Affine Hecke Algebra: Talk 1

February 17, 2023

Contents

Cherednik's polynomial representation of the affine Hecke algebra [7] (Chap-ter 4 up to an including (4.3.10))

1 Notes AH talk 1

These are some rough notes that are made when preparing for the talk, they are far from polished and may contain some errors.

2 Introduction

The main goal of this talk is to introduce the Cherednik's polynomial representation of the affine Hecke algebra and go over some results concerning the structure of these algebras. Affine Hecke algebras and the basic representation are of key importance for the existence of the Macdonald polynomials. The references of this talk are mostly [HMC03] but occasionally we use [HSE17].

2.1 Structure of the talk

We first recall some notions and fix notation regarding affine root systems and affine Hecke algebra. Essentially most of the theory relies on Lustig's relation, we show the relation andconsider it's implication to the structure of the affine Hecke algebra. Having this in place we introduce Cherednik's basic representation and show some first properties.

3 Reminder on affine root systems and their affine Hecke algebras

We recall that affine root systems come in duality. Let V be a real finite dimensional vector space and let us fix a triple of pairs

$$(S, S'), \qquad (L, L'), \qquad (R, R').$$

Where S and S' are in duality. We know that S equal $S(R)$ or $S(R)^\vee$ for some finite root system R. Fix the parameters; $q \in (0,1)$ and $\{\tau_i\}_{i\in I}$ with with $\tau_i = \tau_j$ is s_i is conjugated to s_j in W and a field K containing these parameters. We denote by $W_S = W_0 \ltimes t(\mathbb{Q}^\vee)$ the *affine Weyl group* associated to S and $W = W_0 \rtimes t(L')$ the *extended affine Weyl group* associated to S. The *braid group* $\mathfrak{B}$ associated to W is the group with generators $T(w)$, $w \in W$ relations

$$T(w)T(w') = T(ww') \quad if \quad l(w) + l(w') = l(ww') \qquad w, w' \in W.$$

We will refer to these relation as the *braid relations.* We use the notation $J \cong t(L')/t(Q^\vee)$. The braid group can also be presented in another following two ways. The first one; it has generators U_j, T_i with $i \in I$ and $j \in J$ and relations; $U_jU_k = U_{j+k}, \quad U_jT_iU_{-j} = T_{i+j}$ with $j \in J$ and $i \in I$, together with the braid relations.

Now for the second presentation, let $\mathfrak{B}_0 \subset \mathfrak{B}$ be the subgroup generated by T_i with $i \in I_0$. Then $\mathfrak{B}$ is presented by generators Y^λ and $\mathfrak{B}_0$ subject to the relation

$$T_i^\epsilon Y^{s_i\lambda'} T_i = Y^{\lambda'},$$

whenever $\langle \lambda', \alpha_i \rangle = 0$ or 1, where

$$\epsilon = \begin{cases} +1 & \text{if} \langle \lambda', \alpha_i \rangle = 1, \\ -1 & \text{if} \langle \lambda', \alpha_i \rangle = 0. \end{cases}$$

Lastly we consider two group algebras, $A = K[L]$ and $A' = K[L']$. Let us consider the group algebra A, it's elements are denoted by $e^{\lambda'}$ with $\lambda' \in L'$. The affine Weyl group acts on A by $t(\lambda')v \cdot e^\mu = q^{-\langle \lambda', \mu \rangle} e^{v\mu}$ for $t(\lambda')v \in W$, the action is faithful. Furthermore A and A' contain multiplication operators, for multiplication operators with single elements we denote them by X^λ and $Y^{\lambda'}$, $\lambda \in L$ and $\lambda' \in L'$, and of course we can linearly extend these multiplication operators.

3.1 Hecke algebras.

Definition 3.1 (Hecke algebra)**.** The *Hecke algebra* $\mathfrak{H}$ corresponding to the extended affine Weyl group W, is the group algebra $K[\mathcal{B}]$ quotiented by the ideal generated by the elements

$$(T_i - \tau_i)(T_i + \tau_i^{-1}), \qquad i \in I.$$

Remark 3.2. A calculation shows that this is equivalent to the relation

$$T_i - \tau_i = T_i^{-1} - \tau_i^{-1}, \qquad i \in I.$$

We call this relation the *Hecke relation.*

The first question we want to answer, regarding the structure of the Hecke algebra, is how it is generated as a vector space.

Lemma 3.1. *The elements $T(w)$, $w \in W$ form a $K-$basis of $\mathfrak{H}$.*

Proof. Let $\mathfrak{H}_1$ be the subspace spanned by the elements $T(w)$ with $w \in W$. By the Hecke relations we have $T_i\mathfrak{H} \subset \mathfrak{H}_1$ for all $i \in I$, furthermore $U_jT(w) = T(u_jw)$ so $\mathfrak{H}_1\mathfrak{H} \subset \mathfrak{H}_1$. As $1 \in \mathfrak{H}$ this shows that $\mathfrak{H} = \mathfrak{H}_1$. Now in a sense the proof of linear independence relies on the fact that there is a faithfull representation of $\mathfrak{H}$ on $K[\mathfrak{B}]$, such proves are standard and for the sake of time we skip these proves. They can by found in [HMC03] (4.1.4), (4.1.5) & (4.1.6). □

Theorem 3.1. *The elements $T(w)Y^{\lambda'}$ where $\lambda' \in L'$ and $w \in W_0$ form a $K-$basis of $\mathfrak{H}$.*

Proof. Can be found in [HMC03] (4.2.7) □

As a result we obtain the vector space isomorphism $\mathfrak{H} \cong A' \otimes_K \mathfrak{H}_0$

4 The Lusztig's relation

The first part of mathematics we are going to show it Lusztig's relation, we can think of this relation as how the T_i and $Y^{\lambda'}$ commute with respect to the action of s_i on A'. It has important implication with respect to our knowledge of the structure of the Hecke algebra and Cherednik's basic representation.

Lemma 4.1 (Lusztig relation)**.** *For any $i \in I$ and $\lambda' \in L'$, the following relation holds;*

$$Y^{\lambda'}T_i - T_iY^{s_i\lambda'} = (\tau_i - \tau_i^{-1})\frac{s_i - 1}{Y^{-\alpha_i^\vee} - 1}Y^{\lambda'}.$$

Definition 4.1. We introduce the following notation;

$$b(t, u; x) = \frac{t - t^{-1} + (u - u^{-1})x}{1 - x^2}, \qquad b_i = b(\tau_i, \tau_i; e^{a_i}), \qquad b_i(X) = b(\tau_i, \tau_i; X^{a_i}).$$

Proof. A key observation is that if the relation holds for fixed T_i and two elements $Y^{\lambda'}$ and $Y^{\mu'}$, that this relation also holds for T_i and $Y^{\lambda'+\mu'}$. So we want to choose an effective generating set of $K[L']$. Let us for simplicity assume that $\langle L', \alpha_i\rangle = \mathbb{Z}$ under this consideration L' is generated by the elements for which $\langle \lambda', \alpha_i\rangle = 0$ and $\langle \mu', \alpha_i\rangle = 1$. So it suffices to show the statement for these cases

- **Case 1:** $\langle \lambda', \alpha_i\rangle = 0$. Then the right hand side of the equation vanishes, and the left hand side vanishes because of the defining relations for the braid group; $T_iY^{\lambda'} = Y^{s_i\lambda'}T_i = Y^{\lambda'}T_i$.

- **Case 2:** $\langle \lambda', \alpha_i\rangle = 1$. From [HMC03] (3.2.6) it follows that

  $$T_iY^{s_i\lambda'} = Y^{\lambda'}T_i^{-1} = Y^{\lambda'}(T_i - \tau_i + \tau_i^{-1}),$$

  which implies that

  $$Y^{\lambda'}T_i - T_iY^{s_i\lambda'} = (\tau_i - \tau_i^{-1})Y^{\lambda'}.$$

  Because $s_i\lambda = \lambda' - \alpha_i^\vee$, we have

  $$(\tau_i - \tau_i^{-1})\frac{Y^{\lambda'-\alpha_i^\vee} - Y^{\lambda'}}{Y^{-\alpha_i^\vee} - 1} = (\tau_i - \tau_i^{-1})Y^{\lambda'},$$

  which shows the claim.

By our discussion this finishes the proof. The only thing that we swept under the rug is the generation argument. For the sake of completeness let us record this argument here. Fix $i \in I_0$ and assume that the claim holds for $\lambda', \mu' \in L'$ then

$$\begin{aligned}
Y^{\lambda'-\mu'}T_i - T_iY^{\lambda'-\mu'} &= Y^{\mu'}\left(Y^{\lambda'}T_iY^{s_i\mu'} - Y^{\mu'}T_iY^{s_i\lambda'}\right)Y^{-s_i\mu'} \\
&= Y^{-\mu'}\left(Y^{\lambda'}T_i - T_iY^{s_i\lambda'}\right)Y^{s_i\mu'} \\
&\quad + \left(T_iY^{s_i\mu'} - Y^{\mu'}T_i\right)Y^{s_i\lambda'}Y^{-s_i\mu'} \\
&= Y^{-\mu'}b_i(X)(Y^{\lambda'} - Y^{s_i\lambda'})Y^{s_i\mu'} \\
&\quad - b_i(X)(Y^{\mu'} - Y^{s_i\mu'})Y^{s_i\lambda'}Y^{s_i\mu'} \\
&= b_i(X)(Y^{\lambda'-\mu'} - Y^{s_i(\lambda'-\mu')}).
\end{aligned}$$

□

Remark 4.2. Lustig's relation can be extended to elements of A' as seen inside $\mathfrak{H}$,

$$f(Y)T_i - T_i(s_i f)(Y) = b(\tau_i, \tau_i'; Y^{-\alpha_i^\vee})(f(Y) - (s_i f)(Y)) \tag{1}$$

One shows this, by considering the equation term wise. Furthermore by induction one deduces from (1) that

$$T(w)Y^{\lambda'} = \sum_{v \leqslant w} g_{vw}(Y)T(v), \qquad g_{vw} \in A' \tag{2}$$

with $g_{ww} = wY^{\lambda}$.

5 The polynomial and Cherednik's basic representation

By the decomposition $\mathfrak{H} \cong A' \otimes_K \mathfrak{H}_0$ we allow ourselves to induce left $\mathfrak{H}_0$ modules to left $\mathfrak{H}$ modules. If M is a left $\mathfrak{H}_0$ module we define the left $\mathfrak{H}$ module

$$\mathrm{Ind}(M) := \mathfrak{H} \otimes_{\mathfrak{H}_0} M \cong \mathfrak{H}_0 \otimes_{\mathfrak{H}_0} A' \otimes_K M \cong A' \otimes_K M,$$

where the module action is given by

$$T_i \cdot (g(Y) \otimes m) = g(Y) \otimes T_i \cdot m = T_i g(Y) \otimes m,$$

by the Lusztig relation it follows that $T_i g(Y) = (s_i f)(Y)T_i + (f - s_i f)b(\tau_i; e^{-\alpha_i^\vee})$, which means that

$$\begin{aligned} T_i g(Y) \otimes m &= \Big((s_i f)(Y)T_i + (f - s_i f)b(\tau_i; e^{-\alpha_i^\vee})\Big) \otimes m &(3)\\ &= (s_i f)(Y) \otimes T_i m + (f - s_i f)b(\tau_i; e^{-\alpha_i^\vee}) \otimes m. &(4)\end{aligned}$$

We want to induce a one dimensional representation of $\mathfrak{H}_0$, the following Lemma shows the existence.

Lemma 5.1. *Let $w \in W$ and $s_{i_1} \dots s_{i_p} = w$ be a reduced expression, then $\tau_w := \tau_{i_1} \dots \tau_{i_p}$ does not depend on reduced expression*

Proof. We show this by induction, the case $p = 1$ is clear. Now is $s_{i_1} \dots s_{i_p} = s_{j_1} \dots s_{j_p}$ then $s_{i_1} \dots s_{i_p - 1} = s_{j_1} \dots \widehat{s_{j_r}} \dots s_{j_p}$, meaning that

$$\begin{aligned} s_{j_1} \dots \widehat{s_{j_r}} \dots s_{j_p} s_{i_p} &= s_{j_1} \dots s_{j_p} & \implies\\ s_{j_1} \dots \widehat{s_{j_r}} \dots s_{j_p} s_{i_p} s_{j_p} \dots s_{j_r+1} &= s_{j_1} \dots s_{j_r} & \implies\\ s_{j_r+1} \dots s_{j_p} s_{i_p} s_{j_p} \dots s_{j_r+1} &= s_{j_r} \end{aligned}$$

Hence s_{i_p} and s_{j_r} are conjugate in W, by induction the result follows. □

Seen as a map $\mathfrak{H}_0 \to K$, we note that the Hecke relation is immediate. As a result, the assignment $T_i \mapsto \tau_i$ gives rise to an one-dimensional representation of $\mathfrak{H}_0$ on K that we may induce to $\mathfrak{H}$.

Lemma 5.2. *There is a representation β' of $\mathfrak{H}_0$ on A' that acts as*

$$\beta'(T_i) = \tau_i s_i + b(\tau_i, \tau_i; X^{-\alpha_i^\vee})(1 - s_i),$$

for all $i \in I_0$, where $X^{-a_i^\vee}$ is the multiplication by $e^{-a_i^\vee}$ operator.

Proof. The representation $T_i \mapsto \tau_i$ is a representation of $\mathfrak{H}_0$ onto K and

$$Ind(K) \cong A' \otimes_K K \cong A'.$$

Now looking at equation (3) we see that for all $i \in I_0$

$$\beta'(T_i) = \tau_i s_i + b(\tau_i, \tau_i; X^{-\alpha_i^\vee})(1 - s_i).$$

Which is the action restricted to $\mathfrak{H}_0$. □

The following representation is due to Cherednik, it makes use of the induction and duality of W_0 in an interesting way.

Theorem 5.1 (Cherednik's basic representation)**.** *There exists a representation β of $\mathfrak{H}$ on A such that*

$$\beta(T_i) = \tau_i s_i + b_i(X)(1 - s_i) \tag{5}$$
$$\beta(U_j) = u_j, \tag{6}$$

for $i \in I$ and $j \in J$, where X^{a_i} is the multiplication by e^{a_i} operator.

Proof. By interchanging R with $R^\vee$, we do not change W_0. Hence for $i \neq 0$ it follows that equation (5) gives rise to a representation of $\mathfrak{H}_0$, where we identify R with $R^\vee$ by $\alpha_i \mapsto -\alpha_i^\vee$.

(i) **Claim:** The relation $U_j T_i U_{-j} = T_{i+j}$ is preserved for all $i \in I, j \in J$.
We have

$$\begin{aligned}
\beta(U_j)\beta(T_i)\beta(U_{-j}) &= u_j\left(\tau_i s_i + b_i(X)(1 - s_i)\right) u_{-j} \\
&= \tau_{i+j} s_{i+j} + u_j\left(b(\tau_i; e^{a_i})\right) u_{-j}(1 - s_{i+j}) \\
&= \tau_{i+j} s_{i+j} + u_j\left(b(\tau_i; e^{a_{i+j}})\right)(1 - s_{i+j}) \\
&= \beta(T_{i+j}).
\end{aligned}$$

(ii) **Claim:** Braid & Hecke relations for T_0 hold.
The braid and Hecke relations only involve at most two roots, therefore this is a claim about the root systems generated by a_0 and a_i, for $i \in I$. If this is an rank 1 root system there is nothing to prove, if it is a rank 2 root-system the claim follows from the polynomial representation.

As a result the assignment defined by equations (5) and (6) gives rise to a representation. □

Before we continue we need a preliminary Lemma.

Lemma 5.3. *Let K be a field and $\varphi_1, \ldots, \varphi_n$ distinct automorphisms, then they are linearly independent as functions $K \to K$.*

Proof. Result in Galois theory. □

The next goal is to show that the representation β is faithful. The following Lemma shows even more, and in a sense also motivates the definition of the double affine Hecke algebra.

Lemma 5.4. *The operators $X^\mu \beta(T(w))$ where $\mu \in L$ and $w \in W$ are linearly independent over K*

Proof. Let $w \in W$, then we first want to investigate the expression of $\beta(T(w))$ inside $\mathrm{End}(K[L])$. Let $w = u_j s_{i_1} \ldots s_{i_p}$ be a reduced expression for w then $\beta(T(w)) = u_j \beta(T_{i_1}) \ldots \beta(T_{i_p})$. So we may write

$$\beta(T(w)) = \sum_{v \leqslant w} f_{vw}(X) v.$$

By investigating the highest order therm, we see that

$$f_{ww}(X) = \underbrace{\prod_i \tau_i}_{\in \mathbb{C}} + \underbrace{\prod b_i(X)}_{\notin \mathbb{C}} \neq 0$$

Now suppose that there is a linear relation between the operators $X^{\mu_w} \beta(T(w))$, then there will be a relation of the form

$$\sum_{w \in W} g_w(X) \beta(T(w)) = 0,$$

with $g_w \in A$ and finitely many nonzero. Substituting our expression for $\beta(T(w))$ gives

$$\sum_{v, w \in W,\, v \leqslant w} g_w(X) f_{vw}(X) v = 0$$

Since the element $v \in W$ are distinct automorphisms of the field $K(L)$ it follows from Lemma (5.3) that for each $v \in W$

$$\sum_{v \leqslant w} g_w f_{vw} = 0.$$

Now take v with $g_v \neq 0$ and maximal with this property in the Bruhat ordering, then it follows that $g_v f_{vv} = 0$. Meaning that $f_{vv} \neq 0$, a contradiction. □

Corollary 5.1. *The representation β is faithfull.*

Proof. Take $\mu = 0$ in the previous theorem. □

Theorem 5.2. *The centre of $\mathfrak{H}$ is $A'_0(Y)$*

Proof. Recall the decomposition $\mathfrak{H} \cong \mathfrak{H}_0 \otimes_K A'$, we note that we only have to check that any element $f \in A'_0$ commutes with the T_i with $i \in I_0$. By Remark (4.2) this follows. Next let $f \in A'$, then by Lustig's relation

$$T_i f(Y) - (s_i f)(Y) T_i = g(Y),$$

for some $g \in A'$. If f is central, we have

$$g(Y) = (f(Y) - (s_i f)(Y) T_i = 0,$$

so that $s_i f = f$. In particular f is W_0 invariant. We remain to argue that any central element in fact is an element in $K(L')$. Let

$$z = \sum_{w \in W_0} f_w(Y) T(w),$$

with $f_w \in K[L']$ and let $\lambda' \in L'$ be regular then

$$\sum_{w \in W_0} Y^{\lambda'} f_w(Y) T(v) = \sum_{w \in W_0} f_w(Y) T(v) Y^{\lambda'}$$

since z is central. By (2) $T(w) Y^{\lambda}$ is of the form

$$\sum_{v \leqslant w} g_{vw}(Y) T(v), \qquad g_{vw} \in A'.$$

Substitution this back gives us

$$\sum_{v \in W_0} Y^{\lambda'} f_v(Y) T(v) = \sum_{w \geqslant v} g_{vw}(Y) f_w(Y) T(v),$$

by comparing coefficients we obtain

$$e^{\lambda'} f_v = \sum_{w \geqslant v} g_{vw} f_w, \qquad v \in W_0.$$

One recognises the right hand side of the equation as matrix vector multiplication of $G = (g_{vw})$ and (f_v). Since G is triangular, and λ' is regular all the eigenvalues are distinct and lie on the diagonal of G. Since f is the unique, up to scalar multiple, eigenvector with eigenvalue $e^{\lambda'}$ this implies that $f_v = 0$ if $v \neq 1$ meaning that $z = f_1(Y)$. By the discussion this finishes the proof. □

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