The Fundamental Interactions in Noncommutative Geometry

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1 Introduction

In this report we will try to give a short review of the approach by Noncommutative Geometry to the gauge theories of particle physics and the Standard Model. The main subjects of interest will be the theory of spectral triples and the so called inner fluctuations of the metric.

We will first introduce the basic concepts and formulation of the predominant theory of matter and its fundamental interactions, known as the "Standard Model" of particle physics. This model is based on quantum field theory and consists of a set of fields that propagate in spacetime and interact with each other according to the action principle and Feynman rules for a specific Lagrangian. This action satisfies a number of symmetries, some of which will be referred to as gauge symmetries and will generate the fields that carry the fundamental interactions.

After a short introduction to the basic notions of noncommutative geometry, a few instructive examples of spectral triples and simple noncommutative constructions will be demonstrated. Several ideas and notions of Riemannian geometry are inevitably distorted when expressed in operator algebraic terms, so careful steps should be taken, following the guidelines of basic literature like [4, 8, 7] at the same time.

Then the idea of gauge transformations and gauge invariance can be passed to the context of spectral triples, where the Dirac operator plays a central role. The generalization to noncommutative geometries will allow for a new class of inner fluctuations of the geometry to emerge. This was the missing ingredient that can now give the gauge fields of the Standard Model as inner fluctuations of the noncommutative "metric". The final link that gives the full kinematics and dynamics of the several fields is the spectral action, given in [2] for both fermionic and bosonic sectors. The gravitational aspect will unfortunately not be studied here.

2 The Standard Model

Even before Maupertius' work on the *principle of least action*, mathematical formulations of physical theories have been nothing but action principles. The Lagrangian and Hamiltonian formulation of Newtonian mechanics, has evolved into that of classical field theory which in turn played a crucial role in the formulation of quantum field theory. Again the behavior of quantum fields is ruled by an action principle, which slightly differs from its classical version by allowing "quantum fluctuations" of the fields. However, the role of symmetries remained central throughout the evolution of physics, becoming even more predominant in 20th century theoretical physics.

A quantum field theory is described by a number of fields living on some space and a specific Lagrangian, a simple expression involving the fields and their derivatives. The corresponding action is a functional on the space of field configurations, usually expressed by an integration of the Lagrangian over the underlying space. If the fields live on a flat background spacetime (i.e. the underlying space is Minkowski), the corresponding Lagrangian needs to remain invariant under Lorentz transformations, as dictated by the equivalence principle of special relativity. In a more generic background of General Relativity, if the underlying space is an arbitrary (pseudo-)Riemannian manifold, the theory needs to be invariant under diffeomorphisms, hence each field needs to transform in a certain way under general coordinate transformations. These kind of symmetries that are related to the background metric are called the *external* spacetime symmetries of the theory.

The first example of an Abelian gauge theory is that of QED. One starts from the Dirac equation in Minkowski spacetime, provided by the free massive fermion Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu \partial\mu - m)\psi(x) \quad , \tag{1}$$

with $\bar{\psi} = \psi^* \gamma^0$, noting that this theory has a built in symmetry under phase shifts of the form

$$\psi(x) \to \psi'(x) = e^{i\alpha}\psi(x). \tag{2}$$

These kind of transformations form the group of 1-dimensional unitaries $U(1)$, the unit circle in $\mathbb{C}$. Note that the phase shifting parameter α does not vary from point to point [1], but is a constant in spacetime and therefore, the above symmetry will be referred to as a *global* $U(1)$ symmetry.

One can then try to upgrade this natural global symmetry to a *local* one, by considering an arbitrary spacetime dependence of the phase parameter $\alpha = \alpha(x)$ and requiring the action to remain invariant under the new transformation

$$\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x). \tag{3}$$

It is readily seen that the second term is just fine, since the two prefactors cancel each other on each point. However the derivative in the first term, after applying the Leibnitz rule, gives a nontrivial transformation with the extra term $i\bar{\psi}(x)\gamma^\mu(\partial_\mu \alpha(x))\psi(x)$.

We can still save local invariance by introducing a new *vector gauge field* A_μ which will interact with the Dirac field by the *minimal coupling* term $e\bar{\psi}\gamma^\mu A_\mu \psi$, and will thus take part in a *gauge covariant derivative*, which reads

$$D_\mu \psi = (\partial_\mu - ieA_\mu)\psi \tag{4}$$

and transforms ideally, as

$$D_\mu \psi \to D'_\mu \psi'(x) = e^{i\alpha(x)} D_\mu \psi(x) \quad , \tag{5}$$

if the gauge field A_μ transforms in the following way,

$$A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x). \tag{6}$$

This is exactly the behavior of a field that lives in the adjoint representation.

[1] or more appropriately, from event to event

To make the picture complete, we still need to add the gauge invariant Lagrangian of the vector gauge field. This is simply given by the curvature term

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad , \tag{7}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$ the field's curvature tensor. It is straightforward to see that this expression will be invariant under local gauge transformations. Note here that gauge invariance imposes a *massless* gauge field, since a term of the form $MA^\mu A_\mu$ will not be gauge invariant.

In the path integral formalism, this minimal coupling term is an interaction term that corresponds to the 3-vertex of the photon (A_μ) and the charged fermion line ($\bar{\psi}, \psi$), with the electric charge e as its coupling constant. This simple gauge theory successfully described the full theory of electromagnetism and thus motivated the further investigation of gauge theories in particle physics. The related global and local symmetries are characterized as *internal symmetries* of the theory.

The notion of such symmetries was extended by Yang an Mills (1954) to the more general non Abelian setting, where initially the local internal symmetry group was considered to be $SU(n)$. In the years that followed, a huge amount of phenomenological results, brought by high energy experiments, led to the formulation of the Standard Model of particle physics. The QED gauge group $U(1)$ is a simple one parameter Lie group and similar constructions can be performed for general non-abelian groups such as the Lie groups $SU(2)$ and $SU(3)$ with 3 and 8 generators respectively. These will be the three relevant Lie groups for the Standard Model. A combination of $U(1)$ and $SU(2)$ [2] gives the *electroweak* interactions, the generators of which describe the massless vector field of the photon and the three massive vector bosons of the weak interaction W^+, W^- and Z^0. The interactions between the gauge bosons and fermionic fields are given by Yukawa couplings, whereas for non-Abelian theories such as $SU(2)$, one also gets self-interaction terms between the gauge fields.

An indispensable ingredient of the SM is the Higgs mechanism, that assigns masses to the initially massless fields, by spontaneous symmetry breaking. This type of symmetry breaking involves one or more bosonic fields, whose vector potential obtains a particular form, so that its vacuum state (around which perturbation theory is taken) gives a nonzero expectation value. A typical Higgs-like potential has the form

$$V(H) = -\mu^2|H|^2 + \lambda|H|^4 \quad , \tag{8}$$

which for a range of values in the parameters μ, λ is readily seen to have a family of nontrivial minima. [3] After choosing such a vacuum state (gauge fixing) and rewriting the Lagrangian around it, mass terms appear for all fields that interact with the Higgs bosons.

The last $SU(3)$ non-Abelian gauge symmetry corresponds to the strong nuclear force and is non-trivially represented only on quarks, that have the associated

[2] by a mixing angle θ_W

[3] In fact the parameters may vary dynamically, for example with temperature, in an effective theory and give a phase transition, under which the symmetry is broken.

charge, by a *color* index (r,g,b). The generating non-Abelian gauge fields that carry the strong interaction are called gluons and, apart from being coupled to the fermions, they also self interact by three and four valent vertices, as dictated by the general Yang-Mils theory.

To sum up, the particle artillery of the standard model consists of

- 6 massive charged quarks divided into 3 generations (flavors), namely the couples (u,d) , (c,s) , (t,b) , plus their anti-particles;
- 6 leptons also divided into 3 generations, namely (e, ν_e) , (μ, ν_μ) , (e, ν_e) (also along with their anti-particles);
- gauge bosons for the three interactions, namely the 4 bosons γ and $W^{\pm}, Z^0$ for the broken electro-weak and the 8 gluons for the strong interaction;
- a Higgs boson remaining from the scalar Higgs field after breaking electro-weak symmetry.

There is a grading operator for fermionic fields splitting the particles to left and right handed ones. Right handed neutrinos are absent and all remaining right handed fermions are in the 1-dimensional representation of the weak interaction, while all left handed fermions transform as doublets. This means that there is a rearrangement in isospin doublets and singlets,

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s \end{pmatrix}_L, \quad \begin{pmatrix} t \\ b \end{pmatrix}_L, \quad \begin{matrix} u_R & c_R & t_R \\ d_R, & s_R, & b_R, \end{matrix} \tag{9}$$

and similarly for the leptons,

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, \quad e_R, \quad \mu_R, \quad \tau_R. \tag{10}$$

The full Standard Model Lagrangian is given, along with extensive information, in chapter 9 of [7]. For a more than detailed study on gauge theories and the Standard Model, there is a huge amount of literature that one can consult, including [11, 10, 1].

3 Noncommutative Geometry

The basic idea behind noncommutative geometry is a reformulation and generalization of the notion of geometry in terms of operator algebras on Hilbert spaces.

The *topological* properties of a "geometrical space" X can be captured by the algebra of continuous functions $A = C(X)$ whose spectrum $\Omega(A)$ will be isomorphic to the original space X. This is a well known result for compact Hausdorff topological spaces, due to Gel'fand, which was further developed to the functorial relation between the categories *UCC* and *CH*, of unital commutative C^*-algebras and compact Hausdorff spaces respectively. As a consequence, one can trade any CH space for an algebra that characterizes it without losing any information on its structure since one can fully reconstruct the original space. This characterization leads to a natural generalization to a wider class of *noncommutative spaces*, if one drops the requirement of commutativity of the C^*-algebra that characterizes the space.

It would be nice to extend the above results in a way that incorporates the geometrical structure of a space, in the context of Riemannian geometry. The notion of a geometric space, as formulated in the Riemannian theory, consists of a manifold M equipped with an additional structure that measures distances between points in M, namely the metric g. Having taken care of the topological aspect of the space, which we can encode in the algebra of continuous functions over M, we now seek the additional structure that will hold the *geometric* data of (M, g). It turns out that this task is not as minimal as its predecessor, and led to the definition of a *spectral triple* by Connes in the mid '90s [3].

It is a fact that any $C*$-algebra can be represented as a subspace of $\mathcal{B}(\mathcal{H})$ the bounded linear operators on some Hilbert space $\mathcal{H}$. The new ingredient that will hold Riemannian geometric data (such as the dimension of the space, the metric, a notion of integration, differentiation and smooth structure), will be a self adjoint operator D on that Hilbert space which will be refered to as the *Dirac operator*. This operator will have to satisfy several properties that will be axiomatically incorporated in its definition and will comprise the base of the differential calculus on the spectral triple.

3.1 Spectral Triples

Again, in view of the general strategy of algebraic reformulation of everything, we translate the data in terms of operator algebras acting on Hilbert spaces and then justify the less intuitive ingredients with a reconstruction theorem [3, 5]. The dictionaries of *quantized calculus* that appear as tables in Connes' books, between geometrical objects on classical spaces and algebraic ones, should come in handy.

Definition. *Our initial datum and main framework from now on will be a* **spectral triple**

$$(\mathcal{A}, \mathcal{H}, D)$$

consisting of:

- *an associative *-algebra $\mathcal{A}$, represented faithfully on*
- *a (separable) Hilbert space $\mathcal{H}$ and*
- *a self-adjoint, usually unbounded operator D acting on $\mathcal{H}$, with compact resolvent, i.e.*

$$(D - zI)^{-1} \in \mathcal{K}(\mathcal{H}) \quad , \quad \forall z \in \mathbb{C} \backslash Sp(D)$$

and which "almost commutes" with all $a \in \mathcal{A}$, i.e.

$$[D, a] \in \mathcal{B}(\mathcal{H}) \quad , \quad \forall a \in \mathcal{A}.$$

In particular we assume that $0 \notin Sp(D)$ *and thus* D^{-1} *exists, with* $D^{-1} \in \mathcal{B}(\mathcal{H})$.

For the sake of aesthetics, we will take up the bad habit of not writing explicitly the represented algebra elements as $\pi(a)$ but rather as a. Only when the context causes ambiguity, will π be used. In any case, when writing a commutator with an operator on $\mathcal{H}$ like $[D, a]$, no confusion should be raised.

The first remarkable fact is that, in the special case of a Riemannian (spin) manifold M, the geodesic distance on M can be recovered even after having traded the metric for the operator D. More generally, all differential geometric notions can be translated to fit in the operator algebraic framework.

3.2 Axiomatic definition of commutative geometry

Before studying the interesting aspects of noncommutative spaces, one needs to keep in mind what this construction should reduce to, when one requires the algebra to be commutative. This will be done in the context of spectral triples and will be dictated by the "canonical commutative example" of a compact oriented Riemannian spin manifold.

The ingredients of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ that will define a *commutative geometry* of dimension n, satisfy the requirements in the spectral triple definition and also have the following properties. They are the conditions for a (unital) spectral triple to qualify as a geometric space.

- *Finite summability*
- *Commutativity* property
- *Smoothness* property (as in smooth manifold)
- *Orientability* property (existence of a volume form)
- Existence of a *Real structure*
- *Poincare' duality*
- *Finiteness*

Finite Summability This part has to do with the dimensionality of our space and is a property of the Dirac operator. It states that there exists an integer p called the *degree of summability*, such that D^{-1} as a compact operator lies in the ideal $D^{-1} \in \mathcal{L}^{p+}(\mathcal{H})$. We then say that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is p-summable. This is closely related to the so-called *Dixmier trace*, and is equivalent to saying that the operator $|D|^{-p}$ lies in another ideal of compact operators, the *Dixmier ideal*. Dimensionality is now captured by the set of poles of zeta-functions of the form $\zeta_a : z \mapsto Tr_\omega(a|D|^{-z})$, which define the *dimension spectrum* of the triple [3].

Commutativity This will be a reformulation of the algebraic property satisfied by the infinitesimal,

$$[[f, ds^{-1}], g] = 0 \quad \forall f, g \in C^{\infty}(M) \tag{11}$$

and is simply translated to

$$[[D, a], b] = 0 \quad \forall a, b \in \mathcal{A}. \tag{12}$$

Smoothness One needs to define some sort of smooth structure on the new notion of space. This will be done by noting that D and $|D|$ are in principle unbounded operators of differential order one. Define the derivation,

$$\delta(a) = [|D|, a] \quad , \quad \forall a \in \mathcal{A} \tag{13}$$

Differentiable functions are now elements of the algebra whose representations are operators that lie in the domain of δ,

$$Dom(\delta) = \{a \in \mathcal{B}(\mathcal{H}) : aDom(D) \subset Dom(D)\}. \tag{14}$$

In order for the algebra $\mathcal{A}$ to be one of smooth functions on the space, we require both $\mathcal{A}$ and $[D, \mathcal{A}]$ to fall in the domain of any integer power of δ.

Orientability Among Riemann geometries we want to consider orientable ones. This property is translated in the operator language as the existence of a special operator on $\mathcal{H}$, which we call the *chirality* operator γ, that defines a $\mathbb{Z}_2$-grading on the Hilbert space. This operator should come from a "highest order differential form", a Hochschild cycle of order n.

Spectral triples with a nontrivial chirality are also often refered to as *even* spectral triples, since for odd dimensional manifolds such a cycle can only be represented trivially to the identity operator. When one deals with an even spectral triple, one usually incorporates the chirality γ into the notation as an additional entry and writes $(\mathcal{A}, \mathcal{H}, D, \gamma)$.

Real structure The reality condition is about the existence of another special operator on $\mathcal{H}$ called the *real structure* J. It is an antilinear isometry $J(\lambda\psi) = \bar{\lambda}J\psi$, $J * J = 1$ with $J^2 = \pm 1$ satisfying $JaJ^{-1} = a^*$, for all $a \in A$. Usually, when the above sign is minus, it is called a quaternionic structure, however the term *real* has been established for spectral triples that carry any type of J. This operator will define the charge conjugation of spinor fields in particle physics.

During the study of Clifford algebras and spinor representations, we saw that the existence and type of real structure is completely determined by the dimension n of the manifold. All related properties exhibit a periodicity with respect to n and in fact they only depend on $n(mod8)$. For an even real spectral triple, dimension also determines (anti-)commutation relations between J and the other two important operators, D and γ. If we write all three relations

$$J^2 = \epsilon \tag{15}$$

$$JD = \epsilon' DJ \tag{16}$$

$$J\gamma = \epsilon'' \gamma J \quad , \tag{17}$$

parametrized by three signs, $\epsilon, \epsilon', \epsilon''$, we can sum up all possibilities in Connes' sign table.

n	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

Table 1: The values of $\epsilon, \epsilon', \epsilon''$ depending on dimension n (mod 8).

Poincaré duality This is not the most intuitive property, as it requires a considerable amount of knowledge on K-theory and index theory, in order to be fully understood, but still it deserves to be stated here. It says that the *intersection form*,

$$\begin{array}{rcl} K_*(\mathcal{A}) \times K_*(\mathcal{A}) & \rightarrow & \mathbb{Z} \\ (e, f) & \mapsto & \langle IndD, m_*(e \otimes f) \rangle \end{array} \quad , \tag{18}$$

is invertible.
However hi-tech this statement may seem, it can actually be checked explicitly in some simple spaces, (e.g. finite cases studied below, see [14]).

Finiteness The domain of the differential operator D describes the differentiable states of $\mathcal{H}$. In general we define $\mathcal{H}^k = Dom(|D|^k), \quad k \in \mathbb{N}$ (which can be also extended to real numbers by functional calculus) and the *smooth* subspace

$$\mathcal{H}^\infty = \bigcap_{k \geq 0} Dom(|D|^k). \tag{19}$$

The present axiom states that this space is required to be a finitely generated projective $\mathcal{A}$-module. In a few words, this allows for the defining algebra $\mathcal{A}$ to be not strictly a C^*-algebra, but a pre-C^*-algebra, the norm closure of which will be the C^*-algebra A of continuous functions on some space, namely its spectrum $X = \Omega(A)$. In particular, $\mathcal{A}$ is the dense subalgebra of smooth functions $C^\infty(X)$, and this density also implies that X could be taken as the spectrum of $\mathcal{A}$ in the first place.

3.3 Axiomatic definition of noncommutative geometry

The definition of a noncommutative geometry follows the same lines and in fact, only a few of the axioms defining commutative geometries need to be modified. It is obvious that the modifications will actually be relaxations of the conditions, since we need to keep commutative geometries as a special subset of noncommutative ones.

In particular, the *reality* axiom changes as follows. We define

$$b^0 := Jb^*J^{-1} \quad , \quad b \in \mathcal{A} \quad , \tag{20}$$

thus making b^0 an element of the *conjugate* or *opposite* algebra $\bar{\mathcal{A}}$. Apart from the above sign consistency, the reality axiom imposes the following conditions on the algebra:

- **commutation rule**

$$[a, b^0] = 0 \quad , \quad \forall a, b \in \mathcal{A} \quad , \tag{21}$$

- **order one condition**,

$$[[D, a], b^0] = 0 \quad , \quad \forall a, b \in \mathcal{A}. \tag{22}$$

The last property states that the Dirac operator is seen as an order one differential operator acting on the algebra of smooth functions. This correctly modifies the *commutativity* condition.

Now with the help of J one can assign an $\mathcal{A}$-bimodule structure on $\mathcal{H}$ as follows. Using (20), the representation of $\bar{\mathcal{A}}$ on $\mathcal{H}$ will naturally give the right $\mathcal{A}$-module structure on the Hilbert space:

$$\psi b = b^0 \psi \quad \forall \quad b \in \mathcal{A}, \psi \in \mathcal{H}. \tag{23}$$

This action is *compatible* with the left one, by making use of the commutation rule (21). Thus for any $a, b \in \mathcal{A}$, one can define the bimodule structure,

$$a\psi b = J b^* J^{-1} a \psi \quad \forall \psi \in \mathcal{H}. \tag{24}$$

In the (even case of) *orientability* axiom the Hochschild cycle is now an element $c \in Z_n(\mathcal{A}, \mathcal{A} \otimes \bar{\mathcal{A}})$ such that $\pi(c) = \gamma$ is a projection that commutes with $\mathcal{A}$ and anticommutes with D. There are in fact a couple of more, rather subtle adjustments for the axioms of noncommutative geometry (c.f. [5]), that will not be discussed here.

3.4 The canonical commutative example

As soon as the definition of a spectral triple is given, it is instructive to demonstrate the so called "canonical commutative example" of spectral triples, which makes the transition much smoother and clear. The whole construction will be based on the set of *compact spin manifolds*, a refinement of the set of orientable Riemannian manifolds. Let M be such a manifold. The defining algebra in this commutative case will be none other than the pre-C^*-algebra of smooth complex functions $C^\infty(M)$, whose involution will be defined by complex conjugation ($\psi^*(x) = \bar{\psi}(x)$). For the definition of the Dirac operator we run back to the lecture notes.

First recall that as extra structure for an n-dimensional spin manifold M, we have a principal $Spin(n)$-bundle, which we denote $Spin(M)$, and an explicit isomorphism of the vector bundles:

$$TM \cong Spin(M) \times_{Spin(n)} \mathbb{R}^n \quad , \tag{25}$$

where the right hand side is the *associated vector bundle* over M with $Spin(n)$ as its Lie group.

We also recall that the *Clifford algebra* $\mathbb{C}l_n$ can be represented irreducibly on a complex vector space $\mathbb{S}_n \cong \mathbb{C}^{2^k}$, $k = \lfloor \frac{n}{2} \rfloor$. We can now construct the *spin bundle* to be the associated vector bundle,

$$\mathcal{S}_n = Spin(M) \times_{Spin(n)} \mathbb{S}_n \quad , \tag{26}$$

whose sections $\Gamma(M, \mathcal{S}_n)$ we will call *spinors*. To be more precise, the space of spinors, which we would prefer to be a Hilbert space, will consist of only the square integrable sections, i.e. $\mathcal{H} = \mathcal{L}^2(M, \mathcal{S}_n)$. This will be taken to be the Hilbert space of our spectral triple, and one can already see the connection with particle physics.

Indeed, in a 4-dimensional spacetime we have $n = 4$ (or $k = 2$), hence $\mathbb{C}l_4$ will be represented in $2^k = 4$ dimensions and locally the spinor bundle will be isomorphic to $U \times \mathbb{S}_4 = U \times \mathbb{C}^4$. The fact that in a 4-dimensional spacetime the spinor representation is also 4-dimensional is of course coincidental. This reduces to the usual notion of Dirac fermions on a flat piece of 4-dimensional spacetime and in what follows we will also see the physical meaning of the group action on the spinors.

In this paradigm, we can represent the basis of the Clifford algebra on spinors by the well known chiral γ-matrices which read:

$$\gamma^0 = \begin{pmatrix} 0 & -1_2 \\ -1_2 & 0 \end{pmatrix}, \quad \gamma^j = -i \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3 \quad , \tag{27}$$

where σ_j denote the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{28}$$

Hence the Dirac operator in local *flat* coordinates will be the usual Dirac operator of particle physics $i\,\partial\!\!\!/ = i\gamma^\mu \partial_\mu$.

Now the *chirality element* γ of the Clifford algebra will be represented by $\gamma^5 = (-1)^2\gamma^0\gamma^1\gamma^2\gamma^3$ and in this choice of hermitian matrices will read [4]

$$\gamma = \gamma^5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \tag{29}$$

while the real structure will be the familiar charge conjugation, represented by the anti-unitary

$$J = \gamma^0\gamma^2 \circ cc = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \circ cc. \tag{30}$$

[4]There are different notations for the indexing of the γ matrices, such as $\gamma^a, a = 1, 2, 3, 4$ with the chirality element written either as γ^0 or γ^5. To avoid confusion we stick with γ^5 (and not say γ^4). Moreover, there are different choices for the realization of these matrices. The present one is one of the most common, the *chiral* or *Weyl* basis; the next most popular, for different purposes are the *Dirac* and *Majorana* basis.

But now we need to see how to calculate things on a general curved manifold, and this information will be provided by the connection on the spinor bundle, namely the *spin connection* ∇^S. As promised, the differential structure will be held in the last ingredient of the spectral triple, the Dirac operator and indeed this is where the spin connection comes into play. Recall that the usual Dirac operator on our spin manifold is defined as a linear first order partial differential operator $D : \Gamma(\mathcal{S}_n \to \Gamma(\mathcal{S}_n)$ by means of the spin connection,

$$\not{D} := -ic \circ flip \circ \# \circ \nabla^S \quad , \tag{31}$$

as explained in Landsman's lecture notes and Varilly's notes, along with the more physicist-friendly local expressions.

Note here that we assume M to be a compact manifold, hence all sections of any vector bundle over M are compactly supported. In particular, the space of square integrable sections above is exactly H_0, one of the Sobolev spaces, which have the attractive property of being Banach and thus Hilbert under the inner product defined fiberwise by the regular complex one, i.e. for $\psi \in \mathcal{L}^2(M, \mathcal{S}_n)$.

The canonical commutative paradigm is now clear and, as promised, we will now justify all the above abstractness by the following reconstruction theorem by Connes [5].

Theorem. *Let $\mathcal{A} = C^\infty(M)$, where M is a smooth compact manifold of dimension n. a) Let π be a unitary representation of $\mathcal{A}, ds$ satisfying the axioms of 3.2. Then there exists a unique Riemannian metric g on M such that the geodesic distance between any two points $p, q \in M$ is given by*

$$d(p,q) = \sup \{ |a(p) - a(q)| : a \in \mathcal{A}, \| [D,a] \| \leq 1 \} . \tag{32}$$

b) The metric $g = g(\pi)$ only depends upon the unitary equivalence class of π and the fibers of the map : $\pi \mapsto g(\pi)$ from unitary equivalence classes to metrics form a finite collection of affine spaces $\mathcal{A}_\sigma$ parametrized by the Spin structures σ on M.
c) The action functional $⨍ ds^{n-2}$ is a positive quadratic form on each $\mathcal{A}_\sigma$ with a unique minimum π_σ.
d) π_σ is the representation of $(\mathcal{A}, ds)$ in $\mathcal{L}^2(M, \mathcal{S}_\sigma)$ given by multiplication operators and the Dirac operator of the Levi Civita Spin connection ∇^S.
e) The value of $⨍ ds^{n-2}$ on π_σ is given by the Einstein Hilbert action,

$$-c_n \int R\sqrt{|g|} d^n x, \tag{33}$$

where c_n is a constant depending only on the dimension n coming from Weyl's formula,

$$c_n = \frac{(n-2)}{12} (4\pi)^{n/2} \Gamma \left(\frac{n}{2} + 1 \right)^{-1} 2^{\lfloor \frac{n}{2} \rfloor} \tag{34}$$

We will concentrate on the first point. In Riemannian geometry, distance between points on a manifold was measured by integrating infinitesimals or line elements $\sqrt{g_{\mu\nu} dx^\mu dx^\nu}$ along geodesic curves. These curves were actually *defined* by the property of being the ones that extremize the length, among curves with fixed endpoints. Points are no more, in the noncommutative setting, hence the notion of Riemannian distance needs to be reformulated in a point-free manner. We start from the original expression for distance between two points $p, q \in M$ on a Riemannian manifold M,

$$d(p,q) = \inf_{\gamma} \left\{ L^{\gamma}(p,q) \int_{\gamma(0)=p}^{\gamma(1)=q} \sqrt{g_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds}} ds \right\} \tag{35}$$

and will try to translate it to the language of spectral triples.

What we do know is that the algebra of smooth complex functions on M is $\mathcal{A}$. Now take all such functions on M and see how much they can (smoothly) vary from point p to point q. Obviously, in order for this to be relevant to their distance we need to restrict to the functions that vary in a "controlled" manner with respect to infinitesimal displacements ds, i.e. the ones that satisfy $\|\frac{df}{ds}\| = \|\nabla f\| \leq 1$. Taking the supremum of these differences $|f(p) - f(q)|$ is like adjusting a coordinate along which the smallest length is achieved. The condition $\|\nabla f\| \leq 1$ translates to $\|[\not{D}, f]\| \leq 1$ and it also gives that $|f(p) - f(q)| \leq d(p,q)$. But we also know that there is a such a function for which this value is actually acquired, namely the distance of any point from p, with gradient 1. Hence we can rewrite

$$d(p,q) = \sup |a(p) - a(q) : a \in \mathcal{A}, \|[D,a]\| \leq 1. \tag{36}$$

This expression still involves the points, but only through the evaluation of elements of $\mathcal{A}$. But this can be reconstructed by Gel'fand's theorem, by the characters of the algebra, without any reference to points.

This concludes the reformulation of the notion of Riemannian distance, which in this form continues to make sense for any spectral geometry.

3.5 Elementary geometries

The exploration continues with a some constructions and examples of non-canonical geometries.

3.5.1 Direct sum

In order to widen our collection of noncommutative spaces to a nontrivial set in a minimalistic way, it is useful to demonstrate a simple way of composing a finite number of spectral triples to form a new one. Given the spectral triples [5] $(\mathcal{A}_k, \mathcal{H}_k, \mathcal{D}_k), \quad k = 1 \dots N$, this can be done by taking the direct sum over each component,

$$\mathcal{A} = \oplus_k \mathcal{A}_k, \quad \mathcal{H} = \oplus_k \mathcal{H}_k, D = \oplus_k D_k. \tag{37}$$

The way dimension is defined, not by a natural number but through the dimension spectrum, is appropriate in this context, since no information is lost concerning the different dimensions of the original pieces.

[5] which for generality are not given a real structure or chirality, but this would also work in an obvious way

The simplest example would be a 0-dimensional space with two points, taken as a direct sum over two spaces with one point each, say x_L and x_R (standing for left and right point) [6]. The algebra of functions over each space would simply be the set of values that complex functions can take, $\mathcal{A}_i \cong \mathbb{C}$. The algebra on the space of two points will be exactly the direct sum of the algebras $\mathcal{A} = \mathbb{C}_L \oplus \mathbb{C}_R$, so in a sense direct sums represent the union of disjoint (disconnected) spaces. So any function in $\mathcal{A}$ looks like $f = (f_L, f_R)$. It can be represented on a 2-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$ by the diagonal matrix representation

$$\pi(f) = \begin{pmatrix} f_L & 0 \\ 0 & f_R \end{pmatrix},$$

and the Dirac operator will be the self adjoint

$$D = \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix} \quad , \quad m \in \mathbb{C}.$$

One is free to get rid of diagonal entries in the Dirac matrix since they will be washed out anyway when taking the commutator with any $f \in \mathcal{A}$. As already mentioned, distance as formulated above, continues to make sense and reads

$$d(x_L, x_R) = sup \{ |f_L - f_R| : \|[D, f]\| \leq 1 \} . \tag{38}$$

Notice how D enters (and actually defines) the measuring of distance between two points that were initially foreign to each other! This gives

$$[D, f] = \begin{pmatrix} 0 & m(f_R - f_L) \\ \bar{m}(f_L - f_R) & 0 \end{pmatrix} \tag{39}$$

$$\begin{aligned} \Rightarrow \quad & \|[D, f]\| = |m| \, |f_L - f_R| \leq 1 \\ \Rightarrow \quad & d(x_L, x_R) = sup \left\{ \frac{1}{|m|} \right\} = 1/|m|. \end{aligned} \tag{40}$$

After a closer look, one can say that the choice of notation for the entry m was far from random, since, in terms of dimensionality, $d(x_L, x_R)$ defines a length scale in our space, and thus its inverse m should define some sort of a mass scale.

A nice modification of this example involves the construction of a *real chiral* 0-dimensional space with two points. The same algebra now will be represented on the larger $\mathcal{H} = \mathbb{C}^4$, by

$$\pi(f) = \begin{pmatrix} f_L & 0 & 0 & 0 \\ 0 & f_R & 0 & 0 \\ 0 & 0 & \bar{f}_L & 0 \\ 0 & 0 & 0 & \bar{f}_R \end{pmatrix} \quad , \quad D = \begin{pmatrix} 0 & m & 0 & 0 \\ \bar{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{m} \\ 0 & 0 & m & 0 \end{pmatrix}, \quad m \in \mathbb{C}. \tag{41}$$

The real structure and chiral operator can now be defined as:

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \circ cc \quad , \quad \gamma = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{42}$$

[6] The term 0-dimensional is attributed to finite spaces, in that the spectral algebra is finite dimensional, often denoted by $\mathcal{A}_f$.

Again one can check that this defines a real chiral spectral triple and the distance between the two points/pure-states is $1/|m|$.

One can further play around with these finite spaces to e.g. fit the general properties of the Standard Model and then use it as an "internal finite space" as will become clear in what follows.

3.5.2 Tensor Product of spectral triples

Another useful composition of two spectral triples can be done by more or less tensoring everything. Let $(\mathcal{A}_1, \mathcal{H}_1, D_1)(J_1, \gamma_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)(J_2, \gamma_2)$ be two (real, even) spectral triples that satisfy the axioms of noncommutative spaces. Then we define the *tensor product* as the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with components:

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_1 \otimes \mathcal{A}_2, \\
\mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2, \\
D &= D_1 \otimes 1_2 + \gamma_1 \otimes D_2 \qquad (43) \\
&\cong D_1 \otimes \gamma_2 + 1_1 \otimes D_2, \\
J &= J_1 \otimes J_2, \\
\gamma &= \gamma_1 \otimes \gamma_2. \qquad (44)
\end{aligned}$$

All the above tensor products are with respect to the field $\mathbb{C}$of complex numbers. The representation of the algebra $\mathcal{A}$ and the action of the operators D, J, γ on the Hilbert space $\mathcal{H}$ is defined in the obvious way. Note also that if the factor spectral triples are even, then it can be checked that the resulting product is also an even spectral triple.

This provides a nice playground for the construction of noncommutative spaces, since one can now use elementary examples such as the trivial space of a single point to build larger and richer geometries.

3.6 The "almost commutative" geometry

And now it will all come together with the crucial move, that is to combine commutative geometries, coming from classical spin manifolds, with minimal noncommutative geometries. An **almost commutative geometry** is defined to be a geometry that can be expressed as the spectral triple product of a commutative geometry with a finite 0-dimensional noncommutative geometry. The small finite space could be commutative or not but the most interesting constructions come from noncommutative ones, and of course this will be the case for any Standard Model related attempt.

We write the two factor spaces as

- $(C^\infty(M), \mathcal{L}^2(\mathcal{S}), \not{D}, J, \gamma)$ the spectral triple associated to a (real, even) spin manifold M and
- $(\mathcal{A}_f, \mathcal{H}_f, D_f, J_f, \gamma_f)$ a 0-dimensional noncommutative geometry where f stands for *finite*.

Then the resulting algebra and Hilbert space will be of the form

$$\begin{aligned} \mathcal{A} &= C^\infty(M) \otimes \mathcal{A}_f = C^\infty(M, \mathcal{A}_f) && (45) \\ \mathcal{H} &= \mathcal{L}^2(M, \mathcal{S}) \otimes \mathcal{H}_f = \mathcal{L}^2(M, \mathcal{S} \otimes \mathcal{H}_f). && (46) \end{aligned}$$

If we consider the example of a two point space discussed above, we can see the resulting almost commutative geometry as a spacetime foliated in two parts at a constant distance $1/|m|$, that communicate with each other via particles of mass $|m|$.

That this space describes a spinor field of a particle of mass $|m|$ and its antiparticle, can be observed if one looks at the Dirac operator of the triple. For M a patch of 4-dimensional flat Minkowski spacetime, by (43), this yields

$$D = \partial\!\!\!/ \otimes 1_4 + \gamma^5 \otimes D_f \tag{47}$$

and, when applied on a spinor $\psi \in \mathcal{H}$, gives the massive Dirac equation in 4 dimensions (the second term giving the mass term).

Let us have a quick look at the simplest noncommutative example we can think of. The internal space will be given by the algebra of k-dimensional matrices $M_k(\mathbb{C})$ and the inner Hilbert space will be simply defined as $\mathbb{C}^k$. Now the resulting total algebra of an almost commutative space is $\mathcal{A} = C^\infty(M, M_k(\mathbb{C}))$. As we shall see below this algebra has the smooth sections of unitary matrices $C^\infty(M, U(k))$ as its group of inner automorphisms and actually corresponds to a Yang-Mills theory. This sounds very promising and has indeed been proved fruitful, eventually leading to the construction of the NCG version of the Standard Model.

3.7 NCG Standard Model for beginners: fermionic sector

The noncommutative geometry corresponding to the Standard Model of particle physics is constructed in such a way, after a bit of tweaking with the inner finite dimensional spaces. However nontrivial the input may seem, it turns out to be much more elegant and less arbitrary than the numerous free parameters of the existing model.

In order to find the correct space, we keep in mind that the Hilbert space $\mathcal{H}$ needs to hold the Standard Model fermions. The spinorial nature of these particles is guaranteed by the commutative part of the space, since the commutative Hilbert space is that of spinors living on the manifold. Moreover, the finite space will be responsible for the *internal structure*, providing the different sorts and generations of fermions.

First we take care of handedness, so we need the left and right handed blocks to be independent components of $\mathcal{H}$, distinguished by chirality. After doubling the space by attaching the antiparticles, or charge conjugate states, $\mathcal{H}$ can be written as

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c. \tag{48}$$

The dimensionality of the several components will be determined by the corresponding dimensions of the gauge group representations. In particular the small gauge group for the Standard Model is

$$SU(2) \times U(1) \times SU(3), \tag{49}$$

and three generations of fermions are considered ($N = 3$). So for example, left handed Standard Model quarks will be given by basis vectors of certain weak isospin flavor and color, in $\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3 \cong \mathbb{C}^{6N}$, e.g. $u_r^L = (1,0) \otimes (1,0,0) \otimes (1,0,0)$ and $s_b^L = (0,1) \otimes (0,1,0) \otimes (0,0,1)$. The same structure is repeated for the leptons except that they are in the trivial representation under strong $SU(3)$.

The charge conjugate fields will be obviously represented in the same way, but recall that the right handed quarks are singlets under weak isospin (i.e. transform under trivial representation of $SU(2)$) and also right handed neutrinos are missing. Thus we will have for particles

$$\begin{aligned} \mathcal{H}_L &= (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}) && (50) \\ \mathcal{H}_R &= (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}), && (51) \end{aligned}$$

and similarly for the antiparticles $\mathcal{H}_L^c$ and $\mathcal{H}_R^c$. The internal Hilbert space has dimension $(6N+2N)+(3N+3N+N)+(6N+2N)+(3N+3N+N) = 30N$, hence $\mathcal{H} \cong \mathbb{C}^90$.

The finite algebra represented on $\mathcal{H}_f$ is

$$\mathcal{A}_f = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \tag{52}$$

and was chosen in [6, 5] so that it has the small gauge group (49) as subgroup of its group of unitaries $\mathcal{U}(\mathcal{A}_f)$ and gives the correct rules for the gauge fields, according to what will be discussed in section 4.

For the moment, let us describe the representation on $\mathcal{H}_f$ of an arbitrary element $a = (\lambda, q, m) \in \mathcal{A}_f$. The action $\rho(a)$ will be block diagonal so that it does not mix the terms in (48) and (50)-(51), so we split ρ into the 8-dimensional representations ρ_L, ρ_L^c and the 7-dimensional ρ_R, ρ_R^c[7]. The left $\rho_L(a)$ will be realized on blocks of dimension $6N + 2N$, by the quaternionic component q of a, and the right $\rho_R(a)$ on blocks of dimension $3N + 3N + 2N$ by its complex component λ. In particular,

$$\begin{aligned} \rho_L(a) &= \begin{pmatrix} q \otimes 1_N \otimes 1_3 & 0 \\ 0 & q \otimes 1_N \end{pmatrix} , && (53) \\ \rho_R(a) &= \begin{pmatrix} \lambda \otimes 1_N \otimes 1_3 & 0 & 0 \\ 0 & \bar{\lambda} \otimes 1_N \otimes 1_3 & 0 \\ 0 & 0 & \bar{\lambda} \otimes 1_N \otimes 1 \end{pmatrix}. && (54) \end{aligned}$$

Here the 2-dimensional realization of quaternions as matrices of the form $q \leftrightarrow \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$, with $\alpha, \beta \in \mathbb{C}$ was used.

For the action on the antiparticles, λ and m will be relevant, for the leptons and quarks respectively. The matrix component m will act on the color 3-dimensional space in the obvious way and we write $a\bar{l} = \lambda\bar{l}$ and $a\bar{q} = m\bar{q}$,

[7] Flavor is not mixed either so the second factor will always be trivially 1_N

or

$$\rho_L^c(a) = \begin{pmatrix} 1_2 \otimes 1_N \otimes m & 0 \\ 0 & 1_2 \otimes 1_N \otimes \bar{\lambda} \end{pmatrix}, \tag{55}$$

$$\rho_R^c(a) = \begin{pmatrix} 1 \otimes 1_N \otimes m & 0 & 0 \\ 0 & 1 \otimes 1_N \otimes m & 0 \\ 0 & 0 & 1 \otimes 1_N \otimes \bar{\lambda} \end{pmatrix}. \tag{56}$$

It is the inner space Dirac operator that will fluctuate and give the gauge degrees of freedom, the bosonic fields and through the Higgs mechanism, will assign masses to the particles, so this operator is deeply related to various phenomenological input. Its explicit form is shown in,

$$D_f = \begin{pmatrix} Y & 0 \\ 0 & \bar{Y} \end{pmatrix} \quad , \tag{57}$$

where $Y = Y_q \otimes 1_3 \oplus Y_f$ is the 15-dimensional *Yukawa coupling matrix*, given by

$$Y_q = \begin{pmatrix} 0 & 0 & M_u & 0 \\ 0 & 0 & 0 & M_d \\ \bar{M}_u & 0 & 0 & 0 \\ 0 & \bar{M}_d & 0 & 0 \end{pmatrix} \quad , Y_f = \begin{pmatrix} 0 & 0 & M_e \\ 0 & 0 & 0 \\ \bar{M}_e & 0 & 0 \end{pmatrix}. \tag{58}$$

Obviously, the chirality operator will just give the $\mathbb{Z}_2$-grading of left and right handed particles and thus will have the form

$$\gamma_f = \begin{pmatrix} -1_8 & 0 & 0 & 0 \\ 0 & 1_7 & 0 & 0 \\ 0 & 0 & -1_8 & 0 \\ 0 & 0 & 0 & 1_7 \end{pmatrix}. \tag{59}$$

Finally, charge conjugation will be given by the antilinear real structure J_f that interchanges the upper 15-dimensional block with the lower one, particles with antiparticles,

$$J_f = \begin{pmatrix} 0 & 1_{15} \\ 1_{15} & 0 \end{pmatrix} \circ cc. \tag{60}$$

It is readily checked that this finite geometry satisfies the relations by table 1 for $n = 0$.

It is not easy for one to see at once the motivation behind this particular definitions, but it is enough to mention here that this model can be gradually built up from simpler ones. Of course the construction is also carefully driven by the consistency with the axioms for a noncommutative geometry. These conditions are highly non-trivial and can be readily checked to be satisfied by the above model (c.f. [14]).

We repeat for emphasis the key ingredient that one should never underestimate; it is the existence of the Standard Model group as a subgroup of $\mathcal{U}(\mathcal{A}) = U(1) \otimes SU(2) \otimes U(3)$. This is crucial for the derivation of the Standard Model bosons as gauge fields coming from the inner fluctuation of the Dirac operator D.

The full reconstruction of the Standard Model Lagrangian will be feasible after defining the *spectral action* [2].

4 Gauge symmetries and inner fluctuations

In the framework of a gauge theory on a generally non-flat manifold M we saw that the corresponding Lagrangian (or more accurately, the action functional), will satisfy a number of symmetries, which will form the symmetry group $\mathcal{G}$ of the theory. Of course from a GR point of view, the several fields need to transform in such a way that the theory is generally covariant, i.e. the group of diffeomorphisms $Diff(M)$, apart from being a symmetry subgroup of $\mathcal{G}$, will also act on the gauge group of the matter Lagrangian $\mathcal{G}_{SM}$, since a change of coordinates induced by a diffeomorphism will transform the frame.

This action turns the group of local gauge transformations into a normal subgroup of $\mathcal{G}$ and thus we can write $\mathcal{G}$ as a semi-direct[8] product of the two:

$$\mathcal{G} = \mathcal{G}_{SM} \rtimes Diff(M). \tag{62}$$

In other words, we have the short exact sequence of groups

$$1 \to \mathcal{G}_{SM} \to \mathcal{G} \to Diff(M) \to 1 \tag{63}$$

We now come to the very important notion of algebra automorphisms and some of their properties. Let $\mathcal{A}$ be an involutive unital algebra, and define the set of algebra endomorphisms $End(\mathcal{A})$ as the algebra *-homomorphisms from $\mathcal{A}$ to itself. Then the automorphisms of $\mathcal{A}$ are the bijective endomorphisms and are denoted by $Aut\mathcal{A}$. [9] The set $Aut(\mathcal{A})$ forms a group, called the group of algebra automorphisms, with group operation defined by composition of automorphisms (which gives again an automorphism). The existence of an inverse element is guaranteed by bijectivity of the elements, the identity element is just $1_{Aut(\mathcal{A})} : a \mapsto a \forall a \in \mathcal{A}$ and associativity is clear.

Also note that if $\pi : \mathcal{A} \mapsto \mathcal{B}(\mathcal{H})$ is a faithful representation of the algebra on a Hilbert space $\mathcal{H}$, any automorphism α induces a new representation π_u by pullback:

$$\pi_u(a) := \pi \circ \alpha. \tag{64}$$

In the commutative case, where $\mathcal{A} = C^\infty(M)$ is just an algebra of smooth functions on a manifold M, the algebra automorphisms correspond exactly to the group of diffeomorphisms $Diff(M)$. This comes from a known result for unital C^*-algebras (M is compact), and from the observation that $\Omega(\mathcal{A}) = \Omega(A)$, with $A = \bar{\mathcal{A}}$. For the full proof look at chapter 1 of [8]. It is clear that a diffeomorphism induces an automorphism by pullback but the above result states that every automorphism is of this sort. As we shall see, this is not entirely true for noncommutative spaces.

[8]Recall that a semi-direct product of two groups G and H, given an action ρ of G on H, is defined as the direct product of the two, equipped with the multiplication law

$$(g, h)(g', h') = (gg', h\rho(g)h') \tag{61}$$

[9]Note that as endomorphisms, they are required to respect the involution and of course $\alpha(1_{\mathcal{A}}) = 1_{\mathcal{A}}$

Denote by $\mathcal{U}$ the group of unitary elements of $\mathcal{A}$:

$$\mathcal{U} = \{u \in \mathcal{A} | uu^* = u^*u = 1_{\mathcal{A}}\} . \tag{65}$$

Each of these elements can define an automorphism α_u on $\mathcal{A}$ via conjugation, since for $u \in \mathcal{U}$ we have

$$\alpha_u(a) = uau^{-1} \tag{66}$$

$$\alpha_{uv}(a) = uvav^{-1}u^{-1} = u\alpha_v(a)u^{-1} \tag{67}$$

$$\alpha_u(ab) = uabu^{-1} = uau^{-1}ubu^{-1} = \alpha_u(a)\alpha_u(b) \tag{68}$$

$\forall a, b \in \mathcal{A}$ and also, by unitarity (only),

$$\alpha_u^{-1} = \alpha_{u^{-1}} = \alpha_{u^*} \tag{69}$$

$$\alpha_u(a) = uau^{-1} = uau^* \quad \text{, hence} \tag{70}$$

$$\alpha_u(a^*) = ua^*u^* = (uau^*)^* = \alpha_u(a)^*. \tag{71}$$

These automorphisms form a normal subgroup of $Aut(\mathcal{A})$ denoted by $Inn(\mathcal{A})$ and are called the *inner automorphisms* of $\mathcal{A}$. The fact that $Inn(\mathcal{A}) \triangleleft Aut(\mathcal{A})$ is easily checked, take $u \in \mathcal{U}, \alpha \in Aut(\mathcal{A})$, then

$$\alpha \circ \alpha_u \circ \alpha^{-1}(a) = \alpha(u\alpha(a)u^*) = \alpha(u)a\alpha(u^*) = \alpha(u)a\alpha(u)^*, \tag{72}$$

where we used that α is an algebra *-homomorphism. But this is again an automorphism of the form vav^* with $v = \alpha(u) \in \mathcal{U}$ because $\alpha(u)\alpha(u)^* = \alpha(uu^*) = \alpha(1_{\mathcal{A}}) = 1_{\mathcal{A}} = \alpha(u)^*\alpha(u)$.

The next step of course will be to take the quotient group

$$Out(\mathcal{A}) = Aut(\mathcal{A})/Inn(\mathcal{A}) \tag{73}$$

which corresponds to the classes of "outer automorphisms". Then we can write the above groups in a short exact sequence

$$1_{\mathcal{A}} \to Inn(\mathcal{A}) \to Aut(\mathcal{A}) \to Out(\mathcal{A}) \to 1_{\mathcal{A}} \tag{74}$$

For commutative unitary elements, that lie in the center of the algebra $Z(\mathcal{A})$, we have that $uau^* = uu^*a = a$ and these elements will be mapped by $u \mapsto \alpha_u$ to the identity inner automorphism, so only classes of unitary-modulo-central elements will be relevant.

The crucial point to see is that this subgroup of inner automorphisms is absent in the commutative case, precisely because of the commutativity of the algebra. As in the canonical example, $\mathcal{A}$ is an algebra of smooth functions over some space $\mathcal{A} = C^\infty(M)$ [10]. In this case, the group of inner automorphisms will be trivial, since $\alpha_u = 1_{Aut(\mathcal{C}^\infty(\mathcal{M}))}$ and consequently we will have $Aut(\mathcal{A}) = Out(\mathcal{A}) = Diff(M)$. The last equality also identifies the *outer* automorphisms of the algebra with the group of diffeomorphisms on $M = \Omega(\mathcal{A})$.

This is exactly where the new ingredient of noncommutative geometry comes in: the existence of nontrivial automorphisms of the form uau^{-1}. With respect to

[10] this is always true, due to a reconstruction theorem by Connes

the discussion of section 2, the role of the group of local gauge transformations will now be played by $Inn(\mathcal{A})$ and the big symmetry group $\mathcal{G}$ of the action will correspond to the group of automorphisms thus inheriting the structure of the semi-direct product above. These will become more clear in the following sections.

4.1 Equivalences

In this section we will try to define the analogue of gauge degrees of freedom in the spectral triple formalism. This will in fact involve certain kinds of equivalences between spectral triples, resulting to an ambiguity in the definition of the Dirac operator. Since the metric and all geometric properties are derived from this part of the triple, this ambiguity will be associated to what we will rightfully call the *inner fluctuations of the metric* in noncommutative geometry.

But first a few definitions need to be reminded, in view of the desired foliation of metrics (and thus spectral triples) into equivalence classes.

Unitary Equivalence The first related notion to be reminded is that of *unitary equivalence* between representations of algebras on Hilbert spaces. One can then extend this notion to an equivalence between spectral triples, which one defines as follows:

Definition. *Two (possibly real, even) spectral triples* $(\mathcal{A}_i, \mathcal{H}_i, D_i, J_i, \gamma_i)$, $i = 1, 2$ *with explicit representations* $\pi_i : \mathcal{A}_i \to \mathcal{B}(\mathcal{H}_i)$ *are said to be* unitarily equivalent *if*

- *their Hilbert spaces are isomorphic,* $\mathcal{H}_1 \cong \mathcal{H}_2$*, and there exists a unitary operator* $U : \mathcal{H}_1 \to \mathcal{H}_2$, $UU^* = U^*U = 1$*, such that*
- $U \circ \pi_1 \circ U^* = \pi_2$,
- $UD_1D^* = D_2$ *, and (if applicable)*
- $UJ_1U^* = J_2$,
- $U\gamma_1U^* = \gamma_2$,

i.e. all components of the two spectral triples are unitarily intertwined by U.

Morita Equivalence There is another equivalence relation between algebras called *Morita equivalence* that is almost as strong as isomorphism of algebras but not quite. Here "almost" accounts for the fact that the two relations are equivalent when restricted to commutative algebras yet they differ when considering the general noncommutative case. Morita equivalence will prove to be very useful in spectral geometry since two algebras that are Morita equivalent may not be isomorphic but do look the same in the level of representations, which allows for a new important classification of algebras to emerge. For the details of the definition, one can refer to [12] or the lecture notes of Landsman, where Morita equivalence, as well as the related notions of Hilbert modules, Hilbert bimodules and Clifford modules are neatly defined.

4.2 Fluctuations of the geometry

In the spectral triple formulation of geometry the relevant "gauge" would, roughly speaking, have to come from automorphisms of the algebra, so on a first approach we are looking for a sort of equivalent spectral triples that have the same algebra component. However the equality of the algebras is not in principle necessary in order to talk about equivalence between noncommutative spaces; it should be relaxed to a weaker equivalence relation between the algebras, one that does not affect the geometrical entity of the triple, i.e. the representation, Dirac operator and extra structure of the defining axioms (up to unitary equivalence).

We already have such an equivalence on algebras, namely the Morita equivalence described above, and this will be our starting point. It remains to be proven that one can construct all the necessary components that define a new noncommutative space. Furthermore, since an algebra $\mathcal{A}$ is always equivalent to itself, we can describe the family of equivalent spectral triples when we set $\mathcal{B} = \mathcal{A}$, being particularly interested in such an equality coming from inner automorphisms. So let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple where an explicit representation $\pi : \mathcal{A} \to End(\mathcal{H})$ is understood.

To this end, the following course of action will be taken:

- Demonstrate the construction of a new spectral triple $(\mathcal{B}, \mathcal{H}', D')$ associated to the initial one by a **Morita equivalence** of their spectral algebras $\mathcal{A} = End_{\mathcal{A}}(\mathcal{E})$.
- Reconsider and modify this result in the case of a **real spectral triple** with possible chirality (extra structure).
- Reduce the above to the simplest case where $\mathcal{B} = \mathcal{A}$. See how the new Dirac operator D would look like.
- Again consider a real structure and modify accordingly.
- Finally, consider the explicit occasion where the above is induced by an inner automorphism $\alpha_u \in Inn(\mathcal{A})$. To which unitary intertwiner U does this correspond to? What is the gauge degree of freedom?

So start by letting an algebra $\mathcal{B}$ be Morita equivalent to $\mathcal{A}$, which is known to have the form of the space of $\mathcal{A}$-endomorphisms of some finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$:

$$\mathcal{B} = End_{\mathcal{A}}(\mathcal{E}), \tag{75}$$

where $\mathcal{E}$ being f.g.p. means that, as a right module over $\mathcal{A}$, we can write it in the form $\mathcal{E} = p\mathcal{A}^N$ for some $N \in \mathbb{N}$ and some idempotent $p \in M_N(\mathcal{A})$ [12]. In fact, an important feature of these modules is that they are *Hilbert modules* over $\mathcal{A}$, i.e. they come with an $\mathcal{A}$-valued Hermitian structure, a sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ with all the related properties [11] . This will be the algebra

[11] These are positive definiteness and non-degeneracy of $\langle , \rangle$, and :

$$\langle \eta, \zeta \rangle^* = \langle \zeta, \eta \rangle \quad , \tag{76}$$

$$\langle \eta, \zeta \rangle \, a = \langle \eta, \zeta a \rangle \, . \tag{77}$$

component of the new spectral triple and as required it is unital and involutive, so now we need to determine the new Hilbert space $\mathcal{H}'$ on which it will act. For this we observe that there is a natural action of the algebra $\mathcal{B}$ on the space

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \tag{78}$$

defined as follows. Take an element [12] $\eta \otimes \psi \in \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ and an algebra element $\beta \in \mathcal{B}$. Then define

$$\beta(\eta \otimes \psi) = \beta(\eta) \otimes \psi \tag{79}$$

and this will be well defined with respect to the $\mathcal{A}$-linearity, since β is also $\mathcal{A}$-linear. Hence for $a \in \mathcal{A}$ one gets

$$\beta(\eta a \otimes \psi) = \beta(\eta a) \otimes \psi = \beta(\eta) a \otimes \psi = \beta(\eta) \otimes a\psi = \beta(\eta \otimes a\psi).$$

The product space $\mathcal{H}'$ will need to be equipped with a complete inner product, with respect to which, the above action of the algebra is a *-representation. This comes via the known inner product $\langle \psi, \psi' \rangle_{\mathcal{H}}$ on the original Hilbert space and the Hermitian structure on $\mathcal{E}$:

$$\langle \eta \otimes \psi, \eta' \otimes \psi' \rangle_{\mathcal{H}'} := \langle \psi, \langle \eta, \eta' \rangle_{\mathcal{A}} \psi' \rangle_{\mathcal{H}} . \tag{80}$$

One checks that this is again well defined, positive definite, complete and sesquilinear. Indeed the inner product respects the involution on $End_{\mathcal{A}}(\mathcal{E})$ which is defined by the transposition $< \beta^* \eta, \eta' >_{\mathcal{A}} = < \eta, \beta \eta' >_{\mathcal{A}}, \quad \forall \eta, \eta' \in \mathcal{E}$, since

$$\begin{aligned} \langle \beta^*(\eta \otimes \psi), \eta' \otimes \psi' \rangle_{\mathcal{H}'} &= \langle \beta^*(\eta) \otimes \psi, \eta' \otimes \psi' \rangle_{\mathcal{H}'} \\ &= \langle \psi, \langle \beta^*(\eta), \eta' \rangle_{\mathcal{A}} \psi' \rangle_{\mathcal{H}} \\ &= \langle \psi, \langle \eta, \beta(\eta') \rangle_{\mathcal{A}} \psi' \rangle_{\mathcal{H}} \\ &= \langle \eta \otimes \psi, \beta(\eta' \otimes \psi') \rangle_{\mathcal{H}'} \quad \forall \eta, \eta' \in \mathcal{E}, \psi, \psi' \in \mathcal{H}. \end{aligned} \tag{81}$$

Moreover, linearity is obvious in the second argument as well as anti-linearity in the first, by the respective properties of the inner product on $\mathcal{H}$, or even going through the Hermitian structure first.

The next step is of course the Dirac operator D' acting on $\mathcal{H}'$ using the known Dirac operator on $\mathcal{H}$. The obvious choice $D' = 1 \otimes D \in \mathcal{B}(\mathcal{H}')$ i.e. $D'(\eta \otimes \psi) = \eta \otimes D\psi$, does not work for $\mathcal{A}$-linearity, since

$$D'(\eta a \otimes \psi) = \eta a \otimes D\psi = \eta \otimes aD\psi \neq \eta \otimes Da\psi = D'(\eta \otimes a\psi)$$

for an arbitrary $a \in \mathcal{A}$ that may not commute with D. We will need to introduce the notion of a connection on $\mathcal{E}$ to counteract with this problem. For a more holistic discussion on *universal connections* check section (7.2) in [9] and of course [4].

For now we only need to define a connection as a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ satisfying the Leibnitz rule, [13]

$$\nabla(\eta a) = (\nabla \eta) a + \eta \otimes [D, a] \quad \forall \eta \in \mathcal{E}, a \in \mathcal{A}. \tag{82}$$

Moreover it is required that the sesquilinear map defines a complete norm w.r.t. the C^*-algebra closure $\bar{\mathcal{A}} = A$.

[12] In the calculations we will always suppress the tensoring index for simplicity

[13] any such connection uniquely defines a universal connection on the universal graded algebra $\Omega^\bullet \mathcal{A}$ which raises the order by one

Here some explanation is needed. The set

$$\Omega^1_D = \left\{ \sum_j a_j \, [D, b_j] : a_j, b_j \in \mathcal{A} \right\}$$

has the structure of an $\mathcal{A}$-bimodule, hence the action of a from the right in the first term. The commutator $[D, a]$, often denoted by da or even $\delta(a)$, is an element of this bimodule, so the second term defines an element in the target space $\mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$. Notice the presence of D in the definition, which indicates how the Dirac operator defines the differential forms on a spectral triple. It is worth to mention that a connection on $\mathcal{E}$ as defined above, uniquely determines a universal connection and that such a connection always exists, c.f. [9] chapters (6) and (7). Moreover, ∇ is required to be a *Hermitian connection* in order to guarantee compatibility with the Hermitian structure on $\mathcal{E}$, and therefore needs to satisfy

$$(\nabla \eta, \zeta) - (\eta, \nabla \zeta) = d(\eta, \zeta). \tag{83}$$

Now D' can be defined by combining the naive expression with a term that involves the connection:

$$D'(\eta \otimes \psi) = \eta \otimes D\psi + (\nabla \eta)\psi \tag{84}$$

The second term involves an action of $\nabla \eta \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ on $\psi \in \mathcal{H}$ which is understood by treating Ω^1_D as a subspace of $\mathcal{A}$ and acting by the restriction of the representation, i.e. if $\eta \in \mathcal{E}, \omega \in \Omega^1_D$, then $(\eta \otimes \omega)\psi = \eta \otimes (\omega\psi) \in \mathcal{H}'$. One can now use (82) and check that

$$\begin{aligned} D'(\eta a \otimes \psi) &= \eta a \otimes D\psi + \nabla(\eta a)\psi = \eta \otimes aD\psi + (\nabla \eta) a\psi + (\eta \otimes [D, a])\psi \\ &= \eta \otimes Da\psi + (\nabla \eta) a\psi = D'(\eta \otimes a\psi) \end{aligned} \tag{85}$$

which is exactly what we needed for the Dirac operator to be well defined. Moreover D' as in (84) is self adjoint, by self adjointness of D and the Hermitian property (83) and Leibnitz rule (82) of the connection, and can be checked to also satisfy all the remaining required axioms.

Thus we can now write $(\mathcal{B}, \mathcal{H}', D') = (\mathcal{B}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}, 1_{\mathcal{E}} \otimes D + \nabla \otimes 1_{\mathcal{H}})$.

The situation is slightly different when the triple is equipped with a real structure J. In this case we saw in 24 that J induces an $\mathcal{A}$-bimodule structure on H by $\psi b = b^0 \psi$. This needs to be recovered in a new real structure J' and for this, the new Hilbert space $\mathcal{H}'$ needs to be modified to the more symmetric form

$$\mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \bar{\mathcal{E}} \quad , \tag{86}$$

where $\bar{\mathcal{E}} = \{\bar{\eta} : \eta \in \mathcal{E}\}$ is the *conjugate module* of $\mathcal{E}$ with the left $\mathcal{A}$-module structure

$$\mathcal{A} \times \bar{\mathcal{E}} \to \bar{\mathcal{E}} \quad , \quad a\bar{\eta} = \eta \bar{a}^*, \quad a \in \mathcal{A}, \bar{\eta} \in \bar{\mathcal{E}}. \tag{87}$$

Let $\eta, \zeta \in \mathcal{E}$, $\psi \in \mathcal{H}$. We define the action of $\mathcal{B} = End_{\mathcal{A}}(\mathcal{E})$ on $\mathcal{H}'$ by simply

$$b(\eta \otimes \psi \otimes \zeta) = (b\eta) \otimes \psi \otimes \bar{\zeta} \quad , \quad \forall b \in \mathcal{B}. \tag{88}$$

Then $\mathcal{H}'$ has again a natural inner product, defined in a similar manner to (80), now including the natural Hermitian structure of $\bar{\mathcal{E}}$ and reads

$$\left\langle \eta \otimes \bar{\zeta}, \eta' \otimes \psi' \otimes \bar{\zeta}' \right\rangle_{\mathcal{H}'} := \left\langle \psi, \langle \eta, \eta' \rangle_{\mathcal{A}} \psi' \left\langle \bar{\zeta}', \bar{\zeta} \right\rangle_{\mathcal{A}} \right\rangle_{\mathcal{H}}. \tag{89}$$

Before getting to the expression of D', the new real structure will be given by simply $1 \otimes J \otimes 1$, i.e.

$$J'(\eta \otimes \psi \otimes \bar{\zeta}) = \zeta \otimes J\eta \otimes \bar{\eta} \tag{90}$$

Clearly we will need to define a new Hermitian connection on the conjugate, left $\mathcal{A}$-module, $\bar{\mathcal{E}}$, of the form

$$\bar{\nabla} : \mathcal{E} \to \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \bar{\mathcal{E}}, \tag{91}$$

that also satisfies the Leibnitz rule (but from the left),

$$\bar{\nabla}(a\bar{\eta}) = a\bar{\nabla}\bar{\eta} + [D, a] \otimes_{\mathcal{A}} \bar{\eta} \quad \forall a \in \mathcal{A}. \tag{92}$$

Nothing really prevents us from making use of the unbarred connection in order to define the barred one, by just flipping the objects involved, since $\Omega^1_D(\mathcal{A})$ is an $\mathcal{A}$-bimodule. So we write

$$\bar{\nabla}\bar{\eta} := flip(\nabla\eta) \quad \bar{\eta} \in \bar{\mathcal{E}}, \tag{93}$$

where the map $flip : \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A} \to \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \bar{\mathcal{E}}$ flips the order of the factors in an anti-linear way, like

$$flip(\eta \otimes \omega) = \omega^* \otimes \bar{\eta}, \tag{94}$$

for $\eta \in \mathcal{E}$ and $\omega \in \Omega^1_D(\mathcal{A})$.

Then again compatibility with $\mathcal{A}$-linearity, suggests that the Dirac operator D' should have the form

$$D'(\eta \otimes \psi \otimes \bar{\zeta}) = (\nabla\eta)\psi \otimes \bar{\zeta} + \eta \otimes D\psi \otimes \bar{\zeta} + \eta \otimes \psi(\bar{\nabla}\bar{\zeta}), \tag{95}$$

as before. One can check that the same ϵ' for the reality condition (16) is recovered in the primed spectral triple, as well as all defining axioms.

Finally, in the case of a chiral operator γ in the original space, one can construct such an operator $\gamma' = 1 \otimes \gamma \otimes 1$ on the new space:

$$\gamma'(\eta \otimes \psi \otimes \bar{\zeta}) = \eta \otimes \gamma\psi \otimes \bar{\zeta}. \tag{96}$$

4.3 Inner fluctuations

We will now have a closer look to the transformations induced by a Morita *self-equivalence*, when the spectral algebra is isomorphic to the original. This amounts to setting $\mathcal{E} = \mathcal{A}$ and observing that $\mathcal{B} = End_{\mathcal{A}}(\mathcal{A}) \cong \mathcal{A}$

Concerning the Hilbert space $\mathcal{H}$ it will clearly remain the same (up to isomorphism), since in (78) we have $\mathcal{H}' = \mathcal{A} \otimes_{\mathcal{A}} \mathcal{H} \cong \mathcal{H}$, which is given by the isomorphism

$$\begin{aligned} \phi : \quad \mathcal{A} \otimes_{\mathcal{A}} \mathcal{H} &\rightarrow \mathcal{H} \\ 1_{\mathcal{A}} \otimes_{\mathcal{A}} \psi &\mapsto \psi \quad , \end{aligned} \tag{97}$$

which obviously respects the tensor product structure and is invertible and indeed an isomorphism.

Let's see now what we can say about the fluctuations of the spectral geometry by deriving the possible Dirac operators in this case. We have the freedom of choosing a Hermitian connection on $\mathcal{A}$ but now we only need to define D' on the elements $1 \otimes \psi \in \mathcal{H}'$ since it can then be extended by $\mathcal{A}$-linearity. But

$$D'(1 \otimes \psi) = 1 \otimes D\psi + \nabla(1)\psi \quad , \tag{98}$$

and thus, the new Dirac operator will be uniquely determined by the value of $\nabla(1) \in \Omega^1_D$. We can recall the isomorphism (97) and rewrite $D' : \mathcal{H} \rightarrow \mathcal{H}$ by

$$D' = D + \nabla(1) \tag{99}$$

It is now clear that a Morita self-equivalence will induce a metric fluctuation of the form

$$D \mapsto D + A \tag{100}$$

where A is an element of the bimodule $\Omega^1_D = \left\{ \sum_j a_j \, [D, b_j] : a_j, b_j \in \mathcal{A} \right\}$ that had better be self adjoint, should we want to recognize D' as a Dirac operator. We will call any such one-form $A = A^* \in \Omega^1_D$ a *gauge potential* or *gauge field*.

Things are slightly different when one considers a real spectral triple with a real structure J and a chirality operator γ. According to the derivation in (86) we obtain the Hilbert space

$$\mathcal{H}' = \mathcal{A} \otimes \mathcal{H} \otimes \mathcal{A} \cong \mathcal{H}$$

by recalling the $\mathcal{A}$-bimodule (or equivalently the $\mathcal{A} \otimes \bar{\mathcal{A}}$-module) structure of $\mathcal{H}$ induced by J. The Dirac operator will be of the form (95) and in particular,

$$\begin{aligned} D'\psi &= D'(1 \otimes \psi \otimes \bar{1}) = \nabla(1)\psi + D\psi + \psi\bar{\nabla}(\bar{1}) & (101) \\ &= D\psi + A\psi + \epsilon'\psi A^* = D\psi + A\psi + \epsilon' JAJ^{-1}\psi \Rightarrow & (102) \\ D' &= D + A + \epsilon' JAJ^{-1}, & (103) \end{aligned}$$

which gives the metric fluctuation in this case.

It is now interesting to continue the discussion concerning the group of unitaries $\mathcal{U}$ in (65). Recall that any algebra automorphism α of $\mathcal{A}$ will induce a new representation $\pi_\alpha = \pi \circ \alpha$ and we will consider in particular the case when α is an inner automorphism of the form $u \cdot u^*$, i.e. $\alpha \in Inn(\mathcal{A})$. This will give the gauge or *inner deformations* of a noncommutative geometry, commonly referred to as the *inner fluctuations of the metric*.

Viewing $\mathcal{H}$ as an $\mathcal{A}$-bimodule one defines the action of $\mathcal{U}$ on $\mathcal{H}$ by the "adjoint representation" $Ad : \mathcal{U} \to \mathcal{B}(\mathcal{H})$ given by:

$$Ad(u)\psi = u\psi u^* = uJuJ^{-1}\psi \quad , \forall u \in \mathcal{U}, \psi \in \mathcal{H}. \tag{104}$$

Note that the $*$ on the right is important in order for Ad to be a homomorphism,

$$uv\psi(uv)^* = uv\psi v^* u^* = Ad(u)Ad(v)\psi. \tag{105}$$

From (104) we can see that an inner automorphism $\alpha_u \in Aut\mathcal{A}$ corresponds to a unitary equivalence by the element

$$U = uJuJ^{-1}. \tag{106}$$

First of all, it is clear that

$$Uu * xuU* = x \Rightarrow Ad(U) = Ad(u). \tag{107}$$

Now we have to consider what change this automorphism will bring to the Dirac operator D and call it the related inner fluctuation of the metric. The unitary equivalence suggests that $D_u = UDU^{-1}$ [14] and also for the rest $J_u = UJU^{-1}$, $\gamma_u = U\gamma U^{-1}$. The last two are rather trivial if one uses the defining properties of J and γ:

$$J_u = uJuJ^{-1}Ju^*Ju^*J^{-1} = \epsilon JuJ^{-1}J^{-1}u^* = \frac{\epsilon}{\epsilon}J = J \quad , \tag{108}$$

$$\gamma_u = uJuJ^{-1}\gamma Ju^*J^{-1}u^* = \epsilon'' uJu\gamma u^*J^{-1}u^* = \epsilon^2\gamma = \gamma, \tag{109}$$

where we used the (anti-)commutation relations in table 1, the order one property (22) and commutativity of γ with $u \in \mathcal{A}$.

For the Dirac operator the unitary equivalence yields

$$\begin{aligned} D_u &= UDU^* = uJuJ^{-1}DJu^*J^{-1}u^* = \epsilon' uJuDu^*J^{-1}u^* \\ &= \epsilon' uJu[D,u^*]J^{-1}u^* + \epsilon' uJuu^*DJ^{-1}u^* \\ &= \epsilon' J(u[D,u^*])J^{-1} + \epsilon'^2 uDu^* = \\ &= D + u[D,u^*] + \epsilon' J(u[D,u^*])J^{-1} = D + A + \epsilon' JAJ^{-1}, \end{aligned} \tag{110}$$

where we put $A = u[D,u^*] \in \Omega^1_D$, which we can identify by (101) as the corresponding gauge potential for the inner automorphism of u, and we used $[u, JAJ^{-1}] = 0, A \in \Omega^1_D \subset \mathcal{A}$. A is clearly self-adjoint, by self-adjointness of u, D and $A = uDu^* - D$.

One can easily repeat the above calculations, starting from a generic Dirac operator of the form (101) and see how the gauge field A transforms under the action of an inner automorphism. This will give the transformed A_u, usually denoted by $\gamma_u(A)$:

$$\gamma_u(A) = u[D,u^*] + uAu^*. \tag{111}$$

Note here that the above fluctuations are trivial in the case of a commutative Riemannian geometry therefore returning the initial Dirac operator. In the general noncommutative setting, these special fluctuations will give a classification

[14] here note that $U^{-1} = Ju^*J^{-1}u^* = u^*Ju^*J^{-1}$

of the space of metrics under the action of the nontrivial normal subgroup of inner automorphisms.

Moreover it is almost trivial to check that inner fluctuations of a fluctuated metric also give inner fluctuations. In the first stage where the metric is fluctuated by the self-adjoint gauge $A \in \Omega^1_D(\mathcal{A})$ as in (100), a second fluctuation by another gauge field $B \in \Omega^1_{D'}(\mathcal{A})$ will result to a total fluctuation by $A = A + B$. It remains to be shown that A', or equivalently B, is in fact a gauge field in $\Omega^1_D(\mathcal{A})$, which is obvious by writing any $a[D', b]$ as

$$a[D + A, b] = a[D, b] + a[A, b] \quad \in \Omega^1_D(\mathcal{A}) \tag{112}$$

using the $\mathcal{A}$-bimodule structure of $\Omega^1_D(\mathcal{A})$ in the second term.

Similarly, in the second stage where a real structure is assumed, we have the additive metric fluctuation (101), with a subsequent fluctuation $B \in \Omega^1_{D'}(\mathcal{A})$ giving a total fluctuation

$$D'' = D + (A + B) + \epsilon' J(A + B)J^{-1}.$$

Again we need to verify that $\Omega^1_{D'} \subset \Omega^1_D$, by checking a typical basis element $a[D', b]$ which reads

$$a[D + A \pm JAJ^{-1}, b] = a[D, b] + a[A, b] \pm a[JAJ^{-1}, b], \tag{113}$$

the only new entry being the last term, which actually vanishes by the order one condition (22).

4.4 NCG Standard Model for beginners: bosonic sector

Now that the principles of metric fluctuations are discussed, it is time to see in practice what are the gauge fields of the spectral version of the Standard Model. Recall that in section 3.7 the related almost commutative spectral triple was defined, with the algebra $\mathcal{A}$ having the form (45), where the finite algebra was defined in (52). As mentioned above, in the commutative part of the algebra that corresponds to the spin manifold M, the group of inner automorphisms is trivial, therefore it is only interesting to examine the inner automorphisms of the finite algebra $\mathcal{A}_f$.

The only nontrivial automorphism of the first component $\mathbb{C}$ is complex conjugation, which is not connected to the identity and will therefore be irrelevant in the physical theory. For the quaternions we have that, in their realization as 2x2 matrices, their unitary elements $u \in \mathcal{U}(\mathbb{H})$ will be unitary matrices of the special quaternionic form. The unitarity property automatically gives that $u \in SU(2)$, since

$$uu^* = det(u)1_2 \Rightarrow det(u) = 1 \quad , \tag{114}$$

and in fact the group of unitaries *is* $SU(2)$. The central unitary elements are of course $\pm 1_2$ and therefore we quotient by this abelian group $\mathcal{U}(\mathbb{H}) \cap Z(\mathbb{H}) = \mathbb{Z}_2$ to obtain the elements that contribute to the inner automorphisms, namely $SU(2)/\mathbb{Z}_2$. Finally, the unitaries of the last component, $M_3(\mathbb{C})$ consist of course the group $U(3)$.

In a more general note, any algebra of a finite spectral triple will be made up of components of the three types, $M_n(\mathbb{R}), M_n(\mathbb{C})$ and $M_n(\mathbb{H})$, with unitary groups $O(n), U(n)$ and $USp(n)$ respectively (the last being the group of unitary symplectic 2nx2n matrices). Their groups of central unitaries are $\mathbb{Z}_2 \cong \pm 1_n$, $U(1) \cong U(1) \otimes 1_n$ and $\mathbb{Z}_2 \cong \pm 1_{2n}$. Here it happens that for the algebra $\mathbb{H}$, $USp(1) = SU(2)$.

It follows that for the total Standard Model finite algebra one gets

$$\mathcal{U}(\mathcal{A}_f) = U(1) \times SU(2) \times U(3) \quad , \tag{115}$$

with the central unitaries

$$\mathcal{U}(\mathcal{A}_f) \cap Z(\mathcal{A}_f) = \mathbb{Z}_2 \times U(1) \times U(1). \tag{116}$$

Observe how the $U(1)$ group will not contribute to the inner automorphisms, because of commutativity. One may suspect that this is exactly the reason why Kaluza-Klein theory was partially successful, but only for unifying gravity with electromagnetism. Now we find the group $Inn(\mathcal{A}_f)$ of inner automorphisms by taking the quotient with (116),

$$Inn(\mathcal{A}_f) = (SU(2) \times SU(3)) \,/\, (\mathbb{Z}_2 \times \mathbb{Z}_3) \quad , \tag{117}$$

and observe how the initial $U(3)$ was reduced by using the property $U(n)/U(1) = SU(n)/\mathbb{Z}_n$.

The elements will now be lifted to act on the Hilbert space by the construction of the lifted automorphisms discussed in the appendix. If one wants to include abelian fields such as the E/M or Higgs fields, one needs to *centrally extend* the spin lift, following the calculations in [14]. One can now take the inner automorphisms in their explicit form for the Standard Model algebra and fluctuate the Dirac operator as described in section 4.3.

Of course all these constructions would be incomplete, without the existence of an analogue of the action principle for noncommutative spaces. Such a formulation exists, and was studied by Connes and Chamseddine in 1997 [2] as an attempt to produce the dynamics and all interactions of the Standard Model including the Higgs mechanism. The (fermionic) *spectral action* principle is encoded in the functional

$$S_f[\psi; A] := \langle \psi, D_A \psi \rangle \quad , \tag{118}$$

where D_A is the Dirac operator fluctuated by the gauge field A.

This approach turned out to also predict relations between gauge coupling constants just like in some GUT models. A quite remarkable aspect is that the spectral action provides not only a gravitational plus matter action but also a Higgs mechanism "for free" and all particle masses fall naturally into place. Of course the predictions are yet to be verified or rejected by forthcoming experiments. Due to shortness of spacetime, further details on this subject will not be studied in this essay.

5 Conclusions - Outlook

Despite its celebrated outstanding performance in experimental accuracy and predictive power, the Standard Model has left many questions unanswered. Moreover, the arbitrariness by which the several fields and parameters of the model are introduced cries out for a more justified approach. One would say that the secret desire of particle physicists is to come up with a purely geometrical theory, in which all interactions arise from internal degrees of freedom of the underlying geometry. In this sense, the current approach by noncommutative geometry is extremely promising and, compared to other similar attempts, it also has the advantage of being physically relevant.

Furthermore, the variety of fronts that noncommutative geometry has opened in different fields of mathematics and physics, inspires confidence that as a mathematical construction, it is pointing towards the right direction.

This short essay could only capture a limited amount of remarkable things, among the many that have been coming out of the rich world of noncommutative geometry. The notion of symmetry in theoretical physics is studied in this new framework, obtaining a whole new perspective, as noncommutative spaces provide the ground for a purely geometric interpretation of all fundamental interactions. The idea of an action functional describing the dynamics of the geometry and matter fields is also passed to noncommutative spaces through the spectral action.

It is not clear whether the current theory could evolve into a *fundamental* one, even after a few modifications. There are already enhanced versions of NCG Standard Model including neutrino mixing and other more sophisticated adjustments. It would be extremely interesting though, to see how this approach can become even more fundamental, whether there exist underlying rules that narrow down the choices of input. But on the other end, a more intuitive version of the numerous axioms and properties, for the common mortals, would also be nice. Even only to avoid justified criticism of the type “Come on, Connes, you’re making it up as you go along, aren’t you!”.

Appendix

Spin group and lifted automorphisms In gauge theories we deal with a principal bundle P for some structure group G, while the quotient space P/G is the spacetime manifold M. In this context we want to identify the several diffeomorphism groups and actions. A generic diffeomorphism in $Diff(P)$ will not in principle respect the object's bundle structure [15], a property which ought to be satisfied for an element in the group of automorphisms of P.

Let $\phi \in Diff(P)$. If the points $p, q \in P$ are related via a group element $g \in G$ by $q = pg$, i.e. they lie on the same fiber, then their images under an automorphism need to be related by the same element, $\phi(q) = \phi(p)g$, so that fibers are mapped isomorphically to fibers. This implies the commutativity of the two maps ϕ and R_g and thus we define the automorphisms of P to be

$$Aut(P) = \{\phi \in Diff(P) : \phi \circ R_g = R_g \circ \phi, \forall g \in G\}. \tag{119}$$

It is worth noting that here, we consider diffeomorphisms that are "close to the identity". This means that 1) the diffeomorphism is isotopic to the identity (it is connected to the identity through a family of isomorphisms H_t) and 2) the related isotopy is *compactly supported*, i.e. $\exists K \subset M$, cpt, such that $H_t(x) = x, \forall t \in [0, 1], x \in M \backslash K$.

Then we also have the so called *vertical* gauge transformations which correspond to the *gauge group Gau* consisting of the automorphisms that leave the base points intact, in other words,

$$Gau(P) = \{\phi \in Aut(P) : \pi \circ \phi = \pi\}. \tag{120}$$

In the case of a trivial bundle of the form $M \times G$, or if we only consider a trivializable patch, a global section can be chosen and (only then) the above gauge group will be isomorphic to the space of smooth maps $C^\infty(M, G)$, i.e. the space of sections of the trivialized bundle.

We are particularly interested in spin manifolds, that carry a principal $Spin(n)$-bundle structure. This can be seen as the spin variation of the vielbein (vierbein or tetrad for $n = 4$) formalism of Cartan, where the Lie group was $SO(n)$ and one associates to an observer at a specific point $x \in M$ an oriented orthonormal frame (or tetrad), $\{e_a(x)\}$ on the tangent bundle. The corresponding gauge group in the trivial bundle case $C^\infty(M, SO(4))$, known as the *Lorentz gauge group*, rotates the tetrad field to another oriented orthonormal frame.

Here we will sketch the explicit form of the action of the group of diffeomorphisms on the gauge group and then lift it to the spin gauge group. One can consider for simplicity a local trivializable patch and "fix the gauge" by using the Riemannian metric g on M. Starting with a coordinate system x^μ, one has the basis vector fields $\frac{\partial}{\partial x^\mu}$ and the metric components $g_{\mu\nu} = g(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu})$. These transform in the usual covariant way. Fixing the gauge amounts to making a choice of the tetrad fields, now by choosing their components in the coordinate

[15] In the language of category theory, the object is a G-torsor and automorphisms of P need to be torsor isomorphisms.

vector basis. This is done in a unique way by means of the metric components in that basis, defining

$$(e^{-1})_a{}^\mu(x) = (g^{-1/2}(x))^\mu_a \quad , \quad \mu, a = 0,1,2,3 \tag{121}$$

as the only "symmetric" set of components for the four orthonormal vector fields

$$e_a = (e^{-1})_a{}^\mu \frac{\partial}{\partial x^\mu}. \tag{122}$$

A general transformation of coordinates $\phi \in Diff(M)$, $\phi : x \mapsto \tilde{x}$ (close to the identity) will induce a transformation on the tetrad, by keeping the defining gauge fixing relation (121) unchanged, and plugging in the new transformed components of the metric w.r.t. the new basis vectors. The resulting Lorentz transformation of the four vector fields (c.f. [14]) is given by

$$\Lambda(\phi)(x)^a{}_b = (g^{1/2})^{\tilde{a}}{}_{\tilde{\mu}} \mathcal{J}^{\tilde{\mu}}{}_\mu (g^{-1/2})^\mu{}_b(x), \tag{123}$$

where we denote by $\mathcal{J}$ the Jacobian of the transformation ϕ, $\mathcal{J}^{\tilde{\mu}}_\mu = \frac{\partial \tilde{x}^{\tilde{\mu}}}{\partial x^\mu}$. The map

$$\Lambda : \phi \mapsto (\phi, \Lambda(\phi)) \in Diff(M) \ltimes C^\infty(M, SO(4)) = Aut(C^\infty(M)) \ltimes Gau(P)$$

composed point-wise with the group homomorphism

$$S : SO(4) \to Spin(4)$$

is a double valued [16] group homomorphism $L = S \circ \Lambda$, that lifts the automorphisms of the commutative algebra to the spin group automorphisms. The explicit lift S is given by exponentiation of the algebra isomorphism

$$\begin{aligned} s : so(4) &\to spin(4) \\ \omega &\mapsto \frac{1}{4}\omega_{ab}\gamma^{ab}, \end{aligned} \tag{124}$$

where ω is an anti-symmetric 4x4 matrix and $\gamma^{ab} := [\gamma^a, \gamma^b]$. Now we also have an action of diffeomorphisms on spinor fields by lifting them to the spin group.

This idea will be generalized in Connes' noncommutative geometry, where still a definition of some group of lifted automorphisms that act on the Hilbert space of spinors is needed. So recall that in the commutative case discussed above, we have $Diff M = Aut C^\infty(M)$, and as it turns out, the group that needs to be lifted in order to act on the Hilbert space of the triple will indeed be $Aut(\mathcal{A})$. The receptacle to which the automorphisms will be lifted is defined by Connes to be $Aut_{\mathcal{H}}(\mathcal{A})$ the unitary operators on $\mathcal{H}$ that preserve reality J (charge conjugation) and chirality γ and can be projected to (and therefore may come from) an automorphism by the map

$$\begin{aligned} p : Aut_{\mathcal{H}}(\mathcal{A}) &\to Aut(\mathcal{A}) \\ U &\mapsto \pi^{-1} \circ Ad(U) \circ \pi, \end{aligned} \tag{125}$$

[16] remember that Spin(n) gives a double cover of SO(n)

which for a given lifted automorphism U maps any $a \in \mathcal{A}$ to $p(U)(a) = \pi^{-1}(U\pi(a)U^{-1})$. These properties define the set that will contain the lifted automorphisms as

$$Aut_{\mathcal{H}}(\mathcal{A}) = \{U \in End(\mathcal{H}) : UU^* = U^*U = 1 \quad , \quad UJ = JU, \\ U\gamma = \gamma U \quad , \quad Ad(U) \in Aut(\pi(\mathcal{A}))\} \, , \tag{126}$$

the last condition making sure that such a projection can be defined. Keep in mind that the Hilbert space of interest is the space of spinor fields on M.

It is a non-trivial task to construct the (possibly multivalued) lifting map(s) $L : Aut(\mathcal{A}) \to Aut_{\mathcal{H}}(\mathcal{A})$ such that $p \circ L = Id_{Aut(\mathcal{A})}$, and note that even in the above commutative example things are only treated locally. Some details can be found in [14] and [7].

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