This is Section 6 of the paper
The dual quantum group for the quantum group analogue of the normalizer of SU(1,1) in SL(2,C) by Wolter Groenevelt, Erik Koelink, Johan Kustermans, `arXiv:0905.2830`

6. Results for special functions of basic hypergeometric type

This section is separately readable from the remainder of the paper. This section is meant to give a couple of examples of rather complicated identities for special functions of basic hypergeometric type ${}_1\varphi_1$ and type ${}_2\varphi_1$, see [17]. We assume that the reader of this section is familiar with the notation for basic hypergeometric series [17], but the definition is recalled in Appendix B. In the first subsection we introduce the notation for special functions, and we recall some elementary properties. The first subsection introduces notation and special functions that are used throughout the paper, whereas the following subsections give explicit highly non-trivial results for these special functions. These identities follow from the quantum group theoretic interpretation.

6.1. **Definition of some special functions.** The set of natural numbers (without 0) will be denoted by $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We write, as in Section 3, $I_q = -q^{\mathbb{N}} \cup q^{\mathbb{Z}}$. We use the following functions frequently.

Definition 6.1. *(i)* $\chi\colon -q^{\mathbb{Z}} \cup q^{\mathbb{Z}} \to \mathbb{Z}$ *such that* $\chi(x) = \log_q(|x|)$ *for all* $x \in -q^{\mathbb{Z}} \cup q^{\mathbb{Z}}$*;*
(ii) $\kappa\colon \mathbb{R} \to \mathbb{R}$ *such that* $\kappa(x) = \mathrm{sgn}(x)\, x^2$ *for all* $x \in \mathbb{R}$*;*
(iii) $\nu\colon -q^{\mathbb{Z}} \cup q^{\mathbb{Z}} \to \mathbb{R}^+$ *such that* $\nu(t) = q^{\frac{1}{2}(\chi(t)-1)(\chi(t)-2)}$ *for all* $t \in -q^{\mathbb{Z}} \cup q^{\mathbb{Z}}$*;*
(iv) $s\colon \mathbb{R}_0 \times \mathbb{R}_0 \to \{-1, 1\}$ *is defined such that*

$$s(x,y) = \begin{cases} -1 & \text{if } x > 0 \text{ and } y < 0 \\ 1 & \text{if } x < 0 \text{ or } y > 0 \end{cases}$$

for all $x, y \in \mathbb{R}_0 = \mathbb{R}\backslash\{0\}$.
(v) $\mu : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ *such that* $\mu(y) = \frac{1}{2}(y + y^{-1})$ *for all* $y \in \mathbb{C} \setminus \{0\}$.

For $a, b, z \in \mathbb{C}$, we define

$$\Psi\left(\begin{matrix} a \\ b \end{matrix}\; ; q,\, z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b\,q^n;q)_\infty}{(q\,;q)_n}\,(-1)^n\, q^{\frac{1}{2}n(n-1)}\, z^n \;= (b;q)_\infty\;{}_1\varphi_1\left(\begin{matrix} a \\ b \end{matrix}\; ; q, z\right). \tag{6.1}$$

This is an entire function in a, b and z. Here we have used the standard notation for basic hypergeometric series [17], or see Appendix B.1.

We use the normalization constant $c_q = (\sqrt{2}\, q\, (q^2, -q^2; q^2)_\infty)^{-1}$. Then the following definition is [30, Def. 3.1], and the notations as in Definition 6.1 are used.

Definition 6.2. *If* $p \in I_q$*, we define the function* $a_p\colon I_q \times I_q \to \mathbb{R}$ *such that* a_p *is supported on the set* $\{\, (x,y) \in I_q \times I_q \mid \mathrm{sgn}(xy) = \mathrm{sgn}(p)\,\}$ *and is given by*

$$\begin{aligned} a_p(x,y) \,&= c_q\, s(x,y)\,(-1)^{\chi(p)}\,(-\mathrm{sgn}(y))^{\chi(x)}\,|y|\;\nu(py/x)\;\sqrt{\frac{(-\kappa(p), -\kappa(y); q^2)_\infty}{(-\kappa(x); q^2)_\infty}} \\ &\quad\times \Psi\left(\begin{matrix} -q^2/\kappa(y) \\ q^2\kappa(x/y) \end{matrix}\; ; q^2,\, q^2\kappa(x/p)\right) \end{aligned}$$

for all $(x,y) \in I_q \times I_q$ *satisfying* $\mathrm{sgn}(xy) = \mathrm{sgn}(p)$.

The functions $a_p(x,y)$ for $p,x,y \in I_q$ have been introduced in [30, §3], motivated by their occurrence as Clebsch-Gordan coefficients. Depending on the choices of the sign, these functions can be identified with well-known special functions of basic-hypergeometric type. In particular, for $\mathrm{sgn}(x) = \mathrm{sgn}(y)$ the functions $a_p(x,y)$ can be identified with the q-Laguerre polynomials in case $\mathrm{sgn}(x) = \mathrm{sgn}(y) = -$ and with the associated big q-Bessel functions in case $\mathrm{sgn}(x) = \mathrm{sgn}(y) = +$, see [10]. The q-Laguerre polynomials correspond to an indeterminate moment problem, and the big q-Bessel functions form a complementary orthogonal basis to the orthogonal polynomials for an explicit solution to the moment problem corresponding to Ramanujan's ${}_1\psi_1$-summation formula, see [10] for details. For $\mathrm{sgn}(x) = -\mathrm{sgn}(y)$, the functions $a_p(x,y)$ can be matched with Al-Salam–Carlitz polynomials and q-Charlier polynomials, see [27] for their definition.

For completeness we recall the orthogonality properties of these functions, see [30, Prop. 3.2, 3.3]. For $\theta \in -q^{\mathbb{Z}} \cup q^{\mathbb{Z}}$ we define $\ell_\theta = \{\, (x,y) \in I_q \times I_q \mid y = \theta x \,\}$.

Proposition 6.3. *Consider* $\theta \in -q^{\mathbb{Z}} \cup q^{\mathbb{Z}}$. *Then the family* $\{\, a_p|_{\ell_\theta} \mid p \in I_q$ *such that* $\mathrm{sgn}(p) = \mathrm{sgn}(\theta) \,\}$ *is an orthonormal basis for* $l^2(\ell_\theta)$. *In particular,*

$$\sum_{x \in I_q \text{ so that } \theta x \in I_q} a_p(x,\theta x)\, a_r(x,\theta x) = \delta_{p,r}, \qquad p,r \in I_q.$$

Proposition 6.4. *Consider* $\theta \in -q^{\mathbb{Z}} \cup q^{\mathbb{Z}}$ *and define* $J = q^{\mathbb{Z}} \subset I_q$ *if* $\theta > 0$ *and* $J = -q^{\mathbb{N}} \subset I_q$ *if* $\theta < 0$. *For every* $(x,y) \in \ell_\theta$ *we define the function* $e_{(x,y)} : J \to \mathbb{R}$ *such that* $e_{(x,y)}(p) = a_p(x,y)$ *for all* $p \in J$. *Then the family* $\{\, e_{(x,y)} \mid (x,y) \in \ell_\theta \,\}$ *forms an orthonormal basis for* $l^2(J)$. *In particular,*

$$\sum_{p \in J} a_p(x,\theta x)\, a_p(y,\theta y) = \delta_{x,y}, \qquad x,y \in I_q.$$

For convenience we state the following symmetry relations for the functions $a_p(x,y)$, see [30, Prop. 3.5]:

$$\begin{aligned} a_p(x,y) &= (-1)^{\chi(yp)} \mathrm{sgn}(x)^{\chi(x)} \left|\frac{y}{p}\right| a_y(x,p); \\ a_p(x,y) &= \mathrm{sgn}(p)^{\chi(p)} \mathrm{sgn}(x)^{\chi(x)} \mathrm{sgn}(y)^{\chi(y)} a_p(y,x); \\ a_p(x,y) &= (-1)^{\chi(xp)} \mathrm{sgn}(y)^{\chi(y)} \left|\frac{x}{p}\right| a_x(p,y). \end{aligned} \tag{6.2}$$

6.2. **Summation and transformation formulas for $a_p(x,y)$.** The functions $a_p(x,y)$, which as noted above are closely related to some well-known orthogonal polynomials of basic hypergeometric type, are used in the definition of the so-called multiplicative unitary W, see (7.10). In the general theory of locally compact groups, the multiplicative unitary W plays an important role. In particular, it satisfies the pentagonal equation, a relation that is essential in proving Propositions 4.10 and 4.15. The result in these propositions lead to operator identities in suitable Hilbert spaces, and taking matrix coefficients then essentially lead to Theorems 6.5 and 6.8 in this section. The details of the proofs are given in Section 11.1.

6.2.1. *Representing the structure of $\hat{M}$.* By taking the non-trivial structure constants of Proposition 4.10 and considering matrix coefficients at both sides we obtain the following theorem.

Theorem 6.5. *For $p_1, p_2, r_1, r_2 \in I_q$, $l, n, m \in \mathbb{Z}$, $\varepsilon, \eta \in \{\pm\}$ and with $z \in I_q$ so that $\mathrm{sgn}(z) = \varepsilon$ and $\varepsilon\eta p q^l z \in I_q$ and with $w \in I_q$ so that $\mathrm{sgn}(w) = \varepsilon\mathrm{sgn}(r_1 p_1)$ and $\varepsilon\eta\mathrm{sgn}(r_1p_1r_2p_2)pq^{l+m+n}w \in I_q$ we have*

$$\sum_{\substack{x\in I_q \text{ so that } \mathrm{sgn}(x)=\mathrm{sgn}(r_1p_1)\\ \text{and } |x|\mathrm{sgn}(r_2p_2)pq^{2l+m+n}\in I_q}} a_z(x,w)\, a_x(r_1,p_1)\, a_{|x|\mathrm{sgn}(r_2p_2)pq^{2l+m+n}}(r_2,p_2)$$
$$\times\ a_{\varepsilon\eta pq^l z}(|x|\mathrm{sgn}(r_2p_2)pq^{2l+m+n}, \mathrm{sgn}(r_1p_1r_2p_2)\varepsilon\eta pq^{l+m+n}w) = \delta_{|\frac{r_1}{r_2}|p,q^{-2l-m}}\, \delta_{|\frac{p_1}{p_2}|p,q^{-2l-2m-n}}$$
$$\times \sum_{\substack{u\in I_q \text{ so that } \mathrm{sgn}(u)=\mathrm{sgn}(r_1)\varepsilon\\ \text{and } \varepsilon\eta\mathrm{sgn}(r_1r_2)pq^{l+m}u\in I_q}} a_z(r_1,u)\, a_u(p_1,w)\, a_{\varepsilon\eta pq^l z}(r_2, \varepsilon\eta\mathrm{sgn}(r_1r_2)pq^{l+m}u)$$
$$\times\ a_{\varepsilon\eta\mathrm{sgn}(r_1r_2)pq^{l+m}u}(p_2, \mathrm{sgn}(r_1p_1r_2p_2)\varepsilon\eta pq^{l+m+n}w),$$

where the series on both sides converge absolutely.

Remark 6.6. (i) The formula of Theorem 6.5 contains many special cases involving q-Laguerre polynomials, big q-Bessel functions, Al-Salam–Carlitz polynomials and q-Charlier polynomials as special cases by suitable specializing the signs in the formula. Note moreover that in all cases the sums are essentially sums over $q^{\mathbb{Z}}$ or $q^{\mathbb{N}}$. For each particular choice of the signs the square roots occurring in Definition 6.2 in Theorem 6.5 will cancel or can be taken together. It would be of interest to find a direct analytic proof.
(ii) As stated before, the functions $a_p(x,y)$ can be interpreted as Clebsch-Gordan coefficients related to representations of the quantized function algebra, which has no classical counterpart. For the case of the quantum $SU(2)$ group the corresponding Clebsch-Gordan coefficients are Wall polynomials, which are special cases of little q-Jacobi polynomials and also can be interpreted as q-analogues of Laguerre polynomials, see [35]. The classical Clebsch-Gordan coefficients also satisfy summation formulas involving the product of four Clebsch-Gordan coefficients, see e.g. [55, Ch. 8.7], but the structure of the summations is quite different. Relations as in Theorems 6.5 and 6.8, if proved directly, might give a hint of proving directly that the corresponding q-analogues of the Racah coefficients are zero at the appropriate places, leading to a direct proof of the coassociativity for M, see the discussion [30, p. 289].

Theorem 6.5 can be used to obtain positivity results for sums where the summands have four of the functions $a_p(x,y)$. The result is contained in Corollary 6.7. We give the case corresponding to the q-Laguerre polynomials explicitly, and we refer to Askey [2, Lecture 5] for more information on the related positivity results for the Laguerre polynomials. The q-Laguerre polynomials are defined by,

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\, {}_1\varphi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1}\end{matrix}\ ;q, -q^{1+\alpha}x\right), \tag{6.3}$$

in this application we only consider the case $\alpha = 0$.

Corollary 6.7. *For $r_1, r_2 \in I_q$, $l, m \in \mathbb{Z}$ and with $z \in I_q$ so that $\mathrm{sgn}(z) = \varepsilon$ and $\varepsilon\eta|\frac{r_2}{r_1}|q^{-m-l}z \in I_q$ and we have*

$$
\begin{aligned}
&(-\eta)^{l+m}(\eta\mathrm{sgn}(r_1))^{\chi(r_1)}\,(\eta\mathrm{sgn}(r_2))^{\chi(r_2)}\,(\varepsilon\eta)^{\chi(z)} \\
\sum_{x\in q^{\mathbb{Z}}} &x^2\, a_x(r_1,r_1)\, a_x(z,z)\, a_{xq^{-m}|\frac{r_2}{r_1}|}(r_2,r_2)\, a_{xq^{-m}|\frac{r_2}{r_1}|}(\varepsilon\eta|\frac{r_2}{r_1}|q^{-m-l}z, \varepsilon\eta|\frac{r_2}{r_1}|q^{-m-l}z) > 0
\end{aligned}
$$

and for $a \in \mathbb{Z}$ and $n_1, n_2, n_3, n_4 \in \mathbb{N}_0$ we have

$$
\sum_{k\in\mathbb{Z}} \frac{q^k}{(-q^k, -q^{k+a}; q)_\infty} L^{(0)}_{n_1}(q^k;q)\, L^{(0)}_{n_2}(q^k;q)\, L^{(0)}_{n_3}(q^{k+a};q)\, L^{(0)}_{n_4}(q^{k+a};q) \; > 0.
$$

Note that the sum is closely related to one of the orthogonality measures for the q-Laguerre polynomials, which correspond to an indeterminate moment problem. A similar positivity result can be obtained for the q-Bessel functions involved.

6.2.2. *Representing the comultiplication in $\hat{M}$.* The explicit expression for $\hat{\Delta}$ in the dual quantum group $\hat{M}$ as given in Proposition 4.15, or better the expression (7.23) in the proof of Proposition 4.15, leads to a formula for its matrix elements. The result is the following theorem.

Theorem 6.8. *For fixed $r \in q^{\mathbb{Z}}$, $m_1, m_2, M, n \in \mathbb{Z}$, $p_1, p_2 \in I_q$, $\varepsilon_1, \varepsilon_2, \eta_1, \eta_2, \sigma \in \{\pm\}$ and for $z_1, z_2, w_1, w_2 \in I_q$ satisfying*

$$
\begin{aligned}
&\mathrm{sgn}(z_i) = \varepsilon_i,\ (i=1,2), \quad \varepsilon_1\eta_1 q^{m_1} r z_1 \in I_q, \quad \varepsilon_2\eta_2 q^{-2m_1-m_2-n}\frac{z_2|p_2|}{r|p|} \in I_q, \\
&\mathrm{sgn}(w_1) = \mathrm{sgn}(p_1)\varepsilon_1, \quad \mathrm{sgn}(w_2) = \sigma\varepsilon_2, \qquad \sigma\mathrm{sgn}(p_1)\varepsilon_1\eta_1 q^{m_1+M} r_1 w_1 \in I_q, \\
&\sigma\mathrm{sgn}(p_2)\varepsilon_2\eta_2 q^{-2m_1-m_2-M}\frac{w_2|p_2|}{r|p|} \in I_q
\end{aligned}
$$

and such that $a_{z_1}(p_1, w_1) \neq 0$ we have

$$
\begin{aligned}
&\frac{1}{w_2^2} a_{ep_1\eta_1 q^{m_1} r z_1}(\sigma|p_1|rq^{2m_1+M}, \varepsilon_1\eta_1\sigma\mathrm{sgn}(p_1)w_1 r q^{m_1+M})\, a_{z_2}(\sigma|p_1|rq^{2m_1+M}, w_2) \\
&\qquad \times\ a_{\varepsilon_2\eta_2|\frac{p_2}{p_1}|\frac{z^2}{r}q^{-2m_1-m_2-n}}(p_2, \varepsilon_2\eta_2\sigma\frac{p_2 w_2}{|p_1|r}q^{-2m_1-m_2-M}) = \\
&\qquad\qquad \sum_{\substack{y,x\in I_q \text{ so that } \mathrm{sgn}(y)=\varepsilon_2\eta_1 \text{ and} \\ \mathrm{sgn}(p_1p_2)q^n x w_1/z_1 \in I_q,\ \varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1-m_2} yx/rz_1 \in I_q}} \frac{1}{y^2}\, a_{z_2}(\varepsilon_1\eta_1 q^{m_1} r z_1, y)\, a_{w_2}(\mathrm{sgn}(p_1)\sigma\varepsilon_1\eta_1 q^{m_1+M} r w_1, y) \\
&\times\ a_{\varepsilon_2\eta_2\frac{z_2|p_2|}{r|p_1|}q^{-2m_1-m_2-n}}(x, \varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1-m_2}\frac{yx}{rz_1})\, a_x(p_2, \mathrm{sgn}(p_1p_2)q^n\frac{xw}{z_1}) \\
&\times\ a_{\sigma\mathrm{sgn}(p_2)\varepsilon_2\eta_2\frac{w_2|p_2|}{r|p_1|}q^{-2m_1-m_2-M}}(\mathrm{sgn}(p_1p_2)q^n\frac{xw_1}{z_1}, \varepsilon_1\varepsilon_2\eta_1\eta_2 q^{-m_1-m_2}\frac{yx}{rz_1})
\end{aligned}
$$

where the left-hand-side is considered to be zero in case $\sigma|p_1|rq^{2m_1+M} \notin I_q$. The series converges absolutely.

Remark 6.9. (i) First note that the largest part of Remark 6.6(i) is also applicable to Theorem 6.8, except for the fact that the summation is more involved. Viewing the summation as a sum over an area in $I_q \times I_q \subset \mathbb{R}^2$ (with x on the horizontal axis and y on the vertical axis), we see that the summation area is a subset of $I_q \times I_q$ bounded by a vertical line and a hyperbola. Depending on the sign choices there are eight possibilities for the location of the vertical line and the hyperbola.
(ii) Theorem 6.8 follows from the operator identity in Proposition 4.15, but the single term in the left hand side of Theorem 6.8 corresponds to summation on the left hand side of Proposition 4.15, whereas the double sum on the right hand side of Theorem 6.8 corresponds to the single term on the right hand side of Proposition 4.15.
(iii) Since the results in Theorems 6.5 and 6.8 both reflect the pentagonal equation for the multiplicative unitary, one might expect the resulting identities to be equivalent by using the orthogonality relations of Propositions 6.3 and 6.4. However, this is not the case as follows by considering the dependence of both results on the free parameters.

6.3. **Formulas involving ${}_2\varphi_1$-series.** In Section 9 we show that with respect to the spectral decomposition of the Casimir operator Ω the operators $Q(p_1, p_2, n)$ generating $\hat{M}$, see Proposition 4.9, act by multiplication by a ${}_2\varphi_1$-series up to a sign-change in the argument. Since we also have another explicit expression for the action of $Q(p_1, p_2, n)$ by Lemma 7.1, we have two different explicit expressions for the action of $Q(p_1, p_2, n)$. This leads to the following theorem, where the functions Ψ are essentially ${}_1\varphi_1$-functions as defined in (6.1). Actually, we have written out two of several options depending on several sign choices.

Theorem 6.10. *Let $m, n \in \mathbb{Z}$, $p_1, p_2 \in q^{\mathbb{Z}}$ and $\lambda \in \mathbb{T}$.*

(i) For $k \in \mathbb{Z}$,

$$
\begin{aligned}
&\sum_{l=-\infty}^{\infty} (-1)^{l+k+n} \Big(p_2^2 q^{2n-2k-3}\Big)^l q^{l^2} (-q^{2l-2m} p_2^2/p_1^2; q^2)_\infty \\
&\qquad \times (q^{2-2m-2n}; q^2)_\infty \, {}_2\varphi_1 \left(\begin{matrix} q^{1-n} p_1 \lambda/p_2, q^{1-n} p_1/p_2\lambda \\ q^{2-2m-2n} \end{matrix} ; q^2, -q^{2-2l} \right) \\
&\qquad \times \Psi \left(\begin{matrix} -q^{2-2l} \\ q^{2+2k-2l} \end{matrix} ; q^2, q^{2+2k}/p_1^2 \right) \Psi \left(\begin{matrix} -q^{2-2l+2m} p_1^2/p_2^2 \\ q^{2+2k-2n-2l} \end{matrix} ; q^2, q^{2+2k-2m-2n}/p_1^2 \right) \\
&= p_2^{2k} q^{2n-3k} q^{-k^2} (q^2, -q^2/p_2^2; q^2)_\infty \frac{(p_1 q^{1-n} \lambda/p_2, p_1 q^{1-n}\lambda/p_2; q^2)_n}{(p_2 q^{1-n}\lambda/p_1, p_2 q^{1-n}\lambda/p_1; q^2)_n} \\
&\qquad \times (q^{2-2n}; q^2)_\infty \, {}_2\varphi_1 \left(\begin{matrix} p_2 q^{1-n} \lambda/p_1, p_2 q^{1-n}/p_1\lambda \\ q^{2-2n} \end{matrix} ; q^2, -q^2/p_2^2 \right) \\
&\qquad \times (q^{2-2m}; q^2)_\infty \, {}_2\varphi_1 \left(\begin{matrix} p_1 q^{1+n} \lambda/p_2, p_1 q^{1+n}/p_2\lambda \\ q^{2-2m} \end{matrix} ; q^2, -q^{2-2k} \right)
\end{aligned}
$$

where the sum converges absolutely.

(ii) Assume $q^{-m}p_2/p_1 \le 1$ and $q^{-m-n}p_2/p_1 \le 1$, then for $k \in \mathbb{N}_0$,

$$
\begin{aligned}
&\sum_{l=0}^{\infty} \frac{p_2^{2l} q^{2(k-l)} q^{(l-k)(l-k-1)}}{(q^2;q^2)_l}\, {}_3\varphi_2\left(\begin{matrix} q^{-2l}, q^{1+n}p_2\lambda/p_1, q^{1+n}p_2/p_1\lambda \\ q^{2-2m}p_2^2/p_1^2, 0 \end{matrix}; q^2, q^2\right) \\
&\qquad \times \Psi\left(\begin{matrix} q^{-2l} \\ q^{2+2k-2l} \end{matrix}; q^2, -q^{4+2k}/p_1^2\right) \Psi\left(\begin{matrix} q^{2m-2l}p_1^2/p_2^2 \\ q^{2+2k-2n-2l} \end{matrix}; q^2, -q^{4+2k-2m-2n}/p_1^2\right) \\
&= q^{2n(k-m+1)} q^{-n(n-1)} p_1^{2k-2n} (q^{2m}p_1^2/p_2^2;q^2)_n (q^{2+2k}, -q^2/p_2^2; q^2)_\infty \\
&\qquad \times (q^{2+2n};q^2)_\infty\, {}_2\varphi_1\left(\begin{matrix} q^{1-n}p_2\lambda/p_1, q^{1-n}p_2/p_1\lambda \\ q^{2+2n} \end{matrix}; q^2, -q^2/p_2^2\right) \\
&\qquad \times {}_3\varphi_2\left(\begin{matrix} q^{-2k}, q^{1-n}p_2\lambda/p_1, q^{1-n}p_2/p_1\lambda \\ q^{2-2m-2n}p_2^2/p_1^2, 0 \end{matrix}; q^2, q^2\right)
\end{aligned}
$$

where the sum converges absolutely.

Remark 6.11. (i) The ${}_2\varphi_1$-function inside the sum in Theorem 6.10(i) is essentially the little q-Jacobi function $f_l(\mu(\lambda); q^{2-2m-2n}, q^{1-n}p_1/p_2; -q^2|q^2)$, see (B.28), and the summations formula remains valid if $\mu(\lambda)$ is a discrete mass point of the corresponding orthogonality measure ν, see Appendix B.5. In Theorem 6.10(ii) the ${}_3\varphi_2$-series is essentially an Al-Salam–Chihara polynomial, and the same remark applies using the orthogonality measure described in Appendix B.4. Note that the ${}_3\varphi_2$-series can be transformed to a ${}_2\varphi_1$-series by (B.6).
(ii) If we multiply the formula (i) by $f_{l'}(\mu(\lambda); q^{2-2m-2n}, q^{1-n}p_1/p_2; -q^2|q^2)$ and we use the orthogonality relations, see Appendix B.5, it follows that the above identity is equivalent to an integral identity of the form $\int {}_2\varphi_1\, {}_2\varphi_1\, {}_2\varphi_1\, d\nu = \Psi\,\Psi$. The integral can be written as an integral over $[-1,1]$ plus an infinite sum. The same remark applies for (ii) but this time using the orthogonality relations, see Appendix B.4, for the Al-Salam–Chihara polynomials.
(iii) Note that we can view the Ψ-functions as q-analogues of the Bessel function, cf. the discussion in Section 6.1, and since we can do the same for the ${}_2\varphi_1$-series involved in (i) we may also consider Theorem 6.10(i) as an identity for q-Bessel functions.

The following result follows from the structure constants formula of Proposition 4.10. Note that Theorem 6.5 also follows from Proposition 4.10, but now we use again the fact that we can realize $Q(p_1,p_2,n)$ as multiplication operators by a ${}_2\varphi_1$-series up to a sign-change in the argument.

Theorem 6.12. *Let $\lambda \in \mathbb{T}$, $p_1, p_2, r_1, r_2 \in I_q$, $n, m \in \mathbb{Z}$, and assume that $|\frac{p_2}{p_1}| = q^m$ and $|\frac{r_1}{r_2}| = q^n$. Then*

$$
\begin{aligned}
&\operatorname{sgn}(r_1)^{\frac{1}{2}(1-\operatorname{sgn}(p_1))}\operatorname{sgn}(r_2)^{\frac{1}{2}(1-\operatorname{sgn}(p_2))+n} r_2^m p_2^n |r_1r_2|\nu(r_1)\nu(r_2)\nu(p_1)\nu(p_2)\\
&\times(q^2, -\operatorname{sgn}(r_1)r_1^2, -\operatorname{sgn}(r_2)r_2^2, -\operatorname{sgn}(r_2)q^2/r_2^2, -\operatorname{sgn}(p_2)q^2/p_2^2; q^2)_\infty\\
&\times\frac{(-\operatorname{sgn}(r_1p_1)q^{-m-n-1}/\lambda, -\operatorname{sgn}(r_1p_1)q^{3+m+n}\lambda, -\operatorname{sgn}(r_1r_2)\lambda q^{3-n}/p_1p_2; q^2)_\infty}{(-\operatorname{sgn}(r_1r_2p_1p_2)q^{m+n-1}/\lambda, -\operatorname{sgn}(r_1r_2p_1p_2)q^{1-m-n}\lambda, -\operatorname{sgn}(r_1r_2)p_1|p_2|q^{-n-1}/\lambda, ; q^2)_\infty}\\
&\times\frac{(-\operatorname{sgn}(r_1r_2)p_1p_2q^{n-1}/\lambda, -\lambda q^{3-m}/r_1r_2, -r_1r_2q^{m-1}/\lambda; q^2)_\infty}{(-\operatorname{sgn}(r_1r_2)\lambda q^{3+n}/p_1|p_2|, -r_1|r_2|q^{-m-1}/\lambda, -\lambda q^{m+3}/r_1|r_2|; q^2)_\infty}\\
&\times(\operatorname{sgn}(p_1p_2)q^{2+2n}; q^2)_\infty\, {}_2\varphi_1\left(\begin{matrix}\operatorname{sgn}(r_1r_2)p_2q^{1+n}/p_1\lambda, \operatorname{sgn}(r_1r_2)p_2q^{1+n}\lambda/p_1\\ \operatorname{sgn}(p_1p_2)q^{2+2n}\end{matrix}; q^2, -\operatorname{sgn}(p_2)\frac{q^2}{p_2^2}\right)\\
&\times(\operatorname{sgn}(r_1r_2)q^{2+2m}; q^2)_\infty\, {}_2\varphi_1\left(\begin{matrix}r_2q^{1+m}/r_1\lambda, r_2q^{1+m}\lambda/r_1\\ \operatorname{sgn}(r_1r_2)q^{2+2m}\end{matrix}; q^2, -\operatorname{sgn}(r_2)\frac{q^2}{r_2^2}\right)\\
&= \sum_{(x_1,x_2)\in\mathcal{A}} x_2^{m+n}|x_1|^2\nu(x_1)^2\nu(x_1p_1/r_1)\nu(x_2p_2/r_2)(\operatorname{sgn}(r_2p_2)q^{-2m-2n})^{\chi(x_1)}\\
&\times(-\operatorname{sgn}(r_1p_1)x_1^2, -\operatorname{sgn}(r_2p_2)x_1^2, -\operatorname{sgn}(r_2p_2)q^2/x_1^2, \operatorname{sgn}(r_1r_2p_1p_2)q^{2+2m+2n}; q^2)_\infty\\
&\times {}_2\varphi_1\left(\begin{matrix}\operatorname{sgn}(r_1r_2p_1p_2)q^{1+m+n}/\lambda, \operatorname{sgn}(r_1r_2p_1p_2)q^{1+m+n}\lambda\\ \operatorname{sgn}(r_1r_2p_1p_2)q^{2+2m+2n}\end{matrix}; q^2, -\operatorname{sgn}(r_2p_2)\frac{q^2}{x_1^2}\right)\\
&\times\Psi\left(\begin{matrix}-\operatorname{sgn}(p_1)q^2/p_1^2\\ \operatorname{sgn}(r_1p_1)q^2r_1^2/p_1^2\end{matrix}; q^2, \operatorname{sgn}(p_1)\frac{q^2r_1^2}{x_1^2}\right)\Psi\left(\begin{matrix}-\operatorname{sgn}(p_2)q^2/p_2^2\\ \operatorname{sgn}(r_2p_2)q^2r_2^2/p_2^2\end{matrix}; q^2, \operatorname{sgn}(p_2)\frac{q^2r_2^2}{x_1^2}\right)
\end{aligned}
$$

where the sum converges absolutely. Here $\mathcal{A} \subset I_q \times I_q$ is given by

$$
\mathcal{A} = \Big\{(x_1, x_2) \in I_q \times I_q \mid \operatorname{sgn}(x_1) = \operatorname{sgn}(p_1r_1),\ \operatorname{sgn}(x_2) = \operatorname{sgn}(p_2r_2),\ |x_1| = |x_2|\Big\}.
$$

From Theorem 6.12 we obtain another positivity result.

Corollary 6.13. *Let $p_1, p_2 \in I_q$ and $\lambda \in \mathbb{T}$, then*

$$
\begin{aligned}
0 < \sum_{x\in q^{\mathbb{Z}}}&\nu(x)^2(-x^2; q^2)_\infty\, {}_2\varphi_1\left(\begin{matrix}q/\lambda, q\lambda\\ q^2\end{matrix}; q^2, -\frac{q^2}{x^2}\right)\\
&\times\Psi\left(\begin{matrix}-\operatorname{sgn}(p_1)q^2/p_1^2\\ q^2\end{matrix}; q^2, \operatorname{sgn}(p_1)\frac{q^2p_1^2}{x^2}\right)\Psi\left(\begin{matrix}-\operatorname{sgn}(p_2)q^2/p_2^2\\ q^2\end{matrix}; q^2, \operatorname{sgn}(p_2)\frac{q^2p_2^2}{x^2}\right).
\end{aligned}
$$

6.4. **Biorthogonality relations for ${}_2\varphi_1$-functions.** We have explicit expressions for the matrix elements of the principal series corepresentations $W_{p,x}$, $p \in q^{\mathbb{Z}}$, $x = \mu(\lambda) \in [-1, 1]$, in terms of ${}_2\varphi_1$-functions. Unitarity of $W_{p,x}$ leads to orthogonality relations for the matrix elements. By analytic continuation these orthogonality relations remain valid for other values of λ.

Let $m \in \mathbb{Z}$ and $\lambda \in \mathbb{C} \setminus \{0\}$, and define $s(\,\cdot\,,\,\cdot\,;\lambda, m) : I_q \times I_q \to \mathbb{C}$ by

$$\begin{aligned} s(p_1, p_2; \lambda, m) = & \, p_2^{\chi(p_1p_2)+m} \nu(p_1p_2q^{m+1})|p_1p_2|\, \nu(p_1)\nu(p_2)c_q^2 \sqrt{(-\kappa(p_1), -\kappa(p_2); q^2)_\infty} \\ & \times \frac{(q^2, -q^2/\kappa(p_2), -\lambda q^{3-m}/\kappa(p_1)p_2^2, -\kappa(p_1)p_2^2q^{m-1}/\lambda, \operatorname{sgn}(p_1)q^{1-m}/p_2^2\lambda; q^2)_\infty}{(\operatorname{sgn}(p_2)p_1^2q^{1+m}/\lambda, -\operatorname{sgn}(p_1p_2)q^{-m-1}/\lambda, -\operatorname{sgn}(p_1p_2)\lambda q^{m+3}; q^2)_\infty} \\ & \times (\kappa(p_1p_2)q^{2+2m}; q^2)_\infty \, {}_2\varphi_1\left(\begin{matrix} \operatorname{sgn}(p_1)p_2^2q^{1+m}\lambda, \operatorname{sgn}(p_1)p_2^2q^{1+m}/\lambda \\ \kappa(p_1p_2)q^{2+2m} \end{matrix} ; q^2, -q^2/\kappa(p_2) \right), \end{aligned}$$

for $p_1, p_2 \in I_q$. From this expression it is not clear that the function is defined for all values of $p_2 \in I_q$, but an application of Jackson's transformation formula [17, (III.4)] shows how to extend to all values of $p_2 \in I_q$.

Theorem 6.14. *The following biorthogonality relations hold:*

$$\sum_{p_1 \in I_q} s(p_1, p_2; \lambda, m) s(p_1, p_2'; \lambda^{-1}, m) = \delta_{p_2, p_2'},$$
$$\sum_{p_2 \in I_q} s(p_1, p_2; \lambda, m) s(p_1', p_2; \lambda^{-1}, m) = \delta_{p_1, p_1'}.$$

Remark 6.15. The two biorthogonality relations Theorem 6.14 are actually equivalent. Also, for $\lambda \in \mathbb{T}$ the biorthogonality relations are orthogonality relations.