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Rieffel induction as generalized quantum Marsden–Weinstein reduction

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Abstract

A new approach to the quantization of constrained or otherwise reduced classical mechanical systems is proposed. On the classical side, the generalized symplectic reduction procedure of Mikami and Weinstein, as further extended by Xu in connection with symplectic equivalence bimodules and Morita equivalence of Poisson manifolds, is rewritten so as to avoid the use of symplectic groupoids, whose quantum analogue is unknown. A theorem on symplectic reduction in stages is given. This allows one to discern that the 'quantization' of the generalized moment map consists of an operator-valued inner product on a (pre-)Hilbert space (that is, a structure similar to a Hilbert C^*-module). Hence Rieffel's far-reaching operator-algebraic generalization of the notion of an induced representation is seen to be the exact quantum counterpart of the classical idea of symplectic reduction, with imprimitivity bimodules and strong Morita equivalence of C^*-algebras falling in the right place.

Various examples involving groups as well as groupoids are given, and known difficulties with both Dirac and BRST quantization are seen to be absent in our approach.

Keywords: Quantization; Rieffel induction; Marsden–Weinstein reduction;
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1. Introduction

Marsden–Weinstein reduction [34,36] (alternatively known as Hamiltonian or symplectic reduction) plays a basic role in classical mechanics [1,17,29,31], as well as in pure mathematics. The starting point is a connected symplectic manifold S equipped

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with a right-action of a Lie group H (assumed connected for simplicity), which action we assume to be strongly Hamiltonian for the moment. In that case one has an equivariant moment map $J : S \to (\mathbf{h}^*)^-$, where $\mathbf{h}$ is the Lie algebra of H, and $\mathbf{h}^*$ its dual (that is, J intertwines the co-adjoint action on $\mathbf{h}^*$ and the action on S); here and in what follows the notation P^- stands for a Poisson manifold P, equipped with minus its original Poisson structure. The essential point is that the pull-back $J^* : C^\infty((\mathbf{h}^*)^-) \to C^\infty(S)$ is a morphism of Poisson algebras (relative to the Lie-Poisson structure on $\mathbf{h}^*$ [17,53,29,37]). The choice of a co-adjoint orbit $\mathcal{O} \subset \mathbf{h}^*$ then leads to the reduced space $S^\mathcal{O} = J^{-1}(\mathcal{O})/H$, which inherits a symplectic structure from S. If A is a Poisson subalgebra of $C^\infty(S)$ whose elements are H-invariant (equivalently, they Poisson-commute [2] with $J^*C^\infty(\mathbf{h}^*)$), we obtain a Poisson morphism $\pi^\mathcal{O} : A \to C^\infty(S^\mathcal{O})$. This may be thought of as a 'classical representation' of A on $S^\mathcal{O}$, which is induced from the Poisson morphism (or, once again, 'classical representation') $\pi_\mathcal{O} \equiv i_\mathcal{O}^* : C^\infty(\mathbf{h}^*) \to C^\infty(\mathcal{O})$, where $i_\mathcal{O}$ is the inclusion map of $\mathcal{O}$ into $\mathbf{h}^*$. For example, one is usually given an H-invariant Hamiltonian $H_0 \in C^\infty(S)$, whose representative $\pi^\mathcal{O}(H_0) \in C^\infty(S^\mathcal{O})$ is the reduced Hamiltonian on the reduced phase space. More generally, any symplectic realization $\rho : X \to \mathbf{h}^*$ of the Poisson manifold $\mathbf{h}^*$ (that is, X is symplectic, and ρ is a Poisson map [8]) leads to a classical representation $\pi^X(A)$ on a certain symplectic space S^X, to be detailed below.

The connection with constrained mechanical systems à la Dirac [9] is as follows: one chooses a basis $\{T_i\}_{i=1,\dots,d_H}$ of $\mathbf{h}$, and defines $f_i \in C^\infty(S)$ by $f_i = J^*\tilde{T}_i$; here $\tilde{T}_i \in C^\infty(\mathbf{h}^*)$ is defined by $\tilde{T}_i(\theta) = \langle\theta, T_i\rangle$ for $\theta \in \mathbf{h}^*$. Then pick an arbitrary point $\mu \in \mathcal{O}$, put $\mu_i = \tilde{T}_i(\mu) = \langle\mu, T_i\rangle$, and take the constraints on S to be $\Phi_i = f_i - \mu_i = 0$, $i = 1, \dots, d_H$. These constraints will in general be mixed (that is, of first as well as second class), and one obtains the reduced phase space by quotienting the constraint surface by the foliation defined by the Hamiltonian flows of the first-class constraints [9]. This reduced phase space of Dirac is then symplectomorphic to the Marsden-Weinstein reduced space $S^\mathcal{O}$ mentioned above. The geometric procedure is superior to the 'physicist's' approach just sketched, in that one need not pick a basis of $\mathbf{h}$, an arbitrary point μ, or explicitly classify the functions of constraint Φ_i into first and second class ones.

One would naturally like to generalize this construction to the situation where one has a symplectic space S, a Poisson manifold P [29], and two Poisson maps $J : S \to P^-$ and $\rho : X \to P$, where X is symplectic. This should lead to an 'induced classical representation' π^X of any Poisson subalgebra $A \subset C^\infty(S)$ which Poisson-commutes with $J^*C^\infty(P)$, on some symplectic space S^X. This generalization was partly achieved by Mikami and Weinstein [37] in the special case where P is integrable (in the sense that it is the base space of units of a symplectic groupoid [8,37]), and X is a symplectic leaf of P (with ρ the injection map), and later Xu [56] gave a more general construction avoiding the latter restriction. A slight rewriting of this, finally lifting also the condition

[2] This condition is not strictly necessary, but facilitates the presentation, and is satisfied in generic examples.

that P be integrable (and thereby avoiding constructions involving symplectic groupoids, whose quantization we do not understand), is given in Section 2 below.

From the physical point of view of constrained systems, what this generalization achieves is that now reduced phase spaces obtained from arbitrary Poisson algebras of constraints may be described in a very satisfactory geometric fashion. The physicist's approach would be to choose a basis $\tilde{T}_i$ which generates $C^\infty(P)$ in some appropriate way, and pick a point $\mu \in X$, where X is a symplectic leaf in P. With the f_i (which satisfy the Poisson algebra of P^-) and μ_i defined as above, one then easily finds that the reduced phase space defined by the constraints Φ_i is symplectomorphic to S^X. However, if X in the preceding paragraph is not taken as a symplectic leaf in P, one obtains a symplectic space S^X (and an associated representation of the Poisson algebra A) which cannot be obtained as a reduced phase space in the traditional sense, in any obvious way.

Thus one has a very general method of constructing new symplectic spaces and Poisson morphisms from old ones at one's disposal, which ought to be quantized in some way. While a direct quantization of the reduced symplectic manifolds and concordant induced representations of Poisson algebras may be possible in certain examples, a systematic approach intending to mimic the classical reduction/induction procedure in some quantum fashion ought to start from a quantization of the 'unconstrained' system. Hence we assume we have found two commuting operator algebras $\mathcal{A}$ and $\mathcal{B}$ acting on a complex vector space L from the left and from the right, respectively, as well as a (left) $\mathcal{B}$-module $\mathcal{H}_\chi$; from these data we try to construct an 'induced' representation $\pi^\chi(\mathcal{A})$ on a Hilbert space $\mathcal{H}^\chi$. We denote these data by $\mathcal{A} \to L \leftarrow \mathcal{B}$ and $\mathcal{B} \xrightarrow{\pi_\chi} \mathcal{H}_\chi$. The operator algebras $\mathcal{A}$ and $\mathcal{B}$ are to be seen as (dense subalgebras of) the quantizations of the Poisson algebras A and $C^\infty(P)$, respectively (barring boundedness considerations). Indeed, in all our applications L will be a dense subspace of a Hilbert space $\mathcal{H}$, which in physicists' language plays the role of the state space of the unconstrained system.

For our purpose it does not matter very much what one exactly means by a quantization; the induction procedure may be applied to any data $L, \mathcal{A}, \mathcal{B}, \pi_\chi, \mathcal{H}_\chi$. Ideally, these data correspond to a strict deformation quantization [45] (as redefined in [23]) of the symplectic data, as in some of our examples in Section 4.

We now take our cue from three directions (details to be given later on in this paper):

(i) Take G a locally compact group and $H \subset G$ a closed subgroup. Let $\pi_\chi(H)$ be a unitary representation of H on $\mathcal{H}_\chi$; we may then form the induced representation $\pi^\chi(G)$ on a specific Hilbert space $\mathcal{H}^\chi$, as defined in the Mackey theory [30,50]. Rieffel [43] relates this to the data $C^*(G) \to L^2(G) \leftarrow C^*(H)$ and $C^*(H) \xrightarrow{\pi_\chi} \mathcal{H}_\chi$, where $C^*(G)$ is the group algebra of G [40], which acts on $L^2(G)$ in the left-regular representation, with $C^*(H)$ acting in the right-regular anti-representation (restricted to H).

In case that G and H are Lie groups, it is argued in [20,17,54] that the classical analogue of the Mackey induction procedure is to take $S = T^*G$, $A = C^\infty(\mathbf{g}^*) \simeq C^\infty(T^*G)^G$ (the Poisson algebra of smooth functions on T^*G which commute with the pull-back to T^*G of the right-action of G on itself), and $P = \mathbf{h}^*$; the moment map $J : T^*G \to (\mathbf{h}^*)^-$ comes from the pull-back of the right-action of H on G. A co-adjoint orbit $\mathcal{O} \subset \mathbf{h}^*$ is then analogous to an irreducible unitary representation π_χ, and the Marsden–Weinstein reduced space $J^{-1}(\mathcal{O})/H \equiv (T^*G)^{\mathcal{O}}$, carrying the induced action $\pi^{\mathcal{O}}$ of $C^\infty(\mathbf{g}^*)$, is the symplectic counterpart of the Hilbert space $\mathcal{H}^\chi$ carrying the induced representation π^χ of G (or $C^*(G)$). In our previous notation, this corresponds to the Poisson map $\rho : X = \mathcal{O} \to P = \mathbf{h}^*$ being given by the inclusion map; the general case where X is a Hamiltonian H-space and ρ a moment map is considered in [59] as a classical analogue of induction. To complete the parallel, we recall Rieffel's discovery that the group algebra $C^*(G)$ is a deformation quantization of the Poisson algebra $C^\infty(\mathbf{g}^*)$ [46], which in specific cases is even strict in the sense of [45].

(ii) Let (P, Q, H, pr) be a principal fibre bundle with projection $pr : \mathsf{P} \to Q$ and a compact Lie group H acting on the total space P from the right. The symplectic leaves of the Poisson manifold $(T^*\mathsf{P})/H$ are in one-to-one correspondence with the co-adjoint orbits $\mathcal{O}$ in $\mathbf{h}^*$, and, as originally discovered by Sternberg, a leaf $S^{\mathcal{O}}$ plays the role of the phase space of a particle moving on Q with internal charge $\mathcal{O}$, which couples to a Yang–Mills field with gauge group H [17,52,31]. This is evidently described through Marsden–Weinstein reduction by taking $S = T^*\mathsf{P}$ and $A = C^\infty(T^*\mathsf{P})^H \simeq C^\infty((T^*\mathsf{P})/H)$.

The quantization of this setting was constructed in [23] using some Lie groupoid and algebroid technology. The results were obtained by applying a generalized induction procedure to the quantum data $\mathcal{K}(L^2(\mathsf{P}))^H \to L^2(\mathsf{P}) \leftarrow C^*(H)$ and $C^*(H) \xrightarrow{\pi_\chi} \mathcal{H}_\chi$, thus obtaining irreducible representations of the C^*-algebra $\mathcal{K}(L^2(\mathsf{P}))^H$ of H-invariant compact operators on $L^2(\mathsf{P})$ on spaces $\mathcal{H}^\chi$ analogous to the ones used in the Mackey theory. Indeed, the special case $\mathsf{P} = G$ reproduces the constructions in the previous item. One obtains a sharpened version of a strict deformation quantization even in the general case. A simple special case of this construction appeared in [22].

(iii) It was recognized by Xu [57] that a complete full dual pair $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2^-$ of Poisson manifolds [53] (with connected and simply connected fibers) defines an equivalence bimodule of the corresponding Poisson algebras. Hence there is a bijective correspondence between the categories of symplectic realizations of P_1 and P_2, respectively [56]; from an algebraic point of view this means that the Poisson algebras $C^\infty(P_1)$ and $C^\infty(P_2)$ have equivalent classical representation theories. This equivalence is implemented through a generalized symplectic reduction procedure (see Subsection 2.1 below).

There is an obvious formal analogy between these classical equivalence bimodules, and the imprimitivity bimodules $\mathcal{A} \to L \leftarrow \mathcal{B}$ of operator algebras defined by Rieffel [43]. Under certain conditions, the main one being the existence of compatible rigging maps $\langle\ ,\ \rangle_{\mathcal{B}} : L \times L \to \mathcal{B}$ and ${}_{\mathcal{A}}\langle\ ,\ \rangle : L \times L \to \mathcal{A}$, the representation theories of $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, and the isomorphism is implemented by a generalized induction procedure given in [43] called *Rieffel induction* [13].

Indeed, the term 'Morita equivalence of Poisson manifolds' [57] was clearly inspired by the terminology of (strong) 'Morita equivalence of operator algebras' [43,44]. For example, under certain conditions (cf. Subsection 4.3) the Poisson manifold $(T^*\mathsf{P})/H$ is Morita equivalent to $\mathbf{h}^*$ through the equivalence bimodule $T^*\mathsf{P}$, and on the quantum side we find strong Morita equivalence of the C^*-algebras $\mathcal{K}(L^2(\mathsf{P}))^H$ and $C^*(H)$ through the imprimitivity bimodule $L^2(\mathsf{P})$.

In the light of the above evidence, and more to be given in the main body of the paper, it is not very daring to suggest that the quantum analogue of the generalized symplectic reduction procedure sketched earlier, is provided by Rieffel induction. We will now briefly describe this construction (cf. [43,13] for an exhaustive treatment, or Section 3.1 below for a brief summary of rigging maps and Rieffel induction).

In symplectic geometry, a Poisson map $J : S \to P^-$ plays a double role: it relates S to P, *and* provides a Poisson morphism $J^* : C^\infty(P^-) \to C^\infty(S)$. In operator theory, a (right) action of a *-algebra $\mathcal{B}$ on a Hilbert space $\mathcal{H}$ amounts to a *-anti-homomorphism $\pi^- : \mathcal{B} \to \mathcal{L}(\mathcal{H})$, which is the 'quantum' analogue of J^*. It is now tempting to define a quantum version of J as some map between the projective space $\mathbb{P}\mathcal{H}$ and the state space of $\mathcal{B}$, and construct an induction procedure on this basis, but this appears to lead nowhere unless $\mathcal{B} = C^*(H)$ for compact H.

The correct 'quantization' of the moment map is a so-called rigging map (alternatively called an operator-valued inner product). The starting point is a right $\mathcal{B}$-module L (generally without an inner product). A rigging map $\langle \cdot , \cdot \rangle_{\mathcal{B}}$ is then defined on $L \otimes L$ (algebraic tensor product), and takes values in $\mathcal{B}$. The main property it has to satisfy is $\langle \psi, \varphi B \rangle_{\mathcal{B}} = \langle \psi, \varphi \rangle_{\mathcal{B}} B$ for all $\psi, \varphi \in L$ and all $B \in \mathcal{B}$.

We look at the special case that $\mathcal{B}$ is a suitable dense subalgebra of $C^*(H)$, which leaves a dense subspace L of a Hilbert space $\mathcal{H}$ invariant. Here π^- above is defined through a unitary representation π of H on $\mathcal{H}$. Then the rigging map is defined by $\langle \psi, \varphi \rangle_{C^*(H)} : h \to (\pi(h)\varphi, \psi)$, where $(\cdot , \cdot)$ is the inner product on $\mathcal{H}$. This defines a function $f_{\psi,\varphi}$ on H, and we choose L in such a way that $f_{\psi,\varphi} \in \mathcal{B}$ for all ψ, $\varphi \in L$. If H is compact we can simply take $L = \mathcal{H}$. In the non-compact case, for e.g., $\mathcal{H} = L^2(G)$ with H acting on the right, one may take $\mathcal{B} = C_c(H)$ and $L = C_c(G)$.

Now suppose that another *-algebra $\mathcal{A}$ acts on L, and the condition $\langle A\psi, \varphi \rangle_{\mathcal{B}} = \langle \psi, A^* \varphi \rangle_{\mathcal{B}}$ is satisfied for all $A \in \mathcal{A}$. Under favourable circumstances a representation $\pi_\chi(\mathcal{B})$ on a Hilbert space $\mathcal{H}_\chi$ may then be induced to a representation $\pi^\chi(\mathcal{A})$ on a certain Hilbert space $\mathcal{H}^\chi$. The crucial step in this induction procedure is to start with $L \otimes \mathcal{H}_\chi$, equipped with a sesquilinear form $(\cdot , \cdot)_0$ defined by $(\psi \otimes v, \varphi \otimes w)_0 = (\pi_\chi(\langle \varphi, \psi \rangle_{\mathcal{B}})v, w)_{\mathcal{H}_\chi}$; this is positive semi-definite if the rigging map is positive, and in that case one may quotient $L \otimes \mathcal{H}_\chi$ by the null space of $(\cdot , \cdot)_0$, and complete it into a Hilbert space $\mathcal{H}^\chi$, which inherits the left-action of $\mathcal{A}$ from L.

Forming $L \otimes \mathcal{H}_\chi$ with the given sesquilinear form is the quantum counterpart of taking $J^{-1}(\mu) \subset S$ (for some $\mu \in \mathcal{O}$) with its pre-symplectic form borrowed from S in the Marsden–Weinstein reduction process, and quotienting the null space of $(\cdot , \cdot)_0$ away is obviously the quantum analogue of quotienting $J^{-1}(\mu)$ by its characteristic (null) foliation, thus obtaining a symplectic space symplectomorphic to $J^{-1}(\mathcal{O})/H$. These formal analogies will be more clearly visible in the description of the generalized symplectic reduction procedure defined in Subsection 2.1 below.

In the remainder of this paper we will describe the above ideas in detail, and provide a fair number of examples illustrating why it seems a good idea to quantize the generalized symplectic reduction/induction technique by the Rieffel induction process. For example, we give classical Poisson versions of both the imprimitivity theorem and the theorem on induction in stages [43,13].

We close this Introduction with some loose remarks. These may be skipped without pain.

A point $\mu \in \mathbf{h}^*$ may fail to be a regular value of the moment map [1,31], which leads to some difficulties in the reduction procedure [3,27]. In brief, the reduced space is no longer a symplectic manifold, but a singular space (in the sense of algebraic geometry), which admits a symplectic stratification. This situation is of prime importance for applications in physics, for the physical phase space of Yang–Mills theories [2] as well as of general relativity [14,4] has such singularities at points where the solutions of the field equations possess symmetry. Some consequences of these singularities for the corresponding quantum theories were investigated for gravity by V. Moncrief (unpublished), and for finite-dimensional models in [12].

Such singularities may, of course, equally well occur in the more general symplectic reduction procedure discussed above (and in the next subsection). Our quantization with Rieffel induction suggests a general approach to the quantum situation. The analogue of a singular value μ (or rather its co-adjoint orbit) is a representation $\pi_\chi(H)$ for which $\pi_\chi(\langle\psi,\psi\rangle_B)$ fails to be a positive operator on $\mathcal{H}_\chi$ for certain $\psi \in L$. (Unlike Rieffel [43], we do not require the rigging map to be positive, in the sense that $\langle\psi,\psi\rangle_B \geq 0$ for all $\psi \in L$, as this property is not satisfied in many interesting examples, and the induction procedure can be carried out with a weaker assumption [13], cf. Subsection 3.1.) Therefore, the rigged inner product $(\cdot,\cdot)_0$ is not positive semi-definite on $L \otimes \mathcal{H}_\chi$, and the induced space $\mathcal{H}^\chi$ can only be defined as a Hilbert space if the vectors in $L \otimes \mathcal{H}_\chi$ of negative rigged norm are first removed. Evidently, this problem will not arise if the rigging map is positive. In general, the quantum reduction procedure is better behaved than its classical counterpart, cf. Proposition 12. For we will prove that this non-positivity can only occur if H is not amenable, whereas the classical reduced space may already be singular when H is compact.

The quantization procedure based on Rieffel induction will have to be compared with the fashionable BRST quantization scheme (cf. e.g. [21,19]). For the moment, we just wish to point out that serious difficulties of principle with the latter were spelled out in [11,25], and that on the practical side "at present the computation of BRST-cohomology is an extremely difficult problem" [19]. Moreover, the Rieffel induction process mimics the symplectic procedure more closely than any BRST treatment we are aware of (including the bosonic BRST theory in [10]), and appears to be simpler both conceptually and computationally. On the other hand, the proper domain of the rigging map has to be found case by case, and for C^*-algebras not defined by groupoids even the rigging map itself is not given *a priori*. Finally, all our examples are defined for finite-dimensional Poisson manifolds, and one has yet to see how quantization through Rieffel induction will perform in generic infinite-dimensional situations (where the BRST technique has enjoyed certain success [21]); see [26] for the treatment of free abelian gauge theories.

Finally, the reader may wonder why the generalized moment map $J : S \to P^-$, which is merely a morphism between a symplectic and a Poisson manifold, is quantized by a structure which eventually involves bimodules. However, the generalized moment map is equivalent to the structure $C^\infty(P)' \xrightarrow{i} C^\infty(S) \xleftarrow{J^*} C^\infty(P)$, where $C^\infty(P)'$ is the subset of $C^\infty(S)$ consisting of functions whose Poisson bracket with any J^*f, $f \in C^\infty(P)$, vanishes (and i is the inclusion).

This already looks like the bimodule $\mathcal{L}_\mathcal{B}(L) \to L \leftarrow \mathcal{B}$, where $\mathcal{L}_\mathcal{B}(L)$ is the algebra of linear operators A on L which are compatible with the rigging map [43, Section 2] (our notation is different from this reference). Compatible here means that for such A an A^* must exist such that $\langle A\psi, \varphi\rangle_\mathcal{B} = \langle \psi, A^*\varphi\rangle_\mathcal{B}$ for all $\psi, \varphi \in L$, and that $\langle A\psi, A\psi\rangle_\mathcal{B} \le k_A^2\langle \psi, \psi\rangle_\mathcal{B}$ for all $\psi \in L$, and some constant k_A. (If one quotients $\mathcal{L}_\mathcal{B}(L)$ by the operators A for which $k_A = 0$, one obtains a pre C^*-algebra, whose completion is the so-called C^*-algebra of the given rigged space.)

Moreover, if one insists that the generalized moment map be a morphism, one could further jusitify our quanization proposal by pointing out the close connection between morphisms of operator algebras and so-called correspondences [6]. A correspondence between two von Neumann algebras $\mathfrak{M}$, $\mathfrak{N}$ is a left-$\mathfrak{N}$ right-$\mathfrak{M}$ bimodule where the module is a Hilbert space, and the actions commute. For properly infinite von Neumann algebras a morphism is equivalent to a correspondence. For arbitrary C^*-algebras one may pass from a morphism to an associated correspondence, which however depends on the choice of a faithful state on the source of the morphism.

2. Symplectic induction

2.1. Generalized Marsden–Weinstein reduction

As pointed out in the Introduction, Marsden–Weinstein reduction is a special case of a more general symplectic induction technique. The general procedure described below is essentially due to Xu [56, Prop. 2.1]. By rewriting his construction omitting any reference to symplectic groupoids, we are able to avoid the restriction in [56] to integrable Poisson manifolds (in the sense of [8,37]), while also the parallel with the Rieffel induction technique in the quantum case is more transparent in this way.

Definition 1. Let S and S_ρ be connected symplectic manifolds, P a Poisson manifold, P^- the same manifold as P but equipped with minus its Poisson bracket, and let $J : S \to P^-$ and $\rho : S_\rho \to P$ be Poisson maps. Then $S *_P S_\rho \subset S \times S_\rho$ is defined by

$$S *_P S_\rho = \{(x,y) \in S \times S_\rho \mid J(x) = \rho(y)\}. \tag{2.1}$$

Each $f \in C^\infty(P)$ defines a vector field $\hat{X}_f$ on $S \times S_\rho$ by

$$\hat{X}_f g = \{J^* f - \rho^* f, g\}, \tag{2.2}$$

where the Poisson bracket is the product one on $S \times S_\rho$.

Theorem 2. *$S *_P S_\rho$ is co-isotropically immersed in $S \times S_\rho$. The collection of vector fields $\{\hat{X}_f \mid f \in C^\infty(P)\}$ defines a (generally singular) foliation $\mathcal{F}$, of $S *_P S_\rho$, whose leaf space $S^\rho = S *_P S_\rho/\mathcal{F}$ coincides with the quotient of $S *_P S_\rho$ by its characteristic (null) foliation.*

Proof. The dimension counting argument in the proof comes from [20] and [56].

We write M for $S *_P S_\rho$ for simplicity. Let $X \in T_x S$ and $Y \in T_y S_\rho$; then $X + Y \in T_{(x,y)}M$ iff $J_* X = \rho_* Y$. The dimension of $T_{(x,y)}M$ at any point $(x,y) \in M$ equals $\dim S + \dim S_\rho - (\operatorname{rank} J_*)(x)$, so that the dimension of $T_{(x,y)}M^\perp$ (the symplectic orthoplement of TM in $T(S \times S_\rho)$ at (x,y)) is $(\operatorname{rank} J_*)(x)$. Let $\mathcal{F}_{(x,y)}$ denote the linear span of the collection of vector fields $\hat{X}_f$ taken at (x,y), where f runs through $C^\infty(P)$. Then $\dim \mathcal{F}_{(x,y)} = (\operatorname{rank} J_*)(x)$. We next show that $\mathcal{F}_{(x,y)} \subseteq T_{(x,y)}M^\perp$, so that in fact $\mathcal{F}_{(x,y)} = T_{(x,y)}M^\perp$. Namely, let $X+Y \in T_{(x,y)}M$, as above; then with $\omega = \omega_S + \omega_{S_\rho}$ the symplectic form on $S \times S_\rho$, one has

$$\langle \omega | X + Y, \hat{X}_f \rangle_{(x,y)} = \langle d(J^* f - \rho^* f) | X + Y \rangle_{(x,y)} = 0.$$

Moreover, $\mathcal{F}_{(x,y)} \subset T_{(x,y)}M$ by a similar calculation: if X_g is the Hamiltonian vector field of g, then by Lemma 1.2 in [53] $J_* X_{J^* f} = -X_f$, where X_f is defined w.r.t. the Poisson bracket on P (rather than P^-, hence the sign) and $\rho_* X_{\rho^* f} = X_f$. Thus $\hat{X}_f = X_{J^* f} - X_{\rho^* f} \in TM$. Therefore, M is co-isotropically immersed in $S \times S_\rho$. Furthermore, $[\hat{X}_f, \hat{X}_g] = -\hat{X}_{\{f,g\}}$ (Poisson bracket on P), so that by the Stefan–Sussmann theorems (cf. [29, Thm. 3.9, 3.10, App. 3]) the distribution $\mathcal{F}$ defines a (singular) foliation, called $\mathcal{F}$ as well. □

Under the additional assumption that S^ρ is a manifold, we have accordingly found a new symplectic space S^ρ, which carries a 'classical representation' of certain Poisson subalgebras of $C^\infty(S)$, as follows. We borrow some notation from operator algebras: if B is a subset of $C^\infty(S)$, then B' denotes its Poisson commutant, i.e. the set of all functions in $C^\infty(S)$ whose Poisson bracket with each element of B vanishes. Also, $[x,y] \in S^\rho \equiv (S \times_P S_\rho)/\mathcal{F}$ stands for the equivalence class of a point $(x,y) \in S \times_P S_\rho$ under the foliation $\mathcal{F}$.

Proposition 3. *Let $A \subseteq (J^* C^\infty(P))'$ be a Poisson subalgebra of $C^\infty(S)$. Then the map $\pi^\rho : A \to C^\infty(S^\rho)$, defined by*

$$\pi^\rho(f)([x,y]) = f(x) \tag{2.3}$$

is well-defined, and is a Poisson morphism.

This is obvious. We call π^ρ the classical representation of A induced by the map $\rho : S_\rho \to P$. Suppose that we have a Poisson manifold P_2, and a Poisson map $J_2 : S \to P_2$, such that $J_2^* C^\infty(P_2) \subset C^\infty(S)$ Poisson-commutes with $J^* C^\infty(P)$; then the proposition is equivalent to the production of a Poisson map $J^\rho : S^\rho \to P_2$ defined by $J^\rho([x, y]) = J_2(x)$.

It may be worth spelling out how Marsden–Weinstein reduction emerges as a special case. We take a connected Lie group H acting on S from the right in a strongly Hamiltonian fashion [17,29] (see Subsection 3.3 below for the general case), so that there is an equivariant moment map $J : S \to (\mathbf{h}^*)^-$ (hence $P = \mathbf{h}^*$). (If a left-action with moment map $J^- : S \to \mathbf{h}^*$ is given, simply put $J = -J^-$.) We then take $S_\rho = \mathcal{O}$, a co-adjoint orbit in $\mathbf{h}^*$, and $\rho : \mathcal{O} \to \mathbf{h}^*$ to be the inclusion map, which is evidently a Poisson map if $\mathbf{h}^*$ and $\mathcal{O}$ are endowed with the Lie–Poisson structure. Then clearly $(S \times_{\mathbf{h}^*} \mathcal{O})/\mathcal{F} \simeq J^{-1}(\mathcal{O})/H$, but note that the null foliation of $J^{-1}(\mathcal{O}) \subset S$ does not coincide with the H-foliation (whereas these foliations do coincide on $S \times_{\mathbf{h}^*} \mathcal{O} \subset S \times \mathcal{O}$). Hence the above diffeomorphism is an efficient way of providing $J^{-1}(\mathcal{O})/H$ with its correct symplectic structure (which is usually obtained from the diffeomorphism $J^{-1}(\mathcal{O})/H \simeq J^{-1}(\mu)/H_\mu$, where $\mu \in \mathcal{O}$ is arbitrary, and H_μ is its stabilizer).

Another special case is the Mikami–Weinstein reduction procedure [37]. They assume that P, which is Γ_0 in their notation, is the unit space of a symplectic groupoid, and their reduced space $J^{-1}(u)/\Gamma_u$ emerges from Theorem 2 by taking S_ρ to be the symplectic leaf in Γ_0 containing u, and J_1 the inclusion map in Γ_0.

To close this section, we note that Theorem 2 can be generalized to arbitrary Poisson manifolds (rather than merely symplectic ones).

Generalized Definition 1. Let S and S_ρ be connected Poisson manifolds..., *and continuing as in Definition 1.*

Generalized Theorem 2. *The collection of vector fields $\{\hat{X}_f \mid f \in C^\infty(P)\}$ defines a (generally singular) foliation $\mathcal{F}$, of $S *_P S_\rho$, whose leaf space $S^\rho = S *_P S_\rho/\mathcal{F}$ carries a reduced Poisson structure in the sense of Marsden–Ratiu [32].*

Proof. Making contact with the notation in this reference, we define $M \subset \mathsf{P} \equiv S \times S_\rho$ as $M = S *_P S_\rho$, and the subbundle E as $\{\hat{X}_f \mid f \in C^\infty(P)\}$. Also, the map $B : T^*\mathsf{P} \to T\mathsf{P}$ is defined by the Poisson structure, i.e., $B(df) = X_f$. Finally, $E_x^0 \equiv \{\alpha \in T_x\mathsf{P} \mid \langle \alpha, X \rangle = 0 \, \forall X \in E_x\}$.

Firstly, exactly as in the proof of Theorem 2, it follows that $E \subset TM$. Secondly, $B(E_x^0) \subset T_x M$. To show this local property, take $\alpha = dg_1 + dg_2$, with $g_i = \pi^* h_i$, the natural projections $\pi_1 : S \times S_\rho \to S$ and $\pi_2 : S \times S_\rho \to S_\rho$, $h_1 \in C^\infty(S)$, $h_2 \in C^\infty(S_\rho)$. Then the property that $\alpha \in B(E_x^0)$ is equivalent to the equality $\{J^* f, h_1\} = \{\rho^* f, h_2\} \, \forall f \in C^\infty(P)$. Hence $J_* X_{g_1} = \rho_* X_{g_2}$, which proves the claim.

Generalized Theorem 2 now immediately follows from the Poisson Reduction Theorem in Section 2 of Marsden–Ratiu [32]. □

2.2. Symplectic imprimitivity theorem

The well-known imprimitivity theorem of Mackey [50] has a far-reaching generalization due to Rieffel [43,13,44]. This generalization establishes a bijective correspondence between the respective representation theories of two operator algebras satisfying a certain equivalence relation, known as strong Morita equivalence. A satisfactory 'classical' (that is, Poisson-algebraic) analogue of this equivalence relation and some of its ramifications was recently given by Xu [57]. For the convenience of the reader, we repeat Xu's definition of Morita equivalent Poisson manifolds [Def. 2.1 in [57]; the concept of dual pair, which is central to the definition, is due independently to Weinstein [53] and Libermann [28] (who used the term 'symplectically complete foliation')].

Definition 4. A classical equivalence bimodule of a pair of Poisson manifolds (P_1, P_2) consists of a symplectic manifold S and a pair of Poisson morphisms $J_1 : S \longrightarrow P_1^-$ and $J_2 : S \longrightarrow P_2$, such that $P_2 \xleftarrow{J_2} S \xrightarrow{J_1} P_1^-$ is a complete full dual pair with connected and simply connected fibers. This means that $J_1^* C^\infty(P_1^-)$ and $J_2^* C^\infty(P_2)$ are each other's Poisson commutant in $C^\infty(S)$, that the leaf spaces of the foliations defined by the fibers of J_1 and J_2 are manifolds in the quotient topology, and that J_1 and J_2 are surjective, as well as complete as Poisson maps.

Poisson manifolds P_1 and P_2 are called Morita equivalent if there exists a classical equivalence bimodule in the above sense.

A Poisson map $J : S \longrightarrow P$ is said to be complete if the Hamiltonian vector field $X_{J^* f}$ on S is complete (that is, has a flow defined for all times) if X_f on P is, for all $f \in C^\infty(P)$. This condition is the classical analogue of the requirement that a representation of a $*$-algebra $\mathcal{A}$ on a Hilbert space be $*$-preserving, that is, it is a self-adjointness condition. Namely, in the latter case a self-adjoint element A of $\mathcal{A}$ is mapped into a self-adjoint operator $\pi(A)$ which defines a complete flow on the (projective) Hilbert space carrying the representation, viz. the (projection of) the unitary group generated by $\pi(A)$. The condition that the fibers $J_i^{-1}(x)$ be simply connected for each $x \in P_i$ $(i = 1, 2)$ cannot be omitted, as will become clear from the proof of the next theorem.

We recall that a symplectic realization of a Poisson manifold P consists of a symplectic manifold S and a Poisson map $\rho : S \longrightarrow P$ [53,8]. This leads to a Poisson morphism $\rho^* : C^\infty(P) \longrightarrow C^\infty(S)$, which is the classical analogue of a representation of a $*$-algebra on a Hilbert space [24]. There is an obvious equivalence relation between symplectic realizations, that is, $\rho_1 : S_1 \longrightarrow P$ and $\rho_2 : S_2 \longrightarrow P$ are equivalent if there exists a symplectic diffeomorphism $T : S_1 \longrightarrow S_2$ such that $\rho_1 = \rho_2 \circ T$. In what follows, a realization will mean a symplectic one.

The following theorem was first proved by Xu, who assumed that the Poisson manifolds in question are integrable. However, combining Thms. 4.18 and 5.2 in [55] one infers that this is always the case in the given situation (note that the idea of a symplectic affinoid space introduced in [55] generalizes a full dual pair, so that it would be interesting to find a corresponding generalization of Morita equivalent algebras in operator algebra theory). Xu's proof (which is spread out over Section 4 of [56] and Section 3 of [57]) follows the lines of first showing that integrable Morita equivalent Poisson manifolds have Morita equivalent symplectic groupoids, which in turn have equivalent categories of complete symplectic realizations. Our proof below avoids the use of symplectic groupoids, which may be a loss from a geometric point of view, but has the advantage of being similar in spirit to the proof of the imprimitivity theorem for operator algebras [43,13].

Theorem 5. *Let P_1 and P_2 be Morita equivalent Poisson manifolds. Then there is a bijective correspondence between their respective complete symplectic realizations.*

Proof. Given the equivalence bimodule $P_2 \xleftarrow{J_2} S \xrightarrow{J_1} P_1^-$, there is a second equivalence bimodule $P_1 \xleftarrow{J_1} S^- \xrightarrow{J_2} P_2^-$. Given a realization $\rho : S_\rho \to P_1$, one uses the former equivalence bimodule to obtain a realization $J^\rho : S^\rho \to P_2$, where $S^\rho = S *_{P_1} S_\rho/\mathcal{F}$ is the symplectic space constructed in Theorem 2, and J^ρ is given by $J^\rho([x,y]_1) = J_2(x)$. Here $[(x,y)]_1$ is the equivalence class of $(x,y) \in S \times S_\rho$ under the foliation $\mathcal{F}$, and the map J^ρ is well-defined, because by the theory of full dual pairs [53] the foliation $\mathcal{F}$ restricted to S coincides with the foliation by the fibers of J_2. Also, the same fact combined with the assumption that the quotient of S by the J_2-foliation is a manifold implies that S^ρ is a manifold.

We now relabel S^ρ as S_σ, and J^ρ by σ, and use the second equivalence bimodule to find the corresponding induced realization $J^\sigma : S^\sigma \to P_1$. Below we construct a symplectic diffeomorphism $V : S^\sigma \to S_\rho$, which satisfies $J^\sigma = \rho \circ V$. Since all constructions evidently preserve completeness, this establishes the theorem.

Consider $(S *_{P_1} S_\rho) *_{P_2} S^- \subset S \times S_\rho \times S^-$, that is, the space of triples (x,θ,y) satisfying $J_1(x) = \rho(\theta)$ and $J_2(x) = J_2(y)$. The space S^σ is obtained from this by a double foliation: the first one $\mathcal{F}_1$ on $S \times S_\rho$ generated by the Hamiltonian vector fields defined by the functions $J_1^* f - \rho^* f$, $f \in C^\infty(P_1)$, and the second one $\mathcal{F}_2$ on $S \times S^-$ generated by the Hamiltonian vector fields defined by the functions $\sigma^* g - J_2^* g$, $g \in C^\infty(P_2)$. Let a triple (x,θ,y) as above be given. As above, we denote equivalence classes defined by the first foliation by $[\cdot,\cdot]_1$, and those defined by the second one by $[\cdot,\cdot]_2$.

We now once again exploit the crucial fact from full dual pairs that the foliation of S generated by the the Hamiltonian vector fields defined by the functions $J_1^* f$, $f \in C^\infty(P_1)$, coincides with the foliation by the fibers of J_2. Hence since $J_2(x) = J_2(y)$, if x and y are sufficiently nearby we can find $f \in C^\infty(P_1)$ for which the flow φ_t of $X_{J_1^* f}$ satisfies $\varphi_0(x) = x$, $\varphi_1(x) = y$; in general, the analogous curve connecting x and y is only piecewise smooth, each smooth segment being a trajectory of a vectorfield $X_{J_1^* f_i}$, for some $f_i \in C^\infty(P_1)$, cf. Proposition 1.3 in [53]. Let $\tilde{\varphi}_t$ be the flow of $-X_{\rho^* f}$ on S_ρ; by our assumption that ρ be complete, this flow exists for all times, and we can define $\tilde{\theta} = \tilde{\varphi}_1(\theta)$. By standard foliation theory, $\tilde{\theta}$ only depends on θ and the homotopy class in the fiber $J_2^{-1} \circ J_2(x)$ of the path $\{\varphi_t\}_{t\in[0,1]}$ connecting x and y. But this fiber is assumed to be simply connected, so that $\tilde{\theta}$ is uniquely determined by (x,θ,y).

We now define $V : S^{\sigma} \longrightarrow S_{\rho}$ by $V([[x,\theta]_1, y]_2) = \tilde{\theta}$. This is well-defined, and is a symplectomorphism: given a triple (x,θ,y) we have seen that we may choose a representative $(y,\tilde{\theta},y)$ in the class $([x,\theta]_1, y)$ defined by $\mathcal{F}_1$, and we subsequently note that the foliation $\mathcal{F}_2$ coincides with the foliation by the fibers of J_1. Since $J_1(y) = \rho(\tilde{\theta})$ is determined by $\tilde{\theta}$, it follows that V is a bijection. It is a symplectomorphism by Theorem 2.

Finally,

$$J^{\sigma}([[x,\theta]_1, y]_2) = J_1(y) = \rho(\tilde{\theta}) = \rho \circ V([[y,\tilde{\theta}]_1, y]_2) = \rho \circ V([[x,\theta]_1, y]_2),$$

so that $J^{\sigma} = \rho \circ V$, as announced. □

Note that we could have weakened the definition of a classical equivalence bimodule by omitting the manifold condition on the foliations of S by J_1 and J_2 in Definition 4. In that case we would have obtained a bijection bewteen the set of realizations S_{ρ} of P_1 for which S^{ρ} is a manifold, and the analogous set defined for P_2.

Note, that [20] and [53] already mention the fact that (in modern parlance) the symplectic leaves of Morita equivalent Poisson manifolds are in bijective correspondence. This is obviously a special case of Theorem 5, for the injection of a symplectic leaf into its Poisson manifold is of course a special instance of a symplectic realization (in fact, such realizations play a preferred role, in that they are irreducible in the sense defined in [24]).

2.3. Symplectic induction in stages

After the imprimitivity theorem, the second most important and characteristic result in Mackey's theory of induced group representations is the theorem on induction in stages [30]. This was generalized by Rieffel to his setting of induced representations of C^*-algebras [43,13]. The symplectic counterpart is very easy, and the proof of the following theorem consists of simple bookkeeping, which we leave to the reader.

Let $J : S \longrightarrow P^-$ and $\rho : S_{\rho} \longrightarrow P$ be Poisson maps, with S symplectic, and let $\pi^{\rho} : A \longrightarrow C^{\infty}(S^{\rho})$ be the corresponding induced representation of an appropriate Poisson algebra $A \subset J^*C^{\infty}(P^-)' \subset C^{\infty}(S)$ (cf. Proposition 3). Now assume that the realization ρ is itself induced, in the sense that there are a Poisson manifold $\tilde{P}$ and symplectic manifolds $\tilde{S}$ and S_{σ}, as well as Poisson maps $\tilde{J} : \tilde{S} \longrightarrow \tilde{P}^-$, $\hat{J} : \tilde{S} \longrightarrow P$, and $\sigma : S_{\sigma} \longrightarrow \tilde{P}$, such that $S_{\rho} \simeq \tilde{S}^{\sigma}$ and $\rho \simeq \hat{J}^{\sigma}$ (where $\hat{J}^{\sigma} : \tilde{S}^{\sigma} \longrightarrow P$ is constructed as in Theorem 2 and the text following Proposition 3, with $S, P, P_2, J, J_2, S_{\rho}, \rho$ replaced by $\tilde{S}, \tilde{P}, P, \tilde{J}, \hat{J}, S_{\sigma}, \sigma$, respectively).

Now form the symplectic manifold $S' = (S *_P \tilde{S})/\mathcal{F}$ as in Definition 1 (assuming that the leaf space of the foliation is indeed a manifold) and Theorem 2 (that is, $S *_P \tilde{S}$ consists of those pairs $(x,y) \in S \times \tilde{S}$ for which $J(x) = \hat{J}(y)$, and the foliation $\mathcal{F}$ is generated by $X_{J^*f} - X^*_{\hat{J}}f$, $f \in C^{\infty}(P)$).

Theorem 6. *With the above notation:*

i) There is a well-defined Poisson map $J' : S' \longrightarrow \tilde{P}^-$ *defined by* $J'([x,y]) = \tilde{J}(y)$, *and a Poisson morphism* $\mu : A \to C^\infty(S')$ *given by* $(\mu(f))([x,y]) = f(x)$.

ii) The induced symplectic space $(S')^\sigma = (S' *_{\tilde{P}} S_\sigma)/\tilde{\mathcal{F}}$ *constructed with the maps* J' *and* σ *is symplectomorphic to* S^ρ, *and the corresponding induced representation* $(J')^\sigma \circ \mu$ *of* A *on* $C^\infty((S')^\sigma)$ *is equivalent to* π^ρ *on* $C^\infty(S^\rho)$.

iii) In the special case that one has a Poisson manifold P_2 *and a Poisson map* $J_2 : S \longrightarrow P_2$, *so that* $A = J_2^* C^\infty(P_2)$, *one has thus obtained a symplectic realization* $J^\sigma : (S')^\sigma \longrightarrow P_2$ *which is equivalent to* $J^\rho : S^\rho \longrightarrow P_2$.

It is worth spelling out the special case of Marsden–Weinstein reduction in stages.

Take a connected Lie group G with closed connected subgroup $H \subset G$, and consider the actions $G \longrightarrow T^*G \longleftarrow H$, being the pull-backs of the action of G on itself by left-multiplication, and of the right-action of H on G by right-multiplication. The symplectic form on T^*G is $\omega = d\theta_L$, with θ_L the Liouville form. This leads to two moment maps $\mathbf{g}^* \stackrel{J_L}{\leftarrow} T^*G \stackrel{J_R}{\rightarrow} (\mathbf{h}^*)^-$. Pick a co-adjoint orbit $\mathcal{O} \subset \mathbf{h}^*$, and form the reduced space $(T^*G)^{\mathcal{O}} = J_R^{-1}(\mathcal{O})/H$. This produces a Poisson map $J_L^{\mathcal{O}} : (T^*G)^{\mathcal{O}} \longrightarrow \mathbf{g}^*$, which is just the moment map for the left G-action on $(T^*G)^{\mathcal{O}}$, which is inherited from the left G-action on T^*G.

In the left trivialization of $T^*G \simeq G \times \mathbf{g}^*$ this reads as follows. The Liouville form is $\theta_L(x,p) = p_a\theta^a(x)$ (where $\{\theta_a\}_a$ is a basis of left-invariant one-forms on G), $J_L(x,p) = xp$, $J_R(x,p) = p \restriction \mathbf{h}$, with $x \in G$ and $p \in \mathbf{g}^*$; xp denotes the co-adjoint action of x on p. Hence $(T^*G)^{\mathcal{O}}$ consists of equivalence classes $[x,p]_H$, such that $p \restriction \mathbf{h} \in \mathcal{O}$; the equivalence relation is $(xh, h^{-1}p) \sim (x,p)$ for all $h \in H$. The induced G-action is $y[x,p]_H = [yx,p]_H$, and the moment map is $J_L^{\mathcal{O}}([x,p]_H) = xp$. All this can be found in [33].

Suppose G acts on a symplectic manifold S from the right in strongly Hamiltonian fashion, with associated moment map $J : S \longrightarrow (\mathbf{g}^*)^-$. We then induce from $(T^*G)^{\mathcal{O}}$, obtaining a symplectic space $S^{(T^*G)^{\mathcal{O}}}$, defined as usual: we start with $S *_{\mathbf{g}^*} (T^*G)^{\mathcal{O}} = \{(s,z) \in S \times (T^*G)^{\mathcal{O}} \mid J(s) = J_L^{\mathcal{O}}(z)\}$, and quotient by the characteristic foliation, which in this case coincides with the foliation generated by the G-action ρ given by $\rho_x(s,z) = (sx, x^{-1}z)$. Hence $S^{(T^*G)^{\mathcal{O}}} = (S *_{\mathbf{g}^*} (T^*G)^{\mathcal{O}})/G$.

On the other hand, we may restrict the G-action on S to H, with moment map $J_H : S \longrightarrow \mathbf{h}^*$ simply given by the restriction of J to $\mathbf{h}$. This leads to the reduced space $S^{\mathcal{O}} = J_H^{-1}(\mathcal{O})/H$.

Corollary 7. *With the notations introduced above,* $S^{(T^*G)^{\mathcal{O}}} \simeq S^{\mathcal{O}}$.

Proof. This follows from Theorem 6 above, with $P = \mathbf{g}^*$, $\tilde{S} = T^*G$, $\tilde{P} = \mathbf{h}^*$, $S_\sigma = \mathcal{O}$, and $\sigma = i_{\mathcal{O}}$ (the inclusion map of $\mathcal{O}$ into $\mathbf{h}^*$). To obtain Corollary 7, one only needs to verify that $(S *_{\mathbf{g}^*} T^*G)/G$ is symplectomorphic to S, which is Proposition A4 of [54].

It may be instructive to give a direct proof, too. The induced space $S^{(T^*G)^{\mathcal{O}}}$ consists of equivalence classes ${}_G[s,x,p,\theta]_H$, where the quadruple $(s,x,p,\theta) \in S \times G \times$

$\mathbf{g}^* \times \mathcal{O}$ satisfies $J(s) = xp$ and $p \restriction \mathbf{h} = \theta$. The equivalence relation is $(s, x, p, \theta) \sim (ys, yxh^{-1}, hp, h\theta)$ for all $y \in G$ and $h \in H$. It is then readily verified that $W : S^{(T^*G)^{\mathcal{O}}} \to J_H^{-1}(\mathcal{O})/H$ given by $W({}_G[s, x, p, \theta]_H) = [x^{-1}s]_H$ defines a symplectomorphism. □

This corollary is not as academic as it may appear. As shown in subsections 4.1, 4.2, any co-adjoint orbit of a nilpotent or linear semi-direct product Lie group G is of the form $(T^*G)^{\mathcal{O}}$, so that Marsden–Weinstein reduced spaces with respect to such groups can always be obtained in a substantially simpler fashion by reducing with respect to an appropriate subgroup H. Thus the semidirect product reduction theorem of [33] (also cf. [31]) follows from our theorem on symplectic reduction in stages.

3. Quantization of the symplectic induction procedure

3.1. Rieffel induction

The so-called Rieffel induction process, which we propose as the quantum counterpart of generalized Marsden–Weinstein reduction ("symplectic induction") is discussed in detail in [43,13], so we will just recall the basic definitions and constructions. Let $\mathcal{A}$ and $\mathcal{B}$ be *-algebras which act on a linear space L from the left and from the right, respectively. (In physics, this situation corresponds to having a quantization of the unconstrained system, as well as of the algebra of constraints: then L is a subspace (preferably dense) of a Hilbert space $\mathcal{H}$, which is the state space of the unconstrained system. In that case, L inherits an inner product, which is not essential for the induction process, although its existence may be exploited in particular cases, cf. (3.7) below.) In any case, $\mathcal{B}$ will always, and $\mathcal{A}$ will usually a be pre-C^*-algebra or a C^*-algebra, but it is possible (and necessary for some applications, cf. Subsection 4.5 below) to take $\mathcal{A}$ to be an Op*-algebra of unbounded operators [48], that is, a *-algebra defined on a common dense domain $D \subset \mathcal{H}$; in that case the space $L \subseteq \mathcal{H}$ introduced below will have to lie in D. The key ingredient of the induction process, playing the role of the quantization of the generalized moment map J in symplectic geometry (cf. Def. 1), is a rigging map. This map, denoted by $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ is defined on $L \times L$, takes values in $\mathcal{B}$, and must satisfy the following conditions for all $\psi, \varphi \in L$:

(i) $\langle \lambda\psi, \mu\varphi \rangle_{\mathcal{B}} = \overline{\lambda}\mu \langle \psi, \varphi \rangle_{\mathcal{B}}$ for all $\lambda, \mu \in \mathbb{C}$;
(ii) $\langle \psi, \varphi \rangle_{\mathcal{B}}^* = \langle \varphi, \psi \rangle_{\mathcal{B}}$;
(iii) $\langle \psi, \varphi B \rangle_{\mathcal{B}} = \langle \psi, \varphi \rangle_{\mathcal{B}} B$ for all $B \in \mathcal{B}$;
(iv) $\langle A\psi, \varphi \rangle_{\mathcal{B}} = \langle \psi, A^*\varphi \rangle_{\mathcal{B}}$ for all $A \in \mathcal{A}$.

Thus the rigging map is an operator-valued sesquilinear product; if it is also positive in the sense that $\langle \psi, \psi \rangle_{\mathcal{B}} \geq 0$ for all $\psi \in L$, and if $L = \mathcal{H}$ is a Hilbert space, with $\mathcal{A}$ and $\mathcal{B}$ C^*-algebras, then $\mathcal{H}$ equipped with the rigging map is called a Hilbert C^*-module (for $\mathcal{A}$); see [51] for a 'friendly introduction' to this topic. (In the quantization of constrained systems the main difficulty is to identify L and the rigging map, given $\mathcal{H}$

and the actions of $\mathcal{A}$ and $\mathcal{B}$ on it.)

The aim of the Rieffel induction process is to obtain a representation $\pi^\chi(\mathcal{A})$ on some Hilbert space $\mathcal{H}^\chi$, given a representation $\pi_\chi(\mathcal{B})$ on a Hilbert space $\mathcal{H}_\chi$. This is possible if π_χ is L-positive in the sense that $\pi_\chi(\langle\psi,\psi\rangle_{\mathcal{B}}) \geq 0$ for all $\psi \in L$, as an operator on $\mathcal{H}_\chi$. If so, one obtains π^χ in two steps: firstly, the algebraic tensor product $L \otimes \mathcal{H}_\chi$ is formed, and endowed with a bilinear form $(\cdot,\cdot)_0$, defined by

$$(\psi \otimes v, \varphi \otimes w)_0 = (\pi_\chi(\langle\varphi,\psi\rangle_{\mathcal{B}})v, w)_\chi, \tag{3.1}$$

where $(\cdot,\cdot)_\chi$ is the inner product in $\mathcal{H}_\chi$ (taken linear in the first entry, unlike the rigging map; we follow the conventions of [43,13]). This form is positive semi-definite if π_χ is L-positive. Secondly, one forms the quotient of $L \otimes \mathcal{H}_\chi$ by the subspace $\mathcal{H}_0 \subset L \otimes \mathcal{H}_\chi$ of vectors with vanishing $(\cdot,\cdot)_0$ norm, and completes the quotient (equipped with the form inherited from $(\cdot,\cdot)_0$) into a Hilbert space $\mathcal{H}^\chi$. (To make this procedure resemble the formation of the induced symplectic space S^ρ in Theorem 2 a little bit more, one could follow [43] in introducing the intermediate step of forming the tensor product $L \otimes_{\mathcal{B}} \mathcal{H}_\chi$, which is the quotient of $L \otimes \mathcal{H}_\chi$ by vectors of the type $\psi B \otimes v - \psi \otimes \pi_\chi(B)v$, $B \in \mathcal{B}$, but by (iii) and (3.1) above such vectors are automatically of zero norm, so this intermediate step is incorporated in quotienting $L \otimes \mathcal{H}_\chi$ by its null space. It is the latter step which is obviously the quantum analogue of the third step in forming S^ρ, namely the quotienting by the null foliation of the induced symplectic form on $S *_P S_\rho$).

Denoting the image of an elementary vector $\psi \otimes v \in L \otimes \mathcal{H}_\chi$ in the completion $\mathcal{H}^\chi$ of the quotient $L \otimes \mathcal{H}_\chi / \mathcal{H}_0$ by $\psi \tilde{\otimes} v$, the representation $\pi^\chi(\mathcal{A})$ is then defined on the subspace of $\mathcal{H}^\chi$ of finite linear combinations of such images by

$$\pi^\chi(A)\psi\tilde{\otimes}v = (A\psi)\tilde{\otimes}v, \tag{3.2}$$

compare with (2.3). This representation is well-defined on account of (iii) and (iv) above. If $\mathcal{A}$ is a (pre-)C^*-algebra, then the boundedness of $A \in \mathcal{A}$ does not guarantee that $\pi^\chi(A)$ is a bounded operator on $\mathcal{H}^\chi$. On top of that, it is necessary and sufficient that the bound

$$\pi_\chi(\langle A\psi, A\psi\rangle_{\mathcal{B}}) \leq \|A\|^2 \pi_\chi(\langle\psi,\psi\rangle_{\mathcal{B}}) \tag{3.3}$$

holds for all $\psi \in L$. A stronger condition, implying this bound, is that the maps $T_\psi : \mathcal{A} \to \mathcal{B}$ defined by $T_\psi(A) = \langle A\psi,\psi\rangle_{\mathcal{B}}$ are continuous for each $\psi \in L$. This, in turn, is implied if $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, and T_ψ is positive (that is, $\langle A\psi, A\psi\rangle_{\mathcal{B}} \geq 0$ in $\mathcal{B}$ for all $\psi \in L$ and all $A \in \mathcal{A}$), for a positive map between C^*-algebras is automatically continuous. Of course, this would imply that the rigging map itself is positive (in the sense explained after the list of conditions above), so that any representation $\pi_\chi(\mathcal{B})$ may be used to induce from. In any case, if $\pi^\chi(A)$ is bounded for all A in a pre-C^*-algebra A one may extend the induced representation to the completion of $\mathcal{A}$. See [43, Prop. 4.27], [13, XI.7.11, 12], [44] for more information on these points.

The form of $\mathcal{H}^\chi$ as given is useful for the computation of physical correlation functions (that is, expectation values of (time-ordered) products of the type $(\pi^\chi(A_1(t_1)) \cdots$

$\pi^\chi(A_n(t_n))\Omega, \Omega)$, where $A_i \in \mathcal{A}$, and Ω is some physically relevant state in $\mathcal{H}^\chi$), which can be evaluated in $L \otimes \mathcal{H}_\chi$ on any pre-image of Ω, using the inner product $(\cdot, \cdot)_0$; the contributions of intermediate states with zero norm will automatically drop out. Nonetheless, it is useful to have an alternative realization of $\mathcal{H}^\chi$ [18]. Let $\overline{L}$ be the conjugate space of L (which coincides with L as an additive group, but has the conjugate scalar multiplication), and let $\mathcal{L}(\overline{L}, \mathcal{H}_\chi)$ be the space of linear maps of $\overline{L}$ into $\mathcal{H}_\chi$. Then define $U : L \otimes \mathcal{H}_\chi \to \mathcal{L}(\overline{L}, \mathcal{H}_\chi)$ by

$$(U(\psi \otimes v))(\varphi) = \pi_\chi(\langle\varphi, \psi\rangle_{\mathcal{B}})v. \tag{3.4}$$

One can define an inner product $(\cdot, \cdot)^\chi$ on the image $\Im$ of $L \otimes \mathcal{H}_\chi$ in $\mathcal{L}(\overline{L}, \mathcal{H}_\chi)$ under U by

$$(U(\psi \otimes v), U(\varphi \otimes w))^\chi = (\pi_\chi(\langle\varphi, \psi\rangle_{\mathcal{B}})v, w)_\chi; \tag{3.5}$$

this form is positive definite, and the closure of $\Im$ in this inner product yields a Hilbert space $\tilde{\mathcal{H}}^\chi$. Noticing that U exactly annihilates $\mathcal{H}_0 \subset L \otimes \mathcal{H}_\chi$, it follows that U quotients and extends to a well-defined unitary operator $\tilde{U} : \mathcal{H}^\chi \to \tilde{\mathcal{H}}^\chi$.

We continue by recalling Rieffel's generalized imprimitivity theorem [43,13,44], which we will actually use later on, and whose explicit form will make it clear that the symplectic imprimitivity theorem (Theorem 5 in Subsection 2.2) is indeed a 'classical' version of the former. We assume that $\mathcal{A}$ and $\mathcal{B}$ are (pre-)C^*-algebras, acting on L as above, which is equipped with a rigging map $\langle\cdot, \cdot\rangle_{\mathcal{B}}$ satsfying all properties stated earlier. L is called an $\mathcal{A} - \mathcal{B}$ imprimitivity bimodule (at least in [43,13]; later the terminology 'equivalence bimodule' was adopted [44]) if in addition there is a rigging map ${}_{\mathcal{A}}\langle\cdot, \cdot\rangle : L \times L \to \mathcal{A}$, satisfying the same properties of the $\mathcal{B}$-rigging, but with the roles of $\mathcal{A}$ and $\mathcal{B}$, and left and right interchanged. Moreover, the following conditions must hold:

i) the bounds (3.3), as well as the corresponding ones with $\mathcal{A}$ and $\mathcal{B}$ interchanged, hold;

ii) the linear span of $\{\langle\psi, \varphi\rangle_{\mathcal{B}} \mid \psi, \varphi \in L\}$ is dense in $\mathcal{B}$, and similarly with $\mathcal{B}$ replaced by $\mathcal{A}$;

iii) ${}_{\mathcal{A}}\langle\psi, \varphi\rangle\zeta = \psi\langle\varphi, \zeta\rangle_{\mathcal{B}}$ for all $\varphi, \psi, \zeta \in L$.

The imprimitivity theorem states that if there exists an $\mathcal{A} - \mathcal{B}$ imprimitivity bimodule (in which case $\mathcal{A}$ and $\mathcal{B}$ are called strongly Morita equivalent) then there is a bijective correspondence between the set of L-positive representations of $\mathcal{A}$ and $\mathcal{B}$ (which bijection preserves a number of properties of representations, such as direct integrals and weak containment, but upsets others, such as cyclicity [43,13]). The representation of $\mathcal{A}$ associated with $\pi_\chi(\mathcal{B})$ is simply π^χ, given by the Rieffel induction process. To go in the opposite direction, one makes the conjugate space $\overline{L}$ into a right-$\mathcal{A}$-module and left-$\mathcal{B}$-module by conjugating the respective actions on L, and induces using $\overline{L}$ and the $\mathcal{A}$-rigging map ${}_{\mathcal{A}}\langle\cdot, \cdot\rangle$. This conjugation is analogous to the step in the proof of the symplectic imprimitivity theorem where one passes from S to S^-.

More generally, there is a striking formal correspondence between (quantum) imprimitivity bimodules and classical equivalence bimodules (cf. Definition 4). As already mentioned, the rigging map corresponds to the moment map, and the compatibility condition iii) (which implies that the actions of $\mathcal{A}$ and $\mathcal{B}$ on L commute [13, XI.6.2]) replaces the symplectic assumption that $J_1^*C^\infty(P_1)$ Poisson commutes with $J_2^*C^\infty(P_2)$. Assumption ii) is the quantum analogue of the part of the definition of a full dual pair which states that J_1 and J_2 are surjective. The symplectic assumption that the leaf spaces of the foliations defined by the fibers of J_1 and J_2 are manifolds has its analogue in a condition which we omitted in order to state the imprimitivity theorem in its fullest generality; we could add

iv) the $\mathcal{A}$- and $\mathcal{B}$-rigging maps are positive

(that is, $\langle\psi,\psi\rangle_\mathcal{B} \geq 0$ in $\mathcal{B}$ and ${}_\mathcal{A}\langle\psi,\psi\rangle \geq 0$ in $\mathcal{A}$ for all $\psi \in L$); if this condition is added the imprimitivity theorem evidently states that there is a bijective correspondence between all representations of $\mathcal{A}$ and $\mathcal{B}$. Conversely, the imprimitivity theorem following from i)–iii) alone is analogous to the weakened version of Theorem 5 stated following its proof.

If only a right $\mathcal{B}$-module L is given, together with a positive rigging map whose image is dense in $\mathcal{B}$, one can always find a C^*-algebra $\mathcal{A}$ acting on L so that L becomes an $\mathcal{A} - \mathcal{B}$ imprimitivity bimodule [43,13]. This algebra $\mathcal{A}$ (called the imprimitivity algebra of $(L,\mathcal{B})$) is generated by operators of the form $T_{(\psi,\varphi)}$, whose action on $\zeta \in L$ is defined by $T_{(\psi,\varphi)}\zeta = \psi\langle\varphi,\zeta\rangle_\mathcal{B}$. Similarly, given a Poisson manifold P_1 which is one half of a full dual pair with equivalence bimodule S, one can find the manifold P_2 completing the dual pair by taking the Poisson commutant of $J^*C^\infty(P_1)$ in $C^\infty(S)$, which is necessarily of the form $J_2^*C^\infty(P_2)$, at least in the finite-dimensional case. However, the imprimitivity algebra $\mathcal{A}$ only coincides with the commutant of $\mathcal{B}$ if L is finite-dimensional (in general it is not even a von Neumann algebra). This dichotomy between the classical and the quantum settings will presumably disappear if one studies infinite-dimensional Poisson manifolds and their Morita equivalence.

3.2. Quantum Marsden–Weinstein reduction

We first apply the above framework to the quantization of the symplectic reduction procedure in its original version, where one reduces by a group action (cf. the Introduction, and the paragraph following Proposition 3). Hence we assume that the classical data consisting of a symplectic manifold S, a strongly Hamiltonian (right) action of a Lie group H on S, a Poisson algebra $A \subset C^\infty(S)$ of functions which are invariant under the group action, and a co-adjoint orbit $\mathcal{O} \subset \mathbf{h}^*$, have been quantized as a Hilbert space $\mathcal{H}$, a unitary representation $\pi(H)$ on $\mathcal{H}$, a representation of a C^*-algebra $\mathcal{A}$ on $\mathcal{H}$, which commutes with π, and an irreducible unitary representation $\pi_\chi(H)$ on a Hilbert space $\mathcal{H}_\chi$, respectively. (At no cost one may replace the co-adjoint orbit $\mathcal{O}$ and the irreducible representation π_χ by an arbitrary symplectic space with a strongly Hamiltonian H-action and an arbitrary unitary representation of H, respectively. Moreover, in what follows H does not need to be a Lie group; local compactness suffices.) Of course, the right H-action on S amounts to a Poisson morphism $J^* : C^\infty(\mathbf{h}^*)^- \to C^\infty(S)$, and the representation $\pi(H)$ on $\mathcal{H}$ corresponds to an anti-representation (called π^-) of the group algebra $C^*(H)$ [40], defined by

$$\pi^-(f) = \int_H dh\, f(h)\pi(h^{-1}), \tag{3.6}$$

where dh is the Haar measure on H (assumed unimodular for notational simplicity), and $f \in C_c(H)$. Thus the C^*-algebra $C^*(H)$, being the appropriate completion of the convolution algebra $C_c(H)$ (playing the role of $\mathcal{B}$ of the preceding subsection), is to be seen as the quantization of the Poisson algebra $C^\infty(\mathbf{h}^*)$, a point of view first stated by Rieffel [46]. We remark that it has been proved that $C^*(H)$ is a strict deformation quantization [45] of $C^\infty(\mathbf{h}^*)$ for H nilpotent [46] or compact [23], and we expect it to be true for any amenable group.

Let us first assume that H is compact, with Haar measure normalized to unity. We then take $L = \mathcal{H}$, $\mathcal{B} = C_c(H)$, and define the rigging map by

$$\langle\psi, \varphi\rangle_{C_c(H)} : h \to (\pi(h)\varphi, \psi), \tag{3.7}$$

utilizing the inner product in $\mathcal{H}$. This is easily shown to satisfy all conditions stated in the previous subsection, and it is positive as well:

Lemma 8. *For H compact, $\langle\psi, \psi\rangle_{C_c(H)} \geq 0$ as an element of $C^*(H)$ for all $\psi \in \mathcal{H}$.*

Proof. Let $\mathbf{1}_H$ denote the function on H which is identically equal to one. Then $\mathbf{1}_H^* * \mathbf{1}_H = \mathbf{1}_H$ (where $*$ is the convolution product on $C(H)$), so that $\mathbf{1}_H$ is a positive element of $C^*(H)$. Hence for any representation $\tilde{\pi}(H)$ on $\tilde{\mathcal{H}}$ (with inner product $(\cdot,\cdot)_\sim$) $\int_H dh\, (\tilde{\pi}(h)\tilde{\psi}, \tilde{\psi})_\sim = (\tilde{\pi}(\mathbf{1}_H)\tilde{\psi}, \tilde{\psi})_\sim \geq 0$ for all $\tilde{\psi} \in \tilde{\mathcal{H}}$. Now choose π_1 an arbitrary unitary representation of H on $\mathcal{H}_1$ (with inner product $(\cdot,\cdot)_1$). Using the previous argument with $\tilde{\pi} = \pi \otimes \pi_1$ and $\tilde{\psi} = \psi \otimes \psi_1$, we find that $(\pi_1(\langle\psi, \psi\rangle_{C_c(H)})\psi_1, \psi_1)_1 \geq 0$ for all $\psi_1 \in \mathcal{H}_1$. Since π_1 was arbitrary, this proves the lemma. □

Therefore, any unitary representation π_χ of H may be used to induce from. This is remarkable, for it implies that for compact Lie groups there is no quantum analogue of singular values of the moment map. Moreover, any C^*-algebra contained in the commutant $\pi(H)'$ of $\pi(H)$ on $\mathcal{H}$ is represented by bounded operators in the representation π^χ on the Hilbert space $\mathcal{H}^\chi$. This follows from

Lemma 9. *If $A \in \pi(H)'$ and H is compact then $\langle A\psi, A\psi\rangle_{C_c(H)} \leq \|A\|^2 \langle\psi, \psi\rangle_{C_c(H)}$ in $C^*(H)$ for all $\psi \in \mathcal{H}$.*

Proof. Notation as in the proof of the previous lemma. That proof showed that the operator $P = \int_H dh\, \pi \otimes \pi_1(h)$ is positive on $\mathcal{H} \otimes \mathcal{H}_1$. Clearly, P commutes with $A \otimes \mathbb{I}$ if $A \in \pi(H)'$. Hence with $\tilde{\psi} = \psi \otimes \psi_1$ and $(\cdot,\cdot)_\otimes$ the inner product in $\mathcal{H} \otimes \mathcal{H}_1$,

$$(PA \otimes \mathbb{I}\tilde{\psi}, A \otimes \mathbb{I}\tilde{\psi})_\otimes \leq \|A\|^2 \|P^{1/2}\tilde{\psi}\|^2 = \|A\|^2 (P\tilde{\psi}, \tilde{\psi})_\otimes. \tag{3.8}$$

If ω denotes the state on $C^*(H)$ defined by ψ_1, then this inequality reads

$$\omega(\langle A\psi, A\psi\rangle_{C_c(H)}) \le \|A\|^2\, \omega(\langle \psi, \psi\rangle_{C_c(H)}),$$

which proves the lemma. □

Let us see what the trivially induced representation looks like. We take $\mathcal{H}_\chi = \mathbb{C}$,carrying the trivial representation of H, so that the space $L\otimes\mathcal{H}_\chi$ used in the construction is simply $\mathcal{H}$. Using (3.1) and (3.7), we find that $(\psi,\psi)_0 = (P_{\rm id}\psi, P_{\rm id}\psi)$, where $P_{\rm id}$ is the orthogonal projector on the subspace $\mathcal{H}_{\rm id} \subset \mathcal{H}$ (which may be empty) of vectors which are invariant under H. The null space $\mathcal{H}_0$ is the orthogonal complement of $\mathcal{H}_{\rm id}$, and the final induced space $\mathcal{H}^{\rm id} = \mathcal{H}/\mathcal{H}_0$ is simply $\mathcal{H}_{\rm id}$, with the original inner product of $\mathcal{H}$. This space is invariant under $\pi(H)'$, so we find that $\pi^\chi(A)$ on $\mathcal{H}_{\rm id}$ is just the restriction of A to $\mathcal{H}_{\rm id}$. This is, of course, nothing but Dirac's prescription [9] for first-class constraints (it goes without saying that the above procedure quantizes the Marsden–Weinstein reduced space at zero, so that all the classical constraints are indeed first-class).

The Dirac procedure breaks down if zero is not in the discrete spectrum of each of the constraints, a situation which may arise when H is non-compact. The Rieffel induction procedure can still be used in that case, the main problem being the identification of an appropriate subspace $L \subset \mathcal{H}$. This will have to be done case by case, cf. the example surrounding (3.9) below. There is no guarantee that a dense L may be found; if the trivial representation of H properly occurs in $\mathcal{H}$ one has to exclude its carrier space from L.

In the following proposition, the assumption of unimodularity is only made for convenience (in the general case the rigging map and the convolution product would contain the modular function of H).

Proposition 10. *Let H be locally compact and unimodular, and let L be such that (3.7) defines a function in $C_c(H)$ for all $\psi, \varphi \in L$. Then $\langle\cdot,\cdot\rangle_{C_c(H)}$ is a rigging map, which is positive if H is amenable. Whether or not H is amenable, every representation of H weakly contained in the regular one is L-positive (so that it may be used to induce from); this is true in particular if the representation is square-integrable.*

Proof. The verification of properties (i)–(iv) of a rigging map (cf. previous subsection) is trivial. As to the positivity, the proof of Lemma 8 clearly breaks down in the noncompact case, as the function $\mathbf{1}_H$ is not in $C_c(H)$ (or, indeed, in $C^*(H)$). However, if H is amenable it has a family of subsets called $\{U_j\}_{j\in J}$ in [16, 3.6] (where our H is called G). Here J is a directed index set, and the U_j eventually fill up H. Each U_j is measurable and has finite Haar measure $\mu(U_j)$, and one has the following property. We define a family of functions $g_j \in L^1(H) \subset C^*(H)$ by $g_j = (\mu(U_j))^{-1/2}\chi_{U_j}$ (with χ_E the characteristic function of a Borel set E). Then $\lim_j g_j * g_j^* = \mathbf{1}_H$ pointwise on H. Hence for any $f \in L^1(H)$ one has by the bound $g_j * g_j^* \le \mathbf{1}_H$ and the Lebesgue dominated convergence theorem that $\lim_j \int_H dh\, f(h) g_j * g_j^*(h) = \int_H dh\, f(h)$. Clearly, each $g_j * g_j^*$ is a positive element of $C^*(H)$. (These results easily follow from [16, 3.6], and are even given as the definition of amenability in [42, II.3], specializing the groupoids in this ref. to groups.) Using the notation and strategy of the proof of Lemma 8, we now take $f(h) = (\pi \otimes \pi_1(h)\tilde{\psi}, \tilde{\psi})_\sim$. Then

$$\int_H dh\, f(h) g_j * g_j^*(h) = (\pi \otimes \pi_1(g_j * g_j^*)\tilde{\psi}, \tilde{\psi})_\sim \geq 0$$

for all j. Therefore, $\int_H dh\, f(h) \geq 0$. As in the proof of Lemma 8, we conclude that $\langle \psi, \psi \rangle_{C_c(H)} \geq 0$ in $C^*(H)$.

If H is not amenable, the family $\{U_j\}_{j\in J}$ with the desired properties does not exist. However, in that case $\langle \psi, \psi \rangle_{C_c(H)}$ is a positive element of the reduced group algebra $C_r^*(H)$ by an argument due to Rieffel [47] (in particular his calculation (1.1), specialized to $A = \mathbb{C}$). The proposition follows (the last claim is a consequence of the well-known fact that square-integrable representations are properly contained in the regular representation). □

Proposition 11. *In Lemma 9 above one may replace 'compact' by 'amenable'.*

Proof. This can be proved in a similar way as Lemma 9, replacing the operator P by $P_j = \int_H dh\, \pi \otimes \pi_1(h) g_j * g_j^*(h)$, which is well-defined since $g_j * g_j^*$ has compact support. One then obtains (3.8) with P replaced by P_j, and taking the limit in j yields the proposition. □

This proposition cannot be further improved, in the sense that for nonamenable groups H functions of the type (3.7) exist, for which $\pi_{\text{id}}(\langle \psi, \psi \rangle_{C_c(H)})$ (where π_{id} is the trivial representation of H) is strictly negative. This follows from Rieffel's argument on p. 146 of [47], which shows that a function g of postitive type on H exists, for which $\int_H dh\, g(h) < 0$. But by the well-known characterization of functions of positive type on locally compact groups [40], such a g is necessarily of the form $g(h) = (\pi(h)\psi, \psi)$ for some unitary representation π of H on $\mathcal{H}$, and $\psi \in \mathcal{H}$. (Rieffel assumed H to be discrete for simplicity, but his argument is easily extended to the general case by replacing his delta-function δ_e by an approximate unit of $C^*(H)$ which lies in $C_c(H)$.)

Assuming, instead, that we are in the regular case (that is, $\pi_\chi(H)$ is L-positive), we are now in a position to illustrate (3.4) and (3.5). Namely, let a Lie group H act continuously on a manifold M; the pull-back action on T^*M is then automatically strongly Hamiltonian [1] with moment map J. For any realization $\rho : S_\rho \to \mathbf{h}^*$ one may define the induced space $(T^*M)^\rho$ constructed in Subsection 2.1. In the special case where $\mathcal{O}$ is a co-adjoint orbit in $\mathbf{h}^*$ we thus obtain the Marsden–Weinstein reduced space $J^{-1}(\mathcal{O})/H$. To guarantee that this is a manifold, it suffices to assume that the action of H on M is proper (e.g., [1, p. 264], or [47]), and that any point in $\mathcal{O}$ is a regular value of J.

The quantization of this setting is to take $\mathcal{H} = L^2(M)$ (the Mackey Hilbert space of a manifold [1]), carrying the obvious unitary representation $\pi(H)$ derived from the (right) action of H on M. For simplicity, we assume that M has an H-invariant measure

ν (if not, one works with half-densities on M), so that $\mathcal{H} = L^2(M,\nu)$. Tentatively choosing $L = C_c(M)$, the rigging map (3.7) is then simply given by

$$\langle \psi, \varphi \rangle_{C_c(H)} : h \to \int_M d\nu(m)\, \varphi(mh)\overline{\psi(m)}; \tag{3.9}$$

however, this is guaranteed to indeed take values in $C_c(H)$ only if the H action on M is continuous and proper, thus providing a nice analogy with the classical situation. We now pick an L-positive representation π_χ of H, defined on a Hilbert space $\mathcal{H}_\chi$ (as we saw above, for H amenable any representation will do). If $\psi \otimes v \in C_c(M) \otimes \mathcal{H}_\chi$ then the image $U(\psi \otimes v)$ in $\mathcal{L}(\overline{L}, \mathcal{H}_\chi)$ may be identified with the $\mathcal{H}_\chi$-valued function ψ_v on M defined by

$$\psi_v(m) = \int_H dh\, \psi(mh)\pi_\chi(h)v \tag{3.10}$$

(cf. Thm. 5.12 in [43] for the special case $M = G$ a group with $H \subset G$ a subgroup). This function satisfies the equivariance condition $\psi_v(mh) = \pi_\chi(h^{-1})\psi(m)$ for all $m \in M$ and $h \in H$, and the inner product in $\tilde{\mathcal{H}}^\chi$ of two such functions is given by (3.5). This may be rewritten in terms of a so-called approximate cross-section of M/H in M, that is, a continuous positive function b on M whose support S is such that $S \cap KH$ is compact for any compact $K \subset M$ (here $KH = \{Kh \mid h \in H\}$), and $\int_H dh\, b(mh) = 1$ for all $m \in M$. Such a function is shown to exist in Lemma 1.2 of [38]; for $M = G$, b is the Bruhat approximate cross-section used in Theorem 4.4 in [43]. A short computation then shows that the inner product in $\tilde{\mathcal{H}}^\chi$ is given by

$$(\psi_v, \varphi_w) = \int_M d\nu(m) b(m) (\psi_v(m), \varphi_w(m))_\chi, \tag{3.11}$$

where, as before, $(\cdot,\cdot)_\chi$ is the inner product in $\mathcal{H}_\chi$. Alternatively, this may be written as an integral over M/H in terms of a suitable measure on that space, for $(\psi_v(m), \varphi_w(m))_\chi = (\psi_v(mh), \varphi_w(mh))_\chi$ on account of the equivariance condition stated above. This leads to the generalized induced representations of Moscovici [38] (which were already mentioned in [43] as a special case of the Rieffel induction process). In conclusion, the space $\tilde{\mathcal{H}}^\chi$ consists of H-equivariant functions Ψ on M with values in $\mathcal{H}_\chi$, such that $m \to (\Psi(m), v)_\chi$ is measurable for each $v \in \mathcal{H}_\chi$, and (Ψ, Ψ) defined by (3.11) is finite. Operators A on $L^2(M)$ commuting with $\pi(H)$ are then naturally defined on $\tilde{\mathcal{H}}^\chi$ also, that is, the desired induced representation is defined by $\pi^\chi(A)\psi_v = (A\psi)_v$. Hence we have shown how the Moscovici construction follows from (3.4) and (3.5), and it has been made clear of which symplectic situation it is the quantization.

If we take $M = G$ and H a closed subgroup of G, acting on the latter from the right, we find that the rigging map (3.7), defined on $L = C_c(G)$, is just the convolution (over G) $\psi * \varphi$, restricted to H. The right-action (3.6) is just $\pi^-(f)\psi = \psi * f$ (convolution over H). Hence this rigging map and right-action, which were directly defined by Rieffel [43] in the form just given, are specializations of the general formulae (3.6), (3.7).

As detailed in [43,13], the Rieffel induction procedure applied to this special case is equivalent to Mackey's formalism of induced group representations [30,50]. Note, that in this case the rigging map is positive even if H is not amenable (a fact [43] not covered by our Proposition 10).

3.3. Quantization of symplectic group actions which are not strongly Hamiltonian

What happens when the moment map $J : S \longrightarrow (\mathbf{h}^*)^-$ is not equivariant with respect to the co-adjoint representation π_{co}? (In the literature, one finds the notation $\text{Ad}^*_{h^{-1}}$ for our $\pi_{\text{co}}(h)$.) Equivalently, the pull-back $J^* : C^\infty(\mathbf{h}^*)^- \longrightarrow C^\infty(S)$ fails to be a Poisson morphism with respect to the Lie–Poisson structure on $\mathbf{h}^*$ in that case. It is well known how to handle this situation in the classical case [1,17]. The Lie group H, assumed to act on S from the right, preserving the symplectic form and admitting a moment map, also acts on $C^\infty(S) \otimes \mathbf{h}^*$ by a left-action α defined on $f \in C^\infty(S) \otimes \mathbf{h}^*$ as follows: $(\alpha_h f)(s) = \pi_{\text{co}}(h) f(sh)$. The infinitesimal action $d\alpha$ of $X \in \mathbf{h}$ is then $d\alpha_X f = (\tilde{X} + d\pi_{\text{co}}(X)) f$, where $\tilde{X}$ is the vector field on S defined by $(\tilde{X} f)(s) = d/dt\, f(s\exp(tX))|_{t=0}$. Subsequently, define an element $\Sigma \in \mathbf{h}^* \otimes \mathbf{h}^*$ by $\Sigma(X,Y) = \langle (d\alpha_X J)(s), Y\rangle$, which is independent of $s \in S$ (assuming S connected). Moreover, Σ turns out to be antisymmetric, and defines a 2-cocycle on $\mathbf{h}$. Hence one may define a new Poisson bracket $\{\cdot,\cdot\}^\Sigma$ on $C^\infty(\mathbf{h}^*)$ by putting

$$\{\tilde{X}, \tilde{Y}\}^\Sigma = \widetilde{[X,Y]} + \Sigma(X,Y) 1_{\mathbf{h}^*}; \tag{3.12}$$

here $\tilde{X} \in C^\infty(\mathbf{h}^*)$ is defined by $\tilde{X}(\theta) = \langle \theta, X\rangle$ (giving the Poisson bracket on such functions determines it completely), and $1_{\mathbf{h}^*}$ is the function which is identically 1 on $\mathbf{h}^*$. Then J is a Poisson map with respect to this modified Poisson structure of $\mathbf{h}^*$, and in addition is equivariant relative to the originally given H-action on S, and the new H-action π^Σ_{co} on $\mathbf{h}^*$ defined by

$$\pi^\Sigma_{\text{co}}(h)\theta = \pi_{\text{co}}(h)(\theta + J(s)) - J(sh^{-1}), \tag{3.13}$$

which is independent of s. Clearly, if J was π_{co}-equivariant (that is, $\alpha_h J = J$ for all $h \in H$) then $\Sigma = 0$, and (3.12) reduces to the Lie–Poisson bracket.

The essential point is that the Poisson structure on $C^\infty(\mathbf{h}^*)$, originally defined by the Lie bracket on $\mathbf{h}$, is modified by a certain central extension Σ of $\mathbf{h}$; the moment map remains the same. Also, the Marsden–Weinstein reduction with respect to a point $\mu \in \mathbf{h}^*$ of S is practically unmodified (cf. exercise 2.4.3D in [1]), and is a special case of the general procedure described in Subsection 2.1, taking S_ρ to be the symplectic leaf of $\mathbf{h}^*$ containing μ (relative to the Σ-Poisson bracket), or equivalently, the orbit of μ under the H-action (3.13).

This remark suggests how the situation should be quantized. Firstly, the quantum analogue of a symplectic group action which is not strongly Hamiltonian is a projective unitary representation on a Hilbert space $\mathcal{H}$, for by Wigner's theorem [50] that is the most general structure which quotients to a group action on the state space of $\mathcal{H}$ (i.e., the corresponding projective space), preserving the symplectic structure of the latter (defined by the inner product on $\mathcal{H}$ [1,49,24]). Thus we assume that for each $h \in H$ we are given a unitary operator $\pi(h)$ on $\mathcal{H}$, such that $\pi(h_1)\pi(h_2) = c(h_1, h_2)\pi(h_1h_2)$, where $|c(h_1, h_2)| = 1$ and the identity $c(h_1, h_1)c(h_1h_2, h_3) = c(h_1, h_2h_3)c(h_2, h_3)$ is satisfied (this is the equation one obtains by demanding associativity of the $\pi(h)$). We say that π has multiplier c [50]. If this is seen as the quantization of the H-action on S, one expects that the infinitesimal version of c, that is, the 2-cocycle on h derived from it, coincides with $-\Sigma$. Conversely, starting from $-\Sigma$ one may attempt to find a 2-cocycle c on H satisfying this property, which is always possible if H is simply connected (in general, a certain quantization condition must be satisfied by Σ [49]).

We recall the definition of the twisted group algebra $C^*(H,c)$ of H, which has a product (see below for the initial domain of definition of (3.14)–(3.16))

$$(f *_c g)(h) = \int_H dk\, f(hk^{-1})g(k)c(hk^{-1}, k), \tag{3.14}$$

and involution

$$(f^{*_c})(h) = c(h, h^{-1})\overline{f(h^{-1})}. \tag{3.15}$$

The quantum analogue of $C^\infty(\mathbf{h}^*)$ equipped with the Poisson bracket (3.12) is the twisted group algebra $C^*(H,\overline{c})$. We obtain a right-representation π^- of $C^*(H,\overline{c})$ by

$$\pi^-(f) = \int_H dh\, f(h)\overline{c}(h, h^{-1})\pi(h^{-1}), \tag{3.16}$$

where π has multiplier c, as above. There are some subtle differences with the untwisted case. Firstly, there one can find both a representation of $C^*(H)$ on $\mathcal{H}$ (obtained by replacing h^{-1} in (3.6) by h), and a right-representation, given by (3.6). In the twisted case, one obtains a representation of $C^*(H,c)$ (rather than $C^*(H,\overline{c})$) by omitting $\overline{c}$ and changing h^{-1} to h in (3.16). This is just as well, as we will see in Proposition 12 below. Secondly, the multiplier is not necessarily continuous (cf. [50,49] for conditions when it is), so that $C_c(H)$ is not closed under multiplication and taking the adjoint. Hence we largely follow Rieffel (Example 4.21 of [43]), who took $\mathcal{B}$ to be $K(H,\overline{c})$, the collection of bounded measurable functions of compact support on H, but we exploit the fact that H is a Lie group, and add the condition that elements of $\mathcal{B}$ are continuous in a neighbourhood of the identity. Then (3.14)–(3.16)) may be defined on $\mathcal{B}$, and extended by continuity to the whole twisted group algebra.

We can then define the rigging map by (3.7), as in the untwisted case, and (repeatedly using the cocycle identity on c) easily check it satisfies all conditions (assuming that an appropriate subspace L can be found). Moreover:

Proposition 12. *Let the locally comapct group H be amenable. Then the rigging map (3.7) is positive (i.e., $\langle\psi,\psi\rangle_{K(H,\overline{c})} \geq 0$ in $C^*(H,\overline{c})$).*

Proof. Also in the twisted case there exists a one-to-one correspondence between representations π_1 of $C^*(H,\overline{c})$ on a Hilbert space $\mathcal{H}_1$ and projective unitary representations (called π_1 as well) of H with multiplier $\overline{c}$ [15,39]; the correspondence is $\pi_1(f) = \int_H dh\, f(h)\pi_1(h)$, as in the untwisted case. Hence it is sufficient to prove that $(\pi_1(\langle\psi,\psi\rangle_{K(H,\overline{c})})\psi_1,\psi_1)_1 \geq 0$ for all $\psi \in L$ and $\psi_1 \in \mathcal{H}_1$. As we remarked above, π is a representation of H with multiplier c, whereas π_1 has multiplier $\overline{c}$. Hence $\pi\otimes\pi_1$ is a representation of H and $C^*(H)$, without any multiplier. Therefore, the argument used to prove Lemma 8 and Proposition 10 applies. Taking the compact case for simplicity, we can write

$$(\pi_1(\langle\psi,\psi\rangle_{K(H,\overline{c})})\psi_1,\psi_1)_1 = (\pi\otimes\pi_1(\mathbf{1}_{\mathbf{h}^*})\psi\otimes\psi_1,\psi\otimes\psi_1) \geq 0,$$

now regarding $\psi\otimes\psi_1$ as a representation of $C^*(H)$, in which $\mathbf{1}_{\mathbf{h}^*}$ is a positive element. The noncompact case is handled exactly as in the proof of Proposition 10. □

It is interesting to exhibit Rieffel's treatment of induced projective representations [43] as (almost) a special case of the above (cf. the discussion closing the previous subsection). Namely, assume that $H \subset G$ is a closed subgroup, with a multiplier c given, whose restriction to H is what we called c before. Now take $\mathcal{H} = L^2(G)$ with $L = C_c(G)$ as the dense subspace on which the rigging map is defined. Then $(\pi(h)\psi)(x) = c(x,h)\psi(xh)$ defines a projective unitary representation of H on $\mathcal{H}$ with multiplier c. Then (3.16) specializes to $\pi^-(f)\psi = \psi *_{\overline{c}} f$ (convolution over H), whereas the rigging map (3.7) becomes $\langle\psi,\varphi\rangle_{K(H,\overline{c})} = \psi^{*_c} *_{\overline{c}} \varphi$ (convolution over G). By associativity of $*_{\overline{c}}$, the condition $\langle\psi,\pi^-(f)\varphi\rangle_{K(H,\overline{c})} = \langle\psi,\varphi\rangle_{K(H,\overline{c})}) *_{\overline{c}} f$ is manifestly satisfied. Rieffel's right action of $C^*(H,\overline{c})$ is the one given above, while his rigging map is obtained by putting $\langle\psi,\varphi\rangle_{K(H,\overline{c})} = \psi^{*_{\overline{c}}} *_{\overline{c}} \varphi$, which is positive even if H is not amenable (although not manifestly so, despite appearances, for the convolution product is in $C^*(G,\overline{c})$ rather than $C^*(H,\overline{c})$).

For completeness, we mention another approach to handle the situation studied above; this is based on the result in [35] that symplectic leaves in $\mathbf{h}^*$ with respect to the Poisson structure (3.12) are symplectomorphic to ordinary co-adjoint orbits of the so-called splitting group $\overline{H}$ of H; this is a group whose unitary representations include all projective representations of H. Hence Marsden–Weinstein reduction in the situation studied in this subsection can be replaced by ordinary reduction with respect to $\overline{H}$. Quantization then proceeds as explained in the preceding subsection, featuring the ordinary (untwisted) group algebra of $\overline{H}$. Of course, in this elegant approach one faces the problem of having to find $\overline{H}$ as well as the particular co-adjoint orbit corresponding to the cocycle Σ in (3.12), and subsequently the particular unitary representation of $\overline{H}$ that quotients to the projective representation.

3.4. Induction with groupoid algebras

So far, the general formalism to quantize constrained systems has only been illustrated for the case that the Poisson algebra of the constraints is essentially a Lie algebra, perhaps with central extension. In other words, we took the Poisson algebra $C^\infty(P)$ to be $C^\infty(\mathbf{h}^*)$, where $\mathbf{h}$ is a Lie algebra; the quantization involved the group algebra $C^*(H)$ (perhaps twisted). A much more general situation that we are able to handle, in the sense that an explicit formula for the rigging map can be given, arises when we merely assume that $\mathbf{h}$ is a Lie algebroid $\mathcal{L}(\Gamma)$ of a Lie groupoid Γ [8], and $C^\infty(P) \equiv C^\infty(\mathcal{L}(\Gamma)^*)$ the Poisson algebra canonically associated to $\mathcal{L}(\Gamma)$ [8]. For we know [23] that the quantization of the Poisson algebra $C^\infty(\mathcal{L}(\Gamma)^*)$ is the groupoid C^*-algebra $C^*(\Gamma)$. (Cf. [42] for information on groupoid C^*-algebras; in the present case, $C^*(\Gamma)$ is canonically defined without reference to a left Haar system, since Γ is a manifold and one may use half-densities rather than functions as elements of the algebra. Alternatively, the same algebra may be defined with respect to a left Haar system, each of whose measures is equivalent to the Lebesgue measure in any local co-ordinate system on the relevant fiber, which is a manifold. For convenience we will choose the latter option.) The quantization of $C^\infty(\mathbf{h}^*)$ by $C^*(H)$ is a special case of the groupoid situation, as is the quantization of $C^\infty(T^*M)$ by the C^*-algebra of compact operators on $L^2(M)$, with a strict deformation quantization of Weyl type given in [23].

We use the following notation: the base space of Γ is called B, the source and target projections are s and t, respectively, and the left Haar system consists of measures μ_b on $t^{-1}(b)$, $b \in B$. The convolution product on $C^*(\Gamma)$ is given (firstly on $C_c(\Gamma)$) by

$$f * g(x) = \int_{t^{-1}(s(x))} d\mu_{s(x)}(y)\, f(xy)g(y^{-1}), \tag{3.17}$$

and the involution is

$$f^*(x) = \overline{f(x^{-1})}. \tag{3.18}$$

We assume that a right-representation π^- of $C^*(\Gamma)$ on a Hilbert space $\mathcal{H}$ is given. By a theorem of Renault [42, II.1.21], this representation corresponds to a representation π of Γ itself on $\mathcal{H}$ (to apply this theorem, we need to assume that Γ is 2nd countable; the other assumptions stated in [42] are automatically satisfied for Lie groupoids). Thus there is a measure ν on B, and a Hilbert space $\mathcal{H}_b$ for (ν-almost) every $b \in B$, so that $\mathcal{H} = \int_B^\oplus d\nu(b)\mathcal{H}_b$. The representative $\pi(x)$ of $x \in \Gamma$ is then a unitary map from $\mathcal{H}_{s(x)}$ to $\mathcal{H}_{t(x)}$; note that $\pi(x)$ is not defined as an operator on $\mathcal{H}$. Assuming that Γ with given left Haar system is unimodular in the sense of [42, I.3] (this assumption is satisfied in all examples [23,24]), the right-representation π^- is given on $f \in C_c(\Gamma)$ by

$$(\pi^-(f)\psi)(b) = \int_{t^{-1}(b)} d\mu_b(y) f(y^{-1})\pi(y)\psi(s(y)). \tag{3.19}$$

Using (3.17) and the left-invariance of the Haar system (which means that $\mu_{s(x)}(E) = \mu_{t(x)}(xE)$ for each Borel set $E \subset t^{-1} \circ s(x)$) it indeed follows that $\pi^-(f)\pi^-(g) = \pi^-(g * f)$. We now define the rigging map on an appropriate subspace $L \subset \mathcal{H}$ by

$$\langle\psi,\varphi\rangle_{C_c(\Gamma)} : x \rightarrow (\pi(x)\varphi(s(x)),\psi(t(x)))_{t(x)}, \tag{3.20}$$

where the inner product on the right-hand side is the one in $\mathcal{H}_{t(x)}$. Clearly, the rigging map (3.7) is a special case of (3.20). By L being 'appropriate' we simply mean that it be chosen such that the rigging map indeed takes values in $C_c(\Gamma)$; as in the group case, one may not be able to find a dense L. Checking the properties (i)–(iv) stated at the beginning of Subsection 3.1 is an easy matter, given (3.17)–(3.20); one only needs the properties $\pi(x)\pi(y) = \pi(xy)$, $s(xy) = s(y)$, $t(xy) = t(x)$, and $s(x^{-1}) = t(x)$, $t(x^{-1}) = s(x)$. Of course, the algebra $\mathcal{A}$ should be contained in the commutant of $\pi^{-}(C^*(\Gamma))$.

Proposition 13. *The rigging map (3.20) is positive if Γ is amenable.*

Proof. The notion of amenability of a groupoid is defined in [42, II.3]. We can simply copy the proof of Proposition 10, the functions g_j being given by the functions f_i of Definition II.3.1 of [42]. □

4. Some examples

4.1. Co-adjoint orbits and unitary representations of semidirect products

An important special case of symplectic reduction arises when one reduces T^*G with respect to the right-action of a subgroup $H \subset G$; as we mentioned in the Introduction, it was already pointed out in [20,17,54] that this reduction is the classical analogue of Mackey's construction of induced group representations (which in itself is a special case of Rieffel induction, cf. [43,13], or the end of Subsection 3.2 above). As a neat illustration of the general analogy between symplectic reduction and Hilbert space induction, we will now spell out how the representation theory of regular semidirect product Lie groups of the type $G = L \ltimes V$, with V abelian, may be seen in this light. By the Mackey theory [30,50], all unitary irreducible representations of G are induced from subgroups of the type $H = S \ltimes V$, where $S \subset L$ is the stability group of a point $\mathring{p} \in V^*$ under the dual action of L. If π_σ is a unitary irreducible representations of S, one then induces from representations $\pi_{(\sigma,\mathring{p})}$ defined by $\pi_{(\sigma,\mathring{p})}(s,v) = \exp(i\langle\mathring{p},v\rangle)\pi_\sigma(s)$.

The classical counterpart of this result of Wigner and Mackey would be that all co-adjoint orbits in $\mathbf{g}^*$ are (symplectomorphic to) Marsden–Weinstein reduced spaces of the form $(T^*G)^{\mathcal{O}} \equiv J^{-1}(\mathcal{O})/H$, with H as above, $\mathcal{O} \equiv \mathcal{O}_{(\sigma,\mathring{p})} = \mathcal{O}_\sigma \oplus \mathring{p}$ a co-adjoint orbit in $\mathbf{h}^* = \mathbf{s}^* \oplus V^*$, and $\mathcal{O}_\sigma$ a co-adjoint orbit in $\mathbf{s}^*$. Here $J \equiv J_R : T^*G \rightarrow (\mathbf{h}^*)^-$ is the moment map derived from the pull-back of the right-action of H on G, cf. Subsection 2.3 (paragraph following Theorem 6), whose notation and results we will freely use below. We will now verify that this is indeed the case.

Firstly, we need to check that $\mathcal{O}_{(\sigma,\mathring{p})}$ (which we again will simply call $\mathcal{O}$ in what follows) as defined above is indeed a co-adjoint orbit of H; this follows from the explicit action of a semidirect product group on its dual, given in [17, I.19]. Secondly, we must demonstrate that the map $J_L^{\mathcal{O}} : (T^*G)^{\mathcal{O}} \to \mathbf{g}^*$, which we already know to be symplectic and equivariant (intertwining the left-action of G on $(T^*G)^{\mathcal{O}}$ and the co-adjoint action on $\mathbf{g}^*$), is injective (so that $(T^*G)^{\mathcal{O}}$ is symplectomorphic to its image under $J_L^{\mathcal{O}}$), and thirdly, it should follow that any orbit in $\mathbf{g}^*$ is such an image for appropriately chosen H and $(\sigma, \mathring{p})$.

Using the left-trivialization of T^*G, we have seen that $J_L^{\mathcal{O}}([x,p]_H) = xp$; putting $x = (l,v) \in G$, and $p = (\theta,\mathring{p}) \in g^*$, where $\theta \in \mathbf{l}^*$ is such that $\theta \restriction \mathbf{s}$ lies in $\mathcal{O}_\sigma$, we find $J_L^{\mathcal{O}}([(l,v),(\theta,\mathring{p})]_H) = l\theta + l\mathring{p}$. Here the right-hand side was calculated using the formula for the co-adjoint action of G given in [17, I.19], and (following this reference) we have written the co-adjoint action of l on θ simply as $l\theta$, and the dual action of l on $\mathring{p} \in V^*$ as $l\mathring{p}$. Now use the fact that the right-action ρ of $(s,w) \in H = S \ltimes V$ on the point $((l,v),(\theta,\mathring{p})) \in T^*G \simeq G \times \mathcal{G}^*$ is given by

$$\rho_{(s,w)}((l,v),(\theta,\mathring{p})) = ((ls^{-1}, v - ls^{-1}w),(s\theta,\mathring{p}))$$

to conclude that the map $J_L^{\mathcal{O}}$ is indeed well-defined and injective on the quotient of $J^{-1}(\mathcal{O}) \subset T^*G$ by H. Finally, the fact that any co-adjoint orbit in $\mathbf{g}^*$ is obtained in this way follows from the classification of these orbits in [17, I.19].

This result is closely related to a theorem in [33], which states that each co-adjoint orbit in $\mathbf{g}^*$ is symplectomorphic to a symplectic leaf in $(T^*L)/S$ for suitable S, which S is exactly what we used above. In addition, we mention the work of Rawnsley [41], who related the Wigner–Mackey representation theory of semidirect products to the geometric quantization of certain of their co-adjoint orbits. This is quite different in spirit from our approach, which in this situation does not use any explicit correspondence between co-adjoint orbits and irreducible unitary representations (let alone geometric quantization), but rather emphasizes the fact that both are obtained by an induction procedure, which even employs the same class of subgroups H in the classical and the quantum case. Moreover, even leaving quantum representation theory aside, the results of this subsection and the next, taken together with Corollary 7, considerably simplify the study of of actions of semidirect product or nilpotent Lie groups on symplectic manifolds.

4.2. Co-adjoint orbits and unitary representations of nilpotent Lie groups

A similar result holds when G is nilpotent. Assuming G to be connected and simply connected for simplicity, the Dixmier–Kirillov theory [7] establishes a bijective correspondence between the co-adjoint orbits of G and its irreducible unitary representations. For us, the main point is that all unitary representations are obtained by Mackey induction from certain subgroups H, and this inspires us to demonstrate that all co-adjoint orbits of G are Marsden–Weinstein reduced spaces induced by the same H's.

Pick a point $\mathring{p} \in \mathbf{g}^*$, and take $G_0 \subset G$ the stability group of $\mathring{p}$ under the co-adjoint action. The essential implication of the nilpotence of G is the existence of a so-called polarizing subalgebra $\mathbf{h}$, where $\mathbf{g}_0 \subseteq \mathbf{h} \subseteq \mathbf{g}$, and $\langle \mathring{p}, [X,Y] \rangle = 0$ for all X, Y in $\mathbf{h}$. With

$H = \exp \mathbf{h}$, one then induces from the representation $\pi_{\overset{\circ}{p}}(H)$ given by $\pi_{\overset{\circ}{p}}(\exp(X)) = \exp(i\langle \overset{\circ}{p}, X\rangle)$. Representations thus obtained are unitarily equivalent iff the various $\overset{\circ}{p}$ one starts from lie in the same orbit, and all irreducible unitary representations of G are obtained in this way.

To find the classical analogues of these statements, we first notice that $\mathbf{h}$ being polarizing relative to $\overset{\circ}{p}$ simply means that $\overset{\circ}{p}_r \equiv \overset{\circ}{p} \upharpoonright \mathbf{h} \in \mathbf{h}^*$ is stable under the co-adjoint action of H. Hence $\overset{\circ}{p}_r$ is a co-adjoint orbit in $\mathbf{h}^*$, and we are done if we can show that $G\overset{\circ}{p} \simeq J^{-1}(\overset{\circ}{p}_r)/H$ as symplectic spaces (here $J \equiv J_R : T^*G \to (\mathbf{h}^*)^-$, as in the previous subsection). This is indeed the case.

First, note that $J^{-1}(\overset{\circ}{p}_r) = G \times (H\overset{\circ}{p})$ (in the left-trivialization of $T^*G \simeq G \times \mathbf{g}^*$). To prove this, observe that the set $\Sigma = \{p \in \mathbf{g}^* \mid p \upharpoonright \mathbf{h} = \overset{\circ}{p}_r\} \subset \mathbf{g}^*$ is a copy of $\mathbb{R}^n$ in $\mathbf{g}^*$, with $n = \dim \mathbf{g} - \dim \mathbf{h}$. On the other hand, H, being connected, simply connected, and nilpotent, acts unipotently on $\mathbf{g}^*$, so that by [7, Cor. 3.1.5] the orbit $H\overset{\circ}{p}$ is homeomorphic to $\mathbb{R}^m$, with $m = \dim H - \dim G_0$. But H is a polarizing subgroup of G, hence $n = m$ by [7, Thm. 1.3.3]. Secondly, if $p = h\overset{\circ}{p}$ for some $h \in H$ then $p \upharpoonright \mathbf{h} = \overset{\circ}{p} \upharpoonright \mathbf{h} = \overset{\circ}{p}_r$, since the map $p \to p \upharpoonright \mathbf{h}$ intertwines the co-adjoint action of H on $\mathbf{g}^*$ with its action on $\mathbf{h}^*$. Hence $G \times (H\overset{\circ}{p}) \subseteq J^{-1}(\overset{\circ}{p}_r)$. The claim follows.

As $H\overset{\circ}{p} = H/G_0$, we have $J^{-1}(\overset{\circ}{p}_r)/H = (G \times (H/G_0))/H$, with the right H-action ρ defining the quotient given by $\rho_h(x,p) = (xh, h^{-1}p)$. Hence $(G \times (H/G_0))/H \simeq G/G_0 = G\overset{\circ}{p}$, and this is a symplectomorphism implemented by the map $J_L^{\mathcal{O}} : J^{-1}(\mathcal{O})/H \to \mathbf{g}^*$ (cf. the previous subsection), with $\mathcal{O} = \overset{\circ}{p}_r$. Since $\overset{\circ}{p}$, and hence the co-adjoint orbit $G\overset{\circ}{p}$, was arbitrary, we have indeed established that any co-adjoint orbit in a connected and simply connected nilpotent Lie group is obtained by Marsden–Weinstein reduction from a polarizing subgroup and a zero-dimensional orbit. This establishes a perfect correspondence between classical and quantum induction in this case.

4.3. The generalized Yang–Mills construction

Let (P, H, Q, pr') be a principal fiber bundle with connected compact gauge group H and projection $pr' : \mathsf{P} \to Q = \mathsf{P}/H$; we assume P connected as well. Then H acts from the right on $T^*\mathsf{P}$ by pull-back with moment map J, and we have a full dual pair $(T^*\mathsf{P})/H \overset{pr}{\leftarrow} T^*\mathsf{P} \overset{J}{\rightarrow} (\mathbf{h}^*)^-$, where pr is the canonical projection onto the given quotient space [53]. With extra assumptions on simple connectedness, one even obtains a classical equivalence bimodule, so that $\mathbf{h}^*$ and $(T^*\mathsf{P})/H$ are Morita equivalent Poisson manifolds with $T^*\mathsf{P}$ as their equivalence bimodule, cf. [57] or Definition 4 in Subsection 2.1 above. However, by the argument in [53, Section 8], there is a bijective correspondence between the symplectic leaves in $(T^*\mathsf{P})/H$ and $\mathbf{h}^*$, which is given explicitly in [17,31]. There it is shown that the leaves of $(T^*\mathsf{P})/H$ are fiber bundles over T^*Q with a co-adjoint orbit in $\mathbf{h}^*$ as fiber. This suggests that $(T^*\mathsf{P})/H$ and $\mathbf{h}^*$ are Morita-equivalent without any further assumption; their equivalence bimodule may be different from $T^*\mathsf{P}$ in general.

In any case, the essential point is that $pr^*C^\infty((T^*\mathsf{P})/H)$ Poisson commutes with $J^*C^\infty(\mathbf{h}^*)$ in $C^\infty(T^*\mathsf{P})$, so that, starting from any given realization $\rho : S_\rho \to \mathbf{h}^*$, we obtain an induced representation $\pi^\rho : C^\infty((T^*\mathsf{P})/H) \to C^\infty(S^\rho)$ by the construction in Subsection 2.1 (or, equivalently, we find a Poisson map $J^\rho : S^\rho \to (T^*\mathsf{P})/H$ for each such ρ). If we take S_ρ to be a co-adjoint orbit in $\mathbf{h}^*$ then S^ρ is a symplectic leaf of $(T^*\mathsf{P})/H$, which plays the role of the phase space of a particle in a Yang-Mills field, as originally observed by Sternberg (cf. [52,17,31] for a comprehensive discussion), whence the name 'Yang-Mills construction'. Inducing from an arbitrary realization S_ρ leads to the 'generalized Yang-Mills construction' [54,58].

The Yang-Mills construction was quantized in [23], where we exploited the fact that $C^\infty((T^*\mathsf{P})/H)$ is the Poisson algebra canonically associated to the Lie algebroid $(T\mathsf{P})/H$ [8]. Here we wish to briefly give a general construction based on Rieffel induction. Namely, the quantum analogue of the full dual pair mentioned above is the imprimitivity bimodule $\mathcal{K}(L^2(\mathsf{P}))^H \rightarrow L^2(\mathsf{P}) \leftarrow C^*(H)$, which involves the H-invariant compact operators on $L^2(\mathsf{P})$. To see this, one may start from the right-action π^- of $C_c(H) \subset C^*(H)$ on $C_c(\mathsf{P}) \subset L^2(\mathsf{P})$, provided (via (3.6)) by the unitary representation π of H on $L^2(\mathsf{P})$, which comes from the right-action defining the principal bundle. For simplicity, we put an H-invariant measure μ on P (always possible, as H is compact), which defines $L^2(\mathsf{P})$. Then $(\pi(h)\psi)(x) = \psi(xh)$.

The rigging map into $C^*(H)$ is given by (3.7), and is positive by Lemma 8. Using the fact that the C^*-norm $\|f\|$ of $f \in C_c(H)$ is dominated by its L^1-norm, as well as the Cauchy-Schwartz inequality and $\int_H dh = 1$, one finds that $\|\langle\psi,\varphi\rangle_{C_c(H)}\| \leq \|\psi\|_2\,\|\varphi\|_2$, so that one can extend (3.7) by continuity to a rigging map defined on $\mathcal{H}$ with values in $C^*(H)$. Note that, since H acts freely on P, we can choose a (discontinuous) cross-section $s : Q \to \mathsf{P}$, which leads to a natural isomorphism $L^2(\mathsf{P}) \simeq L^2(Q) \otimes L^2(H)$, where $\pi(H)$ acts trivially on the first factor and via the right-regular representation on the second one. Since the rigging map then amounts to convolution over H (times an inner product in $L^2(Q)$), this implies that the image of the rigging map defined on $C_c(\mathsf{P})$ is dense in $C_c(H)$, hence in $C^*(H)$.

Now consider the imprimitivity algebra [43,13] (also cf. Subsection 3.1) $\mathcal{A}$ defined by $\mathcal{H} = L^2(\mathsf{P})$, $\mathcal{B} = C^*(H)$, and the rigging map (3.7). $\mathcal{A}$ is generated by operators of the form $T_{(\psi,\varphi)}$, whose action on $\zeta \in L$ is defined by $T_{(\psi,\varphi)}\zeta = \pi^-(\langle\varphi,\zeta\rangle_{C^*(H)})\psi$. Starting with $\psi,\varphi,\zeta \in C_c(\mathsf{P})$, and using (3.6), (3.7), we find that $T_{(\psi,\varphi)}$ is Hilbert-Schmidt with kernel given by $K_{(\psi,\varphi)}(x,y) = \int_H dh\,\psi(xh)\overline{\varphi(yh)}$. From the property $K(xh,yh) = K(x,y)$ for all $h \in H$ and $x,y \in \mathsf{P}$ we infer that $T_{(\psi,\varphi)}$ commutes with all $\pi(h)$. Hence the C^*-algebra generated by these operators is clearly $\mathcal{A} = \mathcal{K}(L^2(\mathsf{P}))^H$. The $\mathcal{A}$-rigging map is defined by ${}_{\mathcal{A}}\langle\psi,\varphi\rangle = T_{(\psi,\varphi)}$, and all relevant conditions are now automatically satisfied (cf. [43, Section 6]) for $L^2(\mathsf{P})$ to become a $\mathcal{K}(L^2(\mathsf{P}))^H$ – $C^*(H)$ imprimitivity bimodule. (Incidentally, this construction also shows how to handle the case where H is noncompact: in that case $\mathcal{A}$ is no longer $\mathcal{K}(L^2(\mathsf{P}))^H$, but it may simply be defined as the imprimitivity algebra defined by $\mathcal{H}$, $\mathcal{B}$, and (3.7).)

Physically, $\mathcal{A}$ is the 'universal algebra of observables' of a particle in a Yang-Mills field with gauge group H [23], and is the quantum counterpart of the Poisson algebra $C^\infty((T^*\mathsf{P})/H$, which plays this role in classical mechanics [52]. By the Rieffel imprimitivity theorem [43] (also cf. Subsection 3.1 above) combined with the strong Morita equivalence between $\mathcal{A}$ and $\mathcal{B} = C^*(H)$ established above, all its representations are induced by representations of $C^*(H)$, hence by unitary representations of H. The explicit form of these induced representations is then given by the Moscovici induction technique discussed at the end of Subsection 3.2 above as a special case of the Rieffel process. Starting from a unitary representation $\pi_\chi(H)$ on a Hilbert space $\mathcal{H}_\chi$, one finds that the Hilbert space $\tilde{\mathcal{H}}^\chi$ carrying the induced representation $\tilde{\pi}^\chi(\mathcal{A})$ is just the L^2-closure of the space Γ_χ of smooth compactly supported cross-sections of the vector bundle $\mathsf{P} \times_H \mathcal{H}_\chi$ associated to the principal bundle (P, H, Q, pr'). This realization was previously found by different means [23]; we note that the space Γ_χ is a useful domain of essential self-adjointness of various unbounded operators of physical relevance.

4.4. *The illusion of time*

The classical relativistic particle in Minkowski space–time is discussed in an elegant covariant symplectic formalism in [17]. As we failed to find a convincing quantization of this approach in the literature, we here discuss this system using Rieffel induction.

The classical setup consists of the cotangent bundle $S = T^*\mathbb{R}^4$ and the group $H = \mathbb{R}$, which acts on S by generating geodesic motion on the flat space-time $\mathbb{R}^4$. If we write $(x(\tau), p(\tau))$ for the result of the action of $\tau \in \mathbb{R}$ on $(x,p) \in S$, we thus have

$$(x^\mu(\tau), p_\nu(\tau)) = (x^\mu + p^\mu\tau, p_\nu), \tag{4.1}$$

where $p^\mu = g^{\mu\nu}p_\nu$, with $g^{\mu\nu}$ the metric $\mathrm{diag}(1,-1,-1,-1)$. If the symplectic form is taken to be $dx^\mu \wedge dp_\mu$, this action corresponds to the moment map $J : T^*\mathbb{R}^4 \to \mathbf{h}^* = \mathbb{R}$ defined by $J(x,p) = g^{\mu\nu}p_\mu p_\nu/2$. The observables on S are the functions $f \in C^\infty(S)$ which Poisson-commute with J, that is, satisfy $p^\mu \partial f/\partial x^\mu = 0$. We now reduce S with respect to the co-adjoint orbit $S_\rho = \{m^2/2\} \in \mathbf{h}^*$, and find that the reduced phase space S^ρ consists of two disconnected copies, one with $p_0 > 0$ and one with $p_0 < 0$; the latter may consistently be ignored by imposing the additional constraint $p_0 > 0$. Each copy consists of equivalence classes of points in $S *_\mathbb{R} \{m^2/2\} \equiv \{(x,p) \in S \mid p^2 = m^2\}$, where points are in the same equivalence class iff they are connected by the flow (4.1). Therefore, a point in the physical (i.e., reduced) phase space, identified with a physical state of the relativistic particle, consists of an entire particle trajectory through space–time.

Using the prescription proposed in this paper, it is completely straightforward to quantize this model. We take $\mathcal{H} = L^2(\mathbb{R}^4)$ (regarded as functions on space-time), which carries a representation of $H = \mathbb{R}$ given by $\pi(\tau)\psi = \exp(i\tau\Box)\psi$ (with $\Box = g^{\mu\nu}\,\partial/\partial x^\mu\,\partial/\partial x^\nu$). We define the rigging map (3.7) on $L = C_c^\infty(\mathbb{R}^4)$, and induce from the irreducible unitary representation $\pi_{m^2} : \tau \to \exp(i\tau m^2)$ on $\mathcal{H}_m = \mathbb{C}$. With (3.1), we find that the form $(\cdot,\cdot)_0$ of the Rieffel induction process is given by (note that $L \otimes \mathcal{H}_\chi = L$ in the present case)

$$(\psi,\varphi)_0 = \int_{\mathbb{R}} d\tau\, e^{i\tau m^2} \int_{\mathbb{R}^4} d^4x\, (e^{i\tau\Box}\psi)(x)\overline{\phi(x)}$$
$$= \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^3}\delta(p^2 - m^2)\hat{\psi}(p)\overline{\hat{\varphi}(p)}. \tag{4.2}$$

Hence the final representation space $\mathcal{H}^{m^2}$ consists of solutions ψ of the Klein-Gordon equation $(\Box + m^2)\psi = 0$, with either positive or negative energy p_0, whose Fourier transforms are square-integrable with respect to the measure d^3p/p_0, which one finds by integrating the delta function in (4.2). Alternatively, one may follow the construction of $\tilde{\mathcal{H}}^\chi$ explained after (3.5), with $\chi = m^2$, and arrive at the same result. The quantum observables are those bounded operators which commute with the multiplication operator p^2.

The interpretation of the quantum states is similar to the classical ones: each vector in $\mathcal{H}^{m^2}$ consists of a wave function on space-time. The propagation of states in time, familiar from non-relativistic mechanics, here has to be derived from external considerations.

4.5. Finite W-algebras

Inspired by developments in conformal field theory and integrable systems, the concept of a finite W-algebra was recently introduced [5]. This subject provides an illustration of Rieffel induction applied to an algebra of unbounded operators, and it appears to us that our quantization method applied here is simpler than the BRST and Lie algebra cohomology techniques used in [5].

The setting is a Lie group G with Lie subgroup H. In the context of W-algebras, G is semi-simple and H is nilpotent, but these assumptions hardly play a role in our discussion. H acts on $\mathbf{g}^*$ by restriction of the co-adjoint representation. This action preserves the Lie-Poisson structure, and the corresponding generalized moment map j is simply given by $j(\theta) = \theta \restriction \mathbf{h}$. Picking an orbit $\mathcal{O} \subset \mathbf{h}^*$, we can define the Poisson reduced space $j^{-1}(\mathcal{O})/H$ [32], and the corresponding classical finite W-algebra is the space of real polynomials $W_c(G,H,\mathcal{O}) = \mathrm{pol}_{\mathbb{R}}[j^{-1}(\mathcal{O})/H]$, equipped with the reduced Poisson structure.

To quantize, it is convenient to have an equivalent definition at hand. Recall [1] that $G\backslash T^*G \simeq \mathbf{g}^*$, so that ${}^GC^\infty(T^*G)$ (the space of left G-invariant smooth functions on T^*G) is Poisson-isomorphic to the Lie-Poisson algebra $C^\infty(\mathbf{g}^*)$, whose subspace of polynomials $\mathrm{pol}_{\mathbb{R}}[\mathbf{g}^*]$ is well defined. The right H-action on T^*G quotients to the co-adjoint action on $\mathbf{g}^*$. Hence the space ${}^GC^\infty(T^*G)^H$ may be restricted to the space $A \subset C^\infty(T^*G)$ of H-invariant polynomials on $\mathbf{g}^*$. A is a Poisson algebra which inherits the canonical Poisson bracket on T^*G: since this bracket is left and right G-invariant, it can consistently be restricted to A. If we now take $S = T^*G$, $S_\rho = \mathcal{O}$ a co-adjoint orbit in $\mathbf{h}^*$, and $\rho = i_{\mathcal{O}}$ the injection of $\mathcal{O}$ in $\mathbf{h}^*$, we obtain an induced representation $\pi^{\mathcal{O}}(A)$ on the Marsden-Weinstein reduced space $S^{\mathcal{O}} = J^{-1}(\mathcal{O})/H$, cf. Subsection 2.1 (we here write $\pi^{\mathcal{O}}$ etc. for $\pi^{i_{\mathcal{O}}}$). As before, $J : T^*G \to (\mathbf{h}^*)^-$ is the moment map coming from the right-action of H on T^*G. It is easily seen that $\pi^{\mathcal{O}}(A) \simeq W_c(G, H, \mathcal{O})$ as Poisson algebras.

In this formulation, quantization is a piece of cake. Firstly, the quantization of the Lie–Poisson algebra $\mathrm{pol}_{\mathbb{R}}[\mathbf{g}^*]$ is the operator algebra $\mathcal{U}(\mathbf{g})_{\mathrm{sa}}$, which consists of the symmetric elements of the universal enveloping algebra of $\mathbf{g}$ (hence the quantization of the complexified Poisson algebra $\mathrm{pol}_{\mathbb{C}}[\mathbf{g}^*]$ is $\mathcal{U}(\mathbf{g})$ itself). We here regard $\mathcal{U}(\mathbf{g})$ as the Op*-algebra of left-invariant differential operators defined on the common dense domain $C_c^\infty(G) \subset L^2(G)$ [48] (to be compared with $\mathrm{pol}_{\mathbb{C}}[\mathbf{g}^*]$ being the left-invariant polynomials on T^*G). This quantization is just the infinitesimal and unbounded version of Rieffel's deformation quantization of $C_0(\mathbf{g}^*)$ by the group algebra $C^*(G)$ [46].

To see the connection between the two, we start by taking a smooth function f on $\mathbf{g}^*$ whose Fourier transform $\check{f}$ has compact support on $\mathbf{g}$. Thus a function $\tilde{f}$ may be defined for sufficiently small $\hbar$ on a neighbourhood of the identity of G, by $\tilde{f}(\exp(-\hbar X)) = \hbar^{-n}\check{f}(X)$ (with $n = \dim G$). Then $\tilde{f}$ becomes a smooth function on G by putting it equal to zero elsewhere. We then define the deformation quantization $Q_\hbar(f)$ as an operator on $L^2(G)$ by $Q_\hbar(f) = \pi_R(\tilde{f})$, where π_R is the right-regular representation of G and $C^*(G)$ on $L^2(G)$ (hence $(Q_\hbar(f)\psi)(x) = \int_G dy\, \tilde{f}(y)\psi(xy)$ for $\psi \in L^2(G)$). If we restrict ψ to lie in $C_c^\infty(G)$, then a short formal computation shows that this quantization may be extended to any $f \in \mathrm{pol}_{\mathbb{C}}[\mathbf{g}^*]$, and that the final result is defined for arbitrary values of $\hbar$. Explicitly, one finds that a monomial $\tilde{X}_1 \dots \tilde{X}_l \in \mathrm{pol}_{\mathbb{R}}[\mathbf{g}^*]$ (where, as before, $\tilde{X} \in \mathbf{g} \subset C^\infty(\mathbf{g}^*)$ is defined by $\tilde{X}(\theta) = \langle \theta, X \rangle$) is quantized by $(-i\hbar)^l \lambda(X_1 \dots X_l) \in \mathcal{S}(\mathbf{g})$. Here λ is symmetrization, and this quantization is identified with the corresponding element in the right-regular representation, as before.

We now follow the Rieffel induction process with $L = C_c^\infty(G)$, $\mathcal{A} = \mathcal{U}(\mathbf{g})^H$ (consisting of invariants under the adjoint action of H), and $\mathcal{B} = C_c^\infty(H)$. $\mathcal{A}$ acts on L as indicated above, and $\mathcal{B}$ acts on L by $\pi^-(f)\psi = \psi * f$ (convolution over H). We now exploit the fact that H is nilpotent, which implies that there is an irreducible unitary representation $\pi_{\mathcal{O}}$ on a Hilbert space $\mathcal{H}_{\mathcal{O}}$ defined by the orbit $\mathcal{O}$ [7] (if H is not simply connected, this holds provided that the orbit satisfies a suitable integrality condition). The quantum W-algebra $W_q(G, H, \mathcal{O})$ is then simply the induced representation $\pi_q^{\mathcal{O}}(\mathcal{A})$, which is an algebra of unbounded operators acting on the dense domain $D = C_c^\infty(G)\tilde{\otimes}\mathcal{H}_{\mathcal{O}} \subset \mathcal{H}^{\mathcal{O}}$ (see (3.2), with $\chi = \mathcal{O}$). Explicitly, $\mathcal{H}^{\mathcal{O}}$ is of course just the representation space obtained by inducing $\pi_{\mathcal{O}}(H)$ to $\pi^{\mathcal{O}}(G)$ by the Mackey procedure.

In the Blattner realization $\tilde{\mathcal{H}}^{\mathcal{O}}$ of H-equivariant functions $\psi : G \to \mathcal{H}_{\mathcal{O}}$ (that is, $\tilde{\psi}(xh) = \pi_{\mathcal{O}}(h^{-1}\tilde{\psi}(x))$ for which $(\tilde{\psi}, \tilde{\psi})_{\mathcal{O}}$ (inner product in $\mathcal{H}_{\mathcal{O}}$) is square-integrable on G/H, the corresponding domain $\tilde{D}$ consists of those functions in $\tilde{\mathcal{H}}^{\mathcal{O}}$ which are smooth and the projection of whose support on G onto G/H is compact. Note, that $d\tilde{\pi}^{\mathcal{O}}(\mathcal{U}(\mathbf{g}))$ acts on $\tilde{\psi} \in \tilde{\mathcal{H}}^{\mathcal{O}}$ by hitting the argument of $\tilde{\psi}$ from the left (e.g., $d\tilde{\pi}^{\mathcal{O}}(X)\tilde{\psi}(x) = d/dt\, \tilde{\psi}(\exp(tX)x)|_{t=0}$ for $X \in \mathbf{g}$), which trivially preserves H-equivariance of $\tilde{\psi}$, whereas $\tilde{\pi}_q^{\mathcal{O}}$ maps $\mathcal{U}(\mathbf{g})^H$ into differential operators hitting this argument from the right. This still preserves the equivariance on account of the H-invariance of elements of $\mathcal{A}$.

4.6. Reduction by a groupoid algebra

Our final example probably provides the simplest illustration of the use of groupoids in constrained systems. The classical system has $S = T^*\mathbb{R}^m$, and the aim is to eliminate one degree of freedom. This may be done by imposing the single constraint $p_1 = 0$, and reduce with respect to the corresponding action of $H = \mathbb{R}$ on S. However, it is more instructive to start from a Poisson map $J : T^*\mathbb{R}^m \to (T^*\mathbb{R})^-$. The observables have to commute with the constraints $J^*C^\infty(T^*\mathbb{R})$, and are just the functions which do not depend on x^1, p_1. With $S_\rho = T^*\mathbb{R}$ and ρ the identity map, the reduction procedure of Subsection 2.1 then painlessly leads to the reduced phase space $S = T^*\mathbb{R}^{m-1}$, with the obvious action of the observables.

Since $C^\infty(T^*\mathbb{R}^n)$ is the Poisson algebra defined by the Lie algebroid $T\mathbb{R}^n$, its quantization is the groupoid algebra $C^*(\mathbb{R}^n \times \mathbb{R}^n) = \mathcal{K}(L^2(\mathbb{R}^n))$ [23] (also cf. Subsection 3.4 above). Therefore, taking $n = m$, the quantization of the unconstrained system is given by the defining representation of $\mathcal{K}(L^2(\mathbb{R}^m))$ on $\mathcal{H} = L^2(\mathbb{R}^m)$, and the quantum algebra of the constraints is (put $n = 1$) $\mathcal{K}(L^2(\mathbb{R}))$. To use the procedure of Subsection 3.4, we identify $\mathcal{H}$ as the direct integral $\int_{\mathbb{R}}^{\oplus} dx\, \mathcal{H}(x) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^{m-1})$, with $\mathcal{H}(x) = L^2(\mathbb{R}^{m-1})$. It is convenient to work with suitable dense subspaces, so we take $L = C_c(\mathbb{R}^m)$, $\mathcal{A} = \mathbb{I} \otimes \mathcal{K}_c(L^2(\mathbb{R}^{m-1}))$, and $\mathcal{B} = \mathcal{K}_c(L^2(\mathbb{R}))$. Here $\mathcal{K}_c(L^2(\mathbb{R}^n))$ consists of the Hilbert-Schmidt operators whose kernel is in $C_c(\mathbb{R}^n, \mathbb{R}^n)$. If we identify a Hilbert-Schmidt operator f with its kernel, then the representation (3.19) reads

$$(\pi^-(f)\psi)(x^1, \ldots, x^m) = \int_{\mathbb{R}} dx\, f(x, x^1)\psi(x, x^2, \ldots, x^m). \tag{4.3}$$

The rigging map (3.20) is

$$\langle \psi, \varphi \rangle_{\mathcal{B}} : (x, y) \to \int_{\mathbb{R}^{m-1}} dx^2 \cdots dx^m\, \varphi(y, x^2, \ldots, x^m)\overline{\psi(x, x^2, \ldots, x^m)}. \tag{4.4}$$

We now induce from the identity representation π_{id} of $\mathcal{B}$ on $\mathcal{H}_{\mathrm{id}} = L^2(\mathbb{R})$. The space $L \otimes \mathcal{H}_{\mathrm{id}}$ may be identified with a space of functions in $m+1$ variables, so that the form (3.1) becomes

$$(\Psi, \Phi)_0 = \int_{\mathbb{R}^{m+1}} dx^0 \cdots dx^m\, \Psi(x^0, x^0, x^2, \ldots, x^m)\overline{\Phi(x^1, x^1, x^2, \ldots, x^m)}. \tag{4.5}$$

In particular, $(\Psi, \Psi)_0 = \int dx^2 \cdots dx^m \,|\int dx\, \Psi(x, x, x^2, \ldots, x^m)|^2$, so that $(\cdot, \cdot)_0$ is positive semi-definite, as it should be by Proposition 13. The closure $\mathcal{H}^{\mathrm{id}}$ of the quotient of $L \otimes \mathcal{H}_{\mathrm{id}}$ by the null space $\mathcal{H}_0$ of $(\cdot, \cdot)_0$ is naturally realized as $L^2(\mathbb{R}^{m-1})$: if we define $U : L \otimes \mathcal{H}_{\mathrm{id}} \to L^2(\mathbb{R}^{m-1})$ by $(U\Psi)(x^2, \ldots, x^m) = \int dx\, \Psi(x, x, x^2, \ldots, x^m)$ then U exactly annihilates $\mathcal{H}_0$, and quotients to a unitary map $\tilde{U}$ from $\mathcal{H}^{\mathrm{id}}$ to $L^2(\mathbb{R}^{m-1})$. The corresponding representation $\tilde{U}\pi^{\mathrm{id}}\tilde{U}^{-1}$ of the algebra of observables $\mathcal{A}$ is simply the identity representation of $\mathcal{K}_c(L^2(\mathbb{R}^{m-1}))$ on $L^2(\mathbb{R}^{m-1})$.

This result may be much ado about nothing, but we wish to point out that this extremely simple constrained system cannot be quantized by the BRST method without serious *ad hoc* modifications [25].

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