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Strict deformation quantization of a particle in external gravitational and Yang–Mills fields

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An adaptation of Rieffel's notion of "strict deformation quantization" is applied to a particle moving on an arbitrary Riemannian manifold Q in an external gauge field, that is, a connection on a principal H-bundle P over Q. Hence the Poisson algebra $\mathcal{A}_0 = C_0((T^*P)/H)$ is deformed into the C^*-algebra $\mathcal{A} = \mathcal{K}(L^2(P))^H$ of H-invariant compact operators on $L^2(P)$, which is isomorphic to $\mathcal{K}(L^2(Q))\otimes C^*(H)$, involving the group algebra of H. Planck's constant $\hbar$ is a genuine number rather than a formal expansion parameter, and in the limit $\hbar \to 0$ commutators and anti-commutators converge to Poisson brackets and pointwise products, respectively, in a well-defined analytic sense. This deformation can be interpreted in terms of Lie groupoids and algebroids, as $\mathcal{A}_0$ is the Poisson algebra of the Lie algebroid $(TP)/H$, whereas $\mathcal{A}$ is the C^*-algebra of the gauge groupoid of the bundle (P, Q, H). Other topics we discuss from the point of view of our formalism are Wigner functions, and the quantization of the Hamiltonian as well as position and momentum (including their domains).

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1. Introduction

Building upon previous leads [5,3], M. Rieffel recently proposed a mathematically satisfactory framework of quantization [36,37]. The main point of his approach is to make precise the intuitive idea, due to Dirac, that quantum commutators (times $i/\hbar$) should converge to Poisson brackets in the "classical limit" $\hbar \to 0$. We combine this with an analogous requirement that the anti-commutator converge to the pointwise product, and with a physically self-evident reality condition. Thus we are led to the following procedure of "strict

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deformation quantization". (For standard results and terminology in the theory of C^*-algebras we refer to ref. [33], and for Poisson algebras and geometry we recommend refs. [43,8].)

Given a Poisson manifold S, form the commutative C^*-algebra $\mathcal{A}_0 = C_0(S)$ of continuous complex-valued functions on S which vanish at infinity, and single out a dense subalgebra $\overline{\mathcal{A}_0}$ on which the Poisson bracket $\{\,,\,\}$ is well defined (note that $\overline{\mathcal{A}_0}$ is not necessarily the maximal subalgebra of $\mathcal{A}_0$ on which the Poisson bracket is defined). Then find a non-commutative C^*-algebra $\mathcal{A}$ and a family of linear maps $Q_\hbar : \overline{\mathcal{A}_0} \to \mathcal{A}$, defined for $\hbar \in (0, \hbar_0)$, where $\hbar_0 > 0$ may depend on the argument of $Q_\hbar$, so that the following conditions are satisfied for all f, f_1, f_2 in $\overline{\mathcal{A}_0}$ (for simplicity, we denote the norm and the adjoint in both $\mathcal{A}_0$ and $\mathcal{A}$ by $\| \ \|$ and *, respectively; we recall that $\|f\| = \sup|f|$ and $f^* = \overline{f}$ for $f \in \mathcal{A}_0$):

1. $Q_\hbar(f^*) = Q_\hbar(f)^*$;
2. the function $\hbar \to \|Q_\hbar(f)\|$ is continuous on $[0, \hbar_0)$ (with $Q_0 \equiv \mathrm{id}$);
3. $\lim_{\hbar\to 0} \|\frac{1}{2}(Q_\hbar(f_1)Q_\hbar(f_2) + Q_\hbar(f_2)Q_\hbar(f_1)) - Q_\hbar(f_1 f_2)\| = 0$;
4. $\lim_{\hbar\to 0} \|(\mathrm{i}/\hbar)(Q_\hbar(f_1)Q_\hbar(f_2) - Q_\hbar(f_2)Q_\hbar(f_1)) - Q_\hbar(\{f_1, f_2\})\| = 0$.

Of course, 3 and 4 together imply that $Q_\hbar$ is an "almost homomorphism", in the sense that $Q_\hbar(f)Q_\hbar(g) - Q_\hbar(fg)$ tends to zero in norm (this notion plays a major role in the generalized KK-theory of Connes and Higson [10]). The physical meaning of this setting is as follows: the real part of $\mathcal{A}_0$ is supposed to be the classical algebra of observables of some physical system, and the self-adjoint elements in $\mathcal{A}$ form the "corresponding" quantum algebra. The observables that can be quantized a priori are contained in $\overline{\mathcal{A}_0}$, and $Q_\hbar(f)$ should be the quantum observable with the same physical interpretation as its classical counterpart f. Thus condition 1 guarantees that observables are mapped into observables. Conditions 2–4 are a way of expressing the fact that, at the level of observables (as opposed to states), quantum mechanics is a continuous deformation of classical mechanics. The anti-commutator governs the spectral content of the quantum theory, and the pointwise product plays the same role in the classical theory. The commutator and the Poisson bracket express the role observables play as generators of transformations of the state space. For a more detailed discussion, cf. ref. [26]. One has to remark that the maps $Q_\hbar$ for fixed $\hbar$ are not continuous (at least in the examples known to the author), so that the quantization cannot be extended from $\overline{\mathcal{A}_0}$ to $\mathcal{A}_0$ in any obvious way.

Our interest in strict deformation quantization lies in the fact that it provides a bridge between the modern geometric theory of symmetry and reduction in classical mechanics [31], and the algebraic theory of superselection rules in quantum mechanics. The latter originated in quantum field theory, and is based on the identification of superselection sectors with certain inequivalent representations of the algebra of observables. One finds that the superselection sectors of a generic local quantum field theory with short-range forces in four dimensions correspond to the inequivalent unitary representations of a compact Lie group H (the gauge group of the theory) [17]. Remarkably, an analogous result applies to the quantum mechanics of a single particle. This was first observed for a particle moving on a homogeneous configuration space $Q = G/H$ [23], which was further studied in the context of strict deformation quantization in ref. [25]. The identification of H in G/H with the gauge group whose representation theory labels the superselection sectors is accidental, and in the present paper we shall describe the general situation.

Thus we consider a principal fibre bundle $(P, Q, H, \mathrm{pr}_{P\to Q})$ with total space P (assumed to be paracompact, as we will employ a partition of unity on it), base space Q, projection $\mathrm{pr}_{P\to Q} : P \to Q$, and a compact Lie group H with right action R_h on P. We will denote a generic point of P by x (or y), and write $R_h(x) \equiv xh$. The H-action on P pulls back to an action R_h^* on T^*P, so that we can form the quotient $S = (T^*P)/H$. The Poisson structure on S is inherited from the canonical symplectic form on T^*P. Specifically, we identify $C((T^*P)/H)$ with $C(T^*P)^H$ (the continuous functions on T^*P which are invariant under the pullback of R_h^* for all $h \in H$); it is easily checked that the symplectic form on T^*P is H-invariant, so that the Poisson bracket of two functions in $C^2(T^*P)^H$ is in $C(T^*P)^H$, and this defines the Poisson structure of $(T^*P)/H$.

The physical interpretation (originally due to Sternberg) of S is well known [16,42,32,31]: it is the "universal phase space" of a particle moving on Q which couples to a Yang–Mills field with gauge group H. S is fibered over T^*Q, with fibers isomorphic to $\mathbf{h}^*$ (the dual of the Lie algebra $\mathbf{h}$ of H), and the symplectic leaves $P_{\mathcal{O}}$ of S are in one-to-one correspondence with co-adjoint orbits $\mathcal{O}$ in $\mathbf{h}^*$. The leaf $P_{\mathcal{O}}$ is fibered over T^*Q, too, with fibre $\mathcal{O}$. Each orbit $\mathcal{O}$ in its entirety plays the role of a possible classical charge the particle may have, so that S is the union of the phase spaces $P_{\mathcal{O}}$ corresponding to all possible charges.

In what follows, we shall study the classical algebra of observables $\mathcal{A}_0 = C_0((T^*P)/H)$. If we assume that P has a Riemannian structure, with an H-invariant metric, so that $Q = P/H$ inherits a metric from P, we see that $\mathcal{A}_0$ is the algebra of observables of a charged particle moving in an external gravitational field, as well as a possible external Yang–Mills field (whose explicit form determines the metric on P, or, vice versa, is determined by it [8], also cf. subsection 3.4 below). We assume that P is (geodesically) complete.

The quantization of $\mathcal{A}_0$, i.e., the construction of the quantum algebra $\mathcal{A}$ and the maps $Q_\hbar$, is studied in section 2. It follows from a straightforward generalization of the Weyl prescription from $\mathbb{R}^n$ to arbitrary curved spaces. Although we work in a functional-analytic setting, the procedure itself is entirely geometric, cf. subsection 2.1. Conditions 1, 3, and 4 above on a strict deformation quantization are proved in subsection 2.2. Condition 2 is most easily proved using the auxiliary device of a Wigner function, constructed in subsection 2.3. The result is that, as shown in subsection 3.1, $\mathcal{A}$ is isomorphic to $\mathcal{K}(L^2(Q)) \otimes C^*(H)$, which stands for the tensor product of the C^*-algebra of compact operators on $L^2(Q)$ with the group algebra of H. This immediately leads to the conclusion that the dual of $\mathcal{A}$ (that is, the set of equivalence classes of its irreducible representations) is equal to the dual of H. Hence each unitary irreducible representation of H corresponds to an irreducible representation of $\mathcal{A}$, and is interpreted as a quantum charge of the particle. In subsection 3.2 we give a geometric realization of these representations of $\mathcal{A}$ on Hilbert spaces of sections of certain vector bundles over Q, which are associated to P. This leads to a close parallel between the classical theory, with the symplectic leaves $P_{\mathcal{O}}$ being fibered over T^*Q, and the quantum theory. This realization will prove to be useful in subsection 3.4, in which, following an algebraic discussion of classical dynamics in subsection 3.3, it is shown that the smooth sections of the vector bundles in question form a natural domain of essential self-adjointness of the quantum Hamiltonian. A similar statement holds for position and momentum observables, whose theory is non-trivial in curved space and in a gauge field. The classical situation is examined in subsection 3.5, and the quantization is in subsection 3.6.

While the quantization procedure has a clear geometric interpretation, it is somewhat ad hoc from an algebraic point of view. Also to shed more light on the particular Poisson and C^*-algebras that appear, we reformulate the results in terms of groupoids in section 4. The classical analogue of the C^*-algebra of a (Lie) groupoid is the Poisson algebra of a Lie algebroid, and this point of view leads to a simple algebraic and geometric interpretation of the deformation quantization constructed in this paper.

Although we will not discuss it here, one may replace the compact Lie group H by a discrete group, through which our formalism may describe a system of n identical particles [which is in fact a special case of a single particle moving on a multiply connected configuration space Q, with $H = \pi_1(Q)$]. Another generalization is to start from a given Hamiltonian, rather than from a Riemannian structure on P. All geometric notions, including the connection on the bundle P, then appear as derived objects, subordinate to the dynamics [31].

2. Generalized Weyl quantization on Riemannian spaces

2.1. CONSTRUCTION OF THE DEFORMATION

We start by defining a Fourier transform mapping functions on T^*P to those on TP. In general, whenever it is easier to employ co-ordinates, we will use them. Hence we cover P with open sets $\{U_a\}_{a\in I}$ for some index set I, with each U_a homeomorphic to $\mathbb{R}^n$, and co-ordinate functions $x_a^\mu : U_a \to \mathbb{R}^n$. If no confusion arises we take a fixed, and drop the lower index on x^μ. This leads to canonical co-ordinates (x^μ, p_μ) on T^*P, and $(x^\mu, \dot{x}^\mu)$ on TP. We write x for the point of P with co-ordinates x^μ in a given patch, and $\partial_\mu \equiv \partial/\partial x^\mu$ (so that, for example, $(x^\mu, \dot{x}^\mu)$ are the co-ordinates of $\dot{x}^\mu \partial_\mu \in T_x P$). We denote the metric on P by $\mathbf{g}$, with determinant g in given co-ordinates. Recall our assumption that $\mathbf{g}$ is invariant under the right action of H on P. The invariant measure on P is called μ, the one on the fibre $T_x P$ is μ_x, and the measure on $T_x^* P$ is $\hat{\mu}_x$. In co-ordinates one has

$$d\mu(x) = d^n x \sqrt{g(x)}, \quad d\mu_x(\dot{x}) = d^n \dot{x} \sqrt{g(x)}, \quad d\hat{\mu}_x(p) = \frac{d^n p}{(2\pi)^n \sqrt{g(x)}}, \tag{2.1}$$

where n is the dimension of P. The fibrewise Fourier transform of a suitable function f on T^*P is defined by

$$(\mathcal{F} f)(X) \equiv \check{f}(X) = \int_{T_x^* P} d\hat{\mu}_x(\theta)\, e^{i\langle \theta, X \rangle} f(\theta), \tag{2.2}$$

where $X \in T_x X$. "Suitable" here means that f should be measurable on T^*P, and integrable over each fibre. In co-ordinates, this simply amounts to

$$\check{f}(x, \dot{x}) = \int d\hat{\mu}_x(p)\, e^{i p_\mu \dot{x}^\mu} f(x, p). \tag{2.3}$$

As is clear from the co-ordinate free definition (2.1), $\check{f}$ is a well-defined scalar function on TP. The inverse transform is obvious, and involves the measure μ_x on $T_x P$ without further normalization constants. The convolution $\check{f}_1 * \check{f}_2$ is the Fourier transform of the pointwise product $f_1 f_2$, which gives

$$\check{f}_1 * \check{f}_2(x, \dot{x}) = \int d\mu_x(\dot{x}')\, \check{f}_1(x, \tfrac{1}{2}\dot{x} - \dot{x}') \check{f}_2(x, \tfrac{1}{2}\dot{x} + \dot{x}'). \tag{2.4}$$

We recall that $\mathcal{A}_0$ is the Poisson algebra $C_0((T^*P)/H)$, realized as the H-invariant part of $C_0(T^*P)$. We take the dense subalgebra of quantizable functions to be $\overline{\mathcal{A}_0} = \mathrm{PW}(\theta)$. These are the functions $f \in C_0((T^*P)/H)$ whose Fourier transform $\check{f}$ is in $C_c^\infty(TP)$ (here PW stands for Paley–Wiener, and $\overline{\mathcal{A}_0}$ consists of those H-invariant C_0^∞ functions on T^*P which at each fixed $x \in P$, i.e., as functions on $T_x^* P \simeq \mathbb{R}^n$, satisfy the conditions of the Paley–Wiener theorem, which characterizes functions whose Fourier transform is in $C_c^\infty(\mathbb{R}^n)$ [34, thm. IX.11]). The motivation for this choice will become clear shortly. We easily infer from the H-invariance of the metric $\mathbf{g}$ that $((R_h)_*)^* \circ \mathcal{F} = \mathcal{F} \circ (R_{h^{-1}}^*)^*$, where $(R_h)_*$ and R_h^* are the lift of the right action R_h on P to TP and T^*P, respectively. Hence $\overline{\mathcal{A}_0}$ is isomorphic, by the Fourier transform (2.2), to $\check{\mathcal{A}}_0 = C_c^\infty(TP)^H$, the algebra of H-invariant C_c^∞ functions on TP, equipped with the convolution product (2.4).

The deformation of $C_c^\infty(TP)^H$ into a non-commutative operator algebra is based on the following geometric fact. Consider the diagonal embedding Δ of P into $P \times P$ [that is, $\Delta(x) = (x, x)$]. The normal bundle N^Δ of this embedding may be taken to be a subbundle of $T(P \times P)$ by equating the fibre $N^{\Delta}_{(x,x)}$ at (x,x) with $\{X \oplus -X \mid X \in T_xP\} \subset T_{(x,x)}(P \times P) \simeq T_xP \oplus T_xP$. Hence N^{Δ} is isomorphic to TP, under the isomorphism $\tilde{\varphi} : TP \to N^{\Delta}$, given on $X \in T_xP$ by $\tilde{\varphi}(X) = X/2 \oplus -X/2 \in T_{(x,x)}(P \times P)$. [Many other isomorphisms exist between TP and N^{Δ}, and it will become clear that any particular choice determines an operator ordering prescription in quantum mechanics. The one above corresponds to (generalized) Weyl ordering, and in case of $P = \mathbb{R}^n$ eventually leads to the Weyl calculus of pseudo-differential operators [14]. The alternative choice $\tilde{\varphi}_{\mathrm{KN}}(X) = 0 \oplus -X$ leads to the Kohn–Nirenberg calculus; the corresponding deformation quantization violates the reality condition 1 in the introduction.] By the tubular neighbourhood theorem there exists an open neighbourhood V_1 of P in TP (where $P \subset TP$ is the zero section) which is diffeomorphic to an open neighbourhood V_2 of P in $P \times P$ [here identifying $P \subset P \times P$ with $\Delta(P)$]. Utilizing the metric on P, we take $V_1 = \bigcup_x B(x)$, where $B(x) \subset T_xP$ is the largest open ball on which the exponential map $e_x \equiv \exp_x : T_xP \to P$ is a diffeomorphism. We take V_2 the image of V_1 under the map $\varphi : V_1 \to P \times P$ defined by $\varphi(X) = (\exp_x(X/2), \exp_x(-X/2))$, for $X \in T_xP$.

For any pre-compact set K in TP there will exist a $\hbar_0 > 0$ so that $\hbar K \subset V_1$ for all $\hbar \in (0, \hbar_0)$. Define $\varphi_\hbar(X) = \varphi(\hbar X)$; we have $\varphi_\hbar(K) \subset V_2$ in that case. Now consider a fixed $\check{f} \in C_c^\infty(TP)$, with $K = \mathrm{supp}\, \check{f}$, and $\hbar_0$ as above. We may then define a function $Q_\hbar(f) \in C_c^\infty(P \times P)$ for all $\hbar \in (0, \hbar_0)$ by putting $Q_\hbar(f) = \hbar^{-n}(\varphi_\hbar^{-1})^* \check{f}$ on V_2, and setting $Q_\hbar(f) = 0$ outside V_2. More concretely, for $X \in T_xP$

$$Q_\hbar(f)(e_x^{\hbar X/2}, e_x^{-\hbar X/2}) = \hbar^{-n} \check{f}(X). \tag{2.5}$$

This defines an operator, called $Q_\hbar(f)$ as well, on $L^2(P) \equiv L^2(P, \mu)$ by

$$(Q_\hbar(f)\psi)(x) = \int_P \mathrm{d}\mu(x')\, Q_\hbar(f)(x, x')\psi(x'). \tag{2.6}$$

This is well defined: the value of $Q_\hbar(f)$ at (x, x') is zero when no X exists for which (2.5) holds. Clearly, $Q_\hbar(f)$ is a Hilbert–Schmidt operator, hence compact.

If the exponential map is a diffeomorphism on all of TP, then $\hbar_0 = \infty$, and the quantization map is defined for all $\hbar$. There is, of course, an artificial way of defining $Q_\hbar(f)$ for all $\hbar$ on arbitrary manifolds P, by introducing a smooth cutoff function to the effect that $Q_\hbar(f)$ is defined but vanishes for $\hbar \geq \hbar_0$, and coincides with the expression above for $\hbar < 0.99\hbar_0$ (say). The quantization prescription (2.5) is different from previous attempts at a generalized Weyl quantization on manifolds [41,27], although the use of the exponential map is the same. The difference is most easily seen from the Wigner function (2.31) below: in ref. [41] the Jacobian J is absent (which still leads to a self-adjoint quantization), whereas in ref. [27] a different Jacobian is used, which entails the loss of self-adjointness on non-homogeneous spaces (these references were pointed out to the author after submission of this paper).

We now examine what the H-invariance of $f \in \overline{\mathcal{A}_0}$ means for $Q_\hbar(f)$. Since each R_h defines an isometry on P, it maps geodesics into geodesics. This implies that $R_h(\exp_x(Y)) = \exp_{xh}((R_h)_* Y)$. Define a unitary representation π_R of H on $L^2(P)$ by

$$(\pi_R(h)\psi)(x) = \psi(xh). \tag{2.7}$$

The previous equation then easily implies, for f not necessarily H-invariant,

$$\pi_R(h) Q_\hbar(f) \pi_R(h)^* = Q_\hbar((R^*_{h^{-1}})^* f). \tag{2.8}$$

Therefore, if f is in $\overline{\mathcal{A}_0} \subset C_0(T^*P)^H$ then $Q_\hbar(f)$ commutes with all $\pi_R(h)$, $h \in H$. We express this by saying that $Q_\hbar(f) \in \mathcal{K}(L^2(P))^H$, the H-invariant compact operators on $L^2(P)$. Hence we take $\mathcal{A} = \mathcal{K}(L^2(P))^H$ as the non-commutative C^*-algebra into which $\mathcal{A}_0$ is to be deformed. The $Q_\hbar$ are henceforth regarded as maps from $\overline{\mathcal{A}_0}$ to $\mathcal{A}$.

2.2. CLASSICAL LIMIT OF (ANTI-)COMMUTATORS

In this subsection we shall prove that conditions 3 and 4 (cf. section 1), necessary for a strict deformation quantization, are met. Condition 1 is obviously satisfied by (2.5), as $\check{f}^*(X) = \check{f}(-X)$ if f is self-adjoint in $\mathcal{A}_0$. Condition 4 will be dealt with in the next subsection.

Let f_1, f_2 be quantizable for $\hbar \in (0, \hbar_0)$, that is, the operators $Q_\hbar(f_i)$ $(i = 1, 2)$ are well defined in this interval. We will first analyse the expression

$$N(\hbar) = \|\tfrac{1}{2}[Q_\hbar(f_1), Q_\hbar(f_2)]_+ - Q_\hbar(f_1 f_2)\|, \tag{2.9}$$

where the anti-commutator is written out in condition 3 (section 1), and the norm is the usual operator norm on $L^2(P)$. By linearity of $Q_\hbar$, it is sufficient to prove conditions 3, 4 for real f_i. As condition 1 is satisfied, we may therefore assume that the operator $A(\hbar) = \frac{1}{2}[Q_\hbar(f_1), Q_\hbar(f_2)]_+ - Q_\hbar(f_1 f_2)$ is self-adjoint. Since it is also compact, it has a normalized eigenvector $\Omega_\hbar$, for which $\|A(\hbar)\| = |(A(\hbar)\Omega_\hbar, \Omega_\hbar)|$. It easily follows from this expression with (2.5) and (2.6) that $\lim_{\hbar\to 0} N(\hbar) = 0$ if the f_i have disjoint support. Using a partition of unity on P, we may therefore assume that the projection (from T^*P to P) of the support of both f_1 and f_2 is contained in a pre-compact geodesically convex set $U \subset P$, on which we use some co-ordinate scheme $\{x^\mu\}$.

To relate an integral over $U \times U \subset P \times P$ to one over $TU \subset TP$ we use the formula

$$\int_{U\times U} \mathrm{d}\mu(x_1)\,\mathrm{d}\mu(x_2)\,F(x_1,x_2) = \int_{TU} \mathrm{d}\mu(x)\,\mathrm{d}\mu_x(\dot{x})\,J(x,\dot{x};\tfrac{1}{2})F(\gamma(x,\dot{x};\tfrac{1}{2}),\gamma(x,\dot{x};-\tfrac{1}{2})), \quad (2.10)$$

valid for arbitrary $F \in C(P\times P)$ whose support lies inside $U\times U$. Here $\gamma(x,\dot{x};\cdot)$ is the affinely parametrized geodesic starting at x with tangent vector $\dot{x}$ at x, and the Jacobian is

$$J(x,\dot{x},\lambda) = |\lambda^{-n}|[g(\gamma(x,\dot{x};\lambda))g(\gamma(x,\dot{x};-\lambda))]^{1/2}g(x)^{-1}|\det H(x,\dot{x};\lambda)|. \quad (2.11)$$

$H(x,\dot{x};\lambda)$ is a $2n\times 2n$ matrix

$$\begin{pmatrix} h(x,\dot{x};\lambda) & \tilde{h}(x,\dot{x};\lambda) \\ h(x,\dot{x};-\lambda) & \tilde{h}(x,\dot{x};-\lambda) \end{pmatrix}, \quad (2.12)$$

where h and $\tilde{h}$ are $n\times n$ matrices, defined as follows. Let $h^{\mu}_{(\nu)}(x,\dot{x};\cdot)\partial_\mu$ be the solution of the equation of geodesic deviation [7] at the geodesic $\gamma(x,\dot{x};\cdot)$ with initial conditions

$$h^{\mu}_{(\nu)}(x,\dot{x};0) = \delta^{\mu}_{\nu}, \qquad ((\mathrm{d}/\mathrm{d}\lambda)h^{\mu}_{(\nu)})(x,\dot{x};0) = 0. \quad (2.13)$$

The entries of the matrix h are the components $h^{\mu}_{(\nu)}$. Similarly, the matrix $\tilde{h}$ is defined by replacing the initial conditions (2.13) by

$$\tilde{h}^{\mu}_{(\nu)}(x,\dot{x};0) = 0, \qquad ((\mathrm{d}/\mathrm{d}\lambda)\tilde{h}^{\mu}_{(\nu)})(x,\dot{x};0) = \delta^{\mu}_{\nu}. \quad (2.14)$$

It is clear from (2.10) that $J(\cdot,\cdot,\lambda)$ is a scalar function on TU. To derive (2.10), one passes from the co-ordinates (x_1^μ, x_2^μ) to $x^\mu, \dot{x}^\mu$ via the expression $x_1 = \gamma(x,\dot{x};\lambda)$, $x_2 = \gamma(x,\dot{x};-\lambda)$ (where $\lambda = 1/2$ in the special case above). The definition of the equation of geodesic deviation implies that

$$\begin{aligned} \partial x_1^\mu(x,\dot{x};\lambda)/\partial x^\nu &= h^{\mu}_{(\nu)}(x,\dot{x};\lambda), & \partial x_1^\mu(x,\dot{x};\lambda)/\partial \dot{x}^\nu &= \tilde{h}^{\mu}_{(\nu)}(x,\dot{x};\lambda), \\ \partial x_2^\mu(x,\dot{x};\lambda)/\partial x^\nu &= h^{\mu}_{(\nu)}(x,\dot{x};-\lambda), & \partial x_2^\mu(x,\dot{x};\lambda)/\partial \dot{x}^\nu &= \tilde{h}^{\mu}_{(\nu)}(x,\dot{x};-\lambda), \end{aligned} \quad (2.15)$$

which leads to (2.10)–(2.14). One analogously derives the equation, valid for any $G \in C(P)$ whose support is in U, and an arbitrary point $x \in U$ satisfying the condition that U be contained in the image of the exponential map on T_xP,

$$\int_U \mathrm{d}\mu(x')\,G(x') = \int_{T_xP} \mathrm{d}\mu_x(\dot{x}')\,\tilde{J}(x,\dot{x}';1)G(\gamma(x,\dot{x}';1)), \quad (2.16)$$

with Jacobian

$$\tilde{J}(x,\dot{x}';\lambda) = |\lambda^{-n}|[g(\gamma(x,\dot{x}';\lambda))/g(x)]^{1/2}|\det\tilde{h}(x,\dot{x}';\lambda)|. \quad (2.17)$$

Note that

$$\tilde{J}(x,\dot{x};1) = \Delta(x,\gamma(x,\dot{x};1))^{-1}, \tag{2.18}$$

where Δ is the Van Vleck determinant [7,15]. Using (2.14) and the connection between Δ and Jacobi fields [7], this easily follows in Riemann normal co-ordinates around x, and it must therefore be true in general since $\tilde{J}$ and Δ are geometric objects which are independent of the co-ordinates used to define them.

From (2.9) and following text, (2.4), (2.5), (2.10) and (2.16), and the property $\gamma(x,\dot{x},\hbar\lambda) = \gamma(x,\hbar\dot{x},\lambda)$ we then find

$$N(\hbar) = \left| \int_U \mathrm{d}\mu(x) \int_{T_xP} \mathrm{d}\mu_x(\dot{x}) \int_{T_xP} \mathrm{d}\mu_x(\dot{x}') \right. \\ \left. \times\, \overline{\Omega_\hbar(\gamma(x,\dot{x};\hbar/2))}\Omega_\hbar(\gamma(x,\dot{x};-\hbar/2))F(\hbar,x,\dot{x},\dot{x}') \right|, \tag{2.19}$$

with

$$F(\hbar,x,\dot{x},\dot{x}') = J(x,\dot{x};\hbar/2)\left[\tfrac{1}{2}\hbar^{2n}\tilde{J}(x,\dot{x}';\hbar) \right. \\ \times \{Q_\hbar(f_1)(\gamma(x,\dot{x};\hbar/2),\gamma(x,\dot{x}';\hbar))Q_\hbar(f_2)(\gamma(x,\dot{x}';\hbar),\gamma(x,\dot{x};-\hbar/2)) \\ + Q_\hbar(f_2)(\gamma(x,\dot{x};\hbar/2),\gamma(x,\dot{x}';\hbar))Q_\hbar(f_1)(\gamma(x,\dot{x}';\hbar),\gamma(x,\dot{x};-\hbar/2))\} \\ \left. - \check{f}_1(x,\tfrac{1}{2}\dot{x}-\dot{x}')\check{f}_2(x,\tfrac{1}{2}\dot{x}+\dot{x}')\right]. \tag{2.20}$$

We now analyse F for fixed x. F is invariant under co-ordinate transformations, so we may choose Riemann normal co-ordinates (RNC) based at x. The following order estimates hold pointwisely in $x,\dot{x}\,\dot{x}'$. By evaluating the geodesic deviation equation in RNC [7] at $\lambda = 0$ it follows immediately that $\tilde{h}^\mu_\nu(x,\dot{x},\hbar) = \hbar\delta^\mu_\nu[1+\mathcal{O}(\hbar^2)]$. The initial conditions (2.13) show that $h^\mu_\nu(x,\dot{x},\hbar) = \delta^\mu_\nu[1+\mathcal{O}(\hbar^2)]$. Combined with the explicit form of the metric in RNC [7], we thus infer from (2.11) and (2.17) that (in any co-ordinates)

$$J(x,\dot{x};\hbar/2) = 1+\mathcal{O}(\hbar^2), \qquad \tilde{J}(x,\dot{x}';\hbar) = 1+\mathcal{O}(\hbar^2). \tag{2.21}$$

To deal with the $Q_\hbar(f_i)$ terms in (2.20) we write $\gamma^\mu(x,\dot{x},\lambda) = \lambda\dot{x}^\mu$ etc., and perform a Taylor expansion of $Q_\hbar(f_1)$ around the point $(\gamma(x,\frac{1}{2}\dot{x}-\dot{x}';\hbar/2),\gamma(x,\frac{1}{2}\dot{x}-\dot{x}';-\hbar/2))$, and of $Q_\hbar(f_2)$ around $(\gamma(x,\frac{1}{2}\dot{x}+\dot{x}';\hbar/2),\gamma(x,\frac{1}{2}\dot{x}+\dot{x}';-\hbar/2))$. The result is then rewritten in terms of the f_i themselves using (2.5). If one includes the $\mathcal{O}(\hbar)$ term, which will be needed later on, one encounters expressions of the type

$$\left[\left(\frac{\partial}{\partial x_1^\mu}+\frac{\partial}{\partial x_2^\mu}\right)Q_\hbar(f_1)\right]\ (x_1 = \gamma(x,\tfrac{1}{2}\dot{x}-\dot{x}';\hbar/2), x_2 = \gamma(x,\tfrac{1}{2}\dot{x}-\dot{x}';-\hbar/2)),$$

to be expressed in RNC as indicated above. This is done by inverting (2.15), and expanding in powers of $\hbar$. The result is that $\hbar^n$ times the expression displayed above equals $(\partial\check{f}_1/\partial x^\mu)((x,\frac{1}{2}\dot{x}-\dot{x}'))+\mathcal{O}(\hbar)$.

The conclusion is that pointwisely

$$\lim_{\hbar\to 0} F(\hbar, x, \dot{x}, \dot{x}') = 0. \tag{2.22}$$

Now return to (2.19). We use (2.1) and majorize the x-integral by taking $\sup_x gF$ out of it. We then use the Cauchy–Schwarz inequality to find

$$N(\hbar) \leq K(\hbar) \int d^n\dot{x}\, d^n\dot{x}' \sup_{x\in U} |g(x)F(\hbar, x, \dot{x}, \dot{x}')|, \tag{2.23}$$

with

$$K(\hbar) = \sup_{\dot{x}\in C} \int_U d\mu(x)\, |\Omega_\hbar(\gamma(x, \dot{x}; \hbar/2))|^2 \int_U d\mu(x)\, |\Omega_\hbar(\gamma(x, \dot{x}; -\hbar/2))|^2, \tag{2.24}$$

where C is a compact set in which the support of F in $\dot{x}$ is contained. Since $\Omega_\hbar$ has norm 1 in $L^2(P;\mu)$, $K(\hbar)$ is uniformly bounded by a constant K, which is equal to the supremum over $x, \dot{x}$, and $\hbar$ of the square of the Radon–Nikodym derivative $d\mu(x)/d\mu(\gamma(x, \dot{x}, \hbar/2))$. This is a continuous function of x, with continuous parametric dependence on $\dot{x}$ and $\hbar$, and the supremum is taken over a compact set. Hence $K(\hbar)$ in (2.23) may be replaced by the finite constant K. We recall that F is C_c^∞ in $x, \dot{x}, \dot{x}'$ since $\check{f}_1$ and $\check{f}_2$ are. The dependence on $\hbar$ is continuous as well, and elementary analysis shows that the pointwise result (2.22) implies the corresponding result uniformly in x. Finally, the compact support in $\dot{x}, \dot{x}'$ allows the use of the Lebesgue dominated convergence theorem to interchange $\lim_{\hbar\to 0}$ with the integrations in (2.23). Hence by (2.22) and (2.9)

$$\lim_{\hbar\to 0} \|\tfrac{1}{2}[Q_\hbar(f_1), Q_\hbar(f_2)]_+ - Q_\hbar(f_1 f_2)\| = 0 \tag{2.25}$$

for all f_i in $\mathcal{A}_0$. An entirely analogous computation yields the classical limit of the commutator (cf. condition 4, section 1)

$$\lim_{\hbar\to 0} \|(i/\hbar)[Q_\hbar(f_1), Q_\hbar(f_2)]_- - Q_\hbar(\{f_1, f_2\})\| = 0. \tag{2.26}$$

Here one needs the $\mathcal{O}(\hbar)$ terms in the Taylor expansion of $Q_\hbar(f_i)$, discussed after (2.21). The Poisson bracket is the canonical one, which in canonical coordinates (x, p) simply reads

$$\{f_1, f_2\}(x, p) = \left(\frac{\partial f_1}{\partial p_\mu}\frac{\partial f_2}{\partial x^\mu} - \frac{\partial f_2}{\partial p_\mu}\frac{\partial f_1}{\partial x^\mu}\right)(x, p). \tag{2.27}$$

The Fourier transform of (2.27), which enters the proof of (2.26) is

$$\{\check{f}_1, \check{f}_2\}(x, \dot{x}) = i \int_{T_xP} d\mu_x(\dot{x}')\, \check{f}_1(x, \tfrac{1}{2}\dot{x} - \dot{x}') \times \left[(\tfrac{1}{2}\dot{x} + \dot{x}')^\mu \frac{\overleftarrow{\partial}}{\partial x^\mu} + (-\tfrac{1}{2}\dot{x} + \dot{x}')^\mu \frac{\overrightarrow{\partial}}{\partial x^\mu}\right] \check{f}_2(x, \tfrac{1}{2}\dot{x} + \dot{x}'). \tag{2.28}$$

2.3. WIGNER FUNCTION AND NORM CONTINUITY

We will prove property 2 of a strict deformation quantization using a (lengthy) argument, which we do not reproduce here (cf. theorem 4 in ref. [25]), that reduces the proof of continuity of all functions $\hbar \to \|Q_\hbar(f)\|$ on $[0,\hbar_0)$ ($f \in \overline{\mathcal{A}_0}$, and $\hbar_0$ defined prior to (2.5); recall that $Q_0 \equiv$id) to the construction of sufficiently many "classical germs". A classical germ is a collection of states $\{\omega_\hbar\}_{\hbar\in[0,\varepsilon)}$, for some $\varepsilon > 0$, so that $\omega_\hbar$ is a state on $\mathcal{A}$ for $\hbar > 0$ and ω_0 is a state on $\mathcal{A}_0$, with the property that the function $\hbar \to \omega_\hbar(Q_\hbar(f))$ is continuous for $\hbar \geq 0$ wherever it is defined (the only essential point is continuity on $[0,\varepsilon')$ for arbitrary $\varepsilon' > 0$, e.g., the infimum of ε and $\hbar_0$ for the given function f). We say that a classical germ $\{\omega_\hbar\}$ converges to ω_0. By "sufficiently many" (to prove property 2) we mean that for any pure state ω_0 we need a classical germ converging to it.

To construct these germs we employ a Wigner function formalism, well known for $P = \mathbb{R}^n$ [14] (and generalized to arbitrary homogeneous spaces in ref. [25], with a further extension to arbitrary Riemannian manifolds in ref. [27]; the latter is slightly different from our procedure below). The idea is to express expectation values of quantum observables as phase space integrals, that is,

$$(Q_\hbar(f)\Omega,\Omega) = \int_{(T^*P)/H} f\,W^\hbar_\Omega \tag{2.29}$$

should hold for all $f \in \overline{\mathcal{A}_0}$, with a certain (Wigner) function $W^\hbar_\Omega$ on $(T^*P)/H$, which of course depends on $\Omega \in L^2(P)$. More generally, one may construct a Wigner function for a pair of vectors, so that

$$(Q_\hbar(f)\Omega_1,\Omega_2) = \int_{(T^*P)/H} f\,W^\hbar_{\Omega_1\Omega_2}, \tag{2.30}$$

so that $W^\hbar_\Omega \equiv W^\hbar_{\Omega\Omega}$. To accomplish this, we define a function on T^*P, depending on arbitrary $\Omega_i \in L^2(P)$ $(i = 1,2)$ by

$$\tilde{W}^\hbar_{\Omega_1\Omega_2}(x,p) = \int_{B_\hbar(x)} \mathrm{d}\mu_x(\dot{x})\,J(x,\dot{x};\hbar/2) \times \mathrm{e}^{\mathrm{i}p_\mu \dot{x}^\mu}\,\Omega_1(\gamma(x,\dot{x};-\hbar/2))\overline{\Omega_2(\gamma(x,\dot{x};\hbar/2))}, \tag{2.31}$$

with γ defined after (2.10), and J given in (2.11). Here $B_\hbar(x) \equiv B(x)/\hbar$, with $B(x)$ the largest ball in T_xP on which the exponential map is a diffeomorphism. Clearly, $\tilde{W}^\hbar_{\Omega_1\Omega_2}$ is well defined, i.e., independent of the co-ordinates used in (2.31); this may even be more apparent from the geometric expression (2.38) below. Subsequently, we define the Wigner function on $(T^*P)/H$ by

$$W^\hbar_{\Omega_1\Omega_2}([\theta]) = \int_H \mathrm{d}h\,\tilde{W}^\hbar_{\Omega_1\Omega_2}(R^*_h(\theta)), \tag{2.32}$$

where $\theta \in T^*P$, and $[\theta]$ its equivalence class in $(T^*P)/H$ (and $\mathrm{d}h$ is the Haar measure on H). A simple computation, using (2.10), shows that $W^{\hbar}_{\Omega_1\Omega_2}$ indeed does the job, in the sense that the equality in (2.30) is valid exactly when the quantization $Q_\hbar(f)$ is defined. That is, it holds for $\hbar \in (0,\hbar_0)$, with $\hbar_0$ so that the support of $\check{f}$ in each fibre T_xP of TP is contained in $B_\hbar(x)$ for all $\hbar$ in this interval. Evidently, for exponential manifolds $B(x) = T_xP$ and (2.30) is valid for any $\hbar$. Note that a different attempt to construct a Wigner function in curved space is made in ref. [18].

As an application of the Wigner function technique we will now construct a classical germ $\{\omega_\hbar\}$ converging to an arbitrarily chosen pure state ω_0 on $C_0(T^*P)$, that is, a point $\theta \in T^*P$. Regarded as states on $\mathcal{A}$ and $\mathcal{A}_0$ by restriction, this classical germ then converges to the pure state $[\theta]$ on $\mathcal{A}_0$, so that this construction provides a proof of property 2.

We use Riemann normal co-ordinates x^μ around $\mathrm{pr}_{T^*P\to P}(\theta)$ ($\mathrm{pr}_{T^*P\to P}$ being the projection from T^*P to P), so that θ has canonical co-ordinates $(0,\overline{p_\mu})$ for some $\overline{p_\mu}$. Define $\Omega_\hbar \in L^2(P)$ by

$$\Omega_\hbar(x) = (\pi\hbar)^{-n/4}K_\hbar(x)\mathrm{e}^{\mathrm{i}\overline{p_\mu}x^\mu/\hbar}\mathrm{e}^{-x^2/2\hbar}, \tag{2.33}$$

where $K_\hbar$ is a smooth cutoff function restricting the support of $\Omega_\hbar$ to the RNC patch, and normalizing it to unity; we assume $K_\hbar(0) = 1$ (for $P = \mathbb{R}^n$ one has $K_\hbar = 1$), and (2.33) is the wavefunction of a coherent state with position $x = 0$ and momentum $\overline{p_\mu}$). The $\Omega_\hbar$ define vector states $\omega_\hbar$ by $\omega_\hbar(A) = (A\Omega_\hbar, \Omega_\hbar)$ for $A \in \mathcal{A}$, $\hbar > 0$. We wish to prove that these states converge to $\omega_0 = \theta$ as a classical germ. As we see from (2.29) and the definition of a classical germ, this is the case if the Wigner functions $W^{\hbar}_{\Omega_\hbar}$ converge to $\delta_{(0,\overline{p})}$, the Dirac distribution on T^*P at $\theta = (0,\overline{p})$, in the weak topology defined by $\overline{\mathcal{A}_0}$ (regarded as a space of test-functions on T^*P).

To compute the Wigner function (2.31) (with $\Omega_1 = \Omega_2 = \Omega_\hbar$) we need to compute the RNC of $\gamma(x,\dot{x};\pm\hbar/2)$, for arbitrary x and $\dot{x}$ (expressed in RNC) to $\mathcal{O}(\hbar)$. To $\mathcal{O}(0)$ clearly $\gamma^\mu(x,\dot{x};0) = x^\mu$. Consider the family of geodesics $\{\gamma_\hbar\}_\hbar(\cdot)$, defined by $\gamma_\hbar(0) = 0$ (i.e., the fixed point $\mathrm{pr}_{T^*P\to P}(\theta)$ we assigned RNC $x^\mu = 0$ to), and $\gamma_\hbar(1) = \gamma(x,\dot{x};\hbar/2)$. This leads to a Jacobi field (a solution of the equation of geodesic deviation [7]) h along the geodesic $\gamma^x(\cdot)$ (for which $\gamma^x(0) = 0$ and $\gamma^x(1) = x$), defined by $h = \mathrm{d}\gamma_\hbar/\mathrm{d}\hbar|_{\hbar=0}$. The RNC $\gamma^\mu(x,\dot{x};\pm\hbar/2)$ are then equal to $x^\mu \pm \frac{1}{2}\hbar\,\mathrm{d}h^\mu/\mathrm{d}\lambda|_{\lambda=0}$. The field h satisfies the Dirichlet boundary conditions (in RNC) $h^\mu(0) = 0$; $h^\mu(1) = \dot{x}^\mu$. In general, a Jacobi field with Dirichlet data satisfies [7, V. problem 3, p. 346] $\mathrm{d}h^\mu/\mathrm{d}\lambda|_{\lambda=0} = g^{\mu\nu}(0)\sigma_{\nu\rho}(0,x)h^\rho(1)$, where σ is the Van Vleck matrix. In RNC one simply has $\sigma_{\nu\rho}(0,x) = \delta_{\nu\rho}$, so that we conclude that

$$\gamma^\mu(x,\dot{x};\pm\hbar/2) = x^\mu \pm \tfrac{1}{2}\hbar\dot{x}^\mu + \mathcal{O}(\hbar^2). \tag{2.34}$$

From (2.31), (2.33) and (2.34) we thus find

$$\tilde{W}^{\hbar}_{\Omega_\hbar}(x,p) = (\pi\hbar)^{-n/4} \int_{B_\hbar(x)} \mathrm{d}\mu_x(\dot{x})\, \mathrm{e}^{\mathrm{i}\dot{x}^\mu(p_\mu - \overline{p_\mu})} \mathrm{e}^{-x^2/2\hbar} K_\hbar(x)\,[1 + \mathcal{O}(\hbar)]. \tag{2.35}$$

From (2.29), (2.35) and the Lebesgue dominated convergence theorem (justified by the properties of f, and allowing us to ignore the $\mathcal{O}(\hbar)$ term in (2.35) when taking the limit) we obtain

$$\lim_{\hbar\to 0}(Q_\hbar(f)\Omega_\hbar, \Omega_\hbar) = f(\theta). \tag{2.36}$$

Since the continuity of the function $\hbar \to (Q_\hbar(f)\Omega_\hbar, \Omega_\hbar)$ for $\hbar > 0$ is trivial to establish, we conclude that the collection of states $\{\omega_\hbar\}_{\hbar\geq 0}$ consisting of vector states defined by the $\Omega_\hbar$ for $\hbar > 0$ [cf. text after (2.33)], and $\omega_0 = \theta$, is a classical germ converging to θ. As remarked above, this proves that the deformation we have constructed satisfies condition 2 in the introduction. To summarize:

Theorem 1. *Let a paracompact manifold P be the total space of a principal fibre bundle over Q with compact structure group H, and let $\mathbf{g}$ be a H-invariant metric on P. Equip $(T^*P)/H$ with the quotient Poisson structure derived from the canonical symplectic structure of T^*P. Let $\mathcal{A}_0$ be the commutative C^*-algebra $C_0((T^*P)/H)$, and $\overline{\mathcal{A}_0}$ its dense subalgebra consisting of those functions whose fibrewise Fourier transforms (2.2) are in $C_c^\infty(TP)$. Let $\mathcal{A} = \mathcal{K}(L^2(P))^H$ be the C^*-algebra of H-invariant compact operators on $L^2(P)$ (L^2 defined w.r.t. the measure associated to $\mathbf{g}$). Then the "quantization map" $Q_\hbar(f)$, constructed for each $f \in \overline{\mathcal{A}_0}$ by (2.5) with (2.6), is defined on a strip $(0,\hbar_0)$, where $\hbar_0 > 0$ depends on f [and equals ∞ if the manifold is exponential in the sense explained after (2.32)], and satisfies the conditions 1–4 in the introduction (section 1 above) of a strict deformation quantization.*

The proof that conditions 1, 3, 4 are satisfied is in subsection 2.2 above, and the proof of condition 2 has been the contents of the present subsection.

Although it is only of parenthetical relevance to the main subject of this paper, we give

Theorem 2. *If P is compact the Wigner functions (2.31), (2.32) are continuous for any $\Omega_1, \Omega_2 \in L^2(P)$ (and $\hbar > 0$). For non-compact P they are continuous and vanish at infinity if the following condition is satisfied by the metric $\mathbf{g}$ on P: the constant*

$$\rho = \sup_{x_1,x_2} \mathrm{d}\mu(R(x_1;x_2))/\mathrm{d}\mu(x_2) \tag{2.37}$$

should be finite. Here the supremum is taken over all pairs of points which can be connected by a unique geodesic, and $R(x_1;x_2)$ is the geodesic reflection of x_1 in

x_2 [that is, $R(x_1;x_2) = \gamma(1)$, where γ is the affinely parametrized geodesic for which $\gamma(0) = x_1$ and $\gamma(1/2) = x_2$]. The Radon–Nikodym derivative in (2.37) (which is a function of x_2 with x_1 as a parameter) relates to the Riemannian measure μ on P.

Proof. A geometric expression of (2.31) is

$$\tilde{W}^{\hbar}_{\Omega_1\Omega_2}(\theta) = \hbar^{-n} \int_{\exp(B(x_2))} d\mu(x_1)\, \frac{d\mu(R(x_1;x_2))}{d\mu(x_2)} \times e^{(i/\hbar)\langle\theta,\dot{R}(x_1;x_2)\rangle}\Omega_1(x_1)\overline{\Omega_2(R(x_1;x_2))}, \tag{2.38}$$

where $x_2 = \mathrm{pr}_{T^*P\to P}(\theta)$, and $\dot{R}(x_1;x_2)$ is the tangent vector to γ at x_2. To derive (2.38), choose a regularization $\delta^{\varepsilon}_{x_2}$ of the (covariant) Dirac distribution on P supported at x_2, so that

$$\tilde{W}^{\hbar}_{\Omega_1\Omega_2}(x_2,p) = \lim_{\varepsilon\to 0}\int_P d\mu(x)\, \tilde{W}^{\hbar}_{\Omega_1\Omega_2}(x,p)\delta^{\varepsilon}_{x_2}(x).$$

Substitute (2.31), scale $\hbar$ out, and use (2.10). Then take the limit $\varepsilon \to 0$, and (2.38) follows. Using the Cauchy–Schwarz inequality one then finds

$$|\tilde{W}^{\hbar}_{\Omega_1\Omega_2}(\theta)|^2 \leq \hbar^{-n}\rho^2\|\Omega_1\|^2\,\|\Omega_2\|^2, \tag{2.39}$$

with the vector norms in $L^2(P)$. Hence if $\rho < \infty$ and $\hbar \neq 0$ the map Φ : $L^2(P)\otimes L^2(P) \to L^{\infty}(T^*P)$ defined by $\Phi(\Omega_1,\Omega_2) = \tilde{W}^{\hbar}_{\Omega_1\Omega_2}$ is continuous, since we have shown that it is bounded with norm less than $\hbar^{-n}\rho^2$. Now take both Ω_i in $\mathcal{D}(P)$. It then easily follows from the Riemann–Lebesgue lemma that $\tilde{W}^{\hbar}_{\Omega_1\Omega_2}$ is C^{∞}_c in x and C_0 in p. Now $\mathcal{D}(P)$ is dense in $L^2(P)$, so that Φ maps a dense subspace of $L^2(P)\otimes L^2(P)$ into $C_0(T^*P)$, which is norm-closed; hence Φ, being continuous, maps all of $L^2(P)\otimes L^2(P)$ into $C_0(T^*P)$. The theorem follows. The analogous statement on $(T^*P)/H$ is immediate as H is compact. Compare with prop. 1.92 in ref. [14]. □

We close this subsection with an aside. The quantization rule (2.5) is, of course, far from unique, and corresponds to a particular operator ordering prescription, but one may modify it in a certain obvious way. The motivation for this is that in flat space $P = \mathbb{R}^n$ (and $H = \{e\}$), one has for the transition probability between two vectors $\Omega_i \in L^2(\mathbb{R}^n)$ the remarkable formula

$$|(\Omega_1,\Omega_2)|^2 = \hbar^n \int_{T^*P} W^{\hbar}_{\Omega_1} W^{\hbar}_{\Omega_2}. \tag{2.40}$$

If we now regard the Wigner function $W^{\hbar}_{\Omega}$ as a classical observable (which on the one hand is somehow justified by theorem 2, and by the correspondence between states and observables in quantum mechanics, where a state is at the same time an observable via its density matrix, but on the other hand is slightly perverse, as the Wigner function depends on $\hbar$), then (2.29) and (2.40) show that

$$[\Omega] = Q_\hbar(\hbar^n W_\Omega^\hbar), \tag{2.41}$$

showing that the orthogonal projector $[\Omega]$ on a wave function Ω is precisely the quantization of the Wigner function of this wave function (times $\hbar^n$). Using (2.5), one sees that this property no longer holds for general curved spaces due to the factor J in (2.31). However, (2.40) and (2.41) are valid if we multiply the right-hand side of (2.5) by $J(X;\hbar/2)^{-1/2}$, and replace J in (2.31) by its square root. This is still a strict deformation quantization due to (2.21).

3. Representation theory and dynamics

3.1. STRUCTURE OF THE QUANTUM ALGEBRA OF OBSERVABLES

We have seen that the quantum algebra of observables is $\mathcal{A} = \mathcal{K}(L^2(P))^H$, the compact operators on $L^2(P)$ which commute with the representation $\pi_R(H)$ [cf. (2.8)]. This C^*-algebra has a very transparent structure, as we will now spell out (what follows is completely standard). Consider its dense subalgebra $\overline{\mathcal{A}} = \mathrm{HS}(L^2(P))^H \simeq L^2(P \times P)^H$ of H-invariant Hilbert–Schmidt operators. An element K of $\overline{\mathcal{A}}$ is determined by (and determines) a kernel $K \in L^2(P\times P)^H$, satisfying

$$K(x_1 h, x_2 h) = K(x_1, x_2) \tag{3.1}$$

almost everywhere (w.r.t. the H-invariant Riemannian measure μ on P). Since μ is H-invariant it determines a measure ν on Q; moreover, H has a unique Haar measure with total mass 1. These are related by

$$\int_P \mathrm{d}\mu(x)\, f(x) = \int_Q \mathrm{d}\nu(q) \int_H \mathrm{d}h\, f(s(q)h) \tag{3.2}$$

for any $f \in L^1(P)$ and any measurable section $s : Q \to P$. The Hilbert spaces $L^2(Q)$ and $L^2(H)$ are defined w.r.t. these respective measures. We perform a unitary transformation $T : L^2(P) \to L^2(Q \times H) \simeq L^2(Q) \otimes L^2(H)$ as follows. Choose a measurable section $s : Q \to P$ leading to a unique factorization $x = s(\overline{x})h(x)$ of $x \in P$ into $\overline{x} \equiv \mathrm{pr}_{P\to Q}(x) \in Q$ and $h(x) \in H$. Put

$$(T\psi)(q,h) = \psi(s(q)h) \tag{3.3}$$

for $\psi \in C_c(P)$, and extend T to $L^2(P)$ by continuity. T is unitary, and has inverse T^* given on $\tilde{\psi} \in C_c(Q \times H)$ by

$$(T^*\tilde{\psi})(x) = \tilde{\psi}(\overline{x}, h(x)), \tag{3.4}$$

with $h(x)$ determined by the factorization mentioned above.

Now consider the group algebra $C^*(H)$. This is defined as the C^*-envelope of the Banach algebra $L^1(H)$ [33], equipped with the convolution product

$$(f_1 f_2)(h) = \int_H dk\, f_1(hk^{-1}) f_2(k). \tag{3.5}$$

We recall the left-regular representation π_L of $C^*(H)$ given on $\psi \in L^2(H)$ by

$$(\pi_L(f)\psi)(h) = \int_H dk\, f(k)\psi(k^{-1}h). \tag{3.6}$$

This leads to a faithful representation of the C^*-algebra $\mathcal{A}' = \mathcal{K}(L^2(Q)) \otimes C^*(H)$ on $L^2(Q \times H)$ which is the tensor product of the defining representation of $\mathcal{K}(L^2(Q))$ and the left-regular representation of $C^*(H)$. We identify $\mathcal{A}'$ with its representative. We construct a map $\rho : \mathcal{A} \to \mathcal{A}'$, by defining it on the dense subalgebra $L^2(P \times P)^H \subset \mathcal{A}$ by

$$(\rho(K))(q_1, q_2, h) = K(s(q_1)h, s(q_2)), \tag{3.7}$$

where, as explained above, we identify both $K \in \mathrm{HS}(L^2(P))^H$ and $\rho(K) \in \mathcal{A}'$ with their respective kernel functions (the latter tensored with a function on H). It follows from (3.1), (3.3), and (3.4) that $\rho(K) = TKT^*$, so that ρ is a *-homomorphism. The image of $\mathrm{HS}(L^2(P))^H$ is $L^2(Q \times Q \times H)$, which is dense in $\mathcal{A}'$. Hence the extension of ρ to $\mathcal{A}$ defines an isomorphism between $\mathcal{A}$ and $\mathcal{A}'$. In conclusion,

$$\mathcal{A} = \mathcal{K}(L^2(P))^H \simeq \mathcal{K}(L^2(Q)) \otimes C^*(H). \tag{3.8}$$

This allows allows us to immediately classify the irreducible representations of $\mathcal{A}$, that is, the quantum superselection sectors of the system whose algebra of observables is $\mathcal{A}$. These are determined by $C^*(H)$, as the compacts $\mathcal{K}$ only have one irreducible representation. There is a one-to-one correspondence between non-degenerate representations of $C^*(H)$, and unitary representations of H [33], hence each unitary irreducible representation of $C^*(H)$ is determined by a unitary irreducible representation π_χ of H on a Hilbert space $\mathcal{H}_\chi$. This, in turn, induces an irreducible representation π^χ of $\mathcal{A}$ on $\mathcal{H}^\chi = L^2(Q) \otimes \mathcal{H}_\chi$.

3.2. REALIZATION OF THE IRREDUCIBLE REPRESENTATIONS OF $\mathcal{A}$

In the preceding argument the section s was just measurable. To study the self-adjointness of certain unbounded operators relevant to physics, as well as to get a clear-cut geometric structure, it is useful to use smooth local sections. Hence we cover Q with open sets $\{U_\alpha\}_{\alpha \in J}$ so that smooth sections $s_\alpha : U_\alpha \to P$ exist, cf. ref. [7]. On overlap regions $U_\alpha \cap U_\beta$ one has $s_\beta(q) = s_\alpha(q)h_{\alpha\beta}(q)$ with some smooth gauge transformation $h_{\alpha\beta} : U_\alpha \cap U_\beta \to H$. These sections plus the gauge transformations incorporate the topology of the principal fibre bundle

(P, Q, H). Each representation $\pi_\chi(H)$ leads to an associated vector bundle E^χ, whose space of smooth cross-sections is denoted by Γ^χ. An element ψ consists of a collection $\{\psi_\alpha\}_{\alpha\in J}$ of smooth functions $\psi_\alpha : U_\alpha \to \mathcal{H}_\chi$, which on overlap regions $U_\alpha \cap U_\beta$ are related by

$$\psi_\alpha(q) = \pi_\chi(h_{\alpha\beta}(q))\psi_\beta(q). \tag{3.9}$$

The space of compactly supported smooth cross-sections of E^χ is denoted by Γ_c^χ; this space is equipped with the obvious inner product

$$(\psi, \psi') = \sum_{\alpha\in J} \int_Q d\nu(q)\, P_\alpha(q)\,(\psi_\alpha(q), \psi'_\beta(q))_{\mathcal{H}_\chi}, \tag{3.10}$$

where $\{P_\alpha\}_{\alpha\in J}$ is a partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha\in J}$, and the inner product on the right-hand side is the one in $\mathcal{H}_\chi$. The closure of Γ_c^χ in this inner product is $\mathcal{H}^\chi \equiv L^2(E^\chi)$.

It follows from (3.7) and harmonic analysis on H that $\mathcal{H}^\chi$ carries an irreducible representation π^χ of $\mathcal{A}$, which on the dense subalgebra $C_c^\infty(P\times P)^H$ is given by

$$(\pi^\chi(K)\psi_\beta)(q) = \sum_{\alpha\in J} \int_Q d\nu(q')\, P_\alpha(q') \times \int_H dh\, K(s_\beta(q)h, s_\alpha(q'))\pi_\chi(h)\psi_\alpha(q'), \tag{3.11}$$

where we assume that $q \in U_\beta$. Strictly speaking, this is defined on Γ_c^χ as a bounded operator, and then extended to $\mathcal{H}^\chi$ by continuity. It follows from (3.1) and (3.9) (together with the gauge transformation rule of the sections s_α) that (3.11) is well defined: firstly $\pi^\chi(K)\psi_\beta$ transforms like ψ_β under gauge transformations (3.9), and secondly on the right-hand side one could choose, say, γ rather than α on an overlap region $U_\alpha \cap U_\gamma$ without affecting the result.

It is convenient to have a different, unitarily equivalent realization of the Hilbert space of sections at our disposal, which we call $\tilde{\mathcal{H}}^\chi$. This is the Hilbert space closure of $\tilde{\Gamma}_c^\chi$, which is the space of H-equivariant C_c^∞ functions on P with values in $\mathcal{H}_\chi$, that is, $\tilde{\psi} \in \tilde{\Gamma}_c^\chi$ satisfies

$$\tilde{\psi}(xh) = \pi_\chi(h^{-1})\psi(x). \tag{3.12}$$

The inner product is

$$(\tilde{\psi}, \tilde{\psi}') = \int_Q d\nu(\overline{x})\, (\tilde{\psi}(x), \tilde{\psi}'(x))_{\mathcal{H}_\chi}, \tag{3.13}$$

where $\overline{x} = \mathrm{pr}_{P\to Q}(x)$ as before; as P/H is diffeomorphic to Q, it carries the measure ν (note that due to (3.12) the $\mathcal{H}_\chi$ inner product on the right-hand side of (3.13) is a function on P/H). A unitary transformation $\tilde{T} : \tilde{\mathcal{H}}^\chi \to \mathcal{H}^\chi$ is given by

$$(\tilde{T}\tilde{\psi})_\alpha(q) = \tilde{\psi}(s_\alpha(q)), \tag{3.14}$$

with inverse

$$(\tilde{T}^{*}\psi)(x) = \pi_{\chi}(h_{\alpha}(x))^{-1}\psi_{\alpha}(\overline{x}), \tag{3.15}$$

where $h_\alpha(x)$ is defined by the factorization $x = s_\alpha(\overline{x})h_\alpha(x)$; in overlap regions the right-hand side is independent of α due to (3.9). These transformations map $\tilde{\Gamma}_c^\chi$ and Γ_c^χ into each other, so that the former is just the space of C_c^∞ cross-sections of E^χ in the given realization. We then have $\tilde{\pi}^\chi(K) = \tilde{T}^*\pi^\chi(K)\tilde{T}$ on $\tilde{\mathcal{H}}^\chi$ given by

$$(\tilde{\pi}^{\chi}(K)\tilde{\psi})(x) = \int_{Q} \mathrm{d}\nu(\overline{y}) \int_{H} \mathrm{d}h\, K(xh, y)\pi_{\chi}(h)\tilde{\psi}(y); \tag{3.16}$$

the integrand is indeed a function on $P/H \simeq Q$ due to (3.1) and (3.12). For later use, we display an isomorphism between $\tilde{\mathcal{H}}^\chi$ and a Hilbert subspace $\check{\mathcal{H}}^\chi$ of $L^2(P)$. We take a unit vector $u \in \mathcal{H}_\chi$, and define the projector P_u^χ by

$$(P_u^{\chi}\psi)(x) = d_{\chi} \int_{H} \mathrm{d}h\, (\pi_{\chi}(h)u, u)_{\mathcal{H}_{\chi}}\psi(xh), \tag{3.17}$$

where d_χ is the dimension of π_χ; the orthogonality relations for compact groups imply that P_u^χ is indeed a projector, and we call its image $\check{\mathcal{H}}^\chi$. It follows from (3.1) and the H-invariance of the measure μ on P that P_u^χ commutes with all elements of $\mathcal{A}$. Subsequently, we define a partial isometry $\tilde{V}_u^\chi : L^2(P) \to \tilde{\mathcal{H}}^\chi$ by

$$(\tilde{V}_u^{\chi}\psi)(x) = \sqrt{d_{\chi}} \int_{H} \mathrm{d}h\, \psi(xh)\pi_{\chi}(h)u. \tag{3.18}$$

This is unitary on $\check{\mathcal{H}}^\chi$ and annihilates its orthogonal complement. Moreover, from (3.2), (3.16) and (3.18)

$$\tilde{V}_u^{\chi} \circ A = \tilde{\pi}^{\chi}(A) \circ \tilde{V}_u^{\chi} \tag{3.19}$$

for all $A \in \mathcal{A}$ [with A in its defining representation on $L^2(P)$]. This shows that $\pi^\chi(\mathcal{A}) \simeq P_u^\chi \mathcal{A}$. The intertwining property (3.19) shows that $L^2(P)$ contains d_χ copies of each irreducible representation $\pi^\chi(\mathcal{A})$ (one for each unit vector u in an orthogonal basis of $\mathcal{H}_\chi$).

The physical relevance of the trivial exercise in differential geometry given above is that the smooth cross-sections $\tilde{\Gamma}_c^\chi$ are a natural domain of essential self-adjointness of position, momentum, and Hamiltonian in the representation (superselection sector) χ. Being unbounded, these operators are not of the form $\pi^\chi(Q_\hbar(f))$ for any $f \in \mathcal{A}_0$; moreover, the Hamiltonian is only defined up to a constant, and accordingly is not an observable in the proper sense. We now proceed to explain these points in more detail.

3.3. CLASSICAL DYNAMICS

We will analyse the dynamics defined by the classical Hamiltonian function on T^*P equal to

$$H_0(x,p) = \tfrac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu \tag{3.20}$$

in canonical co-ordinates; it is invariantly defined as $-1/2$ times the (principal) symbol of the Laplacian on $(P,\mathbf{g})$, and many of its properties are explored in ref. [1]. Let j be the isomorphism between T^*P and TP defined by the metric (so that $\mathbf{g}(j(\theta),X) = \langle\theta,X\rangle$). Then the flow ρ_t of H_0 is as follows: $\rho_t(\theta)$ is the one-form obtained by parallel transporting θ along the affinely parametrized geodesic $\gamma(\mathrm{pr}_{T*P\to P}(\theta), j(\theta);t)$ [recall our previous notation, according to which $\gamma(x,Y;0) = x$ and $(\mathrm{d}/\mathrm{d}t)\gamma(x,Y;0) = Y$]. The corresponding flow $\check{\rho}_t$ on TP maps $X \in T_xP$ into $\check{\rho}_t(X)$, which is the parallel transport of X along the geodesic $\gamma(x,X;t)$.

The classical Hamiltonian defines a one-parameter group of *-automorphisms α_t^0 on $C_0(T^*P)$ by

$$\alpha_t^0[f] = f\circ\rho_t. \tag{3.21}$$

This automorphism group quotients well to $\mathcal{A}_0 = C_0((T^*P)/H)$. With $(R_h^*)^*$ being the pull-back of R_h^* (which is itself the pull-back of R_h to T^*P) to the continuous functions on T^*P, one easily derives

$$(R_h^*)^*\circ\alpha_t^0 = \alpha_t^0\circ(R_h^*)^*. \tag{3.22}$$

This follows, since R_h is an isometry of P, so that $(R_{h^{-1}})_*\circ j = j\circ R_h^*$; moreover, if γ is a geodesic with initial tangent $j(\theta)$ then $R_{h^{-1}}(\gamma)$ is a geodesic with initial tangent $j(R_h^*\theta)$, and the parallel transport of $R_h^*\theta$ along the latter equals R_h^* of the parallel transport of θ along the former geodesic γ. Regarding $C_0((T^*P)/H)$ as the H-invariant subalgebra of $C_0(T^*P)$ as before, we see from (3.22) that α_t^0 quotients to a *-automorphism of $\mathcal{A}_0$, which we call by the same name for simplicity. Apart from being automorphisms of $\mathcal{A}_0$ in the C^*-sense, the α_t^0 are Poisson morphisms as well.

We recall that $\mathcal{A}_0$ is isomorphic by Fourier transform (2.2) to $\mathcal{F}\mathcal{A}_0$, which has a dense subalgebra $\check{\mathcal{A}}_0 = C_c^\infty(TP)^H$, cf. the text following (2.4). The automorphism α_t^0 is then equivalent to $\check{\alpha}_t^0$ on $\mathcal{F}\mathcal{A}_0$, defined by

$$\check{\alpha}_t^0[\check{f}] = \mathcal{F}\circ\alpha_t^0[f], \tag{3.23}$$

where $\check{f} = \mathcal{F}f$. For future use, we claim that $\check{\alpha}_t^0$ on $\check{\mathcal{A}}_0$ may be written as

$$\check{\alpha}_t^0[\check{f}](X) = (2\pi)^{-n}\int_{T_xP}\int_{T_xP}\mathrm{d}\mu_x(Y)\,\mathrm{d}\mu_x(Z)\,\mathrm{e}^{\mathrm{i}\mathbf{g}(Z,X-Y)}f(Y_Z(t)), \tag{3.24}$$

where $X\in T_xP$, and $Y_Z(t)$ is the parallel transport of Y along the geodesic emanating from x with initial tangent Z. The main ingredient in the proof of (3.24) is the fact that the measures μ_x are invariant under parallel transport. A geodesic

γ from x to y induces an isomorphism between T_xP and T_yP through parallel transport. Let $y = \gamma(x, Z; t)$, and define the Radon–Nikodym derivatives $\nu_t(Y) = \mathrm{d}\mu_{\gamma(x,Z;t)}(Y_Z(t))/\mathrm{d}\mu_x(Y)$. Clearly $\nu_0 = 1$, and a computation in co-ordinates using (2.1) and the geodesic equation reveals that $\mathrm{d}\nu_t/\mathrm{d}t(t = 0) = 0$. But ν_t satisfies a first-order differential equation in t which follows from the geodesic equation and the condition for parallel transport, and this equation supplied with the two initial conditions listed implies that $\nu_t = 1$ for all t. This easily leads to (3.24); similarly, the measures $\mathrm{d}\hat{\mu}_x$ on T_x^*P are invariant under parallel transport. Hence an alternative way of defining the measures μ_x and $\hat{\mu}_x$ would be to take a Lebesgue measure on some fiducial fibre, and translate this measure to the other fibres of $T^{(*)}P$ by parallel transport along geodesics. This is globally well defined, because the holonomy of Riemannian manifolds takes values in $\mathrm{SO}(n)$, which leaves the Lebesgue measure invariant.

The symplectic leaves $P_\mathcal{O}$ of $(T^*P)/H$ are labelled by the co-adjoint orbits $\mathcal{O}$ in $\mathbf{h}$, and are described in refs. [16,32]. The Hamiltonian H_0 is H-invariant, and quotients to a reduced Hamiltonian $H_\mathcal{O}$ on $P_\mathcal{O}$, which describes the motion of a charged particle on $Q = P/H$ in an external Yang–Mills field. The "charge" of the particle is the orbit $\mathcal{O}$, and the Yang–Mills field is the one encoded in the metric $\mathbf{g}$ on P by standard Kaluza–Klein ideology [8]; the resulting equations of motion on $P_\mathcal{O}$ are the Wong equations [16,42,32,31]. This Yang–Mills field also allows one to define a projection from $P_\mathcal{O}$ to T^*Q, which makes $P_\mathcal{O}$ a fibre bundle over T^*Q with fibre $\mathcal{O}$.

To stress the analogy with the quantized situation, it is helpful to look at the reduced dynamics in a more representation-theoretic way. The embedding $i_\mathcal{O} : P_\mathcal{O} \to (T^*P)/H$ defines a Poisson morphism $i_\mathcal{O}^* : C_0((T^*P)/H) \to C_0(P_\mathcal{O})$, which may be regarded as an irreducible realization ("classical representation") $\pi^\mathcal{O} \equiv i_\mathcal{O}^*$ of the Poisson algebra $\mathcal{A}_0$. One may then ask whether the automorphisms α_t^0 can be implemented in the representation $\pi^\mathcal{O}$, that is, is there a flow $\rho_t^\mathcal{O}$ on $P_\mathcal{O}$ so that

$$\pi^\mathcal{O}(\alpha_t^0[f]) = \pi^\mathcal{O}(f) \circ \rho_t^\mathcal{O} \tag{3.25}$$

for all $f \in \mathcal{A}_0$? The answer is yes of course, and the generating function of the flow is precisely the reduced Hamiltonian $H_\mathcal{O}$.

3.4. QUANTUM DYNAMICS

We now wish to define a one-parameter $*$-automorphism group $\alpha_t^\hbar$ on the quantum algebra of observables $\mathcal{A}$ which in some sense is the quantization of the classical time evolution α_t^0 on $\mathcal{A}_0$. Let

$$H_\hbar = -\tfrac{1}{2}\hbar^2 \Delta_{\mathrm{LB}} \tag{3.26}$$

in terms of the Laplace–Beltrami operator Δ_{LB} on $L^2(P)$, which is essentially self-adjoint on $C_c^\infty(P)$ [12] (recall our assumption that $(P, \mathbf{g})$ is complete).

With $A \in \mathcal{A}$ in its defining representation on $L^2(P)$, define

$$\alpha_t^\hbar[A] = e^{itH_\hbar/\hbar} A e^{-itH_\hbar/\hbar}. \tag{3.27}$$

This indeed defines an automorphism of $\mathcal{A}$: $\alpha_t^\hbar[A]$ is compact and H-invariant if A is, because the compacts form a two-sided ideal in $\mathcal{B}(\mathcal{H})$, and as H is an isometry group of $\mathbf{g}$, $\exp(-i\hbar t\Delta_{\mathrm{LB}}/2)$ commutes with $\pi_R(H)$ [cf. (2.8)].

For $P = \mathbb{R}^n$ with its flat metric it is easy to check that $\alpha_t^\hbar \circ Q_\hbar = Q_\hbar \circ \alpha_t^0$, but this is not the case on general Riemannian manifolds. Experience with scattering theory suggests that even uniform convergence of quantum to classical dynamics [that is, $\|\alpha_t^\hbar \circ Q_\hbar(f) - Q_\hbar \circ \alpha_t^0(f)\|$ going to zero with $\hbar$] cannot be expected. Instead, we will prove convergence of certain expectation values. This involves the notion of a classical germ, which was introduced in ref. [25] as a generalization of a coherent state; the definition is recalled in subsection 2.3 above.

Theorem 3. *Let $\{\omega_\hbar\}$ be an arbitrary classical germ, defined for $\hbar \geq 0$, and define the classical and quantum dynamics by the Laplacian on a complete Riemannian manifold P [cf. (3.20), (3.21) and (3.26), (3.27), respectively]. Then for each observable $f \in \overline{\mathcal{A}_0}$*

$$\lim_{\hbar\to 0} \omega_\hbar(\alpha_t^\hbar[Q_\hbar(f)] - Q_\hbar(\alpha_t^0[f])) = 0. \tag{3.28}$$

Proof. We assume that the classical germ $\{\omega_\hbar\}$ consists of vector states $\Omega_\hbar$ in $L^2(P)$ [that is, $\omega_\hbar(A) = (A\Omega_\hbar, \Omega_\hbar)$ for $\hbar > 0$]; the general case follows by decomposition [any state on $\mathcal{A} \subset \mathcal{K}(L^2(P))$ is given by a density operator, which has a discrete decomposition into vector states]. Define $\Omega_\hbar(t) = \exp(-itH_\hbar/\hbar)\Omega_\hbar$ [cf. (3.26)] and $\omega_\hbar(t) = (\cdot\, \Omega_\hbar(t), \Omega_\hbar(t))$ for $\hbar > 0$. If $\{\omega_\hbar\}$ is a classical germ then so is $\{\omega_\hbar(t)\}$: continuity of the function $\hbar \to (\omega_\hbar(t))(Q_\hbar(f))$ for $\hbar > 0$ follows from the strong continuity of $\hbar \to \exp(-it\hbar\Delta_{\mathrm{LB}}/2)$, and the existence of the limit $\hbar \to 0$, already suggested by the bound $|(\omega_\hbar(t))(Q_\hbar(f))| \leq \|Q_\hbar(f)\|$ and the continuity of $\|Q_\hbar(f)\|$ in $\hbar$ on some strip $[0, \varepsilon)$, is shown below.

Obviously, for $\hbar > 0$ we have from (2.29) (which we rewrite as an integral over T^*P, using the H-invariance of f)

$$\omega_\hbar(\alpha_t^\hbar[Q_\hbar(f)]) = \int_{T^*P} f\, \tilde{W}^\hbar_{\Omega_\hbar(t)}, \tag{3.29}$$

$$\omega_\hbar(Q_\hbar(\alpha_t^0[f])) = \int_{T^*P} f\, \tilde{W}^\hbar_{\Omega_\hbar}(t), \tag{3.30}$$

where the time dependence of $\tilde{W}^\hbar_{\Omega_\hbar}(t)$ is given by the classical Liouville equation, i.e.,

$$\partial \tilde{W}^\hbar_{\Omega_\hbar}(t)/\partial t = \{\tilde{W}^\hbar_{\Omega_\hbar}(t), H_0\}, \tag{3.31}$$

where H_0 is the Hamiltonian (3.20) on T^*P, and $\{\,,\,\}$ is the canonical Poisson bracket on T^*P, which in general is to be evaluated in the sense of weak (distributional) derivatives with regard to the test function f. On the other hand, the time dependence of $\tilde{W}^{\hbar}_{\Omega_\hbar(t)}$ follows from (2.31) and the Schrödinger equation. This holds on the domain of Δ_{LB}, so to be precise we should proceed on the assumption that $\Omega_\hbar$ is in this domain for all $\hbar > 0$.

A straightforward computation, expressing the Laplacian in local co-ordinates, and not overlooking the $\sqrt{g}(x)$ in the measure $d\mu_x(\dot{x})$ then shows that $\partial \tilde{W}^{\hbar}_{\Omega_\hbar(t)}/\partial t - \partial \tilde{W}^{\hbar}_{\Omega_\hbar}(t)/\partial t$ evaluated at $t = 0$ weakly tends to zero for $\hbar \to 0$. The main ingredient of this calculation is to convert derivatives of $\Omega_\hbar(\gamma(x, \dot{x}; \pm\hbar/2))$ w.r.t. $\dot{x}$ into those w.r.t. x, which can be done using (2.15). The Jacobi fields $\tilde{h}$ that enter the resulting expressions may be evaluated for small $\hbar$ using the expressions given after (2.20), whereas J and its derivatives approach unity [cf. (2.21)]. Finally, in converting p_μ into $i\partial/\partial\dot{x}^\mu$ one generates boundary integrals over $\partial B_\hbar(x) \subset T_xP$ [cf. (2.31)], which multiply $\check{f}$ evaluated on this boundary; such terms go to zero for sufficiently small $\hbar$ because of the compact support of $\check{f}$ (particularly in the fibre direction, in this case).

This computation shows that

$$\varphi_0(t) \equiv \lim_{\hbar\to 0}(\omega_\hbar(t))(Q_\hbar(f)) = \lim_{\hbar\to 0}\omega_\hbar(\alpha_t^\hbar[Q_\hbar(f)]) \tag{3.32}$$

has the same derivative at $t = 0$ as $\tilde{\varphi}_0$ defined by $\tilde{\varphi}_0(t) = \omega_0(\alpha_t^0[f])$. Since their values at $t = 0$ obviously coincide, too, and both functions satisfy a first-order autonomous differential equation in t which has a solution for all t (recall our assumption that P be complete), we conclude that $\varphi_0(t) = \tilde{\varphi}_0(t)$ for all t, provided that the $\omega_\hbar$ ($\hbar > 0$) are vector states built from vectors in the domain of Δ_{LB}. But the definition (3.32) of φ_0 itself (as opposed to its derivative) involves only bounded operators, so that the conclusion may be extended to any classical germ. This proves the theorem. □

Alternative proof. To further our insight into the way the quantum time evolution approaches the classical one, we will sketch an instructive alternative proof of theorem 3. This proof uses heat kernels, and only works for P compact. The initial detour is included in order to exploit the pleasant properties of Euclidean (as opposed to real-time) heat kernels.

For fixed $f \in \overline{\mathcal{A}_0}$, define for $\hbar > 0$

$$\varphi_\hbar(z_1, z_2) = (Q_\hbar(f)e^{-\hbar z_1\Delta_{\text{LB}}/2}\Omega_\hbar, e^{-\hbar\overline{z_2}\Delta_{\text{LB}}/2}\Omega_\hbar). \tag{3.33}$$

Since Δ_{LB} is a positive operator, this function is holomorphic for $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$, hence by ref. [39, thm. 14.18] $\phi_\hbar$ has a continuous extension to the real axis. The uniform bound $|\varphi_\hbar(z_1, z_2)| \le \|Q_\hbar(f)\| \le$ const. for these values of its argument implies, by a well-known theorem in complex variables (cf. ref.

[39, thms. 14.6 and 14.18]), that $\lim_{\hbar\to 0}\varphi_\hbar \equiv \varphi_0$ is holomorphic in the same region, with a continuous extension to its boundary. Hence for t real

$$\varphi_0(t) = \lim_{\substack{z_1\to \mathrm{i}t,\\ z_2\to -\mathrm{i}t}} \varphi_0(z_1,z_2), \tag{3.34}$$

that is, the limits where the z_i approach the imaginary axis and $\hbar \to 0$ may be interchanged.

For real and positive τ, the heat kernel is defined by

$$(\exp(-\tau\Delta_{\mathrm{LB}}/2)\Omega)(x) = \int_P \mathrm{d}\mu(y)\, K(\tau;x,y)\Omega(y), \tag{3.35}$$

and K is C^∞ in x and y for $\tau > 0$ [12]. It follows from ref. [38, thm. 5.16] that $K = K_0(1+R)$, where

$$K_0(\tau;x,y) = \frac{(\Delta(x,y))^{1/2}}{(2\pi\tau)^{n/2}}\mathrm{e}^{-\sigma(x,y)/\tau}, \tag{3.36}$$

where $\sigma(x,y)$ is one-half times the distance between x and y (along a minimal geodesic), and Δ is the Van Vleck–Morette determinant (cf. refs. [15,7]). The remainder (which is positive definite, like K itself [12]) is uniformly bounded for sufficiently small τ:

$$R(\tau;x,y) \le C\tau \tag{3.37}$$

for some constant C. Writing $K_0(\tau)\Omega$ for (3.35), with K replaced by K_0, we thus have from (3.34), (3.37), and the analyticity of $\varphi_\hbar$

$$\varphi_0(t) = \lim_{\substack{z_1\to \mathrm{i}t,\\ z_2\to -\mathrm{i}t}} \lim_{\hbar\to 0}(Q_\hbar(f)K_0(\hbar z_1)\Omega_\hbar, K_0(\hbar z_2)\Omega_\hbar). \tag{3.38}$$

However, K_0 generates a holomorphic semigroup, just like K [i.e., $\exp(z\Delta_{\mathrm{LB}})$], hence the same reasoning that led to (3.34) allows us to interchange the limits in (3.38), to conclude that for real t

$$\lim_{\hbar\to 0}\omega_\hbar(\alpha_t^\hbar[Q_\hbar(f)]) = \lim_{\hbar\to 0}(K_0(-\mathrm{i}\hbar t)Q_\hbar(f)K_0(\mathrm{i}\hbar t)\Omega_\hbar,\Omega_\hbar). \tag{3.39}$$

We now take arbitrary $(x_1,x_2) \in P\times P$, except that x_1 and x_2 should not be conjugate points [7]; this only removes a set of measure zero on $P\times P$ from consideration. By a theorem of de Rham, there then exists a minimal geodesic γ connecting x_1 and x_2, which does not have any pair of conjugate points on it. Accordingly, we can write $x_1 = \gamma(x,\dot{x};1/2)$ and $x_2 = \gamma(x,\dot{x};-1/2)$ for some x and $\dot{x}$. By the theory relating conjugate points to Jacobi fields [7], we can find an open set $U \subset P\times P$ containing (x_1,x_2), which is diffeomorphic to an open set $V\subset TP$ which contains $\dot{x}\in T_xP$ (but it does not necessarily contain $0\in T_xP$). We write the right-hand side of (3.39) as an integral over $P\times P$, and examine the contribution from U. A formula analogous to (2.10), but restricted to U, rewrites this integral as an integral over V. Rescaling by a factor $\hbar$ in the fibre direction, this leads to the expression

$$\int_U d\mu(x_1)\, d\mu(x_2)\, \overline{\Omega_\hbar(x_1)}\Omega_\hbar(x_2)\,(K_0(-i\hbar t)Q_\hbar(f)K_0(i\hbar t))(x_1,x_2)$$

$$\begin{aligned} &= \hbar^n \int_V d\mu(x)\, d\mu_x(\dot{x}) \int_{TP} d\mu(y)\, d\mu_y(\dot{y})\, \check{f}(y,\dot{y}) J(x,\dot{x};\hbar/2) J(y,\dot{y};\hbar/2) \cdot \\ &\quad \times K_0(-i\hbar t; \gamma(x,\dot{x};\hbar/2), \gamma(y,\dot{y};\hbar/2)) \\ &\quad \times K_0(i\hbar t; \gamma(y,\dot{y};-\hbar/2), \gamma(x,\dot{x};-\hbar/2)). \end{aligned} \tag{3.40}$$

Using (3.36), and some properties of σ [15] this can be computed for small $\hbar$; for example, one has

$$\begin{aligned} &\sigma(\gamma(x,\dot{x};\hbar/2), \gamma(y,\dot{y};\hbar/2)) \\ &\quad = \sigma(x,y) - \tfrac{1}{2}\hbar[\mathbf{g}(\dot{x}, \dot{\gamma}_{x\to y}) + \mathbf{g}(\dot{y}, \dot{\gamma}_{y\to x})] + \mathcal{O}(\hbar^2), \end{aligned} \tag{3.41}$$

where $\gamma_{x\to y}$ is a minimal geodesic between x and y (which is affinely parametrized so that $\gamma(0) = x$ and $\gamma(1) = y$), and $\dot{\gamma}_{x\to y}$ is its tangent at the origin. Subsequently, one uses (2.16) (with $x' = y$) and one sees that the factor $\tilde{J}$ cancels the VanVleck–Morette determinants in (3.40) for $\hbar = 0$. After a simple further rearrangement we may then use (3.24) and (3.39) to conclude that

$$\begin{aligned} &\lim_{\hbar\to 0} \int_U d\mu(x_1)\, d\mu(x_2)\, \overline{\Omega_\hbar(x_1)}\Omega_\hbar(x_2) \\ &\quad \times \left(\alpha_t^\hbar[Q_\hbar(f)](x_1,x_2) - Q_\hbar(\alpha_t^0[f])(x_1,x_2)\right) = 0. \end{aligned} \tag{3.42}$$

Since x_1 and x_2 were arbitrary (a.e.) this proves (3.28). □

Note that a rather different approach to the classical limit of the dynamics (3.26) is presented in ref. [40]; the first rigorous results on the classical limit of quantum correlation functions were obtained by Hepp [19]. He used coherent states, of which our classical germs are a generalization.

To sum up, we have a satisfactory quantum dynamics on $\mathcal{A}$ defined by (3.27). We may now ask if and how $\alpha_t^\hbar$ is implemented in the irreducible representations π^χ of $\mathcal{A}$ (cf. subsection 3.2). That is, we look for a self-adjoint operator $H_\hbar^\chi$ on $\mathcal{H}^\chi$ which satisfies

$$\pi^\chi(\alpha_t^\hbar[A]) = e^{itH_\hbar^\chi/\hbar}\pi^\chi(A)e^{-itH_\hbar^\chi/\hbar} \tag{3.43}$$

for all $A \in \mathcal{A}$. Such an operator (defined up to a constant) clearly plays the role of the quantum Hamiltonian in the sector χ, cf. the classical case (3.25). Before giving the answer, we recall [8] that the H-invariant metric $\mathbf{g}$ on P determines a connection A on the bundle (P, Q, H) as well as a metric $\mathbf{g}_Q$ on Q: the horizontal subspace $H_xP \subset T_xP$ is defined as the orthogonal complement of the intrinsically defined vertical subspace $V_xP \subset T_xP$, and the value of $\mathbf{g}_Q(X,Y)$ at $q \in Q$ is taken to be $\mathbf{g}(\lambda(X),\lambda(Y))$, where $\lambda(X)$ is the horizontal lift of X at an arbitrary point in the fibre above q. Conversely, a metric on Q plus an invariant metric on H and a connection on P determine an H-invariant metric on P.

Theorem 4. *The Hamiltonian in the sector χ is given by the gauge-covariant Laplacian on E^χ [the vector bundle associated to P by the representation $\pi_\chi(H)$] with respect to the metric $\mathbf{g}_Q$ on Q and the connection A on P which are canonically associated to the H-invariant metric* $\mathbf{g}$ *on P; with $A^\chi = \mathrm{d}\pi_\chi(A)$, and otherwise self-evident notation*

$$H_\hbar^\chi = -\tfrac{1}{2}\hbar^2\nabla^{A_\chi}\cdot\nabla^{A_\chi}. \tag{3.44}$$

This operator is defined and essentially self-adjoint on the space $\Gamma_c^\chi \subset \mathcal{H}^\chi$ of compactly supported smooth cross-sections of E^χ.

Proof. For simplicity, we omit the factor $-\hbar^2/2$ throughout this proof, as well as the $1/\hbar$ appearing in (3.27) and (3.43). As shown after (3.16), the Hilbert space $\tilde{\mathcal{H}}^\chi$ is naturally isomorphic to $\check{\mathcal{H}}^\chi \subset L^2(P)$. It is clear from (3.17), (3.27), and the H-invariance of Δ_{LB} that the unitary group $\exp(\mathrm{i}t\Delta_{\mathrm{LB}})$, restricted to $\check{\mathcal{H}}^\chi$ (which subspace it leaves invariant), implements time evolution on $P_u^\chi\mathcal{A}$. We wish to identify the generator of this group, and this amounts to answering the question whether Δ_{LB}, restricted to $P_u^\chi L^2(P) = \check{\mathcal{H}}^\chi$, is essentially self-adjoint on the domain $P_u^\chi\mathcal{D}(P)$ [with the Schwartz space $\mathcal{D}(P) \equiv C_c^\infty(P)$]. This question is more easily analysed by transferring the situation to $\tilde{\mathcal{H}}^\chi$ using (3.18). Indeed, $\tilde{V}_u^\chi$ maps $\mathcal{D}(P)$ onto $\tilde{\Gamma}_c^\chi \equiv (\mathcal{D}(P)\otimes\mathcal{H}_\chi)^H$ [cf. (3.12)]; the notation X^H stands for the H-equivariant subspace of some space X of functions or distributions; in this case the equivariance condition is given by (3.12). Consider the inclusions

$$\mathcal{D}(P)\otimes\mathcal{H}_\chi \subset L^2(P)\otimes\mathcal{H}_\chi \subset \mathcal{D}'(P)\otimes\mathcal{H}_\chi.$$

The Laplacian Δ_{LB} is defined on $\mathcal{D}'(P)$ (using weak derivatives), and acts componentwise on $\mathcal{D}'(P)\otimes\mathcal{H}_\chi$ (i.e., it does not touch $\mathcal{H}_\chi$); in what follows we call this extended operator still Δ_{LB}. Thus Δ_{LB} acts on the two subspaces listed by restriction, its action on $\mathcal{D}(P)\otimes\mathcal{H}_\chi$ being given through ordinary (strong) derivatives. Since Δ_{LB} is essentially self-adjoint on $\mathcal{D}(P)\subset L^2(P)$ [12] (recall our assumption that P be complete), it is essentially self-adjoint on $\mathcal{D}(P)\otimes\mathcal{H}_\chi \subset L^2(P)\otimes\mathcal{H}_\chi$. If $D(\Delta_{\mathrm{LB}}) = \mathcal{D}(P)$ is the domain of Δ_{LB} on $L^2(P)$ then the domain of its adjoint $D(\Delta_{\mathrm{LB}}^*)$ consists of those elements φ of $L^2(P)$ for which $\Delta_{\mathrm{LB}}\varphi$ is in $L^2(P)$, and the action of Δ_{LB}^* coincides with that of Δ_{LB} on distributions (all this tensored with arbitrary vectors in $\mathcal{H}_\chi$). This is easily seen to be true also in the equivariant case. The space of distributions $\mathcal{D}'(P)$ carries a representation π'_R of H by duality [that is, $\langle\pi'_R(h)(\varphi),f\rangle = \langle\varphi,\pi_R(h^{-1})f\rangle$ for $\varphi\in\mathcal{D}'(P)$ and $f\in\mathcal{D}(P)$, cf. (2.7) for π_R]. Hence it makes sense to speak of the subspace $(\mathcal{D}'(P)\otimes\mathcal{H}_\chi)^H$ of H-equivariant distributions on P with values in $\mathcal{H}_\chi$. This leads to the triple

$$(\mathcal{D}(P)\otimes H_\chi)^H \subset \tilde{\mathcal{H}}^\chi \subset (\mathcal{D}'(P)\otimes\mathcal{H}_\chi)^H.$$

Since Δ_{LB} is H-invariant it is well defined on $(\mathcal{D}'(P)\otimes\mathcal{H}_\chi)^H$, and we denote the restriction of Δ_{LB} to $(\mathcal{D}'(P)\otimes\mathcal{H}_\chi)^H$ by $\Delta^\chi_{\mathrm{LB}}$. If we put $D(\Delta^\chi_{\mathrm{LB}}) = (\mathcal{D}(P)\otimes\mathcal{H}_\chi)^H$ then the domain $D((\Delta^\chi_{\mathrm{LB}})^*)$ of its adjoint in $\tilde{\mathcal{H}}^\chi$ consists of those elements φ of $\tilde{\mathcal{H}}^\chi$ for which $\Delta^\chi_{\mathrm{LB}}\varphi$ is in $\tilde{\mathcal{H}}^\chi$. Hence $D((\Delta^\chi_{\mathrm{LB}})^*)\subset D(\Delta^*_{\mathrm{LB}})$, and this immediately implies that $\Delta^\chi_{\mathrm{LB}}$ is essentially self-adjoint; namely, suppose that there exists a $\psi\in D((\Delta^\chi_{\mathrm{LB}})^*)$ for which $((\Delta^\chi_{\mathrm{LB}})^*\pm \mathrm{i})\psi = 0$ then this ψ is in $D(\Delta^*_{\mathrm{LB}})$ also, and because Δ^*_{LB} coincides with $(\Delta^\chi_{\mathrm{LB}})^*$ on $(\mathcal{D}'(P)\otimes\mathcal{H}_\chi)^H$ we would contradict the essential self-adjointness of Δ_{LB} on $\mathcal{D}(P)$.

We now notice that $\tilde{V}^\chi_u\circ\Delta_{\mathrm{LB}} = \Delta^\chi_{\mathrm{LB}}\circ\tilde{V}^\chi_u$, including their domains, so it follows that $\Delta^\chi_{\mathrm{LB}}$ generates time evolution on $\tilde{\mathcal{H}}^\chi$. Finally, we use the identity, valid on $\tilde{\Gamma}^\chi_c$ [6] (cf. ref. [24] for the special case $P = G$)

$$\Delta^\chi_{\mathrm{LB}} = \nabla^{A_\chi}\cdot\nabla^{A_\chi} - C^\chi_2(H), \tag{3.45}$$

where $C^\chi_2(H) = \sum_i T_i^2$ is the second-order Casimir operator of H in the representation π_χ. Since π_χ is irreducible, the last term is a constant which can be omitted from the Hamiltonian. Using the unitary intertwiner $\tilde{T}$ [cf. (3.14)], we can transfer the whole situation from $\tilde{\mathcal{H}}^\chi$ to $\mathcal{H}^\chi$, and this proves the theorem.□

This theorem provides some motivation for the use of vector bundles and their smooth cross-sections in quantum mechanics. Also, it gives a technique for proving (essential) self-adjointness of Schrödinger operators in external (non-abelian) magnetic fields: given the metric on Q and the connection on P defining the field, one constructs the metric $\mathbf{g}$ on P and finds out whether $(P,\mathbf{g})$ is geodesically complete, which is a classical problem. Further information on operators of the type (3.44) may be found in, e.g., ref. [40].

3.5. POSITION AND MOMENTUM OBSERVABLES: CLASSICAL THEORY

As to position and momentum, in the absence of global co-ordinates on Q one cannot expect operators of the form "q^μ" or "$-\mathrm{i}\hbar\partial/\partial q^\mu$ to be meaningful. In the absence of gauge fields, one can recover position and momentum from the group $\mathrm{Diff}(Q)\rhd C^\infty_c(Q)$ [1,16,22]. This is the semi-direct product of the diffeomorphism group of Q and the group of smooth functions on Q with compact support; the latter is an abelian group under addition. A diffeomorphism $\overline{\varphi}$ acts on a function $F\in C^\infty(Q)$ by sending it to $F\circ\overline{\varphi}^{-1}$, which defines the semi-direct product structure.

However, in an external gauge field [described as a connection on the bundle

(P, Q, H)] the correct group to use is

$$\mathcal{G} = \mathrm{Aut}(P) \rhd C_c^\infty(Q). \tag{3.46}$$

Here $\mathrm{Aut}(P)$ is the group of smooth bundle automorphisms of P, that is, the restriction of $\mathrm{Diff}(P)$ to those C^∞ diffeomorphisms φ that satisfy $\varphi(xh) = \varphi(x)h$ for all $x \in P$ and $h \in H$. The relation between $\mathrm{Aut}(P)$ and $\mathrm{Diff}(Q)$ is described by the exact sequence of groups

$$1 \to \mathrm{Gau}(P) \to \mathrm{Aut}(P) \to \mathrm{Diff}(Q) \to 1, \tag{3.47}$$

which is the global version of the so-called Atiyah exact sequence of Lie algebras (cf. ref. [28])

$$0 \to \mathcal{X}_V^H(P) \to \mathcal{X}^H(P) \to \mathcal{X}(Q) \to 0. \tag{3.48}$$

The gauge group $\mathrm{Gau}(P)$ consists of those bundle automorphisms of P which do not map points out of the fibre they are in, and the epimorphism of $\mathrm{Aut}(P)$ onto $\mathrm{Diff}(Q)$ is the obvious one; our notation will be that $\varphi \in \mathrm{Aut}(P)$ is mapped onto $\overline{\varphi} \in \mathrm{Diff}(Q)$; this defines the semi-direct product structure of $\mathcal{G}$ as in the $\mathrm{Diff}(Q)$ case. We assume that all diffeomorphisms featuring in (3.47) are equal to the identity outside some compact set: if we interpret (3.48) as the Lie algebra version of (3.47), the smooth vector fields occurring in it have compact support, so that their flows are automatically complete, and they are closed under addition and taking Lie brackets. Hence $\mathcal{X}_V^H(P)$ consists of H-invariant vertical vector fields on P, $\mathcal{X}^H(P)$ is the Lie algebra of all H-invariant vector fields on P, and $\mathcal{X}(Q)$ are the vector fields on Q. The connection A on P is equivalent to a splitting of the sequence (3.48), since $\xi \in \mathcal{X}(Q)$ has a horizontal lift $\lambda(\xi) \in \mathcal{X}^H(P)$; we will denote the horizontal lift of $Y \in T_qQ$ to T_pP by $\lambda_p(Y)$ [$\mathrm{pr}_{P\to Q}(p) = q$].

In our approach, the relevance of the group $\mathcal{G}$ is that it naturally acts as a *-automorphism group on both $\mathcal{A}_0$ and $\mathcal{A}$, and that the quantization maps $Q_\hbar$ asymptotically intertwine this action. We start by explaining how $\mathcal{G}$ acts on $S = (T^*P)/H$; this is a straightforward extension of the symplectic case with $\mathrm{Diff}(Q) \rhd C_c^\infty(Q)$ [16,22]. The action is called ρ^0, and is given by

$$(\rho^0(\varphi))([\theta]) = [(\varphi^{-1})^*\theta] \tag{3.49}$$

for $\varphi \in \mathrm{Aut}(P)$, and

$$(\rho^0(F))([\theta]) = [\theta + d\tilde{F}(\mathrm{pr}_{T^*P\to P}(\theta))] \tag{3.50}$$

for $F \in C_c^\infty(Q)$. The action of $(\varphi, F) \in \mathcal{G}$ is then given by $\rho^0((\varphi, F)) = \rho^0(F) \circ \rho^0(\varphi)$. Here $[\theta] \in (T^*P)/H$ is the image of $\theta \in T^*P$ under taking H-equivalence classes, $\tilde{F}$ is the lift of $F \in C^\infty(Q)$ to $C^\infty(P)$ (that is, the pull-back of F for the bundle projection $\mathrm{pr}_{P\to Q}$); the result is independent of the particular choice of θ in its equivalence class. As in the symplectic case, it is easily checked that the ρ^0 are Poisson morphisms. Thus we may define an action α^0 of $\mathcal{G}$ on $\mathcal{A}_0$ as the pull-back of ρ^0, which is *-automorphic as well as Poisson, viz.,

$$\alpha^0_{(\varphi,F)}[f] = f \circ \rho^0((\varphi,F)^{-1}). \tag{3.51}$$

The Fourier transform (2.2) then yields an action $\check{\alpha}^0$ on $\check{\mathcal{A}}_0 = C_c^\infty((TP)/H)$, through $\check{\alpha}^0[\check{f}] = \mathcal{F}\alpha^0[f]$. If $[X] \in (TP)/H$ is the equivalence class of $X \in T_xP$, then

$$(\check{\alpha}^0_\varphi[\check{f}])([X]) = \frac{\mathrm{d}\mu(\varphi^{-1}(x))}{\mathrm{d}\mu(x)}\check{f}([\varphi_*^{-1}X]), \tag{3.52}$$

$$(\check{\alpha}^0_F[\check{f}])([X]) = \mathrm{e}^{-\mathrm{i}(X\tilde{F})(x)}\check{f}([X]), \tag{3.53}$$

which are independent of the choice of X in $[X]$.

Let $\pi^{\mathcal{O}}$ be the irreducible "representation" of $\mathcal{A}_0$ on the symplectic leaf $P_{\mathcal{O}} \subset (T^*P)/H$ [cf. text prior to (3.25)]. We ask whether the automorphism α^0 is implemented in $\pi^{\mathcal{O}}$, that is, is there a symplectomorphism $\rho^{\mathcal{O}}((\varphi,F))$ for which

$$\pi^{\mathcal{O}}(\alpha^0_{(\varphi,F)}[f]) = \pi^{\mathcal{O}}(f) \circ \rho^{\mathcal{O}}((\varphi,F)^{-1})\,? \tag{3.54}$$

The answer is clearly yes, as $\rho^{\mathcal{O}}$ is simply given by the restriction of ρ^0 to $P_{\mathcal{O}}$ (note that the ^{0} in ρ^0 stands for the value $\hbar = 0$ rather than the zero orbit $\mathcal{O} = \{0\}$). If φ_t is a one-parameter subgroup of $\mathrm{Aut}(P)$ with generator $\xi \in \mathcal{X}^H$, then the generating function $\sigma^{\mathcal{O}}(\xi)$ of the flow $\rho^{\mathcal{O}}(\varphi_t)$ on $P^{\mathcal{O}}$ is given by the symbol $\sigma(\xi)$ quotiented to a function on $P^{\mathcal{O}}$ [the symbol $\sigma(\xi)$ of a vector field ξ on P is the C^∞ function on T^*P defined by $(\sigma(\xi))(\theta) = \langle\theta,\xi\rangle(\mathrm{pr}_{T^*P\to P}(\theta))$; if $\xi \in \mathcal{X}^H$ then $\sigma(\xi)$ is H-invariant, so by ref. [1, thm. 4.3.5] it quotients to a function on any symplectic leaf of $(T^*P)/H$]. Similarly, the generating function of the flow $\rho^{\mathcal{O}}(tF)$ is F itself (more precisely, the quotient of $\tilde{F}$ to $P^{\mathcal{O}}$).

A momentum variable on Q is a vector field $\mathcal{X}$; in the presence of a gauge field the only intrinsic object on $P^{\mathcal{O}}$ to be associated to ξ is $\sigma^{\mathcal{O}}(\lambda(\xi))$. Locally $P^{\mathcal{O}} \simeq T^*Q \times \mathcal{O}$, on which we choose canonical co-ordinates (q,p,θ) relative to a given trivialization (gauge) s, in which the connection A has components $s^*A = A^i_\alpha\,\mathrm{d}q^\alpha\,T_i$, and $\xi = \xi^\alpha\,\partial/\partial q^\alpha$; one finds (cf. ref. [16, III.36])

$$\sigma^{\mathcal{O}}(\lambda(\xi))(q,p,\theta) = \xi^\alpha(q)(p_\alpha - \theta_i A^i_\alpha(q)). \tag{3.55}$$

We obtained this expression from the "classical" representation theory of the classical algebra of observables $\mathcal{A}_0$ and its automorphism group $\mathcal{G}$; for a different perspective cf. refs. [16, ch. III] and [31, 3.3].

3.6. QUANTUM POSITION AND MOMENTUM

To quantize the structures of the preceding subsection, we start by constructing an automorphic action of $\mathcal{G}$ on the quantum algebra of observables $\mathcal{G}$. The first step is to define a unitary representation $\rho^\hbar$ (for fixed $\hbar > 0$) of $\mathcal{G}$ on $L^2(P)$ by

$$(\rho^\hbar(\varphi)\psi)(x) = \left[\mathrm{d}\mu(\varphi^{-1}(x))/\mathrm{d}\mu(x)\right]^{1/2} \psi(\varphi^{-1}(x)), \tag{3.56}$$

$$(\rho^\hbar(F)\psi)(x) = \mathrm{e}^{-\mathrm{i}\tilde{F}(x)/\hbar}\psi(x), \tag{3.57}$$

and $\rho^\hbar((\varphi,F)) = \rho^\hbar(F)\rho^\hbar(\varphi)$. This representation commutes with $\pi_R(H)$ [cf. (2.7)], so it defines a *-automorphic action of $\mathcal{G}$ on $\mathcal{A}$ by

$$\alpha^\hbar_{(\varphi,F)}[A] = \rho^\hbar((\varphi,F))A\rho^\hbar((\varphi,F))^*. \tag{3.58}$$

For $K \in C_c^\infty(P\times P)^H \subset \mathcal{A}$ the kernel (called K as well, as before) transforms as

$$(\alpha^\hbar_\varphi[K])(x_1,x_2) = \left(\frac{\mathrm{d}\mu(\varphi^{-1}(x_1))}{\mathrm{d}\mu(x_1)}\frac{\mathrm{d}\mu(\varphi^{-1}(x_2))}{\mathrm{d}\mu(x_2)}\right)^{1/2} \times K(\varphi^{-1}(x_1),\varphi^{-1}(x_2)), \tag{3.59}$$

$$(\alpha^\hbar_F[K])(x_1,x_2) = \mathrm{e}^{\mathrm{i}(\tilde{F}(x_2)-\tilde{F}(x_1))/\hbar}K(x_1,x_2). \tag{3.60}$$

In analogy with theorem 3 we now have

Theorem 5. *For each classical germ* $\{\omega_\hbar\}_{\hbar\geq 0}$ *and for each* $f \in \overline{\mathcal{A}_0}$ *as well as any* $(\varphi,F)\in\mathcal{G}$

$$\lim_{\hbar\to 0}\omega_\hbar(\alpha^\hbar_{(\varphi,F)}[Q_\hbar(f)] - Q_\hbar(\alpha^0_{(\varphi,F)}[f])) = 0. \tag{3.61}$$

Proof. This is very similar to the proof of theorem 3, and we leave the details to the reader. All analysis is trivial since $\check{f}$ has compact support, so that we may make Taylor expansions of the kernels of the operators appearing in (3.61), and conclude that $\mathcal{O}(\hbar)$ terms in the integrand do not contribute in the limit $\hbar \to 0$. The result then easily follows from (3.59), (3.60), (2.5), (3.52), and (3.53). One here needs the fact that the point $a = \varphi(\exp_x(\pm\hbar X/2))$ is equal to $b = \exp_{\varphi(x)}(\pm\varphi_* X/2)$ up to $\mathcal{O}(\hbar^2)$, in the sense that $g(a) = g(b) + \mathcal{O}(\hbar^2)$ for all $g \in C^2(P)$. □

Note that if φ is an isometry of P then (3.61) holds without the lim and the $\omega_\hbar$, i.e., in that case the quantization $Q_\hbar$ exactly intertwines $\alpha^\hbar_\varphi$ and α^0_φ. An interesting case where this happens is when P is a Lie group G and $\mathbf{g}$ is a left-invariant metric on G. Then left multiplication by a fixed $x \in G$ is an isometry. If we furthermore take $H = G$ [so that $(T^*P)/H \simeq \mathbf{g}^*$ and $\mathcal{A} = C^*(G)$] we recover Rieffel's result that the deformation of $C_0(\mathbf{g}^*)$ to $C^*(G)$ is G-invariant [37].

Let $\tilde{\pi}^\chi$ be the irreducible representation of $\mathcal{A}$ on $\tilde{\mathcal{H}}^\chi$ given by (3.16); one has a unitary representation $\tilde{\rho}^\chi$ of $\mathcal{G}$ on $\tilde{\mathcal{H}}^\chi$, which is given by the same formulae as

(3.56), (3.57), with ψ now taking values in $\mathcal{H}_\chi$. Passing to $\mathcal{H}^\chi$ via the unitary map (3.14), we obtain $\rho^\chi = \tilde{T}\tilde{\rho}^\chi\tilde{T}^*$:

$$(\rho^\chi(\varphi)\psi)_\alpha(q) = \left[d\nu(\overline{\varphi}^{-1}(q))/d\nu(q)\right]^{1/2} \times \pi_\chi((h_\beta[\varphi^{-1}(s_\alpha(q))])^{-1})\psi_\beta(\overline{\varphi}^{-1}(q)), \quad (3.62)$$

$$(\rho^\chi(F)\psi)_\alpha(q) = e^{-iF(q)/\hbar}\psi_\alpha(q), \quad (3.63)$$

where h_β is as defined after (3.15): it is the element of H satisfying $s_\beta(\overline{\varphi}^{-1}(q))h_\beta = \varphi^{-1}(s_\alpha(q))$. We here assume that $q \in U_\alpha$ and $\overline{\varphi}^{-1}(q) \in U_\beta$. This representation $\rho^\chi(\mathcal{G})$ implements the automorphisms α^χ, for, defining $\rho^\chi((\varphi, F)) = \rho^\chi(F)\rho^\chi(\varphi)$ it may be shown that for all $A \in \mathcal{A}$ and any $(\varphi, F) \in \mathcal{G}$

$$\pi^\chi(\alpha_{(\varphi,F)}[A]) = \rho^\chi((\varphi, F))\pi^\chi(A)\rho^\chi((\varphi, F))^*. \quad (3.64)$$

An illuminating special case is $P = G$ and $\varphi(x) = yx$ for some fixed $y \in G$. In that case ρ^χ is just the representation of G induced by $\pi_\chi(H)$, the argument of π_χ in (3.62) reducing to the Wigner cocycle $s_\alpha(q)^{-1}ys_\beta(y^{-1}q)$. The pair (3.62), (3.63) is then a system of imprimitivity based on $Q = G/H$ (see refs. [29,13], or [23] for a discussion of induced representations in connection with quantization, fibre bundles and transformation group C^*-algebras).

Following physicists' conventions, we define a representation dU of a Lie algebra $\mathbf{g}$ derived from a unitary representation U of its Lie group G by

$$dU(T) = i\hbar(d/dt)U((e^{tT}))|_{t=0}, \quad (3.65)$$

which is defined and essentially self-adjoint on a suitable domain (for example, the Gårding domain if G is locally compact, which our group $\mathcal{G}$ is not, see below). For $\xi \in \mathcal{X}^H$ [the Lie algebra of $\mathrm{Aut}(P)$] and $F \in C_c^\infty(Q)$ [identifying the abelian group $C_c^\infty(Q)$ with its Lie algebra] we find

$$(d\rho^\chi(\xi)\psi)_\alpha(q) = \left(-i\hbar\left[\nabla_{\overline{\xi}}^{A_\chi} + \tfrac{1}{2}\mathrm{div}\overline{\xi}(q)\right] + d\pi_\chi(\langle A, \xi\rangle(s_\alpha(q)))\right)\psi_\alpha(q), \quad (3.66)$$

$$(d\rho^\chi(F)\psi)_\alpha(q) = F(q)\psi_\alpha(q). \quad (3.67)$$

Here $\overline{\xi} = (\mathrm{pr}_{P\to Q})_*(\xi)$, and div is defined, e.g., in ref. [1]. The expression (3.66) follows from (3.62) by direct computation plus a rearrangement bringing in the connection A, but a more efficient derivation is to start from

$$(d\tilde{\rho}^\chi(\xi)\psi)(x) = -i\hbar((\xi + \tfrac{1}{2}\mathrm{div}\xi(x))\psi)(x). \quad (3.68)$$

One then decomposes $\xi = \mathrm{hor}(\xi) + v(\langle A, \xi\rangle)$, where $\mathrm{hor}(\xi)$ is the horizontal part of ξ, and $v(T)$ is the vertical vector field on P canonically associated to any $T \in \mathbf{h}$, and uses the fact that

$$\mathrm{hor}(\xi)\psi = \nabla_{\overline{\xi}}^{A_\chi}\psi.$$

The equivariance condition (3.12) allows one to convert $v(T)\psi$ into $[(\mathrm{i}\hbar)^{-1}$ times$]$ $\mathrm{d}\pi_\chi(T)\psi$.

The operators (3.66), (3.67) are defined and essentially self-adjoint on the domain Γ_c^χ of compactly supported cross-sections of the vector bundle E^χ; this is the same domain of definition of the Hamiltonian (3.44). In fact, the proof of the essential self-adjointness of $\mathrm{d}\rho^\chi(\xi)$ follows exactly the same lines as the proof of theorem 4. The analogue of the essential self-adjointness of the Laplacian on $C_c^\infty(P) \subset L^2(P)$ in that proof is here the fact that ξ is essentially self-adjoint on $C_c^\infty(P)$ if its flow is complete [1, 5.4.2], a condition automatically satisfied in our case as we assumed that ξ has compact support. In the special case $P = G$ (eq. 3.66) was obtained in ref. [24], where the reader will also find a physical interpretation of the third term in its right-hand side [viz., in case that ξ is a symmetry of the dynamics then this term is the contribution of the external gauge field to the conserved operator $\mathrm{d}\rho^\chi(\xi)$].

Finally, if $\overline{\xi} \in \mathcal{X}(Q)$ then the quantum momentum operator associated with this vector field is $\mathrm{d}\rho^\chi(\lambda(\overline{\xi}))$, which is given by the right-hand side of (3.66) without the third term. If the connection A is flat this yields a representation of Lie(Diff(Q)).

4. Groupoids and algebroids

4.1. QUANTUM THEORY

The passage from the classical algebra of observables $\mathcal{A}_0 = C_0((T^*P)/H)$ to its quantum analogue $\mathcal{A} = \mathcal{K}(L^2(P))^H$ has a transparent description in terms of Lie groupoids and Lie algebroids (general references on these structures are refs. [35,28,11]). The precise definition of a Lie groupoid is given in ref. [28, III.1]; the ingredients are a total space Ω, a base space B (both assumed to be manifolds, all the maps occurring in what follows being smooth), and two projections s (source) and t (target) of Ω onto B, as well as an immersion $i : B \to \Omega$. An element $g \in \Omega$ can be interpreted as an arrow between the objects $s(g)$ and $t(g)$, and such arrows can be composed if their endpoints match [that is, $g_1 g_2$ is defined iff $s(g_1) = t(g_2)$], in which case we write $(g_1, g_2) \in \Omega^2$; moreover, each g has an inverse $I(g) \equiv g^{-1}$ as an arrow in the opposite direction. The subspace $G^0 \equiv i(B) \subset \Omega$ is called the unit space of Ω; it consists of all elements of the type gg^{-1}, which are obviously closed arrows. The multiplication (by composition) of three arrows is associative whenever it is defined. An arrow is not in general uniquely determined by its endpoints, and the unit space does not generally exhaust the set of closed arrows. A transparent way to look at (or, indeed, define) a groupoid is as a (small) category whose arrows all have inverses. The arrows are the elements of Ω and the objects are the points of B.

We will consider the so-called gauge groupoid [28,11] $\Omega_H = P \times_H P$ related to the principal bundle (P, Q, H), which is a quotient by H of the coarse groupoid $\Omega = P \times P$. The latter has total space $\Omega = P \times P$, base space $B = P$, and source and target projections $s((x,y)) = y$, $t((x,y)) = x$. The inclusion is $i(x) = (x,x)$, the inverse is $(x,y)^{-1} = (y,x)$, and the composition rule is $(x_1,y)(y,x_2) = (x_1,x_2)$. Its quotient $P \times_H P$ is obtained by imposing the equivalence relation $(x_1,x_2) \sim (y_1,y_2)$ iff $(x_1,x_2) = (y_1 h, y_2 h)$ for some $h \in H$; we denote the equivalence class of (x,y) by $[x,y]$. Accordingly, $B = Q = P/H$, the inverse is $[x,y]^{-1} = [y,x]$, the projections are $s([x,y]) = \mathrm{pr}_{P\to Q}(y)$, $t([x,y]) = \mathrm{pr}_{P\to Q}(x)$, the inclusion is $i(q) = [s(q), s(q)]$ (for an arbitrary section s of P), and multiplication $[x_1,y_1] \cdot [y_2,x_2]$ is defined iff $y_2 = y_1 h$ for some $h \in H$, and the composition equals $[x_1 h, x_2]$ in that case.

A representation of Ω [35] is a covariant functor π mapping Ω into the category whose objects are Hilbert spaces and whose arrows are unitary maps. Hence $\pi(b) \equiv \mathcal{H}_b$ is a Hilbert space for each $b \in B$, and $\pi(g)$ is a unitary operator from $\mathcal{H}_{s(g)}$ to $\mathcal{H}_{t(g)}$, with $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ whenever $g_1 g_2$ is defined. If one in addition has a measure ν on B, and the $\mathcal{H}_b$ form a measurable field, then one may form the direct integral $\mathcal{H}_\pi = \int_B^\oplus \mathrm{d}\nu(b)\, \mathcal{H}_b$. If all the $\mathcal{H}_b$ are identical to a fixed $\mathcal{H}_B$ then simply $\mathcal{H}_\pi = L^2(B,\nu) \otimes \mathcal{H}_B$. This Hilbert space $\mathcal{H}_\pi$ is useful, because it carries a representation of the groupoid C^*-algebra $C^*(\Omega)$, and for a large class of groupoids (including those of the type we use) there is a bijective correspondence between representations of $C^*(\Omega)$ on $\mathcal{H}_\pi$ and representations π of Ω itself (provided that π satisfies an obvious measurability condition) [35]. For example, a faithful representation of the coarse groupoid $P \times P$ is obtained by putting $\mathcal{H}_p = \mathbb{C}$ for all $p \in P$, and defining $\pi((x,y))$ to be the unitary "operator" mapping $1 \in \mathcal{H}_y$ to $1 \in \mathcal{H}_x$. One may choose ν to be the Riemannian measure μ on P, and $\mathcal{H}_\pi = L^2(P)$.

A faithful representation π of $P \times_H P$ follows from taking $\mathcal{H}_q = L^2(H)$, and $\pi([x,y]) = \pi_L(h_\alpha(x)(h_\beta(y))^{-1})$, regarded as a map from $\mathcal{H}_{\overline{y}}$ to $\mathcal{H}_{\overline{x}}$, with $\overline{x} = \mathrm{pr}_{P\to Q}(x)$, etc.; π_L is the left-regular representation of H [cf. (3.6)], and h_α etc. is defined after (3.15). Alternatively, a single measurable section s may be used, cf. subsection 3.1. We now equip Q with the measure ν [see (3.2)], and $\mathcal{H}_\pi$ is $L^2(Q) \otimes L^2(H)$, which is naturally isomorphic to $L^2(P)$, cf. (3.3).

To define $C^*(\Omega)$ one chooses a measure μ_b on each fibre $\Omega^b \equiv t^{-1}(b)$ of Ω, and this system of measures should be invariant in the following sense. Each $g \in \Omega$ defines a map $L_g : \Omega^{s(g)} \to \Omega^{t(g)}$ by $L_g(g') = gg'$, which is bijective as g has an inverse. All such maps are to be measure preserving. For $\Omega = P \times_H P$ the fibres F_q are all isomorphic to P, and one may take μ_b to be the Riemannian measure μ. Given such an invariant collection of measures (called a Haar system on Ω), one defines a product on $C_c(\Omega)$ by

$$(f_1 * f_2)(g) = \int_{\Omega^{s(g)}} \mathrm{d}\mu_{s(g)}(g')\, f_1(gg') f_2((g')^{-1}), \tag{4.1}$$

and an involution by

$$f^*(g) = \overline{f(g^{-1})}. \tag{4.2}$$

Note that the measure ν on B combines with the Haar system $\{\mu_b\}_{b\in B}$ to a measure μ_Ω on Ω in the obvious way [35, I.3], putting $\mu_\Omega(f) = \int_B \mathrm{d}\nu(b) \times \int_{\Omega^b} \mathrm{d}\mu_b(g)\, f(g)$ for $f \in C_c(\Omega)$. For $\Omega = P \times P$ with $\nu = \mu$ on $B = P$ and $\mu_b = \mu$ on $\Omega^b = P$ this yields $\mu_\Omega = \mu \otimes \mu$, whereas on $\Omega = P \times_H P$ we obtain the quotient of this product measure by H; in fact, realizing $C_c(P \times_H P)$ as $C_c(P \times P)^H$ [the functions f satisfying $f(xh, yh) = f(x,y)$ for all $x, y \in P$, $h \in H$], we can put $\mu_{P\times_H P}(f) = \mu \otimes \mu(f)$, as H is compact.

The choice of the measures μ_b makes the passage from Ω to $C^*(\Omega)$ non-canonical, but in many cases (including those of interest here, cf. the remark at the end of this subsection) one can show that C^*-algebras defined using different measure systems are isomorphic [35]. (The choice of these measures is reminiscent of a corresponding situation is geometric quantization theory, where one has to choose a measure on the space of leaves of a polarization on a symplectic manifold.) In case that Ω is a differentiable groupoid [28,11] (as in all our examples), one may entirely forego the choice of a Haar system on Ω, by using half-densities rather than functions on Ω (this was pointed out by Connes for the special case $\Omega = P \times P$ [9], and in general by Weinstein [46]). The details are as follows.

For any $g \in \Omega$, the manifold Ω has a submanifold $T_g \equiv t^{-1}(t(g))$, which is the fibre of the target projection, and evidently passes through g. Similarly, one has submanifolds $S_g \equiv s^{-1}(s(g))$, which are the fibres of the source projection. Let $\Lambda^{1/2}(\ker t_*)_g$ ($\Lambda^{1/2}(\ker s_*)_g$) be the fibre at g of the bundle of half-densities over T_g (S_g). This leads to a line bundle Λ over Ω, whose fibre at g is $\Lambda_g \equiv \Lambda^{1/2}(\ker s_*)_g \otimes \Lambda^{1/2}(\ker t_*)_g$. With $L_g : t^{-1}(s(g)) \to t^{-1}(t(g))$ defined as above, and $R_g : s^{-1}(t(g)) \to s^{-1}(s(g))$ given by $R_g(g') = g'g$, we have natural isomorphisms for any composable pair $(g, g') \in \Omega^2$,

$$\begin{aligned} (R_{g'})_* &: (\ker s_*)_g \to (\ker s_*)_{gg'}, \\ (L_g)_* &: (\ker t_*)_{g'} \to (\ker t_*)_{gg'}, \\ (L_{(g')^{-1}g^{-1}})_* &: (\ker t_*)_g \to (\ker t_*)_{(g')^{-1}}, \\ I_* &: (\ker t_*)_{g'} \to (\ker s_*)_{(g')^{-1}}. \end{aligned} \tag{4.3}$$

Note, that the push-forward $(L_g)_*$ of a map like L_g, which is only partially defined, is not, in general, defined on the entire tangent space at a point, but it is precisely defined on the the domain given [and similarly for the other maps occurring in (4.3)]. Pulling back, this leads to an isomorphism

$$\lambda^*_{g,g'} : \Lambda^{1/2}(\ker s_*)_{gg'} \otimes \Lambda^{1/2}(\ker t_*)_{gg'} \otimes \Lambda^{1/2}(\ker s_*)_{(g')^{-1}} \otimes \Lambda^{1/2}(\ker t_*)_{(g')^{-1}}$$
$$\to \Lambda^{1/2}(\ker s_*)_g \otimes \Lambda^1(\ker t_*)_{g'} \otimes \Lambda^{1/2}(\ker t_*)_g, \tag{4.4}$$

where $\Lambda^1(\ker t_*) \simeq \Lambda^{1/2}(\ker t_*) \otimes \Lambda^{1/2}(\ker t_*)$ stands for the bundle of densities over a fibre of the target projection. If $f_i \in \Gamma_c(\Lambda)$ are compactly supported continuous sections of the bundle Λ, we may define a product by

$$(f_1 * f_2)(g) = \int_{\Omega^{s(g)}} \lambda^*_{g,g'}(f_1(gg') \otimes f_2((g')^{-1})), \tag{4.5}$$

cf. (4.1). The involution is now given by

$$f^*(g) = \tau \circ I^*(\overline{f(g^{-1})}), \tag{4.6}$$

where τ is the operation which interchanges the order of two factors in a tensor product. In case that $\Omega = G$ is a Lie group, the manifolds T_g and S_g coincide with G, so that Λ is simply the bundle of densities over G, and we see that the convolution algebra may be defined using densities rather than functions, with no need for a Haar measure [46].

We now return to scalar functions on Ω, which are a bit easier to use in practice. A general procedure to construct a norm on $C_c(\Omega)$ is given in ref. [35], but it is easier to define it in a faithful representation. Any representation $\pi(\Omega)$ corresponds to a representation (called π as well) of $C_c(\Omega)$ on $\mathcal{H}_\pi$: for $\psi, \phi \in \mathcal{H}_\pi$ one has

$$(\pi(f)\psi, \phi) = \int_\Omega \mathrm{d}\mu_\Omega(g)\, f(g)\, (\pi(g)\psi(s(g)), \phi(t(g)))_{\mathcal{H}_{t(g)}}. \tag{4.7}$$

[This formula is correct if μ_Ω is invariant in the sense of ref. [35, I.3], which is the case in our examples; otherwise a Radon–Nikodym derivative enters (4.7).] If $\pi(C_c(\Omega))$ is faithful, one may define the norm by $\|f\| = \|\pi(f)\|$, and close $C_c(\Omega)$ in this norm to obtain the groupoid C^*-algebra $C^*(\Omega)$ [note that faithfulness of $\pi(\Omega)$ does not imply the faithfulness of $\pi(C_c(\Omega))$].

Applying this procedure to the coarse groupoid $P \times P$, one easily finds that $\pi(P\times P)$ on $\mathcal{H}_\pi = L^2(P)$ (constructed above) is faithful, and that $C^*(P\times P) = \mathcal{K}(L^2(P))$. Similarly, using $\pi(P \times_H P)$ above as well as the analysis in subsection 3.1 one finds that $C^*(P \times_H P) = \mathcal{A} = \mathcal{K}(L^2(P))^H$. One may ask which representation $\pi^\chi(\Omega)$ the irreducible representation $\pi^\chi(\mathcal{A})$ (cf. subsection 3.2) corresponds to; the answer is that $\pi^\chi(q) = \mathcal{H}_\chi$ for all $q \in Q$, with $\pi^\chi([x,y]) = \pi_\chi(h_\alpha(x)(h_\beta(y))^{-1})$.

In the special case $P = G$ (a Lie group) one has $P \times_H P \simeq G \times (G/H)$ by the isomorphism $P \times_H P \ni [x,y] \leftrightarrow (xy^{-1}, xq_0) \in G \times (G/H)$, where q_0 is the coset $\{H\}$ (hence $(x,q) \leftrightarrow [s(q), x^{-1}s(q)]$). The groupoid $G \times (G/H)$ has $B = G/H$, immersion map $i(q) = (e,q)$, projections $s((x,q)) = x^{-1}q$

and $t((x,q)) = q$, inversion $(x,q)^{-1} = (x^{-1}, x^{-1}q)$, and multiplication $(x,q)(y,x^{-1}q) = (xy,q)$. This is a special case of a semi-direct product groupoid [35]. The fibres Ω^q are all isomorphic to G, which one equips with a Haar measure. The groupoid C^*-algebra $C^*(G, G/H)$ is then the transformation group C^*-algebra corresponding to the canonical action of G on G/H, or, equivalently, the crossed product of G with $C_0(G/H)$ for the natural automorphism group of $C_0(G/H)$ defined by G. Using various mathematical settings and motivations, it was independently proposed in refs. [13,2,30,23] that $C^*(G, G/H)$ be the algebra of observables of a particle moving on the homogeneous configuration space $Q = G/H$, and that the Hamiltonian be a quadratic Casimir operator of G. This was a natural algebraic reformulation of Mackey's quantization on G/H using systems of imprimitivity [29]. It was realized in refs. [24,25] that this construction quantizes the "universal" phase space of a particle in a Yang–Mills field [16,42,32] in the special case $P = G$. We now see that this quantization is a special case of the groupoid construction presented above. The further specialization $H = G$ (so that $C^*(G,e) = C^*(G)$ and $(T^*G)/G \simeq \mathbf{g}^*$ with the Lie–Kirillov Poisson structure) then reproduces Rieffel's deformation of $C_0(\mathbf{g}^*)$ into $C^*(G)$ [37] (at least when G is compact, but we believe that our construction generalizes to the non-compact case).

4.2. CLASSICAL THEORY

In the discussion of the quantum theory above we have exhibited the algebra of observables $\mathcal{A}$ as a derived object, namely the C^*-algebra of a certain Lie groupoid. This is not always possible, because there are many C^*-algebras which are not related to groupoids. A similar special situation prevails classically, for the classical algebra of observables $\mathcal{A}_0 = C_0((T^*P)/H)$ belongs to a limited class of Poisson algebras, which are derived from an underlying Lie algebroid [44, 4.2.2.c], [11]. This leads to a neat parallel with the quantum theory, which we wish to point out.

A Lie algebroid A is a vector bundle over a base manifold B, so that the space of smooth sections $\Gamma(A)$ is equipped with a Lie bracket [,]. Apart from $\mathrm{pr}_{A\to B}$ there is another projection $\mathrm{pr}_{A\to TB}$, the "anchor" of A, which defines a Lie algebra homomorphism of $\Gamma(A)$ to $\Gamma(TB)$ (see ref. [28, III.2] for details). The simplest example is $A = TP$ with the obvious projection onto $B = P$, and the identity projection to $TB = TP$, the Lie bracket on $\Gamma(TP)$ being the usual commutator of vector fields on P. One can pass from Lie groupoids to Lie algebroids [28, III.3], and $A = TP$ is the Lie algebroid of the coarse Lie groupoid $P \times P$. Quotienting by H, one may define the Lie algebroid $(TP)/H$ [28, A2] over Q. The space of sections $\Gamma((TP)/H)$ is isomorphic to the space $\mathcal{X}^H(P)$ of H-invariant vector fields on P [cf. (3.48)], which isomorphism defines the Lie bracket on $\Gamma((TP)/H)$. The projection onto TQ is the obvious one, and $(TP)/H$ is the Lie algebroid of the gauge groupoid $P \times_H P$ [28,11].

It was pointed out by Weinstein [44] that the category of Lie algebroids is isomorphic to the category of vector bundles with additive Poisson structures. Hence we can associate a Poisson algebra $C^\infty(A^*)$ to a Lie algebroid A, which consists of the smooth functions on the dual bundle A^*. The Poisson bracket is initially constructed on elements of $\Gamma(A)$ (identified with the subspace of $C^\infty(A^*)$ of functions linear on the fibres) simply by the Lie bracket. This bracket may subsequently be extended to a dense subset of $C^\infty(A^*)$ (in a suitable topology) by imposing the Leibniz rule (derivation property) on products of linear functions. On $A = TP$ this procedure is equivalent to imposing the identity $\{\sigma(\xi_1), \sigma(\xi_2)\} = \sigma([\xi_1, \xi_2])$, where $\sigma(\xi) \in C^\infty(T^*P)$ is the symbol of the vector field ξ on P [cf. text following (3.55)]. A more intrinsic construction of this Poisson structure is given in ref. [11, II.4.2]: in our case of a gauge groupoid this amounts to the following. $(T^*P)/H$ is isomorphic to the conormal bundle of the inclusion of the base space Q into $P \times_H P$, and is accordingly itself the unit space of the symplectic groupoid $T^*(P \times_H P)$ (whose groupoid structure is spelled out in ref. [11, II.4], and which is a symplectic manifold as a cotangent bundle in the usual way). The Poisson structure on $(T^*P)/H$ is then given by requiring the source projection of $T^*(P \times_H P)$ (with the canonical Poisson structure) onto its unit space to be a Poisson morphism. The dual of $(T^*P)/H$ is $(TP)/H$ (the normal bundle of the embedding), which is precisely the Lie algebroid of $P \times_H P$. Hence the ensuing Poisson algebra is just $C^\infty((T^*P)/H)$ with its canonical Poisson structure. If we restrict the Poisson bracket to its subalgebra $C_c^\infty((T^*P)/H)$ and close in the uniform topology, we obtain the classical algebra of observables $\mathcal{A}_0$ of a particle on Q moving in a Yang–Mills field [44, 4.2.2.c].

To complete the analogy with the quantum situation, we need to develop a suitable representation theory of Lie algebroids, and show that there is a bijective correspondence between representations of a given Lie algebroid, and realizations ("classical representations") of its associated Poisson algebra. The definition of a representation of a Lie algebroid given in ref. [28, III.2.9] is too general for this purpose, and we leave the construction of a suitable theory to the future.

4.3. QUANTIZATION AS DEFORMATION AND GLUEING OF GROUPOIDS

We see that the step of quantizing $\mathcal{A}_0$ by $\mathcal{A}$ is equivalent to the passage from the Lie algebroid $(TP)/H$ to the Lie groupoid $P \times_H P$, that is, to the integration of a given Lie algebroid, and to subsequently finding the irreducible representations of the Poisson algebra and the C^*-algebra associated to the algebroid and the groupoid, respectively (note that it is not always possible to integrate a given

Lie algebroid [28]). From the general point of view of groupoids, the theory in this paper only covers the transitive differentiable case. Hence in ref. [26] we conjecture that there exists a deformation of the Poisson algebra associated to any integrable Lie algebroid into the C^*-algebra of a corresponding differentiable groupoid (this generalization was independently suggested by the referee of the present paper). Further speculation then leads to the hope of associating a Poisson algebra to a dense subalgebra of any C^*-algebra of the type studied in non-commutative geometry, as some form of an "infinitesimal invariant", similar to the Lie algebra attached to a Lie group.

The relation between this program and the theory of quantizing arbitrary Poisson algebras using symplectic groupoids, which is developed in refs. [21,46,47] is not immediately clear. Given a Poisson manifold M, one looks for a symplectic groupoid $S(M)$, of which M is the base space. For $M = (T^*P)/H$ one finds $S(M) = T^*(P \times_H P)$, so that, in particular, $S(\mathbf{g}^*) = T^*G$. The next step is to apply the usual geometric quantization algorithm to $S(M)$, in order to obtain a Hilbert space $\mathcal{H}(M)$. The crucial move is now to exploit the groupoid structure of $S(M)$ to define an associative multiplication and an involution on $\mathcal{H}(M)$, thus producing a *-algebra $\tilde{\mathcal{A}}(M)$. Finally, one needs to introduce a polarization to reduce the size of $\tilde{\mathcal{A}}(M)$, ending up with a quantum algebra of observables $\mathcal{A}(M)$ (cf. refs. [46,47] for details). Even the simple case $M = \mathbf{g}^*$ has yet to be worked out in detail, but one may expect that $\mathcal{A}(\mathbf{g}^*)$ is some dense subalgebra of $C^*(G)$, and, more generally, that the groupoid C^*-algebras used in this paper eventually emerge. Of course, the symplectic groupoid program is intended to quantize even more general Poisson algebras than those obtained from Lie algebroids.

Another way of looking at the deformation of $\mathcal{A}_0$ into $\mathcal{A}$ is to regard $(TP)/H$ not as an algebroid but as a groupoid, with base space Q and the vector bundle projection of $(TP)/H$ onto Q. This move is equivalent to ignoring the Poisson structure of $\mathcal{A}_0$, and just using its C^*-structure; indeed, $\mathcal{A}_0 = C^*((TP)/H)$. The elements of $(TP)/H$ are all closed arrows, and the groupoid operation is addition inside a fibre. One may then regard $P \times_H P$ as a deformation of $(TP)/H$ in the following sense: the space $\Omega_{\text{Planck}} \equiv \{(TP)/H\} \times \{0\} \cup \{P \times_H P\} \times (0,1]$, which glues $(TP)/H$ to $P \times_H P$, may be given the structure of a manifold (with boundary). Two equivalent ways of accomplishing this are given in a more general context in refs. [20, III.1] and [45, 4.1]; the essence is to shrink an open set in TP by a factor $\hbar \in (0,1]$ in such a way that the shrunken set lies in a tubular neighbourhood of $P \subset TP$, so that it is diffeomorphic to an open set in $P \times P$. The procedure is entirely analogous to the construction of the quantization maps $Q_\hbar$ in subsection 2.1 above.

The manifold Ω_{Planck} is itself a Lie groupoid, with base space $Q \times [0,1]$, and all arrows leaving each $\hbar \in [0,1]$ inert, so that the groupoid structures in $(TP)/H$ and in each copy of $P \times_H P$ do not interfere with each other, and are as given before. This groupoid is the quotient of the so-called tangent groupoid of P by H; the construction of the tangent groupoid of a manifold is due to Connes (unpublished), and is described in ref. [20, III.19]. It is a special case of the normal groupoid $G \equiv G(G_1, G_2)$ corresponding to a groupoid homomorphism $\varphi : G_1 \to G_2$, which is also an immersion of manifolds. To obtain Ω_{Planck} as a normal groupoid, one has to perform the construction described in ref. [20, ch. III] with $G_1 = Q$, regarded as a trivial groupoid (i.e., $B = \Omega$), and $G_2 = P \times_H P$, with the immersion φ given by the inclusion i of $B = Q$ into $\Omega = P \times_H P$ specified in the preceding subsection. As already mentioned, the normal bundle of this immersion is $(TP)/H$, and this fact easily leads to the identification of G with Ω_{Planck}. Its C^*-algebra $C^*(\Omega_{\text{Planck}})$ (improperly) contains the classical algebra of observables $\mathcal{A}_0$ as well as a copy of its quantum counterpart $\mathcal{A}$ for each value of $\hbar \in (0, 1]$, and appears to be a fascinating tool for studying the classical limit of quantum mechanics (cf., in a different context, ref. [4]).

In conclusion, we would like to thank R. Nest and G. Skandalis, without whose comments on ref. [25] the preceding subsection could not have been written; in particular, the identification of the continuous field of C^*-algebras used in ref. [25] (which is essentially $C^*(\Omega_{\text{Planck}})$ in the particular case that P is a Lie group) as being a special case of a C^*-algebra of a normal groupoid of the type constructed in ref. [20] (in connection with KK-theory) was explained to the author by Skandalis.

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