (7.16) is a q-periodic function. In particular, the weight function $W(x)$ differs from $\Delta(x)$ only by a q-periodic function, but this factor introduces additional poles which arise in the orthogonality (or spectral) measure.

Let S be the discrete subset

$$\begin{aligned} S &= \{x \in \mathbb{C} \mid |x| > 1, c(x) = 0\} = S_+ \cup S_-, \\ S_+ &= \{aq^k \mid k \in \mathbb{N},\ aq^k > 1\}, \\ S_- &= \{dtq^k \mid k \in \mathbb{Z},\ dtq^k < -1\}. \end{aligned} \tag{7.17}$$

By $\widetilde{S}$ and $\widetilde{S}_\pm$ we denote the subsets S and $S_\pm$ with respect to dual parameters. We define a measure $\nu = \nu(\cdot; a; b, c; d; t; q)$ by

$$\begin{aligned} \int f(x)d\nu(x) &= \frac{K}{4\pi i}\int_{x\in\mathbb{T}} f(x)W(x)\frac{dx}{x} \\ &+ \frac{K}{2}\sum_{x\in S} f(x)\operatorname*{Res}_{y=x}\left(\frac{W(y)}{y}\right) - \frac{K}{2}\sum_{x\in S^{-1}} f(x)\operatorname*{Res}_{y=x}\left(\frac{W(y)}{y}\right), \end{aligned} \tag{7.18}$$

where the positive constant K is given by

$$K = \big(ab, ac, bc, qa/d, q; q\big)_\infty \sqrt{\frac{\theta(qt)\theta(adt)\theta(bdt)\theta(cdt)}{qabcdt^2}}. \tag{7.19}$$

This particular choice of normalization constant for the measure ν is justified in Theorem 7.7, since the corresponding transform is made an isometry.

In view of (7.16), we can relate the discrete masses $\nu(\{x\})(= -\nu(\{x^{-1}\}))$ for $x \in S_+$ to residues of the weight function $\Delta(\cdot)$, which were written down explicitly in [6], see also [18, (7.5.22)] in order to avoid a small misprint in [6]. Explicitly, we obtain for $x = aq^k \in S_+$ with $k \in \mathbb{N}$ the expression

$$\nu(\{aq^k\}) = \frac{\big(qa/d, q/ad, 1/a^2; q\big)_\infty}{\big(q, ab, b/a, ac, c/a; q\big)_\infty \theta(adt)\theta(dt/a)} \frac{(1-a^2q^{2k})}{(1-a^2)} \frac{K}{2\tilde{a}^{2k}} \tag{7.20}$$

for the corresponding discrete weight. For fixed $k \in \mathbb{N}$, the right hand side of (7.20) gives the unique continuous extension of the discrete weight $\nu(\{aq^k\})$ and $-\nu(\{a^{-1}q^{-k}\})$ to all parameters $(a,b,c,d,t) \in V$ satisfying $aq^k > 1$. Furthermore, the (continuously extended) discrete weight $\nu(\{aq^k\})$ is strictly positive for these parameter values. Note that S_+ gives a finite number of discrete mass points in the measure ν.

A similar argument can be applied for the discrete weights $\nu(\{x\})(= -\nu(\{x^{-1}\}))$ with $x \in S_-$. Explicitly we obtain for $x = dtq^k \in S_-$ with $k \in \mathbb{Z}$,

$$\begin{aligned} \nu(\{dtq^k\}) =& \frac{\big(qt, q/d^2t; q\big)_\infty}{\big(q, q, a/dt, b/dt, c/dt, adt, bdt, cdt; q\big)_\infty} \\ &\times \frac{\big(1/t, a/dt, b/dt, c/dt; q\big)_{-k}}{\big(q/adt, q/bdt, q/cdt, q/d^2t; q\big)_{-k}} \left(1 - \frac{1}{d^2t^2q^{2k}}\right) \frac{K\tilde{a}^{2k}}{2}. \end{aligned} \tag{7.21}$$

As for $\nu(\{x\})$ with $x \in S_+$, we use the right hand side of (7.21) to define the strictly positive weight $\nu(\{dtq^k\})(= -\nu(\{d^{-1}t^{-1}q^{-k}\}))$ for all $(a,b,c,d,t) \in V$ satisfying $dtq^k < -1$. Note that S_- gives an infinite number of discrete mass points in the measure ν.

We see that the definition (7.18) of the measure ν can be extended to arbitrary parameters $(a,b,c,d,t) \in V$ using the continuous extensions of its discrete weights in (7.20), (7.21). The resulting measure ν is a positive measure for all $(a,b,c,d,t) \in V$.

Definition 7.6. *Let $\mathcal{H} = \mathcal{H}(a;b,c;d;t;q)$ be the Hilbert space consisting of L^2-functions f with respect to ν which satisfy $f(x) = f(x^{-1})$ ν-almost everywhere.*

We write $\widetilde{\nu}$ for the measure ν with respect to dual parameters $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{t})$, and $\widetilde{\mathcal{H}}$ for the associated Hilbert space $\mathcal{H}$.

Let $\mathcal{D} \subset \mathcal{H}$ be the dense subspace of functions f with compact support, i.e.

$$\mathcal{D} = \{f \in \mathcal{H} \mid f(dtq^{-k}) = 0,\ k \gg 0\},$$

and define

$$(\mathcal{F}f)(\gamma) = \int f(x)\overline{\phi_\gamma(x)}d\nu(x), \qquad f \in \mathcal{D} \tag{7.22}$$

for generic $\gamma \in \mathbb{C} \setminus \{0\}$.

We write $\widetilde{\mathcal{D}} \subset \widetilde{\mathcal{H}}$ (respectively $\widetilde{\mathcal{F}}$) for the dense subspace $\mathcal{D}$ (respectively the function transform $\mathcal{F}$) with respect to dual parameters $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{t})$.

Theorem 7.7. *Let $(a,b,c,d,t) \in V$. The transform $\mathcal{F}$ extends to an isometric isomorphism $\mathcal{F}\colon \mathcal{H} \to \widetilde{\mathcal{H}}$ by continuity. The inverse of $\mathcal{F}$ is given by $\widetilde{\mathcal{F}}\colon \widetilde{\mathcal{H}} \to \mathcal{H}$.*

The isometric isomorphism $\mathcal{F}\colon \mathcal{H} \to \widetilde{\mathcal{H}}$ is called the *Askey-Wilson function transform*.

We will not prove Theorem 7.7 in these notes, but we refer to the original proof in [44].

7.3. Exercises.

1. Give a factorisation for the second order q-difference operator (7.16) in terms of a lowering and raising operator.

Notes. The Askey-Wilson polynomials have been introduced by Askey and Wilson in [6], and these polynomials are on top of the continuous part of the q-analogue of the Askey scheme, see [6], [31], [32]. The discrete counterpart, the q-Racah polynomials are on top of the discrete part of the q-analogue of the Askey scheme. The solutions of the second order q-difference equation as considered here were studied by Ismail and Rahman [27], where they studied the associated Askey-Wilson polynomials. The corresponding Askey-Wilson function transform as in Theorem 7.7 is due to [44]. It is remarkable that the $q = 1$ analogue of this transform, the Wilson function transform of Groenevelt [19], is only established after the q-analogue of the Askey-Wilson function transform. Moreover, Groenevelt's Wilson transform comes into two versions, which also map Wilson polynomials to Wilson polynomials (with dual parameters). The q-analogue of this statement is related to the expansion of the Askey-Wilson function in terms of a series involving a product of the Askey-Wilson polynomials and the Askey-Wilson polynomials with dual parameters, see Stokman [62]. Other approaches to the same or closely related functions can be found in e.g. Haine and Iliev [24], Ruijsenaars [59] and Suslov [64]. As to multivariable extensions, Stokman [63] established the analogue of Proposition 7.5. For Exercise 1 one can e.g. consult [18], [32]. There seems not to be an appropriate analogue of the Frobenius method for the Askey-Wilson difference equation, see [28] for related Taylor expansions.

8. Matrix-valued extensions

The hypergeometric differential equation has a matrix-valued analogue, studied by Tirao [66]. This matrix-valued differential equation plays an important role in the study of matrix-valued spherical functions on (especially compact) symmetric spaces and extensions of these result to more general matrix-valued orthogonal polynmials, see e.g. [23] [37], [38], [39]. See also [12] and the lecture notes [36], as well as references given there, for more information on matrix-valued orthogonal polynomials. This section is of a preliminary nature and based on the Bachelor thesis of Nikki Jaspers [29] and finds its origin in the paper [2]. The results can be considered as q-analogues of the solutions of the matrix-valued hypergeometric series at 0 and ∞, see [66] and [56].

8.1. Vector-valued basic hypergeometric q-difference equation.

Recall the basic hypergeometric equation (3.8), which we now adapt to

$$(q-z)\mathrm{Id} f(q^{-1}z) + \big((A+B)z - C - q\mathrm{Id}\big) f(z) + (C - ABz) f(qz) = 0 \tag{8.1}$$

where $A, B, C \in \mathrm{End}(\mathbb{C}^N)$, i.e. linear maps from $\mathbb{C}^N$ to $\mathbb{C}^N$, and $f\colon \mathbb{C} \to \mathbb{C}^N$ is the unknown vector-valued function, which we want to satisfy (8.1). Note that in particular in (8.1) the identity $\mathrm{Id} \in \mathrm{End}(\mathbb{C}^N)$ and $0 \in \mathbb{C}^N$.

Note that $N = 1$ brings us back to (3.8) and the results presented in these lecture notes, but also the case of commuting diagonalizable A, B and C bring us back to (3.8).

Generically the dimension of the solution space of the vector-valued basic q-difference equation (8.1) is $2N$.

Remark 8.1. More generally, we can consider (8.1) with $A+B$ and AB replaced by more generally U and V. In the case $N = 1$ this is equivalent, but in the vector-valued case this more general. In the case of the hypergeometric differential operator this is dicussed by Tirao [66]. We will not discuss this case, see [29] for the q-case.

Note that we can generalize (8.1) to

$$(q-z)\mathrm{Id} F(q^{-1}z) + \big((A+B)z - C - q\mathrm{Id}\big) F(z) + (C - ABz) F(qz) = 0 \tag{8.2}$$

where $A, B, C \in \mathrm{End}(\mathbb{C}^N)$ and $F\colon \mathbb{C} \to \mathrm{End}(\mathbb{C}^N)$ is the unknown matrix-valued function, which we want to satisfy (8.2). Note that in (8.2) now $0 \in \mathrm{End}(\mathbb{C}^N)$ is the zero-matrix. Moreover, if $F(z)$ is any solution to (8.2), then $f(z) = F(z)f_0$ for a fixed vector $f_0 \in \mathbb{C}^N$ satisfies (8.1).

8.2. **Solutions of matrix-valued q-hypergeometric equation.** The Frobenius method can be extended to case of (8.1). We first consider expansions around $z = 0$. For a matrix $C \in \mathrm{End}(\mathbb{C}^N)$ we let $\sigma(C)$ denote its spectrum, i.e. the zeros of the characteristic polynomial of C. For $A, B, C \in \mathrm{End}(\mathbb{C}^N)$ with $q^{-\mathbb{N}} \cap \sigma(C) = \emptyset$ we define for $n \in \mathbb{N}$ the product

$$\big(A, B; C; q\big)_n = \overleftarrow{\prod_{k=0}^{n-1}} (I - q^k C)^{-1}(I - q^k A)(I - q^k B)$$

where

$$\overleftarrow{\prod_{k=0}^{n-1}} a_k = a_{n-1} \cdots a_0$$

for non-commuting elements a_k. In case A, B, C are 1×1-matrices, this reduces to $\frac{(A;q)_n (B(;q)_n}{(C;q)_n}$.

For $\alpha \in \mathbb{C}$ we define the matrix-valued basic hypergeometric series

$$ {}_2\Phi_1^{\alpha}(A, B; C; q, z) = \sum_{n=0}^{\infty} \big(\alpha A, \alpha B; \alpha C; q\big)_n \frac{z^n}{(\alpha q; q)_n} \tag{8.3}$$

assuming $q^{-\mathbb{N}} \cap \sigma(\alpha C) = \emptyset$. In case $\alpha = 1$, we drop it from the notation, i.e. ${}_2\Phi_1(A, B; C; q, z) = {}_2\Phi_1^1(A, B; C; q, z)$.

Note that obvious symmetry $A \leftrightarrow B$ of the scalar case no longer holds, since $AB \neq BA$ in general.

We can now describe the solutions to (8.1) in the generic case, i.e. when the eigenvalues of C are sufficiently generic.

Theorem 8.2. *Assume C is diagonalizable, so that $\sigma(C) = \{c_1, \cdots, c_N\}$ with $c_i \neq c_j$ for $i \neq j$. Assume $c_i \neq 0$ for all i. Let $f_i \neq 0$ be the corresponding eigenvectors of C; $Cf_i = c_i f_i$, $1 \leq i \leq N$. Assume furthermore that $\sigma(C) \cap q^{\mathbb{Z}} = \emptyset$ and that $c_i/c_j \notin q^{\mathbb{Z}}$ for $i \neq j$. Then the vector-valued functions*

$${}_2\Phi_1(A, B; C; q, z)e_i$$

for any basis $\{e_1, \cdots, e_N\}$ of $\mathbb{C}^N$ and the vector-valued functions

$$z^{1-\log_q(c_i)}\, {}_2\Phi_1^{q/c_i}(A, B; C; q, z) f_i, \qquad 1 \leq i \leq N$$

span the solution space of (8.1) (over the q-periodic functions).

Proof. The proof follows the Frobenius method for the basic hypergeometric q-difference equation. So we assume that we have a solution of the form $\sum_{n=0}^{\infty} f_n z^{n+\mu}$, $f_n \in \mathbb{C}^N$, where $f_0 \neq 0$. Plugging this Ansatz into (8.1) we get

$$\begin{aligned} 0 = (q - z)\mathrm{Id} \sum_{n=0}^{\infty} f_n (q^{-1}z)^{n+\mu} + \big((A + B)z - (C + q\mathrm{Id})\big) \sum_{n=0}^{\infty} f_n z^{n+\mu} + (C - ABz) \sum_{n=0}^{\infty} f_n (qz)^{n+\mu} \\ = \big(f_0 q^{1-\mu} - (C + q)f_0 + q^{\mu} C f_0\big) z^{\mu} + \\ \sum_{n=1}^{\infty} z^{\mu+n} \Big(\big(Cq^{\mu+n} - (C + q) + q^{1-\mu-n}\big) f_n + \big(-ABq^{\mu+n-1} + A + B - q^{1-\mu-n}\big) f_{n-1}\Big) \end{aligned}$$

So in particular, the vector-valued coefficient needs to vanish, and this gives the indicial equation

$$\big((q^{1-\mu}-q)-(1-q^{\mu})C\big)f_0=(1-q^{\mu})(q^{1-\mu}-C)f_0=0$$

This gives $2N$ for q^{μ}, namely $q^{\mu}=1$ and $f_0\in\mathbb{C}^N$ arbitrary and $Cf_0=q^{1-\mu}f_0$, i.e. $f_0=f_i$ and $q^{1-\mu}=c_i$ for $1\le i\le N$.

With each of these solutions we then need to solve recursively

$$\begin{aligned}\big(Cq^{\mu+n}-(C+q)+q^{1-\mu-n}\big)f_n&=\big(ABq^{\mu+n-1}-(A+B)+q^{1-\mu-n}\big)f_{n-1}\quad\Longrightarrow\\ -(1-q^{\mu+n})(C-q^{1-\mu-n})f_n&=(Aq^{\mu+n-1}-1)(B-q^{1-n-\mu})f_{n-1}\quad\Longrightarrow\\ (1-q^{\mu+n-1}C)f_n&=\frac{1}{(1-q^{\mu+n})}(1-q^{\mu+n-1}A)(1-q^{n+\mu-1}B)f_{n-1}\end{aligned}$$

since $q^{\mu+n}\neq 1$ under the assumptions on C and $q^{1-\mu}\in\sigma(C)$.

In case $q^{\mu}=1$, we find

$$f_n=\frac{1}{(1-q^n)}(1-q^{n-1}C)^{-1}(1-q^{n-1}A)(1-q^{n-1}B)f_{n-1}=\frac{1}{(q;q)_n}(A,B;C;q)_nf_0$$

without condition on f_0. This gives the first set of solutions by taking $\mu=0$. All other solutions of $q^{\mu}=1$ lead to the same solution up to q-periodic functions, cf. the proof of Proposition 3.1.

In the other case, we have $f_0=f_i$ and $q^{1-\mu}=c_i$ for some $1\le i\le N$. Take $\mu=1-\log_q(c_i)$, so $q^{\mu}=q/c_i$. Then the recurrence is

$$\begin{aligned}f_n&=\frac{1}{(1-q^{1+n}/c_i)}(1-q^nc_i^{-1}C)^{-1}(1-q^nc_i^{-1}A)(1-q^nc_i^{-1}B)f_{n-1}\\ &=\frac{1}{(q/c_i;q)_n}(qc_i^{-1}A,qc_i^{-1}B;qc_i^{-1}C;q)_nf_i\end{aligned}$$

and this gives the other set of solutions. Again, choosing a different solution of $q^{\mu}=q/c_i$ leads to the same solution up to a q-periodic function.

Since the space of solutions is $2N$ dimensional, and the set of solutions are linearly independent we have obtained all solutions. The linear independence follows since the first set is linearly independent as analytic solutions with linearly independent values at $z=0$, and the other solutions all have different behaviour as $z\to 0$. □

In order to describe the solutions at ∞ we introduce the notation

$$\begin{aligned}[A,B;C;\alpha;q]_n&=\overleftarrow{\prod_{k=0}^{n-1}}(A-\alpha q^k)^{-1}(B-\alpha q^k)^{-1}(C-\alpha q^k)\\ \Theta^{\alpha}(A,B;C;q,z)&=\sum_{n=0}^{\infty}(\alpha/q;q)_n[A,B;C;\alpha;q]_nz^n\end{aligned}\tag{8.4}$$

where we assume that all inverses of the matrices involved exist.

Theorem 8.3. *Assume that A and B are diagonalizable with non-zero eigenvalues and such that the following genericity conditions on the spectra $\sigma(A)$, $\sigma(B)$ hold;*

$$\sigma(A)\cap\sigma(B)=\emptyset,\quad \sigma(A)\cap\sigma(B)q^{1+\mathbb{N}}=\emptyset,\quad \sigma(B)\cap\sigma(B)q^{1+\mathbb{N}}=\emptyset,$$

Moreover, let $\sigma(A) = \{a_1, \cdots, a_N\}$, $\sigma(B) = \{b_1, \cdots, b_N\}$, *with* $Af_i^A = a_i f_i^A$, $Bf_i^A = b_i f_i^B$ *then the solutions*

$$z^{-\log_q(b_i)}\Theta^{qb_i}(B, A; C; q, qz^{-1})f_i^B, \qquad 1 \le i \le N,$$
$$z^{-\log_q(a_i)}\Theta^{qa_i}(B, A; C; q, qz^{-1})(a_i - B)^{-1}f_i^A, \qquad 1 \le i \le N,$$

are linearly independent solutions of (8.1).

Note that in case $N = 1$, this leads to the solutions u_3 and u_4 of Proposition 3.1.

Proof. Now assume that a solution at ∞ has the expansion $\sum_{n=0}^{\infty} f_n z^{-n-\mu}$ for $f_n \in \mathbb{C}^N$. Plugging this expression in in (8.1) and rearranging terms we find

$$0 = z^{-1-\mu}\big(-q^\mu + (A + B) - q^{-\mu}AB\big)f_0 +$$
$$\sum_{n=0}^{\infty} z^{-n-\mu}\Big(\big(q^{n+\mu+1} - (C + q) + q^{-n-\mu}C\big)f_n + \big(-q^{n+\mu+1} + (A + B) - q^{-1-n-\mu}AB\big)f_{n+1}\Big).$$

In order to have a solution, the coefficients of the powers of z have to be zero. The first equation is the indicial equation (for $z = \infty$)

$$-q^{-\mu}(q^\mu - A)(q^\mu - B)f_0 = \big(-q^\mu + (A + B) - q^{-\mu}AB\big)f_0 = 0$$

Since A and B are diagonalizable with N different non-zero eigenvalues and $\sigma(A)\cap\sigma(B) = \emptyset$, we have $2N$ solutions for the indicial equation. In the first case, f_0 is an eigenvector for B with eigenvalue $b_i = q^\mu$, say $f_0 = f_i^B$. So

$$q^\mu = b_i, \qquad f_0 = f_i^B, \qquad 1 \le i \le N.$$

In the second case, we find that $(q^\mu - B)f_0$ is an eigenvector of A for the eigenvalue $a_i = q^\mu$, i.e.

$$q^\mu = a_i, \qquad f_0 = (a_i - B)^{-1}f_i^A, \qquad 1 \le i \le N,$$

where $Af_i^A = a_i f_i^A$.

In the first case we find the recursion

$$\begin{aligned}
\big(b_i q^{n+1} - (A + B) + b_i^{-1}q^{-1-n}AB\big)f_{n+1} = \big(b_i q^{n+1} - (C + q) + q^{-n}b_i^{-1}C\big)f_n \quad &\Longrightarrow \\
(b_i q^{n+1} - A)(1 - b_i^{-1}q^{-1-n}B)f_{n+1} = (b_i q^{n+1} - C)(1 - b_i^{-1}q^{-n})f_n \quad &\Longrightarrow \\
(A - b_i q^{n+1})(B - b_i q^{n+1})\, f_{n+1} = q\,(1 - q^n b_i)(C - b_i q^{n+1})f_n \quad &\Longrightarrow \\
f_{n+1} = q\,(1 - q^n b_i)(B - b_i q^{n+1})^{-1}(A - b_i q^{n+1})^{-1}(C - b_i q^{n+1})f_n& \\
= q^{n+1}\,(b_i; q)_{n+1}[B, A; C; qb_i; q]_{n+1}f_0 = q^{n+1}\,(b_i; q)_{n+1}[B, A; C; qb_i; q]_{n+1}f_i^B,&
\end{aligned}$$

and this gives the first set of solutions. Other choices of μ lead to the same solution up to a q-constant function.

In the second case we find the recursion

$$\begin{aligned}
\big(a_i q^{n+1} - (C + q) + a_i^{-1}q^{-n}C\big)f_n = \big(a_i q^{n+1} - (A + B) + a_i^{-1}q^{-1-n}AB\big)f_{n+1} \quad &\Longrightarrow \\
f_{n+1} = q(1 - a_i q^n)(B - a_i q^{n+1})^{-1}(A - a_i q^{n+1})^{-1}(C - a_i q^{n+1})f_n = &\\
q^{n+1}\,(a_i; q)_{n+1}[B, A; C; qa_i; q]_{n+1}(a_i - B)^{-1}f_i^A&
\end{aligned}$$

which gives the second solution.

Since the singularities of the solutions are all different by the genericity assumptions on the eigenvalues, linear independence follows. □

It is now a natural question to ask if one can develop an analoguous theory for matrix-valued little q-Jacobi polynomials using the solutions developed in this section. A first attempt is in [2]. For this one needs to study when a matrix-valued basic hypergeometric series terminates, and we can directly see that

$$ {}_2\Phi_1^1(A,B;C;q,z)f $$

is a polynomial of degree l if $f \notin \mathrm{Ker}\big((1-q^kA)(1-q^kB)\big)$, $1 \leq k < l$ and $f \in \mathrm{Ker}\big((1-q^lA)(1-q^lB)\big)$.

For an analogous theory of the little q-Jacobi function, we need to connect the solutions of Theorem 8.2 with the solutions of Theorem 8.3, i.e. we need the matrix-valued analogue of Watson's formulas (2.29), (2.30). First results on this approach can be found in [29].

8.3. **Exercises.**

1. Prove that the series in (8.3) converges in $\mathrm{End}(\mathbb{C}^N)$ (equipped with the operator norm) for $|z| < 1$, so that it defines an analytic function. Similarly for the series in (8.4).
2. Show that the series in (8.3) and (8.4) can be written in terms of standard basic hypergeometric series if we assume that the matrices A, B and C pairwise commute.
3. Determine more generally solutions in power series at 0 and ∞ for the equation
$$ (q-z)\mathrm{Id}f(q^{-1}z) + \big(Uz - C - q\mathrm{Id}\big)f(z) + (C - Vz)f(qz) = 0 $$
which reduces to (8.1) in case $V = AB$ and $U = A + B$.

Notes. A slightly more general situation is considered in [2, §4], but then only the analytic solutions are considered. Conflitti and Schlosser [11] consider also matrix-valued basic hypergeometric q-difference equations and hypergeometric differential equation analogues of Tirao [66], but the approach in [11] is different to ours. In [2] a family of 2×2-matrix valued little q-Jacobi polynomials is considered. In [2] matrix-valued analogues of the Askey-Wilson polynomials (of the subclass of the Chebyshev polynomials of the 2nd kind) are constructed using representation of quantum symmetric pairs as a q-analogue of [37], [38]. Using non-symmetric Askey-Wilson polynomials a 2×2-matrix-valued orthogonality is constructed by Koornwinder and Mazzoco [50]. It is not clear if these two approaches can be combined to study the matrix-valued Askey-Wilson polynomials more generally. A suitable spectral analysis of the matrix-valued q-difference operator has not been developed.

References

[1] N.I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner, 1965. 21, 23

[2] N. Aldenhoven, E. Koelink, A.M. de los Ríos, *Matrix-valued little q-Jacobi polynomials*, J. Approx. Theory **193** (2015), 164–183. 41, 45

[3] G.E. Andrews, R. Askey, *Enumeration of partitions: the role of Eulerian series and q-orthogonal polynomials*, pp. 3–26 in "Higher combinatorics", NATO Adv. Study Inst. Ser., Ser. C: Math. Phys. Sci. 31, Reidel, 1977. 8, 22

[4] G.E. Andrews, R.A. Askey, R. Roy, *Special Functions*, Cambridge Univ. Press, 1999. 2, 9

[5] R. Askey, M.E.H. Ismail, *Recurrence relations, continued fractions, and orthogonal polynomials*, Mem. Amer. Math. Soc. **49** (1984), no. 300. 21

[6] R. Askey, J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **54** (1985), no. 319. 35, 36, 40, 41

[7] W.N. Bailey, *Generalized Hypergeometric Series*, Cambridge Univ. Press, 1935, reprinted Hafner, 1964. 2, 4, 9

[8] J.M. Berezanskiĭ, *Expansions in Eigenfunctions of Selfadjoint Operators*, Transl. Math. Monographs 17, Amer. Math. Soc., 1968. 29

[9] H. Buchwalter, G. Cassier, *La paramétrisation de Nevanlinna dans le problème des moments de Hamburger*, Exposition. Math. **2** (1984), 155–178. 23

[10] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Math. and its Appl. 13, Gordon and Breach, 1978. 21

[11] A. Conflitti, M. Schlosser, *Noncommutative hypergeometric and basic hypergeometric equations*, J. Nonlinear Math. Phys. **17** (2010), 429–443. 45

[12] D. Damanik, A. Pushnitski, B. Simon, *The analytic theory of matrix orthogonal polynomials*, Surveys in Approx. Th. **4** (2008), 1–85. 41

[13] P. Deift, *Orthogonal Polynomials and Random Matrices: a Riemann-Hilbert Approach*, Courant Lect. Notes Math. 3, Courant Inst. Math. Sciences, NY University, 1999. 21

[14] N. Dunford, J.T. Schwartz, *Linear Operators II: Spectral Theory*, Interscience, 1963. 21, 22, 23, 24, 25

[15] A. Fitouhi, L. Dhaouadi, *On a q-Paley-Wiener theorem* J. Math. Anal. Appl. **294** (2004), 17–23. 29

[16] A. Fitouhi, L. Dhaouadi, *Positivity of the generalized translation associated with the q-Hankel transform*, Constr. Approx. **34** (2011), 453–472. 29

[17] G. Gasper, *q-Extensions of Erdélyi's fractional integral representations for hypergeometric functions and some summation formulas for double q-Kampé de Fériet series*, Contemp. Math. **254** (2000), 187–198. 32, 35

[18] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, 2nd ed., Cambridge Univ. Press, 2004. 2, 8, 9, 36, 37, 38, 39, 40, 41

[19] W. Groenevelt, *The Wilson function transform* Int. Math. Res. Not. **2003** (2003), 2779–2817. 41

[20] W. Groenevelt, *Bilinear summation formulas from quantum algebra representations*, Ramanujan J. **8** (2004), 383–416. 22

[21] W. Groenevelt, *The vector-valued big q-Jacobi transform*, Constr. Approx. **29** (2009), 85–127. 22

[22] W. Groenevelt, E. Koelink, J. Kustermans, *The dual quantum group for the quantum group analog of the normalizer of $SU(1,1)$ in $SL(2,\mathbb{C})$*, Int. Math. Res. Not. IMRN **2010** (2010), no. 7, 1167–1314. 29

[23] F.A. Grünbaum, I. Pacharoni, J. Tirao, *Matrix valued spherical functions associated to the complex projective plane*, J. Funct. Anal. **188** (2002), 350–441. 41

[24] L. Haine, P. Iliev, *Askey-Wilson type functions with bound states*, Ramanujan J. **11** (2006), 285–329. 41

[25] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge Univ. Press, 2009. 2, 9, 21

[26] M.E.H. Ismail, *A brief review of q-series*, lecture notes 2016 OPSFA summerschool. 9, 36

[27] M.E.H. Ismail, M. Rahman, *The associated Askey-Wilson polynomials*, Trans. Amer. Math. Soc. **328** (1991), 201–237. 36, 37, 38, 41

[28] M.E.H. Ismail, D. Stanton, *q-Taylor theorems, polynomial expansions, and interpolation of entire functions*, J. Approx. Theory **123** (2003), 125–146. 41

[29] N. Jaspers, *Connection formulas for hypergeometric series*, BSc-thesis, Radboud Universiteit, 2017. 2, 41, 45

[30] T. Kakehi, *Eigenfunction expansion associated with the Casimir operator on the quantum group $SU_q(1,1)$*, Duke Math. J. **80** (1995), 535–573. 26, 29

[31] R. Koekoek, P.A. Lesky, R.F. Swarttouw, *Hypergeometric Orthogonal Polynomials and their q-Analogues*, Springer, 2010. 2, 35, 41

[32] R. Koekoek, R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, online at `http://aw.twi.tudelft.nl/~koekoek/askey.html`, Report 98-17, Technical University Delft, 1998. 2, 20, 21, 35, 36, 41

[33] H.T. Koelink, *A basic analogue of Graf's addition formula and related formulas*, Integral Transform. Spec. Funct. **1** (1993), 165–182. 29

[34] E. Koelink, *One-parameter orthogonality relations for basic hypergeometric series*, Indag. Math. (N.S.) **14** (2003), 423–443. 29

[35] E. Koelink, *Spectral theory and special functions*, pp. 45–84 in "Laredo Lectures on Orthogonal Polynomials and Special Functions" (eds. R. Álvarez-Nodarse, F. Marcellán, W. Van Assche), Nova Science Publishers, 2004. 21, 22, 23, 24, 25, 27, 29

[36] E. Koelink, *Applications of spectral theory to special functions*, to appear in Lecture Notes of the London Math. Soc. (ed. H. Cohl, M.E.H. Ismail), Cambridge U. Press, lecture notes for OPSFA Summerschool 6, July 2016, `arXiv:1612.07035`. 41

[37] E. Koelink, M. van Pruijssen, P. Román, *Matrix valued orthogonal polynomials related to* $(\mathrm{SU}(2)\times\mathrm{SU}(2), \mathit{diag})$, Int. Math. Res. Not. **2012** (2012), 5673–5730. 41, 45

[38] E. Koelink, M. van Pruijssen, P. Román, *Matrix valued orthogonal polynomials related to* $(\mathrm{SU}(2)\times\mathrm{SU}(2), \mathit{diag})$*, II*, Publ. RIMS Kyoto **49** (2013), 271–312. 41, 45

[39] E. Koelink, A.M. de los Ríos, P. Román, *Matrix-valued Gegenbauer-type polynomials*, Constr. Approx. **46** (2017), 459–487. 41

[40] E. Koelink, H. Rosengren, *Transmutation kernels for the little q-Jacobi function transform*, Rocky Mountain J. Math. **32** (2002), 703–738. 2, 35

[41] E. Koelink, J.V. Stokman, *The big q-Jacobi function transform*, Constr. Approx. **19** (2003), 191–235. 22, 29

[42] E. Koelink, J.V. Stokman, with an appendix by M. Rahman, *Fourier transforms on the quantum $SU(1,1)$ group*, Publ. Res. Inst. Math. Sci., Kyoto Univ. **37** (2001), 621-715. 26, 28, 29, 30

[43] E. Koelink, J.V. Stokman, *The Askey-Wilson function transform scheme*, pp. 221-241 in "Special Functions 2000: Current Perspective and Future Directions" (eds. J. Bustoz, M.E.H. Ismail, S.K. Suslov), NATO Science Series II, Vol. 30, Kluwer, 2001. 26, 29

[44] E. Koelink, J.V. Stokman, *The Askey-Wilson function transform*, Intern. Math. Res. Notices **2001**, 22, 1203-1227. 2, 40, 41

[45] H.T. Koelink, R.F. Swarttouw, *On the zeros of the Hahn-Exton q-Bessel function and associated q-Lommel polynomials* J. Math. Anal. Appl. **186** (1994), 690–710. 29

[46] S. Kolb, Quantum symmetric Kac-Moody pairs, Adv. Math. **267** (2014), 395–469. 20

[47] T. Koornwinder, *A new proof of a Paley-Wiener type theorem for the Jacobi transform*, Ark. Mat. **13** (1975), 145–159. 31, 32

[48] T.H. Koornwinder, *Jacobi functions and analysis on noncompact semisimple Lie groups*, pp. 1–85 in "Special Functions: Group Theoretical Aspects and Applications" (eds. R.A. Askey, T.H. Koornwinder, W. Schempp) Math. Appl., Reidel, 1984. 10, 29, 31, 35

[49] T.H. Koornwinder, *Compact quantum groups and q-special functions*, p. 46–128 in "Representations of Lie groups and quantum groups" (eds. V. Baldoni, M.A. Picardello), Pitman Res. Notes Math. Ser., 311, Longman Sci. Tech., 1994. 22

[50] T.H. Koornwinder, M. Mazzocco, *Dualities in the q-Askey scheme and degenerated DAHA*, `arXiv:1803.02775`. 20, 22, 45

[51] T.H. Koornwinder, R.F. Swarttouw, *On q-analogues of the Fourier and Hankel transforms*, Trans. Amer. Math. Soc. **333** (1992), 445–461. 7, 29

[52] D.R. Masson, J. Repka, *Spectral theory of Jacobi matrices in $\ell^2(\mathbb{Z})$ and the $\mathfrak{su}(1,1)$ Lie algebra*, SIAM J. Math. Anal. **22** (1991), 1131–1146. 23, 24

[53] M. Noumi, J.V. Stokman, *Askey-Wilson polynomials: an affine Hecke algebra approach*, pp. 111–144 in "Laredo Lectures on Orthogonal Polynomials and Special Functions" (eds. R. Álvarez-Nodarse, F. Marcellán, W. Van Assche), Nova Science Publishers, 2004. 36

[54] F.W.J. Olver, *Asymptotics and Special Functions*, AKP Classics, A.K. Peters, 1997. 21

[55] E.D. Rainville, *Special Functions*, Macmillan, 1960. 2, 9

[56] P. Román, S. Simondi, *Solutions at infinity of the generalized matrix-valued hypergeometric equation*, Appl. Math. Lett. **23** (2010), 39–43. 41

[57] H. Rosengren, *A new quantum algebraic interpretation of the Askey-Wilson polynomials*, Contemp. Math. **254** (2000), 371–394. 22

[58] W. Rudin, *Functional Analysis*, McGraw-Hill, 1973. 25

[59] S.N.M. Ruijsenaars, *A generalized hypergeometric function satisfying four analytic difference equations of Askey-Wilson type*. Comm. Math. Phys. **206** (1999), 639–690; *A generalized hypergeometric function. II. Asymptotics and D_4 symmetry, III. Associated Hilbert space transform*, Comm. Math. Phys.**243** (2003), 389–412, 413–448. 41

[60] B. Simon, *The classical moment problem as a self-adjoint finite difference operator*, Adv. Math. **137** (1998), 82–203. 21, 22, 23

[61] L.J. Slater, *Generalized Hypergeometric Functions*, Cambridge Univ. Press, 1966. 2, 5, 9

[62] J.V. Stokman, *An expansion formula for the Askey-Wilson function*, J. Approx. Theory **114** (2002), 308–342. 41

[63] J.V. Stokman, *The c-function expansion of a basic hypergeometric function associated to root systems* Ann. of Math. (2) **179** (2014), 253–299. 41

[64] S.K. Suslov, *Some orthogonal very-well-poised ${}_8\phi_7$-functions that generalize Askey-Wilson polynomials*, Ramanujan J. **5** (2001), 183–218. 41

[65] N.M. Temme, *Special Functions*, Wiley, 1996. 2, 8, 9

[66] J. Tirao, *The matrix-valued hypergeometric equation*, Proc. Natl. Acad. Sci. USA **100** (2003), 8138–8141. 41, 45

[67] L.L. Vaksman, *Quantum Bounded Symmetric Domains*, Transl. Math. Monographs 238, AMS, 2010. 22

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