Indistinguishability in Quantum theory

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Abstract

This report describes the states of indistinguishable particles by using the representation theory of the symmetric group. The main research question is why in reality there only seem to be boson and fermion states for indistinguishable particles, while mathematically speaking there also are the so called parastatistics as theoretical possibilities. To this effect the last chapter proposes an argument against these parastatistics similar to the one given by Alexander Bach (1997), which is based upon the (im)possibility of extending n particle states to $n+m$ particle states.

Contents

1 Introduction

Around 1925 physicists came to realize that something like indistinguishable particles existed [1]. The assumption of indistinguishability led, for example, to a successful derivation of the entropy of a quantum system [3] (which is a measure of the number of possible states of a system), which illustrates the physical relevance of the concept of indistinguishablility.

The effect of indistinguishability on the entropy is illustrated by the following example: When one assumes two particles to be distinguishable, one can label them for example as red and blue. In this case there are, as shown in figure 1, four ways to distribute these particles over two separate boxes: Particles red and blue in the box on the left, particles red and blue in the box on the right, particle red in the left and particle blue in the right or particle red in the right and blue in the left. However, when we assume the particles to be indistinguishable we cannot label the particles. this means that there is no difference between the last two possibilities for the two distinguishable particles. We therefore only find the three possible ways to distribute the particles over the two boxes pictured in figure 1: Two in the box on the right, two in the box on the left or one in each. The fact that in the case of indistinguishable particles we find fewer possibilities then in the distinguishable case now causes the entropy to decrease.

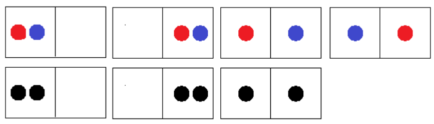


Figure 1: Distinguishable case (above) and Indistinguishable case (below).

This reasoning has a direct physical consequence. Namely, in the case of distinguishable particles the probability of finding both particles in a separate box is one in two (under the assumption hat each distinguishable configuration is equally probable), whereas in the case of indistinguishable particles there is a one in three probability.

In quantum theory the idea that there is no way to distinguish between two identical particles leads to the following well-known (but incorrect) argument about the state of n indistinguishable particles [9]:

We know that a state of n distinguishable particles, of which the first is in the state ψ_1, the second in the state ψ_2 etc., can be written as a linear combination of elementary tensorproducts of these states:

$$\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n. \tag{1}$$

We write $v(\sigma)$ for the operator that permutes the n particles (permuting the entries of the tensor product), according to σ. We find that the most general n particle in which each each paritcle is in a certain state ψ_i is given by:

$$\psi = \sum_{\sigma \in S_n} \lambda_\sigma v(\sigma) \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n. \tag{2}$$

For a state of n indistinguishable particles we now we argue that since the particles are indistinguishable the physical properties should not change when we permute two particles. From this we conclude that the state ψ should remain the same after a permutation, hence for a permutation of two particles σ we find:

$$v(\sigma)\psi = \lambda\psi, \tag{3}$$

with λ an arbitraty phase. Since permuting two particles twice is not permuting at all we argue that λ in equation (3) should satisfy $\lambda^2 = \lambda$ and so should be 1 or -1. If we require the state (wave-function) ψ to be normalized we find two solutions for the wavefunction ψ. For the plus sign we find the boson state, which is symmetric for the permutation of two particles:

$$\psi = \frac{1}{\sqrt{n}} \sum_{\sigma \in S_n} v(\sigma) \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n. \tag{4}$$

Whereas for the minus sign we find the fermion state, which is antisymmetric for the permutation of two particles:

$$\psi = \frac{1}{\sqrt{n}} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) v(\sigma) \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n. \tag{5}$$

This reasoning, however, is not as clean as it may appear at first sight. The statement that the physical properties should remain the same when two particles are permuted is by definition right. However, the conclusion that therefore $v(\sigma)\psi = \pm\psi$ is false. Instead, the conclusion should be that the outcome of all observables of n indistinguishable particles should remain the same. That is to say that the equality,

$$\forall_{\sigma,\psi} \left\langle \psi \right| a \left| \psi \right\rangle = \left\langle \psi \right| v(\sigma^{-1}) a v(\sigma) \left| \psi \right\rangle, \tag{6}$$

should hold for each operators a belonging to an observable of n indistinguishable particles. This means that the observables should intertwine with the action of a permutation, so:

$$v(\sigma^{-1}) a v(\sigma) = a, \tag{7}$$

should hold for all for allowed observables, leaving the statement of equation (3) empty.

To find the true states of indistinguishable particles [2] we should now find all possible states for the subset of all allowed observables $\mathcal{A}_{sym_n}$ for n indistinguishable particles, which is by definition the subset of operators that satisfy equation (7).

In contrast to most introductions to quantum theory this alternative reasoning takes the observables as a starting point instead of the states. This may seem to be unconventional, but in physics one should reason from what one can observe. One should first determine what the observables are and hereafter find the possibilities for states to behave under these observables, and not the other way around.

Chapter 2 will give an introduction to the formalism of quantum theory with as a starting point a certain set of allowed operators $\mathcal{A}$ that commute with a certain symmetry (unitary group representation). Using the well-known representation theory of the symmetric/permutation group [14] reviewed in chapter 3, chapter 4 derives the possible states on this set of allowed operators for indistinguishable particles. This set of states will appear to have a much wider range than only the boson and fermion states.

This leads to the question why it is that we only see boson and fermion states in physics. Chapter 5 tries to answer this question by looking at the way that a n particle state can be seen as a marginalised/extended $n \pm m$ particle state. It appears that the boson and fermion states are the only states that behave 'nicely' under adding and removing particles, as first suggested by A. Bach [7].

We analyse the marginalisation and extension by putting the set of operators of n indistinguishable particles $\mathcal{A}_{sym_n}$ in a chain of the operators belonging to the set of operators of $n \pm m$ indistinguishable particles:

$$\cdots \subset \mathcal{A}_{sym_{n-2}} \subset \mathcal{A}_{sym_{n-1}} \subset \mathcal{A}_{sym_n} \subset \mathcal{A}_{sym_{n+1}} \subset \mathcal{A}_{sym_{n+2}} \subset \cdots, \qquad (8)$$

in which $\mathcal{A}_{sym_{n-m}}$ ignores the states $\psi_n, \psi_{n-1} \cdots, \psi_{n-m+1}$, whilst on the other hand $\mathcal{A}_{sym_{n+m}}$ acts on extra states $\psi_{n+1}, \psi_{n+2}, \cdots, \psi_{n+m}$.

This paper assumes that the reader is familiar with linear algebra and representation theory. In principle it should be readable without any knowledge of quantum theory, but in practice it might be helpful.

2 The algebraic formalism of quantum theory

This chapter defines and characterizes the basic concepts of quantum theory (without equation of motion), namely: observables, states, pure states and transitions, in a way fit for the discussion of indistinguishable particles to come.

2.1 Observables

2.1.1 Definition

An algebra $\mathcal{A}$ of operators (sometimes called a *-algebra) on a Hilbert space H is defined by the following properties:

$$\begin{aligned}
&\mathcal{A} \subset B(H),\\
&a \in \mathcal{A} \to a^* \in \mathcal{A},\\
&a, b \in \mathcal{A} \to ab \in \mathcal{A},\\
&a, b \in \mathcal{A} \text{ and } \lambda \in \mathbb{C} \to a + \lambda b \in \mathcal{A},\\
&\mathbb{I} \in \mathcal{A}.
\end{aligned} \tag{9}$$

2.1.2 Definition

Given such an algebra of operators $\mathcal{A}$ we define the subset of observables as the set of hermitian elements of $\mathcal{A}$,

$$\mathcal{A}_h = \{a \in \mathcal{A} \big| a = a^*\}. \tag{10}$$

2.1.3 Remark

Throughout this paper we will assume that the Hilbert space H is of the form $\otimes^n \mathbb{C}^m$ for certain $n, m \in \mathbb{N}$. This automatically implies that H is finite-dimensional, so that $B(H)$ simply consits of all linear maps a: $h \to H$.

2.1.4 Remark

An algebra of operators is automatically a vector space and the invertible elements of an algebra form a group. This also means that a subalgebra $\mathcal{A}_2 \subset \mathcal{A}_1$ is a linear subspace of $\mathcal{A}_1$ and that the invertible elements of $\mathcal{A}_2$ are a subgroup of the invertible elements of $\mathcal{A}_1$.

2.1.5 Definition

We say that an observable $a \in \mathcal{A}_h$ is finer than $b \in \mathcal{A}_h$ if all eigenspaces of a are contained in the eigenspaces of b. $a \in \mathcal{A}_h$ is a maximal measurement if there is no finer operator in $\mathcal{A}_h$.

2.1.6 Remark

Since $\dim(H)$ is finite each opperator can be refined to a maximal measurement..

2.1.7 Definition

The eigenspaces of the maximal operators of $\mathcal{A}$ are called the minimal eigenspaces of $\mathcal{A}$.

2.1.8 Lemma

When $a \in \mathcal{A}_h$ we can write

$$a = \sum_{i=1}^{n} \lambda_i P_i, \tag{11}$$

with $\lambda_i \in \mathbb{R}$ and the P_i a projections on mutualy perpendicular minimal eigenspaces-paces of $\mathcal{A}$.

Proof:
This Lemma is simply the basic spectral theorem.

2.1.9 Definition

We call equation (11) a spectral decomposition of a.

2.1.10 Theorem

A set of operators on H is an algebra if and only if it is a set of intertwiners for a certain unitary representation $u_{\mathcal{A}}$ of a group G on H.

Proof:
Given a group G that acts on the space H. All operators in the set of all intertwiners of this action satisfy the properties in equation (2.1.1) and therefore form an algebra.

Now we prove the converse. We write the set of operators that commute with all operators in $\mathcal{A}$ as $\mathcal{A}'$ and denote the set of operators that again commute with all operators in $\mathcal{A}'$ as $\mathcal{A}''$. The subset $u_{\mathcal{A}}$ of unitary operators of $\mathcal{A}'$ is a group. This group induces a representation on H by simply applying the operator itself. The set of intertwiners of this representation is now exactly $\mathcal{A}''$, which coincides with $\mathcal{A}$ by the bicommutant theorem of van Neumann [16]. This means that if we take the group to be $u_{\mathcal{A}}$, the unitary operators in $\mathcal{A}'$ with the construction above, the algebra $\mathcal{A}$ indeed consists of all intertwiners of $u_{\mathcal{A}}$.

2.1.11 Remark

All sets of observables are the hermitian intertwiners of a certain symmetry $u_{\mathcal{A}}$. This result will be used throughout this paper to describe observables and states.

2.1.12 Definition

Two irreducible subspaces U and U' are called equivalent when there exists a non-zero intertwiner ϕ between the two spaces. Two vectors $\psi \in U$ and $\psi' \in U'$ are called equivalent when an intertwiner ϕ exists such that: $\phi(\psi) = \psi'$.

2.1.13 Definition

Write X for some set of mutually perpendicular irreducible subspaces of the representation $u_{\mathcal{A}}$, $\{U_i\}$, for wich $H = \oplus_{i=0}^{n} U_i$.

2.1.14 Lemma

Any $a \in \mathcal{A}_h$ can be written as:

$$a = \sum_{U \in X} \lambda_U P_U, \text{ for a certain } X. \tag{12}$$

Proof:
Since $a \in \mathcal{A}_h$, we have that a is an intertwiner of $u_{\mathcal{A}}$ and so we find by applying Schur's Lemma [8] and the spectral decomposition from equation (11), that for all irreducible subspaces U of $u_{\mathcal{A}}$:

$$a|_U = \sum_{i=1}^{n} \lambda_i P_i|_U = \lambda_U \mathbb{I}|_U. \tag{13}$$

For equation (13) to hold for all U we find that a should be of the form described in equation (12).

2.1.15 Remark

Lemma 2.1.14 states that the minimal eigenspaces of an algebra $\mathcal{A}$ are the irreducible subspaces of $u_{\mathcal{A}}$.

2.1.16 Lemma

The projections P_U on irreducible subspaces of a representation u are intertwiners of u.

Proof:
We write $\psi = \psi_1 + \psi_2$, with $\psi_1 \in U$ and $\psi_2 \in U^\perp$. We know that $U^\perp$ is an invariant subspace since U is irreducible. So we know that $u(g)\psi_2 \in U^\perp$ and thus $P_U u(g)\psi_2 = 0$. We also know that $u(g)\psi_1 \in U$ and thus $P_U u(g)\psi_1 = u(g)\psi_1$. Hence we can write:

$$u(g)P_U\psi = u(g)P_U(\psi_1 + \psi_2) = u(g)\psi_1 = P_U u(g)(\psi_1 + \psi_2) = P_U u(g)\psi. \tag{14}$$

So P_U is an intertwiner.

2.1.17 Proposition

The set $\mathcal{A}_h$ is the set:

$$\left\{ \sum_{U \in X} \lambda_U P_U \mid \lambda_U \in \mathbb{R} \right\}. \tag{15}$$

Proof:
From Lemma 2.1.14 we know that all operators are of this form. On the other hand we know from Lemma 2.1.16 we know that P_U is an intertwiner of $u_{\mathcal{A}}$ and so $P_U \in \mathcal{A}$. Now we conclude that all elements in the set of equation (15) are in the algebra, since $\mathcal{A}$ is a vector space. Because the elemenst in the set of equation (15) are hermitian, since the subspaces are mutually perpendicular, they are also contained in $\mathcal{A}_h$. This proves that the set in equation (15) coincides wiht $\mathcal{A}_h$.

2.2 States

2.2.1 Definition

A state ω is a functional: $\mathcal{A}_h \to \mathbb{R}$, with the following properties:

$$\begin{aligned} &\omega(\mathbb{I}) = 1, \\ &\omega(a^2) \geqslant 0. \end{aligned} \tag{16}$$

2.2.2 Remark

We use the so called Bra-ket notation in which:

$$\begin{aligned} ||\psi|| &= 1, \text{ a norm one vector.} \\ |\psi\rangle &= \psi, \text{ the vector.} \\ \langle\psi| &= \psi^*, \text{ the dual/conjugate vector.} \end{aligned} \tag{17}$$

2.2.3 Lemma

Any functional on $\mathcal{A}_h$ can be written as:

$$\omega(a) = \sum_{i=1}^{n} \lambda_i \langle\psi_i| a |\psi_i\rangle \tag{18}$$

Proof:
Any functional on an algebra of operators is linear in all entries of a matrix representation of these operators. This means that all functionals can be written as follows:

$$\omega(a) = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{i,j} \langle\psi_i| a |\psi_j\rangle, \ \lambda_{i,j} \in \mathbb{C}. \tag{19}$$

Since a is hermitian we find that:

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{i,j} \langle\psi_i| a |\psi_j\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{i,j} \frac{1}{2}\Big(\langle\psi_i| a |\psi_j\rangle + \langle\psi_j| a |\psi_i\rangle \Big) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{i,j} \frac{1}{2}\Big(\langle\psi_i + \psi_j| a |\psi_i + \psi_j\rangle - \langle\psi_i| a |\psi_i\rangle - \langle\psi_j| a |\psi_j\rangle \Big). \end{aligned} \tag{20}$$

Making the subsitution $\psi' = \psi_i + \psi_j$ we can write equation (20) in the form of equation (18).

2.2.4 Definition

A density operator is an operator that satisfies:

$$\begin{aligned} \mathrm{Tr}(\rho) &= 1 \\ \forall_{\psi \in H} \langle\psi|\, \rho\, |\psi\rangle &\geq 0 \end{aligned} \tag{21}$$

2.2.5 Proposition

All states can be written as:

$$\omega_\rho(a) = \mathrm{Tr}(a\rho), \text{ with } \rho \text{ a hermit density operator.} \tag{22}$$

Proof:
Equation (18) can be written in a more illuminating form:

$$\omega(a) = \sum_{i=1}^{n} \lambda_i \langle\psi_i|\, a\, |\psi_i\rangle = \mathrm{Tr}\left(\sum_{i=1}^{n} \lambda_i |\psi_i\rangle \langle\psi_i|\, a\right) = \mathrm{Tr}(\rho a), \tag{23}$$

with

$$\rho = \sum_{i=1}^{n} \lambda_i |\psi_i\rangle \langle\psi_i|\,. \tag{24}$$

Since ω is a state we know by definition that equation (16) holds this implies:

$$\begin{aligned} \mathrm{Tr}(\rho) &= \mathrm{Tr}(\rho\mathbb{I}) = 1 \\ \langle\psi|\, \rho\, |\psi\rangle &= \mathrm{Tr}\bigg(\rho\, |\psi\rangle \langle\psi|\bigg) = \mathrm{Tr}\bigg(\rho(|\psi\rangle \langle\psi|)^2\bigg) \geq 0. \end{aligned} \tag{25}$$

Hence equation (21) holds. Futhermore equation (24) shows ρ to be hermit.

2.2.6 Remark

When we call a density operator ρ a state, we refer to the functional ω as given in equation (22). When we call a projection P a state we refer to the state $\rho = \frac{1}{\mathrm{Tr}(P)}P$. When in turn we call a linear subspace U a state we refer to the state corresponding to the projection P_U on this space.

2.2.7 Definition

For $a \in \mathcal{A}_h$ the value $\omega(a)$ is called the expectation value and $\omega(a^2)$ is called the mean squared value of the observable a. The experession $\omega(a^2) - \omega(a)^2$ is therefore the variance of the observable.

2.2.8 Lemma

The variance defined in Definition 2.2.7 is positive.

Proof:
The variance is given by:

$$\omega(a^2) - \omega(a)^2 = \omega\left((a - \omega(a)\mathbb{I})^2\right). \tag{26}$$

Where we used Definition 2.2.1 to say $\omega(\mathbb{I}) = 1$. If we now substitute $b = (a - \omega(a)\mathbb{I})$ and again use Definition 2.2.1 to say $\omega(b^2) \geq 0$, we see that equation (26) and hence the variance is positive.

2.3 Pure states

2.3.1 Lemma

When $\{\psi_i \mid 1 \leq i \leq n\}$ is a certain set of equivalent vectors as defined in definition 2.1.12 we find that $\psi = \sum_{i=1}^{n} \lambda_i \psi_i$ generates again an irreducible subspace. In other words taking the span of the vectors that are the outcome of a certain group element applied to ψ is an irreducible linear subspace equivalent to the one from which the ψ_i originateted.

Proof:
This is clear since the vectors ψ_i are equivalent with respect to the group action.

2.3.2 Lemma

For all unitvectors ψ in an irreducible subspace U of $u_{\mathcal{A}}$ and $a \in \mathcal{A}$ the values

$$\langle\psi| a |\psi\rangle, \tag{27}$$

coincide.

Proof:
We use the spectral decompositionfrom equation (12) of a to rewrite equation (27) as:

$$\langle\psi| a |\psi\rangle = \sum_{U \in X} \mu_U \langle\psi| P_U |\psi\rangle \tag{28}$$

Since we know from Lemma 2.1.16 that P_U is an intertwiner we can conclude from Schurs lemma that: $P_U\psi = \lambda_U\psi'$, with ψ_i' a norm one vector equivalent to ψ_i under the group action. So using the fact that $P_U^2 = P_U^* = P_U$ we can continue equation (28):

$$\sum_{U \in X} \mu_U \left(\langle\psi| P_U^*\right)\left(P_U |\psi\rangle\right) = \sum_{U \in X} \mu_U \lambda_U^2 \langle\psi'||\psi'\rangle = \sum_{U \in X} \mu_U \lambda_U^2. \tag{29}$$

The right hand side of equation (29) is independent of the $\psi \in U$ and therefore the same for all vectors within the same irreducible subspace.

2.3.3 Proposition

The set of all distinct states on an algebra $\mathcal{A}$ is the set:

$$\left\{\sum_{U \in X} \mu_U P_U \mid, \text{ with } \sum_{U \in X} \mu_U = 1 \text{ and } \mu_U \geq 0\right\}. \tag{30}$$

Proof:
We know from Lemma 2.2.3 that each functional can be written as:

$$\omega(a) = \sum_{i=1}^{n} \lambda_i \langle \psi_i | \, a \, | \psi_i \rangle \tag{31}$$

We can write each ψ_i in equation (31) as a sum of vectors that lie in irreducible subspaces to obtain:

$$\begin{aligned} &\sum_{i=1}^{n} \lambda_i \left\langle \sum_{j=1}^{m} \mu_j \psi_{i,j} \middle| \, a \, \middle| \sum_{j=1}^{m} \mu_j \psi_{i,j} \right\rangle = \\ &\quad \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j^2 \langle \psi_{i,j} | \, a \, | \psi_{i,j} \rangle + \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{j' \neq j} \lambda_i \mu_j \mu_{j'} \langle \psi_{i,j} | \, a \, | \psi_{i,j'}, \rangle \, . \end{aligned} \tag{32}$$

with all $\psi_{i,j}$ in an irreducible subspace $U_{i,j}$. Equation (32) splits in the right diagonal part and the right cross-term part. It follows from Lemma 2.3.2 that instead of writing $\langle \psi_{i,j} | \, a \, | \psi_{i,j} \rangle$ we could just as well use another ψ in the same irreducible subspace $U_{i,j}$ as $\psi_{i,j}$. In particular we could choose an orthonormal basis $\psi_{1,i,j}, \psi_{2,i,j}, \cdots, \psi_{k,i,j}$ of $U_{i,j}$, and write for each individual diagonal term of equation (32):

$$\langle \psi_{i,j} | \, a \, | \psi_{i,j} \rangle = \frac{1}{n} \sum_{l=1}^{k} \langle \psi_{l,i,j} | \, a \, | \psi_{l,i,j} \rangle = \frac{1}{\mathrm{Tr}(P_{U_{i,j}})} \mathrm{Tr}(P_{U_{i,j}} a). \tag{33}$$

Now we consider the cross-terms. Let us restrict ourselves to one term in the sum over the cross-terms: $\langle \psi_{i,j} | \, a \, | \psi_{i,j} \rangle$. If we now use the spectral decomposition of a we find for these individual terms:

$$\sum_{U \in X} \nu_U \langle \psi_{i,j} | \, P_U \, | \psi_{i,j} \rangle = \sum_{U \in X} \nu_U \left(\langle \psi_{i,j} | \, P_U^* \right) \left(P_U \, | \psi_{i,j} \rangle \right) \tag{34}$$

Since P_U is an intertwiner we either find $P_U \psi = \kappa_U \psi'$ with $\psi' \in U$ equivalent to ψ, or zero. So we can continue equation (34) as follows:

$$\sum_{U \in X} \nu_U \kappa_{1,U} \kappa_{2,U} \left\langle \psi'_{i,j,U} \middle| \middle| \psi'_{i,j',U} \right\rangle . \tag{35}$$

The innerproduct does not depend on U, since the vectors $\psi'_{i,j,U}$ with the same i and j are equivalent and so are all $\psi'_{i,j,U'}$ for the same i and j. We may therefore asume the inner product in equation (35) to be $C_{i,j}$, and continue as follows:

$$\sum_{U \in X} \nu_U \kappa_{1,U} \kappa_{2,U} C_{i,j}. \tag{36}$$

Now we can choose two equivalent vectors ψ_1 and ψ_2, respectively, in the irreducible subspace of $\psi_{i,j}$ and $\psi_{i,j'}$. We have: $\langle\psi_1| P_U |\psi_2\rangle = \kappa_{1,U}\kappa_{2,U}$ and therefore can continue equation (36) as:

$$C_{i,j}\sum_{U\in X}\nu_U \langle\psi_1| P_U |\psi_2\rangle = \frac{C_{i,j}}{2}\langle\psi_1+\psi_2| a |\psi_1+\psi_2\rangle \\ - \langle\psi_1| a |\psi_1\rangle - \langle\psi_2| a |\psi_2\rangle\,, \tag{37}$$

where ψ_1 and ψ_2 respectively lie in the irreducible subspaces U_1 and U_2, and, as seen in Lemma 2.3.1, $\psi = \psi_1 + \psi_2$ also lies in an irreducible subspace say U_3. Now we can rewrite equation (37) as we did in equation (33):

$$\sum_{l=1}^{3}\frac{C_{i,j}}{2\cdot \mathrm{tr}(P_{U_l})}\mathrm{Tr}(P_{U_l}a). \tag{38}$$

This Means that the left-terms in equation (32) can as well as the diagonal terms (as seen in equation (33)) be written as a linear combination of terms of the form of equation (38). Since ρ is hermit and therefore can be writen as a linear combination of projections on perpendicular eigenspaces, this implies that every functional can be written as in equation (30).

2.3.4 Theorem

The set of all states on $\mathcal{A}_h$ is a convex set with as extreme points the states P_U.

Proof:
First we proof the set of states to be convex that is to show that each functional on the line between two states is again a state. That is to say:

$$\lambda\omega_1 + (1-\lambda)\omega_2, \tag{39}$$

is a state for all $\lambda\in[0,1]$. It is easily verified that the functional in equation (39) satisfies the required condition in equation (16).

When $\rho = \sum_{U\subset}\lambda_U P_U$ with two λ_{U_1} and λ_{U_2} non-zero we also know from equation (30) that both should also be smaller then one since the sum should be one. We find that ther exists an $\epsilon > 0$ such that $0 < \lambda_{U_1}\pm\epsilon < 1$ and $0 < \lambda_{U_2}\pm\epsilon < 1$. Henc we find that ρ lies on the line between the states:

$$\sum_{U\in X\setminus\{U_1,U_2\}}\lambda_U P_U + (\lambda_{U_1}-\epsilon)P_{U_1} + (\lambda_{U_2}+\epsilon)P_{U_2} \\ \text{and} \\ \sum_{U\in X\setminus\{U_1,U_2\}}\lambda_U P_U + (\lambda_{U_1}+\epsilon)P_{U_1}(\lambda_{U_2}(\lambda_{U_2}-\epsilon)P_{U_2}. \tag{40}$$

and is therefore not an extreme point.

Now consider a certain projection on an irreducible subspace $P_{U'}$. We write for $P_{U'}$:

$$P_{U'} = \sum_{U\in X}\lambda_U P_U. \tag{41}$$

Because of Proposition 2.1.17 $P_{U'} \in \mathcal{A}$ so we may apply the state of equation (41) to this observable $P_{U'}$. Taking the trace with $P_{U'}$ of the left hand side of equation (41) results in one and so taking the trace of $P_{U'}$ with the right hand side should also result in one. This can only hold if $\lambda_{U'} = 1$ and therefore all othere $\lambda_U = 0$. Meaning that the equality of equation (41) can only hold if the right hand side is equal to $P_{U'}$ and hence this is an extreme point.

2.3.5 Definition

We call the extreme points of the set of states the pure states.

2.3.6 Corollary

As an immediate consequence of Theorem 2.3.4 we find that the pure states are in bijective correspondance with the projections on irreducible subspaces for $U_{\mathcal{A}}$.

2.3.7 Corollary

We also find from the result of Lemma 2.3.2 that all projections on subspaces of irreducible subspaces are also pure states (namely the same pure state as the projection on the whole irreducible subspace).

2.3.8 Definition

If we write the decomposition of ω in pure states:

$$\omega = \sum_{U \in X} \lambda_U P_U. \tag{42}$$

We call the λ_U the weight of P_U in the decomposition of ω. If P_U has a weight $\lambda_U > 0$ it is said to be contained in the decomposition.

2.3.9 Remark

Note that the weight λ_U in which P_U is contained in the decomposition of a certain state should ba a real number between zero and one to satisfy equation (21).

2.3.10 Proposition

A state ω is pure if and only if there is a maximal operator $a \in \mathcal{A}_h$ as defined in Definition 2.1.5 for which the variance is zero.

Proof:
We can write for the spectral decomposition of a and a^2:

$$\begin{aligned} a &= \sum_{U \in X} \lambda_U P_U, \\ a^2 &= \sum_{U \in X} \lambda_U^2 P_U, \end{aligned} \tag{43}$$

where all λ_U difer. The variance of the observable a for the state $\sum_{U' \in X'} \lambda_{U'} P_{U'}$ is zero if and only if:

$$\omega(a)^2 = \omega(a^2)$$

$$\left(\mathrm{Tr} \left(\sum_{U' \in X'} \frac{\lambda_{U'}}{\mathrm{Tr}(P_{U'})} P_{U'} \sum_{U \in X} \lambda_U P_U \right) \right)^2 = \mathrm{Tr} \left(\sum_{U' \in X'} \frac{\lambda_{U'}}{\mathrm{Tr}(P_{U'})} P_{U'} \sum_{U \in X} \lambda_U^2 P_U \right).$$

$$\left(\sum_{U \in X} \lambda_U \sum_{U' \in X'} \left(\frac{\lambda_{U'}}{\mathrm{Tr}(P_{U'})} \mathrm{Tr}(P_U P_{U'}) \right) \right)^2 = \sum_{U \in X} \lambda_U^2 \sum_{U' \in X'} \left(\frac{\lambda_{U'}}{\mathrm{Tr}(P_{U'})} \mathrm{Tr}(P_U P_{U'}) \right) \tag{44}$$

If we write C_U for the outcome of the sum over the U' we can write the equality of equation (44) as follows:

$$\left(\sum_{U \in X} \lambda_U C_U \right)^2 = \sum_{U \in X} \lambda_U^2 C_U. \tag{45}$$

We know from remark 2.3.9 that C_U should be a positive number and the sum over this C_U should be one, hence the equality can hold if and only if there is only one C_U that is non-zero And this is the case if and only if

$$\sum_{U' \in X'} \left(\frac{\lambda_{U'}}{\mathrm{Tr}(P_{U'})} \mathrm{Tr}\Big(P_U P_{U'} \Big) \right), \tag{46}$$

is zero for all but one U, say only non-zero for U_1. This implies that we should have that $\sum_{U' \in X'} \frac{\lambda_{U'}}{\mathrm{Tr}(P_{U'})} = \frac{1}{\mathrm{Tr}(P_{U_1})} P_{U_1}$. This means that the state should be P_{U_1}, which is a pure state.

2.3.11 Example

If we assume no symmetry that is to say we have the algebra of intertwiners of representation of the group is $\{e\}$ which is $\mathbb{I}$. We find, since all operators intertwine with the identity, that $\mathcal{A}_h$ is the set of all hermitian operators on H. The irreducible subspaces of this representation are all one dimensional subspaces, $\mathbb{C}\psi$. The set of pure states is therefore:

$$\{P_\psi \mid \psi \in H\}. \tag{47}$$

2.4 Transitions

2.4.1 Definition

We define the set of pure states $J(a)$ belonging to an operator a as all states P_j that are projections on minimal eigenspaces contained in the spectral decomposition of a.

2.4.2 Definition

States are functionals on operators but the operators also act on the states. Namely, a acts on a state ρ as follows:

$$\begin{aligned} &a : \{\omega\} \to \{f : J(a) \to [0,1]\} \\ &a(\rho) = f \\ &f(j) = \mathrm{Tr}(P_j \rho) \end{aligned} \tag{48}$$

2.4.3 Proposition

The function $a(\omega) = f$ defined by an operator a as in Definition 2.4.2 is a probability distribution on $J(a)$.

Proof:
First we verify that $f(j)$ is positive for all $j \in J(a)$. By writing $P_j = \sum_{i=1}^{n} |\psi_i\rangle \langle\psi_i|$, for some orthonormal vectors ψ_i, we find:

$$f(j) = \mathrm{Tr}\Big(P_j \rho\Big) = \mathrm{Tr}\Big(\sum_{i=1}^{n} |\psi_i\rangle \langle\psi_i| \rho\Big). \tag{49}$$

Since we know form Proposition 2.2.5 that $\langle\psi| \rho |\psi\rangle \geq 0$ we find $f(j)$ to be positive.

Now we prove that $\sum_{j=1}^{n} f(j) = 1$. Because a is hermitian the minimal eigenspaces are mutually perpendicular and so:

$$\sum_{j=1} nf(j) = \sum_{j=1}^{n} \mathrm{Tr}\Big(P_j \rho\Big) = \mathrm{Tr}\left(\sum_{j=1}^{n} P_j \rho\right) = \mathrm{Tr}\Big(\mathbb{I}\rho\Big) = \omega(\rho) = 1. \tag{50}$$

2.4.4 Definition

We call $a(\rho)(j)$ the probability of a transition from the state ρ to the state P_j after applying an observable a:

$$a(\rho)(j) = p(\rho \to P_j). \tag{51}$$

A transition $\rho \to P_j$ is said to be possible if $P(\rho \to P_j)$ is non-zero for a certain observable $a \in \mathcal{A}_h$ and impossible if the probability is zero for all observables $a \in \mathcal{A}_h$.

2.4.5 Definition

When the transition to P_j occurs after conducting the observable a we call the λ_j in the spectral decomposition of a the outcome of the observable.

2.4.6 Remark

The probability that a state ρ, that contains P_U in its decomposition with weight λ_U, transits to the pure state P_U after applying the observable P_U is exactly the weight λ_U.

2.4.7 Lemma

Two inequivalent irreducible subspaces are perpendicular.

Proof:
Since P_U is an intertwiner we find from schurs lemma that $P_U|_{U'}$ is zero if $U \not\simeq U'$. Therefore $P_U U' = 0$, hence $U \perp U'$.

2.4.8 Corollary

We find from Lemma 2.4.7 that if $H = \oplus_{i=1}^{n} U_i$ all irreducible subspaces U are genarated by a ψ that is a linear combination of equivalent vectors in the irreducible subspaces $U_i \simeq U$. In other words we find that the new constructed irreducible subspaces in Lemma 2.3.1 are all irreducible subspaces.

2.4.9 Proposition

When a system is in a state U a transtition to a state U' is possible (non-zero probability) if and only if $U \simeq U'$.

Proof:
Recall from Definition 2.4.4 that the probability for a transition is given by:

$$p(U \to U') = \frac{1}{\mathrm{Tr}(P_U)} \mathrm{Tr}\Big(P_U P_{U'} \Big). \tag{52}$$

When the two subspaces are inequivalent and therfore by Lemma 2.4.7 perpendicular this probability is zero and therefore impossible.

When two spaces are equivalent, we can take two equivalent vectors $\psi \in U$ and $\psi' \in U'$ and construct the irreducible subspace U'' generated by $\frac{1}{\sqrt{2}}(\psi + \psi')$, as was done in Lemma 2.3.1. This U'' is non-perpendicular to both U and U', thus $\mathrm{Tr}(P_U P_{U''})$ and $\mathrm{Tr}(P_{U'} P_{U''})$ are non-zero. Hence a transition via the state $P_{U''}$ has a non-zero probability and is therefore possible.

2.5 Marginalisation & extension

2.5.1 Definition

Given a chain of algebras in which the previous is contained in the others:

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \cdots \tag{53}$$

A i-step marginalisation of the algebra $\mathcal{A}_n$ is a projection M^i that projects $\mathcal{A}_n$ on the linear subspace $\mathcal{A}_{n-i}$. The marginalisation of a state ω to a marginal state $M_*^i(\omega)$ is defined as the functional ω restricted to the operators in $M^i(\mathcal{A}_n) = \mathcal{A}_{n-i}$. A pure state is said to have a pure i-step marginalisation if the state is also pure when restricted to $\mathcal{A}_{n-i}$.

2.5.2 Definition

An i-step extension of a state $\omega \in \mathcal{A}_n$ is a state $E_*^i(\omega) = \omega' \in \mathcal{A}_{n+i}$ such that: $\omega = M_*^i(\omega')$. An i-step extension of a pure state is called a pure extension if it is pure on $\mathcal{A}_{n+i}$.

2.5.3 Remark

In contrast to a marginalisation an extension is, in general, not unique.

2.5.4 Definition

A pure state ω on $\mathcal{A}_n$ is called expandable through the chain of algebras in equation (53), if all i-step marginalisations are pure and there exist pure i-step extensions for all i. The chain of pure states via which a state expands is called the path of the expansion.

3 Representation theory of the symmetric group

This chapter summerizes the representation theory of the permutation group, which is essential for the theory of indistinguishable particles.

3.1 Representation of the symmetric group

3.1.1 Definition

We define the representation v of S_n on $\otimes^n H$ by permuting the vectors within the entries according to σ. When we apply $v(\sigma)$ to $\psi = \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n$ we permute each vector $\psi_l \in H$ in the i'th entry of ψ to the entry $\sigma(i)$.

3.1.2 Example

$$v(1,2)\psi_3 \otimes \psi_1 \otimes \psi_2 = \psi_1 \otimes \psi_3 \otimes \psi_2 \tag{54}$$

3.1.3 Definition

Construct $H_{S_n} \subset \otimes^n H$ as follows: Take n arbitrary unit vectors in H: $\psi_1, \psi_2, \cdots, \psi_n$. Now construct H_{S_n} as the linear space generated by the representation v of S_n defined in Definition 3.1.1 starting with $\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n$.

If the vectors $\psi_1, \psi_2, \cdots, \psi_n$ are linearly independent, wel call the set:

$$\{v(\sigma)\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n \mid \sigma \in S_n\}, \tag{55}$$

the standard basis of H_{S_n}.

3.1.4 Definition

We call the vectors $\psi_1, \psi_2, \cdots, \psi_n$ from which H_{S_n} is constructed in definition 3.1.3. The individual vectors.

3.1.5 Definition

Another group action u of S_n on H_{S_n} can be defined on the standard basis vectors defined in Definition 3.1.3 as follows:

$$u(\sigma)\psi_i \otimes \psi_j \otimes \cdots \otimes \psi_k = \psi_{\sigma(i)} \otimes \psi_{\sigma(j)} \otimes \cdots \otimes \psi_{\sigma(k)} \tag{56}$$

This automatically defines a group action on all vectors in H_{S_n}.

3.1.6 Example

$$u(1,2)\psi_3 \otimes \psi_1 \otimes \psi_2 = \psi_3 \otimes \psi_2 \otimes \psi_1 \tag{57}$$

3.1.7 Lemma

If the n vectors chosen in Definition 3.1.3 are linearly independent, the representations u and v of S_n are the (unitary equivalent) right and left regular representations on H_{S_n} respectively.

Proof:
H_{S_n} is the vector space with as a basis:

$$\begin{array}{l} \{u(\sigma)\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n \mid \sigma \in S_n\} = \{e_\sigma \mid \sigma \in S_n\} \\ \{v(\sigma)\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n \mid \sigma \in S_n\} = \{d_\sigma \mid \sigma \in S_n\} \end{array}. \tag{58}$$

u and v now act on this space as:

$$\begin{array}{l} u(\sigma')e_\sigma = e_{\sigma'\sigma}, \\ v(\sigma')d_\sigma = d_{\sigma\sigma'^{-1}}, \end{array} \tag{59}$$

which make it the right and left regular representation.

3.1.8 Corollary

We conclude from Lemma 3.1.7 that the representations u and v are unitary equivalent, since an isomorphism ϕ is given by:

$$\phi(e_\sigma) = d_{\sigma^{-1}} \tag{60}$$

3.1.9 Lemma

The group actions v and u of S_n on H_{S_n} commute. This means that each $u(\sigma)$ is an unitary intertwiner on H_{S_n} for the v representation of S_n and vice versa.

Proof:
This follows from the fact that relabeling the entries does not affect the relabeling of the states within these entries.

3.2 Irreducible subspaces of the regular representation

3.2.1 Definition

Let λ_1 be a partition of n, that is, $\lambda_1 = (k_1, k_2, \cdots, k_l)$, $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_l$ and $\sum k_i = n$. This defines a new partition $\lambda_2 = (c_1, c_2, \cdots, c_t)$ of n by $c_i = \#\{k_j \in \lambda_1 \mid k_j - i \geqslant 0\}$. One can picture these two partitions by a diagram of n boxes with decreasing length of rows, called a tableau. Here λ_1 indicates the length of each row and λ_2 indicates the length of the columns.

3.2.2 Example

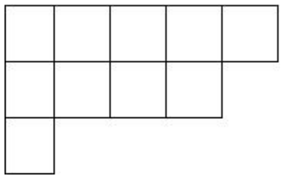


Figure 2: Young tableau beloning to $\lambda_1 = (5, 4, 1)$ and $\lambda_2 = (3, 2, 2, 2, 1)$.

3.2.3 Definition

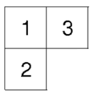
We call the number of boxes of a tableau the size of a tableau.

3.2.4 Definition

When we fill a tableau of size n defined in Definition 3.2.1 with the numbers $1, 2 \cdots, n$, we call it a Young diagram Y_t of form t. We construct the vector $\psi_{Y_t} \in H_{S_n}$ belonging to a Young diagram Y_t, with t of size n, by placing the entries of the n-fold tensor product with $\psi_1, \psi_2, \cdots, \psi_n$ chosen in section 3.1.3, in the in the same order the the numbers $1, 2 \cdots, n$ occur in Y_t from left to right and from top to bottom. On the other hand, we construct $Y_{\psi,t}$ the Young diagram Y_t belonging to a standard basis vector ψ of H_{S_n} defined in Definition 3.1.3 by placing the numbers $1, 2, \cdots, n$ in the boxes from right to left and from top to bottom in the same order as the vectors $\psi_1, \psi_2, \cdots, \psi_n$ occur in the tensorpoduct.

3.2.5 Example

Let Y be the Young diagram:



.

We then find:

$$\psi_Y = \psi_1 \otimes \psi_3 \otimes \psi_2. \tag{61}$$

Vice versa we would regain Y from ψ_Y if we set t to be the triangle.

3.2.6 Definition

Each Young diagram Y_t defines two subgroups of S_n (depending on Y_t). Let K be the subgroup of S_n that only permute numbers in the same row of the Young diagram and let G be the group that permutes the numbers in the same columns. This defines two projections:

$$P_K = \frac{1}{\#K} \sum_{\sigma \in K} u(\sigma), \tag{62}$$

$$P_G = \frac{1}{\#G} \sum_{\sigma \in G} \operatorname{sgn}(\sigma) u(\sigma). \tag{63}$$

Write:

$$P_Y = \lambda_Y P_G P_K, \tag{64}$$

in which Y stands for the Young diagram defining the subgroups K and G and λ_Y is scalar that renormalizes.

3.2.7 Example

For example, let Y again be:

1	3
2	

.

Then we find:

$$P_Y(\psi_1 \otimes \psi_2 \otimes \psi_3) = \frac{1}{2}(\psi_1 \otimes \psi_2 \otimes \psi_3 + \psi_2 \otimes \psi_1 \otimes \psi_3 - \psi_3 \otimes \psi_2 \otimes \psi_1 - \psi_2 \otimes \psi_3 \otimes \psi_1). \tag{65}$$

3.2.8 Definition

We denote the linear span:

$$L\left\{P_{Y_{\psi,t}}\psi \mid \psi \text{ a standard basis vector of } H_{S_n} \text{ and } t \text{ of size } n\right\}, \tag{66}$$

by U_t.

3.2.9 Theorem

The set of all U_t, contains all inequivalent irreducible subspaces of H_{S_n}. If all vectors $\psi_1, \psi_2, \cdots, \psi_n$ that define H_{S_n} are linearly independent, then all U_t are mutually inequivalent and all irreducible subpsaces of S_n occur.

Proof:
We show that the set of all U_t is precisely the set of inequivalent irreducible subspaces of the regular representation. In case that $\psi_1, \psi_2 \cdots, \psi_n$ are independent this is H_{S_n}. The general case can be obtained from this result.
First of all we note that $P_{Y_{\psi,t}}\psi$ generates U_t:

$$u(\sigma)P_{Y_{\psi,t}}\psi = u(\sigma)P_G P_K \psi = P_{Y_{u(\sigma)\psi,t}} u(\sigma)\psi = P_{Y_{\psi',t}}\psi'. \tag{67}$$

The ψ' of equation (67) are all standard basis vectors of H_{S_n}, therefore these vectors in equaiton (67) span the whole space U_t.

We now use an operator $\kappa_{\psi,t}$, beloning to a standard basis vector ψ of H_{S_n} and a tableau t. This $\kappa_{\psi,t}$ acts as follows on a standard basis vector ψ' of H_{S_n}. It first applies the $P_{K'}$, defined in Definition 3.2.6, with K' the subgroup that preserves the elements in the row of $Y_{\psi',t}$. Secondly, it applies the P_G, defined in section 3.2.6, with G the subgroup that preserves the elements in the column of $Y_{\psi,t}$. The first thing that we note about $\kappa_{\psi,t}$ is that it is completely defined by linear combinations of $u(\sigma)$, hence:

$$\kappa_{\psi,t}(X) \subset X, \text{for all invariant subspaces } X. \tag{68}$$

Now we will show that the image of $\kappa_{\psi,t}$ is the linear span of the vector $P_{Y_{\psi,t}}\psi$. If K' would contain a permutation (i,j) that is also contained in G, the function $\kappa_{\psi,t}$ would take it to zero, since G antisymmetrizes the i and j and K' symmetrizes i and j. This means that if the result would be non-zero, K' can only contain one element per column of $Y_{\psi,t}$. We also have that K has cycle lengths that are equal to the partition λ_1 defined by the tableau t. Since G permutes elements within the columns, this implies that there is an operator

$$\tau = \prod_{i=1}^{n} u(\sigma_i), \text{ with all } \sigma_i \in G, \tag{69}$$

for which $\tau K' = K$ with K the subgroup defined by $Y_{\psi,t}$. But this means that $P_K = \pm P_{K'}$. It follows that:

$$\kappa_{\psi,t}\psi' = \pm P_G P_K \psi = \pm P_{Y_{\psi,t}}\psi. \tag{70}$$

This implies that $P_{Y_{\psi,t}}\psi'$ is either zero or equals $\pm P_{Y_{\psi,t}}\psi$. Since $P_{Y_{\psi,t}}\psi$ generates the irreducible subspace U_t we obtain from equation (68) that

$$U_t \subset X \text{ or } U_t \perp X. \tag{71}$$

This implies that U_t does not contain a strict invariant subspace X, therofore U_t is irreducible.

We now show that the U_t are distinct. When two Young diagram Y_t and $Y_{t'}$ have different forms ($t \neq t'$), there have to be two numbers i and j occurring in the same column of Y_t and in the same row of $Y_{t'}$ (or the other way around). This implies that there are for all $P_{Y_t}\psi_{Y_t}$ and $P_{Y'_{t'}}\psi_{Y'_{t'}}$ ther are i and j such that we find that one is symmetric under $u(i,j)$ and the other antisymmetric. This implies that one can not find an intertwiner and hence that all U_t are mutually inequivalent.

We can also show that these U_t are all non-equivalent subspaces.The number of subspaces U_t is the number of partitions of n, which in turn is the number of conjugate classes of S_n and therefore the number of inequivalent irreducible subspaces of S_n. A more detailed proof can be found in [4] and more about the conjugacy classes of S_n can be found in [8].

3.2.10 Definition

When some irreducible subspace of S_n is equivalent to U_t, we denote it by U_T.

3.2.11 Corollary

Because of Theorem 3.2.9 every irreducible subspace equals some U_T.

3.2.12 Definition

A Young diagram in which the numbers are increasing in each row as well as in each column is called a standard Young diagram.

3.2.13 Lemma

The linear span of $\{P_{Y_t}\psi_{Y_t} \mid Y_t \text{ standard}\}$ equals U_t. If the $\psi_1, \psi_2, \cdots, \psi_n$ in section 3.1.3 were chosen to be independent, this set would ba a basis for U_t.

Proof:
The proof of this well-known result is quite elaborate and can be found in [15].

3.2.14 Proposition

If the vectors in Definition 3.1.3 are linearly independent, H_{S_n} can be decomposed as a direct sum of irreducible subspaces isomorphic to the U_t defined in Definition 3.2.8 with multiplicity equal to the number of standard Young diagrams of form t, that is,

$$H_{S_n} = \oplus_{\text{standard } Y_t} U_T. \tag{72}$$

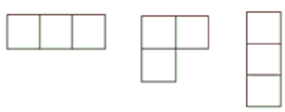
Proof:
This follows from using the results of Lemma 3.2.13 and Theorem 3.2.9 and the fact that the regular representation contains all irreducible subspaces with multiplicity equal to the dimension of the irreducible subspace.

3.2.15 Remark

If the n vectors in Proposition 4.1.1 were not linearly independent then H_{S_n} can still be decomposed into irreducible subspaces isomorphic to the U_t, but the multiplicity of these spaces is in general not given by the number of standard Young diagrams of that form.

3.2.16 Example

If ψ_1, ψ_2, ψ_3 are linearly independent, the representation u on H_{S_3} is the regular representation of the group S_3. H_{S_3} can therefore be decomposed as the direct sum of subspaces equivalent with the U_t corresponding to the three possible tableaux belonging to the partitions of 3, with multiplicity equal to the dimension of the subspace:



.

The tableau with only one row and the tableau with only one column have one and the triangular tablau has two possibilities to fill it with $1, 2, 3$ to make it a standard Young diagram. This means by applying Lemma 3.2.13 that the multiplicity of U_t with t the tableau with only one column or row is one and for t the triangular tableau it is two.

3.3 Representation of subgroups

3.3.1 Definition

The representations u and v of S_n on H_S induc unitary representations u and v of S_{n-1}, respectively, simply by restricting to the subgroup $S_{n-1} \subset S_n$ of elements that leave the number n fixed. The same goes for S_{n-2}, S_{n-3}, etc.

3.3.2 Theorem

Each space U_T as defined in Definition 3.2.10 contains an irreducible subspace $U_{T'}$ of S_{n-1} at most once and exactly once if and only if for the two partitions λ and λ', as defined by the tableau t and t',we have $\lambda' = (k_1, \cdots, k_i - 1, \cdots, k_n)$, with $\lambda = (k_1, \cdots, k_i, \cdots, k_n)$.

Proof:
We prove the claim for U_t. The case for general U_T follows from the fact that $U_T \simeq U_t$. If one can omit the last entry in the i'th row to obtain a tableau t', we definie an operator ϕ_i on the basis vectors $\{P_{Y_t}\psi_{Y_t} \mid Y_t \text{ standard}\}$ defined in Definition 3.2.13. If we write Y' for the tableau given by Y_t with the last box in the i'th row removed, ϕ_i acts on the standard basis vectors as:

$$\phi_i P_{Y_t}\psi_Y = \begin{cases} P_{Y'}\psi_{Y'} \text{ iff } n \text{ stands in the } i\text{'th row,} \\ \\ 0 \text{ iff } n \text{ does not stand in the } i\text{'th row.} \end{cases} \tag{73}$$

It is not hard to see that ϕ_i respects the action of all $u(\sigma)$, with $\sigma \in S_{n-1}$. We therefore find that ϕ_i is an intertwiner of the representation of the group S_{n-1}. We also know a basis of the image of ϕ_i: $\{P_{Y'}\psi_{Y'} \mid Y' \text{ of the form } t'\}$, which is $U_{t'}$. Therefore, U_t contains a subspace that is equivalent to the subspace $U_{t'}$.

To prove that U_t is exactly the direct sum of all these $U_{t'}$ with multiplicity one, we show that the dimension of the subspaces $U_{t'}$ combined is the same as the dimension of U_t. From Lemma 3.2.13 we know that this can be done by comparing the number of standard Young diagrams of form t with the sum of the numbers of standard Young diagrams of the forms t', which are all possible tableaux obtainable from t by removing a box.

To show that these two are equal notice that the number of all standard Young diagrams of form t is the sum over all places in which the number n can stand of all possiblilities to futher fill in the tableau to a standard Youngdiagram starting with the number n in this box. The number of ways to futher fill in a tableau with the number n in a certain box equals the number of standard Youngdiagrams of a tableau t' that is t with the box in which the number n stands is removed. This implies that the number of standard Youngdiagrams of all t' is equal to the number of standard Youngdiagrams of t.

3.3.3 Remark

Theorem 3.3.2 is equivalent to the following statement: an irreducible subspace U_T of S_n contains an irreducible subspaces $U_{T'}$ of S_{n-1} at most once, and exactly once if and only if t' is obtained from t by removing one box of t.

3.3.4 Remark

The containment of the representation of the subgroup as described in Theorem 3.3.2 can be pictured as in figure 3.3.4, in which each upward arrow means: 'being contained in'.

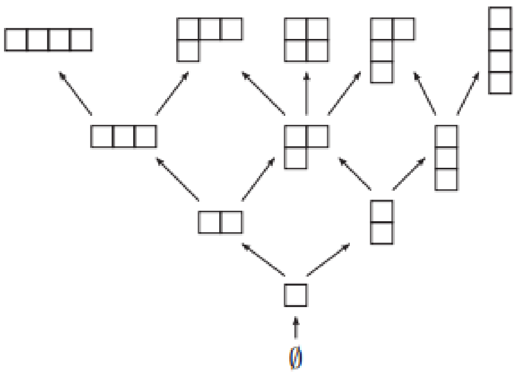


Figure 3: containment of subrepresentations [13]

3.3.5 Definition

To each upward path from $\emptyset$ to a certain tableau t in figure 3.3.4, we can assign the unit vector in U_T that simultaneously lies in all irreducible subspaces corresponding to the tableaux on this path.

3.3.6 Proposition

The vectors that correspond to all paths upwards to a certain tableau t form an orthonormal basis for U_T.

Proof:
When ψ_1 and ψ_2 in H correspond to two different paths upward to t, the path of ψ_1 passes through at least one n-sized tableau through which the path of ψ_2 does not pass. This means that ψ_1 is contained in a subspace that is inequivalent and therefore perpendicular to the subspace in which ψ_2 lies. We also have that the number of upward paths is equal to the number of standard Young diagrams of form t. This means that the set in question is an orthonormal basis.

3.3.7 Definition

We call the basis consisting of the vectors in Proposition 3.3.6 the Yamanouchi basis of U_T.

4 Indistinguishable particles

This chapter uses the theory described in chapter 2 and chapter 3 to derive the pure states and marginal pure states of indistinguishable particles.

4.1 States of indistinguishable particles

4.1.1 Definition

We define the algebra of n indistinguishable particles $\mathcal{A}_{sym_n}$ to be the subset of all operators that act on $\otimes^n H$ that commute with the representation v of S_n defined in definition 3.1.1.

$$v(\sigma^{-1})av(\sigma) = a \leftrightarrow a \in \mathcal{A}_{sym}. \tag{74}$$

4.1.2 Remark

Definition 4.1.1 has constructed an algebra that is ivariant for a permutation of paritcles $v(\sigma)$.

4.1.3 Lemma

We can write $E_{sym} : B(\otimes^n H) \to \mathcal{A}_{sym_n}$, as the conditional expectation onto the subalgebra $\mathcal{A}_{sym_n}$ as follows:

$$E_{sym}(a) = \frac{1}{\#S_n} \sum_{\sigma \in S_n} v(\sigma^{-1})av(\sigma) \tag{75}$$

Proof:
It is easily verified that E_{sym} is norm one since $E_{sym}(\mathbb{I}) = 1$. It is also not too hard to show that $E_{sym}^2 = E_{sym}$. This implies that E_{sym} is indeed a conditional expectation. We now need to show that E_{sym} indeed maps onto $\mathcal{A}_{sym}$. We find for every intertwiner, a, of v:

$$E_{sym}(a) = \frac{1}{\#S_n} \sum_{\sigma \in S_n} v(\sigma^{-1})av(\sigma) = \frac{1}{\#S_n} \sum_{\sigma \in S_n} v(\sigma^{-1})v(\sigma)a = a. \tag{76}$$

We also find that each $E_{sym}(a)$ is an intertwiner:

$$v(\sigma'^{-1})E_{sym}(a)v(\sigma') = \frac{1}{\#S_n} \sum_{\sigma \in S_n} v((\sigma\sigma')^{-1})av(\sigma\sigma') = E_{sym}(a). \tag{77}$$

This proves that the map E_{sym} is indeed onto $\mathcal{A}_{sym_n}$.

4.1.4 Theorem

All pure states of n indistinguishable particles with certain individual vectors are given by the irreducible subspaces U_T from definition 3.2.10

Proof:
This is simply applying Corollary 2.3.6 and Theorem 3.2.9.

4.1.5 Remark

From now on for simplicity we wil restrict ourselves to states the are irreducible subspaces contained in a certain H_{S_n}.

4.1.6 Example

Following Remark 4.1.5 we will for example discard the pure state that projects on:

$$\psi_1 \otimes \psi_2 + \psi_2 \otimes \psi_1 + \psi_3 \otimes \psi_4 + \psi_4 \otimes \psi_3 \not\subset H_{S_n}, \tag{78}$$

with ψ_i in irreducible subspaces of d. But we do consider:

$$\psi_1 \otimes \psi_2 + \psi_2 \otimes \psi_1 \subset H_{S_n}. \tag{79}$$

4.1.7 Remark

We can obtain the case of equation (78) by considering the linear combination of two vectors of the form of equation (79). In general all states can be seen to be linear combinations of states within certain H_{S_n}. Meaning that results can be generalized from H_{S_n} to the whole $\otimes^n H$.

4.2 Some properties of pure states

4.2.1 Definition

A state U_T corresponding to a tableau t that has only one row is is called bosonic. A state that corresponds to the tableau that has only one column is caledl fermionic. The remaining pure states are called parastatistic. For a further classification of parastatistics see [11] and [12].

4.2.2 Remark

The fermion and boson states are the only states that are also pure for distinguishable particles ($\mathcal{A} = B(\otimes^n H)$).

Proof:
The analysis of the number of standard Young diagrams gives that the only one-dimensional irreducible subspaces are given by the one row and one column tableaux. As seen in Example 2.3.11 these are the only states in which distinguishable particles without a symmetry can be.

4.2.3 Example

We can continue Example 3.2.16 to find the inditinguishable states of 3 particles with the three linearly independent individual vectors ψ_1, ψ_2 and ψ_3. It can be seen from Lemma 2.3.1 that there an infinite number of ways to choose the two irreducible subspaces that correspond to the triangle and only one for the tableau with only one row/column. This means that we find one state corresponding to the tableau with only one row, one state corresponding to the tableau with only one column and infintely many states corresponding to the triangular tableau.

4.2.4 Lemma

The bosonic and fermionic states for n indistinguishable particles with specified individual vectors are unique.

Proof:
The bosonic and fermionic states correspond with a projection on a one dimensional subspace. Since the regular representation contains each irreducible subspace as often as its dimension we find that H_{S_n} contains the given irreducible subspace at most once and therefore is uniquely defined.

4.2.5 Proposition

Indistinguishable particles in a state U_T can transit to another state $U_{T'}$ if and only if $t = t'$.

Proof:
Realizing that $U_T \simeq U_{T'}$ if and only if $t = t'$, this is just a reformulation of Proposition 2.4.9.

4.2.6 Proposition

If n particles are in a state $U_T \subset H_{S_n}$ and t is a tableau with q columns, then one can have a maximum of q equal individual vectors.

Proof:
Assume there more than q individual vectors were the same. We prove that the state U_T, in which t has q columns, is the subspace $\{0\}$. The representations u and v are equivalent on H_{S_n}, so we could just as well prove that the irreducible subspace $U_t \simeq U_T$ of the representation u is $\{0\}$.

Having more then q equal individual vectors means that we have chosen the vectors in Definition 3.1.3 such that $\psi_1 = \psi_2 = \cdots = \psi_{q+1}$. If t only has q columns, two of the numbers $1, 2, \cdots, q+1$ will occur in the same column, say i and j. This means that the vector in U_T corresponding with this Young diagram should be minus one times itself when $u(i, j)$ is applied. On the other hand, $\psi_i = \psi_j$ would imply that the vector should remain the same under this permutation. So this vector is zero. This implies that U_T only contains the zero. Though $U_T = \{0\}$ does not have the required trace 1 propery of Definition 2.2.1, since $P_{\{0\}}(\mathbb{I}) = 0$, therefore this is not a state.

4.2.7 Remark

For the case of a fermionic state, one obtains the Pauli exclusion principle [9] from Proposition 4.2.6.

4.3 Marginal states

4.3.1 Definition

Define the subset $M(\mathcal{A}_{sym_n})$ of operators of $\mathcal{A}_{sym_n}$, as all operators of the form:

$$\sum_{i=1}^{m} E_{sym}\left(a_{i_1} \otimes a_{i_2} \otimes \cdots \otimes \mathbb{I} \otimes \cdots \otimes a_{i_n}\right) \text{ with } \forall_{i,j} a_{i_j} \psi_n = 0. \tag{80}$$

4.3.2 Definition

Call the hilberts space $H' \subset \otimes^n H$ the linear space that is the direct sum of the linear spaces of the form:

$$\psi_n^{\perp} \otimes \psi_n^{\perp} \cdots \otimes \mathbb{C}\psi_n \otimes \cdots \otimes \psi_n^{\perp}, \tag{81}$$

with ψ_n occuring in one of the entries of the tensor product.

4.3.3 Lemma

$M(\mathcal{A}_{sym_n})$ as defined in Definition 4.3.1 is an algebra that acts on the space H' defined in Definition 4.3.2
Proof:
It is easily verified that all $a \in M(\mathcal{A}_{sym_n})$ map all vectors in H' to a vector in H'. We now only need to show that the set $M(\mathcal{A}_{sym_n})$ satisfies the four properties on an algebra stated in definition 2.1.1. Only the third point is non-trivial. We need to verify that $ab \in M(\mathcal{A}_{sym_n})$ when $a, b \in M(\mathcal{A}_{sym_n})$. This composition is the sum of terms of the following form:

$$(a_{i_1} \otimes a_{i_2} \otimes \cdots \otimes \mathbb{I}_k \otimes \cdots \otimes a_{i_n})(b_{j_1} \otimes b_{j_2} \otimes \cdots \otimes \mathbb{I}_l \otimes \cdots \otimes b_{j_n}) \tag{82}$$

If $l = k$ equation (82) has the required form to be in the algebra. If $k \neq l$ then if one evaluates this operator for an arbitrary vector $\psi \in H'$ it returns zero, since we have $a_{i_t} b_{j_t} \psi_n = 0$ for all i, j, t. This means that the operator in eqaution (82) is zero on H' and therefore in the algebra. Now the general case follows from taking the sum of terms of equation (82).

4.3.4 Definition

We call the algebra $M(\mathcal{A}_{sym_n})$ from Definition 4.3.1 the marginal algebra mentioned in Definition 2.5.1 with respect to ψ_n. The algebra can be further marginalized by putting in an extra identity and ignoring ψ_{n-1}.

4.3.5 Lemma

The identity $\mathbb{I}$ in equation (80) can be chosen to be in the last entry.

Proof:
We show that an arbitrary operator with the identity in the i'th entry is the same as a certain operator with the identity in the last entry:

$$\begin{aligned}
&\sum_{i=1}^{n} E_{sym}\Big(a_{i_1}\cdots\otimes\mathbb{I}\otimes\cdots\otimes a_{i_n}\Big)\\
=&\sum_{i=1}^{n}\frac{1}{\#S_n}\sum_{\sigma\in S_n} v(\sigma^{-1})\Big(a_{i_1}\cdots\otimes\mathbb{I}\otimes\cdots\otimes a_{i_n}\Big)v(\sigma)\\
=&\sum_{i=1}^{n}\frac{1}{\#S_n}\sum_{\sigma\in S_n} v(\sigma'^{-1})v(i,n)\Big(a_{i_n}\cdots\otimes\mathbb{I}\otimes\cdots\otimes a_{i_n}\Big)v(i,n)v(\sigma')\\
=&\sum_{i=1}^{n}\frac{1}{\#S_n}\sum_{\sigma'\in S_n} v(\sigma'^{-1})\Big(a_{i_1}\cdots\otimes a_{i_n}\otimes\cdots\otimes\mathbb{I}\Big)v(\sigma')\\
=&\sum_{i=1}^{n} E_{sym}\Big(a_{i_1}\cdots\otimes a_{i_n}\otimes\cdots\otimes\mathbb{I}\Big).
\end{aligned} \tag{83}$$

4.3.6 Lemma

The space $M(\mathcal{A}_{sym_n})$ is precicely given by operators of the form:

$$a=\frac{1}{n}\sum_{i=1}^{n} v(i,n)(a'\otimes\mathbb{I})v(i,n), \tag{84}$$

where $a'\in\mathcal{A}_{sym_{n-1}}$ acting on $\otimes^{n-1}\psi_n^{\perp}$.

Proof:
From Lemma 4.3.5 we know that the identity can be chosen in the last entry. Now applying E_{sym} and separating S_{n-1} and $S_n\backslash S_{n-1}$ gives:

$$\begin{aligned}
&E_{sym}\left(\sum_{j=1}^{m} a_{j_1}\otimes a_{j_2}\otimes\cdots\otimes\mathbb{I}\right)=\\
&\frac{1}{\#S_n}\left(\sum_{\sigma\in S_{n-1}}+\sum_{\sigma\in S_n\backslash S_{n-1}}\right)v(\sigma^{-1})\left(\sum_{j=1}^{m} a_{j_1}\otimes a_{j_2}\otimes\cdots\otimes\mathbb{I}\right)v(\sigma).
\end{aligned} \tag{85}$$

Realizing that elements of $S_n\backslash S_{n-1}$ consist of the elements of S_{n-1} composed with a permutation of the form (i,n), with $i\neq n$ and that $(n,n)S_{n-1}=S_{n-1}$, we can rewrite equation (85) as,

$$\frac{1}{n}\sum_{i=1}^{n} v(i,n)\frac{1}{\#S_{n-1}}\sum_{\sigma\in S_{n-1}} v(\sigma^{-1})\left(\sum_{j=1}^{m} a_{j_1}\otimes a_{j_2}\otimes\cdots\otimes\mathbb{I}\right)v(\sigma)v(i,n). \tag{86}$$

Since the sum over S_{n-1} and the multiplication by $\frac{1}{\#S_{n-1}}$ projects on $\mathcal{A}_{sym_{n-1}}$, we obtain an operator $a'\in\mathcal{A}_{sym_{n-1}}$. Finally, we find:

$$\frac{1}{n}\sum_{1\le i\le n} v(i,n)(a'\otimes\mathbb{I})v(i,n), \text{with } a'\in\mathcal{A}_{sym_{n-1}} \text{ on } \otimes^{n-1}\psi_n^{\perp}. \tag{87}$$

On the other hand we find that if $a' \in \mathcal{A}_{sym_{n-1}}$ that the operator stated in equation (84) is indeed in the algebra $\mathcal{A}_{sym_n}$.

4.3.7 Proposition

the algebra $M(\mathcal{A}_{sym_n})$ is isomorphic to $\mathcal{A}_{sym_{n-1}}$. In the sense that there is a bejective map ϕ that satisfies:

$$\begin{aligned} \phi : \ & M(\mathcal{A}_{sym_n}) \to \mathcal{A}_{sym_{n-1}} \\ & \phi(a)\phi(b) = \phi(ab) \end{aligned} \tag{88}$$

Proof:
We use Lemma 4.3.6 that all opperators in $M(\mathcal{A}_{sym_n})$ have the form of equation (84) this gives us the following isomorphism:

$$\begin{aligned} \phi : M(\mathcal{A}_{sym_n}) \to \mathcal{A}_{sym_{n-1}} \\ \phi\left(\frac{1}{n}\sum_{i=1}^{n} v(i,n)(a' \otimes \mathbb{I})v(i,n)\right) \to a' \end{aligned} \tag{89}$$

Since Lemma 4.3.6 states that $a' \in \mathcal{A}_{sym_{n-1}}$, ϕ is well defined. Futhermore it is easily verified that ϕ is indeed bijective.

Keeping the argument made for equation (82) in mind we find that:

$$\begin{aligned} &\left(\frac{1}{n}\sum_{i=1}^{n} v(i,n)(a' \otimes \mathbb{I})v(i,n)\right)\left(\frac{1}{n}\sum_{i=1}^{n} v(i,n)(b' \otimes \mathbb{I})v(i,n)\right) \\ &= \left(\frac{1}{n}\sum_{i=1}^{n} v(i,n)(a'b' \otimes \mathbb{I})v(i,n)\right). \end{aligned} \tag{90}$$

Which proves that $\phi(a)\phi(b) = \phi(ab)$.

4.3.8 Definition

Inspired on Proposition 4.3.7 we call the marginalisation M of $\mathcal{A}_{sym_n}$ a marginalisation to $n-1$ particles.

4.3.9 Theorem

The pure states of $M(\mathcal{A}_{sym_n})$ are given by the projections on subpaces of the form $U' \otimes \mathbb{C}\psi_n$, with U' an irreducible subspace of S_{n-1}. So for all pure states ω:

$$\omega(a) = \frac{1}{\mathrm{Tr}(P_{U' \otimes \mathbb{C}\psi_n})}\mathrm{Tr}\Big(aP_{U' \otimes \mathbb{C}\psi_n}\Big). \tag{91}$$

Proof:
Because of Lemma 4.3.6 we know that a functional on $a \in M(\mathcal{A}_{sym_n})$ can be regarded as a functional on the $\frac{a'}{n}$ in the expression:

$$a = \frac{1}{n} \sum_{1=1}^{n} v(i,n)(a' \otimes \mathbb{I}) v(i,n). \tag{92}$$

As was shown in Theorem 4.1.4, the pure states on the a' which form the set $\mathcal{A}_{sym_{n-1}}$ acting on $\otimes^{n-1}\psi_n^{\perp}$, are given by projections $P_{U'}$ on irreducible subspaces of S_{n-1}. The only thing we have to show is that the functionals defined in equation (91) are precicely these pure states on the a'. This follows from the fact that ψ_n should be in the last entry to give a non-zero outcome:

$$\begin{aligned} &\frac{1}{\mathrm{Tr}(P_{U \otimes \mathbb{C}\psi_n})} \mathrm{Tr}\left(\sum_{1=1}^{n} \frac{1}{n} v(i,n)(a' \otimes \mathbb{I}) v(i,n) P_{U' \otimes \mathbb{C}\psi_n} \right) \\ &= \frac{1}{n \cdot \mathrm{Tr}(P_{U \otimes \mathbb{C}\psi_n})} \mathrm{Tr}\Big((a' \otimes \mathbb{I}) P_{U' \otimes \mathbb{C}\psi_n} \Big). \end{aligned} \tag{93}$$

Now we realize that since ψ_n is bound to be in the last entry, we could just as well omit the last entry. This gives the desired pure state:

$$\frac{1}{\mathrm{Tr}(P_{U'})} \mathrm{Tr}\left(\frac{1}{n} a' P_{U'} \right). \tag{94}$$

4.3.10 Remark

As we could have expected, the states on $M(\mathcal{A}_{sym_n})$ are precicely the states on $\mathcal{A}_{sym_{n-1}}$, since $M(\mathcal{A}_{sym_n}) \simeq \mathcal{A}_{sym_{n-1}}$.

4.3.11 Definition

When we call $U_{T'}$ with t' a $n-1$ sized tableau a state on $M(\mathcal{A}_{sym_n})$, we refer to the state $P_{U_{T'} \otimes \mathbb{C}\psi_n}$ in Theorem 4.3.9.

5 Marginalisation & extension

In this chapter we calculate the marginal states and conclude with the expandabillity or non-expandabillity of the pure states.

5.1 Marginalisation of the states

5.1.1 Definition

We write P_j for the projection of $\otimes^n H$ on the space:

$$\otimes^{n-1} H \otimes \mathbb{C}\psi_j. \tag{95}$$

5.1.2 Lemma

The projection P_j defined in Definition 5.1.1 is an intertwiner of the representation v of S_{n-1} on $\otimes^{n-1} H$.

Proof:
The operator P_n leaves the first $n-1$ entries fixed or takes the vector to zero, depending on where ψ_n occurs in the tensorproduct. So this projection is an intertwiner.

5.1.3 Lemma

If $P_{U'}$ is the projection on some irreducible subspace $U' \subset \otimes^n H$ of S_{n-1}, we find

$$P_n P_{U'} P_n = \lambda_{U'} P_{U''}, \text{ with } U'' \simeq U' \text{ and } U'' \subset \otimes^{n-1} H \otimes \mathbb{C}\psi_j. \tag{96}$$

Proof:
We use the orthonormal Yamanouchi basis $\psi_1, \psi_2, \cdots, \psi_m$ of U' defined in Definition 3.3.7 to write

$$P_n P_{U'} P_n = P_n \frac{1}{m} \sum_{i=1}^{m} |\psi_i\rangle \langle\psi_i| \, P_n \tag{97}$$

Now we recall that P_n is an intertwiner of S_{n-1} by Lemma 5.1.2. This means that $P_n \psi_i = \lambda_{U'} \psi_i'$, with ψ_i' vectors in an equivalent subspace U'' that behave in the same way under the group action of v. Because the Yamanouchi basis is completely defined by the group action we recover the Yamanouchi basis of U''. Therefore we can continue equation (97) as follows:

$$\lambda_{U'}^2 \frac{1}{m} \sum_{i=1}^{m} |\psi_i'\rangle \langle\psi_i'| = \lambda_{U'}^2 P_{U''}. \tag{98}$$

Here $U'' \subset \otimes^{n-1} H \otimes \mathbb{C}\psi_n$, because one ends in equation (97) with applying P_n on the left, which is the projection on this space. By renaming $\lambda_{U'}^2$ as $\lambda_{U'}$, the claim follows.

5.1.4 Proposition

When $U \subset H_S$, the marginal state of P_U on $M(\mathcal{A}_{sym_n})$ has the following form in terms of the pure states given in Theorem 4.3.9:

$$\sum_{U' \subset U} \lambda_{U''} P_{U''}, \text{ with } U' \text{ irreducible for } S_{n-1} \text{ and } U' \cong U'' \subset \otimes^{n-1} H \otimes \mathbb{C}\psi_n. \tag{99}$$

Proof:
Using Lemma 4.3.6, we find:

$$\begin{aligned}
\omega_{P_U}(a) &= \frac{1}{\mathrm{Tr}(P_U)} \mathrm{Tr}\Big(P_U a\Big) \\
&= \frac{1}{\mathrm{Tr}(P_U)} \mathrm{Tr}\left(P_U \frac{1}{n} \sum_{1=1}^{n} v(i,n)(a' \otimes \mathbb{I}) v(i,n)\right) \\
&= \frac{1}{n \cdot \mathrm{Tr}(P_U)} \sum_{i=1}^{n} \mathrm{Tr}\Big(v(i,n) P_U v(i,n)(a' \otimes \mathbb{I}\Big) \\
&= \frac{1}{n \cdot \mathrm{Tr}(P_U)} \sum_{i=1}^{n} \mathrm{Tr}\Big(P_U(a' \otimes \mathbb{I})\Big) \\
&= \frac{1}{\mathrm{Tr}(P_U)} \mathrm{Tr}\Big(P_n P_U P_n (a' \otimes \mathbb{I})\Big) = \frac{n}{\mathrm{Tr}(P_U)} \mathrm{Tr}\Big(P_n P_U P_n a\Big).
\end{aligned} \tag{100}$$

We now use Theorem 3.3.2 to write P_U as the sum of all inequivalent (and therefore perpendicular) irreducible subspaces of S_{n-1}:

$$\frac{n}{\mathrm{Tr}(P_U)} P_n P_U P_n = \frac{n}{\mathrm{Tr}(P_U)} \sum_{U' \subset U} P_n P_{U'} P_n. \tag{101}$$

Finally, from Lemma 5.1.3 we know that this can be written as

$$\sum_{U' \subset U} \lambda_{U'} \frac{n}{\mathrm{Tr}(P_U)} P_{U''}, \text{ with } U'' \cong U', \tag{102}$$

where $P_{U''}$ are pure states on $M(\mathcal{A}_{sym_n})$.

5.1.5 Corollary

In view of Theorem 3.3.2 it follows from Proposition 5.1.4 that the state $U_{T'}$ can only be contained in the decomposition $M_*(U_T)$, when t' can be obtained from t by removing a box.

5.1.6 Corollary

When t' can be obtained from t by removing a box, there is no $U_{T'}$ corresponding to the tableau t' contained in the decomposition of $M_*(U_T)$ if and only if $U'_{T'} \perp \otimes^{n-1} H \otimes \mathbb{C}\psi_n$ for the $U'_{T'} \subset U_T$

Proof:
It follows from Proposition 5.1.4 that the expression $P_n P_{U'_{T'}} P_n$ in Proposition 5.1.4 would be zero if it were perpendicular and non-zero if it were not perpendicular.

5.1.7 Remark

A vector $\psi \in U$ is perpendicular to $\otimes^{n-1} H \otimes \mathbb{C}\psi_n$ if and only if we can write ψ in the following manner as a linear combination of the standard basis vectors defined in Definition 3.1.3:

$$\psi = \sum_{\sigma \in S_n} \lambda_\sigma v(\sigma) \psi_1 \otimes \psi_2 \otimes \cdots \otimes \mathbb{C}\psi_n, \tag{103}$$

with λ_σ zero whenever ψ_n is in the last entry of the the tensor product.

5.1.8 Corollary

The proof of Proposition 5.1.4 describes how to write a marginal state in terms of the pure states on $M(\mathcal{A}_{sym_{n-1}})$ The general case follows by taking linear combinations of these pure states U, namely:

$$M_*(\omega) = M_* \left(\sum_{U \subset H_{S_n}} \lambda_U P_U \right) = \sum_{U \subset H_{S_n}} \lambda_U M_*(P_U). \tag{104}$$

5.2 Expandability of the pure states

5.2.1 Lemma

Every state has an i-step extension.

Proof:
This is clear, as we can extend every functional ω on a finite dimensional linear subspace $M(\mathcal{A}_{sym_n}) \subset \mathcal{A}_{sym_n}$ to a functional ω' on the whole $\mathcal{A}_{sym_n}$. For example by choosing ω' equal to ω on all $\psi \in M(\mathcal{A}_{sym_n}) \subset \mathcal{A}_{sym_n}$ and zero on $\psi \in M(\mathcal{A}_{sym_n})^\perp$. These two sets form the whole algebra $\mathcal{A}_{sym_n}$ since it is finite dimensional. Repeating this argument i times gives an i-step extension.

5.2.2 Proposition

A pure state has a pure i-step extension (not necessarily unique).

Proof:
It follows from Proposition 5.2.1 that for every marginal state $P_{U'}$ there is a certain mixture of pure states $\sum_U \lambda_U P_U$ that marginalizes to this state:

$$P_{U'} = M_* \left(\sum_{U \subset H} \lambda_U P_U \right) = \sum_{U \subset H} \lambda_U M_*(P_U). \tag{105}$$

This can only be a pure state if all $M(P_U)$ marginalize to $P_{U'}$. But this means that each P_U contained in the decomposition is a pure extension of $P_{U'}$. Repeating this argument i times gives a pure i-step extension.

5.2.3 Lemma

If n particles are in the bosonic/fermionic state then marginalizing to a state with $n-1$ particles once again gives a bosonic/fermionic state.

Proof:
If n particles are in a bosonic/fermionic state their state corresponds with a tableau that only has one row/column. The only tableaux t' that can be obtained from this tableau by removing one box is again the row/column tableau with $n-1$ boxes. This implies that the only irreducible subspace of S_{n-1} contained in the irreducible subspace of S_n corresponding to the one row/column is again the one row/column, this time with $n-1$ boxes. So in the light of Proposition 5.1.4, marginalizing a bosonic/fermionic state again results in a bosonic/fermionic state.

5.2.4 Remark

We find from Lemma 5.2.3 that the bosonic/fermionic state has a pure i-step extension that is again the bosonic/fermionic state, since the i-step marginalisation of this the bosonic/fermionic state delivers the bosonic/fermionic state. As is shown explicitly in [10] one can also extend the bosonic/fermionic state to a state that corresponds to a triangular state. This gives rise to at least four paths that expand through the whole chain of algebras. The branch of all bosonic states, the brach of all fermionic states and the paths from the one particle state to the two sized bosonic/fermionic state to the triangular and from there on a path of pure extensions that is by Proposition 5.2.2 guaranteed to exist.

5.2.5 Lemma

For each t and t' there exist states U_T on $\mathcal{A}_{sym_n}$ and $U_{T'}$ on $M(\mathcal{A}_{sym_n})$, with the following property: $U_{T'}$ is contained in the decomposition of $M_*(U_T)$ if and only if t' can be obtained from t by removing one box.

Proof:
We recall from Corollary 5.1.5 that a certain state $U_{T'}$ is contained in the decomposition of $M_*(U_T)$ if and only if t' is obtained from t by removing a box and $P_n U'_{T'}$, with $U_{T'} \simeq U'_{T'} \subset U'_T$, is non-zero. So we need to prove that there is a subspace U_T that contains an $U'_{T'}$ for which $P_n U'_{T'}$ is non-zero. To this end we use the P_j defined in Definition 5.1.2, for which we know that $\sum_{i=1}^n P_j$ is the identity. We find that there is a j for which we have that $P_j U'_{T'}$ is non-zero, for $U'_{T'} \subset U'_T$. Now using the representation u of S_n from Definition 3.1.5 we know that there is a j such that: $P_n u(j,n) U'_{T'}$ is non-zero. We know from Lemma 3.1.9 that $u(j,n)$ is a unitary intertwiner, so $u(j,n)U'_T$ is still an irreducible subspace corresponding to t. We now take the state U_T we were looking for to be:

$$U_T = u(j,n)U'_T. \tag{106}$$

This irreducible subspace contains the subspace $u(j,n)U'_{T'}$ just constructed to satisfy:

$$P_n u(j,n)U'_{T'} \neq 0. \tag{107}$$

Hence we we know that there is a subspace $U_{T'}$ contained in the decomposition of the marginal state $M_*(U_T)$, with U_T as in equation (106).

5.2.6 Lemma

For each non-rectangular tableau t there exists a subspace U_T, such that the pure state U_T does not marginalize to a pure state.

Proof:
Since t is non-rectangular, there are at least two distinct tableaux t', t'' that can be obtained from t by removing a box. We can now use Lemma 5.2.5 to the effect that there are at least two states U'_T and U''_T such that the decomposition of $M_*(U'_T)$ contains $U_{T'}$ and the decomposition $M_*(U''_T)$ contains $U_{T''}$. If these marginal states are not pure there is nothing left to show, so we assume the states $M_*(U'_T)$ and $M_*(U''_T)$ to be pure. Take two equivalent vectors $\psi' \in U'_T$ and $\psi'' \in U''_T$ and take U_T to be the irreducible subspace generated by $\frac{1}{\sqrt{2}}(\psi' + \psi'')$. The state U_T now marginalizes to a mixture of pure states that contains both $U_{T'}$ and $U_{T''}$ with weights $\frac{1}{2}$.

5.2.7 Theorem

In view of Definition 2.4.9 the fermionic and bosonic states are the only states that only allow transitions to states that are expandable through the entire chain of algebras:

$$\mathcal{A}_{sym_1} \subset \mathcal{A}_{sym_2} \subset \mathcal{A}_{sym_3} \subset \cdots . \tag{108}$$

Proof:
We know From Proposition 2.4.9 that a bosonic/fermionic state can only have a transition to another bosonic/fermionic state, and from Lemma 5.2.3 and Remark 5.2.4 we know that all these bosonic/fermionic states expand through the whole chain of algebras. We now show that a parastatistic allows a transition to a non-expandable state. Let U_T be a n-particle state, with t not the row or column (in other words not the bosonic or fermionic state). If t is non-rectangular we know from Lemma 5.2.6 that there is a state U_T' such that $M(U_T')$ is not pure. This means that U_T' cannot be expanded throug the whole chain of algebras. Since $U_T' \simeq U_T$, we know from Proposition 2.4.9 that a transition to this state is possible. If t would be rectangular, then a one-step marginalisation of this state would give a state corresponding to a non-rectangular tableau, on which one can again apply the argument for non-rectangular tableaux.

6 Conclusion

From Theorem 5.2.7 we find a property that distinguishes the bosonic and fermionic states from the parastatistics. When we consider n electrons, these particles are actually in an entangled state with all electrons in the universe. One might expect that therefore a pure n particles state can always be seen as a pure $n \pm m$ particle state, for an arbitrary $m \in \mathbb{N}$. This is indeed the case for the bosonic and fermionic states, however this fails if one allows parastatistics. A parastatistic state always allows a transition to a state that is not expandable and therefore cannot be seen as a pure state for all $n \pm m$ particles.

Remark 5.2.4 yields to the existence of at least two expandable parastatistic states for $n > 2$ indistinguishable particles. This means that the argument against parastatistics is not as strong as found in [7], which claims that each parastatistic state is not one-step extendable to a pure state, which is in contradiction with Proposition 5.2.2.

A question that this paper leaves unanswered is whether or not for each path upwards from $\emptyset$ in figure 3, exists a pure state that expands via this path.

Furthermore are the bosonic and fermionic states the only states that can stay in the same symmetry type when one extends or marginalizes a state, namely fully symmetric or antisymmetric. A parastatistic state always marginalizes or extends to a state of another symmetry type. One always disturbs the symmetry type by adding or removing particles.

The equation of motion in quantum theory is not treated in this paper. If we would include it, we find that the hamiltonian has to be, as all other observables, an intertwiner of v. This means that the equation of motion respects the action $v(\sigma)$, so the symmetry type U_T of a state should be conserved in time. Meaning that introducing the equation of motion would not chage that much, it will only cause the individual vectors of the particles to gradually change in time, but not the permutation symmetry. The only thing one has to realize in this case is that the conditions on $\psi_1, \psi_2, \cdots, \psi_n$ have to be verified for each time t_0 seprately. It may for example occur that the states of the individual particles come to overlap after a certain amount of time and hence a marginalisation to $n-1$ particles gets impossible.

As a final remark it may be pointed out that the content of this paper is actually very general, it can easily be used to analyse other phenomena. If one is dealing with a situation with a certain symmetry (for example rotational) one should take the algebra the intertwiners of this action. Secondly, one should proceed to find the irreducible subspaces of this symmetry, as done in chapter 3 for S_n and conclude that the pure states are given by the projections on these spaces, similar to chapter 4.

References

[1] S.N Bose, Plancks gesetz und lichtquantenhypothese, Z. Physik, 26:178-181, 1924

[2] A.M.L Messiah and O.W. Greenberg, Symmetrization postulate and its experimental foundation, Phys. Rev. 136, B248B267, 1964

[3] Roger Bowley, *Introductory Statistical Mechanics*, Oxford Science Publications, 2011

[4] Jeremy Booher, Representations of the symmetric group via Young tableaux, stanford.edu, 2009

[5] Hermann Weyl, *The Classical Groups : Their Invariants and Representations*, Princeton University Press, 1997

[6] Hermann Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Publications, 1950

[7] Alexander Bach, *Lecture Notes In Physcis*, Springer, 1997

[8] Frans Clauwens, Symmetrie en trransformatiegroepen, Radboud, 2005

[9] David Griffiths, *Introduction To Quantum Mechanics*, Pearson Education, 2005

[10] Robert Stolt and John Taylor, The quantummechanics of paraparticles, Physical Review,2043- 2051 Vol. 1, 1967

[11] Robert Stolt and John Taylor, Clasiffication of paraparticles, Physical Review, 2226-2228 Vol. 1, 1970

[12] Robert Stolt and John Taylor, Paraparticles of infinite order, Physical Review, 1759-1760 Vol. 2, 1979

[13] Jin-Quan Chen, *Group Representation Theory for Physicists*, World Scientific, 1987

[14] Yufei Zhao, Young Tableaux and the Representations of the Symmetric Group, The Harvard College Mathematics Review, 2008

[15] B. E. Sagan: *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Springer, 2001

[16] W.B. Arveson, *An Invitation To C*-algebras*, Springer, 1976.