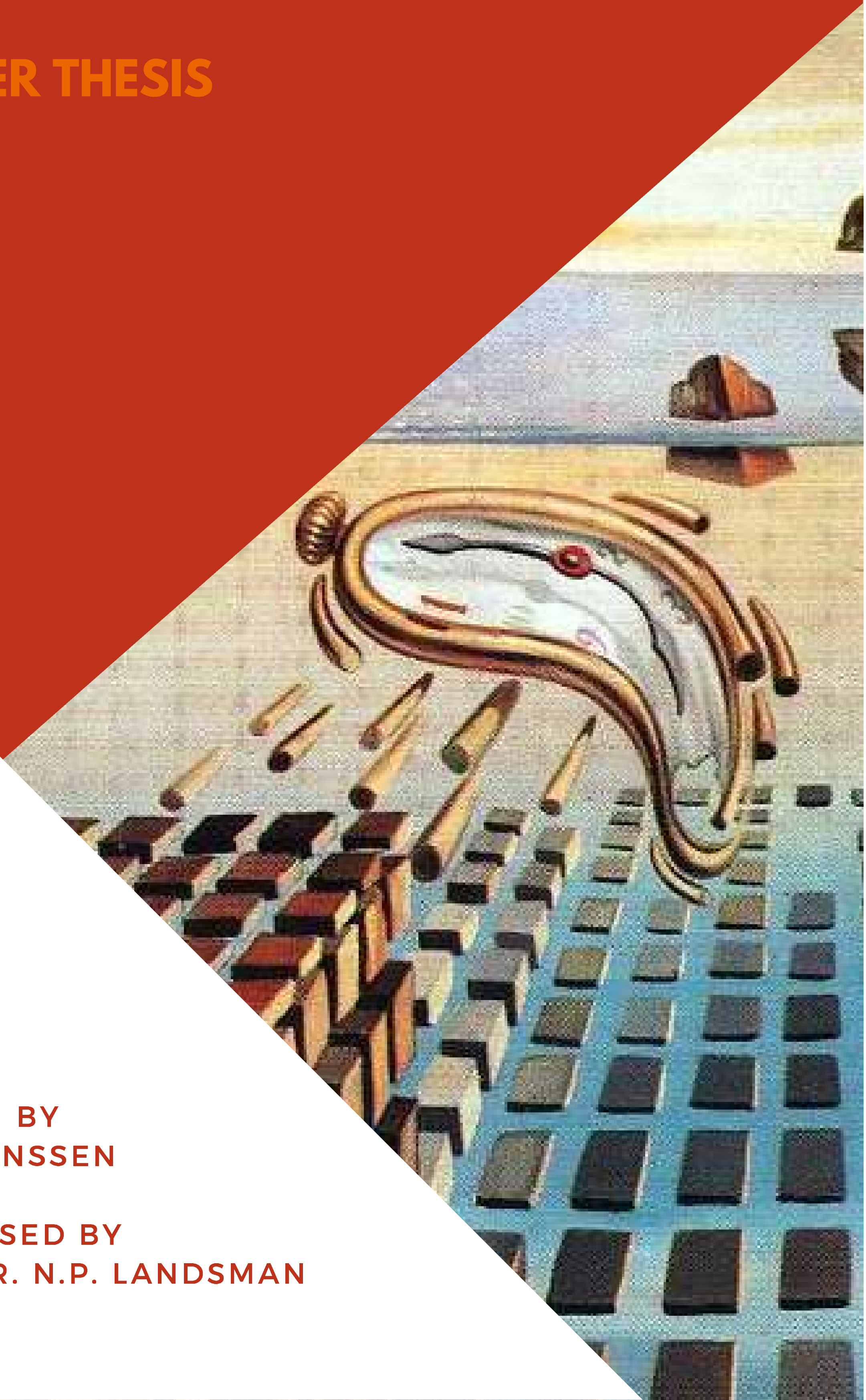
QUANTUM FIELDS ON NON-GLOBALLY HYPERBOLIC SPACE-TIMES AND THE INFORMATION LOSS PARADOX

MASTER THESIS



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Cover art: The Disintegration of the Persistence of Memory - Salvador Dalí

Abstract

The information loss paradox has been a source of controversy ever since Stephen Hawking predicted black hole evaporation in 1975. The suggestion that when quantum fields and (classical) black holes interact, a pure initial state evolves into a mixed final state has many physicists up the fence, in particular when this theorized effect is extrapolated to the realm of quantum gravity. There are those who are more willing to accept information loss, based on insights from algebraic quantum field theory. However, we argue that quantum field theories are not well understood on non-globally hyperbolic space-time such as the black hole evaporation space-time. In this thesis, we study linear scalar algebraic quantum field theories on a class of not necessarily globally hyperbolic space-times, which we dub semi-globally hyperbolic space-times and construct a concrete quantum field theory on the black hole evaporation space-time. While it was originally believed that any pure (or mixed) initial state consistent with black hole formation would evolve to a mixed but uniquely determined final state, we show that in our constructed theory, this is not the case. One either has to impose additional conditions on the state-space, or assume that quantum gravity corrections will sufficiently alter the theory, if one wants to hold on to the idea that a final state should be uniquely fixed by an initial state in black hole formation and evaporation.

Contents

Chapter 1

Introduction to the paradox

In 1975, Stephen Hawking first predicted black hole evaporation as a result of a black hole losing energy via thermal radiation due to quantum effects (Hawking, 1975). It is fair to say that this prediction has sparked controversy that lasts until this very day. Since Hawking's original paper, many have replicated his results on thermal radiation from black holes, now called Hawking radiation, using arguably more sophisticated arguments (e.g. Kay and Wald, 1991; Fredenhagen and Haag, 1990). While the validity of these arguments has been called into question (Helfer, 2003; Gryb, Palacios, & Thebault, 2018), mostly on grounds that quantum gravity effects may significantly alter the radiation spectrum, the hypothesis that black holes lose energy via radiation is widely accepted among the physics community. An important motivation for accepting the hypothesis of Hawking radiation is that it completes an analogy of black holes with thermodynamical systems, allowing a well-defined temperature and entropy to be associated with the black hole such that the laws of thermodynamics can be applied to these gravitational systems (Wall, 2018). As is the case with the behaviour of classical thermodynamical systems finding its origin in statistical/quantum mechanical underpinnings, one may assume that black hole thermodynamics provides clues to an underlying theory of quantum gravity. From this perspective, Hawking radiation is a very appealing phenomenon to a theoretical physicist. However, as mentioned earlier, it has also been the cause of much debate and controversy.

Suppose a black hole emits thermal radiation. This radiation carries energy away from the black hole to infinity. The radiation carries no information about the matter that has formed the black hole, but only about its mass, charge and angular momentum, in accordance with the no hair/black hole uniqueness theorems (e.g. Wald, 1984b). Since the black hole loses energy, it will shrink, or partially evaporate. If we assume this process to continue all the way till the point that the black hole has radiated away all of its mass and has completely evaporated, the information on what originally made up the black hole, such as the state of this matter and the entanglement that correlated matter inside and outside the event horizon, cannot be inferred from the final state. all that is left after evaporation is radiation in a thermal/mixed state. The information that was once contained within the event horizon is lost to us. This supposedly contradicts very foundational assumptions about the nature of our universe, like the retrodictability of time-evolution, suggesting that either we should abandon some of these assumptions, or we should somehow circumvent the conclusion that information is lost. This tension between strongly held beliefs in physics is *the black hole information loss paradox.*

1.1 Setting the stage

1.1.1 The radiating black hole

Before we go into the paradox in more detail, we review some arguments for black hole evaporation. We will only cite some of the key results.

If one puts a quantum field theory on a static (Schwarzschild) black hole background such that an observer at infinity measures the field to initially be in a vacuum state, after some time the observer will register radiation, known as Hawking radiation. By making certain assumptions, one can calculate that at late times, this radiation has a thermal spectrum, up to some grey-body factor due to reabsorbsion of radiation by the black hole, with Hawking temperature $T_H = (4\pi R_s)^{-1}$ where R_S is the Schwarzschild radius (Hawking, 1975; Fredenhagen & Haag, 1990). Since then, many techniques have been developed to study radiation effects of more general black holes. For instance Kay and Wald (1991) prove that under certain assumptions one can assign a Hawking temperature

$$T_H = \frac{\kappa}{2\pi}, \tag{1.1}$$

to space-times with a so-called bifurcate Killing horizon where κ is the surface gravity at the Killing horizon. For the Schwarzschild black hole this reduces to Hawking's original result. It should be noted that less is known about the grey-body factors for a general black hole. The results mentioned so far are mostly concerned with what is measured by an observer at infinity.[1] Far less is understood about the Hawking radiation that would be measured at a finite distance from the black hole, which would tell us where the radiation actually originates. This was for instance studied by Davies, Fulling, and Unruh (1976) and Unruh (1977).

As mentioned before, the validity of the assumptions on which both Hawking's original calculation and subsequent ones are based, are not uncontroversial. We will further discuss this in section 2.1.2. We also have no direct experimental evidence that would back-up the hypothesis that black holes radiate. Measuring the Hawking radiation of an astrophysical black hole is rather unfeasible. A Schwarzschild black hole of solar mass would have a temperature of about 6×10^{-8} K. Comparing this to the fluctuations in the cosmic microwave background, which are of the order 10^{-4} K around a mean temperature of around 2.8 K, it is clear that directly observing Hawking radiation from black holes is not possible.[2] As the temperature of black holes increases as they get smaller,

[1] In calculating the Hawking spectrum, one considers the radiation that is emitted to future null infinity.

[2] Astrophysical black holes are typically much heavier than our sun, so they will also be much colder.

one could hope to observe Hawking radiation from microscopic black holes,[3] yet neither these objects, nor the thermal radiation that might be attributed to them have so far been discovered (e.g. Chatrchyan et al., 2013). Does this mean that there is no experimental support for Hawking radiation? Unruh (1981) proposed that a sonic analogue of a black hole may also produce an effect analogous to Hawking radiation, which can be measured in a laboratory. Since then, various experiments on systems analogous to black holes (sonic or otherwise) have been proposed and conducted. Unfortunately, none of these experiments have given direct evidence of spontaneous Hawking radiation, though in a recent paper (Drori, Rosenberg, Bermudez, Silberberg, & Leonhardt, 2019) it is claimed that stimulated (as opposed to spontaneous) Hawking radiation has been measured in an optical analogue. As argued by Unruh (2014), experiments like these cannot directly prove anything about radiation from actual black holes. Nonetheless, if the derivations of thermal radiation effects in gravitational and analogue systems match up very closely, one can take this as circumstantial evidence for the claim that black holes radiate.

We have seen that experimental evidence for Hawking radiation is at the very best scarce and that the calculations from which the effect is derived, are somewhat rocky. Nevertheless, (Hawking, 1975) is one of Hawking's most famous papers. Why is it that an effect that has never been observed and which does not follow from well established theory has made such a large impact on the physics community?[4] Although there is not one single answer to this question, I will highlight three reasons why Hawking radiation has been getting so much attention in physics (and philosophy) literature for over the past 40 years.

First of all, Hawking radiation in one form or the other is one of the only concrete predictions we have from the interplay of quantum field theory and gravitation. Quantum field theory has been shown to be very successful in describing matter and their interactions. The Standard Model of particle physics, which is constructed in the QFT framework, is a well established description of our universe at very small scales with great predictive power. The most notable shortcoming of this model is that it does not incorporate gravity as an interaction between matter fields. Typically, a quantum field theory is defined on some fixed classical background space-time geometry (usually Minkowski space-time). In classical physics, in particular in general relativity, the geometry of space-time, which affects the dynamics of matter that lives on it, is itself dynamical and is directly influenced by that same matter, making the Einstein equations and the equations of motion for the matter fields a system of coupled partial differential equations. If the space-time is taken as fixed, and thus the backreaction of matter on geometry via the Einstein equations is ignored, this means that matter fields will not interact gravitationally with each other. A theory of quantum gravity should incorporate gravity into the quantum theory framework.[5] Often this involves attempting to quantize geometry, such as in canonical quantum gravity, or introducing more fundamental degrees of free-

[3]These could have been formed in the early universe or in high energy particle collisions

[4]A quick search on the arXiv shows that multiple articles related to this effect are uploaded every week.

[5]or quantum fields into general relativity/some alternative theory of space-time geometry, depending on your perspective

dom, such as string theory (Kiefer, 2004). Unfortunately, despite decades of research that have gone in to this field, no candidate theory for quantum gravity has yet been shown to meet the standards of a trustworthy description of our universe. It should come as no surprise, then, that the Hawking effect as currently understood is not a result of a theory of quantum gravity, or at least, not in the first place. It comes about when placing a quantum field theory on a fixed classical black hole background.[6] Nevertheless, it has given us a better understanding of the interplay between gravity and quantum fields, and it continues to raise questions which push our understanding on this further to this very day.

The second reason for the interest in Hawking radiation is the aforementioned analogy between black holes and thermodynamical systems. In (Bekenstein, 1973) and references therein, it was argued that the area of the event horizon of a black hole will not decrease by any classical process, nor will the total area of a black hole that has formed from the merging of other black holes be smaller than the sum of the areas of the originals. This was taken to suggest that black holes could be studied from a thermodynamical point of view, as in this theory it is the entropy of a system that will never decrease, as stated in the second law of thermodynamics. This led to a formulation of a black hole version of the first law of thermodynamics (Wall, 2018). Given a Kerr-Newman black hole of (effective) mass M, angular momentum $\vec{J}$ and charge Q, the following equation holds:

$$dM = \frac{1}{8\pi}\kappa dA + \vec{\Omega}\cdot d\vec{J} + \Phi dQ, \tag{1.2}$$

with κ the surface gravity at the event horizon, A the area of the horizon, Ω the angular velocity of the horizon, and Φ the electric potential.[7] It is clear that the mass of the black hole is a good analogue of the internal energy of the system, while $\Omega dJ + \Phi dQ$ can be associated with the work done on the system. It is therefore tempting to identify $\frac{1}{8\pi}\kappa dA$ with the heat exchange TdS, where T is the temperature and S the entropy of the system. At the time where (1.2) was first discovered, black holes had not been assigned any finite temperature. Since

[6] Sometimes one imposes the semi-classical Einstein equations $R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R = \langle \hat{T}_{\mu\nu}\rangle$ to estimate the backreaction of quantum fields on the space-time, however, this comes with its own problems, in particular the renormalization ambiguity of the expectation value of the energy-momentum tensor $\langle \hat{T}_{\mu\nu}\rangle$. More details can be found in (Wald, 1994).

[7] For a Kerr-Newman black hole, the surface gravity is given by

$$\kappa = \frac{\sqrt{M^2 - Q^2 - \frac{J^2}{M^2}}}{2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - \frac{J^2}{M^2}}},$$

the angular velocity is

$$\vec{\Omega} = \frac{\vec{J}}{M\left(2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - \frac{J^2}{M^2}}\right)},$$

and the electric potential is

$$\Phi = \frac{Q\left(M + \sqrt{M^2 - Q^2 - \frac{J^2}{M^2}}\right)}{2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - \frac{J^2}{M^2}}},$$

(Bekenstein, 1973).

classically black holes do not radiate, the only sensible physical temperature that could be associated with a black hole was $T = 0$. Nevertheless, the analogy between entropy and area of the event horizon could be taken seriously, while the analogy between temperature and surface gravity was at that point only formal. This changed when Hawking radiation was first derived. From then on, it made sense to associate a finite temperature to a black hole, which happened to be proportional to the surface gravity of a black hole, as can be seen in (1.1). This means that the analogy between black hole dynamics and thermodynamics suddenly became much more direct, and allowed to fix the value of the black hole/Bekenstein entropy as

$$S_B = \frac{A}{4}. \tag{1.3}$$

Since thermodynamics can be seen as a macroscopic theory with microscopic origins via (quantum) statistical mechanics, many believe that a similar origin can be found for black hole mechanics (Wallace, 2017). In particular, a theory of quantum gravity that underlies general relativity and black hole mechanics should associate an entropy to macroscopic black holes, which could for instance be defined by counting microstates, just as in ordinary statistical mechanics, that matches the Bekenstein entropy. Such a requirement could place serious bounds on what a sensible theory of quantum gravity can be. Calculations in effective field theory (e.g. Gibbons and Hawking, 1977), loop quantum gravity (e.g. Rovelli, 1996) and string theory for so called extremal black holes (e.g. Strominger and Vafa, 1996) have indeed reproduced the Bekenstein entropy. Black hole thermodynamics has also played a major role in motivating the holographic principle, culminating in the AdS/CFT correspondence (Bousso, 2002). Using this correspondence, it has been shown that one can calculate the entanglement entropy of certain conformal field theories by applying the Bekenstein entropy formula (1.3) to certain surfaces in anti-de Sitter space-time (Ryu & Takayanagi, 2006), providing further ground for the relevance of black hole thermodynamics.

The third reason for the major role that Hawking radiation plays in physics discourse (and the most relevant reason for this thesis), is that it leads to black hole evaporation, which in turn leads to the famous information loss paradox.

1.1.2 The evaporating black hole

Already in Hawking's original paper (Hawking, 1975) it was noted that the Hawking radiation produced by a black hole results in an outgoing energy flux near infinity and an ingoing negative energy flux at the event horizon. This led Hawking to the conclusion that black holes lose their energy due to Hawking radiation and shrink, after all the Schwarzschild radius is proportional to the mass/energy of the black hole. If one assumes black holes of any size radiate their energy, this means that the black hole will continue to shrink untill it eventually disappears. The evaporation process as Hawking envisioned it is given by the Penrose diagram in figure 1.1. It should be noted that, since the Hawking effect is in a sense semi-classical,[8] we cannot conclude much about the evaporation

[8] We treat the space-time as a classical object while describing the matter content of the universe by quantum fields.

of Planck scale black holes (assuming such an object makes physical sense). This is simply because we have no clue on what a good theory of Planck scale phenomena (i.e. quantum gravity) is. Therefore, we cannot trust the Hawking effect beyond a low curvature regime, at least compared to the Planck scale. This is already clear from Hawking's result on the black hole temperature (1.1), which diverges as the size of the black hole goes to 0. Typically, divergences in physics signal a breakdown of a theory.[9] We should therefore expect that in a fully developed theory of quantum gravity, the Hawking temperature will at the very least be modified for Planck scale black holes, such that it remains finite. Such a modification could completely alter the evaporation process, as this might mean that black holes will not fully evaporate. We will expand on this in chapter 2. For the purposes of the present chapter, we will assume that a black hole will continue to lose energy (though the Hawking temperature or radiation spectrum may be modified), such that the black hole will entirely disappear after a finite amount of time (as measured by a stationary observer at the asymptotically flat infinity). Even then we cannot say with certainty what the causal structure of the resulting space-time will be, as we will also explore further in section 2.1.3. However, for now we also assume that a fully evaporating (Schwarzschild) black hole has a causal structure as in figure 1.1.

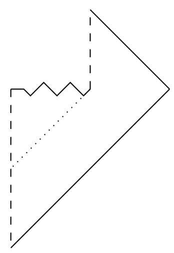

Figure 1.1: Penrose diagram of a fully evaporating astrophysical black hole

How should we read figure 1.1?[10] This conformal diagram is a reduced version of a four dimensional diagram, where each point in the diagram represents a 2-sphere, with the exceptions of points on the dashed line, which represent points on $r = 0$, the axis of rotational symmetry. The zigzagged line represents the black hole singularity and is not part of the smooth space-time, whereas the solid line represents conformal infinity, which is not part of the space-time either. Finally, the dotted line represents the event horizon. It should be noted that this diagram is not an official Penrose diagram such as it is defined in the appendix, as it is not compact. After all, due to the causal structure of the space-time it is not possible to include a point where the singularity and the event horizon meet (informally referred to as the evaporation event) as a point on the conformal boundary (Manchak & Weatherall, 2018).[11] Other than this

[9]Examples of this are singularities, non-renormalizability etcetera.

[10]An introduction to Penrose diagrams can be found in appendix A.2.

[11]One could try to see what this 'point' looks like on the c-boundary of the space-time, as defined by Flores, Herrera, and Sanchez (2011), in this case one finds that the evaporation event is not a single point, but rather a collection of points with a shared future, but differing pasts (the amount of which depends on the space-time dimension, for instance for 1+1 di-

peculiarity, we can treat this diagram as if it were a Penrose diagram and we will further explore its properties in section 4.2. We should nevertheless take the diagram with a grain of salt, as it is rather schematic, heavily based on semi-classical arguments and, since we lack a good method of dynamically evolving space-time coupled to a quantum field theory, there is also no rigorously derived space-time that solves any appropriate field equations and corresponds to the diagram. On the other hand, the evaporation process has been studied by modeling Hawking radiation as pressureless null dust originating near the apparent horizon, carrying negative energy through the horizon and positive energy to infinity, which allows the geometry in the vicinity of the horizon and far away from the black hole to be modeled by the ingoing and outgoing Vaidya metrics respectively. This is known as the Hiscock model (Hiscock, 1981) and allows for a more explicit construction of Penrose diagrams for black hole evaporation (Schindler, Aguirre, & Kuttner, 2019). However, these models are only an approximation, as we do not know the exact behaviour of Hawking radiation at finite distance from the black hole. Another dubious feature of the diagram in figure 1.1 is that it exhibits a naked singularity as the 'evaporation event'. One may question if it is realistic to try to represent such a structure by a classical geometry. This is because we expect that in quantum gravity the nature of singularities is changed.[12] In a space-time where the entire singularity is hidden behind an event horizon, the exact structure of the space-time around the singularity is not expected to have great influence on global causal properties. In the case of a naked singularity however, this is a different matter. Here we may expect that global causal properties are at the mercy of the local causal structure around the naked singularity.[13] However, until we have a satisfying theory of quantum gravity, we have not much more to go on than figure 1.1.

We have seen that Hawking radiation, and in particular the thermodynamics that it suggests, may guide us in uncovering a theory of quantum gravity. On the other hand, a semi-classical treatment of Hawking radiation suggests that black holes, when left undisturbed, lose all of their energy over time and disappear. It is this last fact that has led many to believe that the existence of Hawking radiation, at least in the semi-classical framework, presents us with a paradox. Principles in physics that we hold very dear, turn out to be violated when black holes evaporate. In the following sections we explain the nature of this paradox.

1.2 Stating the paradox

Continuing his work on black hole evaporation, Stephen Hawking published the article "Breakdown of predictability in gravitational collapse" (Hawking, 1976), in which he argues that when a black hole evaporates due to Hawking radiation, the state of the quantum fields on the geometry will evolve from an initially pure state into a mixed state after full evaporation. Taking a look

mensions there are two points on the c-boundary that can be associated with the evaporation event, one for each 'half' of the space-time represented by figure 1.1 after symmetry reduction. It is clear that these points on the c-boundary cannot represent any points in a space-time.

[12]After all, the existence of singularities in general relativity is used to partially motivate the need for quantum gravity.

[13]Though we should note that we do not actually know the meaning of causal structure in quantum gravity.

at figure 1.2, suppose the quantum fields on the black hole evaporation spacetime are in a pure state at hypersurface Σ_1. This evolves to a pure state at Σ_2,[14] which has components inside and outside the event horizon. Due to the "particle" production in the Hawking effect,[15] entangled pairs of (positive energy) particles emitted towards infinity and (negative energy) particles passed through the event horizon cause the degrees of freedom on the external and internal parts of Σ_2 to be highly correlated.[16] This means that the state on the internal and external parts separately are mixed.[17] In other words, the Hawking radiation of a black hole is in a mixed state, while the system as a whole is in a pure state. So far so good; this is the case for any body emitting thermal radiation. The essential difference in this scenario is that whereas for an ordinary radiating body the internal degrees of freedom continue to exist, and as such the full state of the system that contains both the radiation and the radiator remains pure, in black hole evaporation the black hole will disappear and the degrees of freedom inside the black hole will at some point no longer coexist with the Hawking radiation. In figure 1.2 we consider the surface Σ_3, which cannot be extended to a hypersurface that crosses the black hole region (see Manchak and Weatherall, 2018), which means that on this surface we only register the mixed state of the Hawking radiation. Therefore, if we consider evolution from Σ_1 to Σ_3, we have gone from a pure state to a mixed state and thus we have lost information (Unruh & Wald, 2017).

Unlike in closed quantum systems in quantum mechanics or quantum field theory in flat space, the time-evolution is therefore not unitary, since such an evolution would not allow pure to mixed transitions. Neither is it invertible, as we cannot uniquely recover the state of the quantum fields before evaporation from the state after evaporation.[18] This is phrased by Hawking as the claim that there is no (unitary) S-matrix that relates the initial and final state.

1.2.1 A clash of assumptions

Following (Manchak & Weatherall, 2018), we consider something a (seeming) paradox if there are two or more strongly held beliefs/assumptions that (seemingly) lead to a contradiction. A paradox must either be *resolved* by adjusting or giving up one or more of the underlying assumptions such that there is no contradiction anymore, or *solved* by showing that the arguments that derive the contradiction are wrong. It must be said that the distinction between resolving and solving is not as clearcut as it seems. Often, next to the conflicting beliefs that are presented when stating a paradox, there are numerous presupposed assumptions that motivate for instance the consistency of a calculation or define the framework in which the paradox takes place. Therefore, a paradox may be "solved" by challenging some hidden assumption, so that it is unclear how such

[14] This is because they share a domain of dependence.

[15] The notion of a particle is not so clearcut in curved space QFT's, hence the scare quotes.

[16] Note furthermore that matter that falls into the black hole can also be entangled with matter that stays outside the event horizon during evaporation, causing even more entanglement between the internal and external degrees of freedom.

[17] As we will note in section 2.2.1, this is a very general feature of quantum field theory.

[18] In particular, we cannot reconstruct the internal state of the black hole and the correlations between the external and internal state.

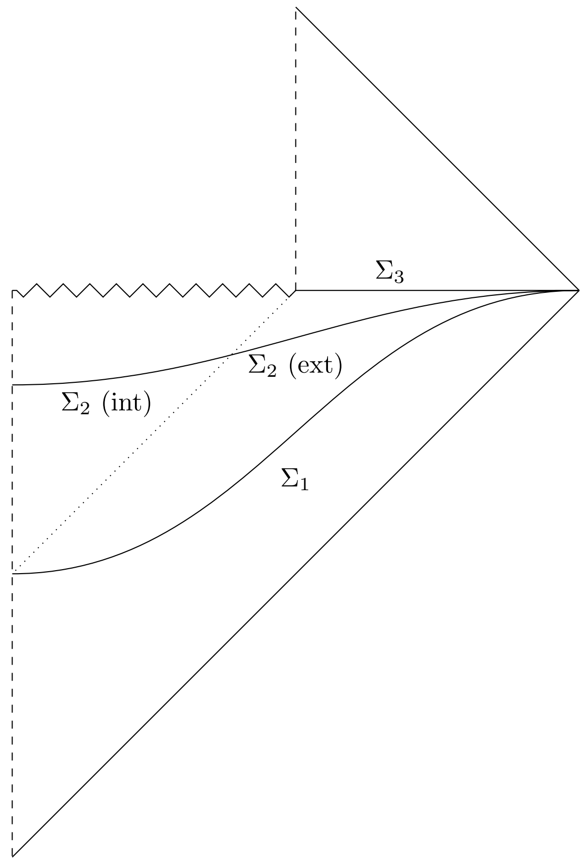

Figure 1.2: Penrose diagram of a fully evaporating black hole with space-like hypersurfaces Σ_1, Σ_2 and Σ_3.

a solution differs essentially from resolving the paradox by challenging one of the visible assumptions. Keeping this subtlety in mind, we may now state a preliminary version on the paradox, based on the discussion above.

The information loss paradox, standard version
If we believe that time-evolution in quantum theory is unitary and that black holes fully evaporate as described in section 1.1.2, we arrive at a contradiction.

This version of the information paradox is perhaps the most common formulation, but in our view it also leads to confusion. In particular, in this common phrasing it is unclear what system we expect to evolve unitarily. The discussion above only suggests that when we view the gravitational field as a classical background, the state of the quantum fields that live on this space-time evolve from a pure state to a mixed state. It does not directly say anything about unitarity of a theory of quantum gravity. This distinction should be made more clearly, as one may believe that unitarity of a quantum gravity theory is fundamental, while this would not have to imply that the semi-classical theory is unitary.[19] We should therefore specify that the information paradox only concerns time-evolution of matter fields and not of the gravitational field.

Furthermore, even in the semi-classical theory, the term ‘unitarity’ can be interpreted in multiple ways. This is because of the fact that the operators of quantum field theory, which has an infinite number of degrees of freedom, have no unique representation on a Hilbert space (up to unitary equivalence). Therefore, two (pure) states of a quantum field may not be representable as vectors in the same Hilbert space. Therefore, if one state evolves into another over time, there is no reason to assume that these states are related by a unitary operator on one Hilbert space. However, in flat space-time, or more specifically, in space-times with a time-translation symmetry (i.e. an (asymptotically) time-like Killing field), time-evolution along this field *can* be described by such a unitary operator; see chapter 5 of (Brunetti, Dappiaggi, Fredenhagen, & Yngvason, 2015). Therefore, the fact that time evolution is given by unitary operators on a Hilbert space is only true in special cases and is not a general feature of quantum theory.[20] What we can say, however, is that similarly to time-evolution being described by unitary operators, time-evolution of quantum fields on curved space-times preserves probability and, for a time-evolution of which the equal-time surfaces are Cauchy,[21] maps pure states to pure states. As noted by Unruh and Wald (2017), black hole evaporation does not result in a loss of probability, but only in an evolution of pure states into mixed states. Let us therefore reformulate the information loss paradox to avoid further confusion.

The information loss paradox, revised version
If we believe that there should be a global notion of time for which time-evolution of matter fields maps pure states to pure states and is invertible and that black holes fully evaporate as described in section 1.1.2, we arrive at a contradiction.

Note that both assumptions on time-evolution in this revised statement are satisfied on globally hyperbolic space-times. However, as can be seen in figure 1.1, the space-time associated with full black hole evaporation is not globally hyperbolic, or in fact not even causally continuous (Lesourd, 2019). In general it has many advantages to assume global hyperbolicity. It allows for well defined initial value formulations of both hyperbolic PDE's on a space-time (Bär, Ginoux, & Pfaeffle, 2007) as well as for the Einstein equations themselves (Choquet-Bruhat & Geroch, 1969). As mentioned, one can also define quantum field theories on these space-times such that they are well behaved with respect

[19]We hope that to some extent quantum fields on a classical curved space-time may be regarded as a semi-classical limit of quantum gravity. That is, for a theory of quantum gravity we hope we can take some limit in the theory or trace out quantum degrees of freedom of the quantum space-time (or whatever structure implements gravity in the quantum theory) resulting in a theory of quantum (matter) fields on a classical space-time background. This limit may give some effective interactions between matter fields or result in extra terms in the Einstein equation that couples matter to gravity, but for now we assume these can be neglected, at least in low-curvature regimes. This is why we refer to quantum fields on curved classical space-time as a semi-classical theory.

[20]Time-evolution being described by a unitary operator means that for each initial state at time t_0 represented by some density matrix ρ on a Hilbert space H there is a family of unitary operators $U(t) : H \to H$ such that at time t the state is given by $\rho(t) = U(t)\rho(U(t))^*$.

[21]See section 2.2.1 and 3.1.

to time-evolution.[22] While there have been proposals for generalizing quantum field theory constructions to non-globally hyperbolic space-times,[23] results on this have so far been rather scarce.[24] Therefore, it is not uncommon to assume that only globally hyperbolic space-times should exist in nature. In particular, the maximal Cauchy development of generic initial data to the Einstein equations (coupled to some equation of motion for matter fields) should give a space-time that cannot be extended any further as a Lorentzian manifold.[25] This assumption is known as *strong cosmic censorship* (Christodoulou, 2008), as opposed to weak cosmic censorship, which 'only' prohibits the existence of naked singularities.[26] Since the maximal Cauchy development of initial data is always globally hyperbolic, strong cosmic censorship implies that no non-globally hyperbolic space-times occur in nature. This leads us to another statement that we could call the information paradox, which we will refer to as the *geometric information paradox*. It is in the same spirit as how the information loss paradox is discussed by Manchak and Weatherall (2018).

The information loss paradox, geometric version
If we believe the strong cosmic censorship hypothesis and we believe that black holes fully evaporate as described in section 1.1.2, we arrive at a contradiction.

We note that the geometric version and the revised version of the information paradox are not equivalent, nor does one imply the other. We can imagine that there exists some generalization of quantum field theory on a non-globally hyperbolic space-time that has (in some cases) well behaved time-evolution. On the other hand, there could exist a sensible theory, though maybe not a quantum field theory that we are used to, on globally hyperbolic space-times with time-evolution that is not invertible, or pure-to-pure. Whatever the resolution of the information paradox will turn out to be, it should (re)solve both of the latter two formulations of the paradox. Assuming black holes do in fact evaporate, this means that we have to show that this still results in pure-to-pure and invertible time-evolution, or show that it is reasonable to discard the assumption of pure-to-pure and invertible time-evolution and we have to show that this still results in a globally hyperbolic space-time,[27] or show that we can reasonably discard strong cosmic censorship. It should be noted that discarding strong cosmic censorship, which leaves room for accepting that the space-time of black hole evaporation may be accurately described by figure 1.1, presents us with a potential difficulty. Let us explain this difficulty in the next section.

[22] This is done both by directly constructing these theories or by using an axiomatic system (Brunetti et al., 2015).

[23] See section 3.2 or for example (Kay, 1992) and (Yurtsever, 1994).

[24] Extending quantum field theory constructions to certain classes of non-globally hyperbolic space-times will be a major part of this thesis, in particular of chapter 4.

[25] To be more precise, there should not be an extension to the maximal Cauchy development of generic initial data that has locally square integrable Christoffel symbols.

[26] Over the years, many formulations of the cosmic censorship hypothesis have been put forward, with varying degrees of mathematical formality. The first version was proposed by Penrose (1969).

[27] This means that we have to show that the Penrose diagram of figure 1.1 is not a correct description of the evaporating black hole space-time.

1.2.2 The problem of predictability

One of the main motivations for accepting strong cosmic censorship is that we desire there to be some concept of time-evolution such that the future can, in an appropriate sense, be determined from the past. That is, when we provide enough initial data for both the space-time geometry and the matter fields that live on this space-time, the (classical) field equations should uniquely determine the configuration of space-time and matter fields in the future (and past) of the initial (Cauchy) surface. This is sometimes referred to as *Laplacian determinism* (Earman, 1986). We should note that when we consider quantum fields coupled to a classical space-time via (for instance) the semi-classical Einstein equations, the notion of determinism becomes trickier. In principle, one would hope that such a theory is also deterministic in the sense that given some geometric initial data and an initial state for a quantum field on the space-time, one could uniquely determine the future geometry as well as the state of the matter fields, given that the quantum field evolves via some dynamical principle (i.e. an equation of motion, a path integral, etc.). However, when observers are introduced into the game, this seems to change. After all, we know that in quantum theory the outcome of a measurement on a quantum system is not uniquely determined by the state of the system: the state only gives some probability on outcomes of measurements. Furthermore, in standard interpretations of quantum physics, such as the Copenhagen interpretation, a measurement changes the state of the system in an indeterministic way.[28] Therefore, in this interpretation quantum theories are not deterministic in the sense described above, at least when it comes to doing actual measurements. Since we do not wish to discuss determinism in quantum physics in detail, we circumvent these issues by not considering any observers at all and focus only on time-evolution governed by field equations.[29] If we do not have strong cosmic censorship, this may imply that after a certain amount of time the state of the system (i.e. a classical or quantum field on a classical background coupled via the Einstein equations) is no longer uniquely determined by the state at some initial point. We point out that this form of indeterminism is in a sense stronger than the indeterminism of the wave-function collapse scenario due to a measurement. After all, in the wave-function collapse scenario there is a probability distribution on the space of possible final states via the Born rule, while in the scenario of an undetermined final state due to a space-time background that violates strong cosmic censorship, there is no probability distribution on the set of allowed final states. Therefore one could argue that the latter form of indeterminism is in a sense stronger than the former.

One might argue that there is no problem with a final state not being uniquely determined from an initial state. After all, if we accept information loss, we accept that an initial state is not uniquely determined by a final state either, so why should we expect the reverse to be true. We do not have a definitive counterargument for this. In principle, one could accept that even if we understand all the local dynamics of our universe, this would not uniquely fix global dynamics. Our universe (past, present and future) may just be one of many solutions

[28]This is known as the collapse solution to the measurement problem.

[29]For those that do wish to read a philosophical discussion of determinism in physics, one could for instance read (Earman, 1986).

to satisfy some dynamical laws and some initial conditions (for instance at the big bang). However, I think that while this is an interesting way to look at our universe, it is also unsatisfying. One would hope that physical laws allow us to determine the future, or in particular, the outcome of an experiment.[30] If we cannot do this, then there is also no way to test theories by experiments, which would mean that we have to completely rethink the way we do physics, or science in general. Therefore, if only from a pragmatic point of view, we think it is undesirable to have a theory in which a final state is not uniquely determined from an initial state in an observer. Let us therefore formulate the following principle.

Principle of predictability
There exists a global notion of time in our system such that from generic initial data on some fixed time we can, as long as the system remains unobserved by an external observer, uniquely derive the state of the system at later times.

If we accept full black hole evaporation as described above, this means we have to give up strong cosmic censorship. We therefore risk that the principle of predictability is not satisfied. This is what we refer to as the *problem of predictability*. If the principle of predictability is indeed violated for a fully evaporating black hole, one may wonder how we could draw a Penrose diagram of figure 1.1 in the first place. After all, this figure suggests that we do know the final state of evaporation, while we just argued that this final state cannot be derived from an initial state. How then do we know what the geometry after evaporation will look like?[31] In drawing figure 1.1, we made the assumption that after evaporation there was no mass left at $r = 0$, the axis of symmetry. Intuitively, this assumption seems to make sense; after all, if the entire mass of the black hole is radiated away by Hawking radiation, there should be no mass left. However, we point out that conservation of energy is not in any way guaranteed on a general curved space-time. Even though one can make some asymptotic statements on conservation laws for an asymptotically flat globally hyperbolic space-time, such as a partially evaporating black hole, using the ADM formalism, these considerations will not generalize to non-globally hyperbolic space-times, such as the fully evaporating black hole. Therefore, the mass configuration after evaporation is in general not be determined from the initial state. It seems that if we want to overcome the problem of predictability, we need to reevaluate our theory. In particular there may be some physical input/law missing that does fix a final state from the initial state. This was also pointed out by Maudlin (2017).[32] Such additional physical input may be given by a theory of quantum

[30] At the very least, we hope that we can predict the statistics of the outcome of an experiment.

[31] We refer to a point in space-time as 'after evaporation' if there is a past directed inextendible causal curve that goes through that point and ends in the (naked) singularity known as the evaporation event. The notion of a singularity as endpoint of a curve can be made precise by introducing the notion of a causal boundary, see (Flores et al., 2011).

[32] Maudlin argues that one may add the evaporation event to the space-time as a single point through which one can continue causal curves, which means that this space-time is globally hyperbolic in a loose sense, as it seems to admit a Cauchy surface, however, this space-time is no longer a Lorentzian manifold, but rather some more general (but not clearly defined) set. Therefore, it is not clear what the status of global hyperbolicity is on such a generalized space-time. Furthermore it is not clear how the physics of the evaporation event, which would supposedly fix a final state of evaporation from an initial state, may be implemented on such a space-time. See (Manchak & Weatherall, 2018) for a discussion.

gravity that becomes relevant in the high curvature regime, replacing the classical theory that gave rise to the singularity.

It should be mentioned that the variations of the information loss paradox we have addressed above are not the only paradoxes associated with black hole evaporation and Hawking radiation. The so called Page-time paradox, which revolves around a contradiction between statistical mechanical underpinnings of black hole thermodynamics and the thermal nature of Hawking radiation is sometimes also referred to as the information loss paradox (Marolf, 2017; Wallace, 2017). It is important not to confuse these two paradoxes. This thesis is only concerned with the paradox as it has been outlined above.

We hope to have explained the information loss paradox and its origins. The issue of how it should be resolved has been long standing. Lack of any experimental data and a well established theory of quantum gravity makes it very difficult to determine which assumption underlying the paradox should be given up. We should note that many of these assumptions are controversial to some, so there is also no consensus on which assumption should be kept safe. In the next chapter we will outline some of the (re)solutions to the paradox that have been put forward over the years.

Chapter 2

Possible resolutions to the paradox

Focussing on the revised version of the information loss paradox, and putting the geometric version apart for a moment, we have to (re)solve the paradox by one of the following strategies:

- Argue that it is *acceptable* that information is lost and time-evolution can take pure states to mixed states and may not be invertible.
- Show that full black hole evaporation *does not occur* in the way we discussed it in the previous chapter.
- Introduce some mechanism such that even in the case of full black hole evaporation we still have pure-to-pure and invertible time-evolution.

It should be noted that historically, many physicists have been adamant to prevent information loss and have therefore pursued one of the two latter strategies. In the next section we will lay out some of these attempts.

2.1 How not to lose information

We categorize some of the attempts to resolve the paradox that aim to 'save' conservation of information. We do not claim that this list is exhaustive, yet hope to paint a good picture of some popular and/or interesting escapes to the information paradox. For other listings of these escapes, see for instance (Belot, Earman, & Ruetsche, 1999), (Unruh & Wald, 2017) and (Curiel, 2019). We will mainly focus on arguments against full black hole evaporation (i.e. the second strategy of the three listed above), which we subdivide into three categories. These categories are 'There are no black holes', 'there is no Hawking radiation' and 'black holes do not fully evaporate'. We only briefly touch upon the third bullet via the scenario 'information escapes from the black hole'.

2.1.1 There are no black holes

The first scenario to escape from information loss would be that black holes never actually form in the first place, which would certainly punch a hole in the paradox, so to speak. Of course, we have in the past observed astrophysical objects that behave very much like black holes, for instance the recent Event Horizon Telescope measurements (see e.g. Akiyama et al., 2019) certainly seems to suggest that black holes exist, but there mayconceivably be objects that appear (effectively) the same as a normal black hole to an outside observer, but lack a singularity at the center. Most of these alternatives appear outside the theory of general relativity, either in some classical alternative to GR or in theories of quantum gravity, but let us first look at a proposal that stays within the framework of GR.

Horizon avoidance

The proposal of horizon avoidance has been popping up under different names since the paradox was first introduced. Standard calculations on the Hawking effect, such as (Hawking, 1975) and (Fredenhagen & Haag, 1990), assume a black hole background. Such a space-time already contains a singularity and an apparent horizon. The conjecture of horizon avoidance states that matter in gravitational collapse already emits radiation (pre-Hawking radiation), as suggested by (Barcelo, Liberati, Sonego, & Visser, 2011). This radiation would carry away all energy from matter in gravitational collapse before the apparent horizon forms (Baccetti, Mann, & Terno, 2017).[1] After all, forgetting about the Hawking effect for a moment, during gravitational collapse an outside observer will never see the collapsing matter cross its apparent horizon, due to gravitational time-dilation. This means that no matter how weak the pre-Hawking radiation is as measured by an outside observer, as long as all energy can radiate away to infinity in a finite time with respect to, for instance, the Schwarzschild time-parameter, the collapsing matter will evaporate before horizon crossing. Therefore, black holes do not actually form. Instead, we have something called an incipient or asymptotic black hole, which for an outside observer is very difficult to distinguish from a real black hole. This would dispel the information loss paradox. Usually, this conjecture is studied using thin shell collapse models, see for instance (Vachaspati, Stojkovic, & Krauss, 2007), (Kawai, Matsuo, & Yokokura, 2013) and (Ho, 2016).[2]

The main opposition to this proposal comes from (Chen, Unruh, Wu, & Yeom, 2018). Using a dust shell collapse model it is argued that the proposed pre-Hawking radiation cannot take away all energy from the collapsing matter without it becoming tachyonic, which would be nonphysical. It is argued in (Baccetti, Murk, & Terno, 2018) that this problem can for instance be remedied by allowing a buildup of pressure. In the my opinion, whether horizon avoidance is a viable way to escape information loss can only be determined after a study of more realistic collapse models. The fact that this proposal does not seem to work for collapse of pressureless dust is a cause for worry, but not enough to discard this proposal entirely. After all, realistic matter models do

[1] In the case of a spherically symmetric collapse of electrically neutral matter, this means that all energy is radiated away from the collapsing object (and this object has thus disappeard) before it crosses its Schwarzschild radius.

[2] We note that this proposal only resolves the information loss paradox if the pre-Hawking radiation is not in a thermal/mixed state, but rather in a pure state from which we can deconstruct the initial state of the collapsing matter. This latter point is supported by Dai and Stojkovic (2016).

allow for pressure. A more dubious part of this proposal is the actual existence of pre-Hawking radiation. So far, results on this have been conflicting (see for instance (Vachaspati et al., 2007), (Unruh, 2018) and (Juárez-Aubry & Louko, 2018)). A more rigorous study of quantum fields on space-times associated with gravitational collapse should be undertaken to settle this point.

Let us note that the above references mostly concerned electrically neutral and spherically symmetric collapsing matter. In (pre-)Hawking radiation, the massless fields dominate the radiations spectrum. It is therefore argued that pre-Hawking radiation cannot efficiently carry away information about the quantum-numbers of collapsing matter. However, we note that in the 'classical limit' of, for instance, the standard model of high energy physics, the only intrinsic quantum number to survive is electric charge. We also note that in the normal picture of black hole evaporation, charged black holes can have a stable endstate, i.e. the extremal black hole, which has a Hawking temperature of 0. Whether this is the actual end state, or the black hole fully evaporates, depends on the radiation spectrum, in particular, on the ratio of charge and energy carried away by Hawking radiation. In the scenario of Horizon avoidance we have a similar situation: either all charge is carried away by the pre-Hawking radiation, or we end up with a stable final state similar to the extremal black hole. The latter scenario is supported by (Wang, 2018). Either way, the region outside the event horizon (if the horizon exists; otherwise, the entire space-time) is globally hyperbolic and we have global charge conservation. Therefore, also in the case of charged collapsing matter, pre-Hawking radiation may resolve the paradox.

Horizon avoidance is an example of how to avoid the information loss paradox without major departments from the semi-classical theory. Let us now look at a recent proposal of how black holes are replaced by entirely different objects in certain theories of quantum gravity, namely fuzzballs.

Fuzzball conjecture

Fuzzballs are conjectural objects that might replace 'classical' black holes within the context of string theory (Mathur, 2005). As string theory is beyond the scope of this text, we shall not go into much detail on this. Nevertheless we hope to present the idea of the fuzzball conjecture in a somewhat understandable, yet very handwavy fashion.

In string theory it is necessary to assume that space-time has more than 4-dimensions, the exact number varying from 10 in superstring theory to 26 in bosonic string theory . In order to regain a space-time that is 4 dimensional at a macroscopic level, these extra dimensions need to be compactified. Typically this means that the space-time on which strings live, is some product manifold of a four dimensional (macroscopic) space-time and some compact manifold(s), which represent the microscopical dimensions. The fuzzball model as studied in (Mathur, 2005) takes a superstring theory to live on a background space-time of

$$M_{4,1} \times T^4 \times S^1$$

i.e. the product of 5-dimensional Minkowski space, the 4-torus, and the circle, so a total number of 10 dimensions. Now a fuzzball state is represented by a string that winds around S^1 and carries a momentum charge as (transverse) waves along the string. We can further dress this state by adding 5-branes (higher dimensional objects that appear in string theory) that wrap around $T^4 \times S^1$. If one compares the microscopic entropy of these states with the Bekenstein entropy of a black hole that corresponds to the energy, momentum and gauge charges of the fuzzball state, it is found that these match (Callan & Maldacena, 1996).

The key difference between these fuzzball states and a classical point mass located at the singularity (which is a way to view a classical black hole) is that strings are extended objects, where the momentum carried by strings comes in the form of transverse waves. This means that a string is not located at a single point, but rather that it is spread out over a (microscopic) region as a 'fuzz'. This fuzz typically extends all the way to where in the classical case the event horizon would be situated, such that the horizon is also 'fuzzed' and does not exist as such. Therefore, unlike for a black hole, the matter inside the fuzzball, i.e. some string in a fuzzball state, is in causal contact with the outer region and therefore information can escape the fuzzball. This would mean that the fuzzball version of Hawking radiation (if this exists) may carry away all information besides just energy, charge and angular momentum as the fuzzball evaporates. Therefore, if full evaporation occurs, this means that one can deduce the initial (fuzzball) state of matter inside the black hole from the (pure) state of the outgoing radiation.

Using this model as an inspiration, the *fuzzball conjecture* states that as black holes form by gravitational collapse (and start evaporating), the infalling matter 'stabilizes' into some fuzzball type state and the resulting object is a fuzzball instead of a classical black hole, allowing an escape to the information loss paradox.

As of yet, many things remain unclear about the fuzzball conjecture. Most importantly, the mechanism that ensures that collapsing matter ends up in a fuzzball state needs to be made explicit. This would also give more information about the time-scale at which this stabilization phase takes place. As noted in (Mathur, 2009), the time scale at which this process takes place should be somewhere between the 'crossing time scale' of the collapse phase ($\sim GM$) and the 'evaporation time-scale' ($\sim GM^3/M_{\text{planck}}^2$). One might deduce a smaller upper bound on this by entropy considerations, related to the Page time (Wallace, 2017). The time-scale of stabilization also relates to the size of a typical fuzzball, as during the stabilization time, ordinary Hawking radiation is already expected to carry away energy, momentum and charge. Therefore, the longer stabilization will take, the smaller fuzzballs will typically be. Besides the lack of clarity on the stabilization process, we should notice that the fuzzball models presented so far live on a space-time with 5 macroscopic dimensions instead of 4. For the fuzzball conjecture to be of relevance to our universe, fuzzball states should also be identified in a 4-dimensional theory.

The fuzzball conjecture is criticized in (Unruh & Wald, 2017) on the basis that for the conjecture to hold, one would have to considerably depart from classical physics at a relatively low curvature regime for the collapsing matter to stabilize into a macroscopic fuzzball. In my view, this need not be the case if stabilization only takes place after most of the evaporation time has already elapsed. This means that stable fuzzballs will always be small, possibly near the Planck scale. However, objections may hold to this scenario that are similar to the objections to remnant scenarios, which we will touch upon later in this chapter.

2.1.2 There is no Hawking radiation

In section 1.1.1 we briefly discussed Hawking radiation, but also mentioned that its existence is not undisputed. The main issue is the so called trans-Planckian problem (Helfer, 2003). While dominant frequencies of Hawking radiation as measured at late times at 'infinity' are relatively small, as we can see from the fact that the Hawking temperature is typically low with respect to the CMB temperature, these modes find their origin in high frequency modes at early time affected by some (exponential) red shift. In particular, the energy scales of these early time modes are beyond the Planck scale. Therefore, the spectrum of Hawking radiation, or even its existence in the first place, may be very sensitive to Planck scale physics, i.e. quantum gravity. When trying to derive the Hawking effect, one is forced to make assumptions about Planck scale physics. Often these assumptions are made somewhat implicitly. If physics from quantum field theory is extrapolated to the trans-Planck scale modes (as done in the original calculation by Hawking), one finds the familiar Hawking temperature. However, various (speculative/toy) models for Planck scale physics have been put forward in which one finds either strong deviations from the spectrum predicted by Hawking or no radiation at all.

Of course there have been many attempts to remedy the trans-Planckian problem, often on the basis of some universality argument. Gryb et al. (2018) review some of these arguments and compare them to Wilsonian universality arguments for the universal behaviour of phase transitions in condensed matter systems.[3] The article identifies six criteria on which one can judge the strength of a universality argument and motivates from those that the arguments for the universality of the Hawking radiation (i.e. that the macroscopical physics, the Hawking effect, is not (very) sensitive to microscopical physics, the underlying theory of quantum gravity) is at best not as convincing as the Wilsonian arguments and at worst not convincing at all. This places further doubt on the universality of the Hawking spectrum. We seem to end up with the rather unsatisfying conclusion that we cannot make any robust statements about Hawking radiation without a well established theory of quantum gravity.

How much would the Hawking spectrum have to be altered by Planck-scale physics to escape the information paradox? Trivially, if there is no Hawking radiation, there is no information loss either, as this means that black holes do not evaporate. However, if Planck-scale physics merely alters the spectrum, the trans-Planckian problem does not spell the end of the paradox. After all, it was not so much the thermal nature of the Hawking radiation that was key to

[3] These arguments are taken as an example of convincing universality arguments.

information loss, but only the fact that the radiation does not carry any information about the black hole interior, other than mass, charge and momentum. A (possibly heavily) altered radiation spectrum that causes full evaporation of a black hole may still not allow all or some of the information to 'leak out' of the black hole, such that the radiation left after a black hole that was initially in a pure state has evaporated, is itself in a mixed state. Therefore, the paradox would still stand.

2.1.3 Black holes do not fully evaporate

If black holes do not fully evaporate, or in particular, if there is no point in the space-time that could be regarded as 'after the full evaporation', there is no paradox. After all, when a black hole does not fully evaporate, the full state of the system at late times specifies the state of both internal (i.e. inside the black hole) and external degrees of freedom and their correlations. Whilst the state of only the external degrees of freedom may be mixed (thus with all internal d.o.f.'s traced out), the full state is pure as long as the internal degrees of freedom still exist.

Remnant scenarios

Note that the Hawking temperature for a Schwarzschild black hole, $T_H = (4\pi R_s)^{-1}$, diverges as the Schwarzschild radius decreases. When an evaporating black hole approaches Planck length dimensions, the mean energy associated with the radiation derived from Hawkings 'semi-classical' calculation (i.e. QFT on a classical background) also approaches the Planck scale. At this regime, we would not expect a semi-classical treatment to model the physics well. After all, at the Planck scale, we expect quantum gravity to play a crucial role. Therefore, at this regime we might see a clear deviation from Hawkings prediction. For instance, the Hawking temperature may be bounded or even go to 0 for Planck scale black holes. The fate of these tiny black holes depends in an essential way on quantum gravity corrections to the Hawking temperature. Whilst the semi-classical model suggests that these black holes evaporate almost in an instant, leading to the scenario where all that is left from the black hole is the mixed state Hawking radiation, an altered model may give rise to long-lived or even stable Planck scale black holes. It was speculated in (Aharonov, Casher, & Nussinov, 1987) that such remnants (or as they called it, Planckons) should exist in order to resolve the information loss paradox. Over the years many pros and cons of this idea have been studied. Various proposals for how remnants could form have been put forward, as well as various counter arguments for their existence. In (Chen, Ong, & Yeom, 2015) the various remnant scenarios as well as their challenges are reviewed.

One of the major challenges of remnant scenarios that has often been brought up is the fact that a remnant may be too small to contain all the information that it would need to in order to have pure-to-pure time-evolution. While there is *a priori* no limit in quantum field theory to the amount of information that is contained in a certain volume, it has been argued using the Bekenstein entropy that a black hole of a certain size should only be able to contain a limited amount of information. After all, following the proposed generalized second law by Bekenstein, stating that the total entropy in the universe containing a black hole with surface area A, given by

$$S_{\text{total}} = \frac{A}{4} + S_{\text{external matter}}, \tag{2.1}$$

should increase, we can derive a bound on the entropy of matter fields contained in a bounded region. This bound,

$$S \leq 2\pi RE, \tag{2.2}$$

where R is the radius of the region and E the energy of the matter field, known as the *Bekenstein bound*, implies that microscopic (near static) black holes are only able to contain very few bits of information, assuming this entropy to have some statistical mechanical underpinning (Bekenstein, 1994).[4] This is worrisome if we want to claim that remnants prevent information loss by effectively storing all information that would otherwise be lost. Of course the relation of the Bekenstein entropy to the actual information content of a black hole is debatable, as long as there is no full theory of quantum gravity from which we can derive this correspondence to Bekenstein entropy and show that the generalized second law indeed holds. It is noted in (Chen et al., 2015) that if instead of attributing the information content of the entire black hole, it could be related to only near-horizon degrees of freedom, leaving the interior of the remnant free to contain as much information as necessary. This is known as the weak interpretation of the Bekenstein entropy.

Remnant scenarios are relatively conservative when it comes to introducing new physics. After all, the Hawking spectrum only needs to deviate from the semi-classical result at the Planck scale for the Remnant scenario to work. Furthermore, to an outside observer it is probably very hard to distinguish between a fully evaporating black hole and a black hole evaporating to a Planck size remnant. After all, due to its size, the latter would hardly interact with its environment.[5] However, there have been suggestions that Planck scale remnants have been produced in large amounts over the history of the universe, after evaporation of primordial black holes. Though possible over-production of these remnants would place doubts on their existence, as their collective gravitational effect should be measureable, it has been suggested on numerous occasions that black hole remnants may contribute to the dark matter content of our universe (Carr, Kuhnel, & Sandstad, 2016).

2.1.4 Information escapes from the black hole

Lastly, if we assume that black holes fully evaporate, we could hope that information may still be retrievable in some way , for instance via radiation emitted from the black hole during evaporation, or only at the end of the evaporation process.

[4] This means that the entropy can be related to the number of microstates of the black hole.

[5] Even though the tidal forces near the horizon will probably still be very violent, the chances of any physical object falling into a Planck size black hole should be quite small.

Black hole complementarity

One such proposal is known as black hole complementarity, introduced by Susskind, Thorlacius, and Uglum (1993), which tries to descibe an evaporating black hole as an ordinary quantum system. In particular, evolution from an initial state to Hawking radiation should, according to Susskind et al., be described by a unitary S-matrix.[6] We will not delve deeply into this proposal, as there is a lot to unpack, which would go beyond the scope of this thesis. What we do note is that it has been argued that complementarity is at odds with Einstein's equivalence principle. In particular, if black hole complementarity is true, an observer crossing the horizon will most likely encounter a 'firewall' and will burn up (Almheiri, Marolf, Polchinski, & Sully, 2013). The equivalence principle suggests that a local observer should perceive nothing out of the ordinary at the horizon compared to other regions in space-time, so this leads to yet another paradox.

Black-hole-to-white-hole tunneling

The last proposal on how information loss may be prevented that we discuss is a scenario where at the end of the evaporation process a black hole turns into a white hole via a tunneling effect in quantum gravity, which allows the information in the black hole to escape from the resulting white hole. This proposal is discussed in (Bianchi, Christodoulou, D'Ambrosio, Haggard, & Rovelli, 2018). Arguably, one could also view this white hole as a remnant, though in this case one that will disappear when all matter/information has escaped from the white hole.

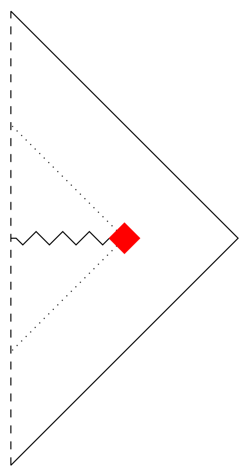

Figure 2.1: Penrose diagram of a black hole formed by a collapsing null shell transitioning to a white hole with outgoing null shell. The red region violates the vacuum Einstein equations.

[6]Note that this is at odds with the observation that time-evolution of quantum fields on a non-stationary background need not be unitary, but at most pure-to-pure for an appropriate notion of time.

It is possible to have a space-time of gravitational collapse into a black hole transitioning to a time-reversed 'collapse' of a white hole (i.e. matter coming out of the white hole) that solves the Einstein field equations everywhere except for a small neighbourhood of the singularity, as drawn in figure 2.1. This picture can be made more realistic by including Hawking radiation effects via an adaption of the Hiscock evaporation model (Martin-Dussaud & Rovelli, 2019). This suggests that it is possible for a theory of quantum gravity (where one expects the Einstein equations to be violated in high curvature regimes) to result in space-times of this form. Indeed, loop quantum gravity suggests that it is possible for a black hole geometry to tunnel into a white hole geometry and that the tunneling amplitude reaches order unity as the black hole becomes of Planck mass (Bianchi et al., 2018). Therefore, a black hole will always tunnel to a white hole at the end of its life-time and information can escape.

2.2 Allowing information loss

Above we have seen proposals of how to escape the conclusion that information is lost due to black hole evaporation, or, more accurately, that pure states evolve into mixed states. Not all of these ideas are as popular or well studied as the other. It is fair to say that without a full theory of quantum gravity supported by experiments,[7] we will probably not be able to determine which of these proposals is 'the one'.[8]

We conclude that there have been many proposals to reject information loss, but naturally we may also resolve the paradox by accepting the conclusion that information is lost. This would entail that we give up on the assumption that time evolution is pure to pure and invertible. This would involve deconstructing why we originally believed in pure to pure time-evolution and explain why information loss may still be consistent with our current knowledge. We will follow an argument for this laid out in (Unruh & Wald, 2017).

2.2.1 The Unruh-Wald argument

Let us first very briefly review some quantum field theory in a mathematically loose way; we will return to this in more detail and rigour in section 3.1. A quantum field living on some space-time M assigns to each point on this manifold an operator $\hat{\phi}(x)$.[9] These operators live in some algebra and the state of the quantum field is then given by a functional on this algebra. Let us for simplicity assume that the operator algebra in question acts on some Hilbert

[7]Due to the large gap between the energy scale that we can probe with particle accelerators at this point in time and the Planck scale energy, there have not been any experiments that truly probe quantum gravity. However, we hope that via quantum gravity phenomenology, we will at some point gain some experimental data, for instance via cosmological observations, that will help us in our search for a correct theory of quantum gravity (Amelino-Camelia, 2013).

[8]Of course it is also an options that none of the scenarios above are realized in nature.

[9]In the next chapter we will see that it is more accurate to view $\hat{\phi}$ as an operator valued distribution rather than a function on M.

space H and that the state is given by a unit vector $|\Psi\rangle$.[10] In the spirit of the Heisenberg picture of quantum mechanics, the dynamics of the theory is due to the fact that the operator $\hat{\phi}$ depends both on time and space coordinates, rather than that the state is time-dependent. The state is a global object, both in space and time, while the quantum field is local.

Usually when studying quantum field theory, we assume our background space-time to be globally hyperbolic. That is to say, the space-time on which the quantum field lives should admit a Cauchy surface, see A.1.1 for definitions. Just as initial data on a Cauchy surface define a unique solution on a globally hyperbolic space-time for certain PDE's, the behaviour of a quantum field on a globally hyperbolic space-time is uniquely determined by the behaviour around a Cauchy surface.[11] In terms of n-point functions $\langle\Psi|\,\hat{\phi}(x_1)...\hat{\phi}(x_n)\,|\Psi\rangle$, this means that these functions can be derived from initial data on some Cauchy surface.[12] Furthermore, if the state of the field in a neighbourhood of a Cauchy surface is pure, the global state will be pure, and *vice versa*. This means that the 'time evolution' from one Cauchy surface to the other is pure-to-pure and invertible.[13] In the next chapter we will give a more algebraic and rigorous treatment on these features, but for now this slightly loose description is sufficient.

Unruh and Wald note that typical states in quantum field theory are highly entangled. Let us recall that in the case of quantum mechanics (with finite number of degrees of freedom) entanglement can be characterized as follows.

Suppose the degrees of freedom of quantum system is described by a finite dimensional Hilbert space $H = H_1 \otimes H_2$, i.e. we can separate the degrees of freedom into 2 independent groups characterized by 2 Hilbert spaces. A (pure) state $|\Psi\rangle \in H$ then takes the form

$$|\Psi\rangle = \sum_j c_j\,|\Psi_{1,j}\rangle \otimes |\Psi_{2,j}\rangle\,,$$

with $|\Psi_{i,j}\rangle \in H_i$ and $c_j \in \mathbb{C}$. Similarly, a linear operator H, $\mathcal{O} \in L(H)$, is some linear combination of operators of the form $\mathcal{O}_1 \otimes \mathcal{O}_2$ with $\mathcal{O}_i \in L(H_i)$ a linear operator on H_i. A way to determine (or even define) if a state $|\Psi\rangle \in H$

[10] In fact it is known that if the quantum field lives in some C*-algebra, then for any state there is a Hilbert space representation of the operator algebra such that the state is given by a unit vector in that algebra (we omit some thechnical details). This is known as the GNS construction. A state is pure if and only if the representation is irreducible. Note however that not all pure states will give the same Hilbert space representation.

[11] In axiomatic quantum field theory this is known as the time-slice axiom.

[12] To be more accurate, one has to use some distributional version of the initial value problem.

[13] Admittedly, the terminology pure-to-pure time evolution is rather confusing in this setting, as we stated earlier that it is not the state that evolves, but the operators. The point is that when we have a (pure) state $\omega : \mathcal{A} \to \mathbb{C}$ on some operator algebra $\mathcal{A}$ and a subalgebra $\mathcal{B} \subset \mathcal{A}$ such that $\omega_{\mathcal{B}}$ is a state on $\mathcal{B}$, this reduced state need not be pure (as an element of the state-space of $\mathcal{B}$). In the case of quantum field theory, we say the full operator algebra is generated by $\hat{\phi}(x)$ for x ranging over M, while if we let x range over a subset $U \subset M$, this gives a subalgebra. Therefore a global pure state on M could be mixed on some $U \subset M$. However, due to the time-slice axiom, if U contains a Cauchy surface for M, then the algebra on U is already the full algebra, and therefore a pure state on M is pure on U, and *vice versa*.

is entangled with respect to the decomposition $H = H_1 \otimes H_2$ is to show that there is an operator $\mathcal{O} = \mathcal{O}_1 \otimes \mathcal{O}_2$ such that

$$\langle\Psi| \mathcal{O} |\Psi\rangle \neq \langle\Psi| \mathcal{O}_1 \otimes 1_{H_2} |\Psi\rangle \langle\Psi| 1_{H_1} \otimes \mathcal{O}_2 |\Psi\rangle .$$

In other words, the full state caries more information than just the configurations of the degrees of freedom associated H_1 and H_2 separately. The state also encodes correlations between these degrees of freedom. An equivalent way to determine if a state $|\Psi\rangle \in H$ is entangled is to see if the reduced state on H_1 or H_2 is pure or mixed. In the language of density matrices, a pure state $|\Psi\rangle$ is represented by the one dimensional projection

$$\rho = |\Psi\rangle \langle\Psi| .$$

The reduced state ρ_1 on H_1 is given by tracing the degrees of freedom associated with H_2 out of ρ,

$$\rho_1 = Tr_{H_2}(\rho).$$

Here $Tr_{H_2} : L(H) \to L(H_1)$ is the partial trace.[14] For a pure state of the form $|\Psi\rangle = \sum_j c_j |\Psi_{1,j}\rangle \otimes |\Psi_{2,j}\rangle$, this means that the reduced state on H_1 will be

$$\rho_1 = \sum_{i,j} c_i c_j^* \langle\Psi_{2,j}|\Psi_{2,i}\rangle |\Psi_{1,i}\rangle \langle\Psi_{1,j}| .$$

A pure state on H is entangled with respect to the decomposition $H = H_1 \otimes H_2$ if and only if the reduced state ρ_1 is mixed. In other words, ρ is entangled if ρ_1 is *not* a one dimensional projection matrix, $\rho_1 \neq |\Psi_1\rangle \langle\Psi_1|$ for some unit vector $|\Psi_1\rangle \in H_1$.

Even though for quantum field theory, which deals with an infinite number of degrees of freedom, the story of entanglement becomes more complicated, it is very helpful to keep the finite-dimensional case described above in mind. Recall that the fundamental operators of a quantum field theory are given by the field $\hat{\phi}(x)$. Just as in the finite d.o.f. case, we can define entanglement by looking at expectation values of products of these observables (thus looking at 2-point functions). Suppose $U_1, U_2 \subset M$ are two disjoint regions. Then we say these regions are entangled by a state $|\Psi\rangle$ if there are $x_1 \in U_1$, $x_2 \in U_2$ such that

$$\langle\Psi| \hat{\phi}(x_1)\hat{\phi}(x_2) |\Psi\rangle \neq \langle\Psi| \hat{\phi}(x_1) |\Psi\rangle \langle\Psi| \hat{\phi}(x_2) |\Psi\rangle .$$

Typically, we are only interested in entanglement between regions that are not causally related, as the fact that space-time points that are causally related are correlated seems almost trivial.[15] Let us the therefore assume that U_1 and U_2

[14]The partial trace $Tr_{H_2} : L(H) \to L(H_1)$ is defined as the unique linear map such that for all $\mathcal{O}_i \in L(H_i)$ we have

$$Tr_{H_2}(\mathcal{O}_1 \otimes \mathcal{O}_2) = \mathcal{O}_1 Tr(\mathcal{O}_2).$$

[15]In fact on a globally hyperbolic space-time the operators near a Cauchy surface already generate the full operator algebra (time-slice axiom). Therefore, an arbitrary operator can always be expressed in terms of these Cauchy neighbourhood operators. This means that the correlations between points that are causally related are essentially due to inequalities like $\langle\Psi| A^2 |\Psi\rangle \neq \langle\Psi| A |\Psi\rangle^2$ for some operator A (or more dressed versions of this). However, we do not associate these inequalities with entanglement, but rather with some uncertainty in a measurement.

are space-like separated. In (Unruh & Wald, 2017) it is argued that although in the finite d.o.f. case entanglement is an important feature of quantum mechanics, if two systems do not interact they will typically not become entangled. In quantum field theory on the other hand entanglement is unavoidable and essential. We will illustrate this by the example of the massless Klein-Gordon field. When we look at the vaccuum state on flat space-time, $|0\rangle$, the 1-point functions satisfy $\langle 0| \hat{\phi}(x) |0\rangle = 0$. Now suppose that x, y are space-like separated by Minkowski distance r, then

$$\langle 0| \hat{\phi}(x)\hat{\phi}(y) |0\rangle = -\frac{i}{(2\pi r)^2} \neq 0,$$

which can just be read off from the Feynman propagator.

Unruh and Wald discuss more general states for more general theories,[16] but the message is the same. In quantum field theory there will typically be correlations that spread over spatial distances. Using the intuition from the case of finite d.o.f., this means that if we take a region of space-time $U \subset M$ such that there is a region $V \subset M$ with U and V space-like related (and thus U cannot contain a Cauchy surface), there will typically be entanglement between U and V and thus a typical (pure) state $|\Psi\rangle$ of the quantum field will give a mixed reduced state on U.

The argument above suggests that, whereas if $U \subset M$ contains a Cauchy surface, a pure state restricted to U is pure, if U does not contain a Cauchy surface, there is no guarantee that the state on U will be pure and in general it is not. This is due to the fact that degrees of freedom in- and outside of U are correlated and thus tracing out degrees of freedom outside U results in a mixed state. Of course, this was argued from an intuition that we got from the finite-dimensional Hilbert space case, and hence need not be applicable here, but when one does explicit calculations, we indeed find that our intuitive result was correct (Unruh & Wald, 2017).

What does this tell us? Pure-to-pure invertible time evolution can only be expected when the evolution is from one Cauchy surface (or rather, a neighbourhood thereof) to another. Therefore, if we look at time-evolution of a quantum field with respect to a notion of time that does not foliate the space-time into Cauchy surfaces, we cannot expect pure-to-pure time evolution. This brings us to the observation that for a space-time that is not globally hyperbolic, such as the space-time associated with full black hole evaporation, there is in general no notion of time for which it is natural to expect pure-to-pure time-evolution. Therefore, since the space-time associated with black hole evaporation is not globally hyperbolic, information loss is to be expected. Thus the fact that information is lost in black hole evaporation is not paradoxical at all. This view has also been advocated by Maudlin (2017).

[16] They make use of so-called *Hadamard states*, which form a generalization of vaccuum states for curved space-time.

2.2.2 Challenges to the Unruh-Wald argument

As explained above, the Unruh-Wald argument states that information loss is not paradoxical at all in a space-time associated with black hole evaporation and that it is fully consistent with quantum field theory as we know it on other space-times. This might sound pretty convincing, but if so, why is a majority group of physicists still arguing for escapes from the information paradox a mentioned in the previous section? Some of these objections are already countered in (Unruh & Wald, 2017), which we will discuss first.

Violation of energy conservation or locality

If we assume that a full theory of quantum gravity still has information loss, i.e. pure to mixed evolution, one may expect this not only to happen in the evaporation of macroscopic black holes, but also in lower energy scale physics. We know from quantum field theory that in ordinary physical processes, high energy physics still plays a role due to virtual particles. In quantum gravity we might therefore expect the physics of (virtual) black holes to still play a role even in lower energy processes, which may mean that there is also information loss in these processes, as argued by Hawking (1982).[17] Building on this idea, we may expect that even on the scale of laboratory physics, quantum gravity effects will cause pure states to evolve into mixed states for closed quantum systems.[18] Some have even speculated that this is the origin of the second law of thermodynamics, i.e. that the entropy of any system must stay constant or increase (Kay, 2018).[19] Information loss for closed quantum systems at a laboratory scale has proved to be controversial, since it has been suggested that such a dynamics would either violate conservation of energy[20] or locality, at least when we assume that time-evolution is Markovian (Banks, Susskind, & Peskin, 1984). However, it is shown in (Unruh & Wald, 1995) that deviations in energy conservation at laboratory scale can be arbitrarily small. Furthermore, it is shown in (Unruh, 2012) that when we allow for a more general time-evolution law, there can be information loss and exact energy conservation. This is shown by coupling a quantum mechanical system to a spin-bath. Therefore, information loss is in principle not in conflict with (approximate) energy conservation and locality.

[17]However, it is entirely unclear if concepts like virtual particles generalize to quantum gravity and if so, what role black holes will play here. It should be noted that Hawking radiation is a semi-classical effect and we can so far only speculate about if virtual black holes, which are firmly in the realm of quantum gravity, actually radiate.

[18]In principle this does not go against any experimental observations, as even in the best experiments we practically never measure all relevant degrees of freedom. After all, in 'normal' quantum physics we only expect pure-to-pure time-evolution in a closed quantum system, in practice experiments will never involve perfectly closed quantum systems, so we do not know for sure if, even at laboratory scale, time-evolution does not involve information loss.

[19]The standard view on the second law is that it has a statistical origin: the information loss that causes entropy to increase is due to the fact that a quantum system getting correlated with the environment is much more likely than that the system and the environment will not be correlated.

[20]Of course, this is generally also violated for an open quantum system, as there will be an energy exchange between the system and the environment, so also in this setting information loss and energy fluctuations go hand in hand. Unfortunately, this also means that, without being absolutely sure that the system we are measuring is closed, we cannot really make a distinction on the basis of energy fluctuations between the different possible origins of information loss in an experiment.

Non-global hyperbolicity of the space-time

While on the one hand the fact that the space-time of full black hole evaporation is non-globally hyperbolic is an essential ingredient for the Unruh-Wald argument, i.e. without Cauchy surfaces one should not expect pure-to-pure time evolution, it also presents us with further problems. Recall that in section 1.2 we formulated the geometric version of the information loss paradox. This version stresses the contradiction between strong cosmic censorship and black hole evaporation. Obviously, the scenario that Unruh and Wald sketch still violates strong cosmic censorship. In section 1.2.2 we have argued why strong cosmic censorship, or in particular why space-times occurring in nature being globally hyperbolic, is often assumed and that when cosmic censorship is violated, one runs into the problem of predictability.

We first note that there is no guarantee that we can construct a consistent quantum field theory on a non-globally hyperbolic space-time, assuming that we have even agreed upon a suitable definition of what a quantum field theory on a non-globally hyperbolic space-time is. As we will see in section 3.1, constructing a linear scalar quantum field theory on a globally hyperbolic space-time makes use of the well-posedness of the Cauchy problem of the classical field theory to yield a quantum theory with well-posed dynamics (such that the time-slice axiom holds). However, as there have been examples of quantum field theories being constructed on certain non-globally hyperbolic space-times (Yurtsever, 1994; Kay, 1992; Fewster & Higuchi, 1996), we are hopeful that this problem can be overcome in that we can define a quantum field theory on the evaporating black hole space-time. Only when we can indeed show that such a QFT exists, the Unruh-Wald argument really makes sense.

Now we recall the problem of predictability that we introduced in section 1.2.2. Suppose we have shown that there exists a QFT on the black hole evaporation background. Such a theory suffers information loss if the initial state (i.e. a state of the quantum field prior to evaporation) cannot be reconstructed from a final state (a state after evaporation). On the other hand, a final state may not be uniquely determined from an initial state either. In that sense, non-global hyperbolicity of a space-time is a double edged sword. Of course the fact that global hyperbolicity implies that the principle of predictability holds, does not mean that all theories on non-globally hyperbolic space-times violate this principle. It remains to be seen how a quantum field theory on an evaporating black hole behaves. Using a continuity argument, Wald (1984a) has suggested that the principle of predictability may still be satisfied for a such a theory. However, as was also pointed out by Maudlin (2017), this argument is not rigorous at all and we cannot be sure if the principle of predictability holds until we actually construct the quantum field theory in question.

Let us for now entertain the possibility that the principle of predictability does not hold for a quantum field on an evaporating black hole background. How should we interpret this? As we discussed earlier, one might accept that the principle of predictability is violated. After all, the time-reversed version of this principle is also violated if we assume information loss takes place in black hole evaporation. If we want the laws of physics to satisfy time-reversal (or CPT) symmetry, it makes sense to either reject information loss and embrace the principle of predictability, or accept information loss and reject predictability. On the other hand, we do experience an arrow of time, so maybe we should not expect CPT symmetry for a theory of quantum gravity or for its semi-classical limit (i.e. a classical space-time with quantum fields living on it). Therefore, we are comfortable with accepting information loss, while still viewing the principle of predictability as something that should not be violated by a sensible physical theory. This is for the simple reason that experiments tell us that the physical laws have predictive power, at least in a statistical sense due to the probabilistic nature of quantum theory. Of course, one could say that we have not done any experiments/observations that involve fully evaporating black holes. Therefore, violation of the principle of predictability in black hole evaporation is not at all at odds with experiments, since these experiments usually only take place in a globally hyperbolic neighbourhood far from the black hole singularity. So maybe we shouldn't be so fazed by this potential breakdown of predictability. However, we recall the observation from earlier in this section, based on (Hawking, 1982), that due to virtual black holes in quantum gravity, information loss may occur in every physical process, even at low energy scales. This may imply that also in laboratory scale physics, an additional factor of unpredictability is introduced (on top of the uncertainty of a measurement in quantum mechanics and the pure-to-mixed time evolution). Naturally, it is possible that these effects are too small to measure, but let us for now assume that these effects are not present. Therefore, we stick to our view that a sensible physical theory should satisfy the principle of predictability and that, if it turns out that quantum field theory on an evaporating black hole background does not satisfy this principle, there is something either wrong with or missing from this theory.

In the next chapters we will investigate whether a quantum field theory on a fully evaporating black hole background exists and if such a theory violates the principle of predictability or not. We confine ourselves to studying free/linear real field theories. We shall first review how to construct these theories on a globally hyperbolic space-time. Then we will try to generalize these theories to a larger class of space-times backgrounds that includes the fully evaporating black hole space-time. We investigate how to tell if such a theory is predictable (and retrodictable) and lastly we try to apply what we have learned to the fully evaporating black hole.

Chapter 3

Quantum fields on Curved Space-times

Before we construct a quantum field theory on a fully evaporating black hole background, which is a non-globally hyperbolic space-time, we first review how to construct a Klein-Gordon quantum field theory on globally hyperbolic space-times. The usual constructions of quantum field theory are done assuming a Minkowski space background geometry. After all, QFT first came about as a way of making quantum mechanics compatible with special relativity, necessitating the Poincaré group to be a symmetry of the theory. This group is the symmetry of Minkowski space and plays a vital role in standard quantization procedures used to construct a quantum field theory. Of course, we know the universe is not flat. We know from General Relativity that matter curves space-time and this curvature affects the movement of the matter, which is observed as the gravitational force. This also means that the Poincaré group is not a fundamental symmetry. Therefore, standard constructions of QFT do not always generalize well to curved space-times, let alone to non-globally hyperbolic curved space-times. Luckily, there is an approach to quantum field theory that does lend itself well to curved space-time, and this is known as *algebraic quantum field theory* (Brunetti et al., 2015).

After we have reviewed the globally hyperbolic case, we will explore some work that has been done on extending the notion of algebraic quantum field theory beyond globally hyperbolic space-times. In particular, we look at a class of theories that are known as *F-local quantum field theories* (Kay, 1992).

3.1 A QFT on globally hyperbolic space-times

While algebraic quantum field theory can be studied from a very axiomatic point of view, we will mostly focus on (generalizations of) a particular construction of an AQFT, namely the quantized real Klein-Gordon field theory. Given a Lorentzian manifold (M, g), the Klein-Gordon equation is given by

$$(\Box^2 - m^2)F = 0 \tag{3.1}$$

with $\square^2 = \nabla_\mu \nabla^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)$, and m a (real) scalar that we take to be some fixed value for the rest of this thesis. Note that we are using the $(-,+,+,+)$ sign convention for the metric. The goal of this section is to define a quantum field theory associated with (3.1) on globally hyperbolic space-times. These theories are discussed extensively in (Wald, 1994; Brunetti et al., 2015), though we will mostly adhere to the point of view taken in the latter. Our goal is to construct an operator algebra that will be some more rigorous version of an algebra containing $\hat{\phi}(x)$ that formally satisfies (3.1) and (given some particular foliation of the space-time) whose commutation relations are determined by the canonical equal time commutation relations of $\hat{\phi}_\Sigma(x)$ and its conjugate $\hat{\pi}_\Sigma(x)$ on some Cauchy surface Σ, that is

$$[\hat{\phi}_\Sigma(x), \hat{\phi}_\Sigma(x')] = 0, \quad [\hat{\pi}_\Sigma(x), \hat{\pi}_\Sigma(x')] = 0, \quad [\hat{\phi}_\Sigma(x), \hat{\pi}_\Sigma(x')] = i\delta^{(3)}(x, x'). \tag{3.2}$$

In particular, since we are constructing the theory via an operator algebra in which we have built in a time-dependence, this way of constructing a quantum theory is very similar to the way one constructs quantum mechanics using the Heisenberg picture. However, since the Stone-von Neumann theorem fails in the case of quantum fields, as these have an infinite number of degrees of freedom, this formulation is not equivalent to a Schrödinger formulation. In order to construct the operator algebra, let us first study the Klein-Gordon equation in a more classical context.

3.1.1 Classical solution spaces to the Klein-Gordon equation

We first note that we should specify the space of solutions F of equation (3.1). Obviously this will have to be a class of functions on which the differential equation is well defined. A minimal choice would be that $F \in C^2(M, \mathbb{R})$, yet for our purposes we will often restrict to the safe choice $F \in C^\infty(M, \mathbb{R})$, unless otherwise noted. Since we are working with a real field theory, we will often work with real functions, therefore often referring to $C^\infty(M, \mathbb{R})$ as $C^\infty(M)$. The same goes for other function spaces.

If (3.1) is interpreted more formally, one could significantly broaden the class of solutions to the space of *distributions* of M, which are continuous linear functionals on the space of *test functions* on M. Let us first define the space of test functions on subsets of $\mathbb{R}^n$.

Definition 3.1.1. *Let $U \in \mathbb{R}^n$ open. A* test function *is a smooth function $f \in C^\infty(U)$ such that there is a compact set $K \subset U$ with* $\mathrm{supp}(f) \subset K$*, i.e. f has compact support.*[1] *The space of test functions on U is denoted by $C_c^\infty(U)$.*

We now promote this vector space to a topological space in the following way.

Definition 3.1.2. *Let $U \in \mathbb{R}^n$ open. The topological space $\mathcal{D}(U)$ is the vector space of test functions $C_c^\infty(U)$ endowed with the coarsest topology such that for*

[1]For a manifold M and $f \in C^\infty(M)$, the support of f is given by $\mathrm{supp}(f) = \overline{\{x \in M : f(x) \neq 0\}}$.

each multi-index α taking values in 1 to n, the semi-norm $p_\alpha : \mathcal{D}(U) \to \mathbb{R}_{\geq 0}$ is continuous, where for $f \in \mathcal{D}(U)$

$$p_\alpha(f) = \sup_{x \in \operatorname{supp}(f)} |\partial_\alpha f(x)|.$$

We now define distributions on these sets.

Definition 3.1.3. *Let $U \in \mathbb{R}^n$ open. A* distribution *on U is a continuous linear map $T : \mathcal{D}(U) \to \mathbb{R}$. The space of distributions on U is denoted by $\mathcal{D}'(U)$.*

One can show that a linear map $T : C_c^\infty(U) \to \mathbb{R}$ is a distribution by finding for each compact $K \subset U$ a $C > 0$ and a $k \in \mathbb{N}$ such that for each $f \in C_c^\infty(U)$ with $\operatorname{supp}(f) \subset K$ we have

$$|T(f)| \leq C \sum_{|\alpha| \leq k} p_\alpha(f),$$

where α is a multi-index (Hörmander, 2015, Ch. 2).

We now generalize this to an arbitrary manifold, the test function space $C_c^\infty(U)$ (not considering their topology for a moment) generalizes trivially to M, so we focus on the distributions.

Definition 3.1.4. *Let M an n-dimensional manifold. A* distribution *on M is a linear map $T : C_c^\infty(M) \to \mathbb{R}$ such that there is an atlas (U_i, φ_i) of M and distributions $T_i \in \mathcal{D}'(\varphi_i(U_i))$ such that for $f \in C_c^\infty(U_i)$ we have*

$$T(f) = T_i\left(\left(\varphi_i^{-1}\right)^* f\right).$$

The set of distributions on M is denoted by $\mathcal{D}'(M)$.

One commonly writes $T(f) = \langle T, f\rangle$ for $f \in C_c^\infty(M)$. Note that $\mathcal{D}'(M)$ on a manifold M is a vector space under pointwise addition and multiplication. There is also a natural topology on the space of distributions, namely the *weak*-topology.*

Definition 3.1.5. *Let M a manifold and $\mathcal{D}'(M)$ the set of distributions. The* weak*-topology *on $\mathcal{D}'(M)$ is defined as the coarsest topology such that for each $f \in C_c^\infty(M)$ the evaluation map*

$$\langle \cdot, f\rangle : \mathcal{D}'(M) \to \mathbb{R}$$

is continuous.[2]

If (M, g) is a Lorentzian manifold, one has a natural measure $dV = \sqrt{-g}d^4x$ and there is a natural embedding of the smooth functions on M into the set of distributions via the following identification. Let $F \in C^\infty(M)$ and $f \in C_c^\infty(M)$, then we define $\langle F, f\rangle = \int_M dV F f$. It is clear that for $\operatorname{supp}(f) \subset K$ with K compact, it follows that

$$|\langle F, f\rangle| \leq \left(\int_K dV F\right) \sup_K |f|,$$

[2]This topology is sometimes also referred to as the *topology of pointwise convergence.*

so we see that this indeed defines a distribution. If F is a solution to (3.1), it follows via integration by parts that this is equivalent to

$$\forall f \in C_c^\infty(M) : \int_M dV F(\Box^2 - m^2) f = 0.$$

We can therefore make the following generalization of (3.1).

Definition 3.1.6. *Let $T \in \mathcal{D}'(M)$. We say that T is a* weak solution *to the Klein-Gordon equation if*

$$\forall f \in C_c^\infty(M) : \langle T, (\Box^2 - m^2) f \rangle = 0. \tag{3.3}$$

The vector space of all weak solutions to the Klein-Gordon equation will be referred to as $S(M)$.

The notion of *weak* solutions will be of use to us later. Let us for now return to *smooth* solutions to (3.1). In particular, we will look at the equation from the perspective of an intial value (Cauchy) problem. As noted in appendix A.1.1, global hyperbolicity is often assumed for a space-time when discussing the Cauchy problem. When a Lorentzian manifold M is globally hyperbolic, it contains a smooth Cauchy surface Σ such that $M \cong \mathbb{R} \times \Sigma$. It follows that when we fix appropriate initial conditions on Σ, this uniquely gives a smooth solution to (3.1). Let us first define what we mean by initial conditions.

Definition 3.1.7. *Let (M, g) be a globally hyperbolic Lorentzian manifold and $\Sigma \subset M$ a Cauchy surface. Define the vector space of initial data on Σ as*

$$\mathfrak{C}(\Sigma) = C_c^\infty(\Sigma) \times C_c^\infty(\Sigma).$$

Elements of this space are usually denoted as $(\phi, \pi) \in \mathfrak{C}(\Sigma)$

We now have the following theorem.

Theorem 3.1.8. *Let M globally hyperbolic and $\Sigma \subset M$ a Cauchy surface, with $n \in T_\Sigma M$ the unit normal future directed vector field along Σ. Then for each $(\phi, \pi) \in \mathfrak{C}(\Sigma)$ and $f \in C_c^\infty(M)$, there is a unique $F \in C^\infty(M)$ such that $(\Box^2 - m^2)F = f$, $F_{|\Sigma} = \phi$ and $n^\mu \partial_\mu F_{|\Sigma} = \pi$. Furthermore,*

$$\text{supp}(F) \subset J(\text{supp}(f) \cup \text{supp}(\phi) \cup \text{supp}(\pi)),$$

where for $U \subset M$ we have $J(U) = J^+(U) \cup J^-(U)$.

A more general statement and its proof can be found in (Bär et al., 2007). We can see that not every smooth solution to the Klein-Gordon equation necessarily has initial data of compact support, yet for our purposes we often consider just the class of solutions that does satisfy this condition. Let us give it a name:

Definition 3.1.9. *Let M be globally hyperbolic and $\Sigma \subset M$ Cauchy. We define the* solving map $\mathfrak{s}_\Sigma : \mathfrak{C}(\Sigma) \to C^\infty(M)$ *as the (linear) function that associates to each initial data on Σ a unique smooth solution to the Klein-Gordon equation (without source) as given by theorem 3.1.8. We refer to the range of this map as the* strong solutions $S_c(M) = \mathfrak{s}_\Sigma(\mathfrak{C}(\Sigma))$ *to the Klein-Gordon equation. Note that this is also a vector space under pointwise addition and multiplication.*

We can see from the last line of theorem 3.1.8 that a strong solution has compactly supported initial data on any Cauchy surface, hence the definition of $S_c(M)$ is independent of the choice of Σ. We can therefore introduce a "time evolution" of Cauchy data from one Cauchy surface Σ_1 to another Σ_2, $\mathfrak{s}_{\Sigma_1\to\Sigma_2} : \mathfrak{C}(\Sigma_1) \to \mathfrak{C}(\Sigma_2)$ via $\mathfrak{s}_{\Sigma_1\to\Sigma_2} = \mathfrak{s}_{\Sigma_2}^{-1} \circ \mathfrak{s}_{\Sigma_1}$.

Let us now turn to symplectic geometry as a means of studying the classical Klein-Gordon theory. We define a symplectic form on $\mathfrak{C}(\Sigma)$, making it a symplectic vector space.

Definition 3.1.10. *Let (M, g) globally hyperbolic, $\Sigma \subset M$ Cauchy and h the induced metric on Σ.*[3] *The symplectic vector space $(\mathfrak{C}(\Sigma), \Omega_\Sigma)$ is given by the symplectic form*[4] *$\Omega_\Sigma : \mathfrak{C}(\Sigma)^2 \to \mathbb{R}$, where*

$$\Omega_\Sigma((\phi_1, \pi_1), (\phi_2, \pi_2)) = \int_\Sigma d^3x\sqrt{h}(\pi_1\phi_2 - \pi_2\phi_1), \tag{3.4}$$

which is indeed antisymmetric and nondegenerate.

This symplectic form can be lifted to $S_c(M)$ via $\mathfrak{s}_\Sigma$. We can easily see that for $(\phi_1, \pi_1), (\phi_2, \pi_2) \in \mathfrak{C}(\Sigma)$ and $F_i = \mathfrak{s}_\Sigma((\phi_i, \pi_i)) \in S_c(M)$ we have

$$\Omega_\Sigma((\phi_1, \pi_1), (\phi_2, \pi_2)) = \int_\Sigma d^3x\sqrt{h}n^\mu(F_2\partial_\mu F_1 - F_1\partial_\mu F_2). \tag{3.5}$$

In fact, if we evaluate the right hand side of this equation at some other Cauchy surface Σ', we find that this gives us the same value. Indeed, let $U \subset M$ a submanifold such that $\partial U = \Sigma \cup \Sigma'$, then

$$\begin{aligned}
&\int_\Sigma d^3x\sqrt{h}n^\mu(F_2\partial_\mu F_1 - F_1\partial_\mu F_2) - \int_{\Sigma'} d^3x\sqrt{h}n^\mu(F_2\partial_\mu F_1 - F_1\partial_\mu F_2) \\
&= \int_U d^4x\sqrt{-g}\nabla^\mu(F_2\partial_\mu F_1 - F_1\partial_\mu F_2) = \int_U d^4x\sqrt{-g}(F_2\Box^2 F_1 - F_1\Box^2 F_2) \\
&= \int_U d^4x\sqrt{-g}(F_2(\Box^2 - m^2)F_1 - F_1(\Box^2 - m^2)F_2) = 0.
\end{aligned}$$

This allows us to make the following definition.

Definition 3.1.11. *For M globally hyperbolic, $(S_c(M), \Omega)$ is a symplectic vector space with symplectic form $\Omega : S_c(M)^2 \to \mathbb{R}$ given by*

$$\Omega(F_1, F_2) = \int_\Sigma d^3x\sqrt{h}n^\mu(F_2\partial_\mu F_1 - F_1\partial_\mu F_2), \tag{3.6}$$

for Σ some Cauchy surface of M (on which this definition does not depend).

Since this definition is independent of the choice of Σ, a theorem immediately follows that will be of use later when we start to quantize the theory.

Theorem 3.1.12. *Let M be globally hyperbolic, with $\Sigma, \Sigma' \subset M$ Cauchy surfaces. Then the solution and time evolution maps $\mathfrak{s}_\Sigma$ and $\mathfrak{s}_{\Sigma\to\Sigma'}$ are symplectomorphisms from $(\mathfrak{C}(\Sigma), \Omega_\Sigma)$ to $(S_c(M), \Omega)$ and $(\mathfrak{C}(\Sigma'), \Omega_{\Sigma'})$ respectively.*

[3] For a Lorentzian manifold M with metric g that at each point $p \in M$ defines a bilinear map $g_p : T_pM \times T_pM \to \mathbb{R}$, with T_pM the tangent space of M at p, a submanifold $N \subset M$ has an induced metric h with $h_p : T_pN \times T_pM \to \mathbb{R}$ defined by $h_p(v, w) = g_p(v, w)$ for each $p \in N$, which is well defined since there is a natural embedding $T_pN \subset T_pM$.

[4] This structure is a generalization of the symplectic form on the even dimensional linear phase space associated with the Poisson structure of classical mechanics; $\Omega((\vec{q}_1, \vec{p}_1), (\vec{q}_2, \vec{p}_2)) = \vec{p}_1 \cdot \vec{q}_2 - \vec{p}_2 \cdot \vec{q}_1$. See (Wald, 1994) for more details.

3.1.2 Construction of the Klein-Gordon algebra

We are now ready to take a first stab at quantizing the theory. For a general symplectic vector space we can do a construction which generates the CCR (Canonical Commutation Relation) *-algebra (Brunetti et al., 2015).[5]

Definition 3.1.13. *Given a real symplectic vector space (V, Ω), we first define the free *-algebra of V. Let $A = \bigoplus_{k\in\mathbb{N}_0} V_{\mathbb{C}}^{\otimes k}$ with $\oplus$ the direct sum and $\otimes$ the (complex) tensor product. $V_{\mathbb{C}}$ is the complexification of V, i.e. $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$, with $\otimes_{\mathbb{R}}$ the real tensor product, such that we set $V_{\mathbb{C}}^{\otimes 0} = \mathbb{C}$.*

Elements of $a \in A$ can be written as

$$a = (a_1, ..., a_N, 0, ...),$$

for N finite with

$$a_k = \sum_{i\in I} c_i(v_{i,1} \otimes ... \otimes v_{i,k}),$$

with $c_i \in \mathbb{C}$ and $v_{i,j} \in V$ and $\#I < \infty$.

*The free *-algebra of V is given by $(A, \cdot, *)$ where $\cdot : A^2 \to A$ is an associative multiplication and $* : A \to A$ is an involution. The multiplication is given by*

$$(a_1, ..., a_M, 0, ..) \cdot (b_1, ..., b_N, 0, ...) = ((a \cdot b)_1, ..., (a \cdot b)_{M+N}, 0, ...),$$

where

$$(a \cdot b)_k = \sum_{i+j=k} a_i \otimes b_j.$$

This allows us to write

$$a = \sum_{i\in I} c_i(v_{i,1} \cdot ... \cdot v_{i,k_i}),$$

for any $a \in A$, where I is some finite index set and $k_i \in \mathbb{N}$.

The involution is given by

$$\left(\sum_{i\in I} c_i(v_{i,1} \cdot ... \cdot v_{i,k_i})\right)^* = \sum_{i\in I} c_i^*(v_{i,k_i} \cdot ... \cdot v_{i,1}).$$

*The CCR algebra is then defined by imposing canonical commutation relations on A. Let $I \subset A$ be the smallest two-sided *-ideal containing all elements of the form*

$$v \cdot w - w \cdot v - i\Omega(v, w),$$

with $v, w \in V$. The CCR algebra of (V, Ω) is then given by $\mathcal{A} = A/I$.

[5] While we are quantizing a real scalar field, the resulting algebra is over the complex numbers. This is necessary in order to implement that the commutation relations.

We can now apply this construction to $(S_c(M), \Omega)$. The construction above is very concrete. More abstractly, we could characterize the CCR algebra as the unital *-algebra that is generated by elements $\hat{\phi}(F)$ for each $F \in S_c(M)$ such that the following relations hold (Hollands & Wald, 2015):

$$\forall F, G \in S_c(M), a \in \mathbb{R} : \hat{\phi}(aF + G) = a\hat{\phi}(F) + \hat{\phi}(G) \tag{3.7}$$

$$\forall F \in S_c(M) : \hat{\phi}(F)^* = \hat{\phi}(F) \tag{3.8}$$

$$\forall F, G \in S_c(M) : [\hat{\phi}(F), \hat{\phi}(G)] = i\Omega(F, G)\mathbb{1}. \tag{3.9}$$

Notationally, this is already closer to how we would write observables in 'ordinary' quantum field theory. In principle, we can view $\hat{\phi}$ as the linear map that embeds the vector space V, or in this case $S_c(M)$, into $\mathcal{A}$.

While this is a perfectly valid way of constructing an operator algebra and it will turn out to be exactly the algebra that we are looking for, it is not immediately clear how this algebra relates to the formal local field operators $\hat{\phi}(x)$ and whether this gives the right canonical commutation relations. It turns out that there is a completely equivalent construction of the Klein-Gordon CCR algebra, using a slightly different vector space, which makes this connection evident.

We will show that

$$S_c(M) \cong C_c^\infty(M)/(\Box^2 - m^2)C_c^\infty(M).$$

This allows us to associate $\hat{\phi}(F)$ for $F \in S_c(M)$ with $\hat{\phi}(f)$ for $f \in C_c^\infty(M)$ such that we can interpret $\hat{\phi}$ as an "operator valued distribution" formally satisfying

$$\hat{\phi}(f) = \int_M dV \hat{\phi}(x) f(x). \tag{3.10}$$

where $\hat{\phi}((\Box^2 - m^2)f) = 0$, which would make the statement that $\hat{\phi}(x)$ solves the Klein-Gordon equation precise.

Let us first make some more definitions.

Definition 3.1.14. *For a globally hyperbolic M we define the* retarded *and* advanced Green's function, *$R, A : C_c^\infty(M) \to C^\infty(M)$. Given a test function $f \in C_c^\infty(M)$, let Rf be the unique smooth function on M such that*

$$(\Box^2 - m^2)Rf = f,$$

where Rf vanishes in the past of $\mathrm{supp}(f)$*, or in other words it has vanishing Cauchy data on some Σ such that* $\mathrm{supp}(f) \subset D^+(\Sigma)$*. The advanced Green's function A is defined analogously, but Af vanishes in the future of* $\mathrm{supp}(f)$*. Now we define the* causal map *$\Delta : C_c^\infty(M) \to S_c(M)$ by*

$$\Delta = R - A.$$

The causal map has some nice properties, summed up in the following theorem.

Theorem 3.1.15. *Let M globally hyperbolic, $(S_c(M), \Omega)$ the symplectic vector space of strong solutions, and $\Delta : C_c^\infty(M) \to S_c(M)$ the causal map. Then the following four properties hold:*

1. *Δ is linear;*
2. *Δ is onto, $\forall F \in S_c(M) \exists f \in C_c^\infty(M) : F = \Delta f$;*
3. *$\ker(\Delta) = (\Box^2 - m^2)C_c^\infty(M)$;*
4. *$\forall G \in S_c(M), f \in C_c^\infty(M) : \Omega(\Delta f, G) = \int_M dV fG$.*

Proofs of these properties can be found in (Wald, 1994).

From theorem 3.1.15 and the first isomorphism theorem we can directly conclude:

Corollary 3.1.16. *Let M globally hyperbolic, $(S_c(M), \Omega)$ the symplectic vector space of strong solutions, and $\Delta : C_c^\infty(M) \to S_c(M)$ the causal map. Then the map*

$$\Omega(\Delta\cdot, \Delta\cdot) : (C_c^\infty(M)/((\Box^2 - m^2)C_c^\infty(M)))^2 \to \mathbb{R}$$

is a symplectic form on the vector space $C_c^\infty(M)/((\Box^2 - m^2)C_c^\infty(M))$ such that

$$\Delta : C_c^\infty(M)/((\Box^2 - m^2)C_c^\infty(M)) \to S_c(M)$$

is a symplectomorphism.

Due to property 4 from theorem 3.1.15, we can write, for arbitrary $f, g \in C_c^\infty(M)$,

$$\Omega(\Delta f, \Delta g) = \int_M dV g\Delta f = \langle \Delta f, g \rangle. \tag{3.11}$$

Note that this defines a presymplectic form on $C_c^\infty(M)$.[6]

Definition 3.1.17. *Let M be globally hyperbolic with associated causal map $\Delta : C_c^\infty(M) \to S_c(M)$. We define the* causal propagator *as the anti-symmetric linear map*

$$\Delta : C_c^\infty(M)^2 \to \mathbb{R},$$
$$\Delta(f, g) = \langle \Delta f, g \rangle.$$

The distinction between the causal map and the causal propagator is to be understood from context.

We saw that the causal propagator $\Delta : C_c^\infty(M)^2 \to \mathbb{R}$ is a presymplectic form on $C_c^\infty(M)$. The degenerate elements (i.e. the test functions $f \in C_c^\infty(M)$ such that $\forall g \in C_c^\infty(M) : \Delta(f, g) = 0$) are given by

$$\mathrm{dgn}(\Delta) = (\Box^2 - m^2)C_c^\infty(M),$$

as can be inferred from property 3 of theorem 3.1.15. We can conclude that by symplectic reduction of this presymplectic vector space, we exactly regain the

[6] A presymplectic form $\Omega : V^2 \to \mathbb{R}$ on a vector space V is an anitsymmetric linear map. $\Omega(v, w) = -\Omega(w, v)$ for $v, w \in V$. The difference symplectic and presymplectic form is that a presymplectic form can be degenerate, i.e. there may be a $v \in V$ such that for all $w \in V$ we have $\Omega(v, w) = 0$. We refer to the space of these degenerate elements as $\mathrm{dgn}(\Omega)$.

symplectic space from corollary 3.1.16.[7]

Doing a CCR construction with a symplectic vector space (V', Ω') that is the symplectic reduction of some presymplectic space (V, Ω) is equivalent to doing the CCR construction with the presymplectic space and imposing the relation

$$\forall f \in \mathrm{dgn}(\Omega) \subset V : \hat{\phi}(f) = 0.$$

We conclude that we can make the following construction.

Definition 3.1.18. *Given a globally hyperbolic M, we define the* Klein-Gordon algebra $\mathcal{A}_{KG}(M)$ *as the unital *-algebra generated by formal elements*

$$\{\hat{\phi}(f) | f \in C_c^\infty(M)\},$$

subject to the following relations:

$$\forall f, g \in C_c^\infty(M), a \in \mathbb{R} : \hat{\phi}(af + g) = a\hat{\phi}(f) + \hat{\phi}(g); \tag{3.12}$$

$$\forall f \in C_c^\infty(M) : \hat{\phi}(f)^* = \hat{\phi}(f); \tag{3.13}$$

$$\forall f, g \in C_c^\infty(M) : [\hat{\phi}(f), \hat{\phi}(g)] = i\Delta(f, g)\mathbb{1}; \tag{3.14}$$

$$\forall f \in C_c^\infty(M) : \hat{\phi}((\Box^2 - m^2)f) = 0. \tag{3.15}$$

We refer to $(M, \mathcal{A}_{KG}(M), \hat{\phi})$ as the Klein-Gordon triple, *where we interpret $\hat{\phi}$ as the linear map*

$$\hat{\phi} : C_c^\infty(M) \to \mathcal{A}_{KG}(M).$$

We note that this algebra is isomorphic to the CCR algebra of $(S_c(M), \Omega)$, since the underlying symplectic space is symplectomorphic to the quotient space

$$(C_c^\infty(M)/((\Box^2 - m^2)C_c^\infty(M)), \Delta).$$

This algebra reminds us of the formal operators $\hat{\phi}(x)$ that we are familiar with from standard approaches in flat space quantum field theory. We note that we can interpret the condition

$$\forall f \in C_c^\infty(M) : \hat{\phi}((\Box^2 - m^2)f) = 0$$

as the fact that the algebra generators weakly solve the Klein-Gordon equation as "operator valued distributions". On flat space, the commutation relations

$$[\hat{\phi}(f), \hat{\phi}(g)] = i\Delta(f, g)\mathbb{1}$$

also corresponds to 'standard' QFT. In fact, one can view these commutation relations as the solution to the Klein-Gordon equation subject to initial values given by (3.2) (Brunetti et al., 2015). Therefore, we conclude that the construction above indeed gives the free scalar quantum field theory with mass m on curved space-time. The upshot of defining the Klein-Gordon algebra in this abstract way, in contrast to defining them as operators on a Hilbert or Fock space as is done in standard flat space quantization, is that we have a much larger class of states available to us.

[7] In general, given a presymplectic vector space (V, Ω), we can define the symplectic reduction of this space as the symplectic space (V', Ω') where $V' = V/\mathrm{dgn}(V)$ and $\Omega' : V'^2 \to \mathbb{R}$ defined by $\Omega'([v], [w]) = \Omega(v, w)$ for $v, w \in V$. Usually we omit the $'$ in Ω' and refer to the symplectic and presymplectic form by the same symbol.

3.1.3 Time evolution and the Klein-Gordon algebra

An important property of this theory (and of more general QFT's on a globally hyperbolic space-time) is that it is satisfies the time-slice axiom, i.e. there is a well-defined time evolution in a sense that mirrors the classical case where a global solution to the Klein-Gordon equation is uniquely fixed from initial data on a Cauchy surface Σ. The global field is entirely determined/predicted by the field configuration in a globally hyperbolic neighbourhood of Σ. Let us make this precise.

Definition 3.1.19. *Let M globally hyperbolic, $\mathcal{A}_{KG}(M)$ the Klein-Gordon algebra on M and $U \subset M$ an open subset. We can define the subalgebra*

$$\mathcal{A}_{KG}(M;U) \subset \mathcal{A}_{KG}(M)$$

as the algebra generated by

$$\{\hat{\phi}(f) | f \in C_c^\infty(U) \subset C_c^\infty(M)\}$$

satisfying relations inherited by $\mathcal{A}_{KG}(M)$.

Note that whenever U is in itself a globally hyperbolic manifold, the algebra $\mathcal{A}_{KG}(U)$ is exactly the same as $\mathcal{A}_{KG}(M;U)$, since the pre-symplectic form $\Delta_U : C_c^\infty(U)^2 \to \mathbb{R}$ coincides with $\Delta : C_c^\infty(M)^2 \to \mathbb{R}$ on the domain $C_c^\infty(U)^2$.

An important consequence of this is that associating a Klein-Gordon algebra to a globally hyperbolic manifold can be seen as a functor between the categories of globally hyperbolic space-times and unital *-algebras, where the functor maps the morphisms of manifold inclusion to algebra inclusion. This motivates an axiomatic formulation of quantum field theory within the language of category theory. See for instance (Brunetti et al., 2015, Ch. 4).

Since an embedding $i : U \to M$ of an open subset U into M naturally gives rise to an embedding of unital * algebras $i : \mathcal{A}_{KG}(M;U) \to \mathcal{A}_{KG}(M)$ (which is a *-homomorphism), one might wonder when a manifold embedding gives rise to a *-isomorphism.

Definition 3.1.20. *Let M be globally hyperbolic and $U \subset M$ an open subset. We call M* detemined *from U if the induced *-homomorphism $i : \mathcal{A}_{KG}(M;U) \to \mathcal{A}_{KG}(M)$ is an isomorphism.*

Now we have the following theorem.

Theorem 3.1.21. *Let M be globally hyperbolic, Σ a Cauchy surface of M, and $U \subset M$ an open subset such that $\Sigma \subset U$. Then M is determined by U.*

Proof. We first prove that

$$C_c^\infty(U)/(\Box^2 - m^2)C_c^\infty(U) = C_c^\infty(M)/(\Box^2 - m^2)C_c^\infty(M),$$

when the former is seen as a subspace of the latter.

Denote $[\cdot]_U$ as the equivalence classes that make up $C_c^\infty(U)/(\Box^2 - m^2)C_c^\infty(U)$. Let

$$[f]_U \in C_c^\infty(U)/(\Box^2 - m^2)C_c^\infty(U),$$

then $f \in C_c^\infty(U) \subset C_c^\infty(M)$, so

$$[f]_M \in C_c^\infty(M)/(\Box^2 - m^2)C_c^\infty(M).$$

Note that if $g \in [f]_U$ then

$$f - g \in (\Box^2 - m^2)C_c^\infty(U) \subset (\Box^2 - m^2)C_c^\infty(M),$$

so $g \in [f]_M$. Therefore we can indeed write

$$C_c^\infty(U)/(\Box^2 - m^2)C_c^\infty(U) \subset C_c^\infty(M)/(\Box^2 - m^2)C_c^\infty(M).$$

Now suppose

$$[f]_M \in C_c^\infty(M)/(\Box^2 - m^2)C_c^\infty(M),$$

we have seen that this can be uniquely associated with a strong solution of the Klein-Gordon equation $F \in S_c(M)$ on M satisfying some initial data $c \in \mathfrak{C}(\Sigma)$. Note that $D(\Sigma) \cap U$ is in itself a globally hyperbolic submanifold of M containing the Cauchy surface Σ. Therefore, we have a unique strong solution to the Klein-Gordon equation with initial data c on the space-time $D(\Sigma) \cap U$ that is also uniquely extended to F. From this and theorem 3.1.15 we see that there is a $g \in C_c^\infty(D(\Sigma) \cap U) \subset C_c^\infty(U) \subset C_c^\infty(M)$ such that $\Delta g = F$. This means that $g \in [f]_M$, so it follows that $[f]_M = [g]_U$, so

$$C_c^\infty(M)/(\Box^2 - m^2)C_c^\infty(M) \subset C_c^\infty(U)/(\Box^2 - m^2)C_c^\infty(U).$$

From this, we can conclude that for all the generators $\hat{\phi}(f) \in \mathcal{A}_{KG}(M)$ there exists a generator $\hat{\phi}(g) \in \mathcal{A}_{KG}(M;U)$ (as embedded in $\mathcal{A}_{KG}(M)$) such that $\hat{\phi}(f) = \hat{\phi}(g)$. It follows that $\mathcal{A}_{KG}(M;U) = \mathcal{A}_{KG}(M)$. This means that the embedding $i : \mathcal{A}_{KG}(M;U) \to \mathcal{A}_{KG}(M)$ is an isomorphism. $\Box$

What does this effectively mean? Let us introduce the notion of a state on the Klein-Gordon algebra. The space of states is defined as follows.

Definition 3.1.22. *Let $\mathcal{A}$ be a unital *-algebra. The* state space *of $\mathcal{A}$ is*

$$\mathcal{S}(\mathcal{A}) = \{\omega \in \mathcal{A}^* \mid \omega(\mathrm{Id}_{\mathcal{A}}) = 1, \forall a \in \mathcal{A} : \omega(a^*a) \geq 0\}, \tag{3.16}$$

where $\mathcal{A}^*$ is the algebraic dual of $\mathcal{A}$, i.e. the complex vector space of complex linear functionals on $\mathcal{A}$.

This allows us to define n-point functions

Definition 3.1.23. *Let M be globally hyperbolic, $\mathcal{A}_{KG}(M)$ the Klein-Gordon algebra on M and $\omega \in \mathcal{S}(\mathcal{A}_{KG}(M))$ a state. We define the* n-point functions *on M associated with ω as $\omega_{(n)} : C_c^\infty(M)^n \to \mathbb{C}$ via*

$$\omega_{(n)}(f_1, ..., f_n) = \omega(\hat{\phi}(f_1), ..., \hat{\phi}(f_n)), \tag{3.17}$$

wheree $f_1, ..., f_n \in C_c^\infty(M)$.

Note that the n-point function $\omega_{(n)}$ can be viewed as an n-distribution that solves the Klein-Gordon equation in all n entries seperately.[8]

We can quite easily see that for some globally hyperbolic M with Klein-Gordon triple $(M, \mathcal{A}_{KG}(M), \hat{\phi})$, where M is determined by $U \subset M$, defining a state on $\mathcal{A}_{KG}(M;U)$ rather trivially fixes the state on $\mathcal{A}_{KG}(M)$, as these algebras are basically the same: the global state is determined by the local state on U. When formulated in terms of n-point functions, we can combine this with 3.1.21 to get the following result.

Corollary 3.1.24. *Let M be globally hyperbolic and Σ a Cauchy surface. Then an n-point function on M is uniquely determined by its behaviour in an arbitrary neighbourhood of Σ.*

From this result we can infer that, given a global time function that foliates the space-time into equal time Cauchy surfaces, a state of the quantum field on some fixed time, uniquely determines the state at all later times (and also at all earlier times). Consequently, the Klein-Gordon QFT on a globally hyperbolic space-time satisfies principle of predictability stated in section 1.2.2.

We can take a slightly different perspective on time evolution of the Klein-Gordon algebra by recalling theorem 3.1.12.[9] The CCR algebra of $(\mathfrak{C}(\Sigma), \Omega_\Sigma)$ for some arbitrary Cauchy surface $\Sigma \subset M$ is isomorphic to the Klein-Gordon algebra by associating initial data $c \in \mathfrak{C}(\Sigma)$ with a test function $f \in C_c^\infty(M)$ via the symplectomorphism $\mathfrak{s}_\Sigma^{-1} \circ \Delta : C_c^\infty(M) \to \mathfrak{C}(\Sigma)$. Now note that two arbitrary (smooth) Cauchy surfaces are diffeomorphic (Bernal & Sanchez, 2003). In fact, we have the following lemma.

Lemma 3.1.25. *Let $\Sigma, \Sigma' \subset M$ Cauchy surfaces with induced metrics h and h' and $\psi : \Sigma \to \Sigma'$ a diffeomorphism. The map $\Psi : \mathfrak{C}(\Sigma') \to \mathfrak{C}(\Sigma)$ defined by*

$$\Psi(\phi, \pi) = \left(\psi^*(\phi), \frac{\psi^*\left(\sqrt{h'}\pi\right)}{\sqrt{h}} \right)$$

is a symplectomorphism, where ψ^* is the pullback and $\sqrt{h}$ and $\sqrt{h'}$ are the square roots of the determinant of the induced metrics evaluated in coordinate charts $x : U \to \mathbb{R}^n$ and $x' : U' \to \mathbb{R}^n$ on Σ and Σ' respectively, such that $x = \psi^*(x')$.

Proof. Let us first prove that the map Ψ is independent of the choice of coordinate systems on Σ and Σ'. Suppose that x', y' are two coordinate maps on some region of $U' \subset \Sigma'$ such that $x = \psi^*(x'), y = \psi^*(y')$ are coordinate maps on $U = \psi^{-1}(U') \subset \Sigma$. Let us make the distinction between the determinant of the metric on Σ evaluated for x and y as h_x and h_y respectively, and similarly

[8] Technically, these are not distributions, as the states as we have introduced then do not need to satisfy any continuity requirements. We could have required our states to be continuous in an appropriate topology, but at this point we will stick to the purely algebraic notion of a state.

[9] In fact, this shift in perspective is due to a switch from a covariant quantization of a global field $\hat{\phi}$ to a canonical quantization, which introduces the fields $(\hat{\varphi}, \hat{\pi})$ defined on a Cauchy surface.

$h'_{x'}$ and $h'_{y'}$. We will show that

$$\frac{\psi^*\left(\sqrt{h'_{x'}}\right)}{\sqrt{h_x}} = \frac{\psi^*\left(\sqrt{h'_{y'}}\right)}{\sqrt{h_y}}.$$

We know that $\sqrt{h}$ transforms as a weight 1 scalar density under coordinate transformations, in other words, $\sqrt{h_y} = (J \circ y)\sqrt{h_x}$, where J is the Jacobian of the map $x \circ y^{-1} : y(U) \to x(U)$. Now we note that

$$x \circ y^{-1} = x \circ \psi \circ \psi^{-1} \circ y^{-1} = x \circ \psi \circ \circ (y \circ \psi)^{-1} = x' \circ y'^{-1},$$

and thus $\sqrt{h'_y} = (J \circ y')\sqrt{h'_x}$. We have

$$(J \circ y) = (J \circ \psi^*(y')) = \psi^*(J \circ y'),$$

therefore

$$\frac{\psi^*\left(\sqrt{h'_{y'}}\right)}{\sqrt{h_y}} = \frac{\psi^*\left((J \circ y')\sqrt{h'_{x'}}\right)}{\sqrt{(J \circ y)h_x}} = \frac{\psi^*(J \circ y')}{\psi^*(J \circ y')}\frac{\psi^*\left(\sqrt{h'_{x'}}\right)}{\sqrt{h_x}} = \frac{\psi^*\left(\sqrt{h'_{x'}}\right)}{\sqrt{h_x}}.$$

This shows that Ψ is well defined.

It then follows from definition 3.1.10 that

$$\Omega_\Sigma(\Psi(\phi_1, \pi_1), \Psi(\phi_2, \pi_2)) = \Omega_{\Sigma'}((\phi_1, \pi_1), (\phi_2, \pi_2)).$$

Thus Ψ is a symplectomorphism. □

From lemma 3.1.25 we can also conclude that a diffeomorphism between two Cauchy surfaces Σ and Σ' induces an isomorphism between their associated CCR algebras. We can therefore identify these algebras with each other. In particular, if we foliate M by (smooth) Cauchy surfaces, so that $M = \mathbb{R} \times \Sigma$, there is an obvious diffeomorphism between two 'time-slices' Σ_t and $\Sigma_{t'}$ for $t, t' \in \mathbb{R}$. Therefore, given a particular notion of time (a global time-function) on M, we have a particular way of identifying the CCR algebras associated with different time-slices. By theorem 3.1.12, which states that the function $\mathfrak{s}_{\Sigma_t \to \Sigma_{t'}}$ is a symplectomorphism, this map also induces an isomorphism of CCR algebras. Hence we obtain:

Theorem 3.1.26. *Given M globally hyperbolic and $T : M \to \mathbb{R}$ a global time-function such that M is foliated by equal time surfaces, which implements a diffeomorphism $M \cong \mathbb{R} \times \Sigma$. Let $\mathcal{A}$ be the CCR algebra associated with Σ. Then for any $t, t' \in \mathbb{R}$ there is an automorphism $S(t', t)$ on $\mathcal{A}$ associated with time evolution from $T = t$ to t' given by*

$$S(t', t)\hat{\phi}(c) = \hat{\phi}(\mathfrak{s}_{\Sigma_t \to \Sigma_{t'}}(c)),$$

where $c \in \mathfrak{C}(\Sigma_t)$, which means $\mathfrak{s}_{\Sigma_t \to \Sigma_{t'}}(c) \in \mathfrak{C}(\Sigma_{t'})$.

Hence we see that given a particular notion of time/foliation, time evolution in algebraic QFT can be associated with *automorphisms* on the Klein-Gordon algebra. This canonical viewpoint on AQFT somewhat reminds us somewhat of the Heisenberg formulation of quantum mechanics. In this formulation, time-evolution is also given by a family of automorphisms on the algebra of observables, however, a key difference here is that while in quantum mechanics these automorphisms have an implementation of unitary operators on a Hilbert space, in the context of quantum fields this is no longer the case. In principle, only when the space-time background has a time-translation symmetry (i.e. when the gradient of the time-function foliating the space-time into Cauchy surfaces is a Killing field) can we find unitary implementations of the automorphism encoding time-evolution of the quantum field, at least within a particular representation of the Klein-Gordon algebra (Brunetti et al., 2015, Ch. 5).

3.2 QFT on non-globally hyperbolic space-times

As we have discussed in section 1.2, the space-time usually associated with black hole evaporation (see figure 1.1) is not globally hyperbolic. We have seen from the example of the Klein-Gordon theory that the standard construction of a quantum field theory relies heavily on the global hyperbolicity of the space-time. When we give up global hyperbolicity, we can typically not find unique solutions to the classical theory from initial data, and therefore there is no unique CCR algebra that can be constructed from such a solution space (if it even exists), as theorem 3.1.8 generally does not hold anymore. Still, people have tried to take some of the intuitions from the globally hyperbolic case, and have made suggestions for what a QFT on a more general curved space-time should look like. These suggestions have not resulted in a unique construction of, for instance, a Klein-Gordon QFT as in the globally hyperbolic case. Historically, there are two approaches to algebraic quantum field theory on non-globally hyperbolic space-times, both proposed in the nineties, viz. the Yurtsever construction (Yurtsever, 1994) and Kay's F-locality approach (Kay, 1992; Fewster & Higuchi, 1996; Kay, 1996). In this section, we will focus on the latter approach.[10]

The F-locality approach was originally introduced in order to study quantum field theories on space-times with closed time-like curves. It was meant to address the question whether one can consistently define a notion of quantum field theory on achronal space-times,[11] or whether some (quantum gravity) mechanism is needed to prohibit the formation of such space-times (Hawking, 1992). We will mostly look at the F-locality proposal in the context of Klein-Gordon theory, as discussed in (Fewster & Higuchi, 1996), though a more general notion of F-locality has been introduced in (Kay, 1996).

The idea of F-locality is that in an arbitrarily small neighbourhood of any point a Klein-Gordon quantum field theory on a non-globally hyperbolic space-time should behave exactly like the Klein-Gordon algebra on globally hyperbolic space-times. Let us formalize this. First, we identify the class of algebras that may be considered to be a quantum field theory, called *pre-field algebras*.

[10] The construction of Yurtsever might also be applicable to the evaporating black hole, but so far we have not found a way to make it work. The Yurtsever construction depends in an essential way on a choice of a 'classical solution space' of (smooth or less constrained) solutions to the Klein-Gordon equation and it is not clear which choice of solution space is the right one for the black hole evaporation space-time.

[11] An achronal space-time has closed causal curves, see appendix A.1.

Definition 3.2.1. *Given a space-time M, a* pre-field triple *is given by* $(M, \mathcal{A}, \hat{\phi})$ *where* $\mathcal{A}$*, the* pre-field algebra*, is a unital *-algebra and* $\hat{\phi} : C_c^\infty(M) \to \mathcal{A}$ *such that*

$$\forall f, g \in C_c^\infty(M), a \in \mathbb{R} : \hat{\phi}(af + g) = a\hat{\phi}(f) + \hat{\phi}(g) \quad (3.18)$$

$$\forall f \in C_c^\infty(M) : \hat{\phi}(f)^* = \hat{\phi}(f) \quad (3.19)$$

$$\forall f \in C_c^\infty(M) : \hat{\phi}((\Box^2 - m^2)f) = 0 \quad (3.20)$$

$$\forall \mathcal{B} \subset \mathcal{A} \text{ unital *-subalgebra with } \hat{\phi}(C_c^\infty(M)) \subset \mathcal{B} : \mathcal{B} = \mathcal{A}. \quad (3.21)$$

Obviously, for M globally hyperbolic, the Klein-Gordon triple $(M, \mathcal{A}_{KG}(M), \hat{\phi}_{KG})$ forms a pre-field algebra. In fact, any pre-field triple on a globally hyperbolic space-time that satisfies the right commutation relations is a Klein-Gordon triple, hence the prefix 'pre', as one can sometimes construct a Klein-Gordon triple out of a pre-field triple by taking a suitable quotient of some pre-field algebra.

In order to define F-locality, we first need to discuss what we mean by the local behaviour of an algebra by generalizing definition 3.1.19.

Definition 3.2.2. *Given a pre-field triple* $(M, \mathcal{A}, \hat{\psi})$*,* $U \subset M$ *open, we define the* local pre-field algebra $\mathcal{A}(U) \subset \mathcal{A}$ *as the smallest unital *-subalgebra such that* $\hat{\psi}(C_c^\infty(U)) \subset \mathcal{A}(U)$.

When considering a pre-field triple $(M, \mathcal{A}, \hat{\phi})$ and $N \subset M$ a globally hyperbolic submanifold, there are two natural pre-field theories on N, the local pre-field triple $(N, \mathcal{A}(N), \hat{\phi}_{|C_c^\infty(N)})$ and the Klein-Gordon algebra $(N, \mathcal{A}_{KG}(N), \hat{\phi}_{KG})$. This brings us to the definition of *F-locality.*[12]

Definition 3.2.3. *A pre-field triple* $(M, \mathcal{A}, \hat{\phi})$ *satisfies* F-locality *if for all* $x \in M$ *there is an* $N \subset M$ *globally hyperbolic such that there is a *-isomorphism*

$$\chi : \mathcal{A}(N) \to \mathcal{A}_{KG}(N),$$

where $\hat{\phi}_{KG} = \chi \circ \hat{\phi}_{|C_c^\infty(N)}$.

In other words, for an F-local pre-field theory any point should have a globally hyperbolic neighbourhood on which the local pre-field triple and the Klein-Gordon triple coincide.

This notion of a quantum field theory will form the starting point for the next chapter, here we will attempt to construct an F-local quantum field theory on a fully evaporating black hole space-time and study some properties of this theory.

[12]The F stands for finite, indicating that F-local quantum fields only have to agree with the globally hyperbolic construction on finitely large globally hyperbolic neighbourhoods.

Chapter 4

A QFT on evaporating black holes

The message that we take away from the end of the previous chapter is that a quantum field theory may be defined as a pre-field triple that locally looks like a quantum field theory on a globally hyperbolic space-time. We have seen that F-locality is, in a sense, a minimal requirement for this to be true. As such it has proven a useful requirement to study the possibility of quantum field theory on space-times that, for instance, have closed time-like loops. However, we want to study space-times similar to the black hole evaporation space-time, which are arguably better behaved when it comes to their global causal structure. For these space-times, we propose a stronger condition than F-locality.

For a globally hyperbolic space-time, the Klein-Gordon algebra of course satisfies the F-locality condition. However, an F-local linear scalar quantum field theory on a globally hyperbolic space-time does not have to be the same as the Klein-Gordon algebra globally, only locally. We find this an undesirable feature of the definition of F-locality. Therefore, we propose the following definition of a linear scalar quantum field theory:

Definition 4.0.1. *Let* $(M, \mathcal{A}, \hat{\phi})$ *a pre-field triple. We say that this is a* quantum field triple *if for any globally hyperbolic submanifold* $N \subset M$ *the triple* $(N, \mathcal{A}(N), \phi_{|C_c^\infty(N)})$ *is isomorphic to the Klein-Gordon triple on* N. *We say that* M *is* quantum compatible *if it admits a quantum field triple.*

This is a stronger requirement than F-locality and therefore many space-times that admit an F-local quantum field theory may not admit a quantum field theory as defined above.[1] However, for a globally hyperbolic space-time the unique quantum field triple is the Klein-Gordon triple, as we desired.

[1] Arguably, the definition for a quantum field triple as we give it is less sophisticated than a pre-fielt triple satisfying F-locality. In fact, this definition is not of much use in the context where F-locality was first introduced, quantum field theories on space-times with closed causal loops (i.e. acausal space-times), as it is immediately clear that our definition places too many constraints on a theory to exist on such a space-time, even for the most simple cases of acausal background space-times. However, as we will see later, we consider a far smaller class of space-times, for which our definition of a quantum field triple is sufficient.

We wish to show that we can define a quantum field triple for a fully evaporating black hole. In order to do this, let us first study slightly more general space-times, namely those that are semi-globally hyperbolic, as we will define in the next section.

4.1 A new causality condition

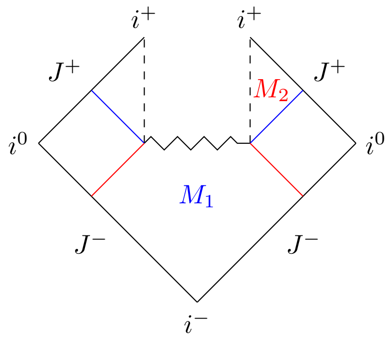

Figure 4.1: Evaporating (2D) black hole as a union of two globally hyperbolic space-times

Though the space-time associated with full black hole evaporation is not globally hyperbolic, it is the union of a finite number of globally hyperbolic space-times, in this case two, i.e. $M = M_1 \cup M_2$, see figure 4.1. It should also be noted that it is stably causal.[2] Even though we cannot foliate M using Cauchy surfaces, we can still find a smooth global function $T : M \to \mathbb{R}$ of which the gradient ∇T is past directing time-like,[3] taking for instance the 'height' of a point in the conformal diagram. The existence of a global time-function is equivalent to causal stability (Minguzzi & Sanchez, 2006). We can thus view this space-time as an example of the following.

Definition 4.1.1. *Let M be a connected space-time. We call M* semi-globally hyperbolic *if M is stably causal (i.e. there exists a global time-function $T : M \to \mathbb{R}$) and there exists a finite cover of open connected globally hyperbolic $M_i \subset M$ with $M = \bigcup_i M_i$, such that for each $U \subset M$ connected and globally hyperbolic there is an $M_i \subset M$ with $U \subset M_i$.*[4]

Since it was shown in (Lesourd, 2019) that the evaporation space-time is not causally continuous, it is clear that semi-global hyperbolicity is not stronger than causal continuity. However, it is also not weaker than causal continuity, as we can easily see that a space-time with a time-like boundary (see for instance figure

[2]See appendix A.1.

[3]∇T should be past directing since we use the (-,+,...,+) signature, if we use the oposite signature, ∇T should be future directing.

[4]For simplicity we only consider connected space-times. However we could generalize to an arbitrary space-time M. Then we call M semi-globally hyperbolic if all the connected components are semi-globally hyperbolic.

4.2) is causally continuous, but not semi-globally hyperbolic. Obviously semi-global hyperbolicity is a stronger property than just causal stability and weaker than global hyperbolicity. Therefore, we cannot simply put the property of semi-global hyperbolicity in the hierarchy of causality conditions as established in appendix A.1 (i.e. the causal ladder). However we could have an alternative hierarchy where the properties of causal continuity and causal simplicity are replaced by semi-global hyperbolicity.

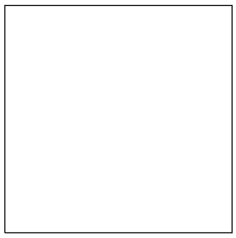

Figure 4.2: A square cut-out of 2D Minkowski space $((0,1)^2, \eta)$ is causally continuous, but not semi-globally hyperbolic.

We can classify semi-globally hyperbolic space-times by the minimal number of globally hyperbolic submanifolds necessary to cover the entire space-time.

Definition 4.1.2. *Let M be a connected semi-globally hyperbolic space-time. We call M of n*th degree *if the minimal number of globally hyperbolic sets covering M as in 4.1.1 is n. Such a set $\{M_i\}_{i\in I}$ with $\#I = n$, is called the* standard cover*.*

We say *the* standard cover instead of a standard cover due to the following lemma.

Lemma 4.1.3. *Let M be a connected semi-globally hyperbolic space-time of nth degree. Let $\{M_i\}_{i\in I}$ and $\{\tilde{M}_j\}_{j\in J}$ be standard covers. Then $\{M_i\}_{i\in I} = \{\tilde{M}_j\}_{j\in J}$.*

Proof. Let $i' \in I$. Then since $M_i \subset M$ is globally hyperbolic there is a $j' \in J$ such that $M_{i'} \subset \tilde{M}_{j'}$. Similarly there is a $i'' \in I$ such that $\tilde{M}_{j'} \subset M_{i''}$. This means $M_i \subset M_{i'}$. Since $\{M_i\}_{i\in I}$ is a standard cover, this means that $i' = i''$, otherwise $M = \bigcup_{i\in I\setminus\{i'\}} M_i$, thus M would at most be of $(n-1)$th degree, which is a contradiction. Therefore, for each $i' \in I$ there exists a $j' \in J$ such that $M_{i'} = \tilde{M}_{j'}$, hence $\{M_i\}_{i\in I} \subset \{\tilde{M}_j\}_{j\in J}$. Since we can show in the same way that $\{\tilde{M}_j\}_{j\in J} \subset \{M_i\}_{i\in I}$, this finishes the proof. □

Now it is clear that a 1st degree semi-globally hyperbolic manifold is globally hyperbolic. Therefore, the first interesting case is 2nd degree, of which we had already seen an example in figure 4.1, the black hole evaporation space-time. Another example is drawn in figure 4.3. One might imagine that a space-time consisting of multiple evaporating black holes is of higher degree, though the exact degree would depend on if the 'evaporation events' are in causal contact or not.

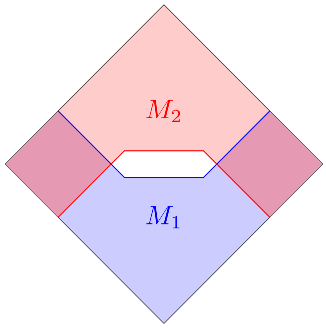

Figure 4.3: A 2nd degree semi-globally hyperbolic space-time

In the next section we discuss how we might construct Klein-Gordon quantum field theories on semi-globally hyperbolic space-times.

4.2 Semi-global hyperbolicity and QFT

As we have seen in section 3.2, it is possible to extend the notion of a quantum field theory to non-globally hyperbolic space-times. However, we have formulated a stricter definition for what a quantum field theory is supposed to be than the F-locality concept introduced in (Kay, 1992). In this section we study these stricter quantum field theories on semi-globally hyperbolic space-times.

4.2.1 The extended causal propagator

Let us for simplicity assume that the space-times that we consider in this section are 2nd degree semi-globally hyperbolic with standard cover $M = M_1 \cup M_2$. We will try to generalize the construction from section 3.1 by defining a pre-symplectic vector space $(C_c^\infty(M), \Delta)$, where Δ is an *extended causal propagator*.

Definition 4.2.1. *Let $M = M_1 \cup M_2$ be a 2nd degree semi-globally hyperbolic space-time with standard cover. An* extended causal propagator *on M is an anti-symmetric bi-distribution $\Delta : C_c^\infty(M)^2 \to \mathbb{R}$ that satisfies the Klein-Gordon equation* (3.3) *in both entries and coincides with the causal propagators Δ_1 and Δ_2 associated with M_1 and M_2 in their respective domains.*

We then apply the CCR construction to the pre-symplectic vector space $(C_c^\infty(M), \Delta)$. This results in a quantum field triple as defined in definition 4.0.1. It should be noted that this is not the most general construction of a pre-field or quantum field triple on a semi-globally hyperbolic space-time. After all, this construction results in $[\hat{\phi}(f), \hat{\phi}(g)] \propto \mathbb{1}$ for all $f, g \in C_c^\infty(M)$ and not just for f and g with support on some globally hyperbolic submanifold. This is not a necessary requirement for a quantum field triple. Nevertheless, it is clear that when such a map Δ exists, we can always construct a quantum field triple. Therefore, we have the following theorem.

Theorem 4.2.2. *Let M be a 2nd degree semi-globally hyperbolic space-time. If there exists an extended causal propagator on M, this space-time is quantum compatible c.f. definition 4.0.1.*

We now ask ourselves how we can construct such an extended causal propagator? Let us first make the following observation.

Lemma 4.2.3. *Let $M = M_1 \cup M_2$ be a manifold with M_1 and M_2 open submanifolds of M. For each $f \in C_c^\infty(M)$, there are $f_1 \in C_c^\infty(M_1), f_2 \in C_c^\infty(M_2)$ such that $f = f_1 + f_2$.*

Proof. There exists a partition of unity subordinate to the cover $M = M_1 \cup M_2$ (Tu, 2011). Thus we have a $\psi_{1,2} \in C^\infty(M)$ such that $\operatorname{supp}\psi_i \subset M_i$ and $\psi_1 + \psi_2 = 1$. Now let $f_i = f\psi_i \in C^\infty(M)$, so $f = f_1 + f_2$. Since $\operatorname{supp} f$ is compact and $\operatorname{supp}\psi_i$ is closed, this means $\operatorname{supp} f_i = \operatorname{supp} f \cap \operatorname{supp}\psi_i \subset M_i$ is compact, thus $f_i \in C_c^\infty(M_1)$. □

Using this, we see that for any extended causal propagator the following should hold:

$$\Delta(f, g) = \Delta_1(f_1, g_1) + \Delta_2(f_2, g_2) + \Delta(f_1, g_2) - \Delta(g_1, f_2),$$

for any $f, g \in C_c^\infty(M)$ where $f_{1,2}, g_{1,2} \in C_c^\infty(M_{1,2})$ such that $f = f_1 + f_2$, $g = g_1 + g_2$. We require Δ to be a solution to (3.3) in both entries. This already holds for $\Delta_{1,2}$, so we still need to require that for any $f_1 \in C_c^\infty(M_1)$, $g_2 \in C_c^\infty(M_2)$ we have

$$\Delta((\Box^2 - m^2)f_1, g_2) = \Delta(f_1, (\Box^2 - m^2)g_2) = 0.$$

This means in particular that for all $f_1 \in C_c^\infty(M_1)$ one must have

$$\Delta(f_1, \cdot)_{|C_c^\infty(M_2)} \in S(M_2).$$

Therefore, we can isolate from Δ a linear map $D : C_c^\infty(M_1) \to S(M_2)$ such that for all $f_1 \in C_c^\infty(M_1)$ we have

$$D((\Box^2 - m^2)f_1) = 0.$$

This map is given by

$$\langle Df_1, g_2\rangle = \Delta(f_1, g_2), \tag{4.1}$$

for any $f_1 \in C_c^\infty(M_1)$ and $g_2 \in C_c^\infty(M_2)$. We now try to do this the other way round: By defining a map $D : C_c^\infty(M_1) \to S(M_2)$ we construct a corresponding extended causal propagator. Here we have to be careful, since the decomposition of a test function $f = f_1 + f_2$ depends on a choice of partition of unity and hence is not unique. Luckily, we have the following theorem.

Theorem 4.2.4. *Let $M = M_1 + M_2$ be a 2nd degree semi-globally hyperbolic space-time with standard cover such that $\Delta_{1,2} : C_c^\infty(M_{1,2})^2 \to \mathbb{R}$ are the causal propagators associated with the globally hyperbolic space-times $M_{1,2}$. Let $D : C_c^\infty(M_1) \to S(M_2)$ be a linear map with $(\Box^2 - m^2)C_c^\infty(M_1) \subset \ker(D)$ such that:*

1. *$\forall f \in C_c^\infty(M_1) : (D - \Delta_1)f_{|C_c^\infty(M_1 \cap M_2)} = 0$;*
2. *$D_{|C_c^\infty(M_1\cap M_2)} - \Delta_{2|C_c^\infty(M_1\cap M_2)} = 0$.*

Then we can define an anti-symmetric bi-distribution $\Delta_D : C_c^\infty(M)^2 \to \mathbb{R}$ by

$$\Delta_D(f, g) = \Delta_1(f_1, g_1) + \Delta_2(f_2, g_2) + \langle Df_1, g_2\rangle - \langle Dg_1, f_2\rangle, \tag{4.2}$$

which solves (3.3) in both entries and coincides with $\Delta_{1,2}$ on their domains.

Proof. We prove that Δ is well defined, i.e. does not depend on the decomposition of test functions on M into test functions on the globally hyperbolic submanifolds M_1 and M_2. Then the theorem follows from the discussion above.

Suppose $g \in C_c^\infty(M)$ with decomposition

$$g = g_1 + g_2 = g_1' + g_2',$$

and $f_1 \in C_c^\infty(M_1)$. We note that

$$\tilde{g} = g_1 - g_1' = g_2' - g_2 \in C_c^\infty(M_1 \cap M_2).$$

Then

$$\begin{aligned} \Delta_1(f_1, g_1) + \langle Df_1, g_2\rangle - (\Delta_1(f_1, g_1') + \langle Df_1, g_2'\rangle) &= \\ \Delta_1(f_1, g_1 - g_1') - \langle Df_1, g_2' - g_2\rangle &= \\ \langle(\Delta_1 - D)f_1, \tilde{g}\rangle &= 0, \end{aligned}$$

by property 1 from the theorem. We show in the same way that for $f_2 \in C_c^\infty(M_2)$ we obtain

$$\begin{aligned} \Delta_2(f_2, g_2) - \langle Dg_1, f_2\rangle - (\Delta_2(f_2, g_2') - \langle Dg_1', f_2\rangle) &= \\ -\Delta_2(g_2 - g_2', f_2) + \langle D(g_1' - g_1), f_2\rangle &= \\ \langle(D - \Delta_2)\tilde{g}, f_2\rangle &= 0, \end{aligned}$$

using property 2. Now we can show that for $f, g \in C_c^\infty(M)$ with $f = f_1 + f_2 = f_1' + f_2'$ and $g = g_1 + g_2 = g_1' + g_2'$ we have

$$\begin{aligned} \Delta_1(f_1, g_1) + \Delta_2(f_2, g_2) + \langle Df_1, g_2\rangle - \langle Dg_1, f_2\rangle &= \\ \Delta_1(f_1, g_1') + \Delta_2(f_2, g_2') + \langle Df_1, g_2'\rangle - \langle Dg_1', f_2\rangle &= \\ -(\Delta_1(g_1', f_1) + \Delta_2(g_2', f_2) + \langle Dg_1', f_2\rangle - \langle Df_1, g_2'\rangle) &= \\ -(\Delta_1(g_1', f_1') + \Delta_2(g_2', f_2) + \langle Dg_1', f_2'\rangle - \langle Df_1', g_2'\rangle) &= \\ \Delta_1(f_1', g_1') + \Delta_2(f_2', g_2') + \langle Df_1', g_2'\rangle - \langle Dg_1', f_2'\rangle&. \end{aligned}$$

Thus we conclude that Δ_D is well defined, which together with the discussion prior to this theorem finishes the proof. □

Due to the importance of this map D for our considerations, let us give it a name.

Definition 4.2.5. *Let $M = M_1 \cup M_2$ a 2nd degree semi-globally hyperbolic space-time with standard cover. We call a map $D : C_c^\infty(M_1) \to S(M_2)$ satisfying the properties from theorem 4.2.4 a* causal map extension.

Since we had already noted that we can also retrieve the map D from Δ via (4.1), we find:

Corollary 4.2.6. *Let $M = M_1 \cup M_2$ quantum compatible. Any extended causal propagator Δ uniquely corresponds to a causal map extension $D : C_c^\infty(M_1) \to S(M_2)$ such that $\Delta = \Delta_D$.*

How should we interpret a causal map extension D? Backtracking a little to the globally hyperbolic case, we found that the causal propagator Δ maps a test function to a strong solution of the Klein-Gordon equation. We could therefore take D as an extension of the map Δ_1 (that maps a test function on M_1 to a strong solution on M_1) to give a global (possibly weak) solution. It is therefore tempting to think that the symplectic space which we are trying to construct, and then quantize, might be equivalent to the space of global solutions given by D, some subset of $S(M)$, but we will see that this is not the case.

At first glance it seems that in our discussion above we have introduced an asymmetry between M_1 and M_2, since, as noted, D can be seen as a choice of how to continue a strong solution on M_1 to a weak solution on all of M. Obviously one might just has well have done the constructions above by defining a causal map extension $D : C_c^\infty(M_2) \to S(M_1)$, swapping M_1 and M_2 in the entire discussion. As is obvious from corollary 4.2.6, there is also a unique correspondence between causal map extensions from M_1 to M_2 and the other way around, by virtue of defining the same extended causal propagator. Let us make this more specific.

Definition 4.2.7. *Let $M = M_1 \cup M_2$ be quantum compatible c.f. definition 4.0.1. Given a causal map extension $D : C_c^\infty(M_1) \to S(M_2)$, we define the* conjugate *causal map extension*

$$D^\dagger : C_c^\infty(M_2) \to S(M_1),$$

by

$$\langle D^\dagger f_2, f_1 \rangle = -\langle D f_1, f_2 \rangle, \tag{4.3}$$

such that $\Delta_D = \Delta_{D^\dagger}$.

It should be noted that, unlike in the globally hyperbolic case, where $\Delta : C_c^\infty(M) \to S_c(M)$ maps test functions into strong solutions to some well defined initial value problem, an extended causal propagator can only be viewed as mapping testfuctions to weak solutions. We will see that some of the notions we introduce on quantum compatible space-times are made simpler when we assume that Δ maps test functions to strong solutions.

Definition 4.2.8. *Let $M = M_1 \cup M_2$ be quantum compatible, and $D : C_c^\infty(M_1) \to S(M_2)$ a causal map extension. We call D* strong *if $D(C_c^\infty(M_1)) \subset S_c(M_2)$.*

It should be noted that if D is strong, this does not mean that $D^\dagger$ is strong.

In the next section we will see some examples of cases where M is semi-globally hyperbolic for which we can define a pre-symplectic form on $C_c^\infty(M)$ using the ideas above. Nevertheless, a causal map extension is not guaranteed to exist for every (2nd degree) semi-globally hyperbolic space-time, nor will it generally be unique. Nevertheless, suppose we have constructed a pre-symplectic vector space $(C_c^\infty(M), \Delta)$ that fits our wishes. We argued that via the CCR construction this gives us a quantum field triple. This construction requires us to take a symplectic reduction. In other words, we should divide out the subspace of degenerate elements of Δ, i.e.

$$\mathrm{dgn}(\Delta) = \{f \in C_c^\infty(M) \mid \forall g \in C_c^\infty(M) : \Delta(f, g) = 0\}. \tag{4.4}$$

The symplectic reduction of $(C_c^\infty(M), \Delta)$ is the symplectic vector space

$$(C_c^\infty(M)/\mathrm{dgn}(\Delta), \Delta).$$

This brings us to the following definition.

Definition 4.2.9. *Let $M = M_1 \cup M_2$ be a semi-globally hyperbolic manifold with causal map extension $D : C_c^\infty(M_1) \to S(M_2)$. We define the associated* Klein-Gordon algebra $\mathcal{A}_{KG}(M, D)$ *as the CCR algebra of $(C_c^\infty(M)/dgn(\Delta_D), \Delta_D)$.*

Note that $(\Box^2 - m^2)C_c^\infty(M) \subset \mathrm{dgn}(\Delta_D)$, since Δ_D is a bi-solution to the Klein-Gordon equation. However, unlike the globally hyperbolic case, the set $\mathrm{dgn}(\Delta_D)$ may in general also contain other elements. We have the following lemma:

Lemma 4.2.10. *Let $M = M_1 \cup M_2$ semi-globally hyperbolic and $D : C_c^\infty(M_1) \to S(M_2)$ a causal map extension. Then $f = f_1 + f_2 \in dgn(\Delta_D)$ if and only if*

$$\Delta_1 f_1 + D^\dagger f_2 = 0; \tag{4.5}$$

$$\Delta_2 f_2 + D f_1 = 0. \tag{4.6}$$

Since the range of $\Delta_{1,2}$ consists of strong solutions on $M_{1,2}$, the condition $f = f_1 + f_2 \in \mathrm{dgn}(\Delta_D)$ implies $Df_1 \in S_c(M_2)$ and $Df_2 \in S_c(M_1)$. In particular, whenever D is strong (i.e. $D(C_c^\infty(M_1)) \subset S_c(M_2)$), we can slightly rewrite the conditions of lemma 4.2.10.

Lemma 4.2.11. *Let $M = M_1 \cup M_2$ semi-globally hyperbolic and $D : C_c^\infty(M_1) \to S_c(M_2)$ a strong causal map extension. Then $f = f_1 + f_2 \in dgn(\Delta_D)$ if and only if*

$$D f_1 + \Delta_2 f_2 = 0; \tag{4.7}$$

$$\forall g_1 \in C_c^\infty(M_1) : \Omega_1(\Delta_1 f_1, \Delta_1 g_1) = \Omega_2(D f_1, D g_1). \tag{4.8}$$

Proof. Note that

$$\Delta_1 f_1 + D^\dagger f_2 = 0 \iff \forall g_1 \in C_c^\infty(M_1) : \Delta_1(f_1, g_1) = \langle D g_1, f_2 \rangle.$$

Since Dg_1 is a strong solution on M_2, by theorem 3.1.15 this is equivalent to

$$\forall g_1 \in C_c^\infty(M_1) : \Omega_1(\Delta_1 f_1, \Delta_1 g_1) = \Omega_2(D g_1, \Delta_2 f_2).$$

Assuming $Df_1 + \Delta_2 f_2 = 0$, this is equivalent to

$$\forall g_1 \in C_c^\infty(M_1) : \Omega_1(\Delta_1 f_1, \Delta_1 g_1) = \Omega_2(D f_1, D g_1).$$

□

Furthermore, note that another part of theorem 3.1.15 stated that the map $\Delta_2 : C_c^\infty(M_2) \to S_c(M_2)$ is onto. Therefore, for any f_1 we can always find an f_2 such that (4.7) holds. This gives us the following corollary, which will be of use when we discuss pre- and retrodictability of the Klein-Gordon algebra in section 4.2.3.

Corollary 4.2.12. *Let $M = M_1 \cup M_2$ be semi-globally hyperbolic, and $D : C_c^\infty(M_1) \to S_c(M_2)$ a strong causal map extension defining a presymplectic form Δ_D on $C_c^\infty(M)$. Let $f_1 \in C_c^\infty(M_1)$. Then there exists a function $h \in \mathrm{dgn}(\Delta_D)$ such that*

$$h_{|M_1 \setminus M_2} = f_{1|M_1 \setminus M_2} \iff \forall g_1 \in C_c^\infty(M_1) : \Omega_1(\Delta_1 f_1, \Delta_1 g_1) = \Omega_2(D f_1, D g_1).$$

Proof. Let $f_2 \in \Delta_2^{-1}(-Df_1)$; then obviously $\Delta_2 f_2 + Df_1 = 0$. Hence we see $h = f_1 + f_2 \in \mathrm{dgn}(\Delta_D)$ iff (4.7) holds. It is clear that $h_{|M_1 \setminus M_2} = f_{1|M_1 \setminus M_2}$. □

4.2.2 Examples of quantum compatible space-times

Just like we cannot say that each space-time is F-quantum compatible, we can at this point not say with certainty that each semi-globally hyperbolic space-time is quantum compatible. We can, however, easily find examples of quantum compatible space-times. Let us discuss some contexts in which we can construct Δ. The most trivial example is when M can be isometrically embedded into a globally hyperbolic space-time $\tilde{M}$. On this space-time we have a naturally defined causal propagator $\tilde{\Delta}$ that coincides with Δ_1 and Δ_2 on the appropriate domains, as M_1 and M_2 are also globally hyperbolic subsets of $\tilde{M}$. It is easy to show that $D : C_c^\infty(M_1) \to S(M_2)$ defined by $Df = \tilde{\Delta} f_{|C_c^\infty(M_2)}$ is a causal map extension. Of course, this embedding is, in general, not unique. This highlights the fact that F-local QFT's on non-globally hyperbolic space-times are generally highly non-unique (Fewster & Higuchi, 1996), as the global dynamics and commutation relations are not uniquely determined by local dynamics and field equations.

Let us now consider a slightly less trivial construction. Consider a semi-globally hyperbolic space-time $M = M_1 \cup M_2$. Suppose there are two Cauchy surfaces Σ_1 and Σ_2 for M_1 and M_2 respectively such that $\Sigma_1 \cap \Sigma_2$ is Cauchy for $M_1 \cap M_2$.[5] Such a space-time is drawn in figure 4.4. We have seen that a causal map extension D could be seen as extending strong solutions on M_1 to M_2. A strong solution on M_1 is uniquely mapped to initial data on a Cauchy surface of M_1, so in particular it maps uniquely to $\mathfrak{C}(\Sigma_1)$. Now suppose we can map initial data in $\mathfrak{C}(\Sigma_1)$ to $\mathfrak{C}(\Sigma_2)$ in an appropriate way; this will give us a strong solution on M_2, which allows us to define a strong causal map extension.

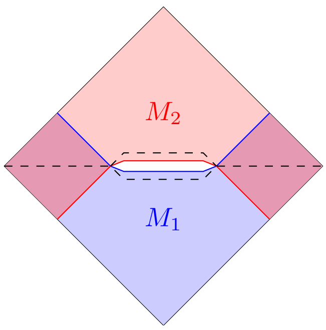

Figure 4.4: A 2nd degree semi-globally hyperbolic space-time on which we can construct a Cauchy data transition function

To make the idea above more concrete, assume we have a linear function $\mathfrak{d}$: $\mathfrak{C}(\Sigma_1) \to \mathfrak{C}(\Sigma_2)$ such that $\forall c \in \mathfrak{C}(\Sigma_1)$ and $p \in \Sigma_1 \cap \Sigma_2$ we have $\mathfrak{d}(c)|_p = c|_p$.

[5] It should be noted that we are glossing over the fact that such surfaces Σ_i will typically not exist if we do not allow it to contain points on the boundary of the space-time. In particular, the points $\partial(\Sigma_1 \cap \Sigma_2)$ in figure 4.4 that are not at infinity will somehow need to be part of the manifolds Σ_i. An important underlying assumption of this construction is therefore that the space-time M can be extended to contain those points. Therefore, it would be more rigorous to assume that there is some (not necessarily semi-globally hyperbolic) space-time $\tilde{M}$ such that $M \subset \tilde{M}$ with slices $\Sigma_1, \Sigma_2 \subset \tilde{M}$ such that $M_i \subset D(\Sigma_i)$ and $M_1 \cap M_2 \subset D(\Sigma_1 \cap \Sigma_2)$.

From this, we construct the map $D_{\mathfrak{d}} : C_c^\infty(M_1) \to S(M_2)$ via

$$\langle D_{\mathfrak{d}} f_1, f_2 \rangle = \int_{M_2} dV f_2 (\mathfrak{s}_{\Sigma_2} \circ \mathfrak{d} \circ \mathfrak{s}_{\Sigma_1}^{-1} \circ \Delta_1)(f_1), \tag{4.9}$$

for every $f_1 \in C_c^\infty(M_1), f_2 \in C_c^\infty(M_2)$. We now claim the following:

Lemma 4.2.13. *Given the assumptions above, $D_{\mathfrak{d}}$ is a causal map extension.*

Proof. We see directly that $D_{\mathfrak{d}}$ is linear and $(\Box^2 - m^2)C_c^\infty(M_1) \subset \ker(D_{\mathfrak{d}})$. So we are left with proving that D and $\Delta_{1,2}$ coincide on the appropriate domains.

We first prove

$$\forall f \in C_c^\infty(M_1) : (D_{\mathfrak{d}} - \Delta_1) f_{|C_c^\infty(M_1 \cap M_2)} = 0.$$

Suppose $f \in C_c^\infty(M_1)$ and $g \in C_c^\infty(M_1 \cap M_2)$. It is clear that $\Delta_1(f,g) = \Delta(f,g)$. In view of the computation

$$\begin{aligned} \langle D_{\mathfrak{d}} f, g \rangle = \int_{M_2} dV g (\mathfrak{s}_{\Sigma_2} \circ \mathfrak{d} \circ \mathfrak{s}_{\Sigma_1}^{-1} \circ \Delta_1)(f) = \\ \Omega_2((\mathfrak{s}_{\Sigma_2} \circ \mathfrak{d} \circ \mathfrak{s}_{\Sigma_1}^{-1} \circ \Delta_1)(f), \Delta_2 g) = \\ \Omega_{\Sigma_2}((\mathfrak{d} \circ \mathfrak{s}_{\Sigma_1}^{-1} \circ \Delta_1)(f), (\mathfrak{s}_{\Sigma_2}^{-1} \circ \Delta_2)(g)) = \\ \Omega_{\Sigma_1 \cap \Sigma_2}((\mathfrak{s}_{\Sigma_1}^{-1} \circ \Delta_1)(f), (\mathfrak{s}_{\Sigma_2}^{-1} \circ \Delta_2)(g)) = \\ \Omega_{\Sigma_1}((\mathfrak{s}_{\Sigma_1}^{-1} \circ \Delta_1)(f), (\mathfrak{s}_{\Sigma_1}^{-1} \circ \Delta_1)(g)) = \Delta_1(f,g), \end{aligned}$$

we conclude $\langle (D_{\mathfrak{d}} - \Delta_1) f, g \rangle = 0$.
We find that $D_{\mathfrak{d}|C_c^\infty(M_1 \cap M_2)} - \Delta_{2|C_c^\infty(M_1 \cap M_2)} = 0$ also easily follows from $\mathfrak{d}(c)|_p = c|_p$. We can thus conclude that $D_{\mathfrak{d}}$ is a causal map extension. $\Box$

From this it follows that, as long as we can define the map $\mathfrak{d}$, the space-time is quantum compatible.

We will apply (an adapted version of) this idea to a space-time with a fully evaporating black hole in section 4.3.

4.2.3 Time evolution on semi-globally hyperbolic space-time

Recall from section 3.1.3 that, in the globally hyperbolic case, for any open subset $U \in M$, where $\Sigma \subset U$ for some Cauchy surface $\Sigma \subset M$, the associated algebra embedding $i : \mathcal{A}_{KG}(M;U) \to \mathcal{A}_{KG}(M)$ is an isomorphism. We took from this a notion of time evolution with a well defined Cauchy problem, in the sense that the full state of the quantum field is determined by the state in an arbitrary neighbourhood of a Cauchy surface. We note that this determination works both forward and backward in time, and therefore the time evolution of a state is both predictable and retrodictable.

We want to extend this notion of time evolution to semi-globally hyperbolic space-times. We are mostly interested in Klein-Gordon algebras as defined in the previous section, but in principle we could write down a definition for more general operator algebras on semi-globally hyperbolic space-times of arbitrary degree. Let us first generalize definition 3.1.20 to arbitrary space-times.

Definition 4.2.14. *Let $(M, \mathcal{A}, \hat{\phi})$ be a pre-field triple. Let $U \subset M$ be an open subset. The algebra $\mathcal{A}(U) \subset \mathcal{A}$ is the smallest subalgebra of $\mathcal{A}$ containing $\hat{\phi}(C_c^\infty(U))$. We say that $(M, \mathcal{A}, \hat{\phi})$ is determined by $U \subset M$ if the embedding $i : \mathcal{A}(U) \to \mathcal{A}$ is a *-isomorphism.*

This definition is very general and $(M, \mathcal{A}, \hat{\phi})$ is not even required to represent any sensible quantum field theory. Note also that, due to its generality, this definition not only applies to the CCR algebra approach as used in the previous sections, but also to the Weyl algebra approach to algebraic quantum field theory, which deals with C*-algebras instead of arbitrary *-algebras (Wald, 1984b). We do not necessarily need this level of generality, but it does show very clearly that our notion of predictability not only applies to globally hyperbolic space-times. What is different from the globally hyperbolic space is that we now don't have theorem 3.1.21. Instead of proper time evolution being a natural feature of quantum field theories, there might be theories that do not have any determinable time evolution, only forward determined evolution (predictability) or backward determined (retrodictability). Let us properly define these notions for a semi-globally hyperbolic space-time.

Definition 4.2.15. *Let M be semi-globally hyperbolic and $T : M \to \mathbb{R}$ a global time-function. Let $(M, \mathcal{A}, \hat{\phi})$ be a pre-field triple. We say that $(M, \mathcal{A}, \hat{\phi})$ is* fully predictable *with respect to T if for all $t \in \mathbb{R}$ and open neighbourhood $U \subset M$ of*

$$\{x \in M : T(x) \leq t\} \subset U,$$

the triple $(M, \mathcal{A}, \hat{\phi})$ is determined by U. Similarly, the theory is fully retrodictable *if for all $t \in \mathbb{R}$ and open neighbourhood $V \subset M$ of*

$$\{x \in M : T(x) \geq t\} \subset V,$$

the triple $(M, \mathcal{A}, \hat{\phi})$ is determined by V.

Looking back at section 2.2.2 we noted that in order for a quantum field theory to be physically meaningful, or rather, useful, it should satisfy some adequate notion of predictability. After all, if we ever want to test a theory by observations, we need to be able to make predictions from a theory. We could imagine a universe that is not predictable, but that would have very serious consequences for the way we do science. Therefore, we do not wish to tamper with the assumption of predictability. We are much more willing to discard retrodictability, which is essentially the same as accepting information loss. In order to show that we can have a consistent quantum field theory with information loss, as argued by Unruh and Wald (2017), we set ourselves the task to construct a Klein-Gordon quantum field theory on a fully evaporating black hole background that is predictable, but not necessarily retrodictable.

In definition 4.2.15 the choice of global time-function is highly relevant. A theory can be fully predictable with respect to one time-function whilest failing to be so for another. This may seem rather problematic, but in fact the same issue already arises in the globally hyperbolic case. Obviously, if we pick a time-function such that the equal-time surfaces are Cauchy, the quantum field theory will be both fully pre- and retrodictable with respect to *that* time-function. However, for any other time-function, this will in general not be true. After all, the Klein-Gordon triple for a globally hyperbolic space-time M is determined by $U \subset M$ if $D(U) = M$, and therefore, if U is some neighbourhood of an equal time surface that is not a Cauchy surface, the triple is in general not determined by U. Because of this, on globally hyperbolic space-times it is natural to consider time-fuctions that foliate the space-time into Cauchy surfaces.

Let us see what definition 4.2.15 means for Klein-Gordon algebras on a semi-globally hyperbolic manifold M of the form $\mathcal{A}_{KG}(M, D)$ with D a causal map extension. Let $M = M_1 \cup M_2$ the standard cover. As mentioned, the choice of time-function matters for our definition of pre- and retrodictability. In order to get some more grip on the situation, we first generalize the natural time-functions on globally hyperbolic space-times (i.e. for which the equal time surfaces are Cauchy and for which the Klein-Gordon triple was both pre- and retrodictable) to the semi-globally hyperbolic case:

Definition 4.2.16. *Let M be a semi-globally hyperbolic and $\{M_i\}_{i \in I}$ its standard cover. We call a time-funtion $T : M \to \mathbb{R}$* natural *if for all $t \in \mathbb{R}$ there is an $I' \subset I$ such that the hypersurface $T = t$ is Cauchy in $\bigcap_{i \in I'} M_i$.*

To our knowledge, semi-globally hyperbolic space-time are not guaranteed to have a natural time-function, but for our purposes it will be a useful assumption that they do (as for instance the black hole evaporation space-time does admit a natural time-function). Let us finally make the following definition:

Definition 4.2.17. *Let M be an nth degree semi-globally hyperbolic space-time, $\{M_i\}_{i=1}^n$ its standard cover, and $T : M \to \mathbb{R}$ a natural time-function. We say $\{M_i\}_{i=1}^n$ is* ordered *with respect to T if the interval $T(M) \subset \mathbb{R}$ can be partitioned into $2n - 1$ intervals I_j with $j = 1, \frac{3}{2}, 2, ..., n - \frac{1}{2}, n$ such that for $x \in I_j, y \in I_{j'}$ we have $x \le y \iff j \le j'$ and*

$$t \in I_j \iff \begin{cases} T = t \text{ Cauchy in } M_j & \text{if } j \text{ is integer} \\ T = t \text{ Cauchy in } M_{j-\frac{1}{2}} \cap M_{j+\frac{1}{2}} & \text{if } j \text{ is not integer} \end{cases}.$$

Here even semi-globally hyperbolic space-times that admit a natural time-function may not admit an orderable standard cover. However, if we do assume this, we may recast definition 4.2.15.

Lemma 4.2.18. *Let M be a connected nth degree semi-globally hyperbolic space-time with natural time-function $T : M \to \mathbb{R}$ and ordered standard cover $\{M_i\}_{i=1}^n$. Then a quantum field triple $(M, \mathcal{A}, \hat{\phi})$ is fully predictable with respect to T iff the pre-field triple is determined by M_1 and fully retrodictable iff the quantum field triple is determined by M_n.*

Proof. Let us only prove predictability, as the case of retrodictability follows the same logic.

Suppose a quantum field triple $(M, \mathcal{A}, \hat{\phi})$ is fully predictable with respect to T. Then in particular there is a $t \in \mathbb{R}$ for which $T = t$ is Cauchy in M_1 and

$$\{x \in M : T(x) \leq t\} \subset M_1,$$

since $\{M_i\}_{i=1}^n$ is ordered with respect to T. By definition of full predictability, this means the triple $(M, \mathcal{A}, \hat{\phi})$ is determined by M_1.

Conversely, suppose $(M, \mathcal{A}, \hat{\phi})$ is determined by M_1. Note that

$$(M_1, \mathcal{A}(M_1), \hat{\phi}_{|C_c^\infty(M_1)})$$

is a Klein-Gordon triple on the globally hyperbolic space-time M_1. Hence $(M_1, \mathcal{A}(M_1), \hat{\phi}_{|C_c^\infty(M_1)})$ is determined by any U containing a Cauchy surface of M_1. Let $t \in \mathbb{R}$. Since the standard cover is ordered, there is a $t' \leq t$ such that $T = t'$ is Cauchy in M_1. Thus for $U \subset M$ for which

$$\{x \in M : T(x) \leq t\} \subset U,$$

we know the surface $T = t'$ is contained in $U \cap M_1$, hence $(M_1, \mathcal{A}(M_1), \hat{\phi}_{|C_c^\infty(M_1)})$ is determined by $U \cap M_1$, and therefore $(M, \mathcal{A}, \hat{\phi})$ is determined by U. Thus $(M, \mathcal{A}, \hat{\phi})$ is fully predictable w.r.t. T. □

Let us now combine the ideas above with the observations from lemma 4.2.10 and the subsequent results to discuss pre- and retrodictability of the Klein-Gordon algebra on 2nd degree globally hyperbolic space-times as defined in section 4.2.

Theorem 4.2.19. *Let $M = M_1 \cup M_2$ be 2nd degree semi-globally hyperbolic. Assume $T : M \to \mathbb{R}$ is a natural time-function such that the standard cover is ordered. For a causal map extension $D : C_c^\infty(M_1) \to S(M_2)$, the quantum field triple $(M, \mathcal{A}_{KG}(M, D), \hat{\phi})$ is fully predictable w.r.t. T if and only if $D^\dagger$ is strong and preserves the symplectic form, i.e.*

$$\forall f_2, g_2 \in C_c^\infty(M_2) : \Delta_2(f_2, g_2) = \Omega_1(D^\dagger f_2, D^\dagger g_2).$$

Similarly, the triple is fully retrodictable if and only if D is strong and preserves the symplectic form.

Proof. For the case of full predictability, it follows from lemma 4.2.18 that $(M, \mathcal{A}_{KG}(M, D), \hat{\phi})$ is fully predictable w.r.t. T iff $(M, \mathcal{A}_{KG}(M, D), \hat{\phi})$ is determined by M_1. This is equivalent to saying that the embedding

$$i : \mathcal{A}_{KG}(M_1) \to \mathcal{A}_{KG}(M, D),$$

is a *-isomorphism.

Note that the Klein-Gordon algebra consists of polynomials in the generators (via the CCR construction). We therefore know that i is a *-isomorphism if and only if $i(\hat{\phi}(C_c^\infty(M_1))) = \hat{\phi}(C_c^\infty(M))$, as i is an injective *-homomorphism by definition.

This means that full predictability is equivalent to the fact that for each $f \in C_c^\infty(M)$ there is a $\tilde{f}_1 \in C_c^\infty(M_1)$ such that $\hat{\phi}(f) = \hat{\phi}(\tilde{f}_1)$, or, in other words,

$$f - \tilde{f}_1 \in \mathrm{dgn}(\Delta_D).$$

Note that $f = f_1 + f_2$ for some $f_1 \in C_c^\infty(M_1)$ and $f_2 \in C_c^\infty(M_2)$. Therefore, we could equivalently state that

$$\text{for all } f_2 \in C_c^\infty(M_2) \text{ there is } \tilde{f}_1 \in C_c^\infty(M_1) \text{ such that } f_2 - \tilde{f}_1 \in \mathrm{dgn}(\Delta_D).$$

For $g \in C_C^\infty(M)$ the equality $f_{2|M_2\setminus M_1} = g_{|M_2\setminus M_1}$ is equivalent to $g = \tilde{f}_1 + f_2$ for some $\tilde{f}_1 \in C_c^\infty(M_1)$. Therefore, $(M, \mathcal{A}_{KG}(M,D), \hat{\phi})$ is fully predictable w.r.t. T if and only if

$$\text{for all } f \in C_c^\infty(M_2) \text{ there is } g \in \mathrm{dgn}(\Delta_D) \text{ such that } f_{|M_2\setminus M_1} = g_{|M_2\setminus M_1}.$$

Note that by lemma 4.2.10 the condition $f_2 - \tilde{f}_1 \in \mathrm{dgn}(\Delta_D)$ is equivalent to

$$\Delta_1 \tilde{f}_1 = D^\dagger f_2;$$
$$\Delta_2 f_2 = D\tilde{f}_1.$$

Therefore, for a fully predictable theory, it follows that $D^\dagger(C_c^\infty(M_2)) \subset S_c(M_1)$, hence the map $D^\dagger$ is strong. Using lemma 4.2.11 we conclude that $(M, \mathcal{A}_{KG}(M,D), \hat{\phi})$ is fully predictable w.r.t. T if and only if for all $f_2 \in C_c^\infty(M_2)$ there is an $f_1 \in C_c^\infty(M_1)$ such that

$$\Delta_1 \tilde{f}_1 + D^\dagger f_2 = 0;$$
$$\forall g_2 \in C_c^\infty(M_2) : \Omega_2(\Delta_2 f_2, \Delta_2 g_2) = \Omega_1(D^\dagger f_2, D^\dagger g_2),$$

i.e. $D^\dagger : C_c^\infty(M_2) \to S_c(M_1)$ is a strong causal map extension that preserves the symplectic form. □

We can find easy examples of fully predictable theories. Returning to the example where M can be embedded into a globally hyperbolic space-time $\tilde{M}$. Suppose M_1 contains a Cauchy surface of $\tilde{M}$. It is easy to see that this gives a predictable algebra. If M_2 contains a Cauchy surface of $\tilde{M}$, it is retrodictable. From these observations we can easily find space-times that admit quantum field triples that are fully predictable but not fully retrodictable, see for instance figure 4.5.

Looking back at lemma 4.2.13, we see that $\mathfrak{d} : \mathfrak{C}(\Sigma_1) \to \mathfrak{C}(\Sigma_2)$ gives a retrodictable theory if $\Omega_{\Sigma_1}(c_1, c_2) = \Omega_{\Sigma_1}(\mathfrak{d}(c_1), \mathfrak{d}(c_2))$.[6] If we recall lemma 3.1.25, this gives us the following result.

Theorem 4.2.20. *Let M be a 2nd degree globally hyperbolic space-time and let Σ_1, Σ_2 be as in lemma 4.2.13. Suppose furthermore that there is a diffeomorphism $\psi : \Sigma_2 \to \Sigma_1$ such that $\psi_{|\Sigma_1\cap\Sigma_2} = \mathrm{Id}$. Then the causal map extension $D_\Psi : C_c^\infty(M_1) \to S_c(M_2)$, with Ψ as in lemma 3.1.25, defines a quantum field theory that is both fully predictable and fully retrodictable.*

[6] A fully predictable theory can be defined via a symplectic form preserving $\mathfrak{d} : \mathfrak{C}(\Sigma_2) \to \mathfrak{C}(\Sigma_1)$ instead.

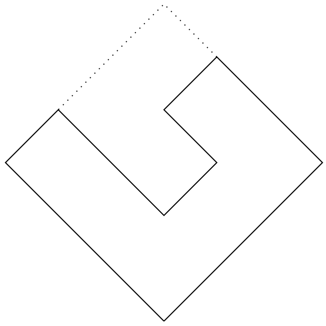

Figure 4.5: Conformal diagram of a semi-globally hyperbolic space-time admitting a causal map extension via embedding in a globally hyperbolic space-time (given by the dotted diagram) such that the resulting QFT is fully predictable but not fully retrodictable.

4.3 Quantum fields on an evaporating black hole

Until now we have concerned ourselves with general features of quantum field theories on semi-globally hyperbolic space-times, with a particular interest in those of 2nd degree. The motivation for this was to find a quantum field theory on fully evaporating black hole space-times. Unfortunately, the examples we have discussed in the previous sections do not directly give a construction on these space-times. For instance, we might hope to apply lemma 4.2.13 to the space-time of figure 4.1. However, this is not possible, as we cannot simply extend this space-time beyond the singularity, which we would need to do in order to find Cauchy surfaces for which a map $\mathfrak{d}$ gives the desired transition of Cauchy data. This is for two reasons. Firstly, the metric blows up at the space-like singularity that forms part of the boundary of M_1, and hence continuing this space-time beyond this singularity is, at best, not trivial. Secondly, the 'evaporation event', i.e. the part of the singularity that is part of the boundary of M_2, has the property that it cannot be added as a point to the space-time without breaking the smooth Lorentzian manifold structure (Manchak & Weatherall, 2018). How then could we still define a Klein-Gordon algebra on this space-time?

4.3.1 An intermezzo on continuity

For reasons that will become clear in the next section, we introduce some additional topology into the game. Given a Cauchy surface on a globally hyperbolic manifold, we will turn the set of all possible Cauchy data on this surface into a topological space by introducing a norm on this space.

Definition 4.3.1. *Let Σ be Cauchy in a globally hyperbolic space-time M. We define a norm $\|.\|_\Sigma : \mathfrak{C}(\Sigma) \to \mathbb{R}_{\geq 0}$ by*

$$\|(\phi, \pi)\|_\Sigma^2 = \int_\Sigma d^3x\sqrt{h}\left(\phi^2 + \pi^2\right), \tag{4.10}$$

where h is the induced metric on Σ.

Note that for $(\phi, \pi) \in \mathfrak{C}(\Sigma)$, we have $\phi, \pi \in L_2(\Sigma)$, the space of square integrable real functions on Σ. Therefore $\mathfrak{C}(\Sigma) \subset L_2(\Sigma) \times L_2(\Sigma)$. The space $L_2(\Sigma)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\Sigma} d^3x \sqrt{h} f g \tag{4.11}$$

for any $f, g \in L_2(\Sigma)$. Denoting the norm on this space by $\|f\|_2 = \sqrt{\langle f, f \rangle}$, we can rewrite

$$\|(\phi, \pi)\|_{\Sigma}^2 = \|\phi\|_2^2 + \|\pi\|_2^2; \tag{4.12}$$
$$\Omega_{\Sigma}((\phi_1, \pi_1), (\phi_2, \pi_2)) = \langle \pi_1, \phi_2 \rangle - \langle \pi_2, \phi_1 \rangle. \tag{4.13}$$

Lemma 4.3.2. *Let Σ be Cauchy in a globally hyperbolic space-time M. Then $(\mathfrak{C}(\Sigma), \Omega_{\Sigma}, \|.\|_{\Sigma})$ is a normed symplectic vector space, with Ω_{Σ} continuous in both entries.*

Proof. Note that from (4.12) it follows that

$$\|(\phi, \pi)\|_{\Sigma} \geq \|\phi\|_2;$$
$$\|(\phi, \pi)\|_{\Sigma} \geq \|\pi\|_2.$$

Using these inequalities and the Cauchy-Schwartz inequality on (4.13) it follows that

$$\begin{aligned} |\Omega_{\Sigma}([\phi_1, \pi_1], [\phi_2, \pi_2])| \leq & \|\pi_1\|_2 \cdot \|\phi_2\|_2 + \|\pi_2\|_2 \cdot \|\phi_1\|_2 \\ \leq & (\|\pi_1\|_2 + \|\phi_1\|_2) \|[\phi_2, \pi_2]\|_{\Sigma}, \end{aligned}$$

from which we can infer that for any $(\phi, \pi) \in \mathfrak{C}(\Sigma)$, the map $\Omega_{\Sigma}((\phi, \pi), .)$ is a bounded linear functional, and hence is continuous. Due to antisymmetry of Ω_{Σ} we see that it is a continuous map in both entries. $\square$

This choice of topology seems rather arbitrary for now. Obviously this normed symplectic vector space is not complete, as we know that compactly supported functions on Σ form a dense subspace of $L^2(\Sigma)$, but do not exhaust it. For many purposes, this incompleteness would be undesireable, but we will use it to our advantage.

Before we do this, let us recall the topology on the space of distributions on M, the weak *-topology. The space of weak solutions to the Klein-Gordon equation $S(M) \subset \mathcal{D}'(M)$ is closed under this topology.[7] The subset topology induced on $S(M)$ gives a notion of convergent sequences on the weak solutions to the Klein-Gordon equation.

Definition 4.3.3. *Let M be a manifold. Pointwise convergence on the space of weak (distributional) solutions to the Klein-Gordon equation $S(M)$ is defined by the following condition. Let $\forall n \in \mathbb{N} : T_n \in S(M)$ and $T \in S(M)$. We say*

$$\lim_{n\to\infty} T_n = T \iff \forall f \in C_c^{\infty}(M) : \lim_{n\to\infty} \langle T_n, f \rangle = \langle T, f \rangle.$$

[7] Recall that by definition, evaluation maps $\langle \cdot, f \rangle : \mathcal{D}'(M) \to \mathbb{R}$ are continuous for each $f \in C_c^{\infty}(M)$. Using this, it is easy to see that $S(M) = \bigcap_{f \in (\Box^2 - m^2) C_c^{\infty}(M)} \langle \cdot, f \rangle^{-1}(\{0\})$ is closed.

Using this notion of convergence, we state the following lemma:

Lemma 4.3.4. *Let Σ be Cauchy in a globally hyperbolic space-time M and the map $\mathfrak{s}_\Sigma : \mathfrak{C}(\Sigma) \to S_c(M)$ the symplectomorphism associating to each Cauchy data the unique associated strong solution. Using the topology on $\mathfrak{C}(\Sigma)$ induced by $\|.\|_\Sigma$ and pointwise convergence on $S_c(M) \subset S(M)$, the map $\mathfrak{s}_\Sigma$ is sequentially continuous.*

Proof. Let $c_n \in \mathfrak{C}(\Sigma)$ such that $\lim_{n\to\infty} c_n = c \in \mathfrak{C}(\Sigma)$. We note that for any $f \in C_c^\infty(M)$ we have

$$\langle \mathfrak{s}_\Sigma(c_n), f\rangle = \Omega(\Delta f, \mathfrak{s}_\Sigma(c_n)) = \Omega_\Sigma(\mathfrak{s}_\Sigma^{-1}(\Delta f), c_n).$$

Since Ω_Σ is continuous, it follows that

$$\lim_{n\to\infty} \langle \mathfrak{s}_\Sigma(c_n), f\rangle = \Omega_\Sigma(\mathfrak{s}_\Sigma^{-1}(\Delta f), c) = \langle \mathfrak{s}_\Sigma(c), f\rangle.$$

It therefore follows that

$$\lim_{n\to\infty} \mathfrak{s}_\Sigma(c_n) = \mathfrak{s}_\Sigma(c),$$

hence $\mathfrak{s}_\Sigma$ is sequentially continuous. □

It should be noted that even though we have proved (sequential) continuity of $\mathfrak{s}_\Sigma : \mathfrak{C}(\Sigma) \to S_c(M)$ for some Cauchy surface Σ, we have not proved continuity for $\mathfrak{s}_{\Sigma\to\Sigma'} : \mathfrak{C}(\Sigma) \to \mathfrak{C}(\Sigma')$. This might cause some worry whether the norm $\|.\|_\Sigma$ is a good choice. However, as we will see later, this does not really matter for our construction of a quantum field theory on an evaporating black hole.

Let us now prove one more lemma, which together with the previous lemmas proves a corollary which will aid us in our construction of a quantum field theory.

Lemma 4.3.5. *Let M a space-time and X a metric vector space. Suppose there is a linear sequentially continuous map $\mathfrak{D}_X : X \to S(M)$. Let $\overline{X}$ be the completion of X. There exists a linear map $\mathfrak{D} : \overline{X} \to S(M)$ such that $\mathfrak{D}_{|X} = \mathfrak{D}_X$.*

Proof. Given $c \in \overline{X}$ we can, by defintion of the closure, find a Cauchy sequence $(c_n)_{n\in\mathbb{N}}$ in X with $\lim_{n\to\infty} c_n = c$. Note that, since we know the map $\mathfrak{D}_X : \mathfrak{C}(\Sigma) \to S(M)$ is linear and sequentially continuous, we know in particular that for all $f \in C_c^\infty(M)$ the map $\langle \mathfrak{D}_X(.), f\rangle : C_c^\infty(\Sigma) \to \mathbb{R}$ is linear and continuous (as sequential continuity and continuity are equivalent if the domain and codomain are metric spaces). Therefore, we know that for all $f \in C_c^\infty(M)$ the sequence $(\langle \mathfrak{D}_X(c_n), f\rangle)_{n\in\mathbb{N}}$ is Cauchy. Hence we can define $\mathfrak{D}(c)$ by the pointwise limit

$$\langle \mathfrak{D}(c), f\rangle = \lim_{n\to\infty} \langle \mathfrak{D}_X(c_n), f\rangle.$$

This limit exists, since $\mathbb{R}$ is complete.

We now show that $\mathfrak{D}(c)$ is well defined. Given a sequence $(b_n)_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} b_n = c$, we know $\lim_{n\to\infty}(c_n - b_n) = 0$. From this it follows that for all $f \in C_c^\infty(M)$ we have

$$\begin{aligned}\lim_{n\to\infty}\langle \mathfrak{D}_X(c_n), f\rangle - \lim_{n\to\infty}\langle \mathfrak{D}_X(b_n), f\rangle &= \\ \lim_{n\to\infty}\langle \mathfrak{D}_X(c_n - b_n), f\rangle &= \\ \lim_{n\to\infty}\langle \mathfrak{D}_X(0), f\rangle &= 0.\end{aligned}$$

Therefore the definition of $\mathfrak{D}(c)$ is independent of the choice of the sequence $(c_n)_{n\in\mathbb{N}}$.

We know that $S(M)$ is (sequentially) closed, from which it is clear that $\mathfrak{D}(c) \subset S(M)$. Furthermore, $\mathfrak{D}$ is linear, since for each $b, c \in \overline{X}$ with $\lim_{n\to\infty} b_n = b$ and $\lim_{n\to\infty} c_n = c$ and $\lambda \in \mathbb{R}$, it follows that $\lim_{n\to\infty} \lambda b_n + c_n = \lambda b + c$. Hence $\mathfrak{D}(\lambda b + c) = \lambda\mathfrak{D}(b) + \mathfrak{D}(c)$. Finally, we can directly see that for $c \in X$ it follows that $\mathfrak{D}(c) = \mathfrak{D}_X(c)$, as $\lim_{n\to\infty} c = c$. □

This brings us to the following result.

Corollary 4.3.6. *Let M a globally hyperbolic space-time with Cauchy surface $\Sigma \subset M$. Then the completion of $\mathfrak{C}(\Sigma)$ is $L^2(\Sigma)^2 \equiv L^2(\Sigma) \times L^2(\Sigma)$ and there exists a linear map $\mathfrak{D} : (L^2(\Sigma))^2 \to S(M)$ such that $\mathfrak{D}_{|\mathfrak{C}(\Sigma)} = \mathfrak{s}_\Sigma$.*

4.3.2 A quantum field on a fully evaporating classical background

How can we use the results from the previous section in constructing a Klein-Gordon algebra for a fully evaporating black hole? Let $M = M_1 \cup M_2$ be the evaporation space-time with standard cover (see figure 4.6). Note that in order to define this algebra, it suffices to define a causal map extension $D : C_c^\infty(M_2) \to S(M_1)$ (note that we have switched around the 'direction' of D compared to section 4.2). If we let $\Sigma_{1\to 2}$ be an acausal Cauchy surface of $M_1 \cap M_2$, we note that any compact subset of $\Sigma_{1\to 2}$ with boundary can be extended to a Cauchy surface Σ_1 of M_1 and Σ_2 of M_2 (Bernal & Sanchez, 2006). Therefore, we make the following definition.

Definition 4.3.7. *Let M globally hyperbolic and Σ a Cauchy surface. We define $\mathfrak{C}_b(\Sigma) \subset \mathfrak{C}(\Sigma)$ by*

$$\begin{aligned}\mathfrak{C}_b(\Sigma) = \{&(\phi,\pi) \in \mathfrak{C}(\Sigma) : \\ &\mathrm{supp}((\phi,\pi)) = \mathrm{supp}(\phi) \cup \mathrm{supp}(\pi) \subset \Sigma \textit{ a manifold with boundary}\}.\end{aligned}$$

Therefore, supposing that we have some $(\phi,\pi) \in \mathfrak{C}_b(\Sigma_{1\to 2})$, we can find Σ_1 Cauchy on M_1 and Σ_2 Cauchy on M_2 that coincide on $\mathrm{supp}((\phi,\pi)) \subset \Sigma_{1\to 2}$. Therefore it is not difficult to see that for any $(\phi,\pi) \in \mathfrak{C}_b(\Sigma_{1\to 2})$ we can find a unique solution to the Klein-Gordon equation on M with support in $J(\mathrm{supp}((\phi,\pi)))$.[8] In particular, this gives us a map $\mathfrak{D}_{\Sigma_{1\to 2}} : \mathfrak{C}_b(\Sigma_{1\to 2}) \to S_c(M_1)$. We now show

[8]For $U \subset M$ one has $J(U) = J^+(U) \cup J^-(U)$, as we also saw in theorem 3.1.8.

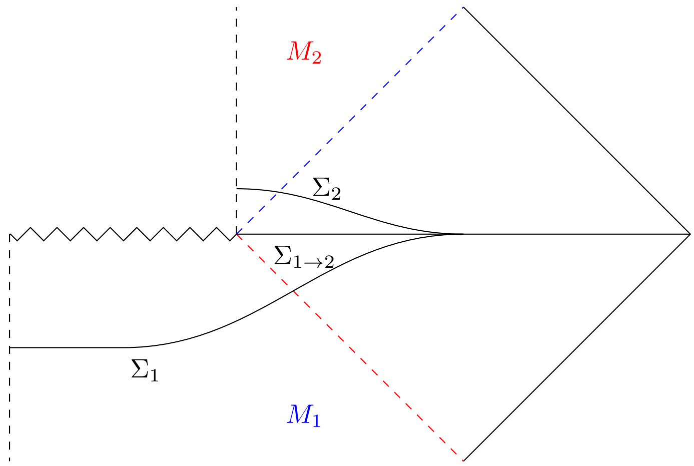

Figure 4.6: A fully evaporating black hole with standard cover $M = M_1 \cup M_2$ and acausal surfaces

Lemma 4.3.8. *Assume that for any test function $f \in C_c^\infty(M_1)$ the quantity*

$$B = \sqrt{\int_{\Sigma_{1\to2}} d^3x\sqrt{h}(n^\mu\partial_\mu\Delta_1 f)^2} + \sqrt{\int_{\Sigma_{1\to2}} d^3x\sqrt{h}(\Delta_1 f)^2}$$

is finite.[9] *Then the map $\mathfrak{D}_{\Sigma_{1\to2}} : \mathfrak{C}_b(\Sigma_{1\to2}) \to S_c(M_1)$ defined above is a sequentially continuous injective linear map between the normed vector space $(\mathfrak{C}_b(\Sigma_{1\to2}), \|.\|_{\Sigma_{1\to2}})$ and $S_c(M_2) \subset S(M_2)$, with respect to pointwise convergence.*

Proof. We will use very similar arguments to the proof of lemma 4.3.4. Linearity of the map is trivial. To show sequential continuity, let $\forall n \in \mathbb{N} : c_n \in \mathfrak{C}_b(\Sigma_{1\to2})$ such that $\lim_{n\to\infty} c_n = c \in \mathfrak{C}_b(\Sigma_{1\to2})$. Note that for each $n \in \mathbb{N}$ we have a Cauchy surface $\Sigma_{1,n}$ of M_1 such that $\Sigma_{1,n}$ coincides with $\Sigma_{1\to2}$ on $\mathrm{supp}(c_n) \cup \mathrm{supp}(c)$. Let $f \in C_c^\infty(M_2)$, we have $\langle\mathfrak{D}_{\Sigma_{1\to2}}(c_n), f\rangle = \Omega_{\Sigma_{1,n}}(\mathfrak{s}^{-1}_{\Sigma_{1,n}}(\Delta_1 f), c_n)$. We do not know yet if we can take a limit here, as we did in lemma 4.3.4, since the choice of Cauchy surface $\Sigma_{1,n}$ depends on n. However, we do note

$$\mathfrak{s}^{-1}_{\Sigma_{1,n}}(\Delta_1 f) = ((\Delta_1 f)_{|\Sigma_{1,n}}, \sqrt{h}(n^\mu\partial_\mu\Delta_1 f)_{|\Sigma_{1,n}}) \in \mathfrak{C}(\Sigma_{1,n}).$$

[9]We do not know when this condition is realized. Further study on partial differential equations on an evaporating black hole background should clarify this.

Now let $c - c_n = (\phi, \pi)$. Then

$$|\Omega_{\Sigma_{1,n}}(\mathfrak{s}^{-1}_{\Sigma_{1,n}}(\Delta_1 f), c_n - c)| =$$
$$\left|\int_{\Sigma_{1,n}} d^3x\sqrt{h}((n^\mu\partial_\mu\Delta_1 f)\phi - \pi(\Delta_1 f))\right| =$$
$$\left|\int_{\mathrm{supp}(c-c_n)} d^3x\sqrt{h}((n^\mu\partial_\mu\Delta_1 f)\phi - \pi(\Delta_1 f))\right| =$$
$$\left|\int_{\Sigma_{1\to 2}} d^3x\sqrt{h}((n^\mu\partial_\mu\Delta_1 f)\phi - \pi(\Delta_1 f))\right| \leq B\|c - c_n\|_{\Sigma_{1\to 2}}.$$

Hence $\langle \mathfrak{D}_{\Sigma_{1\to 2}}(c_n - c), f\rangle$ converges to 0, so that $\mathfrak{D}_{\Sigma_{1\to 2}}$ is sequentially continuous.

Injectivity follows from the fact that for $c \in \mathfrak{C}_b(\Sigma_{1\to 2})$, we find

$$\mathfrak{D}_{\Sigma_{1\to 2}}(c) = 0 \implies c = 0,$$

as $c = (\mathfrak{D}_{\Sigma_{1\to 2}}(c)_{|\Sigma_{1\to 2}}, n^\mu\partial_\mu\mathfrak{D}_{\Sigma_{1\to 2}}(c)_{|\Sigma_{1\to 2}})$. □

From here on we assume that the bound B is always finite for a test function on M_1, ideally we would have some physical motivation for this, but unfortunately we do not. We hope that future research will validate this assumption for the black hole evaporation space-time.

We now shift our attention to M_2. In figure 4.7 we have drawn this globally hyperbolic space-time. We make the following assumption about our evaporation space-time.

Definition 4.3.9. *Given the evaporation space-time $M = M_1 \cup M_2$ we say it has a* weak evaporation event *if there is a Cauchy surface $\Sigma_{1\to 2}$ of $M_1 \cap M_2$ such that M_2 can be isometrically embedded into a globally hyperbolic manifold $\tilde{M}_2$ and this space-time has a smooth Cauchy surface*

$$\Sigma = \Sigma_{1\to 2} \cup \{p\},$$

with $p \in \tilde{M}_2$ the 'evaporation event'.

Note that this assumption is in a sense similar to that of lemma 4.2.13, yet this only concerns M_2 rather than M as a whole. Furthermore, the assumption that we only need to add a single point to the Cauchy surface also sets this assumption apart from previous examples. Though we cannot be completely sure if the evaporation space-time satisfies this assumption, as we don't have a precise calculation of the back-reaction of the Hawking radiation. Therefore, we do not know the precise geometry near the evaporation event. However, several models of black hole evaporation do satisfy this property; this is the case, for instance, when the near-horizon geometry is modeled by the advanced Vaidya metric with decreasing mass (Hiscock, 1981; Schindler et al., 2019).

Let us now prove one more lemma before we are finally ready to define a causal map extension on the evaporation space-time.

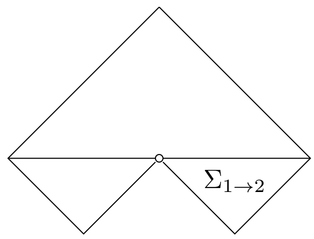

Figure 4.7: A cut-out of M_2 as part of the black hole evaporation space-time

Lemma 4.3.10. *Let M be globally hyperbolic, $\Sigma \subset M$ a Cauchy surface, and $p \subset \Sigma$. Using the topology from definition 4.3.1, $\mathfrak{C}_b(\Sigma \setminus \{p\})$ is dense in $\mathfrak{C}(\Sigma)$.*

Proof. The surface Σ is a smooth m-manifold for some dimension m, so for any $p \in \Sigma$ there exists an open set $p \in U \subset \Sigma$ and a homeomorphism $\psi : U \to \psi(U) \subset \mathbb{R}^m$. A function $f : M \to \mathbb{R}$ is smooth on U iff $f \circ \psi^{-1} : \psi(U) \to \mathbb{R}$ is smooth.

Assume, without loss of generality, that $\psi(p) = 0$. Let $R > 0$ be such that for the open ball of radius R, i.e. $B(R) = \{x \in \mathbb{R}^n | \|x\| < R\}$, we have $B(R) \subset \psi(U)$. Let $0 < r_1 < r_2 < R$ and $\tilde{f}_{r_1,r_2} : \psi(U) \to \mathbb{R}$ be a bump function such that

$$\begin{aligned} \tilde{f}_{r_1,r_2} &\leq 1; \\ \tilde{f}_{r_1,r_2|\overline{B(r_1)}} &= 1; \\ \tilde{f}_{r_1,r_2|\psi(U)\setminus B(r_2)} &= 0. \end{aligned}$$

Then the function $f_{r_1,r_2} : U \to \mathbb{R}$ with $f_{r_1,r_2} = \tilde{f}_{r_1,r_2} \circ \psi$ is smooth on U and the function can be smoothly extended to $f_{r_1,r_2} : M \to \mathbb{R}$ such that $q \in M \setminus U \implies f_{r_1,r_2}(q) = 0$.

Now let $\phi \in C_c^\infty \Sigma$. Clearly, for any $0 < r_1 < r_2 < R$, the function $(1 - f_{r_1,r_2})\phi$ has compact support with boundary in $\Sigma \setminus \{p\}$. We then estimate

$$\begin{aligned} \|\phi - (1 - f_{r_1,r_2})\phi\|_2^2 = \|f_{r_1,r_2}\phi\|_2^2 = \int_\Sigma dV (f_{r_1,r_2}\phi)^2 = \\ \int_{B(r_2)} d^3x \sqrt{h} (f_{r_1,r_2}(\phi \circ \psi^{-1}))^2 \leq \\ \sup_{B(r_2)} ((f_{r_1,r_2}(\phi \circ \psi^{-1}))^2) \mathrm{V}(\psi^{-1}(B(r_2))) \leq \sup_\Sigma(\phi^2) C r_2^m, \end{aligned}$$

where $\mathrm{V}(\psi^{-1}(B(r_2))) \leq C r_2^m$ is the volume (with respect to the metric h on Σ) of the ball of radius r_2 w.r.t. the coördinate chart (U, ψ). Note that C depends on the dimension of the space-time, the metric, and the maximal radius R. From this we conclude that we can find a sequence ϕ_n, having compact support with boundary in $\Sigma \setminus \{p\}$, such that $\lim_{n\to\infty} \phi_n = \phi$. Hence $\mathfrak{C}_b(\Sigma \setminus \{p\})$ is dense in $\mathfrak{C}(\Sigma)$. □

Now we combine the previous two lemmas to prove that we can define a causal map extension for the black hole evaporation space-time. Therefore, this proves that we can define a quantum field triple for an evaporating black hole.

Theorem 4.3.11. *Let $M = M_1 \cup M_2$ be an evaporation space-time with a weak evaporation event as viewed from M_2, c.f. definition 4.3.9. There exists a causal map extension $D : C_c^\infty(M_2) \to S(M_1)$.*

Proof. Let $\Sigma_{1\to 2}$ be a Cauchy surface of $M_1 \cup M_2$ such that M_2 can be extended to a globally hyperbolic space-time $\tilde{M}_2$ with Cauchy surface $\Sigma_{1\to 2} \cup \{p\}$. From lemma 4.3.10 we see that $\mathfrak{C}(\Sigma_{1\to 2} \cup \{p\}) \subset \overline{\mathfrak{C}_b(\Sigma_{1\to 2})}$. Using lemma 4.3.8 and 4.3.5, we know that there is a map $\mathfrak{D} : \mathfrak{C}(\Sigma_{1\to 2} \cup \{p\}) \to S(M_1)$ such that $\mathfrak{D}_{|\mathfrak{C}_b(\Sigma_{1\to 2})} = \mathfrak{D}_{\Sigma_{1\to 2}}$. We now define $D : C_c^\infty(M_2) \to S(M_1)$ by

$$D = \mathfrak{D} \circ \mathfrak{s}^{-1}_{\Sigma_{1\to 2}\cup\{p\}} \circ \tilde{\Delta}_{2|C_c^\infty(M_2)},$$

where $\tilde{\Delta}_2 : C_c^\infty(\tilde{M}_2) \to S_c(\tilde{M}_2)$ is the causal propagator on $\tilde{M}_2$ and

$$\mathfrak{s}_{\Sigma_{1\to 2}\cup\{p\}} : \mathfrak{C}(\Sigma_{1\to 2} \cup \{p\}) \to S_c(\tilde{M}_2)$$

is the function that maps Cauchy data to the associated strong solution to the Klein-Gordon equation. We note that D is linear, and that $(\Box^2 - m^2)C_c^\infty(M_2) \subset \ker(D)$. Furthermore, let $f_1 \in C_c^\infty(M_1)$, $f_2 \in C_c^\infty(M_2)$ and $g \in C_c^\infty(M_1 \cap M_2)$. We now show that $\langle Df_2, g\rangle = \langle \Delta_2 f_2, g\rangle$ and $\langle Dg, f_1\rangle = \langle \Delta_1 g, f_1\rangle$.

Let $c = \mathfrak{s}^{-1}_{\Sigma_{1\to 2}\cup\{p\}} \circ \tilde{\Delta}_2 f_2 \in \mathfrak{C}(\Sigma_{1\to 2} \cup \{p\})$. There is a sequence $c_n \in \mathfrak{C}_b(\Sigma_{1\to 2})$ such that $\lim_{n\to\infty} c_n = c$, and therefore we have

$$\begin{aligned}\langle Df_2, g\rangle = \langle \mathfrak{D}(c), g\rangle = \lim_{n\to\infty} \langle \mathfrak{D}_{\Sigma_{1\to 2}}(c_n), g\rangle = \\ \lim_{n\to\infty} \langle \mathfrak{s}_{\Sigma_{1\to 2}\cup\{p\}}(c_n), g\rangle = \langle \mathfrak{s}_{\Sigma_{1\to 2}\cup\{p\}}(c), g\rangle = \\ \langle \tilde{\Delta}_2 f_2, g\rangle = \langle \Delta_2 f_2, g\rangle.\end{aligned}$$

Now let $c = \mathfrak{s}^{-1}_{\Sigma_{1\to 2}\cup\{p\}} \circ \tilde{\Delta}_2 g$. Since $\mathrm{supp}(g) \subset M_1 \cap M_2$, we have $c = (\phi, \pi) \in \mathfrak{C}(\Sigma_{1\to 2})$. Let $c_n = [\phi_n, \pi_n] \in \mathfrak{C}_b(\Sigma_{1\to 2})$ such that $\lim_{n\to\infty} c_n = c$. Then

$$\begin{aligned}\langle Dg, f_1\rangle = \lim_{n\to\infty} \langle \mathfrak{D}(c_n), f_1\rangle = \\ \lim_{n\to\infty} \int_{\Sigma_{1\to 2}} d^3x((\sqrt{h}n^\mu \partial_\mu \Delta_1 f_1)\phi_n - \pi_n(\Delta_1 f_1)) = \\ \int_{\Sigma_{1\to 2}} d^3x((\sqrt{h}n^\mu \partial_\mu \Delta_1 f_1)\phi - \pi(\Delta_1 f_1)) = \\ \Omega_{1/2}(\Delta_1 f_1, \Delta_2 g) = -\int_{M_1\cap M_2} dV g\Delta_1 f_1 = -\int_{M_1} dV g \Delta_1 f_1 = \\ -\langle \Delta_1 f_1, g\rangle = \langle \Delta_1 g, f_1\rangle\end{aligned}$$

□

We might worry that due to the fact that $\mathfrak{s}_{\Sigma\to\Sigma'} : \mathfrak{C}(\Sigma) \to \mathfrak{C}(\Sigma')$ is not continuous, the construction of the causal map extension D may depend on the choice of the hypersurface $\Sigma_{1\to 2}$. It turns out that this is not the case. Suppose there are $\Sigma_{1\to 2}$ and $\Sigma'_{1/2}$ Cauchy surfaces of $M_1 \cap M_2$ such that D and D' are causal map extensions as given by theorem 4.3.11. Now suppose $f \subset C_c^\infty(M_2)$ and let $c_n \in \mathfrak{C}_b(\Sigma_{1\to 2})$ such that

$$\lim_{n\to\infty} c_n = (\mathfrak{s}^{-1}_{\Sigma_{1\to 2}\cup\{p\}} \circ \tilde{\Delta}_{2|C_c^\infty(M_2)})(f),$$

and $c'_n \in \mathfrak{C}_b(\Sigma'_{1/2})$ such that

$$\lim_{n\to\infty} c'_n = (\mathfrak{s}^{-1}_{\Sigma'_{1/2}\cup\{p\}} \circ \tilde{\Delta}_{2|C_c^\infty(M_2)})(f).$$

This means

$$\begin{aligned} Df &= \lim_{n\to\infty} \mathfrak{D}_{\Sigma_{1\to 2}} c_n; \\ D'f &= \lim_{n\to\infty} \mathfrak{D}_{\Sigma_{1\to 2}} c'_n. \end{aligned}$$

We note that since $\mathfrak{D}_{\Sigma_{1\to 2}}$ and $\mathfrak{D}_{\Sigma'_{1/2}}$ are both continuous injective maps, we have $Df = D'f$ if and only if

$$\lim_{n\to\infty} \mathfrak{s}_{\Sigma_{1\to 2}} c_n = \lim_{n\to\infty} \mathfrak{s}_{\Sigma'_{1/2}} c'_n,$$

as solutions on $M_1 \cap M_2$. Note that by continuity of the solve function,

$$\lim_{n\to\infty} \mathfrak{s}_{\Sigma_{1\to 2}} c_n = \tilde{\Delta}_{2|C_c^\infty(M_2)}(f) = \lim_{n\to\infty} \mathfrak{s}_{\Sigma'_{1/2}} c'_n.$$

Hence $D = D'$. However, the fact that $D : C_c^\infty(M_2) \to S(M_1)$ is independent the choice of Cauchy surface, does not mean it is the only possible causal map extension on M. For example, the topology that we have introduced on Cauchy data is not unique.[10] We therefore do not claim that we have found *the* algebraic quantum field theory for a linear scalar field on an evaporating black hole; we merely found an example.[11] It should also be noted that our definition of a quantum field theory is such that it agrees with the standard construction on any globally hyperbolic open submanifold. This class of theories could be significantly broadened by assuming that the field theory should just be F-local instead. Nevertheless, it should be noted that the theory above has some nice features, most notably that for any $f, g \in C_c^\infty(M)$ such that $\text{supp}(f)$ and $\text{supp}(g)$ are space-like separated, we have $\Delta_D(f, g) = 0$. On the other hand, our theory does present a problem, at least from the perspective of the argument by Unruh and Wald as layed out in section 2.2.1, namely the following.

Corollary 4.3.12. *For M our evaporation space-time and $D : C_c^\infty(M_2) \to S(M_1)$ as constructed in the proof of theorem 4.3.11, the quantum field theory $\mathcal{A}_{KG}(M, D)$ is not fully predictable w.r.t. any natural time-function.*

[10] We could have used a different Sobolev norm instead of the L_2 norm. Here $\mathfrak{C}_b(\Sigma_{1\to 2})$ is in general not dense in $\mathfrak{C}(\Sigma_{1\to 2} \cup \{p\})$. However, when the Sobolev space has an associated inner product, this allows elements of $\mathfrak{C}(\Sigma_{1\to 2} \cup \{p\})$ to be projected onto the closure of $\mathfrak{C}_b(\Sigma_{1\to 2})$, which allows for a definition of D. In this case, the definition of D heavily depends on the choice of inner product, which is not uniquely fixed by the Sobolev topology, illustrating the non-uniqueness of a quantum field triple on a general semi-globally hyperbolic space-time.

[11] Nevertheless, one could argue that the construction above is the most natural, as when one applies is to a semi-globally hyperbolic space-time that is constructed by removing a single point from a globally hyperbolic space-time, the resulting quantum field triples on this semi-globally hyperbolic space-time from the construction above and from its embedding in the globally hyperbolic space-time will be the same.

This follows from the fact that $D : C_c^\infty(M_2) \to S(M_1)$ is not strong, as, unlike $S(M_1)$, the set of strong solutions $S_c(M_1)$ is not sequentially closed under pointwise convergence.

This means that in general an initial state on M_1 does not uniquely determine a final state on M_2. While one could accept this as a reasonable feature of nature, it is, at the very least, a troubling result, as argued in section 2.2.2.

4.3.3 Notes on predictability and the state space

Does corollary 4.3.12 mean that we should doubt our theory? We argue that there is still a way we could make the theory above predictable, by a slight shift of perspective. Note that the definition of full predictability is made at the level of the Klein-Gordon *algebra* instead of at the level of the *states*. Alternatively, we could make the following definition.

Definition 4.3.13. *Let $M = M_1 \cup M_2$ be a 2nd order semi-globally hyperbolic connected space-time that admits a natural time-function and a quantum field triple $(M, \mathcal{A}, \hat{\phi})$. For $\mathcal{S}(\mathcal{A})$ the set of (algebraic) states on $\mathcal{A}$, let $\mathcal{S}' \subset \mathcal{S}(\mathcal{A})$. We say that the state space $\mathcal{S}'$ on $(M, \mathcal{A}, \hat{\phi})$ is* predictable *if*

$$\forall \omega, \omega' \in \mathcal{S}' : \omega_{|\mathcal{A}(M_1)} = \omega'_{|\mathcal{A}(M_1)} \implies \omega = \omega'.$$

Note that our original definition of full predictability implies predictability of the entire algebraic state space $\mathcal{S}(\mathcal{A})$. If one does not need the entire state space to be predictable, but rather only a subset of 'physically allowed' states $\mathcal{S}_{\text{phys}} \subset \mathcal{S}(\mathcal{A})$, we do not require full predictability to ensure predictability of $\mathcal{S}_{\text{phys}}$. Of course to a certain extent we are free to determine what a physical state is and what not. One could, for instance, place some topology on $\mathcal{A}$ and only consider continuous states or place some other regularity conditions on the states. Obviously, we do not want to make the physical state space too small. Even though a physical state space consisting of a single element would automatically give rise to a predictable theory w.r.t. that space, it is clear that this is not a good choice for modeling all relevant physical states. For whatever choice of physical state space we make, it should ultimately be checked if this space is large enough, given that we have some agreement on what this means. Our choice of physical states should ideally also have some physical motivations

Let us give a promising example of a state subspace that is predictable in our theory. Since a state ω is uniquely determined by its n-point functions and vice versa, we will impose conditions on the states by imposing conditions on the n-point functions $\omega_{(n)} : C_c^\infty(M)^n \to \mathbb{C}$. Let us first introduce a new notion of convergence on $C_c^\infty(M)$.[12]

Definition 4.3.14. *Let M globally hyperbolic and $\Delta : C_c^\infty(M) \to S_c(M)$ the associated causal propagator. We define convergence for a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(M)$ by*

$$f_n \xrightarrow{n\to\infty} f \iff \lim_{n\to\infty} \Delta(f_n) = \Delta(f),^{13}$$

[12] This notion of convergence differs from the one given by the topology used to define distributions as continuous functionals in section 3.1.1.

[13] In contrast to pointwise convergence in $S_c(M)$, limits for sequences in $C_c^\infty(M)$ are not unique, as the causal propagator Δ is not injective, hence the change of notation.

where the limit in $S_c(M)$ is given by pointwise convergence.[14]

This notion of convergence has the following property.

Lemma 4.3.15. *Let Σ be a Cauchy surface of the globally hyperbolic manifold M. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C_c^\infty(M)$. Then if $(\mathfrak{s}_\Sigma^{-1}(\Delta(f_n)))_{n\in\mathbb{N}}$ is a convergent sequence in $\mathfrak{C}(\Sigma)$, we have*

$$f_n \xrightarrow{n\to\infty} f \iff \lim_{n\to\infty} \mathfrak{s}_\Sigma^{-1}(\Delta(f_n)) = \mathfrak{s}_\Sigma^{-1}(\Delta(f)).$$

Proof. First note that if

$$\lim_{n\to\infty} \mathfrak{s}_\Sigma^{-1}(\Delta(f_n)) = \mathfrak{s}_\Sigma^{-1}(\Delta(f)),$$

then

$$\lim_{n\to\infty} (\Delta(f_n)) = (\Delta(f)),$$

as $\mathfrak{s}_\Sigma$ is sequentially continuous. Now suppose

$$\lim_{n\to\infty} \mathfrak{s}_\Sigma^{-1}(\Delta(f_n)) \neq \mathfrak{s}_\Sigma^{-1}(\Delta(f)),$$

so by assumption the sequence $(\mathfrak{s}_\Sigma^{-1}(\Delta(f_n)))_{n\in\mathbb{N}}$ converges to some other limit. Therefore,

$$\lim_{n\to\infty} \mathfrak{s}_\Sigma^{-1}(\Delta(f_n)) - \mathfrak{s}_\Sigma^{-1}(\Delta(f)) = \lim_{n\to\infty} \mathfrak{s}_\Sigma^{-1}(\Delta(f_n - f)) \neq 0.$$

Since $\mathfrak{s}_\Sigma$ is injective, this means that $\lim_{n\to\infty} \Delta(f_n - f) \neq 0$, hence $(f_n)_{n\in\mathbb{N}}$ does not converge to f. □

Now we use this notion of convergence to define a restriction on the state space.

Definition 4.3.16. *Let M be a semi-globally hyperbolic space-time and $(M, \mathcal{A}, \hat{\phi})$ a quantum field triple. We call a state ω on $\mathcal{A}$* Δ-regular *if and only if for every globally hyperbolic subset $U \subset M$ the n-point functions are sequentially continuous,*[15] *that is, for any sequence $(f_m)_{m\in\mathbb{N}}$ in $C_c^\infty(U)$ with $f_m \xrightarrow{m\to\infty} f \in C_c^\infty(U)$, we have*

$$\lim_{m\to\infty} \omega_{(n)}(..., f_m, ...) = \omega_{(n)}(..., f, ...).$$

Theorem 4.3.17. *Let $M = M_1 \cup M_2$ be the 2nd degree semi-globally hyperbolic black hole evaporation space-time and $(M, \mathcal{A}_{KG}(M, D), \hat{\phi})$ the quantum field triple defined by theorem 4.3.11. Let $S' \subset S(M)$ the set of Δ-regular states on $\mathcal{A}_{KG}(M, D)$. This state space is predictable.*

[14] This way of defining convergence is similar to how we define the initial topology for maps from a set into a topological space. Indeed, we could alternatively have defined a topology on $C_c^\infty(M)$ as the initial topology defined by Δ and the weak * topology on $S_c(M)$ (or any other topology that is consistent with pointwise convergence of sequences).

[15] We have seen that the limit of a sequence of test functions was not unique, however, this is consistent with the fact that $\omega_{(n)}$ are distributional solutions to the Klein gordon equations in each of their entries.

Proof. Suppose $\omega, \omega' \in S'$ with $\omega_{|\mathcal{A}_{KG}(M_1)} = \omega'_{|\mathcal{A}_{KG}(M_1)}$. This is equivalent to the n-point functions of ω and ω' coinciding on $C_c^\infty(M_1)^n$. Now let $f \in C_c^\infty(M_2)$. Let $\Sigma_{1\to 2}$ a Cauchy surface of $M_1 \cap M_2$ such that there is a globally hyperbolic manifold $\tilde{M}_2$ with $M_2 \subset \tilde{M}_2$ and $p \in \tilde{M}_2$ such that $\Sigma_{1\to 2} \cup \{p\}$ is a Cauchy surface for $\tilde{M}_2$. Let $c = \mathfrak{s}^{-1}_{\Sigma_{1\to 2}\cup\{p\}}(\Delta(f)) \in \mathfrak{C}(\Sigma_{1\to 2} \cup \{p\})$. Then there is a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathfrak{C}_b(\Sigma_{1\to 2}) \subset \mathfrak{C}(\Sigma_{1\to 2} \cup \{p\})$ with $\lim_{n\to\infty} c_n = c$. By lemma 4.3.15 this means there is a convergent sequence $f_n \xrightarrow{n\to\infty} f$ in $C_c^\infty(M_1 \cap M_2) \subset C_c^\infty(M_2)$ with $c_n = \mathfrak{s}^{-1}_{\Sigma_{1\to 2}\cup\{p\}}(\Delta(f_n))$. Therefore, it follows that

$$\omega_{(1)}(f) = \lim_{m\to\infty} \omega_{(1)}(f_m) = \lim_{m\to\infty} \omega'_{(1)}(f_m) = \omega'_{(1)}(f),$$

and similarly for higher n. Hence it follows $\omega = \omega'$. □

Thus we have seen that, given some Δ-regular state at an early time, there is a unique global Δ-regular state that matches this initial state. Some further discussions on state space restrictions can be found in chapter 5.

4.3.4 An alternative approach: the black box black hole

In the previous section we introduced a semi-classical model where a linear quantum field was defined on the entire space-time. We assumed in our model that quantum field theory as we know it is still a viable physical model in high curvature regions of the evaporation space-time. In reality, one would expect quantum gravity effects to play an important role in regions of high curvature and it is questionable if quantum field theory on a classical background is a viable framework to describe even the effective low-energy dynamics in such a region of space-time. Let us therefore propose a different class of models, where we do not aim to define an algebraic quantum field theory on the entire black hole evaporation space-time, but solely outside a certain region of high curvature.[16] Effectively, this means that we cut out a piece of the space-time in the neighbourhood of the singularity and define our quantum field theory on the left-over space-time. However, we should keep in mind that this missing region still needs to be filled in somehow in a final theory of quantum gravity. Therefore the causal relations on the left-over geometry cannot be determined by that geometry alone, as those will depend on whatever geometry fills the high curvature region.

Let us for simplicity assume that the left-over space-time takes a form as in lemma 4.2.13, as drawn in figure 4.8. Recall that by lemma 4.2.13 we could define a quantum field theory on this space-time by defining a linear map $\mathfrak{d} : \mathfrak{C}(\Sigma_2) \to \mathfrak{C}(\Sigma_1)$, where Σ_i is a Cauchy surface of M_i such that $\Sigma_1 \cap \Sigma_2$ is a Cauchy surface of $M_1 \cap M_2$ and $\mathfrak{d}(c)_{|(\Sigma_1\cap\Sigma_2)} = c_{|(\Sigma_1\cap\Sigma_2)}$ for all $c \in \mathfrak{C}(\Sigma_2)$.[17]

Without any extra assumptions, we have considerable freedom in determining $\mathfrak{d}$. The dynamics in the high-curvature region is unknown, so we could view it

[16] In fact, we don't even expect the the space-time in the 'high-curvature' region to be accurately described by a Lorentzian manifold.

[17] Using the insights from lemma 4.3.5, we can generalize the construction of a causal map extension $D_{\mathfrak{d}} : C_c^\infty(M_2) \to S(M_1)$ using a linear map $\mathfrak{d} : \mathfrak{C}(\Sigma_2) \to L_2(\Sigma_1)^2$ satisfying the same properties.

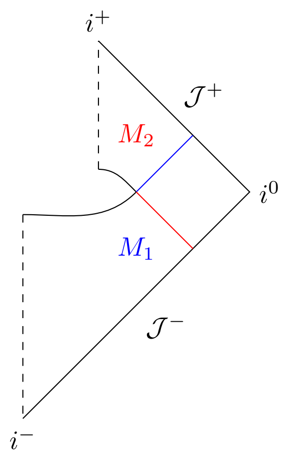

Figure 4.8: Penrose diagram of a fully evaporating black hole with a removed high curvature region

as a black box, with $\mathfrak{d}$ encoding which input (i.e. $f \in C_c^\infty(M_1)$ or, equivalently, $\Delta_1 f \in S_c(M_1)$) is sent to which output ($D_{\mathfrak{d}}^\dagger f \in S(M_2)$).

As we hope to overcome the problem of predictability of the quantum field theory in the construction of the previous section of, this would require us to assume that $\mathfrak{d} : \mathfrak{C}(\Sigma_2) \to \mathfrak{C}(\Sigma_2)$ satisfies

$$\Omega_{\Sigma_2}(c_1, c_2) = \Omega_{\Sigma_1}(\mathfrak{d}c_1, \mathfrak{d}c_2), \tag{4.14}$$

for any $c_1, c_2 \in \mathfrak{C}(\Sigma_2)$. Previously, we had noted in theorem 4.2.20 that if there is an appropriate diffeomorphism between Σ_1 and Σ_2, we could construct a map that satisfies (4.14). However, the resulting quantum field theory would be both fully predictable and retrodictable. Our goal in this section is to show that we can define a theory that is predictable, but not retrodictable (i.e. that it suffers information loss, without having to impose any extra conditions to uniquely determine a final state).

There should be a region in the space-time that is associated with the black hole. Since we have taken out the singularity of the space-time and view this missing region as a 'black box', it has become ambiguous where we should put the event horizon, as this is a global feature of a space-time, which cannot be determined by local geometry. After all, the usual definition of a black hole on some space-time with null infinity $\mathscr{J}^+$ is $B = I^+(\partial I^-(\mathscr{J}^+))$ and as noted the causal relations are not fully determined by the low curvature geometry. However, as soon as we have defined a quantum field theory on the space-time, we can construct causal relations. Recall that in the globally hyperbolic case, if for two functions $f, g \in C_c^\infty(M)$ the regions $\operatorname{supp} f$ and $\operatorname{supp} g$ are not causally related, then $[\hat\phi(f), \hat\phi(g)] = 0$. On the other hand, if we have two open regions U and V that are causally related, then there exist functions $f \in C_c^\infty(U)$ and $f \in C_c^\infty(V)$ such that $[\hat\phi(f), \hat\phi(g)] \neq 0$. Let us therefore define the effective causal relations for some quantum field triple.

Definition 4.3.18. *Let $(M, \mathcal{A}, \hat{\phi})$ be a quantum field triple. We say $x, y \in M$ are* effectively causally related *iff for all $U, V \subset M$ open with $x \in U$ and $y \in V$, there are $f, g \in C_c^\infty(M)$ with $\operatorname{supp} f \subset U$ and $\operatorname{supp} g \subset V$ such that $[\hat{\phi}(f), \hat{\phi}(g)] \neq 0$.*

On globally hyperbolic space-times, the effective causal relations and causal relations as defined in appendix A.1 coincide. However, for a quantum field triple on a semi-globally hyperbolic space-time this will typically not be the case.[18] Let us use this effective causal relation to define the black hole region on the space-time of figure 4.8.

Definition 4.3.19. *Let M be a quantum compatible 2nd degree semi-globally hyperbolic with time-ordered standard cover $M = M_1 \cup M_2$ and let $(M, \mathcal{A}, \hat{\phi})$ be a quantum field triple. We define the effective black hole region by*

$$B = \{x \in M : x \textit{ is not effectively causally related to } M_2\} \subset M_1$$

We will now show that we can construct a predictable quantum field theory on M such that $B \neq \emptyset$. This also implies that M is not retrodictable, which means that we have information loss.

It is clear that if $B \neq \emptyset$, then $B \cap \Sigma_1 \neq \emptyset$. Therefore, if our quantum field triple is of the form $(M, \mathcal{A}_{KG}(M, D_{\mathfrak{d}}), \hat{\phi})$, it must be the case that for all $c \in \Sigma_2$ we have $\mathfrak{d}(c)_{|B\cap\Sigma_1} = 0$.

In principle, we would also want $\mathfrak{d}$ to respect the symmetries of the space-time. For instance, if the space-time has a global rotational symmetry (as in figure 4.8), then $\mathfrak{d}$ should commute with this group acting on Cauchy data (i.e. for a rotation R acting on both $\mathfrak{C}(\Sigma_1)$ and $\mathfrak{C}(\Sigma_2)$, we should have $\mathfrak{d} = R \circ \mathfrak{d} \circ R^{-1}$). Let us give a 'proof of concept' that it is indeed possible to construct a map $\mathfrak{d}$ satisfying the properties above by looking at the 1+1 dimensional case with a reflection symmetry.[19]

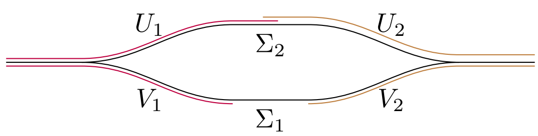

Figure 4.9: Sketch of the hypersurfaces used in the construction of the map $\mathfrak{d}$

Assume for simplicity that Σ_1 and Σ_2 also respect the reflection symmetry. Let $U_1, U_2 \subset \Sigma_2$ be an open cover of Σ_2 (i.e. $\Sigma_2 = U_1 \cup U_2$), such that U_1 and U_2 are each other's mirror image and $U_1 \cap U_2 \cap \Sigma_1 = \emptyset$, hence the two sets only overlap on Σ_2 and not on $\Sigma_1 \cap \Sigma_2$. Now let $V_1, V_2 \subset \Sigma_1$ be disjoint such that $B \cup V_i = \emptyset$, and such that there are diffeomorphisms $\psi_i : V_i \to U_i$ for which $\psi_{i|V_i\cap\Sigma_2} = \mathrm{Id}$, as sketched in figure 4.9. Now let $f_i \subset C^\infty(\Sigma_2, [0,1])$ be a symmetric partition of unity w.r.t. the cover U_i, such that $\operatorname{supp}(f_i) \subset U_i$ and $f_1 + f_2 = 1$. Recall

[18] Though for the quantum field defined in section 4.3.2 these definitions also coincide.
[19] Notice that this space-time has a Penrose diagram identical to figure 4.3.

from lemma 3.1.25 that we can map initial data on $f_i\mathfrak{C}(\Sigma_2) \subset \mathfrak{C}(U_i)$ to initial data on $\mathfrak{C}(V_i) \subset \mathfrak{C}(\Sigma_1)$ via the map Ψ_i which preserves symplectic form. For $c \in \mathfrak{C}(\Sigma_2)$ e can therefore define $\mathfrak{d}(c)$ by

$$\mathfrak{d}(c) = \Psi_1(f_1 \cdot c) + \Psi_2(f_2 \cdot c). \tag{4.15}$$

This argument only works well in 1+1 dimensions, but we expect that a variation of this argument will work in general dimensions and for more general symmetries. We therefore have the following claim.

Claim:
It is possible to construct a fully predictable quantum field theory on an evaporating black hole where we disregard a neighbourhood of the singularity, such that this theory suffers information loss.

Of course, the fact that such a theory exists, does not mean it is realized in nature. One could construct a large range of theories, some fully predictable, some fully retrodictable, some both, some neither. Which theory is actually (approximately, i.e. as some semi-classical limit) realized, can only be decided when we have a good theory of quantum gravity.

Chapter 5

Concluding remarks

Let us quickly recap this thesis. In the first chapter I have recalled why black holes evaporate (at least according to Hawking) and shown that this leads to the information loss paradox. I also argued that the process of black hole evaporation may result in a breakdown of predictability, in the sense that for an initial state that leads to the formation of a fully evaporating black hole, there may be a multitude of final states (pure or mixed) that are consistent with local evolution given by equations of motion. I dubbed this the *problem of predictability*.

In the second chapter I have laid out some responses to the paradox; while there have been many attempts to resolve the paradox by introducing mechanisms that prevent information loss, one can also argue that information loss is acceptable. However, we noted that so far a consistent semi-classical theory (i.e. a quantum field on a classical evaporating black hole background) that indeed suffers from information loss had not been constructed, nor has it been shown how to deal with the problem of predictability in this scenario in a satisfying way. This was the goal I set for myself in rest of the thesis.

In chapter 3 I reviewed how the Klein-Gordon field on a curved background is quantized via the algebraic quantum field theory approach. This standard construction only works for space-times that are globally hyperbolic, which is not the case for a fully evaporating black hole. However, there are ways to generalize concepts from algebraic quantum field theory to more general background space-times, in particular by using the notion of F-locality, though unfortunately this does not give a unique prescription of constructing a quantum field theory. Inspired by these ideas, I have defined a notion of a quantum field theory (which is stronger than F-local QFT's) on space-times that resemble the evaporating black hole in chapter 4, which I have dubbed semi-globally hyperbolic space-times. Using this definition I have shown that there indeed exists a quantum field theory on an evaporating black hole background. This theory has the unfortunate property that it does not overcome the problem of predictability, and we need additional physical input to make the theory complete, at least when we disregard the possibility that a breakdown of predictability in black hole evaporation is an acceptable feature of nature. We have noted that one way to overcome the problem of predictability is to restrict the state space of the theory to a 'physical' state space such that when we only consider global states from this state space, the global dynamics is predictable. One example of such a restriction that works for this are the Δ-regular states. However, so far this example lacks a physical motivation and also places restrictions on the possible initial states (i.e. the state on M_1).

Alternative restrictions will need to be studied further, a particularly interesting question being if there is a state space restriction that does not require any conditions on initial (pre-evaporation) states. We interpret such a restriction as placing a boundary condition on the (weak) evaporation event, though it is arguably quite vague what it means for a boundary condition to be defined on a single point. This also presents us with the following interesting possibility. When we consider the full state space, in general any initial state will allow for a plethora of final states. Most of these possible final states will be mixed, but it is possible that some of these possible final states are pure (as states on M_2). Therefore, one may place a 'boundary condition' on the evaporation event such that the time-evolution of a quantum field on a fully evaporating black hole is still pure-to-pure, perhaps even invertible. This is quite a surprising possibility, as the original motivation for considering quantum fields on non-globally hyperbolic space-times was to show that information loss is not paradoxical, while this discussion seems to suggest that in this setting 'unitarity' of black hole evaporation may be restored. This possibility certainly requires further study. It will first need to be shown that for every pure initial state there is indeed a pure state in the (convex) set of possible final states. Then it remains to be seen if these final states are compatible with the background geometry via the backreaction due to the semi-classical Einstein equations. After all, recall that we constructed our quantum fields on the fixed background, i.e. the black hole evaporation space-time, where the Einstein tensor is 0 near $r = 0$ post-evaporation (see section 1.2.2).[1] Therefore, we should check if the expectation values of the stress-energy tensor (assuming that it is well defined) for the (possibly pure) final states is also 0 in this region, given some initial state that is also consistent with the background geometry (i.e. that it has a stress energy tensor that agrees with spherically symmetric gravitational collapse).

If (some of) the possible final states do not meet this consistency condition, then there are two choices; label these states as non-physical, i.e. using consistency with the background geometry as a (further) selection criterion for physical states, or conclude that the initial choice for background space-time was perhaps not the correct one. For instance, an alternative background may resemble the black hole to white hole transition space-time discussed in chapter 2.

Of course, as mentioned in section 4.3.4, it would be naive to think that quantum fields on curved space-time would be a good description all the way up to the singularity. That is why we have introduced the black box model for black hole evaporation. We argued that in this model, or class of models, it

[1] Actually, we have not uniquely specified which background we use, as we have only used the global causal structure and some generic assumptions on the geometry to construct the quantum field, which are not specific to one particular background.

is possible to have a theory that both satisfies the principle of predictability and suffers information loss. This supports the Unruh-Wald argument that information loss is not *a priori* paradoxical, at least not at the semi-classical level. However, many black box model theories are possible; some will suffer information loss, some will satisfy the principle of predictability, but as long as we do not have a good understanding of what goes on in the high curvature regime, i.e. as long as we do not have a theory of quantum gravity, we have no clue which of these models resembles physical reality.

One might say that this is not a very satisfying conclusion, though we think that it is important to point out where the gaps in our knowledge are that need to be filled in order to solve the information loss parardox. Of course, we are by no means the first to do this, but we still hope that the approaches presented in this thesis shine a somewhat new light on the paradox. At the very least, we have presented a new way of constructing quantum field theories on a specific class of non-globally hyperbolic space-times that does not only include the full black hole evaporation space-time, but also includes space-times such as black box models, black hole to white hole transition space-time and many others.

5.1 Acknowledgements

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Appendices

Appendix A

Lorentzian manifolds and causality

The theory of General Relativity, i.e. the widely accepted classical field theory describing gravity, is formulated within the language of differential geometry, or, to be more specific, Lorentzian geometry. The Lorentzian manifold, in particular its metric, plays the role of the dynamical field in this theory, which will interact with other matter fields defined on the manifold. The metric appears in equations of motion of the matter fields, whilest the equations of motion for the metric (i.e. the Einstein equations) depend on the matter fields as well. For an extensive and mathematically rigorous discussion on general relativity, see for instance (Hawking & Ellis, 1973) and (Wald, 1984b). In this appendix we discuss some of the aspects of Lorentzian geometry and general relativty which are relevant for this thesis. We focus in particular on global causal structure and on black holes.

We assume that the reader is familiar with the concept of a Lorentzian manifold and we will therefore not revise all definitions of a manifold, the tangent space, vector bundles, tensors etc. We just note that a Lorentzian manifold is a $1 + d$ dimensional manifold with metric tensor field g that, in the convention that we will be using, has signature $(-, +, ..., +)$. This defines a vector in a tangent space $v \in T_pM$ at a point p to be either *space-like* ($g_p(v, v) > 0$), *time-like* ($g_p(v, v) < 0$) or *null* ($g_p(v, v) = 0$). This classification, known as the *local light-cone structure* is the basic ingredient for studying the global causal structure of a Lorentzian manifold.

A.1 Causal structure

The notion of causality plays a central role in both classical and quantum physics. Simply put, causality means that an effect can only take place after its cause. To see what that means for, for instance, a quantum field theory defined on a curved space-time, we first need a consistent way to talk about ordering events in time.

We have seen that a Lorentzian metric defines a local light-cone structure on the tangent spaces at each point on a manifold. It also gives us a notion of *causal curves* (all tangent vectors of the curve are time-like or null). When two events on a space-time can be connected by a causal curve, we say that these events are *causally related*, if not, we say that they are *space-like seperated.* We want to define a notion of past and future on our manifold, so that we are able to say which of these two events occurs 'first'. In order to do that, we need to define a time orientation on our manifold, which means that in every tangent space, we want to designate half of the lightcone as 'future directed' and the other half as 'past directed' (these halves are the connected parts of the lightcone, i.e. the vectors that are time-like or null excluding the zero vector). If we can define this designation on all of the tangent bundle in a continuous way, we call the manifold *time-orientable* (for details, see Wald, 1984b). While it is definitely not guaranteed that an arbitrary Lorentzian manifold is time-orientable, we assume (unless stated otherwise) that we are working with time orientable manifolds on which one of the two possible orientations has been chosen.

Given such an orientation, we define a causal curve to be *future or past directed*, if all tangent vectors to this curve are future or past directed respectively. For each point p on our manifold M this allows us to define the following sets:

- The *chronological future* of p:

  $I^+(p) = \{q \in M : \text{there exists a future directed time-like curve from } p \text{ to } q\}$,

- the *causal future* of p:

  $J^+(p) = \{q \in M : \text{there exists a future directed causal curve from } p \text{ to } q\}$.

Similarly, we define the *chronological* or *causal past* $I^-(p)$ and $J^-(p)$. We can also talk about the *chronological future* of a set $S \subset M$

$$I^+(S) = \bigcup_{p \in S} I^+(p),$$

et cetera.

It is shown in (Penrose, 1972) (using smoothing arguments) that for $p, q, r \in M$, one has

$$q \in I^+(p) \wedge r \in I^+(q) \implies r \in I^+(p).$$

Therefore, we can define a transitive relation on M,

$$p \ll q \text{ if } q \in I^+(p),$$

known as the *chronological relation.* We can also define the *causal relation* on M,

$$p \prec q \text{ if } q \in J^+(p) \vee p = q,$$

which is also transitive by the same arguments.

The hierarchy of causality conditions

We use the definitions above to define a hierarchy of causality conditions on a space-time (from lowest to highest). For a proof that the hierarchic structure holds, as well as alternative definitions and more details, see (Minguzzi & Sanchez, 2006).

From lowest to highest in the hierarchy, the causality conditions on a space-time (M, g) are:

- *Non-totally vicious*:
  $\exists p \in M : p \not\ll p$.

- *Chronological*:
  $\forall p \in M : p \not\ll p$, in other words, the chronological relation is irreflexive. Equivalently, there are no closed time-like curves. This establishes $\ll$ as a strict partial order on M.

- *Causal*:
  $\forall p, q \in M : p \prec q \wedge q \prec p \implies p = q$, in other words, the causal relation is antisymmetric. Equivalently there are no closed causal curves. It establishes $\prec$ as a partial order on M.

- *Distinguishing*:
  If $\forall p, q \in M : I^+(p) = I^+(q) \implies p = q$ then we call M *future distinguishing*. The analogous definition holds for *past distinguishing*.

- *Strongly causal*:
  $\forall p \in M$ and neighbourhood U of p, there exists a neighbourhood of p, $V \subset U$, such that all causal curves with start- and endpoints in V are entirely contained in U.

- *Stably causal*:
  Next to the metric g on M, there exists another Lorentzian metric $\tilde{g}$ on M such that $\forall p \in M, v \in T_pM : g(v, v) \leq 0 \implies \tilde{g}(v, v) < 0$ and $(M, \tilde{g})$ is causal.

- *Causally continuous*:
  The space-time is both past- and future distinguishing and *reflecting*, $\forall p, q \in M : I^+(p) \subset I^+(q) \iff I^-(q) \subset I^-(p)$.

- *Causally simple*:
  The space-time is both causal and $\forall p \in M : J^{\pm}(p)$ are closed.

- *Globally hyperbolic*:
  The space-time is both causal and $\forall p, q \in M : J^+(p) \cap J^-(q)$ is compact.

The property of global hyperbolicity is of particular interest to us. Therefore we now take a closer look at this property.

A.1.1 Global hyperbolicity

Global hyperbolicity, introduced in (Leray, 1955), is the most restrictive of the causality conditions that we have introduced, but for that reason it is often the most useful. It is a natural condition to impose when treating the Einstein equations as an initial value problem or when one wants to define a quantum field theory on the curved space-time. This explains why it is relevant to our thesis. There is an alternative definition for global hyperbolicity that is of particular interest to us, in order to give this definition, we first work towards the definition of a Cauchy surface.

Given a future directed causal curve $\gamma : (a, b) \to M$, we say that γ is *future extendible* if $\lim_{t\to b} \gamma(t)$ exists in M. Similarly, we can define a curve to be *past extendible*. If one of these properties does not hold, we call the curve *future* or *past inextendible*. These notions allow us to discuss predictability. Given a set $S \subset M$ we now define the *future domain of dependence*:

$$D^+(S) = \{p \in M : \text{every past inextendible causal curve through } p \text{ intersects } S\}.$$

Of course, a similar definition holds for the *past domain of dependence* $D^-(S)$. The full domain of dependence is given by $D(S) = D^+(S) \cup D^-(S)$. The interpretation of this set is that for a physical system (for example a matter field) on a space-time background of which the dynamics satisfies causality (and is deterministic), the physical state on $D(S)$ is completely determined by the physical state on S. Therefore, the notion of a domain of dependence often comes up when discussing the initial value problem for some (partial) differential equation on the Lorentzian manifold. For a well-posed initial value problem, defining initial values on S should define a unique solution on $D(S)$.

Suppose we have a set $U \subset S$ s.t. $U \subset D^+(S - U)$. It is clear that defining initial conditions on all of S gives an overdetermined initial value problem, as defining initial conditions on $S - U$ fixes the solution on U. To circumvent this problem, we will take S to be an *achronal set*. This means that $S \cap I^+(S) = \emptyset$, or in other words, that every time-like curve starting on S cannot end on S. An achronal set is said to have an *edge* $p \in S$ if for every open neighbourhood U of p, there are $q, r \in O$ s.t. $q \ll p$ and $p \ll r$ and there exists a causal curve from q to r not intersecting S. If a set is achronal, edgeless and closed, it is referred to as a *slice*. It is proved in (Wald, 1984b) that slices of a $1 + d$-dimensional space-time are always d-dimensional embedded C^0 submanifolds of M.

Now we define a *Cauchy surface* Σ as a closed achronal set s.t. $D(\Sigma) = M$. This means that every causal inextendible curve on M intersects Σ. Due to the fact that Σ is achronal, we see that every time-like inextendible curve intersects Σ exactly once. It can also be shown that Σ has no edges (Hawking & Ellis, 1973), so a Cauchy surface is a slice and hence a C^0 hypersurface in M.

This brings us to an alternative definition to global hyperbolicity, equivalent to our initial definition (Hawking & Ellis, 1973).

Theorem A.1.1. *A space-time M is globally hyperbolic if and only if there exists a Cauchy surface $\Sigma \subset M$. Any other Cauchy surface of M is homeomorphic to Σ.*

In fact, a globally hyperbolic space-time can be entirely foliated by Cauchy surfaces, in the sense that given a Cauchy surface $\Sigma \subset M$, it can be shown that M is homeomorphic to $\mathbb{R} \times \Sigma$. This is known as *Geroch's Splitting Theorem*, first proven by Geroch (1970). This result is mostly topological, in the sense that it only shows a continuous foliation of the manifold, rather than a smooth one. However, it was proven in (Bernal & Sanchez, 2003) that a smooth foliation also exists. We thus have the following theorem.

Theorem A.1.2. *A space-time M is globally hyperbolic if and only if it admits a smooth space-like Cauchy surface Σ such that M is diffeomorphic to $\mathbb{R} \times \Sigma$.*

As we noted before, the physical state of some system on the domain of dependence $D(S)$ of some set S should be determined by the state on S (at least for a deterministic theory satisfying causality). Therefore, for a globally hyperbolic space-time M, defining initial values on some Cauchy surface $\Sigma \subset M$ for some suitable differential equation, should uniquely fix the global solution on M. This is true for (for instance) the Klein-Gordon equation (see theorem 3.1.8), but it holds more generally. In particular, the Einstein equations, i.e. the equations of motion for general relativity, also have an initial value formulation, in the sense that when one chooses certain geometric initial data on some Riemannian hypersurface Σ, there is a unique (maximally extended) globally hyperbolic space-time that solves the Einstein equations and has Σ as a Cauchy surface, and satisfies the initial data (Wald, 1984b, ch.10).

A.2 Penrose diagrams

One might have noticed that in the discussion about causal structure, all we needed to define causal and chronological relations were the (local) light-cone structure and time orientation. It is therefore evident that all definitions above can be lifted from a Lorentzian manifold $(\mathcal{M}, g)$ to a *conformal manifold* $(\mathcal{M}, [g])$, a manifold with a *conformal class* of metrics defined by the following equivalence relation. Given Lorentzian manifolds (M, g) and (M, g'), we say that

$$g \sim g' \iff \exists \Omega \in C^\infty(M, \mathbb{R}) : \Omega > 0 \wedge g = \Omega g'.$$

In other words, two metrics are conformally equivalent if they can be transformed into each other by a *conformal transformation*, i.e. a local rescaling of the metric.

Two conformal manifolds $(M, [g])$ and $(N, [h])$ are said to be *conformally isomorphic* if there is a $g' \in [g]$ and $h' \in [h]$ s.t. (M, g') is isometric to (N, h'). This also gives us the notion of a *conformal embedding* $i : M \hookrightarrow N$ where there is a $g' \in [g], h' \in [h]$ s.t. i defines an isometric embedding of (M, g') into (N, h'). These definitions extend naturally to Lorentzian manifolds, where, for instance, (M, g) and (N, h) are conformally isomorphic iff $(M, [g])$ and $(N, [h])$ are.

Taking 1+1 dimensional Minkowski space $(\mathbb{R}^2, \eta)$ as an example. In standard Minkowski coordinates $(t, x) \in \mathbb{R}^2$, the metric is

$$ds^2 = -dt^2 + dx^2.$$

Introducing a coordinate transformation to $U, V \in (-\pi/2, \pi/2)$ with $U = \tan^{-1}(t+x)$, $V = \tan^{-1}(t-x)$, we get

$$ds^2 = -\frac{1}{\cos^2(U)\cos^2(V)} dU dV.$$

This means that $(\mathbb{R}^2, \eta)$ is conformally isomorphic to $((-\pi/2, \pi/2)^2, g)$ where g is given by the flat metric $ds^2 = -dUdV$ using null coordinates. From this, we see that Minkowski space can be conformally embedded into itself via $i : \mathbb{R}^2 \hookrightarrow \mathbb{R}^2$, such that

$$i(\mathbb{R}^2) = \left\{(t,x) \in \mathbb{R}^2 : t+x, t-x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}.$$

It should be noted that $\overline{i(\mathbb{R}^2)}$ is compact. This makes this particular conformal embedding an example of a *Penrose/conformal diagram*.

Definition A.2.1. *Given space-times (M, g) and (N, h) such that there is a conformal embedding $i : M \hookrightarrow N$, we call this embedding a Penrose diagram if $\overline{i(M)}$ is compact.*

The boundary of this embedding, $\partial i(M)$, but often referred to just as ∂M, is known as the *conformal boundary* of M. One can usually distinguish between points on this boundary that are part of some singularity/finite boundary or or at "infinity". In fact, studying these infinities was the main motivation of Penrose to introduce his conformal (Penrose) diagrams (Penrose, 1964). We define infinity in the following way.

Definition A.2.2. *Let (M, g) and (N, h) be space-times space-times such that $i : M \hookrightarrow N$ is a Penrose diagram and let $p \in \partial i(M)$. Also, let $\gamma : I \to M$ be a geodesic such that $i(\gamma)$ has p as an endpoint.*[1] *We say that p is* at infinity *when the affine length of γ in (M, g) is infinite.*

We would like to distinguish different types of infinity based on their causal relations to the space-time.

Definition A.2.3. *For a Penrose diagram of M, $p \in \partial M$, with p at infinity, denote a geodesic in M with endpoint p (in the embedding) by γ_p. We then define future and past null infinity, future and past time-like infinity and space-like infinity as the following sets:*

$$\begin{aligned}
\mathscr{J}^+ &= \{p \in \partial M : p \textit{ at infinity, } \exists \gamma_p \textit{ future directed null}\}, \\
\mathscr{J}^- &= \{p \in \partial M : p \textit{ at infinity, } \exists \gamma_p \textit{ past directed null}\}, \\
i^+ &= \{p \in \partial M : p \textit{ at infinity, } \exists \gamma_p \textit{ future directed time-like}\}, \\
i^- &= \{p \in \partial M : p \textit{ at infinity, } \exists \gamma_p \textit{ past directed time-like}\}, \\
i^0 &= \{p \in \partial M : p \textit{ at infinity, } \exists \gamma_p \textit{ space-like}\}.
\end{aligned}$$

Applying these definitions to our earlier example of 2d Minkowski space, we can see that the entire boundary of the Penrose diagram, as drawn in A.1, can be divided into the different types of conformal infinities.

[1] For $\gamma : (a, b) \to M$, an endpoint of γ is $p = \lim_{t \to b} \gamma(t)$.

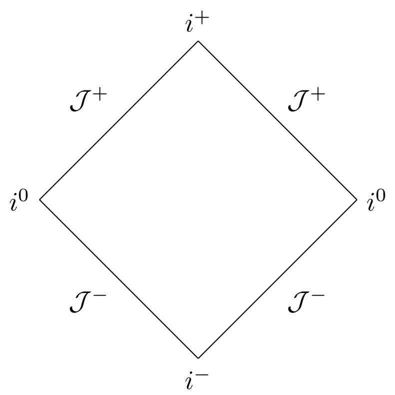

Figure A.1: Penrose diagram of Minkowski space $(\mathbb{R}^2, \eta)$

A similar construction can be applied to Minkowski space of higher dimension; the 1+3 dimensional case can be conformally embedded into the Einstein static universe (Penrose, 1964). Of course, it is not possible to draw a Penrose diagram in arbitrary dimensions on a piece of paper as we did for the two-dimensional case. Therefore, it is common practice that when a space-time has symmetries, the conformal embedding that gives the Penrose diagram is chosen such that it preserves these symmetries. This allows for a suppression of these symmetries in the diagram such that one gets an effectively lower dimensional diagram. In figure A.2 we can see what this looks like for Minkowski space.

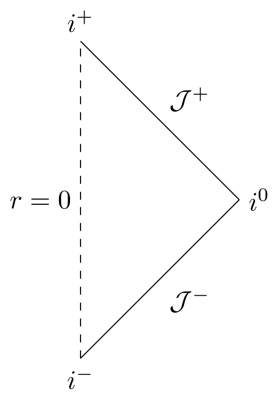

Figure A.2: Penrose diagram of Minkowski space $(\mathbb{R}^n, \eta)$ with suppressed $SO(n-1)$ symmetry, the dashed line is the axis of symmetry

Note that the boundary of the reduced diagram A.2 contains a part that cannot be associated with infinity (the dashed line at $r = 0$). This is because in the original n-dimensional diagram, these points were in the interior. When we refer to the conformal boundary ∂M of an n-dimensional space-time M, we are only considering the boundary in the n-dimensional Penrose diagram rather than the boundary of any reduced diagram.

The reason Penrose diagrams are useful, is that we can read the global causal structure of a space-time directly from its diagram, as the Penrose diagram will satisfy the same causality conditions as the space-time itself.

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