Deformation Quantization and the Baum–Connes Conjecture

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Dedicated to Rudolf Haag on the occasion of his eightieth birthday

Abstract: Alternative titles of this paper would have been "Index theory without index" or "The Baum–Connes conjecture without Baum."

In 1989, Rieffel introduced an analytic version of deformation quantization based on the use of continuous fields of C^*-algebras. We review how a wide variety of examples of such quantizations can be understood on the basis of a single lemma involving amenable groupoids. These include Weyl–Moyal quantization on manifolds, C^*-algebras of Lie groups and Lie groupoids, and the E-theoretic version of the Baum–Connes conjecture for smooth groupoids as described by Connes in his book *Noncommutative Geometry*.

Concerning the latter, we use a different semidirect product construction from Connes. This enables one to formulate the Baum–Connes conjecture in terms of twisted Weyl–Moyal quantization. The underlying mechanical system is a noncommutative desingularization of a stratified Poisson space, and the Baum–Connes Conjecture actually suggests a strategy for quantizing such singular spaces.

1. Introduction

As a tribute to Rudolf Haag, this paper is a double provocation. Firstly, it is about quantization, a concept Haag apparently doesn't like. Indeed, he has always stressed that (local) quantum physics stands on its own, and should not be thought of as the quantization of some classical theory. Secondly, it fits in the ideology of "physical mathematics," in attempting to understand a concept in pure mathematics (viz. the Baum–Connes conjecture), in terms of ideas from physics (namely quantization). Characteristically, there is not a single theorem in this paper. As the founding editor of *Communications in Mathematical Physics*, Haag may well have second thoughts about the seemingly irrepressible development of his journal into a medium for both "mathematical physics" and "physical mathematics." On the positive side, concerning the first point we

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use a formulation of quantization in terms of C^*-algebras, and even manage to relate the Baum–Connes conjecture to the algebraic theory of superselection rules initiated by Haag [30]. With regard to the second, we note that this paper only contains valid mathematics, which everyone can check and understand.

One source of inspiration for this paper is the known relationship between index theory (in the sense of Atiyah and Singer [5]) and quantum physics. This relationship was discovered in the context of anomalies in quantum field theory [1, 3], and is closely related to supersymmetry [2, 28]. See, e.g., [10, 11, 70] for representative mathematical literature generated by this line of research. On a different note, it turns out that index theory is closely linked to deformation quantization [26, 27, 50]. It remains unclear (at least to the author) how the supersymmetric approach to index theory is related to the one based on deformation theory.

A promising way of looking at the relationship between quantization and index theory is to involve the K-theory of C^*-algebras [12, 63]. Pragmatically speaking, K-theory is the (generalized) cohomology theory of algebraic topology that is best adapted to a generalization to noncommutative C^*-algebras. K-theory is defined by functors K_n, $n \in \mathbb{Z}$, from C^*-algebras to abelian groups, which are stable, homotopy invariant, and satisfy Bott periodicity $K_{n+2}(A) \cong K_n(A)$ (natural in A). One therefore simply writes the K-theory of A as $K_*(A)$, where $* = 0, 1$. Bott periodicity leads to a periodic (or cyclic) 6-term exact sequence associated to a short exact sequence, which underlies most explicit computations in K-theory.

K-theory for C^*-algebras is a fundamental tool in noncommutative geometry [19], and also plays a key role in Elliott's classification program for simple nuclear C^*-algebras [45, 64]. In mathematical physics, the best-known applications of noncommutative K-theory have been to the theory of the quantum Hall effect [9] and to the description of quasi-crystals [33]. So far, the use of K-theory in physical mathematics seems limited to the commutative case [72].

The bivariant E-theory of Connes and Higson [19, 20] is a generalization of the K-theory of C^*-algebras, which at the same time provides maps between K-groups. E-theory is based on specific deformations of C^*-algebras, and is closely related to index theory [19, 31]. Thus it seems natural to use E-theory in an attempt to further clarify the relationship between index theory and quantization. However, deformation quantization contains an ingredient that seems to be missing in E-theory, namely the Poisson bracket. This determines the "direction" of a deformation, providing information that could be useful in understanding why certain maps between K-groups defined by E-theory occur naturally. Indeed, this is a guiding thought behind this paper.

One of the main issues in K-theory in the context of noncommutative geometry is the so-called Baum–Connes conjecture, which is closely related to index theory [7, 8, 19] (in this paper, we restrict ourselves to the conjecture "without coefficients"). Here the problem is to give a geometric description of the K-theory of the reduced C^*-algebra $C_r^*(G)$ of a locally compact group [24, 52] or groupoid [55] G. This is potentially interesting for physics, since algebras of observables of a large class of quantum mechanical systems are of the form $C_r^*(G)$ [36], and the K-theory of such algebras is an invariant of the physical description that deserves to be explored.

For a compact group, $K_0(C_r^*(G))$ equals the free abelian group on $\hat{G}$ (the unitary dual of G, which in this case is discrete), whereas K_1 is trivial. The groupoid analogue of a compact group is a proper groupoid; a groupoid G with base $G^{(0)}$ and source and range maps $s, r : G \to G^{(0)}$, respectively, is called proper when $(r, s) : G \to G^{(0)} \times G^{(0)}$ is a proper map. The K-theory of the reduced C^*-algebra of such a groupoid can in

principle be described in terms of the K-theory of the compact stability groups G_u^u of points $u \in G^{(0)}$ [55], combined with the (equivariant) topological K-theory of the orbit space $G^{(0)}/G$ (which is locally compact and Hausdorff). Hence the compact or proper case is fully understood in principle.

One idea behind the Baum–Connes conjecture is to "tame" a noncompact group or nonproper groupoid by letting it act properly on some space. Under a proper action, all stability groups are compact, and the orbit space is locally compact and Hausdorff [4]. Baum and Connes define a computable topological K-theory $K_{\text{top}}^*(G)$ in terms of such proper actions, and relate it to the actual K-theory $K_*(C_r^*(G))$ by a map μ, called the analytic assembly map. The Baum–Connes conjecture states that μ should be an isomorphism. This would, then, render $K_*(C_r^*(G))$ computable as well.

The Baum–Connes conjecture actually enjoys a number of different formulations. For groups, the standard version is that of [8]. Here $K_{\text{top}}^*(G)$ is defined in terms of the G-equivariant K-homology of the classifying space $\underline{E}G$ of G for proper actions. Roughly speaking, elements of $K_{\text{top}}^*(G)$ are equivalence classes of G-invariant operators on some Hilbert space carrying representations of G as well as of $C_0(X)$, where X is some proper G space. These operators have an index taking values in $K_*(C^*(G))$, and μ is essentially this index, composed with the K-theory map π_{r*} induced by the canonical projection $\pi_r : C^*(G) \to C_r^*(G)$. Thus the Baum–Connes conjecture states that, in a suitably injective way, every element of $K_*(C_r^*(G))$ may be represented as an index.

In this form, the Baum–Connes conjecture has been proved for large classes of discrete or algebraic groups (cf. [67, 69]), as well as for all almost connected locally compact groups [17]. There exists an analogous formulation for locally compact groupoids with Haar system, surveyed in [68]. The usual formulation of the Baum–Connes conjecture for both groups and groupoids is based on Kasparov's KK-theory (cf. [12]), which is also the fundamental tool in the extant proofs of special cases of the conjecture.

A different approach to the Baum–Connes conjecture, based on E-theory, was initiated by Connes himself [19, §II.10]. The main purpose of this paper is to make explicit how Connes's E-theoretic formulation of the Baum–Connes conjecture is nothing but the statement that the G-twisted Weyl–Moyal quantization of a certain space preserves K-theory. This is actually closely related to Connes's own way of seeing the Baum–Connes conjecture as a G-equivariant version of Bott periodicity. To accomplish this, we have to slightly modify Connes's construction of the analytic assembly map μ in order to bring it in line with the C^*-algebraic approach to Weyl–Moyal quantization. Moreover, we prove a fundamental and nontrivial continuity property left to the reader in [19]. As suggested above, the use of deformation quantization amplifies E-theory by providing the direction of the deformation defining μ.

When G is a Lie group, the classical mechanical systems underlying the above approach to the Baum–Connes conjecture are Poisson spaces of the type $T^*(P)/G$, where P is a proper G space, and the G action on $T^*(P)$ is the pullback of the one on P. This action automatically preserves the canonical Poisson bracket (or, equivalently, the symplectic form) on $T^*(P)$, which therefore descends to a Poisson structure on $T^*(P)/G$. In case that P is a principal G bundle (i.e., when the G action is free), $T^*(P)/G$ is a manifold, whose physical interpretation is well understood in terms of a particle moving on the configuration space $Q = P/G$, coupled to an external gauge field [47]. The algebra of observables of the corresponding quantum system [36] is the C^*-algebra of the so-called gauge groupoid $(P \times P)/G$ of the principal G bundle P [46]. Such a quantum system has a nontrivial superselection structure, which is fully described by the irreducible unitary representations of G. Similarly, the underlying classical system has

"classical superselection sectors," defined as the symplectic leaves of $T^*(P)/G$ [47]. In analogy to the quantum situation, these turn out to correspond to the coadjoint orbits of G.

However, when the G action on P is not free (and this is the main case of interest in connection with the Baum–Connes conjecture), the quotient $T^*(P)/G$ is no longer a manifold. In fact, the Baum–Connes conjecture for Lie groups à la Connes is formulated in terms of a noncommutative desingularization of $T^*(P)/G$, namely the crossed product C^*-algebra $C_0(T^*(P)) \rtimes G$. The structure of $T^*(P)/G$ as a singular space is well known [44]: its naive symplectic leaves are actually stratified symplectic spaces [66], which further decompose as unions of symplectic manifolds. This introduces additional classical superselection sectors, which should be related to the structure of the desingularization $C_0(T^*(P)) \rtimes G$ in some way. In any case, inspired by Connes's E-theoretic formulation of the Baum–Connes conjecture, we are led to a concrete proposal to quantize the singular Poisson space $T^*(P)/G$ by deforming its noncommutative desingularization.

The plan of this paper is as follows. In Sect. 2 we review the notion of C^*-algebraic deformation quantization, and state the key technical lemma, on which most subsequent arguments will be based. In Sect. 3 we discuss a number of examples relevant to the Baum–Connes conjecture, and in Sect. 4 we turn to the Baum–Connes conjecture itself. Finally, in Sect. 5 we provide the details of the physical interpretation sketched above.

We hope that this expository paper attracts mathematical physicists to the Baum–Connes conjecture, and draws the attention of noncommutative geometers to the problem of quantizing singular symplectic spaces [42].

2. Basic Setting

The C^*-algebraic approach to deformation quantization was initiated in 1989 by Rieffel [59], who observed that a number of examples of quantization could be described by continuous fields of C^*-algebras in a natural and attractive way. We refer to [36, 60] for surveys of the starting period of C^*-algebraic deformation quantization, including references.

We now review the basic definitions pertinent to C^*-algebraic deformation quantization. On the classical side, we have

Definition 1. *A Poisson algebra is a commutative algebra $\tilde{A}$ over $\mathbb{C}$ equipped with a Lie bracket $\{\,,\,\}$, such that for each $f \in \tilde{A}$ the map $g \mapsto \{f, g\}$ is a derivation of $\tilde{A}$ as a commutative algebra. A Poisson manifold P is a manifold equipped with a Lie bracket on $\tilde{A} = C^\infty(P)$, such that it becomes a Poisson algebra with respect to pointwise multiplication.*

On the quantum side, one needs

Definition 2. *A field of C^*-algebras over a compact Hausdorff space X is a triple $(X, \{A_x\}_{x \in X}, A)$, where $\{A_x\}_{x \in X}$ is some family of C^*-algebras indexed by X, and A is a family of sections (that is, maps $a : X \to \coprod_{x \in X} A_x$ for which $a(x) \in A_x$) that is:*

1. A C^*-algebra under pointwise operations and the natural norm

$$\|a\| = \sup_{x \in X} \|a(x)\|_{A_x};$$

2. *Closed under multiplication by $C(X)$;*
3. *Full, in that for each $x \in X$ one has $\{a(x) \mid a \in A\} = A_x$.*

The field is said to be continuous when for each $a \in A$ the function $x \mapsto \|a(x)\|$ is in $C(X)$.

This is equivalent to the corresponding definition of Dixmier [24]; cf. [13, 34]. Such a field comes with a collection of *-homomorphisms $\pi_x : A \to A_x$, defined by $\pi_x(a) = a(x)$. We will use this with $X = I = [0, 1]$, seen as the set of values of Planck's constant $\hbar$.

Poisson manifolds are related to continuous fields through the concept of C^*-algebraic deformation quantization.

Definition 3. *A C^*-algebraic deformation quantization of a Poisson manifold P consists of:*

1. *A continuous field of C^*-algebras $(I, \{A_\hbar\}_{\hbar \in I}, A)$ in which $A_0 = C_0(P)$;*
2. *A Poisson subalgebra $\tilde{A}_0$ of $C^\infty(P)$ that is densely contained in $C_0(P)$;*
3. *A cross-section $Q : \tilde{A}_0 \to A$ of π_0,*

such that, in terms of $Q_\hbar = \pi_\hbar \circ Q$, for all $f, g \in \tilde{A}_0$ one has

$$\lim_{\hbar \to 0} \| \frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f, g\}) \|_\hbar = 0. \qquad (2.1)$$

The idea behind (2.1), which may be traced back to Dirac, is that the Poisson bracket on P determines the direction in which $C_0(P)$ is deformed into a noncommutative C^*-algebra. In any case, this definition (with evident modifications when $I = [0, 1]$ is replaced by a more general index set) seems to cover practically all known examples.

A surprisingly large collection of examples can be constructed from the following data [40, 54]. We refer to [46, 55] for the theory of groupoids. Recall that a Lie groupoid is a groupoid where all spaces and maps are smooth, and s and r are surjective submersions [46].

Definition 4. *A field of groupoids is a triple (G, X, p), with G a groupoid, X a set, and $p : \mathsf{G} \to X$ a surjection such that $p = p_0 \circ r = p_0 \circ s$, where $p_0 = p \upharpoonright \mathsf{G}^{(0)}$. If G is a locally compact groupoid and X is a topological space, one requires that p is continuous and open. When G is a Lie groupoid and X a manifold, p should be a surjective submersion.*

It follows that each $\mathsf{G}_x = p^{-1}(x)$ is a subgroupoid of G over $\mathsf{G}^{(0)} \cap p^{-1}(x)$, so that $\mathsf{G} = \bigsqcup_{x \in X} \mathsf{G}_x$ as a groupoid. This holds algebraically, topologically, or smoothly, as appropriate.

In the context of deformation quantization, the following two cases occur: either G is smooth, or G is étale. In both cases, G and all G_x automatically have a (left or right) Haar system [36, 40, 54, 55]. More generally, one may simply assume that G is a locally compact groupoid with Haar system. One may then form the convolution C^*-algebras $C^*(\mathsf{G})$ and $C^*(\mathsf{G}_x)$, or the corresponding reduced C^*-algebras $C_r^*(\mathsf{G})$ and $C_r^*(\mathsf{G}_x)$ [19, 55]. Each $a \in C_c(\mathsf{G})$ (or $C_c^\infty(\mathsf{G})$) defines $a_x = a \upharpoonright \mathsf{G}_x$ as an element of $C_c(\mathsf{G}_x)$ (etc.). These maps $C_c(\mathsf{G}) \to C_c(\mathsf{G}_x)$ are continuous in the appropriate norms, and extend to maps $\pi_x : C^*(\mathsf{G}) \to C^*(\mathsf{G}_x)$. Hence one obtains a field of C^*-algebras

$$(X, \{A_x = C^*(\mathsf{G}_x)\}_{x \in X}, A = C^*(\mathsf{G})),$$

where $a \in C^*(\mathsf{G})$ defines the section $x \mapsto \pi_x(a)$. A similar statement applies to the corresponding reduced C^*-algebras.

The question now arises when this field is continuous. The answer, generalizing certain results by Rieffel for groups [58], is as follows.

Lemma 1. *The field $(X, \{C^*(\mathsf{G}_x)\}_{x \in X}, C^*(\mathsf{G}))$ is continuous at all points where G_x is amenable [4, 55] (and similarly for the case of reduced C^*-algebras).*

This lemma was first mentioned to the author by Skandalis in 1997; see [36, p. 469]. A complete proof, based on results of Skandalis's student Blanchard [13], appeared in [54], and was repeated in [40]. In our examples of deformation quantization, where $X = I$, two possibilities occur.

In the first situation, all $\mathsf{G}_\hbar$ are amenable, in which case Lemma 1 immediately proves continuity of the field in question. See [14] for a description of the noncommutative tori of Rieffel [59] and of the noncommutative four-spheres of Connes and Landi [21] (and of many other examples) as deformation quantizations along these lines.

In the second situation, typically only G_0 is amenable, and the field is trivial away from $\hbar = 0$ (see below). The former property then yields continuity at $\hbar = 0$ by the lemma, whereas the latter gives continuity on $(0, 1]$. In the context of Definition 3, the reason why G_0 is amenable is that A_0 must be commutative, which implies that G_0 is a bundle of abelian groups. But such groupoids are always amenable [4]. In both cases, one obtains a continuous field.

Here a continuous field $(I, \{A_\hbar\}_{\hbar \in I}, A)$ is said to be trivial away from $\hbar = 0$ when $A_\hbar = B$ for all $\hbar \in (0, 1]$, and one has a short exact sequence

$$0 \to CB \to A \to A_0 \to 0, \tag{2.2}$$

in terms of the so-called cone $CB = C_0((0, 1], B)$. For later use, we recall that such a field induces a map $K_*(A_0) \to K_*(B)$ in the following way [19, 20]. Since the cone CB is contractible, and therefore has trivial K-theory, the periodic 6-term sequence shows that

$$\pi_{0*} : K_*(A) \to K_*(A_0) \tag{2.3}$$

is an isomorphism; here π_{0*} stands for the image of the *-homomorphism $\pi_0 : A \to A_0$ under the K-functor. The K-theory map defined by the field is then simply

$$\pi_{1*} \circ \pi_{0*}^{-1} : K_*(A_0) \to K_*(B). \tag{2.4}$$

3. Examples

3.1. Particle on a manifold. The simplest physically relevant example of this setting is provided by Connes's tangent groupoid G_M of a manifold M; see [19, p. 102]. Here

$$\mathsf{G} = G_M = \coprod_{\hbar \in I} G_\hbar,$$

where $\mathsf{G}_0 = T(M)$ is the tangent bundle of M, seen as a groupoid over M under addition in each fiber, and $\mathsf{G}_\hbar = M \times M$ for all $\hbar \in (0, 1]$ is the pair (or coarse) groupoid on M. The point is, of course, that G has a smooth structure turning it into a Lie groupoid (see below).

The corresponding field of C^*-algebras has fibers

$$
\begin{aligned}
A_0 &= C^*(T(M)) \cong C_0(T^*(M)), \\
A_\hbar &= C^*(M \times M) \cong B_0(L^2(M)) \ \forall \hbar \in (0, 1],
\end{aligned} \tag{3.1}
$$

where $B_0(H)$ is the C^*-algebra of compact operators on H. For later use, it is crucial to remark that the isomorphism in the first equation is given by a fiberwise Fourier transformation. The continuity of this field follows from Lemma 1 as explained above (among many other proofs; cf. [25, 36] and references therein). For the quantization maps $Q_\hbar$ see [36, 37, 53]; these are essentially given by Weyl–Moyal quantization with respect to a Riemannian structure on M. The relationship between the tangent groupoid and quantization was independently noted by Connes during his lectures at Les Houches in 1995; see [16].

Combining the trace tr (to implement the isomorphism $K_0(B_0) \cong \mathbb{Z}$) with the map in (2.4), one obtains a map

$$
\mathrm{ind}_a = \mathrm{tr} \circ \pi_{1*} \circ \pi_{0*}^{-1} : K^0(T^*(M)) \to \mathbb{Z}, \tag{3.2}
$$

which is precisely the analytic index of Atiyah and Singer [5]; cf. Lemma II.5.6 in [19]. For $M = \mathbb{R}^n$, one has $K^0(\mathbb{R}^{2n}) \cong \mathbb{Z}$, and the analytic index is the isomorphism β of the Bott periodicity theorem [6]. The fact that the "classical algebra" $C_0(\mathbb{R}^{2n})$ and the "quantum algebra" $B_0(L^2(\mathbb{R}^n))$ have the same K-theory is peculiar to this special case; for general M this will, of course, fail. The special case $M = \mathbb{R}^n$, however, lies behind the Baum–Connes conjecture; see below.

3.2. Particle with internal degree of freedom. The above example describes the quantization of a particle moving on M, with phase space $T^*(M)$. If, on the other hand, a particle has no kinematic degrees of freedom (in that it does not move on a configuration space), but is only endowed with internal degrees of freedom, described by a Lie group G, its algebra of observables is the group C^*-algebra $C^*(G)$. As first recognized in [61] (under certain assumptions, which later turned out to be unnecessary [40, 54]), this algebra is a deformation quantization in the sense of Definition 3 of the Poisson manifold $\mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of G, and its dual vector space $\mathfrak{g}^*$ is equipped with the so-called Lie–Poisson structure (which on linear functions is just given by the Lie bracket) [36, 47].

The underlying Lie groupoid G has fibers $\mathsf{G}_0 = \mathfrak{g}$ and $\mathsf{G}_\hbar = G$ for $\hbar \in (0, 1]$. Here $\mathfrak{g}$ is regarded as an abelian group, so that it is amenable, and Lemma 1 proves continuity of the associated field of C^*-algebras

$$
\begin{aligned}
A_0 &= C^*(\mathfrak{g}) \cong C_0(\mathfrak{g}^*), \\
A_\hbar &= C^*(G) \ \forall \hbar \in (0, 1].
\end{aligned} \tag{3.3}
$$

Here $\mathfrak{g}$ is treated as an abelian group; once again, the isomorphism in the first equation is given by the Fourier transformation. The quantization maps $Q_\hbar$ are defined in terms of the usual exponential map from $\mathfrak{g}$ to G, and Definition 3 turns out to be satisfied.

3.3. The Connes–Mackey semidirect product deformation. The deformation described by Connes in [19, p. 141] is similar to the preceding example, with the difference that only the "noncompact part" of G is deformed. Let G be a connected Lie group with maximal compact subgroup H. With $\mathfrak{m} = T_e(G/H)$ one has $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and H acts naturally on $\mathfrak{m}$. One then has a Lie groupoid G that is a field of groups with fibers $\mathsf{G}_0 = \mathfrak{m} \rtimes H$ and $\mathsf{G}_\hbar = G$ for $\hbar \in (0, 1]$. Since $\mathfrak{m} \rtimes H$ is amenable, Lemma 1 proves continuity of the associated field of C^*-algebras. Note that, unlike in the previous examples, $A_0 = C^*(\mathfrak{m} \rtimes H)$ is now noncommutative, like $A_\hbar = C^*(G)$ (except in trivial cases).

3.4. Poisson manifolds associated to Lie algebroids. Examples 3.1 and 3.2 are both special cases of a very general construction [36, 37, 40, 54]. A Lie algebroid E is a (real) vector bundle over a manifold M, whose space $\Gamma(E)$ of smooth sections is equipped with a Lie bracket satisfying the Leibniz rule

$$[s_1, f s_2] = f[s_1, s_2] + (\alpha \circ s_1) f \cdot s_2 \tag{3.4}$$

for some vector bundle map $\alpha : E \to T(M)$. Such a map, called the anchor map of the Lie algebroid, is unique when it exists. (This definition, which we learnt from Marius Crainic, is more efficient than the usual one [15, 36, 46].) The simplest example is $E = T(M)$, where α is the identity map.

A Lie groupoid G has an associated Lie algebroid $A(G)$ over the base space $G^{(0)}$ [15, 36, 46]. The dual vector bundle $A^*(G)$ has a canonical Poisson structure, which generalizes both the usual symplectic structure on $T^*(M)$ and the Lie–Poisson bracket on $\mathfrak{g}^*$ [22, 23]. Generalizing Connes's tangent groupoid [32, 71] (which emerges as a special case for $G = M \times M$), there exists a Lie groupoid $\mathsf{G} = \coprod_{\hbar \in I} G_\hbar$, where $\mathsf{G}_0 = A(G)$ (seen as a Lie groupoid over $G^{(0)}$ under addition in each fiber) and $\mathsf{G}_\hbar = G$ for $\hbar > 0$. With abuse of terminology, this is called the tangent groupoid of G.

As noted in [51], the Lie algebroid of G is the so-called adiabatic Lie algebroid associated to $A(G)$. In general, the adiabatic Lie algebroid E_t associated to some Lie algebroid E over M is a vector bundle over $M \times I$ whose total space is the pullback $\mathrm{pr}_1^* E$ of the map $\mathrm{pr}_1 : M \times I \to M$; the Lie bracket is, in obvious notation,

$$[s_1, s_2]_{E_t}(\cdot, \hbar) = \hbar[s_1(\hbar), s_2(\hbar)]_E. \tag{3.5}$$

The tangent groupoid of G is then obtained by applying the integration procedure of [18] to $A(G)_t$; this provides, in particular, the smooth structure.

By our standard argument, the associated field of C^*-algebras

$$\begin{aligned} A_0 &= C^*(A(G)) \cong C_0(A^*(G)), \\ A_\hbar &= C^*(G) \ \forall \hbar \in (0, 1], \end{aligned} \tag{3.6}$$

is continuous, and provides a deformation quantization of the Poisson manifold $A^*(G)$ in the sense of Definition 3. As in the previous examples, the isomorphism in the first equation is given by a fiberwise Fourier transformation. The analogy between the maps $G \mapsto A^*(G)$ and $G \mapsto C^*(G)$ is quite deep; see [38].

3.5. Gysin maps. Certain constructions of Connes in index theory turn out to be special cases of Example 3.4. One instance is the "shriek" map $p! : K^*(F^*) \to K_*(C^*(V, F))$ on p. 127 of [19], which plays a key role both in the longitudinal index theorem for foliations and in the construction of the analytic assembly map for foliated manifolds. Here V is a manifold with foliation $F \subset T(V)$, and $C^*(V, F) = C^*(G(V, F))$ is the canonical C^*-algebra of the holonomy groupoid $G(V, F)$ of the foliation. Now $p!$ is nothing but the K-theory map (2.4) induced by the continuous field (3.6), where $G = G(V, F)$. The analytic index (3.2) corresponds to the special case that $V = M$ is trivially foliated (i.e., $F = T(M)$).

The index groupoid defined in [19, §II.6] is another example of (2.4) with (3.6). Let $L : E \to F$ be a vector bundle map between vector bundles over a common base B. Then one has a Lie groupoid $G = \text{Ind}_L = F \rtimes_L E$ over F, whose Lie algebroid is $F \times_B E$. The latter is a vector bundle over B, and in the formalism of this paper it should be regarded as a groupoid over F under addition in each fiber. Hence $A_0 = C^*(F \times_B E) \cong C_0(F \times E^*)$. The corresponding map (2.4) is basic to Connes's construction of the Gysin or shriek map $f! : K^*(X) \to K^*(Y)$ induced by a smooth K-oriented map $f : X \to Y$ between two manifolds.

4. The Baum–Connes Conjecture

We first recall a generalized semidirect product construction for groupoids, which is necessary to relate the Baum–Connes conjecture to quantization. We then describe the analytic assembly map à la Connes. In what follows, G is a Lie groupoid over $G^{(0)}$.

4.1. On semidirect products. Recall [19, 46] that a (right) G space P is a smooth map $P \overset{\alpha}{\to} G^{(0)}$ along with a map $P \times_{G^{(0)}} G \to P$, where

$$P \times_{G^{(0)}} G = \{(p, \gamma) \in P \times G \mid \alpha(p) = r(\gamma)\}, \tag{4.1}$$

written as $(p, \gamma) \mapsto p\gamma$, such that $(p\gamma_1)\gamma_2 = p(\gamma_1\gamma_2)$ whenever defined, $p\alpha(p) = p$ for all p, and $\alpha(p\gamma) = s(\gamma)$. The action is called proper when α is a surjective submersion and the map $P \times_{G^{(0)}} G \to P \times P$, $(p, \gamma) \mapsto (p, p\gamma)$ is proper (in that the inverse images of compact sets are compact).

In Connes's description of the Baum–Connes conjecture [19], the standard semidirect product construction in groupoid theory is used: if G acts on a space P as above, one forms a groupoid $P \rtimes G$ over P, with total space $P \times_{G^{(0)}} G$, source and range maps $s(p, \gamma) = p\gamma$ and $r(p, \gamma) = p$, inverse $(p, \gamma)^{-1} = (p\gamma, \gamma^{-1})$, and multiplication $(p, \gamma) \cdot (p\gamma, \gamma') = (p, \gamma\gamma')$. However, as we shall see shortly, the use of these semidirect products distorts the relationship between the Baum–Connes conjecture and deformation quantization. For our purposes, we must work with generalized semidirect products (see [4] for the locally compact case and [46] (2nd ed.) for the smooth case).

Let a G space H be a Lie groupoid itself, and suppose the base map $H \overset{\alpha}{\to} G^{(0)}$ is a surjective submersion that satisfies

1. $\alpha_0 \circ s_H = \alpha_0 \circ r_H = \alpha$ (cf. Definition 4); in other words, H is a field of groupoids over $G^{(0)}$, and α is a morphism of groupoids if $G^{(0)}$ is seen as a space (where a groupoid X is a space when $X^{(0)} = X$ and $s = r = \text{id}$).

2. For each $\gamma \in G$, the map $\alpha^{-1}(r(\gamma)) \to \alpha^{-1}(s(\gamma))$, $h \mapsto h\gamma$, is an isomorphism of Lie groupoids; note that for each $u \in G^{(0)}$, $\alpha^{-1}(u)$ is a Lie groupoid over $\alpha^{-1}(u) \cap H^{(0)}$. In other words, one has $(h_1 h_2)\gamma = (h_1\gamma)(h_2\gamma)$ whenever defined.

Under these conditions, one may define a Lie groupoid $H \rtimes G$, called the generalized semidirect product of H and G. The total space of $H \rtimes G$ is $H \times_{G^{(0)}} G$ as in (4.1), the base space $(H \rtimes G)^{(0)}$ is $H^{(0)}$, the source and range maps are

$$s(h, \gamma) = s_H(h)\gamma,$$
$$r(h, \gamma) = r_H(h), \tag{4.2}$$

respectively, the inverse is $(h, \gamma)^{-1} = (h^{-1}\gamma, \gamma^{-1})$ (note that one automatically has $\alpha(h^{-1}) = \alpha(h)$, so that this element is well defined), and multiplication is given by $(h_1, \gamma_1)(h_2\gamma_1, \gamma_2) = (h_1 h_2, \gamma_1\gamma_2)$, defined whenever the product on the right-hand side exists (this follows from the automatic G-equivariance of s_H and r_H). Familiar special cases of this construction occur when H is a space and G is a groupoid, so that $H \rtimes G$ is the usual semidirect product groupoid over H discussed above, and when G and H are both groups, so that $H \rtimes G$ is the usual semidirect product of groups.

Now let P be a G space. Connes [19, §II.10] notes that the tangent bundle $T_G(P)$ of P along α (i.e., $\ker(\alpha_*)$, where $\alpha_* : T(P) \to T(G^{(0)})$ is the derivative of α) is a G space, with base map $\xi_p \mapsto \alpha(p)$ (where $\xi_p \in T_G(P)_p$) and with the obvious push-forward action. He then regards $T_G(P)$ as a space, and forms the standard semidirect product groupoid $T_G(P) \rtimes G$ over $T_G(P)$; to emphasize this, we write the groupoid in question as

$$T_G(P) \rtimes G \overset{\rightarrow}{\rightarrow} T_G(P). \tag{4.3}$$

This groupoid is proper, and therefore its C^*-algebra has computable K-theory. Connes then defines a geometric cycle for G as a proper G space P along with an element of

$$K_*(C^*(T_G(P) \rtimes G \overset{\rightarrow}{\rightarrow} T_G(P))).$$

Alternatively [41], one could work with the generalized semidirect product

$$T_G(P) \rtimes G \overset{\rightarrow}{\rightarrow} P, \tag{4.4}$$

where $T_G(P)$ is seen as a Lie groupoid over P by inheriting the Lie groupoid structure from $T(P)$ (see Example 3.1). This groupoid fails to be proper, but the following property will be sufficient.

Lemma 2. *If P is a proper G space, then the groupoid $T_G(P) \rtimes G \overset{\rightarrow}{\rightarrow} P$ is amenable.*

Proof. Corollary 5.2.31 in [4] states that a (Lie) groupoid H is amenable iff the associated principal groupoid (that is, the image of the map $H \to H^{(0)} \times H^{(0)}$, $h \mapsto (r(h), s(h))$) is amenable and all stability groups of H are amenable. As to the first condition, the principal groupoid of $T_G(P) \rtimes G$ is the equivalence relation on P defined by $p \sim q$ when $q = p\gamma$ for some $\gamma \in G$. This is indeed amenable, because this equivalence relation is at the same time the principal groupoid of $P \rtimes G \overset{\rightarrow}{\rightarrow} P$, which is proper (hence amenable) because P is a proper G space. As to the second condition, the stability group of $p \in P$ in $T_G(P) \rtimes G$ is $T_G(P)_p \rtimes G_p$, where G_p is the stability group of $p \in P$ in $P \rtimes G$. The former is abelian, and the latter is compact by the properness of the G

action, so that $T_G(P)_p \rtimes G_p$ is amenable as the semidirect product of two amenable groups. $\square$

Despite the fact that the groupoids (4.3) and (4.4) are not even equivalent (in the sense of [48]), they have isomorphic C^*-algebras through a Fourier transformation along the fibers of $T_G(P)$ (seen as a vector bundle over P), and the use of (4.3) or (4.4) therefore leads to the same geometric cycles. Hence for the Baum–Connes conjecture it does not matter which of these two groupoids is used. However, for the interpretation of the Baum–Connes conjecture in terms of deformation quantization one has to work with (4.4). To see this, consider the case where G is trivial. The C^*-algebra of (4.3) is $C_0(T(P))$, which is isomorphic to $C_0(T^*(P))$ through the choice of a Riemannian metric on P. On the other hand, the C^*-algebra of (4.4) is isomorphic to $C_0(T^*(P))$ through a fiberwise Fourier transform. It should now be clear from Example 3.1 that (4.4) rather than (4.3) is the correct groupoid to work with, if one is interested in relating the Baum–Connes conjecture to deformation quantization.

Furthermore, the fibered product $P \times_{G^{(0)}} P$ is a G space under the base map $(p, q) \mapsto \alpha(p) = \alpha(q)$ and the diagonal action $(p, q)\gamma = (p\gamma, q\gamma)$. Now $P \times_{G^{(0)}} P$ inherits a Lie groupoid structure from the pair groupoid $P \times P$ over P, becoming a Lie groupoid over P. Hence one has the semidirect product groupoid

$$(P \times_{G^{(0)}} P) \rtimes G \overset{\rightarrow}{\rightarrow} P.$$

The tangent groupoid G_P associated to P has a Lie subgroupoid G'_P over $I \times P$ that by definition contains all points $(\hbar = 0, \xi_p)$ of G_P whose ξ_p lies in $T_G(P)$, and all points $(\hbar > 0, p, q)$ for which $\alpha(p) = \alpha(q)$. It is clear that G'_P is a field of groupoids over I, whose fiber at $\hbar = 0$ is $T_G(P)$, and whose fiber at any $\hbar \in (0, 1]$ is $P \times_{G^{(0)}} P$. Combining the G actions defined in the preceding two cases, there is an obvious fiberwise G action on G'_P with respect to a base map $\tilde{\alpha}(\hbar, \cdot) = \alpha_\hbar(\cdot)$, where $\alpha_\hbar = \alpha_1$ for $\hbar \in (0, 1]$. This action is smooth, so that one obtains a generalized semidirect product groupoid

$$G'_P \rtimes G \overset{\rightarrow}{\rightarrow} I \times P.$$

This groupoid is the main tool in the construction of the analytic assembly map occurring in Connes's version of the Baum–Connes conjecture.

4.2. The analytic assembly map. The following lemma provides the continuity conditions tacitly assumed in §II.10.α in [19].

Lemma 3. *If P is a proper G space, then $C^*(G'_P \rtimes G)$ is the C^*-algebra A of sections of a continuous field of C^*-algebras over I with fibers*

$$
\begin{aligned}
A_0 &= C^*(T_G(P) \rtimes G \overset{\rightarrow}{\rightarrow} P), \\
A_\hbar &= C^*((P \times_{G^{(0)}} P) \rtimes G \overset{\rightarrow}{\rightarrow} P) \;\; \forall \hbar \in (0, 1].
\end{aligned}
\tag{4.5}
$$

This field is trivial away from $\hbar = 0$. The same is true if all groupoid C^*-algebras are replaced by their reduced counterparts.

Proof. The groupoid $G'_P \rtimes G$ inherits the structure of a smooth field of groupoids over I from the tangent groupoid G_P in the obvious way. The claim is then immediate from Lemmas 1 and 2. □

When G is trivial, the continuous field of this proposition is, of course, the one defined by the tangent groupoid of P, which coincides with the field defined by the Weyl–Moyal deformation quantization of the cotangent bundle $T^*(P)$; see Example 3.1. The general case is a G-twisted version of this, which cannot really be interpreted in terms of an underlying Poisson manifold, because the fiber algebra at $\hbar = 0$ is no longer commutative.

Lemma 4. *The C^*-algebras $C^*((P \times_{G^{(0)}} P) \rtimes G \overset{\rightarrow}{\to} P)$ and $C^*(G)$ are (strongly) Morita equivalent, as are the corresponding reduced C^*-algebras.*

Proof. It is easily checked that the map $(p, q, \gamma) \mapsto \gamma$ from $(P \times_{G^{(0)}} P) \rtimes G$ to G is an equivalence of categories. Since this map is smooth, it follows from Cor. 4.23 in [39] that $(P \times_{G^{(0)}} P) \rtimes G$ and G are equivalent as Lie groupoids (and hence as locally compact groupoids with Haar system). The lemma then follows from Thm. 2.8 in [48]. □

We have now provided the background for understanding Connes's amazing construction of the analytic assembly map [19, §II.10]

$$\mu_P : K_*(C^*(T_G(P) \rtimes G)) \to K_*(C^*_r(G)), \tag{4.6}$$

where P is a proper G space. By (2.4), the continuous field of Lemma 3 yields a map

$$\pi_{1*} \circ \pi_{0*}^{-1} : K_*(C^*(T_G(P) \rtimes G)) \to K_*(C^*((P \times_{G^{(0)}} P) \rtimes G)). \tag{4.7}$$

By Lemma 4 and the fact that the K-theories of Morita equivalent C^*-algebras are isomorphic, the right-hand side of (4.7) may be replaced by $K_*(C^*(G))$. The canonical projection π_r from $C^*(G)$ to $C^*_r(G)$ pushes forward to $\pi_{r*} : K_*(C^*(G)) \to K_*(C^*_r(G))$, so that Connes is in a position to define

$$\mu_P = \pi_{r*} \circ \pi_{1*} \circ \pi_{0*}^{-1}. \tag{4.8}$$

When the classifying space $\underline{E}G$ for proper G actions is a smooth manifold (which is true, for example, when G is a connected Lie group [19, §II.10.β], or when G is the tangent groupoid of a manifold), the topological K-theory $K^*_{\text{top}}(G)$ is defined as

$$K^*_{\text{top}}(G) = K_*(C^*(T_G(\underline{E}G) \rtimes G)). \tag{4.9}$$

In that case, Connes's analytic assembly map is

$$\mu = \mu_{\underline{E}G} : K^*_{\text{top}}(G) \to K_*(C^*_r(G)). \tag{4.10}$$

In general, $K^*_{\text{top}}(G)$ is defined by putting a certain equivalence relation on the geometric cycles for G, and μ is given by (4.8) applied to each cycle. In any case, the Baum–Connes conjecture states that μ should be an isomorphism. Connes's interpretation of this conjecture as a G-equivariant version of Bott periodicity [19, §II.10.ϵ] is consistent with the quantization-oriented approach in this paper, since the field (4.5) underlying the Baum–Connes conjecture is a G-twisted version of the field (3.1), which for $M = \mathbb{R}^n$ leads to Bott periodicity. (See [25, 29] for a detailed analysis of the relationship between Bott periodicity and quantization.)

Similarly, the usual interpretation of the analytic assembly map as a generalized index is understandable in the light of the comment below (3.2) and a comparison between (3.1) and (4.5). In fact, the symbol of a G-invariant elliptic pseudodifferential operator D on P [43, 51] defines an element $[\sigma_D]$ of $K_*(C^*(T_G(P) \rtimes G))$, and the image of this element under (4.7) is precisely the $K_*(C^*(G))$-valued index of D. At least when G is a group, this argument also bridges the gap between the usual formulation of the Baum–Connes conjecture in KK-theory [8] and its formulation due to Connes discussed above, for in that case D defines an element of the G-equivariant K-homology $K_*^G(P)$ of P in terms of which $K_{\text{top}}^*(G)$ is usually defined (A. Valette, private communication).

5. Physical Interpretation

5.1. General comments. When (4.9) holds, the Baum–Connes conjecture claims that the G-twisted Weyl–Moyal deformation quantization of the phase space $T^*(\underline{E}G)$ preserves K-theory. This conjecture is a far-reaching generalization of the fact that the deformation quantization of $T^*(\mathbb{R}^n)$ preserves K-theory; as already mentioned, this fact comes down to Bott periodicity. More generally, Connes's Thom isomorphism in K-theory [12, 19], which implies Bott periodicity, can be understood through deformation quantization [25]. The general question whether deformation quantization preserves K-theory has been the subject of some research [49, 62, 65] outside the context of the Baum–Connes conjecture, and there are only a few general results.

We now take a closer look at the continuous field (4.5). Since the C^*-algebra $C^*(T_G(P) \rtimes G)$ is noncommutative (unless G is trivial), it has no immediate underlying Poisson manifold, so that G-twisted quantization cannot itself be seen as quantization. To analyze the situation, for simplicity we assume that G is a Lie group. In that case, the continuous field (4.5) may be written in terms of conventional crossed product C^*-algebras [52] as

$$A_0 = C_0(T^*(P)) \rtimes G,$$
$$A_\hbar = B_0(L^2(P)) \rtimes G \ \forall \hbar \in (0, 1]. \tag{5.1}$$

In the first equation the given G action on P is pulled back first to $T^*(P)$ and subsequently to $C_0(T^*(P))$, and in the second the natural unitary representation of G on $L^2(P)$ defines an associated action on the C^*-algebra $B_0(L^2(P))$ of compact operators by conjugation. We now first make the assumption that the G action on P is free, allowing a clean analysis, to drop it afterwards.

5.2. Free actions and superselection theory. When the G action on P is free (so that P is a principal G space), one has a Morita equivalence

$$C_0(T^*(P)) \rtimes G \overset{M}{\sim} C_0(T^*(P)/G). \tag{5.2}$$

This is a special case of a well-known result of Rieffel [57]; in connection with what follows, another useful proof is to note that one has an equivalence of groupoids (in the sense of [48])

$$T^*(P) \rtimes G \overset{\rightarrow}{\rightarrow} T^*(P) \sim T^*(P)/G \overset{\rightarrow}{\rightarrow} T^*(P)/G \tag{5.3}$$

through the equivalence bibundle $T^*(P)$. By [48], this induces a Morita equivalence of the corresponding groupoid C^*-algebras, yielding (5.2).

Under the freeness assumption one has an analogous Morita equivalence on the quantum side, namely

$$B_0(L^2(P)) \rtimes G \overset{M}{\sim} C^*((P \times P)/G). \tag{5.4}$$

Here

$$(P \times P)/G \overset{\rightarrow}{\rightarrow} P/G$$

is the so-called gauge groupoid of the principal G bundle P [46]. (When G is compact, the corresponding groupoid C^*-algebra $C^*((P \times P)/G)$ consists of the G-invariant compact operators on $L^2(P)$.) To prove (5.4), one starts from the equivalence of groupoids

$$(P \times P) \rtimes G \overset{\rightarrow}{\rightarrow} P \times P \sim (P \times P)/G \overset{\rightarrow}{\rightarrow} P/G, \tag{5.5}$$

through the equivalence bibundle $P \times P$. Compare (5.3). Thus the Morita equivalent counterpart of the continuous field (5.1) is the field

$$\begin{aligned} A'_0 &= C_0(T^*(P)/G), \\ A'_\hbar &= C^*((P \times P)/G) \ \forall \hbar \in (0, 1]. \end{aligned} \tag{5.6}$$

This field is continuous as well: in fact, (5.6) is just a special case of (3.6) in Example 3.4, in which (with abuse of notation) the groupoid G is taken to be the gauge groupoid $(P \times P)/G$. In particular, the continuous field (5.6) is even a C^*-algebraic deformation quantization of the Poisson manifold $T^*(P)/G$ in the sense of Definition 3 (as already mentioned in the Introduction, $T^*(P)/G$ inherits the canonical Poisson structure on $T^*(P)$).

Poisson manifolds of this type [47] and their quantization [36] have been extensively analyzed. The underlying classical mechanical system is a particle moving on the configuration space $Q = P/G$ with an internal degree of freedom coupling to G. The classical phase space $T^*(P)/G$ decomposes as a disjoint union of its symplectic leaves, which may be thought of as the "classical superselection sectors" of the system. Specifically, if $J : T^*(P) \to \mathfrak{g}^*$ is the momentum map of the G action, the symplectic leaves of $T^*(P)/G$ are the connected components of the Marsden–Weinstein quotients $J^{-1}(\mathcal{O})/G$, where $\mathcal{O} \subset \mathfrak{g}^*$ is a coadjoint orbit for G. Locally, such a leaf is of the form $T^*(Q) \times \mathcal{O}$. The first factor is just the usual phase space of a particle moving on Q, and the second is the classical charge of the particle. The latter typically couples to an external gauge field [47].

The fact that the quantum algebra of observables $C^*((P \times P)/G)$ is related to its classical counterpart $C_0(T^*(P)/G)$ by a C^*-algebraic deformation is reflected in the superselection structure of the model. One of Haag's fundamental insights was that superselection sectors of a quantum system may be identified with inequivalent irreducible representations of its algebra of observables (in quantum field theory further selection criteria are needed, though) [30]. By Lemma 4, both sides of (5.4) are Morita equivalent to $C^*(G)$, so that, in particular, the superselection sectors of $C^*((P \times P)/G)$ bijectively correspond to the irreducible unitary representations of G. Of course, this reflects the DHR theory in algebraic quantum field theory [30]. A comparison with the classical situation then confirms Kirillov's general principle that coadjoint orbits should be seen as the classical analogues of irreducible unitary representations [35]; also cf. Example 3.2.

5.3. General actions and singular quantization. When the G action on P is not free (and this is the main case of interest in connection with the Baum–Connes conjecture), the quotient $T^*(P)/G$ is no longer a manifold. Nonetheless, its structure is well understood [44]. Each naive symplectic leaf of the form $J^{-1}(\mathcal{O})/G$ (or rather a connected component thereof) of $T^*(P)/G$ is not a symplectic manifold, but a stratified symplectic space [66]. In particular, the leaf in question itself decomposes as a disjoint union of symplectic manifolds, which are glued together in a certain topological way that one can describe in detail. Compared to the regular situation discussed above, this introduces new classical superselection sectors.

The problem arises how to quantize such singular symplectic spaces; cf. [42] for a survey of what little is known. The noncommutative geometry approach to the situation would be to desingularize $T^*(P)/G$ by starting from $C^*(T^*(P) \rtimes G)$ rather than $C_0(T^*(P)/G)$. Although the former C^*-algebra is noncommutative, it is still a description of $T^*(P)/G$ as a classical space. This is reflected by the fact that $A_0 = C^*(T^*(P) \rtimes G)$ carries a structure analogous to the notion of a Poisson fibered algebra defined in [56]. In the C^*-algebraic context, it is necessary to involve the multiplier algebra to make sense of this idea.

The multiplier algebra of A_0 contains $\tilde{Z} = C_b^\infty(T^*(P))^G$ (where the suffix G denotes the G-invariant functions) in its center, and also contains the subalgebra $\tilde{A}_0$ generated by $\tilde{Z}$ and $C_c^\infty(T^*(P) \rtimes G)$. Then $\tilde{A}_0$ is a Poisson fibered algebra over $\tilde{Z}$, in that one has a bracket $(f, a) \mapsto \{f, a\}$ from $\tilde{Z} \times \tilde{A}_0$ to $\tilde{A}_0$, which restricts to a Poisson bracket on $\tilde{Z}$, and is a derivation on $\tilde{A}_0$ for fixed f and a derivation on $\tilde{Z}$ for fixed a. This bracket is simply given by the one on $T^*(P)$, ignoring the G-dependence of a.

To quantize the desingularized system, one has to deform $C^*(T^*(P) \rtimes G)$. This is precisely what happens in Connes's formulation of the Baum–Connes conjecture described in Sect. 4. The continuous field (4.5) may be seen as an educated guess to quantize the singular Poisson manifold $T^*(P)/G$ by the C^*-algebra $B_0(L^2(P)) \rtimes G$; the direction of the deformation is now determined by the more general notion of a Poisson structure discussed in the previous paragraph.

This proposal should be tested in concrete examples, such as the Universe with a Big Bang singularity. A complete analysis of this case will have to wait for Haag's 90th birthday Festschrift.

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