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Do Hidden Variables Improve Quantum Mechanics?

Bachelor Thesis

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Abstract

Since the dawn of quantum mechanics physicist have contemplated if a hidden variable theory would be able to improve quantum theory. The main goal of this paper is to look at the article "The completeness of quantum mechanics for predicting measurement outcomes" (2012) by Colbeck and Renner. We try to examine the methods used, and the proofs given in this paper. Through this, we try to make an evaluation of the strength of the result obtained in this article. Our conclusion is that one has to make additional assumptions about the hidden variable theory, in order to complete the proof as given in the article.

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1 Introduction

Until the twentieth century, physics in principle provided the exact description of nature. Precise knowledge of all variables that concerned a problem, gave evolution of that physical process in space and time. Think, for example, of Newtonian mechanics. This principle of determinism changed dramatically when quantum mechanics was discovered. Introduced as a method to predict the outcomes of measurements on microscopic systems such as particles, quantum mechanics seems to clash with the notion of determinism.

Quantum mechanics describes only the probability of obtaining a certain measurement outcome. This yields a stark contrast with classical physics. Another important difference between classical physics and quantum mechanics is the influence the observer seems to have in the latter. In most orthodox interpretations of quantum mechanics, performing a measurements causes a collapse of the state of the system. Roughly speaking, the system suddenly changes from a superposition of possible measurement outcomes, to the particular measurement outcome actually obtained.

The birth of quantum mechanics occurred at the beginning of the twentieth century. The probabilistic nature of quantum mechanics caused a great rift between the physicists of that time. A large part of the community saw the future of physics in quantum mechanics, willing to accept the loss of determinism. One of the more famous defendants of quantum mechanics was Niels Bohr. The other part, smaller, of the community agreed with the powerful predictions quantum mechanics provide, but thought quantum mechanics was just an intermediary theory: it had to be incomplete. One of the most famous people on this side of the argument was Albert Einstein. Bohr and Einstein are well known for their (initially oral) discussion concerning the nature and problems of quantum mechanics. This discussion between Bohr and Einstein eventually led Einstein, Podolsky and Rosen to publish a famous paper entitled: "Can Quantum Mechanical Description of Physical Reality Be Considered Complete?" ([9]). This paper contains one of the most elegant and clear ways to illustrate what seems so skewed about quantum mechanics. Through this, the paper also gives a motivation why it should be reasonable and interesting to look at the idea of a hidden variable.

Let's consider an observer measuring the polarization of a photon using a polarized piece of glass. He can measure the polarization of the photon either along a 0 degree axis, or a 90 degree axis. If, when measuring at 0 degrees, a photon passes through the glass, the measurement result is that this photon is polarized along the 0 degree axis. If it does not pass through, the photon is polarized along the 90 degree axis. Nowadays, the EPR thought experiment is usually reformulated in terms of the measuring of the polarization of an entangled pair of photons. This measurement takes place in a configuration with two observers, "Alice" and "Bob". Of the photon pair, one reaches Alice, whilst to other reaches Bob. Alice and Bob agree to measure the polarization of these photons either along a 0 degree axis, or a 90 degree axis. Quantum mechanics can describe a special photon pair (an entangled pair) that either is measured by both observers to have a polarization of 0 degrees (with a probability of $\frac{1}{2}$ of this occurring), or a polarization of 90 degrees. The photon pair before measurement is a superposition of these two measurement outcomes. The surprising results occur when Alice and Bob are extremely far away from each other. If Alice measures her photon to have a polarization of 0 degrees, she instantly knows Bob's photon will also have this polarization. Behold: when Alice has measured her photon as having a polarization of 0 degrees, Bob will indeed **always** measure his photon to have a polarization of 0 degrees as well. It does not matter how far away Alice is. This means that Bob's photon somehow instantaneously knows the measurement result of Alice's photon. Keeping in mind the the theory of relativity assumes the speed of light to be an absolute limit, we are stuck with a problem. But there seems to be a solution: a hidden variable. We can for example postulate that, upon creation of the photon pair, both photons have a polarization along the same axis (either the 0 degree or 90 degree axis). Alice and Bob just aren't aware of which axis it is (it is hidden from them), and neither is quantum mechanics. If the two possible polarizations each have a probability of $\frac{1}{2}$ of occurring, it gives rise to the same behavior, but there is no more need for instantaneous interaction between the photons. We can say the information, of which polarization the photons have, is determined by a variable, that is not accessible by quantum mechanics. A hidden variable. This hidden variable theory seemed to be a viable way to solve the friction between classical physics and quantum mechanics.

In 1964 John Steward Bell published a paper named "On the Einstein-Podolsky-Rosen Paradox" ([8]). In this paper he very elegantly proved that the assumption of a hidden variable that reduces to both determinism and locality leads to a contradiction. Generally speaking, he proved that every theory having the previously mentioned properties of determinism and locality, must satisfy a so called "Bell inequality". Quantum mechanics does not satisfy this inequality. The fact that the inequality is not satisfied has been verified experimentally on numerous occasions.

Similar results such as the Conway-Kochen Free Will Theorem and everything else concerning hidden variables has been the subject of numerous published papers. Quite recently, a new claim was made in this territory by Roger Colbeck and Renato Renner. In two papers "No extension of quantum theory can have improved predictive power"([2], Nature, 02 Aug. 2011) and "The completeness of quantum theory for predicting measurement outcomes" ([3], arXiv, 20 Aug. 2012) the authors make the claim that a theory using hidden variables cannot give better predictions than quantum mechanics, provided the hidden variable theory satisfies a number of properties (notably a "free choice" assumption) that seem weaker than the assmption in either Bell s theorem of the Free Will Theorem etc. Hence, this result would generalize previous results like those of Bell. Both papers have attracted attention, most discussion revolving around the assumption of free choice. The first paper "No extension of quantum theory can have improved predictive power" ([2]) contains little detail on the actual proof. The second paper "The completeness of quantum theory for predicting measurement outcomes" contains more technical details on how to prove the actual claim, but especially the crux of the proof is still largely left to the reader. My goal for this Bachelor thesis is to give the proof of the main claim in a complete and clear way.

The main focus of this paper will be on the third paper published on this subject by Colbeck and Renner ([3]). We have opted to keep both the assumptions made, and the structure of the proof, quite similar to this paper. In this thesis we will look at one hidden variable with a finite number of possible values. This hidden variable describes some property of the particle or system we are measuring. To arrive at the final result, we must first specify what is meant by probabilities given by a physical theory. This analysis is found in the first section. After that, we must make clear when a "higher" theory is considered compatible with quantum mechanics, and when it gives better predictions. We can then proceed by introducing quantities with which we can prove the result for a very specific measurement and state. Finally we will expand this to every state and measurement to obtain the final result mentioned in the article [3] by Colbeck and Renner.

The result, may suggest that the topic of hidden variables is now closed once and for all:

hidden variables do not improve quantum mechanical predictions. However, upon closer inspection one might criticize the methods employed by Colbeck and Renner. Notably, the somewhat dubious notion of free choice, and the way in which measurements are generalized. The result certainly is a step in the right direction, but not as strong and straightforward as previous results such as Bell's Theorem.

2 Probabilities and quantum mechanics

In this paper we mainly make use of five random variables, called A_N, B_N, X, Y and Z ($N \in \mathbb{N}$). We assume there is a probability space Ω with a probability measure μ such that:

$$X, Y : \Omega \to \{0, 1\}, \tag{1}$$
$$A_N : \Omega \to \mathcal{A}_N = \{0, 2 \ldots, 2N - 2\}, \tag{2}$$
$$B_N : \Omega \to \mathcal{B}_N = \{1, 3, \ldots 2N - 1\}, \tag{3}$$
$$Z : \Omega \to \mathcal{Z}. \tag{4}$$

We do not fix the value set $\mathcal{Z}$ of Z. The only restriction is for it to be finite. The probability of general discrete random variable X having a value x is given by

$$P(X = x) = \mu(X^{-1}(x)). \tag{5}$$

A conditional distribution function for two random variables X and Z is given by:

$$P(X = x \mid Z = z) = \frac{P(X = x, Z = z)}{P(Z = z)}, \tag{6}$$

defined whenever $P(Z = z) > 0$. We will introduce some short-hand notation for our distributions. If we want to consider the probability distribution of X as a function (of x) we will write P_X. For the conditional probability we write $P_{X|Z}(\cdot \mid z)$. The conditional probability is defined for $z \in \mathcal{Z}$ such that $P_Z(z) \neq 0$. For the whole collection of probabilities defined this way we write $P_{X|Z}$. If we consider two random variables X and Y, for the joint probability we introduce the following notation:

$$P(X \neq Y) := \sum_{\substack{x,y \\ x \neq y}} P_{X,Y}(x, y). \tag{7}$$

In our application the random variable X describes a measurement by observer A (called Alice) and takes possible values $\{0, 1\}$. Similarly, the random variable Y describes measurements by observer B (called Bob). The probabilities of X or Y taking one of the possible values 0 and 1, then simply describe the probability that a measurement of X or Y has the result 0 or 1, respectively. The random variables A_N and B_N describe the possible settings of the measurement by observers A and B; A_N takes values a in $\mathcal{A}_N$, B_N takes values b in $\mathcal{B}_N$. An important thing to consider is what we mean by P_X and P_Y (or $P_{X,Y}$) without considering the values of A_N and B_N. The observers (A and B) carry out the experiment. For A, measurement of X gives results which the observer can see. The setting of A_N can either be hidden from the observer, giving the distribution P_X, or it can be accessed by the observer, giving the distribution $P_{X|A_N}$. In case the setting of A_N is hidden, in our probabilistic setting the results of P_X are effectively averaged over the values a that A_N can take. The same process applies to observer B.

Now that we have established our notation, we can consider the question how we obtain such probabilities. The main goal of this paper is to compare probabilities given by a hypothetical theory $\mathcal{T}$ having an extra variable Z, with the probabilities given by quantum theory. There is no need to specify how the probabilities for theory $\mathcal{T}$ are obtained, as actual measurements only concern the probabilities it produces. The probabilities quantum theory produce have so far been confirmed by experiments, so it is natural to assume $\mathcal{T}$ is compatible with quantum mechanics (this will be explained in greater detail below), in the sense that averaging over the extra variable Z gives the same predictions as quantum mechanics. From now on, if we consider a probability

P that explicitly uses the hidden variable Z, we assume the probability to be derived using the theory $\mathcal{T}$.

A (pure) quantum-mechanical state ψ is a unit vector in some Hilbert space H. It describes the state of a quantum-mechanical system. An observer can preform measurements on this system. Quantum mechanics gives probabilities for possible outcomes of such measurements for every possible state ψ. Therefore, when we are looking at the probabilities of a specific ψ, we write P^{ψ} both for the quantum mechanical prediction and the predictions given by our theory $\mathcal{T}$. What is actually being measured in a state ψ is some observable O, which corresponds with a hermitian operator $\mathcal{O}$ on H. In the scope of this thesis, the operator $\mathcal{O}$ has a discrete spectrum of eigenvalues $\{\lambda_i\}$. We have $dim(H) < \infty$ typically. Quantum mechanics postulates the following properties:

- The outcome of a measurement of some observable O is one of the eigenvalues λ of $\mathcal{O}$.
- Let P_{λ} be a projection that projects on the eigenspace of $\mathcal{O}$ spanned by the eigenvectors with eigenvalue λ. The probability of measuring λ in a state $\phi \in H$ is given by (the Born rule):

$$P_O^{\phi}(\lambda) = \langle \phi, P_{\lambda}(\phi) \rangle \,. \tag{8}$$

The Born rule is a postulate. At present it does not seem possible to derive the rule without introducing other, often questionable assumptions.

3 Variational distance and correlation measure

We now define two important concepts which will be used to prove the main theorem. First we construct a metric on the space of probabilities with that have the same value set.

Definition 1. *Let $X : \Omega \to \mathfrak{X}$ and $Y : \Omega \to \mathfrak{X}$ be two random variables. For the corresponding probability distributions P_X and P_Y the **variational distance** between P_X and P_Y is given by*

$$D(P_X, P_Y) = \frac{1}{2} \sum_{x \in \mathfrak{X}} |P_X(x) - P_Y(x)|. \tag{9}$$

As the probabilities are functions in L^1 this metric is actually the same as (half) the canonical metric $d_1(.,.)$ on L^1. The fact that the variational distance as defined above is a metric, as well as other important properties, is summarized in the following lemma:

Lemma 1. *The variational distance $D(.,.)$ has the following properties:*

1. *$D(.,.)$ is a metric on the space of probability distributions P_X on $\mathfrak{X}$.*
2. *For all probability distributions P_X and P_Y: $0 \le D(P_X, P_Y) \le 1$.*
3. *Suppose we have random variables $X, X' : \Omega \to \mathfrak{X}$ and $Y, Y' : \Omega \to \mathfrak{Y}$. For the joint probability distributions $P_{X,Y}$ and $P_{X',Y'}$ $D(P_X, P_{X'}) \le D(P_{X,Y}, P_{X',Y'})$.*
4. *$D(.,.)$ is convex: let $\{\alpha_i\}_{i \in I}$ be a finite set satisfying $\forall i \in I : \alpha_i \ge 0$ and $\sum_{i \in I} \alpha_i = 1$. Let $\{P_{X_i}\}_{i \in I}$ and $\{P_{Y_i})\}_{i \in I}$ be sets of probability distributions. Then we have:*
$$D(\sum_{i \in I} \alpha_i P_{X_i}, \sum_{i \in I} \alpha_i P_{Y_i} \le \sum_{i \in I} \alpha_i D(P_{X_i}, P_{Y_i}).$$
5. $$D(P_X, P_Y) \le P(X \ne Y).$$

Proof. 1:

- It is clear that $D(P_X, P_Y) \ge 0$, as it is a sum over positive terms.
- Suppose $D(P_X, P_Y) = 0$. We know that
$$\forall x : |P_X(x) - P_Y(x)| \ge 0$$
$$\Longrightarrow$$
$$\forall x : |P_X(x) - P_Y(x)| = 0.$$
This means $P_X = P_Y$. Suppose $P_X = P_Y$, then
$$\forall x : |P_X(x) - P_Y(x)| = 0$$
$$\Longrightarrow$$
$$D(P_X, P_Y) = 0.$$

- As $|P_X(x) - P_Y(x)| = |P_Y(x) - P_X(x)|$ we have $D(P_X, P_Y) = D(P_Y, P_X)$.
- Suppose we have probability distributions $P_X, P_{X'}$ and $P_{X''}$. We see that:

$$\begin{aligned} D(P_X, P_{X'}) &= \frac{1}{2}\sum_x |P_X(x) - P_{X''}(x) + P_{X''}(x) - P_{X'}(x)| \\ &\leq \frac{1}{2}\sum_x |P_X(x) - P_{X''}(x)| + \frac{1}{2}\sum_x |P_{X''}(x) - P_{X'}(x)|. \end{aligned}$$

  This means $D(P_X, P_{X'}) \leq D(P_X, P_{X''}) + D(P_{X''}, P_{X'})$.

2:
As $\sum_x P_X(x) = 1$ and $\sum_x P_Y(x) = 1$ we can see that:

$$D(P_X, P_Y) \leq \frac{1}{2}\sum_x |P_X(x)| + |P_Y(x)| = 1.$$

3:

$$\begin{aligned} D(P_X, P_{X'}) &= \frac{1}{2}\sum_x |P_X(x) - P_{X'}(x)| \\ &= \frac{1}{2}\sum_x \sum_y |P_{X,Y}(x,y) - P_{X',Y'}(x,y)| \\ &\leq \frac{1}{2}\sum_x \sum_y |P_{X,Y}(x,y) - P_{X',Y'}(x,y)|. \\ &\leq D(P_{X,Y}, P_{X',Y'}). \end{aligned}$$

4:

$$\begin{aligned} D\left(\sum_i \alpha_i P_{X_i}, \sum_i \alpha_i P_{Y_i}\right) &= \frac{1}{2}\sum_x |\sum_i \alpha_i (P_{X_i}(x) - P_{Y_i}(y))| \\ &\leq \frac{1}{2}\sum_i \alpha_i \sum_x |P_{X_i}(x) - P_{Y_i}(y)| \\ &\leq \sum_i \alpha_i D\left(P_{X_i}, P_{Y_i}\right). \end{aligned}$$

5:
See lemma 6 in the Colbeck and Renner article ([3]). □

We now introduce the so-called **correlation measure**.

Definition 2. *If $P_{X,Y|A_N,B_N}$ is a collection of conditional probabilities, in the context of equations (1) -(4), we define, for $N \in \mathbb{N}$, the **correlation measure** I_N as:*

$$I_N(P_{X,Y|A_N,B_N}) = P(X = Y|A_N = 0, B_N = 2N-1) + \sum_{\substack{a\in\mathcal{A}_N, b\in\mathcal{B}_N \\ |a-b|=1}} P(X \neq Y|a,b). \tag{10}$$

We will extensively use this correlation measure on a special state in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$, namely the **maximally entangled state** defined as follows.

Definition 3. *Let $H_A = \mathbb{C}^2$ and $H_B = \mathbb{C}^2$ be two Hilbert spaces (of dimension 2) . The* ***maximally entangled state*** *$\psi_0 \in H_A \otimes H_B$ is defined by*

$$\psi_0 = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right). \tag{11}$$

In a more general sense, if we define a orthonormal basis on $\mathbb{C}^2$ by choosing e_1 and e_2, we can write

$$\psi_0 = \frac{1}{\sqrt{2}} \left(e_1 \otimes e_1 + e_2 \otimes e_2 \right).$$

4 Compatability of theories

We will be comparing the probabilities given by quantum mechanics with those of a theory $\mathcal{T}$ that has access to a hidden variable Z. All experiments so far have shown us that measurements agree with the predictions obtained by quantum theory. So in order to still explain the measurement results, our hidden variable theory $\mathcal{T}$ should be compatible with quantum mechanics. By this we mean that, if we calculate measurement outcomes with our theory $\mathcal{T}$ without having access to the extra parameter Z (effectively summing over all possible values of Z), we should obtain the same predictions as given by quantum theory.

Definition 4. *Suppose we have random variables X, A_N and Z. For measurements on a quantum state ψ, our hidden variable theory is* ***compatible*** *with quantum mechanics if*

$$\forall a, x : P^{\psi}_{X|A_N}(x \mid a) = \sum_{z \in \mathcal{Z}} P^{\psi}_{X,Z|A_N}(x, z \mid a). \tag{12}$$

Here $P^{\psi}_{X,Z|A_N}$ are the probabilities produced by our theory $\mathcal{T}$, whereas $P^{\psi}_{X|A_N}$ is the probability obtained from quantum mechanics.

In order to compare the predictions given by quantum mechanics with those given by the hidden variable theory, we have to specify what it means for our theory $\mathcal{T}$ to be **more informative**.

Definition 5. *Suppose we have random variables x, A_N and Z. Let the hidden variable theory $\mathcal{T}$ be compatible with quantum mechanics. For measurement on a quantum state ψ quantum mechanics is* ***as least as informative*** *as the hidden variable theory $\mathcal{T}$ if*

$$\forall x, a, z \text{(for which } P_{Z|A_N}(z \mid a) > 0\text{)}: P^{\psi}_{X|A_N}(x \mid a) = P^{\psi}_{X|A_N,Z}(x \mid a, z). \tag{13}$$

In other words, knowing z adds nothing to the predictions done on a measurement on ψ. From now on we always assume our hidden variable theory $\mathcal{T}$ to be compatible with quantum mechanics. This means we do not have to worry about distinguishing probabilities given by both theories without using the hidden variable Z, as these probabilities are the same.

5 Free parameters

A theory ($\mathcal{T}$) that describes our (hypothetical) measurements uses certain parameters. Common sense would dictate some of these parameters can be chosen "freely". We expect a theory to predict outcomes for every initial condition. For a physically relevant theory we demand that the parameter A_N is not correlated to other parameters, except to those parameters in the causal future of A_N. If this is the case we can consider A_N to be a free parameter. First we will have to define an semi-order on the parameters of our theory $\mathcal{T}$. A semi-order has all the properties of a normal order, but is not necessarily anti-symmetric.

Definition 6. *Let V be a set. A **semi-order** on V is a binary relation $\leq$ that is both reflexive and transitive. In other words:*

$$\forall v \in V : v \leq v, \tag{14}$$

$$\forall v, w, x \in V : \textit{If } v \leq w \textit{ and } w \leq x \Rightarrow v \leq x. \tag{15}$$

If Γ is the set of all parameters of our theory $\mathcal{T}$, we can define a causal order on this set.

Definition 7. *Let Γ be the set of all parameters of our theory $\mathcal{T}$. A **causal order** is a semi-order $\rightsquigarrow$ on Γ . From now on we say X lies in the causal future of A_N if $A_N \rightsquigarrow X$.*

The use of the the symbol $\rightsquigarrow$ indicates the central idea of ordering the parameters: we would like the causal order to be compatible with relativity. For example, the result of a measurement (X) should be free in relation to the parameters that describe the setting of the experiment (A_N), but the setting could possibly affect the measurement result. So: $\neg(X \rightsquigarrow A_N)$ but $A_N \rightsquigarrow X$. To be compatible with relativity we demand that in the causal order we only have $R \rightsquigarrow S$ if S lies in the future light-cone of R. With this in mind, we define the causal order as follows:

Definition 8. *Let Γ be the set of all parameters with the causal order $\rightsquigarrow$. We say $A_N \in \Gamma$ is a **free parameter** if:*

$$\forall X \in \Gamma \textit{ with } \neg(A_N \rightsquigarrow X) : P_{X,A_N} = P_{A_N} \cdot P_X. \tag{16}$$

6 Bipartite setup

In this paper we consider a specific measurement setup in quantum mechanics. As mentioned before there are two observers A (Alice) and B (Bob). The possible quantum states for each observer individually are described by the Hilbert spaces $H_A = \mathbb{C}^2$ for A, and $H_B = \mathbb{C}^2$ for B. The space $H_A \otimes H_B$ describes the states in the composite system that contains both A and B. As mentioned before, the random variables X and Y describe the measurement results for observers A and B, respectively (with possible results in $\{0, 1\}$). The random variables A_N and B_N describe the possible setting of the experiment. A_N has N possible values a in $\mathcal{A}_N = \{0, 2, \ldots, 2N - 2\}$, whilst B_N has N possible values b in $\mathcal{B}_N = \{1, 3, \ldots, 2N - 1\}$. We refer to this situation as a **bipartite setup**. For a bipartite setup we want the causal order to have the following properties:

$$A_N \rightsquigarrow X, B_N \rightsquigarrow Y, \tag{17}$$
$$\neg(A_N \rightsquigarrow Z), \neg(B_N \rightsquigarrow Z), \tag{18}$$
$$\neg(A_N \rightsquigarrow Y), \neg(B_N \rightsquigarrow X). \tag{19}$$
$$\tag{20}$$

If either A or B preforms a measurement, the resulting quantum state in the total system $H_A \otimes H_B$ will be a projection of our initial state onto a certain subspace, namely for a given vector $v \in H$ we can define a projection onto the subspace spanned by this vector. This projection $P_v : H \to \mathbb{C} \cdot v$ is given by

$$\forall x \in H : P_v(x) = \langle x, v \rangle \, v, \tag{21}$$

where $\langle \ , \ \rangle$ is the inner product of H (taken to be linear in the second variable). We can also project onto a subspace spanned by multiple vectors. If Y is a subspace of H spanned by the vectors $\{\nu_1, \ldots, \nu_n\}$ we write for the projection of x onto the subspace Y

$$Proj_Y = \sum_{i=1}^{n} P_\nu(x). \tag{22}$$

Suppose H_A and H_B are two Hilbert spaces. If P_ν is a projection on H_A and P_ω is a projection on H_B (so $\nu \in H_A, \omega \in H_B$), we can construct a projection $P_\nu \otimes P_\omega : H_A \otimes H_B \to (\mathbb{C} \cdot \nu) \otimes (\mathbb{C} \cdot \omega)$ by taking:

$$P_\nu \otimes P_\omega \left(\sum_{i,j} \alpha_i \otimes \beta_j \right) = \sum_{i,j} (P_\nu(\alpha_i) \otimes P_\omega(\beta_j)). \tag{23}$$

These projections on $H_A \otimes H_B$ are used to find the resulting state after some measurement by A and B. In our bipartite setup we preform a measurement using the following vectors in $\mathbb{C}^2$:

- $e_x^a = \frac{\pi}{2}(\frac{a}{2N} + x)$, $E_x^a = \binom{cos(e_x^a)}{sin(e_x^a)}$;
- $f_x^b = \frac{\pi}{2}(\frac{b}{2N} + y)$, $F_y^b = \binom{cos(f_y^b)}{sin(f_y^b)}$.

For a fixed value of a, the measurement of A can have two possible outcomes $X = 0$ and $X = 1$ which are described by projecting onto E_0^a and E_1^a, respectively. The same goes for B. This means we have a probability distribution for the measurement in our bipartite setup on $\psi \in H_A \otimes H_B$ given by

$$P^\psi_{X,Y|A_N,B_N}(x, y \mid a, b) = \left\langle \psi, \left(P_{E_x^a} \otimes P_{F_y^b} \right) \psi \right\rangle.$$

An important property of this bipartite setup in combination with the maximally entangled state and the correlation measure is given in the following lemma.

Lemma 2. *According to quantum mechanics, the correlation measure of the maximally entangled state ψ_0 in a bipartite setup with fixed $N \in \mathbb{N}$ is equal to:*

$$I_N(P^{\psi_0}_{X,Y|A_N,B_N}) = 2N\sin^2(\frac{\pi}{4N}) \leq \frac{\pi^2}{8N}. \tag{24}$$

Proof. First, we calculate $P^{\psi_0}_{X,Y|A_N,B_N}(x, y \mid a, b)$.

$$\begin{aligned}
\left\langle \psi_0, P_{E^a_x} \otimes P_{F^b_y}(\psi_0) \right\rangle &= \frac{1}{2}\langle [\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}], [\cos(e^a_x)E^a_x \otimes \cos(f^b_y)F^b_y \\
&\quad + \sin(e^a_x)E^a_x \otimes \sin(f^b_y)F^b_y]\rangle \\
&= \left\langle \frac{1}{2}(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}, \cos(e^a_x)E^a_x \otimes \cos(f^b_y)F^b_y \right\rangle \\
&\quad + \left\langle \frac{1}{2}(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}, \sin(e^a_x)E^a_x \otimes \sin(f^b_y)F^b_y \right\rangle \\
&\quad + \left\langle \frac{1}{2}(\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}, \cos(e^a_x)E^a_x \otimes \cos(f^b_y)F^b_y \right\rangle \\
&\quad + \left\langle \frac{1}{2}(\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}, \sin(e^a_x)E^a_x \otimes \sin(f^b_y)F^b_y \right\rangle \\
&= \frac{1}{2}(\cos^2(e^a_x)\cos^2(f^b_y) + 2\cos(e^a_x)\cos(f^b_y)\sin(e^a_x)\sin(f^b_y) \\
&\quad + \sin^2(e^a_x)\sin^2(f^b_y)) \\
&= \frac{1}{2}(\cos(e^a_x)\cos(f^b_y) + \sin(e^a_x)\sin(f^b_y))^2 \\
&= \frac{1}{2}\cos^2(e^a_x - f^b_y) = \frac{1}{2}\cos^2(\frac{\pi}{2}(\frac{a-b}{2N} + x - y)).
\end{aligned} \tag{25}$$

Now we will calculate I_N:

$$\begin{aligned}
I_N(P^{\psi_0}_{X,Y|A_N,B_N}) &= \left\langle \psi_0, P_{E_0^0} \otimes P_{F_0^{2N-1}}(\psi_0) \right\rangle + \left\langle \psi_0, P_{E_1^0} \otimes P_{F_1^{2N-1}}(\psi_0) \right\rangle \\
&\quad + \sum_{|a-b|=1} \left\langle \psi_0, P_{E_0^a} \otimes P_{F_1^b}(\psi_0) \right\rangle + \left\langle \psi_0, P_{E_1^a} \otimes P_{F_0^b}(\psi_0) \right\rangle \\
&= \cos^2(\frac{2N-1}{4N}\pi) \\
&\quad + \sum_{|a-b|=1} \frac{1}{2}(\cos^2(\frac{a-b}{4N}\pi - \frac{1}{2}\pi) + \cos^2(\frac{a-b}{4N}\pi + \frac{1}{2}\pi)) \\
&= \sin^2(\frac{1}{4N}\pi) + \sum_{|a-b|=1} \sin^2(\frac{a-b}{4N}\pi) \\
&= \sin^2(\frac{1}{4N}\pi) + 2N\sin^2(\frac{1}{4N}\pi) - \sin^2(\frac{1}{4N}\pi) \\
&= 2N\sin^2(\frac{1}{4N}\pi) \\
&\leq 2N(\frac{\pi}{4N})^2 = \frac{\pi^2}{8N}. \qquad (26)
\end{aligned}$$

□

7 Main theorem

Our ultimate goal is to prove that the assumptions of free variables and compatibility with quantum mechanics lead to the conclusion that the hidden variable Z does not give improved predictions compared with quantum mechanics. In order to build towards this goal, we will first derive a direct consequence of our free variables. It turns out that free-variables force our higher theory $\mathcal{T}$ to be no-signaling. The concept will be explained in the next subsection. Subsequently we prove our claim for a very specific state and measurement. This is the second subsection. Then, in order to generalize, we need a process called "embezzlement". This process is described in the third subsection. Finally, the last subsection will state the claim, and use all the previous work to prove the main claim of Colbeck and Renner: i.e. that hidden variables do not improve quantum mechanics.

7.1 No-signaling

Lemma 3. *The causal order satisfying the conditions in equations (17), (18) and (19) imply:*

$$P_{X,Z|A_N,B_N} = P_{X,Z|A_N} \text{ and } P_{Y,Z|A_N,B_N} = P_{Y,Z|B_N}.$$

We call a theory with this property a **no-signaling theory**.

Proof.

$$\begin{aligned}
P_{X,Z|A_N,B_N}(x,z,a,b) &= \frac{P_{X,Z,A_N,B_N}(x,z,a,b)}{P_{A_N,B_N}(a,b)} \\
&= \frac{P_{B_N}(b)P_{X,Z,A_N}(x,z,a)}{P_{A_N}(a)P_{B_N}(b)} && \text{(As } B_N \text{ is free w.r.t} \\
& && \text{by the causal order: eq (17), (18), (19))} \\
&= P_{X,Z|A_N}(x,z \mid a).
\end{aligned}$$

The same holds for $P_{Y,Z|A_N,B_N}$. □

7.2 Claim for the entangled state

During this whole subsection, we will be working in the bipartite setup (using the measurement described above). This means we know the value sets of X, Y, A_N, B_N and Z(see equation (1) to (4)). The state for which we calculate all the probabilities and correlations below is ψ_0, the maximally entangled state.

Lemma 4. *Suppose the distribution $P^{\psi_0}_{X,Y|A_N,B_N}$ is a no-signaling theory. We introduce the uniform distribution defined by:*

$$P_{\overline{X}}(0) = \frac{1}{2};$$
$$P_{\overline{X}}(1) = \frac{1}{2}.$$

We then have the following inequality:

$$\forall a,b : \left\langle D(P^{\psi_0}_{X|A_N,B_N,Z}(\cdot \mid a,b,z), P_{\overline{X}}\right\rangle_Z \leq \frac{1}{2} I_N(P^{\psi_0}_{X,Y|A_N,B_N}), \tag{27}$$

where $\langle \ , \ \rangle_Z$ is the average over $z \in \mathcal{Z}$, where the z are distributed according to $P^{\psi_0}_{Z|A_N,B_N}(\cdot \mid a,b)$.

Proof. We will omit the ψ_0 in the probabilities, and write $a_0 = 0$, $b_0 = 2N - 1$.

$$\begin{aligned}
I_N(P_{X,Y|A_N,B_N,Z}(\cdot,\cdot \mid \cdot,\cdot,z)) &= P(X = Y|A_N = a_0, B_N = b_0, Z = z) \\
&+ \sum_{\substack{a\in\mathcal{A}_N, b\in\mathcal{B}_N \\ |a-b|=1}} P(X \neq Y \mid A_N = a, B_N = b, Z = z) \\
&\overset{\text{Lemma 1}}{\geq} D(1 - P_{X|A_N,B_N,Z}(x \mid a_0, b_0, z), P_{Y|A_N,B_N}(y \mid a_0, b_0, z)) \\
&+ \sum_{\substack{a\in\mathcal{A}_N, b\in\mathcal{B}_N \\ |a-b|=1}} D(P_{X|A_N,B_N,Z}(x \mid a, b, z), P_{Y|A_N,B_N,Z}(y \mid a, b, z)) \\
&\overset{\text{Lemma 3}}{=} D(1 - P_{X|A_N,Z}(x \mid a_0, z), P_{Y|B_N,Z}(y \mid b_0, z)) \\
&+ \sum_{\substack{a\in\mathcal{A}_N, b\in\mathcal{B}_N \\ |a-b|=1}} D(P_{X|A_N,Z}(x \mid a, z), P_{Y|B_N,Z}(y \mid b, z)).
\end{aligned}$$

As $D(\ , \ )$ is a metric, we have, using the triangle inequality,

$$\begin{aligned}
&D(1 - P_{X|A_N,Z}(\cdot \mid a_0, z), P_{X|A_N,Z}(\cdot \mid a_0, z)) \leq \\
&D(1 - P_{X|A_N,Z}(\cdot \mid a_0, z), P_{Y|B_N,Z}(\cdot \mid b_0, z)) + D(P_{X|A_N,Z}(\cdot \mid a_0, z), P_{Y|B_N,Z}(\cdot \mid b_0, z)).
\end{aligned}$$

Now for the second term $D(P_{X|A_N,Z}(\cdot \mid a_0, z), P_{Y|B_N,Z}(\cdot \mid b_0, z))$, by using the triangle inequality multiple times can get an expression which only contains a 's and b's which have a distance of one. For $a_0 = 0, b_0 = 2N - 1$ we have:

$$\begin{aligned}
D(P_{X|A_N,Z}(\cdot \mid a_0, z), P_{Y|B_N,Z}(\cdot \mid b_0, z)) &\leq D(P_{X|A_N,Z}(\cdot \mid a_0, z), P_{Y|B_N,Z}(\cdot \mid 1, z)) \\
&+ D(P_{X|A_N,Z}(\cdot \mid 1, z), P_{Y|B_N,Z}(\cdot \mid b_0, z)) \\
&\leq D(P_{X|A_N,Z}(\cdot \mid a_0, z), P_{Y|B_N,Z}(\cdot \mid 1, z)) \\
&+ D(P_{X|A_N,Z}(\cdot \mid 1, z), P_{Y|B_N,Z}(\cdot \mid 2, z)) \\
&+ D(P_{X|A_N,Z}(\cdot \mid 2, z), P_{Y|B_N,Z}(\cdot \mid b_0, z)) \\
&\dots \\
&\leq \sum_{\substack{a\in\mathcal{A}_N, b\in\mathcal{B}_N \\ |a-b|=1}} D(P_{X|A_N,Z}(x \mid a, z), P_{Y|B_N,Z}(y \mid b, z)).
\end{aligned}$$

which gives us

$$\begin{aligned}
I_N(P_{X,Y|A_N,B_N,Z}(\cdot,\cdot \mid \cdot,\cdot,z) &\geq D(1 - P_{X|A_N,Z}(\cdot \mid a_0, z), P_{X|A_N,Z}(\cdot \mid a_0, z)) \qquad (28) \\
&= \frac{1}{2}\sum_x |1 - P_{X|A_N,Z}(x \mid a_0, z) - P_{X|A_N,Z}(x \mid a_0, z)| \\
&= 2(\frac{1}{2}\sum_x |\frac{1}{2} - P_{X|A_N,Z}(x \mid a_0, z)|) \\
&= 2D(P_{X|A_N,B_N,Z}(\cdot \mid a_0, b_0, z), P_{\overline{X}})). \qquad (29)
\end{aligned}$$

Note that the probability $P_{X|A_N,B_N,Z}(\cdot \mid a_0, b_0, z)$ for ψ_0 only depends on the distance of a and b modulus $2N - 1$. For a_0 and b_0 above the distance (modulo $2N - 1$) is 1. This means we can replace a_0 and b_0 for any a and b with distance 1. Eventually we only use a_0 (equation 28), so equation 29 hold for all a and b. We now average both sides of inequality (29) over all $z \in \mathcal{Z}$. Taking the average on the left hand side ($I_N(P_{X,Y|A,B,Z}(\cdot,\cdot \mid \cdot,\cdot,z))$ will complete the proof of the lemma. Note that as A_N and B_N are free variables, we have for all z, a, b:

$$\begin{aligned}
P_{Z|A_N,B_N}(z \mid a,b) &= \frac{P_{Z,A_N,B_N}(z,a,b)}{P_{A_N,B_N}(a,b)} \\
&= \frac{P_{Z,A_N,B_N}(z,a,b)}{P_{A_N}(a)P_{B_N}(b)} && (\text{as } \neg(A_N \rightsquigarrow B_N)) \\
&= \frac{P_Z(z)P_{A_N}(a)P_{B_N}(b)}{P_{A_N}(a)P_{B_N}(b)} && (\text{as } \neg(A_N \rightsquigarrow Z), \neg(A_N \rightsquigarrow B_N) \\
& && \text{and } \neg(B_N \rightsquigarrow Z), \neg(B_N \rightsquigarrow A_N)) \\
&= P_Z(z).
\end{aligned}$$

Using the fact that for all z one has $P_{Z|A_N,B_N}(z \mid a,b) = P_Z(z)$, we can average I_N over z:

$$\begin{aligned}
\langle I_N(P_{X,Y|A_N,B_N,Z}(\cdot,\cdot \mid \cdot,\cdot,z)\rangle_Z :&= \sum_z P_{Z|A_N,B_N}(z \mid a,b) I_N(P_{X,Y|A_N,B_N,Z}(\cdot,\cdot \mid \cdot,\cdot,z) \\
&= \sum_z P_Z(z) I_N(P_{X,Y|A_N,B_N,Z}(\cdot,\cdot \mid \cdot,\cdot,z) \\
&= \sum_z P_{Z|A_N,B_N}(z \mid a_0,b_0) P(X=Y|A_N=a_0,B_N=b_0,Z=z) \\
&+ \sum_{\substack{a\in\mathcal{A}_N,b\in\mathcal{B}_N \\ |a-b|=1}} \sum_z P_{Z|A_N,B_N}(z \mid a,b) P(X \neq Y|A_N=a,B_N=b,Z=z) \\
&= P(X=Y|A_N=a_0,B_N=b_0) \\
&\quad + \sum_{\substack{a\in\mathcal{A}_N,b\in\mathcal{B}_N \\ |a-b|=1}} P(X\neq Y|A_N=a,B_N=b) \\
&= I_N(P_{X,Y|A_N,B_N}).
\end{aligned}$$

This is the expression given in the lemma. □

Now we average over z the right hand side of equation (29):

$$\begin{aligned}
&\left\langle D(P_{X|A_N,B_N,Z}(\cdot \mid a,b,z), P_{\overline{X}}\right\rangle_Z \\
&:= \sum_z P_{Z|A_N,B_N}(z \mid a_0,b_0) D(P_{X|A_N,B_N,Z}(\cdot \mid a,b,z), P_{\overline{X}}) \\
&= \sum_z P_{Z|A_N,B_N}(z \mid a_0,b_0)(\frac{1}{2}\sum_x |P_{X|A_N,B_N,Z}(x \mid a,b,z) - P_{\overline{X}}(x)|) \\
&= \frac{1}{2}\sum_{x,z} |P_{X|A_N,B_N,Z}(x \mid a,b,z) P_{Z|A_N,B_N}(z \mid a_0,b_0) \\
&- P_{\overline{X}}(x) P_{Z|A_N,B_N}(z \mid a_0,b_0)| \\
&= D(P_{X,Z|A_N,B_N}(\cdot,\cdot \mid a,b), P_{\overline{X}} P_{Z|A_N,B_N}(\cdot \mid a_0,b_0)). \qquad (30)
\end{aligned}$$

Now we will use Lemma 2 and Lemma 4 (which we have just proven) on $P^{\psi_0}_{X,Y|A_N,B_N}$.

Lemma 5.

$$\forall x, a, z \text{ (such that } P^{\psi_0}_{Z|A_N}(z \mid a) > 0\text{): } P^{\psi_0}_{X|A_N,Z}(x \mid a, z) = P^{\psi_0}_{X|A_N}(x \mid a). \tag{31}$$

Proof. It is easy to see that $P_{\overline{X}}(x) = P^{\psi_0}_{X|A_N,B_N}(x \mid a, b)$. In combination with the average (30) we just calculated, and the previous lemma (4), this gives the inequality

$$D(P^{\psi_0}_{X,Z|A_N,B_N}(\cdot,\cdot \mid a,b), P^{\psi_0}_{X|A_N,B_N}(\cdot \mid a,b)P^{\psi_0}_{Z|A_N,B_N}(\cdot \mid a_0,b_0)) \leq \frac{1}{2}I_N(P_{X,Y|A_N,B_N}).$$

The idea is to take the limit $N \to \infty$. By doing this we increase the range of possible values taken by A_N and B_N. The quantum-mechanical probabilities $P^{\psi_0}_{X|A_N,B_N}$ are given by projecting onto the vector $E^a_x = \binom{cos(e^a_x)}{sin(e^a_x)}$ (where $e^a_x = \frac{\pi}{2}(\frac{a}{2N} + x)$). If we choose $N' = kN (k \in \mathbb{N})$ and $a' = ak$ we project on the same vector. So

$$P^{\psi_0}_{X|A_N,B_N}(x \mid a, b) = P^{\psi_0}_{X|A_{kN},B_{kN}}(x \mid ak, bk).$$

Increasing N does not change the probability of $P^{\psi_0}_{X|A_N,B_N}(\cdot \mid a, b)$ as long as we scale a and b accordingly. However, for $P_{X,Z|A_N,B_N}(x, z \mid a, b)$ we cannot directly conclude that scaling gives the same probabilities. A condition we have to place on our theory $\mathcal{T}$ is that for the same physical measurement, the probabilities given by $\mathcal{T}$ stay the same. Thus if we scale a and b as above, we are measuring the state ψ_0 along the same angles, therefore physically giving the same measurement. From this we can conclude:

$$P_{X,Z|A_N,B_N}(x, z \mid a, b) = P_{X,Z|A_{kN},B_{kN}}(x, z \mid ak, bk).$$

The last term $P^{\psi_0}_{Z|A_N,B_N}$ can easily be scaled as:

$$P^{\psi_0}_{Z|A_N,B_N} = P^{\psi_0}_{Z} = P^{\psi_0}_{Z|A_{kN},B_{kN}}.$$

Now it follows that:

$$\begin{aligned}
D(P^{\psi_0}_{X,Z|A_N,B_N}&(\cdot,\cdot \mid a,b), P^{\psi_0}_{X|A_N,B_N}(\cdot \mid a,b)P^{\psi_0}_{Z|A_N,B_N}(\cdot \mid a_0,b_0)) \\
&= D(P^{\psi_0}_{X,Z|A_{kN},B_{kN}}(\cdot,\cdot \mid ak,bk), P^{\psi_0}_{X|A_{kN},B_{kN}}(\cdot \mid ak,bk)P^{\psi_0}_{Z|A_{kN},B_{kN}}(\cdot \mid a_0k,b_0k)) \\
&\leq \frac{1}{2}I_N(P_{X,Y|A_{kN},B_{kN}}) \\
&\leq \frac{\pi^2}{16kN}.
\end{aligned}$$

Taking the limit $k \to \infty$ now forces the metric to zero, making the distributions equal, which in turn gives:

$$P^{\psi_0}_{X,Z|A_N,B_N}(\cdot,\cdot \mid a,b) = P^{\psi_0}_{X|A_N,B_N}(\cdot \mid a,b)P^{\psi_0}_{Z|A_N,B_N}(\cdot \mid a_0,b_0);$$
$$\Longrightarrow$$
$$P^{\psi_0}_{X|A_N,Z}(x \mid a, z) = P^{\psi_0}_{X|A_N}(x \mid a) \text{ (if } P^{\psi_0}_{Z|A_N}(z \mid a) \geq 0).$$

□

7.3 Ordering coefficients of states

Before we continue to build a construction to exploit the results we have so far, we need to introduce some notation and facts that are needed in the next sections.

Definition 9. *Let $\phi = \sum_{i=1}^{n} \alpha_i e_i$ be a state in Hilbert space H of dimension m. We define the sequence $\{\alpha_{i_r}^{\downarrow}\}$ (where $1 \leq r \leq n$) to be the sequence of coefficients α_i ordered in descending order.*

To be clear, if for example α_5 is the largest coefficient, we define $i_1 = 5$, such that $\alpha_{i_1}^{\downarrow} = \alpha_5$. We use this idea of rearranging coefficient in descending order to compare different states.

Definition 10. *Suppose we have two states $\phi = \sum_{i=1}^{n} \alpha_i e_i$ and $\psi = \sum_{j=1}^{n} \beta_j e_j$ in Hilbert space H. We write: $\phi \succ \psi$ if*

$$\sum_{r=1}^{k} (|\alpha_{i_r}^{\downarrow}|)^2 \geq \sum_{r=1}^{k} (|\beta_{i_r}^{\downarrow}|)^2 \text{ for all } k \in \{1, \ldots, n\}$$

It turns out maximally entangled states are "minimal" in this sense.

Lemma 6. *Suppose we have a Hilbert space H of dimension m. Let $\psi_m = \sum_{i=1}^{m} \frac{1}{\sqrt{m}} e_i$ be the maximally entangled state of rank m and let $\phi = \sum_{i=1}^{m} \alpha_i e_i$ be another state in H. We have $\phi \succ \psi_m$.*

Proof. We will use induction on the number of elements we sum over. First we claim $|\alpha_{i_1}^{\downarrow}| \geq \frac{1}{m}$. Suppose $|\alpha_{i_1}^{\downarrow}| < \frac{1}{m}$. This implies $|\alpha_{i_r}^{\downarrow}| < \frac{1}{m}$ for all r, as $\alpha_{i_1}^{\downarrow}$ is the largest coefficient. But then we have

$$1 > \sum_{r=1}^{n} |\alpha_{i_r}^{\downarrow}|^2 = \sum_{i=1}^{n} |\alpha_i|^2 = 1.$$

So this is a contradiction with ϕ having norm 1. Suppose now that we know for all $l < k$: $\sum_{r=1}^{l} |\alpha_{i_r}^{\downarrow}|^2 \geq \sum_{i=1}^{l} \frac{1}{m}$. We again prove by contradiction, so suppose $\sum_{r=1}^{k} |\alpha_{i_r}^{\downarrow}|^2 < \sum_{i=1}^{k} \frac{1}{m}$. This means

$$\begin{aligned}
|\alpha_{i_k}^{\downarrow}|^2 &< \frac{1}{m} - \left(\sum_{r=1}^{k-1} |\alpha_{i_r}^{\downarrow}|^2 - \sum_{i=1}^{k-1} \frac{1}{m} \right) \\
|\alpha_{i_r}^{\downarrow}|^2 &< \frac{1}{m} \text{ for all } r > k \\
\sum_{r=1}^{m} |\alpha_{i_r}^{\downarrow}|^2 &< \sum_{r=1}^{k} |\alpha_{i_r}^{\downarrow}|^2 + \sum_{k+1}^{m} \frac{1}{m} \\
\sum_{r=1}^{m} |\alpha_{i_r}^{\downarrow}|^2 &< \sum_{i=1}^{m} \frac{1}{m} = 1
\end{aligned}$$

This is a contradiction with ϕ having norm 1, so $\phi \succ \psi_m$. □

This idea can also be expanded to states which consists of tensor products.

Corollary 1. *Suppose we have state $\phi = \sum_{i=1}^{m} \alpha_i e_i$ and the maximally entangled state $\psi_m = \sum_{i=1}^{m} \frac{1}{\sqrt{m}} e_i$ of rank m in Hilbert space H of dimension m. Let $\psi = \sum_{j=1}^{n} \beta_j$ be another state in Hilbert space H' of dimension n. Then $\phi \otimes \psi \succ \psi_m \otimes \psi$*

Proof. The coefficients of $\phi \otimes \psi$ are of the form $\alpha_i \beta_j$, those of $\psi_m \otimes \psi$ are $\frac{1}{m}\beta_j$. We know from the previous lemma that $\phi \succ \psi_m$. Looking at the sum of the k largest $|\frac{1}{m}\beta_j|$ squared, we know there are k α_i, such that the sum over these α_i squared is greater than $\frac{k}{m}$ (as $\phi \succ \psi_m$). This means the sum over the coefficients which consists of the b_j of the k largest coefficients of $\psi_m \otimes \psi$ paired with these α_i is larger than the sum over the k largest $|\frac{1}{m}\beta_j|$ squared □

7.4 Embezzlement ([5], [6])

To generalize the previous results to the main theorem we are trying to prove, we are going to make use of a construction called the *embezzling state*. These embezzling states are vectors in the space $\bigotimes^n \mathbb{C}^2 \simeq \mathbb{C}^{2^n}$. For $\mathbb{C}^2$ we take the basis vectors $|1\rangle = \binom{1}{0}$ and $|0\rangle = \binom{0}{1}$. A tensor product over n such vectors may, for example, have the form $|1\rangle \otimes |0\rangle \otimes \ldots \otimes |0\rangle \otimes |0\rangle$. Any natural number i such that $1 \leq i \leq 2^n - 1$ can be written in base 2: $b_2(i) = k_{n-1}2^{n-1} + k_{n-2}2^{n-2} + \ldots + k_1 2 + k_0$ ($k_j \in \{0,1\}$). For this decomposition in base two we write $i = (k_n \ldots k_0)_2$. We now define a basis of $\bigotimes^n \mathbb{C}^2$ as

Definition 11. *For $n \in \mathbb{N}$ we decompose any $0 \leq i \leq 2^n - 1$ in base two. If $b_2(i) = (k_n \ldots k_0)_2$ we take:*

$$\bigotimes^n \mathbb{C}^2 \ni e_{i+1} := |k_n\rangle \otimes \ldots \otimes |k_0\rangle \, . \tag{32}$$

The $\{e_i\}_{i=1,\ldots 2^n}$ are pairwise orthonormal and therefore form a basis of $\bigotimes^n \mathbb{C}^n$. Using this basis we now define a special collection of vectors μ_n.

Definition 12. *Take $H_{\tilde{A}} = \bigotimes^n \mathbb{C}^2 \simeq \mathbb{C}^{2^n}$ and $H_{\tilde{A}} = H_{\tilde{B}}$. For all $n \in \mathbb{N}$ we have an* ***embezzling state*** *μ_n defined as:*

$$H_{\tilde{A}} \otimes H_{\tilde{B}} \ni \mu_n := \frac{1}{\sqrt{C_n}} \sum_{j=1}^{2^n} \frac{1}{\sqrt{j}} e_j^{H_{\tilde{A}}} \otimes e_j^{H_{\tilde{B}}}, \tag{33}$$

where $C_n = \sum_{j=1}^{j=2^n} \frac{1}{j}$ (to normalize this μ_n).

We look at $H_{A'} = \bigotimes^m \mathbb{C}^2 = \mathbb{C}^{2^m}$ and $H_{B'} = H_{A'}$. Suppose, for an arbitrary m, we are given a bipartite state in $H_{A'} \otimes H_{B'} = \mathbb{C}^{2^m} \otimes \mathbb{C}^{2^m}$, written in its Schmidt decomposition as $\phi = \sum_{i=1}^{2^m} \alpha_i \theta_i^{H_{A'}} \otimes \theta_i^{H_{B'}}$, where $\theta_i^{H_{A'}} \otimes \theta_i^{H_{B'}} \in H_{A'} \otimes H_{B'}$. The α_i are greater than 0.

The situation for this embezzlement protocol is very similar to the bipartite setup. We have observers A and B. This time, the states belonging to A are in the Hilbert spaces $H_{A'}$ and $H_{\tilde{A}}$. Those of B in spaces $H_{B'}$ and $H_{\tilde{B}}$. The idea behind these embezzling states is to transform the initial state $e_1^{H_{A'}} \otimes e_1^{H_{B'}} \otimes \mu_n$ to a vector arbitrarily close to $\phi \otimes \mu_n$, but doing it without any communication between observers A and B. This means we can only use unitary transformations on $H_{A'}$ and $H_{\tilde{A}}$ for A, and unitary transformation on $H_{B'}$ and $H_{\tilde{B}}$ for B. Such transformations are called **local unitary transformations**. The ability to extract an arbitrary ϕ from the initial state will enable us to apply our previous results for the maximally entangled states on more general measurements.

Definition 13. *Take $H_{\tilde{A}} = H_{\tilde{B}} = \bigotimes^n \mathbb{C}^2 \simeq \mathbb{C}^{2^n}$ and $H_{A'} = H_{B'} = \bigotimes^m \mathbb{C}^2 \simeq \mathbb{C}^{2^m}$. For $\mu_n \in H_{\tilde{A}} \otimes H_{\tilde{B}}$ the embezzling state, and $\phi \in H_{A'} \otimes H_{B'}$ an arbitrary state (ie. unit vector), the vector*

$$\phi \otimes \mu_n = \sum_{j,i} \gamma_{i,j} ((\theta_i^{H_{A'}} \otimes \theta_i^{H_{B'}})) \otimes (e_j^{H_{\tilde{A}}} \otimes e_j^{H_{\tilde{B}}}),$$

(in $H_{A'} \otimes H_{B'} \otimes H_{\tilde{A}} \otimes H_{\tilde{B}}$) gives us coefficients $\{\gamma_{i,j}\}_{i,j}$. Taking the 2^n largest of those and ordering them as a descending sequence we obtain a sequence $\gamma^{\downarrow}_{i_r,j_r}$ such that $\gamma^{\downarrow}_{i_1,j_1} \geq \gamma^{\downarrow}_{i_2,j_2} \geq \ldots \geq \gamma^{\downarrow}_{i_{2^m},j_{2^n}}$. For this sequence and the given unit vector ϕ we define ***the embezzlement of*** *ϕ as*

$$E(\phi)_{n,m} := \frac{1}{\sqrt{C_n}} \sum_{r=1}^{r=2^n} \frac{1}{\sqrt{r}} ((\theta_{i_r}^{H_{A'}} \otimes \theta_{i_r}^{H_{B'}}) \otimes (e_{j_r}^{H_{\tilde{A}}} \otimes e_{j_r}^{H_{\tilde{B}}})). \tag{34}$$

This $E(\phi)_{n,m}$ is also a vector in $H_{A'} \otimes H_{B'} \otimes H_{\tilde{A}} \otimes H_{\tilde{B}}$.

This embezzlement of ϕ is important because we are able to transform μ_n to $E(\phi)_{n,m}$ using only local unitary transformations.

Lemma 7. *Let $\{e_1^{H_{A'}} \ldots \tilde{e}_{2^m}^{H_{B'}}\}$ be an orthonormal basis of $\bigotimes^m \mathbb{C}^2 \simeq \mathbb{C}^{2^m}$. With $e_1^{H_{A'}} \otimes e_1^{H_{B'}}$ we can consider the state*

$$(e_1^{H_{A'}} \otimes e_1^{H_{B'}}) \otimes (\mu_n) = \frac{1}{\sqrt{C_n}} \sum_{j=1}^{n} \frac{1}{\sqrt{j}} (e_1^{H_{A'}} \otimes e_1^{H_{B'}}) \otimes (e_j^{H_{\tilde{A}}} \otimes e_j^{H_{\tilde{B}}}),$$

in $H_{A'} \otimes H_{B'} \otimes H_{\tilde{A}} \otimes H_{\tilde{B}}$. There exist local isometries $U_{n,m;\phi}$(for A) on $H_{A'} \otimes H_{\tilde{A}}$ and $V_{n,m;\phi}$ on $H_{B'} \otimes H_{\tilde{B}}$ (for B) such that these transform $(e_1^{H_{A'}} \otimes e_1^{H_{B'}}) \otimes (\mu_n)$ to $E(\phi)_{n,m}$.

Proof. We will prove this lemma for $U_{n,m;\psi}$ on $H_{A'} \otimes H_{\tilde{A}}$. The proof for $V_{n,m;\psi}$ is exactly the same, only this time taking place on $H_{B'} \otimes H_{\tilde{B}}$ (which is effectively the same space as $H_{A'} \otimes H_{\tilde{A}}$).

The space $H_{A'} \otimes H_{\tilde{A}}$ is isomorphic to the space $\mathbb{C}^{2^{n+m}}$. We have two orthonormal bases on this space. Basis one is $e_i^{H_{A'}} \otimes e_j^{H_{\tilde{A}}}$ with $j = 1, \ldots 2^n$ and $i = 1, \ldots, 2^m$. The vectors $e_1^{H_{A'}} \otimes e_1^{H_{\tilde{A}}}$ are part of this basis. The second base is $\theta_i^{H_{A'}} \otimes e_j^{H_{\tilde{A}}}$ ($\theta_i^{H_{A'}} \in H_{A'}$) with $j = 1, \ldots 2^n$ and $i = 1, \ldots, 2^m$. The vectors $\theta_{i_r}^{H_{A'}} \otimes e_{j_r}^{H_{\tilde{A}}}$ are part of this basis.

A basis transform between two orthogonal basis is unitary. Therefore, we take $U_{n,m;\phi}$ to be the basis transformation that maps basis vector $e_1^{H_{A'}} \otimes e_j^{H_{B'}}$ to $\sqrt{C_n}(\theta_{i_j}^{H_{A'}} \otimes e_{j_j}^{H_{\tilde{A}}})$. We let $V_{n,m;\phi}$ be the basis transform that maps $e_1^{H_{B'}} \otimes e_j^{H_{\tilde{B}}}$ to $\sqrt{C_n}(\theta_{i_j}^{H_{B'}} \otimes e_{j_j}^{H_{\tilde{B}}})$. We want to apply $U_{n,m;\phi}$ and $V_{n,m;\phi}$ to $(e_1^{H_{A'}} \otimes e_1^{H_{B'}}) \otimes \mu_n$. Note that $U_{n,m;\phi} \otimes V_{n,m;\phi}$ is a map on $H_{A'} \otimes H_{\tilde{A}} \otimes H_{B'} \otimes H_{\tilde{B}}$. So to apply this map we have to permute the order of the Hilbert spaces. Let T be such a permutation, i.e.

$$T : H_{A'} \otimes H_{B'} \otimes H_{\tilde{A}} \otimes H_{\tilde{B}} \to H_{A'} \otimes H_{\tilde{A}} \otimes H_{B'} \otimes H_{\tilde{B}}.$$

We can then transform $(e_1^{H_{A'}} \otimes e_1^{H_{B'}}) \otimes \mu_n$ to $E(\phi)_{n,m}$ with aid of the unitary transformation $T^{-1}(U_{n,m;\phi} \otimes V_{n,m;\phi})T$. □

Now that we know how to create such a state $E(\phi)_{n,m}$ from $(e_1^{H_{A'}} \otimes e_1^{H_{B'}}) \otimes \mu_n$, we complete our embezzlement procedure by showing that if we take n to be large enough, $E(\phi)_{n,m}$ is arbitrarily close to $\mu_n \otimes \phi$.

Lemma 8. *Take a fixed $H_{A'} \otimes H_{B'} \ni \phi = \sum_{i=1}^{2^m} \alpha_i \theta_i^{H_{A'}} \otimes \theta_i^{H_{B'}}$ (where $\theta_i^{H_{A'}} \otimes \theta_i^{H_{B'}} \in H_{A'} \otimes H_{B'}$) and a fixed ϵ ($0 < \epsilon < 1$). If $n \geq \frac{m}{\epsilon}$ the following inequality holds:*

$$|\langle \phi \otimes \mu_n, E(\phi)_{n,m} \rangle| \geq 1 - \epsilon. \tag{35}$$

Proof. We will first fully expand the state $\phi \otimes \mu_n$, after which we will give estimates for the inner product $\langle \phi \otimes \mu_n, E(\phi)_{n,m} \rangle$. The proof given here is similar to the proof provided by [5]. We know

$$\begin{aligned}\phi \otimes \mu_n &= \left(\sum_{i=1}^{2^m} \alpha_i \theta_i^{H_{A'}} \otimes \theta_i^{H_{B'}} \right) \otimes \left(\frac{1}{\sqrt{C_n}} \sum_{j=1}^{2^n} \frac{1}{\sqrt{j}} e_j^{H_{\tilde{A}}} \otimes e_j^{H_{\tilde{B}}} \right) \\ &= \frac{1}{\sqrt{C_n}} \sum_{j,i} \frac{\alpha_i}{\sqrt{j}} (\theta_i^{H_{A'}} \otimes \theta_i^{H_{B'}}) \otimes (e_j^{H_{\tilde{A}}} \otimes e_j^{H_{\tilde{B}}}) \quad (j = 1 \ldots 2^n, i = 1 \ldots 2^m).\end{aligned}$$

Remember that $E(\phi)_{n,m} := \frac{1}{\sqrt{C_n}} \sum_{r=1}^{r=2^n} \frac{1}{\sqrt{r}} ((\theta_{i_r}^{H_{A'}} \otimes \theta_{i_r}^{H_{B'}}) \otimes (e_{j_r}^{H_{\tilde{A}}} \otimes e_{j_r}^{H_{\tilde{B}}}))$. Using the fact that the e_j and θ_i are orthonormal we can conclude:

$$\begin{aligned}&|\langle \phi \otimes \mu_n, E(\phi)_{n,m} \rangle| \\ &= \left| \left\langle \frac{1}{\sqrt{C_n}} \sum_{j,i} \frac{\alpha_i}{\sqrt{j}} (\theta_i^{H_{A'}} \otimes \theta_i^{H_{B'}}) \otimes (e_j^{H_{\tilde{A}}} \otimes e_j^{H_{\tilde{B}}}), \frac{1}{\sqrt{C_n}} \sum_{r=1}^{r=2^n} \frac{1}{\sqrt{r}} ((\theta_{i_r}^{H_{A'}} \otimes \theta_{i_r}^{H_{B'}}) \otimes (e_{j_r}^{H_{\tilde{A}}} \otimes e_{j_r}^{H_{\tilde{B}}})) \right\rangle \right| \\ &= \frac{1}{C_n} \sum_{i,j,r} \frac{\alpha_i}{\sqrt{jr}} \left\langle e_j^{H_{\tilde{A}}}, e_{j_r}^{H_{\tilde{A}}} \right\rangle \left\langle e_j^{H_{\tilde{B}}}, e_{j_r}^{H_{\tilde{B}}} \right\rangle \left\langle \theta_i^{H_{A'}}, \theta_{i_r}^{H_{A'}} \right\rangle \left\langle \theta_i^{H_{B'}}, \theta_{i_r}^{H_{B'}} \right\rangle \quad (j, r = 1 \ldots 2^n; i = 1 \ldots 2^m) \\ &= \sum_{r=1}^{2^n} \frac{\alpha_{i_r}}{\sqrt{j_r C_n}} \frac{1}{\sqrt{r C_n}}.\end{aligned}$$

So we have a sum over the 2^n largest coefficients of $\phi \otimes \mu_n$ (which are by definition the 2^n coefficients of $E(\phi)_{n,m}$) times the first 2^n coefficients of μ_n. We will now verify that in fact, the j-th coefficient of $E(\phi)_{n,m}$ is smaller than the corresponding coefficient of μ_n. This observation and the proof come from [5].

For a fixed t and i, we define N_i^t to be the number of coefficients $\frac{\alpha_i}{\sqrt{jC_n}}$ that are stricly greater than $\frac{1}{\sqrt{tC_N}}$. Saying $\frac{\alpha_i}{\sqrt{jC_n}}$ is stricly greater than $\frac{1}{\sqrt{tC_N}}$, is equivalent to taking j 's such that $1 \leq j < a_i^2 t$. From this it follows $N_i^t < \alpha_i^2 t$. As ϕ is a vector of length one, we have $\sum_{i=1}^{2^m} \alpha_i^2 = 1$. This gives us $\sum_{i=1}^{2^m} N_i^t = t$. This is an upper bound on the number of coefficients of $E(\phi)_{n,m}$ that are strictly bigger than $\frac{1}{\sqrt{tC_n}}$. As the coefficients of $E(\phi)_{n,m}$ are in descending order, this means for $1 \leq k \leq 2^n$ we have: $\gamma_{i_k, j_k} < \frac{1}{\sqrt{C_n k}}$. This means

$$|\langle \phi \otimes \mu_n, E(\phi)_{n,m} \rangle| \geq \frac{1}{C_n} \sum_{r=1}^{2^n} \frac{\alpha_{i_r}}{j_r}^2.$$

We have $\phi \otimes \mu_n \succ \psi'_m \otimes \mu_n$ by Corollary 1. Here ψ'_m is the maximally entangled state of rank 2^m in the basis $\theta_i \otimes \theta_i$. This means the sum over the 2^n largest coefficients of $\phi \otimes \mu_n$ squared is greater than or equal too the first 2^n coefficients of $\psi'_m \otimes \mu_n$. We know $\psi_m \otimes \mu_n = \sum_{i=1}^{2^m} \sum_{j=1}^{2^n} \frac{1}{\sqrt{C_n j 2^m}}$.

This gives us:

$$\begin{aligned}\frac{1}{C_n}\sum_{r=1}^{2^n}\frac{\alpha_{i_r}}{j_r}^2 &\geq \sum_{j=1}^{2^{n-m}}\sum_{i=1}^{2^m}\frac{1}{C_n j 2^m}\\ &= \sum_{j=1}^{2^{n-m}}\frac{1}{C_n j}\\ &= \frac{C_{n-m}}{C_n}.\end{aligned}$$

The C_n can be estimated using the natural logarithm. $\sum_{i=1}^{2^n}\frac{1}{i}$ is the Riemann-sum for $\int_1^{2^n+1}\frac{1}{x}dx$ (where we evaluate on the left of every interval). This implies:

$$\begin{aligned}C_n &= \sum_{i=1}^{2^n}\frac{1}{i}\\ &\geq \int_1^{2^n+1}\frac{1}{x}dx\\ &\geq \ln(2^n).\end{aligned}$$

Note that $\sum_{i=1}^{n+1}\frac{1}{i}-\sum_{i=1}^{n}\frac{1}{i}=\frac{1}{n+1}$ and that $\ln(n+1)-\ln(n)=\ln(1+\frac{1}{n})$. We know that $\frac{1}{n-1}\leq\ln(1+\frac{1}{n})$ which, for every n, implies:

$$C_n-\ln(2^n)\geq C_{n+1}-\ln(2^{n+1})\geq 0. \tag{36}$$

Write: $\alpha := \frac{C_{n-m}-\ln(2^{n-m})}{\ln(2)}$ and $\beta := \frac{C_n-\ln(2^n)}{\ln(2)}$. As $\alpha>\beta$ (because of equation (36)) we know

$$\begin{aligned}\alpha n &\geq \beta(n-m)\\ &\Rightarrow\\ n(n-m)+\alpha n &\geq (n-m)n+(n-m)\beta\\ &\Rightarrow\\ n(n-m+\alpha) &\geq (n-m)(n+\beta)\\ &\Rightarrow\\ \frac{n-m+\alpha}{n+\beta} &\geq \frac{n-m}{n}\\ &\Rightarrow\\ \frac{\ln(2^{n-m})+C_{n-m}-\ln(2^{n-m})}{\ln(2^n)+C_n-\ln(2^n)} &\geq \frac{n-m}{n}\\ &\Rightarrow\\ \frac{C_{n-m}}{C_n} &\geq 1+\frac{m}{n}.\end{aligned}$$

So we have proven:

$$|\langle\phi\otimes\mu_n, E(\phi)_{n,m}\rangle|\geq 1-\epsilon. \tag{37}$$

□

7.5 Generalization

Suppose we have a general measurement quantum-system S. We are measuring some observable $\mathcal{M}$ with corresponding operator M. The state of the system prior to measurement is called ψ, which is a unit vector in Hilbert space H_S. We describe the measurement as a projective measurement with K possible outcomes. This means $M = \sum_{i=1}^{K} \lambda_i Proj_{\lambda_i}$. Here, λ_i are the K eigenvalues of this operator. Each $Proj_{\lambda_i}$ projects onto the eigenspace of H_S belonging to the eigenvalue λ_i. With this measurement we have the associated probabilities

$$P_M^{\psi}(\lambda_i) = \langle \psi, Proj_{\lambda_i} \psi \rangle .$$

We write $p_i = P_M^{\psi}(\lambda_i)$. Right after the measurement with result λ_i, the state of the system is given by $\rho_i := \frac{Proj_{\lambda_i}\psi}{\sqrt{p_i}}$. Note that the ρ_i are a set of K orthonormal vectors in H_S. This is what we consider a *general measurement*. With this general measurement we can state the main claim.

Theorem 1. *Let $\mathcal{M}$ be a general measurement on $\psi \in H_S$ with K possible outcomes. Firstly we assume our hidden variable theory $\mathcal{T}$ is compatible with quantum mechanics (as defined in section 4). Secondly, we demand that for the specific measurement on ψ_0 in the bipartite setup, the variables A_N and B_N are free, w.r.t the causal order defined in equations* (17)*,* (18) *and* (19)*. Then quantum mechanics is at least as informative as $\mathcal{T}$, i.e., for any $\lambda_i \in \sigma(M)$ and $z \in \mathcal{Z}$ we have*

$$P_M^{\psi}(\lambda_i) = P_{M|Z}^{\psi}(\lambda_i \mid z).$$

We have the general measurement S on ψ (with ψ in Hilbert space H_S). We will describe this measurement in two alternative ways. First, in a system consisting of the combination of S with a system called the **measurement device** D (specified by yet another Hilbert space H_D). Secondly, we will describe the measurement in an even larger space. On this last space we will use embezzlement together with our previous results to extract the claim. Using this embezzlement, we will finally try to reduce the situation to the original measurement on ψ.

7.5.1 Measurement device

We now describe the measurement $\mathcal{M}$ on the state ψ in an alternative way. Consider a Hilbert space H_D of dimension K. Choose an orthonormal basis given by $e_1^{H_D}, \dots e_N^{H_D}$. Now define

$$H_S \otimes H_D \ni \phi := \sum_{i=1}^{K} \sqrt{p_i} \rho_i \otimes e_i^{H_D}. \tag{38}$$

This, together with the projections $\mathbb{1}_S \otimes P_{e_i^{H_D}}$, gives rise to probabilities:

$$\begin{aligned} P_{M'}^{\phi}(\lambda_i) &= \left\langle \phi, \mathbb{1}_S \otimes P_{e_i^{H_D}} \phi \right\rangle \\ &= p_i \langle \rho_i, \rho_i \rangle \left\langle e_i^{H_D}, e_i^{H_D} \right\rangle \\ &= \langle \psi, Proj_{\lambda_i} \psi \rangle \\ &= P_M^{\psi}(\lambda_i). \end{aligned}$$

After the measurement with result λ_i, the state collapses to

$$\frac{(\mathbb{1}_S \otimes P_{e_i^{H_D}})\phi}{\sqrt{\left\langle \phi, (\mathbb{1}_S \otimes P_{e_i^{H_D}})\phi \right\rangle}} = \frac{\sqrt{p_i}\rho_i \otimes e_i^{H_D}}{\sqrt{p_i}}$$
$$= \rho_i \otimes e_i^{H_D}.$$

This means that after measurement on $H_S \otimes H_D$ with result λ_i the H_S-part of ϕ collapses to ρ_i just as it would when measuring ψ on H_S with result λ_i.

7.5.2 Embezzlement on measurement device

As a last step we consider ϕ as defined above in a larger Hilbert space, on which we use embezzlement. This space is

$$\mathbf{H} := H_S \otimes H_D \otimes H_{S'} \otimes H_{D'} \otimes H_{\tilde{S}} \otimes H_{\tilde{D}}.$$

Looking back to the previous subsection on embezzlement, the space $H_{\tilde{S}}$ and $H_{\tilde{D}}$ correspond to $H_{\tilde{A}}$ and $H_{\tilde{B}}$, whereas $H_{S'}$ and $H_{D'}$ correspond to $H_{A'}$ and $H_{B'}$. For the sizes of these spaces, take $H_{\tilde{S}}$ and $H_{\tilde{D}}$ to be $\bigotimes^n \mathbb{C}^2$. We will take $H_{S'}$ and $H_{D'}$ to be $\bigotimes^r \mathbb{C}^2$. The exact choice for n and r will be determined below, as it is dependent on how close of an approximation we are trying to achieve via embezzlement. We can chose a vector in $H_{S'} \otimes H_{D'}$ to embezzle. We choose a maximally entangled state of rank m, which is $\psi^m = \frac{1}{\sqrt{m}} \sum_{i=1}^m e_i^{H_{S'}} \otimes e_i^{H_{D'}}$ (in $H_{S'} \otimes H_{D'}$). Note that due to the fact that H'_S and H'_D have dimension 2^r, we can have ψ^1 up to ψ^{2^r}.

Corollary 2. *For a fixed ϵ ($0 < \epsilon < 1$) and $n \in \mathbb{N}$: $\forall r \leq n$ there exist local isometrics $U_{n,r;\psi^m}$ and $V_{n,r;\psi^m}$ such that*

$$T^{-1}(U_{n,r;\psi^m} \otimes V_{n,r;\psi^m})T : (e_1^{H_{S'}} \otimes e_1^{H_{D'}}) \otimes \mu_n \mapsto E(\psi^m)_{n,r},$$
$$\text{with: } |\langle \psi^m \otimes \mu_n, E(\psi^m)_{n,r} \rangle| \geq 1 - \epsilon.$$

Here T again permutes the order of the Hilbert spaces to let $U_{n,r;\psi^m} \otimes V_{n,r;\psi^m}$ map $(e_1^{H_{S'}} \otimes e_1^{H_{D'}}) \otimes \mu_n$.

Proof. It is evident that this corollary is just embezzlement applied to the specific state ψ^m. The validity of the inequality is guaranteed by lemma 8. □

With transformations $U_{n,r;\psi_m}$ and $V_{n,r;\psi_m}$, we construct the following map on $\mathbf{H}$:

$$\hat{U} \otimes \hat{V} = \sum_{i=1}^{K} \sum_{j=1}^{K} P_{\rho_i} \otimes P_{e_j^{H_D}} \otimes T^{-1}(U_{n,r;\psi^{m_i}} \otimes V_{n,r;\psi^{m_j}})T. \tag{39}$$

This is a unitary map we can apply to $\mathbf{H} \ni \Phi := \phi \otimes (e_1^{H_{S'}} \otimes e_1^{H_{D'}}) \otimes \mu_n$. For the i-th position in the sum, we embezzle to extract a maximally entangled state of rank m_i. We do this in order to embezzle the Φ to a state that looks very much like a maximally entangled state. This opens up ways to apply the previous results in this more general setting. The next lemma describes which vectors we end up with after embezzlement via $\hat{U} \otimes \hat{V}$.

Lemma 9. *Applying the operator $\hat{U} \otimes \hat{V}$ on Φ gives, for appropriate choices for m_i, r and n, a state arbitrarily close to $\mathbf{H} \ni \Psi := (\frac{1}{\sqrt{2^r}} \sum_{i=1}^N \sum_{j=1}^{m_i} \rho_i \otimes e_i^{H_D} \otimes e_j^{H_{S'}} \otimes e_j^{H_{D'}}) \otimes \mu_n$ in the sense that for every $0 > \epsilon > 1$ we have $|\left\langle (\hat{U} \otimes \hat{V})\Phi, \Psi \right\rangle| \geq 1 - \epsilon$.*

Proof. We want to use the triangle inequality. For this we need to establish a connection between the norm and the inequality assumed in the lemma. The norm $\|\cdot\|$ is given by $\|\psi\| = \sqrt{\langle\psi,\psi\rangle}$. Suppose for a real number ϵ $(0 < \epsilon < 1)$ and two vectors ν and ω of norm 1 we have $\|\nu-\omega\| \leq \sqrt{2\epsilon}$. An easy calculation gives us:

$$\begin{aligned}
\|\nu-\omega\| &= \sqrt{\langle \nu-\omega, \nu-\omega\rangle} \\
&\iff \\
\sqrt{\langle \nu-\omega, \nu-\omega\rangle} &\leq \sqrt{2\epsilon} \\
\langle\nu,\nu\rangle - \langle\nu,\omega\rangle - \langle\omega,\nu\rangle + \langle\omega,\omega\rangle &\leq 2\epsilon \\
2-2\epsilon &\leq \langle\nu,\omega\rangle + \langle\omega,\nu\rangle \\
1-\epsilon &\leq \mathrm{Re}(\langle\nu,\omega\rangle) \\
1-\epsilon &\leq |\langle\nu,\omega\rangle|.
\end{aligned}$$

Thus we can conclude that the following are equivalent:

$$|\langle\nu,\omega\rangle| \geq 1-\epsilon \iff \|\nu-\omega\| \leq \sqrt{2\epsilon}.$$

First we look at the what applying the operator $\hat{U}\otimes\hat{V}$ to Φ does:

$$\begin{aligned}
&(\hat{U}\otimes\hat{V})(\Phi) \\
&= (\sum_{i,j} P_{\rho_i}\otimes P_{e_j^{H_D}} \otimes T^{-1}U_{n,r;\psi^{m_i}} \otimes V_{n,r;\psi^{m_j}}T)((\sum_k \sqrt{p_k}\rho_k \otimes e_k^{H_D} \otimes (e_1^{H_{S'}} \otimes e_1^{H_{D'}}) \otimes \mu_n \\
&= \sum_{i,j,k} \sqrt{p_k}\,\langle\rho_i,\rho_k\rangle\,\rho_i \otimes \left\langle e_j^{H_D}, e_k^{H_D}\right\rangle e_j^{H_D} \\
&\otimes (T^{-1}(U_{n,r;\psi^{m_i}} \otimes V_{n,r;\psi^{m_j}})T((e_1^{H_{S'}} \otimes e_1^{H_{D'}}) \otimes \mu_n) \\
&= \sum_{i=1}^{K} \sqrt{p_i}\rho_i \otimes e_i^{H_D}(T^{-1}(U_{n,r;\psi^{m_i}} \otimes V_{n,r;\psi^{m_i}})T((e_1^{H_{S'}} \otimes e_1^{H_{D'}}) \otimes \mu_n) \\
&=: \hat{\Phi}.
\end{aligned}$$

Now, when we embezzle by applying the operator $U\otimes V$ we obtain something close to: $\sum_i \sqrt{p_i}\rho_i \otimes e_i^{H_D} \otimes (\frac{1}{\sqrt{m_i}}\sum_{j=1}^{m_i} e_j^{H_{S'}} \otimes e_j^{H_{D'}}) \otimes \mu_n$, which we will call $\Phi_{\mathrm{emb}} \in \mathbf{H}$.

$$\begin{aligned}
\Phi_{\mathrm{emb}} &:= \sum_i \sqrt{p_i}\rho_i \otimes e_i^{H_D} \otimes (\frac{1}{\sqrt{m_i}}\sum_{j=1}^{m_i} e_j^{H_{S'}} \otimes e_j^{H_{D'}}) \otimes \mu_n \\
&= \sum_{i=1}^{K}(\sqrt{\frac{p_i}{m_i}}\sum_{j=1}^{m_i} \rho_i \otimes e_i^{H_D} \otimes e_j^{H_{S'}} \otimes e_j^{H_{D'}}) \otimes \mu_n.
\end{aligned}$$

We would like the state Φ_{emb} to be close to a state of this form:

$$\mathbf{H} \ni \Psi := (\frac{1}{\sqrt{2^r}}\sum_{i=1}^{K}\sum_{j=1}^{m_i} \rho_i \otimes e_i^{H_D} \otimes e_i^{H_{S'}} \otimes e_j^{H_{D'}}) \otimes \mu_n.$$

Not surprisingly, this can be accomplished to arbitrary precision by choosing the appropriate m_i and n. Each m_i describes the rank of entanglement we achieve by embezzlement in the $i-th$

position of the sum. The n was a parameter in the embezzlement to control the precision. We want $\frac{p_i}{m_i}$ to be close to $\frac{1}{2^r}$, choosing the m_i such that $\sum_{i=1}^{K} m_i = 2^r$.

This means we have to choose the natural number m_i close to $2^r p_i$. We do this by taking $m_i = \lfloor p_i 2^r \rfloor$ for $i = 1 \dots K-1$ and for $m_K = 2^r - (m_1 + m_2 + \dots + m_{K-1})$. This satisfies the condition that the sum of the m_i is equal to 2^r. Note that m_K is always well defined, as

$$m_1 + m_2 \dots + m_{K-1} \leq (p_1 + \dots + p_{K-1})2^r < 2^r.$$

We know the distance between m_i and $p_i 2^r$ is bounded. For $i = 1 \dots K-1$ we know

$$|m_i - 2^r p_i| \leq 1.$$

For the K-th m we have $|m_K - p_K 2^r| \leq K-1$. Therefore,

$$\begin{aligned} \left| \sqrt{\frac{1}{2^r}} - \sqrt{\frac{p_i}{m_i}} \right| &\leq \sqrt{\left| \frac{1}{2^r} - \frac{p_i}{m_i} \right|} \\ &\leq \sqrt{\left| \frac{m_i - 2^r p_i}{2^r m_i} \right|} \\ &\leq \sqrt{\frac{K-1}{2^r m_i}} \\ &\leq \sqrt{\frac{K-1}{2^r}}. \end{aligned}$$

Looking at $\|\Phi_{\text{emb}} - \Psi\|$, we can now easily see that

$$\|\Phi_{\text{emb}} - \Psi\| \leq \sqrt{K\left(\frac{K-1}{2^r}\right)}. \tag{40}$$

With these estimates, we are able to complete the proof of the lemma. Let ϵ be a real number such that $0 < \epsilon < 1$. Choose an r such that $\|\Phi_{\text{emb}} - \Psi\| \leq \frac{1}{2}\sqrt{2\epsilon}$ (see equation (40)). Then we pick $n \geq \frac{r}{\frac{1}{4}\epsilon}$, and the m_i as defined above. Looking at $\hat{\Phi}$ and Φ_{emb} and using lemma 8 we can conclude they are close as well:

$$\begin{aligned} \left\langle \hat{\Phi}, \Phi_{\text{emb}} \right\rangle &= \sum_{i=1}^{K} p_i \left\langle T^{-1} U_{n,m_i;\psi^m} \otimes V_{n,m_j;\psi^m} T(e_i^{H_{S'}} \otimes e_i^{H_{D'}} \otimes \mu_n), \left(\frac{1}{\sqrt{m_i}} \sum_{j=1}^{m_i} e_j^{H_{S'}} \otimes e_j^{H_{D'}}\right) \otimes \mu_n \right\rangle \\ &\geq \sum_{i=1}^{K} p_i \left(1 - \frac{1}{4}\epsilon\right) \\ &\geq 1 - \frac{1}{4}\epsilon. \end{aligned}$$

This implies $\|\hat{\Phi} - \Phi_{\text{emb}}\| \leq \frac{1}{2}\sqrt{2\epsilon}$, which implies $\|\hat{\Phi} - \Psi\| \leq \|\hat{\Phi} - \Phi_{\text{emb}}\| + \|\Phi_{\text{emb}} - \Psi\| \leq \sqrt{2\epsilon}$. This, implies: $\left\langle \hat{\Phi}, \Psi \right\rangle \geq 1 - \epsilon$. This concludes the proof of the lemma. □

Let us quickly summarize what we have done until now. We have introduced a unitary operator $\hat{U} \otimes \hat{V}$ on $\mathbf{H}$ which transforms Φ to Ψ up to arbitrary precision. Taking a closer look at

$\Psi = (\frac{1}{\sqrt{2^r}} \sum_{i=1}^{N} \sum_{j=1}^{m_i} \rho_i \otimes e_i^{H_D} \otimes e_j^{H_{S'}} \otimes e_j^{H_{D'}}) \otimes \mu_n$, we see it consists of a sum of 2^r orthogonal vectors in $\mathbf{H}$. This is very much like the result of taking the tensor product of the maximally entangled state of order 2 (previously written as ψ_0) r times. This allows us to use our previous results concerning measurements on such a ψ_0.

We now try to make the idea that Ψ looks like a tensor product of maximally entangled states more explicit. Remember that $H_{S'} = H_{D'} = \bigotimes^r \mathbb{C}^2$. Let us look at the tensor product of r such maximally entangled states ψ_0. To make sure we are still distinguishing the Hilbert spaces belonging to S and those belonging to D correctly, we will write $H_{s'} = H_{d'} = \mathbb{C}^2$. Write:

$$\psi_0 = \frac{1}{\sqrt{2}} \left(e_1^{H_{s'}} \otimes e_1^{H_{d'}} + e_2^{H_{s'}} \otimes e_2^{H_{d'}} \right).$$

We look at the tensor product of r such entangled states, which gives

$$\begin{aligned}(H_{s'} \otimes H_{d'}) \otimes \ldots \otimes (H_{s'} \otimes H_{d'}) \ni \bigotimes_{i=1}^{r} \frac{1}{\sqrt{2}} \left(e_1^{H_{s'}} \otimes e_1^{H_{d'}} + e_2^{H_{s'}} \otimes e_2^{H_{d'}} \right) \\ = \sum_{\substack{i_j \in \{1,2\} \\ j=1,\ldots,r}} (e_{i_1}^{H_{s'}} \otimes e_{i_1}^{H_{d'}}) \otimes \ldots \otimes (e_{i_r}^{H_{s'}} \otimes e_{i_r}^{H_{d'}}).\end{aligned}$$

Now introduce an operator T_σ. This T_σ changes the order of all Hilbert spaces in question such that it groups those belonging to S and those belonging to D, i.e.

$$T_\sigma \left((H_{s'} \otimes H_{d'}) \otimes \ldots \otimes (H_{s'} \otimes H_{d'}) \right) = (H_{s'} \otimes \ldots \otimes H_{s'}) \otimes (H_{d'} \otimes \ldots \otimes H_{d'}).$$

This operator is an isometry. Now applying T_σ to the tensor product of maximally entangled states gives the vector we will call $\psi_0^{\otimes r}$:

$$\begin{aligned}\psi_0^{\otimes r} &:= T_\sigma \left[\sum_{\substack{i_j \in \{1,2\} \\ j=1,\ldots,r}} (e_{i_1}^{H_{s'}} \otimes e_{i_1}^{H_{d'}}) \otimes \ldots \otimes (e_{i_r}^{H_{s'}} \otimes e_{i_r}^{H_{d'}}) \right] \\ &= \sum_{\substack{i_j \in \{1,2\} \\ j=1,\ldots,r}} (e_{i_1}^{H_{s'}} \otimes \ldots \otimes e_{i_r}^{H_{s'}}) \otimes (e_{i_1}^{H_{d'}} \otimes \ldots \otimes e_{i_r}^{H_{d'}}) \\ &= \frac{1}{\sqrt{2^r}} \sum_{i=1}^{2^r} e_i^{H_{S'}} \otimes e_i^{H_{D'}}.\end{aligned}$$

The last step going from $(e_{i_1}^{H_{s'}} \otimes \ldots \otimes e_{i_r}^{H_{s'}}) \otimes (e_{i_1}^{H_{d'}} \otimes \ldots \otimes e_{i_r}^{H_{d'}})$ to $e_i^{H_{S'}} \otimes e_i^{H_{D'}}$ is exactly the construction of the basis of $\bigotimes^r \mathbb{C}^2$ introduced in Definition 11.

We want to implement our $\psi_0^{\otimes r}$ in $\mathbf{H}$, so we define the vector: $\mathbf{H} \ni \Psi_{ent} = \rho_1 \otimes e_1^{H_D} \otimes \psi_0^{\otimes r} \otimes \mu_n$. Both Ψ and Ψ_{ent} are vectors in $\mathbf{H}$ consisting of 2^r orthonormal vectors. This means we can easily transform Ψ into Ψ_{ent} using local unitary transformations (just like U and V in the embezzlement procedure). Let S be a unitary operator on $H_S \otimes H_{\tilde{S}'}$ that transforms a vector $\rho_i \otimes e_j^{H_{S'}}$ to $\rho_1 \otimes e_k^{H_{S'}}$. W is a unitary operator on $H_D \otimes H_{D'}$ that transforms $e_i^{H_D} \otimes e_j^{H_{D'}}$ to $e_1^{H_D} \otimes e_k^{H_{D'}}$. Again using the operator T to change the order of the Hilbert-spaces we have

$$\left(T^{-1}(S \otimes W)T \right) \Psi = \Psi_{\text{emb}}.$$

Corollary 3. *For every* $0 < \epsilon < 1$ *there is a* $\tilde{\Phi}$ *with* $|\left\langle \Phi, \tilde{\Phi} \right\rangle| \geq 1 - \epsilon$ *such that*

$$(T^{-1}W \otimes ST)(\hat{U} \otimes \hat{V})\tilde{\Phi} = \Psi_{emb}.$$

Proof. From lemma 9 we know that there is an r such that $|\left\langle (\hat{U} \otimes \hat{V})\Phi, \Psi \right\rangle| \geq 1 - \epsilon$. We know that $T^{-1}S \otimes WT$ is unitary, so

$$|\left\langle (T^{-1}S \otimes WT)(\hat{U} \otimes \hat{V})\Phi, \Psi_{\text{emb}} \right\rangle| \geq 1 - \epsilon.$$

For $\tilde{\Phi}$ we take

$$\tilde{\Phi} = \left[(T^{-1}W \otimes ST)(\hat{U} \otimes \hat{V}) \right]^* (\Psi_{\text{emb}}).$$

It follows that $\tilde{\Phi}$ with $|\left\langle \Phi, \tilde{\Phi} \right\rangle| \geq 1 - \epsilon$. □

There is one last thing we need to consider before we can finally prove our claim. We have a several vectors that are close. It is quite intuitive that, when the vectors are close, projective measurements on such vectors give similar results. This intuitive idea is made precise in the following lemma.

Lemma 10. *Consider a general measurement of some observable* $\mathcal{M}$ *with outcomes* λ_i, *with* $\lambda_i \in \sigma(M)$. *Let* ν *and* ω *be two states. If there is an* $0 < \epsilon < 1$ *such that* $|\langle \nu, \omega \rangle| \geq 1 - \epsilon$, *then for the probability distributions* $P_M^{\phi}(\lambda_i)$ *and* $P_M^{\psi}(\lambda_i)$ *we have*

$$D(P_M^{\nu}, P_M^{\omega}) \leq \sqrt{1 - |\langle \nu, \omega \rangle|} \leq \sqrt{2\epsilon}.$$

Proof. The proof will not be given. See Chapter 9 in the book [4]. □

An important matter is how the hidden variable theory behaves for measurements on states that are close. We assume the hidden variable theory to be compatible with quantum mechanics (as described in section 4), but this alone is not enough to guarantee that the hidden variable theory also given similar probabilities on similar states. To follow the proof as given in the Colbeck and Renner article ([3]), we do need such a property in order to complete the proof. We will say the theory $\mathcal{T}$ exhibits **continuous behavior** if measurements on close states give rise to a probability distributions that are close (in the variational distance).

Definition 14. *Consider a general measurement of some observable* $\mathcal{M}$ *with outcomes* λ_i. *Let* ν *and* ω *be two states. We say the hidden variable theory* $\mathcal{T}$ *has* ***continuous behaviour*** *if the follow holds: for every* $\epsilon > 0$, *there is a* $0 < \delta < 1$ *such that if* $\langle \nu, \omega \rangle \geq 1 - \delta$ *we have*

$$D\left(P_{M|Z}^{\nu}(\cdot \mid z), P_{M|Z}^{\omega}(\cdot \mid z) \right) \leq \epsilon$$

7.6 Proof of Theorem 1

Now that we know how to reach, from a general measurement, something resembling the rank two maximally entangled state, we can introduce a measurement on Φ. First let us introduce some notations. We will name the operator, that transforms Φ to something close to Ψ_{emb}, $\mathbf{U}$. So

$$\mathbf{U} := T^{-1}(W \otimes S)T(\hat{U} \otimes \hat{V}).$$

Next, we would like to make the projections introduced in the bipartite setup r times on the $H_{S'} \otimes H_{D'}$ part of Ψ_{emb}, leaving all the other spaces unaltered. We define

$$\mathbf{O}^r_{\vec{x},\vec{a};\vec{y},\vec{b}} := \mathbb{1}_{H_S \otimes H_D} \otimes T_\sigma \left(\bigotimes_{i=1}^{r} P_{E^{a_i}_{x_i}} \otimes P_{F^{b_i}_{y_i}} \right) T^*_\sigma \otimes \mathbb{1}_{H_{\tilde{S}} \otimes H_{\tilde{D}}}.$$

Again we quickly recap. We start with the state Φ. Using the operator $\mathbf{U}$ (which only uses local unitary transformations), we know $\mathbf{U}(\Phi)$ is close to Ψ_{emb}. This is equivalent to saying there is a $\tilde{\Phi}$ close to Φ which gets mapped to Ψ_{emb} by $\mathbf{U}$ (by corollary 3). On this Ψ_{emb} we can apply $\mathbf{O}^r_{\vec{x},\vec{a};\vec{y},\vec{b}}$ to achieve the projections on the maximally entangled state. We now use the following operator to describe measurements on Φ:

$$\mathbf{U}^* \mathbf{O}^r_{\vec{x},\vec{a};\vec{y},\vec{b}} \mathbf{U}.$$

We first transform to something resembling a maximally entangled state using $\mathbf{U}$, perform projective measurements on this maximally entangled state (with the operator $\mathbf{O}^r_{\vec{x},\vec{a};\vec{y},\vec{b}}$), after which we transform back using $\mathbf{U}^*$.

This measurement is described by random variables $\vec{X}$, $\vec{Y}$, $\vec{A}_N$ and $\vec{B}_N$. The new random variable $\vec{X}$ consists of r copies of X, in the sense that the possible values of $\vec{X}$ are vectors $\vec{x}$ of length r, with $x_i \in \{0, 1\}$. In other words, $\vec{x} \in \{0, 1\}^r$. The same holds for $\vec{Y}$. The variables $\vec{A}_N$ and $\vec{B}_N$ work the same way; they take values $\vec{a}$ (and $\vec{b}$) in the r-fold Cartesian product of $\mathcal{A}_N$ (and $\mathcal{B}_N$) which we write as $\mathcal{A}_N^{\times r}$ (and $\mathcal{B}_N^{\times r}$). Using the operator $\mathbf{U}^* \mathbf{O}^r_{\vec{x},\vec{a};\vec{y},\vec{b}} \mathbf{U}$ as a projective measurement on Φ we obtain the following probabilities:

$$\begin{aligned} P^{\Phi}_{\vec{X},\vec{Y}|\vec{A}_N,\vec{B}_N}(\vec{x}, \vec{y} \mid \vec{a}, \vec{b}) &:= \left\langle \Phi, \mathbf{U}^* \mathbf{O}^r_{\vec{x},\vec{a};\vec{y},\vec{b}} \mathbf{U}(\Phi) \right\rangle \\ &= \left\langle \mathbf{U}\Phi, \mathbf{O}^r_{\vec{x},\vec{a};\vec{y},\vec{b}} \mathbf{U}(\Phi) \right\rangle. \end{aligned}$$

This same measurement on $\tilde{\Phi}$, the state close to Φ gives (using $\mathbf{U}(\tilde{\Phi}) = \Psi_{\text{emb}}$)

$$\begin{aligned} P^{\tilde{\Phi}}_{\vec{X},\vec{Y}|\vec{A}_N,\vec{B}_N}(\vec{x}, \vec{y} \mid \vec{a}, \vec{b}) &= \left\langle \mathbf{U}\tilde{\Phi}, \mathbf{O}^r_{\vec{x},\vec{a};\vec{y},\vec{b}} \mathbf{U}\tilde{\Phi} \right\rangle \\ &= \left\langle \Psi_{\text{emb}}, \mathbf{O}^r_{\vec{x},\vec{a};\vec{y},\vec{b}} \Psi_{\text{emb}} \right\rangle \\ &= \left\langle \psi_0^{\otimes r}, T_\sigma \left(\bigotimes_{i=1}^{r} P_{E^{a_i}_{x_i}} \otimes P_{F^{b_i}_{y_i}} \right) T^*_\sigma \psi_0^{\otimes r} \right\rangle \\ &= \prod_{i=1}^{r} \left\langle \psi_0, P_{E^{a_i}_{x_i}} \otimes P_{F^{b_i}_{y_i}}(\psi_0) \right\rangle \\ &= \prod_{i=1}^{r} P^{\psi_0}_{X,Y|A_N,B_N}(x_i, y_i \mid a_i, b_i). \end{aligned}$$

Let us sum this over all possible $\vec{y} \in \{0,1\}^r$

$$\begin{aligned}
P^{\tilde{\Phi}}_{\vec{X}|\vec{A}_N,\vec{B}_N}(\vec{x} \mid \vec{a},\vec{b}) &= \sum_{\vec{y}\in\{0,1\}} P^{\tilde{\Phi}}_{\vec{X},\vec{Y}|\vec{A}_N,\vec{B}_N}(\vec{x},\vec{y} \mid \vec{a},\vec{b}) \\
&= \sum_{\vec{y}\in\{0,1\}} \prod_{i=1}^{r} P^{\psi_0}_{X,Y|A_N,B_N}(x_i, y_i \mid a_i, b_i) \\
&= \prod_{i=1}^{r} \Big(P^{\psi_0}_{X,Y|A_N,B_N}(x_i, 0 \mid a_i, b_i) + P^{\psi_0}_{X,Y|A_N,B_N}(x_i, 1 \mid a_i, b_i) \Big) \\
&= \prod_{i=1}^{r} P^{\psi_0}_{X|A_N,B_N}(x_i \mid a_i, b_i). \qquad (41)
\end{aligned}$$

Note that, when we assume that A_N and B_N are free variables for the bipartite setup, the probabilities above are independent of $\vec{b}$. Let us look more closely at $P^{\tilde{\Phi}}_{\vec{X}|A_N,B_N}$, for $\vec{A}_N = \vec{a}_0$ with $\vec{a}_0 = (0,\ldots,0)$.
Writing it in a slightly different way, yields

$$\begin{aligned}
P^{\tilde{\Phi}}_{\vec{X}|\vec{A}_N,\vec{B}_N}(\vec{x} \mid \vec{a}_0,\vec{b}) &= \left\langle \psi_0^{\otimes r}, T_\sigma \left(\bigotimes_{i=1}^{r} P_{E^0_{x_i}} \otimes \mathbb{1}_{H_{d'}} \right) T_\sigma^* \psi_0^{\otimes r} \right\rangle \\
&= \left\langle \psi_0, T_\sigma \bigotimes_{i=1}^{r} \left(P_{E^0_{x_i}} \otimes \mathbb{1}_{H_{d'}} \psi_0 \right) \right\rangle \\
&= \left\langle \psi_0, \frac{1}{\sqrt{2^r}} T_\sigma \left[\bigotimes_{i=1}^{r} \left(P_{E^0_{x_i}} \otimes \mathbb{1}_{H_{d'}} e_1^{H_{s'}} \otimes e_1^{H_{d'}} \right) + \left(P_{E^0_{x_i}} \otimes \mathbb{1}_{H_{d'}} e_2^{H_{s'}} \otimes e_2^{H_{d'}} \right) \right] \right\rangle \\
&= \left\langle \psi_0^{\otimes r}, \frac{1}{\sqrt{2^r}} e_k^{H_{S'}} \otimes e_k^{H_{D'}} \right\rangle \quad (\text{for a certain } k \in \{1,\ldots,2^r\}) \\
&= \left\langle \Psi_{\text{emb}}, \frac{1}{\sqrt{2^r}} \rho_1 \otimes e_1^{H_D} \otimes e_k^{H_{S'}} \otimes e_k^{H_{D'}} \otimes \mu_n \right\rangle \\
&= \left\langle \Psi, \frac{1}{\sqrt{2^r}} \rho_i \otimes e_i^{H_D} \otimes e_j^{H_{S'}} \otimes e_j^{H_{D'}} \otimes \mu_n \right\rangle \quad (\text{for certain } i \in \{1,\ldots,K\}, j \in \{1,\ldots m_i\}).
\end{aligned}$$

So the distribution for $\vec{X}$ having a value of $\vec{x}$ corresponds to $\left\langle \Psi, \frac{1}{\sqrt{2^r}} \rho_i \otimes e_i^{H_D} \otimes e_j^{H_{S'}} \otimes e_j^{H_{D'}} \otimes \mu_n \right\rangle$ for certain i and j (on the state $\tilde{\Phi}$). These i and j are uniquely determined by the k because $T^{-1}(S \otimes W)T(e_k^{H_{S'}} \otimes e_k^{H_{D'}}) = \rho_i \otimes e_i^{H_D} \otimes e_j^{H_{S'}} \otimes e_j^{H_{D'}}$. The k is again uniquely determined by how we have constructed the basis on $\bigotimes^r \mathbb{C}^2$. To express this connection between the $\vec{x}$ and the i and j to which $\vec{x}$ corresponds, we will write $\vec{x}^i_j$ for the vector corresponding to $\rho_i \otimes e_i^{H_D} \otimes e_j^{H_{S'}} \otimes e_j^{H_{D'}} \otimes \mu_n$. This relation also allows us to group the 2^r vectors $\vec{x}^i_j$ in K sets, such that every set contains all $\vec{x}^i_j$ with a fixed i and j ranging from 1 to m_i. Such a set we write as

$$S_i = \{\vec{x}^i_j \mid j \in \{1,\ldots,m_i\}\}.$$

In order to keep the notation short, we replace the random variable $\vec{X}$ with S. The variable S takes value λ_i if $\vec{X} \in S_i$. The choice for λ_i will become clear if we look at the probability for

$\vec{X} \in S_i$ for the original state Φ. Looking at the probabilities for $\tilde{\Phi}$, we get

$$\begin{aligned} P^{\tilde{\Phi}}_{S|\vec{A}_N,\vec{B}_N}(\lambda_i \mid \vec{a}_0, \vec{b}) &:= P^{\tilde{\Phi}}(\vec{X} \in S_i \mid \vec{A}_N = \vec{a}_0, \vec{B}_N = \vec{b}) \\ &= \sum_{\vec{x} \in S_i} P^{\tilde{\Phi}}_{\vec{X}|\vec{A}_N,\vec{B}_N}(\vec{x} \mid \vec{a}_0, \vec{b}) \\ &= \left\langle \Psi, \frac{1}{\sqrt{2^r}} \rho_i \otimes e_i^{H_D} \otimes \left(\sum_{j=1}^{m_i} e_j^{H_{S'}} \otimes e_j^{H_{D'}} \right) \otimes \mu_n \right\rangle. \end{aligned}$$

This also implies, by definition of $\tilde{\Phi}$, that

$$\begin{aligned} &P^{\Phi}_{S|\vec{A}_N,\vec{B}_N}(\lambda_i \mid \vec{a}_0, \vec{b}) \\ &\qquad = \sum_{\vec{x} \in S_i} P^{\Phi}_{\vec{X}|\vec{A}_N,\vec{B}_N}(\vec{x} \mid \vec{a}_0, \vec{b}) \\ &\qquad = \left\langle \Phi, \sqrt{p_i} \rho_i \otimes e_i^{H_D} \otimes \left[T^{-1} U_{n,2^r;\psi^{m_i}} \otimes V_{n,2^r;\psi^{m_i}} T \left(e_1^{H_{S'}} \otimes e_1^{H_{D'}} \otimes \mu_n \right) \right] \right\rangle \\ &\qquad = \left\langle \phi, \mathbb{1}_{H_S} \otimes P_{e_i^{H_D}} \phi \right\rangle \\ &\qquad = P^{\psi}_M(\lambda_i) \qquad (42) \\ &\qquad = p_i. \end{aligned}$$

We are now able to prove Theorem 1.

Proof. We are going to use a proof by contradiction. For a general measurement $\mathcal{M}$ we assume the following:

- The hidden variable theory $\mathcal{T}$ is compatible with quantum mechanics
- For the measurement in the bipartite setup, the variables A_N and B_N are free w.r.t the causal order defined by equations (17), (18) and (19).
- The theory $\mathcal{T}$ has continuous behavior (per definition 14)
- For a $z \in \mathcal{Z}$ we have $D(P^{\psi}_M, P^{\psi}_{M|Z}(\cdot \mid z)) > \epsilon$.

As illustrated by equation (42), describing the measurement $\mathcal{M}$ as applying $\mathbf{U}^* \mathbf{O}^r_{\vec{x},\vec{a};\vec{y},\vec{b}} \mathbf{U}$ to Φ (and summing over all $\vec{x}^i_j \in S_i$) is equivalent to applying M to ψ. As both distributions describe the same measurement, we necessarily have

$$P^{\Phi}_{S|\vec{A}_N,\vec{B}_N}(\cdot \mid \vec{a}_0, \vec{b}) = P^{\psi}_M. \qquad (43)$$

and

$$P^{\Phi}_{S|\vec{A}_N,\vec{B}_N,Z}(\cdot \mid \vec{a}_0, \vec{b}, z) = P^{\psi}_{M|Z}(\cdot \mid z). \qquad (44)$$

We have finally established a connection between the original measurement on ψ, and the more complicated measurement (using embezzlement) on Φ.

We assume A_N and B_N to be free variables (in the bipartite setup). This, together with lemma 5, means

$$\begin{aligned} P^{\psi_0}_{X|A_N,B_N}(x \mid a, b) &= P^{\psi_0}_{X|A_N}(x_i \mid a) \\ &= P^{\psi_0}_{X|A_N,Z}(x_i \mid a, z). \end{aligned}$$

We know that $P^{\tilde{\Phi}}_{\vec{X}|\vec{A}_N,\vec{B}_N}(\vec{x} \mid \vec{a}, \vec{b}) = \prod_{i=1}^{r} P^{\psi_0}_{X|A_N}(x_i \mid a_i)$. The distributions for the measurement on the maximally entangled state are independent of Z (as proven in lemma 5). The distribution for Φ is just a product of distributions for ψ_0, so is independent of Z as well. This gives

$$P^{\tilde{\Phi}}_{S|\vec{A}_N,\vec{B}_N}(\lambda_i \mid \vec{a}_0, \vec{b}) = P^{\tilde{\Phi}}_{S|\vec{A}_N,\vec{B}_N,Z}(\lambda_i \mid \vec{a}_0, \vec{b}, z). \tag{45}$$

Using corollary 3, we can conclude that we can choose r_1 (and the m_i associated with it) such that

$$\left\langle \Phi, \tilde{\Phi} \right\rangle \leq 1 - \frac{\epsilon^2}{8}.$$

Using lemma 10 this implies

$$D\left(P^{\Phi}_{S|\vec{A}_N,\vec{B}_N}(\cdot \mid \vec{a}_0, \vec{b}), P^{\tilde{\Phi}}_{S|\vec{A}_N,\vec{B}_N}(\cdot \mid \vec{a}_0, \vec{b})\right) \leq \frac{1}{2}\epsilon.$$

Next, using the constraint that the theory $\mathcal{T}$ has continuous behavior, we know that for $\frac{1}{2}\epsilon$, there is a δ such that when $\left\langle \Phi, \tilde{\Phi} \right\rangle \geq 1 - \delta$ we know

$$D\left(P^{\Phi}_{S|\vec{A}_N,\vec{B}_N,Z}(\cdot \mid \vec{a}_0, \vec{b}, z), P^{\tilde{\Phi}}_{S|\vec{A}_N,\vec{B}_N,Z}(\cdot \mid \vec{a}_0, \vec{b}, z)\right) \leq \frac{1}{2}\epsilon.$$

We can choose r_2, such that $\left\langle \Phi, \tilde{\Phi} \right\rangle \geq 1 - \delta$. We want to use both inequalities, so define r to be the maximum of r_1 and r_2:

$$r = \max(r_1, r_2).$$

Combining the above inequalities, we are able to give an upper bound on the variational distance between P^{ψ}_M and $P^{\psi}_{M|Z}(\cdot \mid z)$.

$$\begin{aligned}
D\left(P^{\psi}_M, P^{\psi}_{M|Z}(\cdot \mid z)\right) &\leq D\left(P^{\psi}_M, P^{\tilde{\Phi}}_{S|\vec{A}_N,\vec{B}_N}(\cdot \mid \vec{a}_0, \vec{b})\right) + D\left(P^{\psi}_{M|Z}(\cdot \mid z), P^{\tilde{\Phi}}_{S|\vec{A}_N,\vec{B}_N}(\cdot \mid \vec{a}_0, \vec{b})\right) \\
&\leq D\left(P^{\psi}_M, P^{\tilde{\Phi}}_{S|\vec{A}_N,\vec{B}_N}(\cdot \mid \vec{a}_0, \vec{b})\right) + D\left(P^{\psi}_{M|Z}(\cdot \mid z), P^{\tilde{\Phi}}_{S|\vec{A}_N,\vec{B}_N,Z}(\cdot \mid \vec{a}_0, \vec{b}, z)\right) \\
&\leq D\left(P^{\Phi}_{S|\vec{A}_N,\vec{B}_N}(\cdot \mid \vec{a}_0, \vec{b}), P^{\tilde{\Phi}}_{S|\vec{A}_N,\vec{B}_N}(\cdot \mid \vec{a}_0, \vec{b})\right) \\
&+ D\left(P^{\Phi}_{S|\vec{A}_N,\vec{B}_N,Z}(\cdot \mid \vec{a}_0, \vec{b}, z), P^{\tilde{\Phi}}_{S|\vec{A}_N,\vec{B}_N}(\cdot \mid \vec{a}_0, \vec{b})\right) \\
&\leq \epsilon.
\end{aligned}$$

We have used equations (45), (43) and (44). This is a contradiction with the assumption that $D\left(P^{\psi}_M, P^{\psi}_{M|Z}(\cdot \mid z)\right) > \epsilon$. As this construction is possible for every $\epsilon > 0$, it follows that

$$P^{\psi}_M = P^{\psi}_{M|Z}(\cdot \mid z).$$

□

8 Conclusions and Discussion

The main goal has been achieved. We have constructed a proof for the claim of Colbeck and Renner that, under some natural assumptions, hidden variables do not improve the predictions made by quantum mechanics. If some hidden variable theory has free choice (to deduce no-signaling) and it is compatible with quantum mechanics, quantum mechanics is at least as informative as the hidden variable theory.

The compatibility states, in more detail, that the predictions given by quantum mechanics should be identical to those provided by a hidden variable theory that is averaged over the hidden variable. As quantum predictions seem sound, this assumption is non-negotiable. The assumption on free variables however, is more controversial. In the paper "About possible extensions of quantum theory" ([7]) by Ghirardi and Romano, a dummy higher theory is suggested which does not conform to the notion of free choice. On the other hand, it does have the no-signaling property and seems to be a legitimate hidden variable theory. This implies the free choice as described by Colbeck and Renner is actually an assumption stronger than no-signaling, and therefore somewhat redundant. Assuming just no-signaling, the proof can be completed in exactly the same way. It can even be slightly simplified by, describing the settings A_N and B_N as fixed parameters in the form of angels, instead of describing the settings by random variables with N possible values. This would make the proof of lemma 5 somewhat easier, as one does not have to use the limit of $N \to \infty$ anymore. But in general, the main proof would remain the same.

A far more pressing matter is contained in the crux of the proof, namely the generalization from one state and one measurement to any state and any measurement. This is achieved by describing the original general measurement (on ψ) as an alternative measurement (on Φ) in a larger Hilbert Space. One can show that the predictions given by both descriptions are the same. After measurement with a certain result, in both cases we collapse to the same eigenstate (in the S part of the system). The question remains though, if this again implies the predictions using the hidden variable theory $\mathcal{T}$ are the same. Intuitively, the predictions made by $\mathcal{T}$ should be the same if we have equivalent descriptions of the same measurement. As the hidden variable is assumed to be a property of the system, one might expect that as long as we are describing the same experiment, equivalent descriptions of this measurement should give rise to the same predictions, even using the hidden variable theory. But it is not really clear that the measurement via embezzlement we perform in the larger Hilbert space is in fact an equivalent description of the general measurement. But this fact is crucial in tying the general measurement and the specific projections on the maximally entangled state together. Therefore, in our view, the result is not complete without a specification on when exactly the theory $\mathcal{T}$ should give the same predictions for alternative descriptions of a measurement.

Secondly, in order to complete the proof, we have to assume an additional property for a hidden variable theory, that is not mentioned in the Colbeck and Renner article. Throughout the proof of the claim, we make use of states that are close. For a given r, the probabilities obtained for $\tilde{\Phi}$ are equal to a product of r probabilities obtained by measurement on ψ_0 in the bipartite setup. But Φ and $\tilde{\Phi}$ are now only guaranteed to be close up to a bound determined by r. If we want them to be closer, we have to choose a larger r, which leads to a different $\tilde{\Phi}$ and operator $\mathbf{U}^*\mathbf{O}^r_{\vec{x},\vec{a};\vec{y},\vec{b}}\mathbf{U}$. There is no reason to assume that the construction for one particular r would yield a $\tilde{\Phi}$ equal to Φ. For quantum mechanics, we know when the states are close, the probability distributions associated with a measurement on these states are close as well.

For $\mathcal{T}$ we have, a priori, no reason to assume the probability distributions for measurements on close states are close. In order to say something about the predictions given by $\mathcal{T}$ for our measurement on Φ (which is equivalent with the general measurement on ψ), we use knowledge about the predictions of $\mathcal{T}$ on $\tilde{\Phi}$. We have assumed the theory $\mathcal{T}$ to have an extra property we have called continuous behavior, which enables us to compare the probabilities given by $\mathcal{T}$ on Φ and $\tilde{\Phi}$, if Φ and $\tilde{\Phi}$ are close. Without this assumption, we cannot compare the predictions given by $\mathcal{T}$ on Φ and $\tilde{\Phi}$.

In conclusion, we have reached our goal. But along the way, we had to make some concessions in the form of extra assumptions on how the hidden variable theory $\mathcal{T}$ relates to quantum mechanics. Therefore, I think the result of Colbeck and Renner is not quite strong enough to exclude the existence of a hidden variable theory which assumes compatibility with quantum mechanics and free choice in a bipartite setup.

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