RADBOUD UNIVERSITY NIJMEGEN

FACULTY OF SCIENCE

Constructivism and the Continuum

A COMPARATIVE STUDY OF INTUITIONISM AND POINTLESS TOPOLOGY

BACHELOR THESIS IN MATHEMATICS

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Introduction

The fact that there are multiple ways to believe in mathematics is something that not everyone is aware of. For a very long time, I was not aware of this either. I simply believed that everyone saw mathematics the same way I did. When I became aware of the several ideas one could - more or less consciously - have about mathematics, I became very interested. How does our view of this science affect its practice?

An important concept where one's view of mathematics is highly relevant is infinity. As a child, my idea of infinity was very basic: I thought of it as the largest number possible. This led to a long project of counting to infinity, which, of course, ended in failure. When I got a bit older, I learned that infinity could never truly be reached. During this period, my understanding of the concept shifted to an infinite *space* or *never-ending line*, since this was the way in which I encountered infinity in daily practice. Later, when I became familiar with Cantor's set theory, I became deeply interested in the idea of *totalities* of infinities. Yet, once I embraced the notion of mathematics as a mental construction, I could no longer accept the idea of a totality of infinity. My experience altered my understanding of mathematics, and with it, the way I practiced it.

For me, mathematics has always been a game of playing with the rules that we came up with by formalizing our intuition. When the intuition develops, so does mathematics. In this thesis, I try to use the intuition I have about the continuum together with my intuition about mathematics, in ways that are more or less formalized. In writing this thesis, it is assumed that the reader is at least at the level of a third-year bachelor Mathematics student. Especially, prior knowledge in logic is recommended.

The topic of this thesis is combining two views of the continuum which are both related to constructivism, an approach in mathematics emphasizing the necessity of constructive proofs and the avoidance of non-constructive methods. The first view is part of intuitionism, a philosophy of mathematics which is a form of constructivism [10]. In this view, the intuition of the continuum as its totality is taken as primal, which results in the interesting mathematical objects of choice sequences. As little as possible is formalized in this intuitionistic theory, staying close to our mathematical intuition. The second view on the continuum is the one of pointless topology. Here, the lattice of open sets is considered instead of the point-set topology itself. Interestingly, this structure can be used to model intuitionistic propositional logic in the form of a Heyting algebra. On top of that, the theory of pointless topology leads to locales, which play an important role in constructive mathematics. Both theories are explained and compared with each other.

The history of intuitionism begins in 1907 when Luitzen Egbertus Jan Brouwer (1881-1966) introduces this philosophy of mathematics in his dissertation [24]. He challenges two major schools of thought, formalism and Platonism, asserting that mathematical objects and proofs are constructions of the human mind. Because of this, intuitionistic logical principles for mathematics need to preserve *constructability* instead of *truth*. This perspective gives rise to a different type of logic, where, for example, the law of the excluded middle ($A \vee \neg A$) does not hold. Brouwer emphasizes that this logic is *descriptive* rather than *creative*. In his view, logic should not be seen as the foundation of mathematics but as a tool to describe it. According to him, mathematical constructions are inherently languageless: language is only used when we *communicate* about mathematics. We can study the linguistic patterns these communications form, which leads to the study of logic. For intuitionists, however, logic is not necessary. This explains why Brouwer never formalized much of intuitionistic logic.

Later, other mathematicians *did* formalize intuitionistic logic. The first to do so was Brouwer's student, Arend Heyting (1898-1980) [24]. In 1928, Heyting publishes an article in which he formalizes, among other things, intuitionistic propositional logic, predicate logic and arithmetic. In the revised version of 1930, he introduces Heyting algebras for the first time. These algebraic structures form the intuitionistic equivalent of Boolean algebras, which were already introduced in 1847 by George Boole. Heyting algebras capture the logical operations within intuitionistic logic, just as Boolean algebras do for classical logic.

The formalization of intuitionism by Heyting was very helpful, since this made intuitionism more accessible and understandable to the mathematical community [23]. There are, namely, some aspects of intuitionism that can be a bit mysterious. One of these mysteries is to find in Brouwer's intuitionistic continuum, for which he based the description on the concept of choice sequences. These objects are essential for the constructive nature of intuitionistic mathematics, since they are created step by step. Choice sequences first appear in 1912 in Brouwer's article "Intuitionism and formalism"[5], where they are described in a vague manner. Most remarkable is the fact that Brouwer admits infinite sequences that are created by free will, which is something that the constructivist would not accept. In the years after that, Brouwer develops the concept, but it remains mysterious for other mathematicians. In 1930, Heyting carefully starts with the formalization [21]. This project has been continued by multiple mathematicians, like Kleene, Kreisel, Troelstra and Moschovakis. Despite these efforts, the concept of choice sequences remains one of the more puzzling aspects of intuitionistic mathematics.

Apart from this, pointless topology was developed. This history begins with topology, which is a result of Hausdorff's act to take open sets as the primitive notion to describe continuity [13]. Already from 1914 it was known that a topological space gives rise to a lattice of open sets, but only in the 1930s this relation became thoroughly researched. This was due to the work of Marshall Stone (1903-1989) on the topological representation of Boolean algebras. Henry Wallman was one of the first to apply lattice theory to topology, creating the "Wallman compactification" of a topological space. A significant shift occurred in the late fifties with Charles Ehresmann's and Jean Bénabou's work, suggesting that a lattice with the right properties should be studied as a generalized topological space. Many topological results were extended to these generalized spaces. By around 1972, John Isbell and André Joyal demonstrate that these generalized spaces could sometimes behave better than traditional topological spaces. After that much work in pointless topology has been done by Peter Johnstone (1948-), for example in proving Tychonoff's theorem constructively for locales.

To my knowledge, the combination of pointless topology and choice sequences has not been made before, in the conceptual sense at least. Brouwer himself, who began his career in topology, never knew of *pointless* topology, as most of the developments occurred after his time. Probably he would not have been interested in it anyway, since it works with classical mathematics. Some work in connecting choice sequences and locales has been done on a more technical level, for example by Van der Hoeven and Moerdijk [9]. For me, the goal is to connect the theories on an elementary and conceptual level.

We start in Chapter 1 by introducing and explaining intuitionism as a philosophy of mathematics. As a part of this, choice sequences are treated, as is the case in the original philosophy. Respecting Brouwer's wishes as much as possible, we try to stay close to our intuition about the mathematical objects and try to formalize the theory as little as possible. In this way, we aim to remain aligned with the original philosophy. Nonetheless, an important technical result for the intuitionistic continuum is proved, namely the fact that every total function on it is continuous.

Chapter 2 covers an introduction into pointless topology in which the relation with intuitionistic logic is specified. We start with introducing lattice theory and we build up to frames and locales. Then, the correspondance between frames and Heyting algebras is shown. On top of that intuitionistic logic is discussed and related to Heyting algebras. At the end of this chapter, the relation between spaces and locales is investigated. In this part, the relationship between pointless topology and constructive mathematics is made explicit.

In the conclusion of this thesis we reflect on our findings by bringing together the intuitionistic and the pointless continuum and discussing the ways in which pointless topology is compatible with intuitionism.

I am very grateful for the help I have received in writing this thesis. First of all, my sincere gratitude to my enthusiastic supervisor, Klaas Landsman, who has encouraged and inspired me from day one. His prompt and thorough responses to my questions, coupled with his dedication and passion, have been invaluable throughout this journey. Secondly, I would like to thank Wim Veldman for his guidance in navigating the philosophy of intuitionism. His insights have been indispensable in shaping my understanding of this subject. Lastly, I extend my thanks to Chris Kooloos, for his proactive assistance and insightful questioning. To all those mentioned, and to anyone else who has guided me in my process, thank you for your support.

Chapter 1

Intuitionism

Intuitionism, a mathematical philosophy introduced by Dutch mathematician L.E.J. Brouwer in the early 20th century, posits that mathematics is a creation of the mind. This perspective carries significant implications for daily mathematical practice. In this chapter, the philosophy and some of the practical consequences of intuitionism are discussed. In the first section of this chapter, we introduce Brouwer's view on mathematics and use this to explain a difference between classical and intuitionistic logic. In the second section, we discuss another way in which intuitionism strongly deviates from classical mathematics: the conception of the continuum. Our main references for this discussion are Veldman [25] and Iemhoff [10].

1.1 Mathematics as a mental creation

The first act of intuitionism, as described in Brouwer [6], is separating mathematics from mathematical language. In this sense, he rejects formalism, a philosophy of mathematics in which it is believed that mathematics should be seen as a game of manipulating symbols according to strict rules. In the mathematics of the formalist, the goal is *correctness* instead of *truth*. Brouwer saw this differently. He wanted mathematical statements to be *meaningful* and to recognize our *conceptual* experience of truth, so he placed the source of mathematical exactness in the human intellect instead of on paper.

To be precise, he viewed mathematics as a languageless activity of human mind, which is based on our perception of time. We break down this description.

Brouwer sees mathematics as an *activity* of the *human mind*. This means, among other things, that one has to *construct* mathematical objects in order for them to 'exist'. According to Brouwer, this construction is done in an idealized mind: *The Creating Subject*. By attributing creation to this abstract entity the influence of individual biases and imperfections is minimized. The truth of mathematical statements is to be thought of as a mental construction that proves it to be true.

By saying that the activity is *languageless*, Brouwer means that the construction itself does not use any language when it takes place in the mind. Only when we want to remember our own ideas or communicate it to others, we have to translate the construction to written or spoken language.

Brouwer's description of mathematics also focuses on our experience of *time*. To be exact, he finds the origin of the natural numbers in the perception of a move in time: first we have one thing, then we have another, then a third thing, then again another... Brouwer connects this to the construction of the natural numbers by seeing the number 1 as the moment we have the first thing, then we have another one, so 2 in total, then we have 3, 4, etcetera. Important in this conception of the natural numbers is that it creates a *potential* infinity, one that is never finished. In intuitionism, all kinds of infinity should be considered as an ongoing process, since at one point in time we can never have taken enough steps to have finished an infinite construction.

This view of mathematics results in some important differences compared to everyday mathematics. An important axiom of classical propositional logic is '$A \vee \neg A$'. This represents the idea that a mathematical proposition is always classifiable as either true or false, that is, we have a *two-valued* logic. In this case, *false* is equivalent to *not true*. This rule also goes by the name of *the law of the excluded middle* and it is rejected by the intuitionist. We can understand this by further analyzing the intuitionistic conception of mathematical truth.

The essential aspect about mathematical constructions is *time*. At one point in time, not everything that *can* be constructed *has* yet been constructed, so we cannot yet determine the truth-status of all mathematical statements. It may even be the case that statements become provable and therefore valid in the course of time, while this may not have been the case before. We look at an example. Consider the real number π, which has an infinite decimal development. We cannot discover any pattern in this decimal development, so we are not able to know all of its decimal places. We may wonder: is there a string of ninety-nine nines in this decimal development? When someone has found this string in the decimal development, we can answer this question by: yes. But if we have not found this yet, there is nothing to say about the truth of the statement $A :=$ 'There is a string of ninety-nine nines in the decimal development of π'. Since at any point in time only finitely many decimals of π have been developed it may be the case that in the future the string of ninety-nine nines will appear. However, we cannot be sure that this will happen.

Classical mathematicians would not see the problem with this, since they have a different idea of truth from the intuitionists. If one believes that mathematics exists independently of us and of our language, as is the case in the classical philosophy of mathematics, truths have to be *discovered* instead of *invented*. The classical mathematician would therefore argue that the truth of either 'A' or '$\neg A$' is predetermined, we only have to find out which one of the two options is the case. It does not matter that we might not know which one is true: since either 'A' or '$\neg A$' is the case anyhow, we have that '$A \vee \neg A$' is (classically) true. However, since in intuitionism mathematics is a product of the human mind, mathematical objects do not have an existence independently of us and mathematical truths have to be constructed in the form of proofs. The intuitionist interprets '$A \vee \neg A$' as: we either have a proof for 'A' or a proof for '$\neg A$'. This view is discussed in more detail in Section 2.3. The crucial aspect here is that a mathematical assertion is considered true by intuitionists only if there is a constructive proof for it. Consequently, in the absence of a proof for either, neither 'A' nor '$\neg A$' is considered true. Therefore, why would '$A \vee \neg A$' be?

For this reason, the law of the excluded middle is rejected by the intuitionists. They believe that there is in fact a *third* possibility, namely the one where neither is true (yet). This means that the intuitionist cannot use '$A \vee \neg A$' as a law in their mathematical reasoning, which creates a difference between classical and intuitionistic logic. We will return to this in the chapter on pointless topology.

Intuitionism is a form of constructivism, the philosophy of mathematics in which the construction of a mathematical object is seen as necessary in order to prove its existence [4]. It is easy to see that this matches the philosophy of intuitionism. In constructive mathematics all non-constructive principles, such as the axiom of choice, are rejected. But, where most constructive theories of mathematics are a *restriction* of classical mathematics, intuitionism *contradicts* it. We see this by turning to the intuitionistic conception of the continuum.

1.2 The intuitionistic continuum

The classical continuum contains an uncountable infinity of infinite objects, such as π and $\sqrt{2}$. From an intuitionistic point of view, infinite structures are always thought of as *constructions* in progress, rather than something that can ever be (let alone have been) completed. While the individual real numbers are easy to define in a constructive manner, for example by using potentially infinite converging sequences of rational numbers, one loses the intuition of the totality of real numbers in this way [2]. The idea of the total continuum should be taken as primary and therefore, the intuitionistic continuum should be considered in this light. Treating the continuum in its totality results in the theory of *choice sequences*, which is an interesting and remarkable part of intuitionism.

Choice sequences were introduced by Brouwer himself to describe the continuum. A **choice sequence** is an infinite sequence created in time via successive choices of new elements. These elements can be chosen from any already constructed domain of mathematical objects, for example from the natural numbers $\mathbb{N}$.

A choice sequence may be given by an *algorithm* or equivalently by a *finite description*. In that case, we speak of a **lawlike** choice sequence. A lawlike choice sequence is predetermined. For example, $\forall n \in \mathbb{N}\ a_{n+1} = a_n + 1$ with $a_0 = 0$ defines a lawlike sequence $\{a_n\}_{n\in\mathbb{N}}$, which corresponds (by definition) to the natural numbers $\mathbb{N}$.

Brouwer also recognized that there has to be room for real numbers that are not given by any algorithm, since otherwise the continuum would be countable and hence discrete. Therefore, he admits choice sequences that are **freely proceeding**. The choices of the elements are made *freely* and *step-by-step*. This means firstly that the values of a freely proceeding choice sequence are not subject to any laws and secondly that at any point in time only a finite initial segment of the choice sequence is known. Later, Kreisel gave choice sequences for which this is the case the appropriate name of **lawless** choice sequences [21]. We say that at the k-th stage of the construction of a sequence the first k values have been determined.

A typical example of a lawless choice sequence is rolling a die an infinite number of times [21]. The values of the choice sequence can be chosen from the set $\{1, 2, 3, 4, 5, 6\}$. The 5-th stage could for example look like: 2 4 3 4 1. At any point in the sequence we cannot predict the outcome of the sequence, since we cannot predict the outcome of a throw.

We have to acknowledge that there are in fact laws in probability theory to which the row of a number of die throws is subject, for example the law of large numbers. Also, theoretically, if one knew every detail about the physical conditions, one could predict the outcome of the die throws. However, in the practical context of our everyday thought experiment, all outcomes are equally likely and we cannot know them in advance. While we may know that some number *has* to appear, we never know with certainty *when* this will be. In practice, it means that we can only know a finite initial segment of this forever unfinished object, so that it creates a lawless sequence. We can also envision a lawless sequence as if a person were randomly selecting values, yet achieving true randomness is notably difficult. Nonetheless, we hope that the idea of a lawless choice sequence in the sense of freely made step-by-step choices is clear.

Of course, there are also possible scenarios where we feel like a sequence falls in between lawless and lawlike, such as a choice sequence on which restrictions on future possibilities are imposed, but where the choices are made freely otherwise. For example, imagine that we are constructing a choice sequence consisting of natural numbers. At a certain stage j we decide that we can only choose even numbers in the future. Now, our sequence is still lawless, since we only have a more limited domain. However, when at a certain stage k we decide that we only will choose prime numbers from now on, our only possibility left is choosing the number 2. At this moment, it looks like our sequence will continue as lawlike. But then, in the l-th stage, we suddenly abolish the restriction of only being able to choose prime numbers. Now we have options again. We have to keep in mind that the Creating Subject does not know which restrictions are imposed or abolished in the future, so we can still not predict the outcome of the sequence. Characteristics like these make choice sequences mysterious objects to many mathematicians.

This points us to something fundamental to the nature of lawless choice sequences: since we can always only know a finite initial segment, we can never know if a lawless sequence will coincide with a lawlike one. As defined by Brouwer, two choice sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are **equal** if $a_n = b_n$ for each $n \in \mathbb{N}$ and **distinct** if there is a natural number m such that $a_m \neq b_m$. In general, checking the equality for all the terms is done step by step. Only when two sequences are given by an algorithm, we can check the equality in a finite amount of time. This means that for a lawless choice sequence α, we can never determine equality to another choice sequence, since we can never have checked the equality of all the terms. This leads to the fact that for choice sequences the law of the excluded middle does not hold. To understand this, we look at an example.

Imagine we are creating a lawless choice sequence α and at one point in time we wonder: does this choice sequence equal the lawlike choice sequence 0, that consists only of zero's? If the finite initial segment of α that we have determined until now only consists of zero's, we have nothing to contradict our suspicion. But, at no moment in time we will be *sure* that they equal each other, since we will never have checked the equality of all the terms of the sequence. Only by finding a term that does not equal zero we can prove something, namely that they are not equal. But again, we cannot be sure that this happens. This means that we are not able to prove either $\alpha = 0$ or $\alpha \neq 0$ for all lawless choice sequences α, so the statement $\forall\alpha(\alpha = 0 \vee \alpha \neq 0)$ is not valid.

In general, at any time we only know an initial segment of a lawless choice sequence α. We denote the initial segment as $\bar{\alpha}n =< \alpha_0, \alpha_1, ..., \alpha_{n-1} >$. Suppose we have a one-place predicate P, also called a *property*, and suppose we have established the truth of $P(\alpha)$ for some lawless choice sequence α. We assume that P only refers to the values of the choice sequence α. Then, since for lawless sequences we can know nothing more than a finite initial segment, the truth of $P(\alpha)$ can only be based upon a finite segment. This results in the fact that every lawless sequence with the same initial segment ought to have the same property. Therefore, the following principle holds for all lawless choice sequences [22]:

The principle of open data: If P is a property of lawless choice sequences, and $P(\alpha)$, then there is an initial segment $\bar{\alpha}n$ such that for every lawless choice sequence β we have

$$\forall m < n(\alpha_m = \beta_m) \to P(\beta).$$

An example of a property $P(x)$ of lawless choice sequences x is $P(x) \coloneqq x \neq 0$. If we have established $P(\alpha)$ for a lawless choice sequence α, this means we have found $\bar{\alpha}k$ such that $\alpha_{k-1} \neq 0$. Then for any lawless choice sequence β with the same initial segment we know $P(\beta)$.

We can also describe a very similar principle, which relies on the same idea [25]. Here, we denote $\mathcal{N}$ as the 'set' of all choice sequences of natural numbers, which makes up the *full continuum.* We define $\sqsubset$ as follows: for all finite sequences of natural numbers s, for all $\alpha \in \mathcal{N}$, $s \sqsubset \alpha \iff \exists n[\bar{\alpha}n = s]$.

Brouwer's Continuity Principle (BCP): for every $R \subseteq \mathcal{N} \times \mathbb{N}$ if $\forall \alpha \in \mathcal{N} \exists n \in \mathbb{N}\ [\alpha R n]$ then

$$\forall \alpha \in \mathcal{N} \exists m \in \mathbb{N} \exists n \in \mathbb{N} \forall \beta \in \mathcal{N}\ [\bar{\alpha}m \sqsubset \beta \to \beta R n].$$

This principle can be justified intuitively in the same way as the principle of open data for lawless choice sequences. Important is that we discuss a relation on *all* choice sequences here, not just lawless ones. However, since we encounter many lawless choice sequences in the 'set' containing all choice sequences, in general we can only know a finite initial part. Using this principle, we can prove that every function $\mathbb{R} \to \mathbb{R}$ is continuous. In this sense, intuitionistic mathematics contradicts classical mathematics.

First, we define the real numbers $\mathbb{R}$ in an intuitionistically acceptable way, which we can subsequently relate to our choice sequences. Here, Veldman [25] is followed almost *verbatim.*

Definition 1.2.1. *A* ***real number*** *is an approximation process, an infinite sequence* $x = x(0), x(1), x(2), ..$ *of pairs* $x(n) = (x'(n), x''(n))$ *of rationals such that*

i) x is shrinking: for all n, $x'(n) \leq x'(n+1) \leq x''(n+1) \leq x''(n)$, and,

ii) x is dwindling: for all m, there exists n such that $x''(n) - x'(n) \leq \frac{1}{2^m}$.

We define $\mathbb{R}$ as the set of all real numbers.

The two conditions for the sequence of intervals tell us that every new interval should lay in the former one(s) and that the intervals become arbitrary small. Remember that we should consider the infinite sequence in the usual intuitionistic way: as an ongoing construction that is never finished.

In the next definition, we define how to compare real numbers with each other.

Definition 1.2.2. *For all real numbers x, y we define*

- *$x <_{\mathbb{R}} y$ if and only if there exists n such that $x''(n) <_{\mathbb{Q}} y'(n)$;*
- *$x \leq_{\mathbb{R}} y$ if and only if for all n we have $x'(n) \leq_{\mathbb{Q}} y''(n)$;*
- *$x \#_{\mathbb{R}} y$ if and only if either $x <_{\mathbb{R}} y$ or $y <_{\mathbb{R}} x$;*
- *$x =_{\mathbb{R}}$ if and only if $x \leq_{\mathbb{R}} y$ and $y \leq_{\mathbb{R}} x$.*

Apart from the set $\mathbb{R}$ of real numbers, we can also consider the set of all choice sequences of natural numbers, denoted by $\mathcal{N}$. Moreover, we can define a function that sends every choice sequence $\alpha \in \mathcal{N}$ to a real number u_α. This is done in the following definition. Here, $s * < n >$ refers to appending the natural number n to the end of the finite sequence s of natural numbers of length m, resulting in a finite sequence of natural numbers of length $m+1$.

Definition 1.2.3. *Let $(r'_0, r''_0), (r'_1, r''_1), (r'_2, r''_2), \ldots$ be a fixed enumeration of all pairs of rationals. We define a mapping associating to each sequence $s \neq 0$ a pair (u'_s, u''_s) of rationals as follows:*

- *For each n, if $0 < r''_n - r'_n \leq 1$, then $(u'_{<n>}, u''_{<n>}) = (r'_n, r''_n)$. If not, $(u'_{<n>}, u''_{<n>}) = (0, 1)$.*
- *For each $s \neq 0$, for each n, if $u'_s < r'_n < r''_n < u''_s$ and $0 < r''_n - r'_n \leq \frac{1}{2}(u''_s - u'_s)$, then $(u'_{s*<n>}, u''_{s*<n>}) = (r'_n, r''_n)$. If not, $(u'_{s*<n>}, u''_{s*<n>}) = (\frac{2}{3}u'_s + \frac{1}{3}u''_s, \frac{1}{3}u'_s + \frac{2}{3}u''_s)$.*

Then, for each α in $\mathcal{N}$ we let u_α be the infinite sequence of pairs of rationals such that for all n,

$$u_\alpha = (u'_{\bar{\alpha}(n+1)}, u''_{\bar{\alpha}(n+1)}) = (u'_{<\alpha_0, \ldots, \alpha_n>}, u''_{<\alpha_0, \ldots, \alpha_n>}).$$

This construction can be understood as follows. The first condition in the definition of u_α tells us that the first rational interval in the approximation that is u_α should not be "too big", that is, at most of length 1. We choose the first interval in a way such that this is the case. Then, the second condition ensures that every new interval is chosen in a way such that it lays inside the former one and such that its length is at most half of the length of the former interval. This guarantees that any choice sequence leads to an approximation of intervals with rational endpoints that is shrinking and dwindling. This is made exact in the following lemma. Here, we show that this construction defines a real number and that every real number can be described in this way.

Lemma 1.2.4. *Let u_α be defined as above. Then:*

i) For every $\alpha \in \mathcal{N}$, u_α is a real number.

ii) For every real number x there exists $\alpha \in \mathcal{N}$ such that $x =_\mathbb{R} u_\alpha$.

Proof. i) By definition, u_α is an infinite sequence of pairs of rationals. We have to check that for each α, u_α is *shrinking* and *dwindling*. The first condition in the definition of u_α is not very relevant for this proof, since it only gives a upper bound for the length of the first rational interval that defines u_α. By looking into the second condition, we see that this infinite sequence indeed defines a real number. For every $s \neq 0$ and n, we have two options.

If we have taken $(u'_{s*<n>}, u''_{s*<n>}) = (r'_n, r''_n)$ then we are in the situation where

$$u'_s < r'_n < r''_n < u''_s \text{ and } 0 < r''_n - r'_n \leq \frac{1}{2}(u''_s - u'_s)$$

by definition of u_α. Then the length of the new interval is at most half the length of the former interval and the new interval lays inside the former. This shows that the sequence is shrinking as well as dwindling.

In the case where $(u'_{s*<n>}, u''_{s*<n>}) = (\frac{2}{3}u'_s + \frac{1}{3}u''_s, \frac{1}{3}u'_s + \frac{2}{3}u''_s)$ it is clear that

$$u'_s < u'_{s*<n>} < u''_{s*<n>} < u''_s,$$

so u_α is shrinking. To see that it is also dwindling, notice that

$$u''_{s*<n>} - u'_{s*<n>} = \frac{1}{3}u'_s + \frac{2}{3}u''_s - (\frac{2}{3}u'_s + \frac{1}{3}u''_s) = -\frac{1}{3}u'_s + \frac{1}{3}u''_s = \frac{1}{3}(u''_s - u'_s) \leq \frac{1}{2}(u''_s - u'_s).$$

Again, this shows that the length of the new interval is less than or equal to half the length of the former. This shows that also in this case a dwindling sequence is created. Therefore, for all α our u_α is a real number.

ii) Now we show that for each real number x we can find an $\alpha \in \mathcal{N}$ such that $x =_{\mathbb{R}} u_\alpha$. Let $x = x(0), x(1), x(2), ...$ be given. First, we have to take the first m such that $x''(m) - x'(m) \leq 1$. This m exists since x is dwindling. Define $\alpha(0) = m$, so that

$$(u'_{<\alpha(0)>}, u''_{<\alpha(0)>}) = (x'(m), x''(m)).$$

Now, take the first $k > m$ such that

$$x''(k) - x'(k) \leq \frac{1}{2}(x''(m) - x'(m)).$$

Set $\alpha(1) = k$. Continue in this way: for each n find the first $p > \alpha(n)$ such that

$$x''(p) - x'(p) \leq \frac{1}{2}(x''(\alpha(n)) - x''(\alpha(n)))$$

and set $\alpha(n+1) = p$. This is always possible, since the real number x is dwindling. Our u_α automatically meets the condition of shrinking, since x meets this condition as well. Now, we have constructed α in a way that $x =_{\mathbb{R}} u_\alpha$. This is possible for every $x \in \mathbb{R}$. □

We see that we can use choice sequences to describe real numbers and vice versa. Although the definition in ii) specifies the choice sequence using an algorithm, we still retain our 'freedom' because we can use *any* real number to define the choice sequence. For instance, a real number that cannot fully be described by a finite algorithm would correspond to a lawless choice sequence. This is because it becomes impossible to predict the outcome of the corresponding choice sequence beyond a finite initial segment. If we can only know a finite initial segment of the real number itself, then we can also only know a finite initial segment of the choice sequence.

Next, we define what it means to be pointwise continuous for a real function. The quantifiers are to be interpreted intuitionistically, so in a constructive way.

Definition 1.2.5. *Let f be a function from $X \subseteq \mathbb{R}$ to $\mathbb{R}$ and let $x \in X$ be given. Then f is continuous at x if and only if*

$$\forall p \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall y \in X[|y - x| < \frac{1}{2m} \to |f(y) - f(x)| < \frac{1}{2p}].$$

A real function f is ***pointwise continuous*** *if and only if f is continuous at every x in X.*

The constructive part here is that given any $p \in \mathbb{N}$ one should be able to *construct* a $m \in \mathbb{N}$ that matches the rest of the conditions. In this definition, we should read $|y - x|$ as the (positive) difference between real numbers x and y. We can prove now that every total function on the intuitionistic continuum as defined by Brouwer is continuous at every point. Here, *total* means that the function is defined and thus computable for every real number $x \in \mathbb{R}$.

Theorem 1.2.6. ***BCP*** $\Longrightarrow$ *Every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous at every point.*

Proof. Let f be a total real function and let a real x be given. We shall prove that f is continuous at x. Let p be given, then we have to find m such that, for every real y,

$$\text{if } |y - x| < \frac{1}{2m} \text{ then } |f(y) - f(x)| < \frac{1}{2p}.$$

By using Lemma 1.2.4. we can find α such that $x =_{\mathbb{R}} u_\alpha$. Also, we have

$$\forall \alpha \exists n [r'_n < f(u_\alpha) < r''_n \text{ and } r''_n - r'_n < \frac{1}{2p}],$$

where $(r'_0, r''_0), (r'_1, r''_1), (r'_2, r''_2), \dots$ is the fixed enumeration of all pairs of rationals as before. By applying BCP we find q, n such that

$$\forall \beta [\bar{\alpha} q \sqsubset \beta \to r'_n < f(u_\beta) < r''_n \text{ and } r''_n - r'_n < \frac{1}{2p}].$$

Find i, j such that $u_\alpha(q-1) = (r'_i, r''_i)$ and $u_\alpha(q) = (r'_j, r''_j)$. Note that

$$r'_i <_{\mathbb{Q}} r'_j <_{\mathbb{Q}} r''_j <_{\mathbb{Q}} r''_i.$$

Now, find m such that $\frac{1}{2m} < \min_{\mathbb{Q}}(r'_j -_{\mathbb{Q}} r'_i, r''_i -_{\mathbb{Q}} r''_j)$. We have that for every real y, if

$$|y - x| < \frac{1}{2m}, \text{ then } r'_i < y < r''_i.$$

This means that there exists β such that $\bar{\alpha} q \sqsubset \beta$ and $y =_{\mathbb{R}} u_\beta$ and, therefore: $r'_n < f(u_\beta) < r''_n$ and $r'_n < f(y) < r''_n$. This means that we can conclude that for every real y, if

$$|y - x| < \frac{1}{2m} \text{ then } |f(y) - f(x)| < r''_n - r'_n < \frac{1}{2p}.$$

Since this works for any real x, we have that f is continuous at every point. □

We take a closer look at what happens here. The crucial argument is that the value of the corresponding real number for any choice sequence with the same initial segment as a choice sequence α must lie within the same rational interval as $f(u_\alpha)$. This is a direct result of Brouwer's continuity principle. We choose our m in a way such that all the choice sequences β with $y =_{\mathbb{R}} u_\beta$ and $|y - x| < \frac{1}{2m}$ have the same relevant initial segment as α. Then, since the length of the interval in which $f(u_\alpha)$ is contained is smaller than $\frac{1}{2p}$, which can be chosen like this since the approximation of $f(u_\alpha)$ is dwindling, we have that the distance between $f(u_\alpha)$ and $f(u_\beta)$ is smaller than $\frac{1}{2p}$.

This theorem is one of the most remarkable results of intuitionistic mathematics. Here, we also see how intuitionistic mathematics *contradicts* classical mathematics, which shows that it is more extreme than other types of constructive mathematics, for example as a view of the continuum. By accepting choice sequences as a way of describing the continuum we have to acknowledge the fact that we only have a finite amount of information to our disposal. In essence, this means that our intuitionistic continuum consists of real numbers with an 'open ending': we can never know the exact description of a real number, since it will always stay unfinished.

In the next chapter, we discuss another theory which closely relates to intuitionism and constructivism as well. This leads to a different view of the continuum: a pointfree one.

Chapter 2

Pointless topology

In the way topology is traditionally done, a *set* is given, on which a *topology* is defined. Together this forms a *topological space.* From the definition of a topology, it becomes clear that any topological space gives rise to a complete lattice of open subsets [13]."Pointless topology" works the other way around: it is the study of topology where lattices of open sets are taken as the primitive notion. While the structure of these lattices is interesting in itself, it also makes it possible to introduce topologies in a new way, namely by starting with such a lattice. This enables the mathematician to view topologies in a more general, algebraic way; one that avoids mentioning points, i.e. the elements in the underlying set. This could be interesting as a *pointless* vision on $\mathbb{R}$. When we think about it, it is in fact not quite that hard to match this attitude with our intuition on the real numbers. As Picardo and Pultr put it: "the space (in our case the line) is a conglomerate of realistic places, spots of non-trivial extent, somehow related with each other; the points are just abstractions of "centers" of diminishing systems of spots." [19] We will see that for our every-day spaces no information is lost by considering these *realistic* spots instead of the points, so we can choose to work in a point-free way. This strategy turns out to have interesting results. For example, from this point-free construction we obtain Heyting algebras, which are the intuitionistic counterparts of Boolean algebras. Moreover, in pointless topology we can prove analogues of theorems which are classically equivalent to their constructive versions. This is interesting to the constructivist.

In this chapter, we show how pointless topology gives rise to intuitionistic logic and how this relates to the continuum. In Section 2.1, we discuss the preliminaries, such as lattices, frames and locales. In this section, we also discuss how any topology leads to a lattice of open sets and how this connects with the theory of frames and locales. Then, in Section 2.2, we establish the correlation to Heyting algebras. In Section 2.3, we dive into intuitionistic logic, connecting this with the structure of Heyting algebras. In Section 2.4, we discuss the relation between topological spaces and locales. Here, we also consider the position of pointfree topology within constructive mathematics.

2.1 Lattices, frames and locales

We can define a lattice both as a partial ordered set or as an algebraic structure. We start with the first. The theory and proofs for the next part are taken from Davey and Priestley [7].

Definition 2.1.1. *Let P be a set. An* **order** *(or* **partial order***) on P is a binairy relation $\leq$*

on P such that, for all $x, y, z \in P$,

(i) $x \leq x$ (reflexivity),

(ii) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry),

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity).

In this case, $(P, \leq)$ is said to be a ***(partially) ordered set*** *(or* ***poset****).*

Example 2.1.2. An example of a poset is the set of natural numbers $\mathbb{N}$ with the usual order $\leq$.

Example 2.1.3. The integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$ with the usual order $\leq$ all form a poset as well.

Example 2.1.4. For any set X, the power set $\mathcal{P}(X)$ ordered by inclusion is an ordered set. We will write $(\mathcal{P}(X), \subseteq)$.

To be able to introduce lattices, we have to define the operations *join* and *meet*. We will see that these respectively generalize the already familiar *supremum* and *infimum*. Before we do this, we need some other important definitions.

Definition 2.1.5. *Let $(P, \leq)$ be a poset and let $S \subseteq P$. If there exists an element $x \in P$ such that $s \leq x$ for all $s \in S$, then x is called an* ***upper bound*** *of S. The set of upper bounds is denoted by S^u:*

$$S^u := \{x \in P \mid (\forall s \in S)\ s \leq x\}.$$

Example 2.1.6. For example, the subset $\{2, 3, 4\}$ of $\mathbb{N}$ has upper bound 4 (in relation to the usual order $\leq$), but also all the numbers higher than 4, so in this case, $\{2, 3, 4\}^u = \{4, 5, 6, ...\}$.

Example 2.1.7. When we take the set

$$A = \{x \in \mathbb{Q} \mid x^2 < 2\} \subseteq \mathbb{Q}$$

it is clearly bounded from above by for example $2 \in \mathbb{Q}$

However, there are many examples where the upper bound of a subset S does not exist, i.e. $S^u = \varnothing$.

Example 2.1.8. Consider the subset $T = \{4, 5, 6, ...\}$ of $\mathbb{N}$. Since T is clearly not bounded from above, we see $T^u = \varnothing$.

Now, we define a *lower* bound:

Definition 2.1.9. *Let $(P, \leq)$ be an poset and let $S \subseteq P$. If there exists an element $y \in P$ such that $y \leq s$ for all $s \in S$, then y is called a* ***lower bound*** *of S. The set of lower bounds is denoted by S^l:*

$$S^l := \{y \in P \mid (\forall s \in S)\ y \leq s\}.$$

Example 2.1.10. The subset $\{2, 3, 4\}$ of $\mathbb{N}$ has lower bound 2 (in relation to the usual order $\leq$), and all the numbers lower than 2 are also lower bounds, so $\{2, 3, 4\}^l = \{0, 1, 2\}$.

Example 2.1.11. The subset $\{..., -2, -1, 0\}$ of $\mathbb{Z}$ does not have a lower bound when we consider the poset $(\mathbb{Z}, \leq)$.

Definition 2.1.12. *Given any poset $(P, \leq)$ we can form a new ordered set $(P^\partial, \leq)$ (the* ***dual*** *of P) by defining $x \leq y$ to hold in P^∂ if and only if $y \leq x$ holds in P.*

We quickly check that this indeed defines a partial order on P^∂.

(i) (Reflexivity) Let $x \in P^\partial$, then $x \leq x$ since $(P, \leq)$ is a partial order.

(ii) (Anti-symmetry) Let $x, y \in P^\partial$ such that $x \leq y$ and $y \leq x$. Since $(P, \leq)$ is a poset we thus have that $y \leq x$ and $x \leq y$ in P, so $x = y$.

(iii) (Transitivity) Let $x, y, z \in P^\partial$ such that $x \leq y$ and $y \leq z$. Then $y \leq x$ and $z \leq y$ in P, so $z \leq x$ in P. By definition we have that $x \leq z$ in P^∂.

So $(P^\partial, \leq)$ is a poset.

Remark 2.1.13. For any statement about the ordered set P there corresponds a statement about the ordered set P^∂. For example, an *upper bound* in P is a *lower bound* in P^∂ and vice versa.

Definition 2.1.14. *Let $(P, \leq)$ be a poset and $S \subseteq P$. If there exists $a \in P$ such that*

1. *For each $s \in S$, $s \leq a$,*
2. *For any t such that $s \leq t$ for each $s \in S$, we have $a \leq t$,*

then we say that this $a \in P$ is a ***join*** *for S, denoted by $a = \bigvee S$.*

We can alternatively refer to this as the *lowest upper bound* or the *supremum* for S. For pairs, we will write $x \vee y$ instead of $\bigvee\{x, y\}$.

Example 2.1.15. In the example of $\{2, 3, 4\}$ it is clear that $\bigvee\{2, 3, 4\} = 4$, where $\leq$ is again the usual order on $\mathbb{N}$.

Not for every set the join exists.

Example 2.1.16. For the poset $(\mathbb{N}, \leq)$ and the subset $S = \{1, 2, 3, ...\}$, the join does not exist. The reason for this is that there is no upper bound of S, that is, $S^u = \varnothing$.

There are also examples of subsets which have an upper bound, but no *lowest* upper bound.

Example 2.1.17. Let us consider Example 2.1.7. again. It is easy to see that $\sqrt{2}$ is the *lowest* upper bound for A, but we have $\sqrt{2} \notin \mathbb{Q}$. However, since $\sqrt{2} \in \mathbb{R}$, the join for A exists in $(\mathbb{R}, \leq)$.

Dually, we can define the *meet*:

Definition 2.1.18. *Let $(P, \leq)$ be a poset and $S \subseteq P$. If there exists $b \in P$ such that*

1. *For each $s \in S$, $b \leq s$,*
2. *For any t such that $t \leq s$ for each $s \in S$, we have $t \leq b$,*

then we say that this $b \in P$ is a ***meet*** *for S, denoted by $b = \bigwedge S$.*

We can alternatively refer to this as the *greatest lower bound* or the *infimum* for S. For pairs, we will write $x \wedge y$ instead of $\bigwedge\{x, y\}$.

Example 2.1.19. We can easily see that $\bigwedge\{2, 3, 4\} = 2$ with the help of Example 2.1.10.

The meet of a set also does not necessarily exist.

Example 2.1.20. Consider the poset $(\mathbb{Z}, \leq)$ and the subset $T = \{..., -2, -1, 0\}$. In this case, it is clear that $\bigwedge T$ does not exist, since there is no lower bound for T.

Example 2.1.21. For $(\mathcal{P}(X), \subseteq)$ we can also define the *meet* and *join* in an already familiar way. Take the *intersection* of elements as the meet of a subset and the *union* of elements as the join. It is easy to see that this matches the definitions.

With the definitions of *join* and *meet*, we can now define lattices.

Definition 2.1.22. *Let $(P, \leq)$ be a poset, with P non-empty. If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a* **lattice**. *If $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq P$, then P is called a* ***complete lattice****.*

Notice that a this means that a poset $(P, \leq)$ is a *lattice* if $\bigvee S$ and $\bigwedge S$ exist for all finite subsets S of P and that it is a *complete lattice* if this is also true for infinite subsets S of P. The poset $(\mathbb{N}, \leq)$ is thus a lattice, since the lowest upper bound for any finite subset is defined by its maximum and the greatest lower bound by its minimum. However, it is not a complete lattice, since there exist infinite subsets for which the join does not exist. We have seen this in Example 2.1.16. The lattice $(\mathbb{R}, \leq)$ is also not complete, since it is not bounded. We do, however, know an example of a complete lattice:

Example 2.1.23. The powerset lattice $(\mathcal{P}, \subseteq)$ with *meet* and *join* defined as in Example 2.1.21 is a *complete* lattice.

The following lemma makes it easier for us to check if a poset is a lattice.

Lemma 2.1.24. *Let $(P, \leq)$ be any ordered set. If $x, y \in P$ with $x \leq y$, then $x \vee y = y$ and $x \wedge y = x$*

Proof. We show both statements seperately.

- $(x \vee y = y)$ Let x, y be any elements of P such that $x \leq y$. Then it is clear that both $x \leq y$ and $y \leq y$, so for all $s \in \{x, y\}$ it is true that $s \leq y$. Now, take any $t \in P$ such that $s \leq t$ for all $s \in \{x, y\}$. Since $y \in \{x, y\}$ it follows that $y \leq t$. So y meets both conditions in the definition of the join for $\{x, y\}$, i.e. $x \vee y = y$.

- $(x \wedge y = x)$ Let x, y be any elements of P such that $x \leq y$. Then it is clear that both $x \leq x$ and $x \leq y$, so for all $s \in \{x, y\}$ it is true that $x \leq s$. Now, take any $t \in P$ such that $t \leq s$ for all $s \in \{x, y\}$. Since $x \in \{x, y\}$ it follows that $t \leq x$. So x meets both conditions in the definition of the meet for $\{x, y\}$, i.e. $x \wedge y = x$. □

Corollary 2.1.25. Since $\leq$ is reflexive, we have $x \vee x = x$ and $x \wedge x = x$.

Corollary 2.1.26. To prove that a poset $(P, \leq)$ is a lattice, we only have to check that $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$ that are non-comparable, i.e. all pairs $(x, y) \in P^2$ such that $x \nleq y$ and $y \nleq x$.

We can even use Lemma 2.1.24 to prove a much stronger statement, which we will call **The Connecting Lemma** (or **TCL**):

Lemma 2.1.27 (The Connecting Lemma)**.** *Let L be a lattice and let $x, y \in L$. Then the following are equivalent:*

(i) $x \leq y$;

(ii) $x \vee y = y$;

(iii) $x \wedge y = x$.

Proof. With Lemma 2.1.24, (i) implies (ii) and (iii). Now assume (ii), then $x \vee y = y$ and so y is an upper bound for x, i.e. $x \leq y$. So (ii) implies (i). By assuming (iii), we have $x \wedge y = x$ and so x is a lower bound for y, i.e. $x \leq y$. Now we have that (iii) implies (i). So we have proven that all three statements are equivalent. □

Definition 2.1.28. *Let $(P, \leq)$ be an ordered set. We say P has a bottom element if there exists $\bot \in P$ (called* **bottom***) with the property that $\bot \leq x$ for all $x \in P$. We say P has a top element if there exists $\top \in P$ (called* **top***) with the property that $x \leq \top$ for all $x \in P$.*

Proposition 2.1.29. *When they exist, the bottom and top elements are unique.*

Proof. First we prove that the *bottom* element is unique in the case that it exists. Assume $\bot_1$ and $\bot_2$ are both bottom elements in a poset $(P, \leq)$, so for all $x \in P$ we have $\bot_1 \leq x$ and $\bot_2 \leq x$. Since $\bot_2 \in P$, we have $\bot_1 \leq \bot_2$. Similarly, $\bot_2 \leq \bot_1$. With anti-symmetry, we can conclude that $\bot_1 = \bot_2$, so the bottom element is unique. The proof for the top element is symmetrical to the proof for the bottom element. □

Remark 2.1.30. Except for a *top* and *bottom* element, we can also define *maximal* and *minimal* elements in a lattice. Consider $(P, \leq)$ a poset and $Q \subseteq P$. If there exists $a \in Q$ such that $a \leq x$ and $x \in Q$ imply $a = x$, then a is called a **maximal** element of **Q**. If there exists $b \in Q$ such that $x \leq b$ and $x \in Q$ imply $b = x$, then b is called a **minimal** element of **Q**. A subset Q of P can have multiple maximal and/or minimal elements. The top and bottom elements are examples of respectively maximal and minimal elements. This follows from the definition of the top and bottom element and the anti-symmetry of a partial order.

Definition 2.1.31. *A lattice $(P, \leq)$ is said to be* **bounded** *if it has both a bottom element $\bot$ and a top element $\top$.*

Remark 2.1.32. Every complete lattice $(P, \leq)$ is bounded, since in complete lattices $\bigwedge \varnothing$ and $\bigvee \varnothing$ exist. In this case, we have $\bigwedge \varnothing = \top$ and $\bigvee \varnothing = \bot$, since for any x in the poset and for any $a \in \varnothing$ we have $x \leq a$ and for all $a \in \varnothing$ we have $a \leq x$. This means that any element of the poset is both a lower and an upper bound for the empty set. Then, the greatest lower bound is equal to the top element, so $\bigwedge \varnothing = \top$, and the lowest upper bound is equal to the bottom element, so $\bigvee \varnothing = \bot$.

As mentioned before, we can also view a lattice with operations *join* and *meet* as an algebraic structure. This structure is specified in the following theorem.

Theorem 2.1.33. *Let L be a lattice. Then $\vee$ and $\wedge$ satisfy, for all $a, b, c \in L$,*

(L1) $(a \vee b) \vee c = a \vee (b \vee c)$;

(L1$^\partial$) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$;

(L2) $(a \vee b) = (b \vee a)$;

(L2$^\partial$) $(a \wedge b) = (b \wedge a)$;

(L3) $a \vee a = a$;

(L3$^\partial$) $a \wedge a = a$;

(L4) $a \vee (a \wedge b) = a$;

(L3$^\partial$) $a \wedge (a \vee b) = a$.

Note that by interchanging $\vee$ and $\wedge$ one obtains the dual of a statement about lattices when this statement is phrased in terms of $\vee$ and $\wedge$. This is called the **Duality principle for lattices** [7].

Proof. Because of the *Duality principle for lattices*, it is enough to consider (L1)-(L4). (L3) is exactly Corollary 2.1.25. (L2) is clearly true since in a set the order in which the elements are listed is irrelevant, i.e. $\{a,b\} = \{b,a\}$. (L4) follows from (L2) and from TCL by noticing that $(a \wedge b) \leq a$. To prove (L1) it is enough to prove that $(a \vee b) \vee c = \bigvee\{a,b,c\}$ This is the case if $\{a \vee b, c\}^u = \{a,b,c\}^u$. We have

$$\begin{aligned} d \in \{a,b,c\}^u &\iff d \in \{a,b\}^u \text{ and } c \leq d \\ &\iff (a \vee b) \leq d \text{ and } c \leq d \\ &\iff d \in \{a \vee b, c\}^u. \end{aligned}$$

So we have now shown (L1)-(L4). □

Starting with a non-empty set L equipped with the two operations $\vee$ and $\wedge$ which satisfy the conditions (L1)-(L4$^\partial$) from Theorem 2.1.33, we can also define a lattice. This is shown in the following theorem.

Theorem 2.1.34. *Let L be a non-empty set equipped with two binary relations, $\vee$ and $\wedge$, that satisfy (L1)-(L4$^\partial$) from Theorem 2.1.33. Define $\leq$ on L by $a \leq b$ if and only if $a \vee b = b$. Then,*

(i) $\leq$ is an order relation,

(ii) $(L, \leq)$ is a lattice in which the original operations agree with the induced operations, that is, for all $a, b \in L$,

$$a \vee b = \bigvee\{a,b\} \text{ and } a \wedge b = \bigwedge\{a,b\}.$$

We introduce and prove the following lemma first, since it shows that we can use duality in the proof.

Lemma 2.1.35. *Let L be a non-sempty set equipped with two binairy relations, $\vee$ and $\wedge$, which satisfy (L1)-(L4$^\partial$) from Theorem 2.1.33. Then for all $a, b \in L$ we have*

$$a \vee b = b \text{ if and only if } a \wedge b = a.$$

Proof. Assume $a \vee b = b$. Then

$$\begin{aligned} a &= a \wedge (a \vee b) && (\text{L4}^\partial) \\ &= a \wedge b. && (\text{assumption}) \end{aligned}$$

Conversely, assume $a \wedge b = a$. Then

$$\begin{aligned} b &= b \vee (b \wedge a) && (\text{L4}) \\ &= b \vee (a \wedge b) && (\text{L2}^\partial) \\ &= b \vee a && (\text{assumption}) \\ &= a \vee b. && (\text{L2}) \end{aligned}$$

So now we have $a \vee b = b$ if and only if $a \wedge b = a$ □

We now prove Theorem 2.1.34.

Proof. (i) We show that $\leq$ is an order.

- Reflexivity: follows from (L3).
- Anti-symmetry: follows from (L2), since $a \leq b$ and $b \leq a$ imply $b = a \vee b = b \vee a = a$.
- Transitivity: follows from (L1), since $a \leq b$ and $b \leq c$ imply $a \vee c = a \vee (b \vee c) = (a \vee b) \vee c = b \vee c = c$, so by definition $a \leq c$.

So since $\leq$ as defined in the theorem matches all the properties, it is indeed an order on L.

(ii) To show that $\bigvee\{a,b\} = a \vee b$ in the poset $(L, \leq)$, we have to show that $a \vee b$ is an upper bound for $\{a,b\}$ and that for any upper bound d of $\{a,b\}$ we have $a \vee b \leq d$. We have

$$\begin{aligned}(a \vee b) \in \{a,b\}^u &\iff a \leq a \vee b \text{ and } b \leq a \vee b \\ &\iff a \vee (a \vee b) = a \vee b \text{ and } b \vee (a \vee b) = a \vee b.\end{aligned}$$

We have $a \vee (a \vee b) = (a \vee a) \vee b = a \vee b$ and

$$b \vee (a \vee b) = (b \vee a) \vee b = (a \vee b) \vee b = a \vee (b \vee b) = a \vee b.$$

So $a \vee b$ is an upper bound for $\{a,b\}$. Let d be another upper bound, then $a \leq d$ and $b \leq d$, so $a \vee d = d$ and $b \vee d = d$. Then

$$(a \vee b) \vee d = a \vee (b \vee d) = a \vee d = d,$$

so $(a \vee b) \leq d$. We can conclude that $\bigvee\{a,b\} = a \vee b$. Now, for $\bigwedge\{a,b\} = a \wedge b$, we only have to apply duality (Lemma 2.1.35). $\square$

We can rephrase the top and bottom element of a lattice from an algebraic standpoint. We do this in the following definition.

Definition 2.1.36. *Let L be a lattice. We say that L has a* ***one*** *if there exists $1 \in L$ such that $a = a \wedge 1$ for all $a \in L$. Dually, L is said to have a* ***zero*** *if there exists $0 \in L$ such that $a = a \vee 0$ for all $a \in L$. A lattice possessing 0 and 1 is called* ***bounded****.*

Remark 2.1.37. By applying TCL, we can easily see that the lattice $(L, \vee, \wedge)$ has a one 1 if and only if $(L, \leq)$ has a top element $\top$. In that case, $1 = \top$. Dually, again by using TCL, we see that $(L, \vee, \wedge)$ has a zero 0 if and only if $(L, \leq)$ has a bottom element $\bot$. In that case, $0 = \bot$.

Remark 2.1.38. A finite lattice is automatically bounded, with $\bigvee L = 1$ and $\bigwedge L = 0$.

For this thesis, we are interested in a certain kind of lattices: distributive ones.

Definition 2.1.39. *Let L be a lattice. L is said to be* ***distributive*** *if it satisfies the* ***distributive law****,*

$$(\forall a, b, c \in L)\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Example 2.1.40. Any powerset lattice $(\mathcal{P}(X), \cup, \cap)$ is distributive, since $\cup$ and $\cap$ are distributive as set-theoretical operations.

Now, frames are introduced. The theory in the next part is mainly based on Picado and Pultr [19] and on Johnstone 1982 [12]. For general lattice theory, we still use Davey and Priestley [7]. It is useful to note that in many sources most of the definitions and results are stated in a categorical language. Since this is not essential for this thesis, the choice was made to formulate it in a way that preliminary knowledge in category theory is not required.

Definition 2.1.41. *A **frame** is a complete lattice L satisfying the infinite distributive law*

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\}$$

for any subset $A \subseteq L$ and any $b \in L$.

We look at an example:

Example 2.1.42. An important example of a frame is the lattice of open subsets of X, denoted by $\Omega(X)$, for a given topology on X. The order on the open subsets of X is given by inclusion $\subseteq$. Let A be a collection of opens of X. We define the join of A to be the *union* of A, that is, $\bigvee A = \cup A$, and the meet of A as the *interior* of the *intersection* of A, that is, $\bigwedge A = (\cap A)^\circ$. Notice that we have to take the interior in the definition for the meet since we only know that a finite intersection of open subsets is open again. Since we want completeness, we also have to be able to take infinite infima. In this way, $\Omega(X)$ is complete as a lattice, since $\bigvee A$ and $\bigwedge A$ exist for any $A \subseteq \Omega(X)$. For the infinite distributive law, notice that

$$(\bigvee A) \wedge b = ((\cup A) \cap b)^\circ = (\cup A) \cap b = \bigcup\{a \cap b \mid a \in A\} =$$
$$\bigcup\{a^\circ \cap b^\circ \mid a \in A\} = \bigcup\{(a \cap b)^\circ \mid a \in A\} = \bigvee\{a \wedge b \mid a \in A\}$$

for any $A \subseteq \Omega(X)$ and any $b \in \Omega(X)$, where we use that for any open subset U we have $U^\circ = U$. Thus we see that $\bigvee$ and $\bigwedge$ satisfy the infinite distributive law. This means that $\Omega(X)$ is a frame.

The maps corresponding to frames are *frame homomorphisms*.

Definition 2.1.43. *Let L, K be lattices. A map $f : L \to K$ is said to be a **homomorphism** (or **lattice homomorphism**) if f is join-preserving and meet-preserving, that is, for all $a, b \in L$,*

$$f(a \vee b) = f(a) \vee f(b) \text{ and } f(a \wedge b) = f(a) \wedge f(b).$$

*A **frame homomorphism** is a map $f : L \to K$ between frames that preserves finite meets and arbitrary joins.*

Let X, Y be two (topological) spaces and let $f : X \to Y$ be a continuous map. Then f^{-1} restricts to a map $\Omega(Y) \to \Omega(X)$, since we know that for f continuous f^{-1} maps open sets to open sets. Clearly, f^{-1} also preserves finite meets and arbitrary joins. This motivates us to introduce **locales**, which are *frames* but where the morphisms go in the opposite direction, so from $\Omega(Y)$ to $\Omega(X)$ where frame homomorphisms go from $\Omega(X)$ to $\Omega(Y)$. These morphisms are called **locale homomorphisms**. Since we are motivated by topology and continuous maps relate to *locale homomorphisms* instead of *frame homomorphisms*, $\Omega(X)$ will therefore often be considered as the *locale* of opens of X instead of the corresponding *frame*. Thus, for any topological space X we call $\Omega(X)$ the **locale of opens** of X. It is important to note that when we are concerned with *objects*, the terms *frame* and *locale* are interchangeable. The difference is only in the morphisms.

There is an interesting connection between frames (or locales) and intuitionism. In the next section we will see that a complete lattice satisfies the infinite distributive law if and only if it is a complete Heyting algebra.

2.2 Heyting algebras

We now define a Heyting algebra. Here, Johnstone 1982 [12] and Mac Lane and Moerdijk [15] are used as sources.

Definition 2.2.1. *A **Heyting algebra** is a bounded lattice H equipped with a binary operation $\rightarrow: HxH \rightarrow H$, called **implication**, such that*

$$x \leq (y \rightarrow z) \text{ if and only if } (x \wedge y) \leq z.$$

*The unary operation $\neg : H \rightarrow H$, called **negation**, is defined by:*

$$\neg x = x \rightarrow 0.$$

*A Heyting algebra is **complete** when it is complete as a lattice.*

For $x \in H$ we say that $\neg x$ is the **pseudo-complement** of x. We now show an important property of Heyting algebras.

Proposition 2.2.2. *A Heyting algebra is distributive.*

Proof. We start by proving $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$. We notice that $(a \wedge b) \leq a$ and $(a \wedge c) \leq a$, so $(a \wedge b) \vee (a \wedge c) \leq a$. Similarly, $(a \wedge b) \leq b \leq (b \vee c)$ and $(a \wedge c) \leq b \leq (b \vee c)$ imply $(a \wedge b) \vee (a \wedge c) \leq b \vee c$. Hence, together we have $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$. Now we prove $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$:

$$\begin{aligned} b \vee c &\leq (a \rightarrow (a \wedge b)) \vee (a \rightarrow (a \wedge c)) \\ &\leq a \rightarrow ((a \wedge b) \vee (a \wedge c)). \end{aligned}$$

So $(b \vee c) \wedge a = a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$. □

Every complete lattice that satisfies the infinite distributivity rule becomes a Heyting algebra. Vice versa, every complete Heyting algebra defines a frame. This shows how these mathematical structures are closely linked. The following two propositions make this concrete.

Proposition 2.2.3. *Let H be a complete Heyting algebra. Then, for every $x, y \in H$,*

$$x \rightarrow y = \bigvee \{z \mid (z \wedge x) \leq y\}.$$

Moreover,

$$(\bigvee A) \wedge x = \bigvee \{a \wedge x \mid a \in A\}$$

for any $x \in H$ and any $A \subseteq H$.

Proof. First, we show $x \rightarrow y = \bigvee \{z \mid (z \wedge x) \leq y\}$. For any $h \in H$, we have

$$\begin{aligned} h \leq x \rightarrow y &\iff h \wedge x \leq y \\ &\iff h \in \{z \mid (z \wedge x) \leq y\} \\ &\iff h \leq \bigvee \{z \mid (z \wedge x) \leq y\}. \end{aligned}$$

For the last equivalence, we use that $z \wedge x \leq y$ and $w \leq z$ imply $w \wedge x \leq y$. This shows that $x \rightarrow y = \bigvee \{z \mid (z \wedge x) \leq y\}$. Now we prove the second equality. For any $y \in H$, we have

$$\begin{aligned} (\bigvee A) \wedge x \leq y &\iff \bigvee A \leq (x \rightarrow y) \\ &\iff a \leq (x \rightarrow y) \text{ for all } a \in A \\ &\iff a \wedge x \leq y \text{ for all } a \in A \\ &\iff \bigvee \{a \wedge x \mid a \in A\} \leq y. \end{aligned}$$

This shows that $(\bigvee A) \wedge x = \bigvee \{a \wedge x \mid a \in A\}$. □

Remark 2.2.4. In the case where the completeness of a Heyting algebra is not ensured, we can still regard $x \to y$ as the greatest element z such that $z \wedge x \leq y$, since $(x \to y) \wedge x \leq y$ if and only if $(x \to y) \leq (x \to y)$ and $z \wedge x \leq y$ has to imply that $z \leq (x \to y)$ for all $z \in H$.

The converse of Proposition 2.2.4 is also true:

Proposition 2.2.5. *Let H be a complete lattice satisfying the infinite distributive law. Then for any $x, y \in H$, defining*

$$x \to y := \bigvee \{z \mid z \wedge x \leq y\}$$

turns H into a complete Heyting algebra.

Proof. We want to show that $h \leq x \to y$ if and only if $h \wedge x \leq y$ for any $h \in H$. By definition this is equivalent to proving that $h \leq \bigvee\{z \mid z \wedge x \leq y\}$ if and only if $h \wedge x \leq y$ for any $h \in H$. So, assume $h \wedge x \leq y$, then $h \in \{z \mid z \wedge x \leq y\}$ so $h \leq \bigvee\{z \mid z \wedge x \leq y\}$. Conversely, assume $h \leq \bigvee\{z \mid z \wedge x \leq y\}$. Then

$$h \wedge x \leq \bigvee\{z \mid z \wedge x \leq y\} \wedge x = x \wedge \bigvee\{z \mid z \wedge x \leq y\} = \bigvee\{x \wedge z \mid z \wedge x \leq y\} \leq \bigvee\{y\} = y.$$

So we have proved the required equivalence. □

With Proposition 2.2.5, we see that $\Omega(X)$ is a complete Heyting algebra. In this way, we have a pointless, algebraic generalization of topological spaces. Notice how we can relate the operations of taking the *pseudo-complement* and *implication* in a Heyting algebra H, as defined in Definition 2.2.1, to the set-theoretical operations in a topological space X.

Let $U \subseteq X$ be open. By the meaning of $\neg U = U \to 0$, using Proposition 2.2.5, $\neg U$ equals the union of all open subsets of X that do not meet U, which is just the interior of the set-theoretic complement of U. Mathematically formulated:

$$\neg U = U \to 0 = U \to \varnothing = \bigcup\{Z \mid (Z \cap U) \subseteq \varnothing\} = (U^c)^\circ = (\overline{U})^c.$$

In other words, $\neg U$ is the set-theoretic complement of the *closure* of U. Then, $\neg\neg U$ equals the interior of the closure of U:

$$\neg\neg U = (\overline{U})^c \to \varnothing = (((\overline{U})^c)^c)^\circ = (\overline{U})^\circ.$$

This means that $\neg\neg U$ may be larger than U, so we can have that $\neg\neg U \neq U$. For example, by taking $U = (0,1) \cup (1,2)$, we see that $\neg\neg U = (0,2) \neq U$.

For the implication, notice that $U \to W$ with $U, W \subseteq X$ open is definable by Proposition 2.2.5 as the union of all the open subsets of X whose intersection with U is a subset of W:

$$U \to W = \bigcup\{Z \mid (Z \cap U) \subseteq W\} = (U^c \cup W)^\circ.$$

We can understand the last equality using our conceptual idea of the supremum: the biggest subset of W is W itself and by taking $Z = (U^c \cup W)$ we have $Z \cap U = (U^c \cup W) \cap U = W$. To assure that we find the biggest *open* set such that the intersection with U is a subset of W, we have to take the interior of Z, so we find $(U^c \cup W)^\circ$.

It is important to note that although complete Heyting algebras and frames are equivalent as *algebraic structures*, they differ in their *morphisms*. We have already seen frame homomorphisms and locale homomorphisms. We now define *Heyting algebra morphisms* [26].

Definition 2.2.6. *Let J, H be Heyting algebras. A* ***Heyting homomorphism*** *is a lattice homomorphism $f : J \to H$ such that for all $a, b \in J$ $f(a \to b) = f(a) \to f(b)$.*

In general, frame homomorphisms do not preserve implication. This means that frames and complete Heyting algebras differ in their homomorphisms. We look at an example where a specific implication, namely taking the pseudo-complement, is not preserved.

Example 2.2.7. Let $U \subseteq \mathbb{R}$ be open such that $\neg\neg U \neq \mathbb{R}$ with $\mathcal{T}$ the standard topology on $\mathbb{R}$. We take $U = (0, \infty)$, then $\neg\neg U = (0, \infty) \neq \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the continuous map

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x & \text{for } x > 0. \end{cases}$$

Notice that $U^c = f^{-1}(\{0\})$. Now, let $\Omega(f) : \mathcal{T} \to \mathcal{T}$ be the induced mapping where $\Omega(f)(V) = f^{-1}(V)$ for each open set V. Then $\Omega(f)$ is a frame homomorphism. Let $W = \mathbb{R} \setminus \{0\}$. Then $\Omega(f)(\neg\neg W) = \Omega(f)(\mathbb{R}) = \mathbb{R}$. However,

$$\neg\neg\Omega(f)(W) = \neg\neg f^{-1}(\mathbb{R} \setminus \{0\}) = \neg\neg(f^{-1}(\{0\}))^c = \neg\neg U \neq \mathbb{R}.$$

Therefore, $\neg\neg\Omega(f)(W) \neq \Omega(f)(\neg\neg W)$.

Heyting algebras were first introduced by Arend Heyting around 1930 to formalize intuitionistic logic [24]. He was a student of L.E.J. Brouwer, the founder of intuitionism. To understand how Heyting algebras were used for this, we first introduce intuitionistic logic.

2.3 Intuitionistic logic

In this section of the thesis, Sørensen and Urzyczyn [20] is followed almost *verbatim.*

Intuitionistic logic is an abstract formal system that reflects the principles of logical reasoning in intuitionistic mathematics. We have seen in the chapter on intuitionism that intuitionistic mathematics diverges from classical mathematics by rejecting the law of the excluded middle. Intuitionistic logic is designed to capture the constructive aspect of mathematics, so it may also be considered as a logical basis of constructive mathematics. It consists of a *language* and a *proof system.*

The *language* of *intuitionistic propositional calculus* **(IPC)** is the same as the one of classical logic and is defined as follows:

Definition 2.3.1. *We assume an infinite set PV of propositional variables and we define the set Φ of formulas as the smallest set such that:*

- *Each propositional variable and the constant $\bot$ are in Φ;*
- *If $\phi, \psi \in \Phi$ then $(\phi \to \psi), (\phi \vee \psi), (\phi \wedge \psi) \in \Phi$.*

While the languages for intuitionistic and classical logic are the same, their logical interpretation is quite different. In classical logic, truth-tables are often used in the interpretation of the logical connectives. In intuitionism, a different approach is used: interpretation of the connectives in terms of their construction.

The Brouwer-Heyting-Kolmogorov Interpretation (BHK-Interpretation) is the standard explanation of the logical symbols $\vee, \wedge, \neg$ and $\to$ in intuitionistic logic. Their interpretation, as described by Troelstra and Van Dalen, is as follows:

(H1) A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.

(H2) A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B (plus the stipulation that we want to regard the proof presented as evidence for $A \vee B$).

(H3) A proof of $A \to B$ is a construction which permits us to transform any proof of A into a proof of B.

(H4) Absurdity $\perp$ (contradiction) has no proof; a proof of $\neg A$ is a construction which transforms any hypothetical proof of A into a proof of a contradiction, i.e. $A \to \perp$.

This is somewhat of an informal description, since words as *construction* and *proof* are quite vague without a formal definition, but it gives us an idea of how the intuitionist would like to see mathematics. Notice that we do not specify what a construction of a propositional variable is. The reason for this is that we only come to know the meaning of a propositional variable when it is replaced by a concrete statement. Only then we can ask about the construction of that statement.

Heyting has proven that none of the logical connectives $\vee, \wedge, \neg$ and $\to$ is definable in terms of the others [1]. In this respect we also see a difference in intuitionistic and classical logic.

We formalize the intuitionistic propositional calculus by defining a proof system, called *natural deduction*, for which the intuitionistic version is denoted by **(NJ)**. The rules make precise the informal semantics as introduced above.

Definition 2.3.2. *Natural deduction is defined as follows:*

(i) A judgement in natural deduction is a pair, denoted $\Gamma \vdash \phi$ (Γ proves ϕ) consisting of a finite set of formulas Γ and a formula ϕ.

(ii) A formal proof or derivation of $\Gamma \vdash \phi$ is a finite rooted tree of judgements satisfying the following conditions:

- *The root label is $\Gamma \vdash \phi$;*
- *All the leaves are axioms, i.e. judgements of the form $\Gamma \cup \{\phi\} \vdash \phi$;*
- *The label of each mother node is obtained from the labels of the daughters using one of the rules in Figure 1.*

*If such a proof exists, we say that the judgement $\Gamma \vdash \phi$ is **provable** or **derivable** and we write $\Gamma \vdash_N \phi$. For infinite Γ we write $\Gamma \vdash_N \phi$ to mean that $\Gamma_0 \vdash_N \phi$ for some finite subset Γ_0 of Γ. However, it is customary to omit the index N in $\vdash_N$, since the meaning of $\Gamma \vdash \phi$ is usually clear from context.*

*(iii) If $\varnothing \vdash \phi$ we say that ϕ is a **theorem**. We write $\vdash \phi$ instead of $\varnothing \vdash \phi$.*

We dive a bit deeper into the concept of a proof tree. A proof tree is a graphical representation of a logical argument or proof. It starts with a root node, representing the statement to be proven. Each branch of the tree represents a step in the logical reasoning process, connecting premises to conclusions through intermediate steps. The leaves of the tree represent the ultimate axioms of the argument. By following the branches from the root to the leaves, one can visually trace the logical structure of the proof, demonstrating the validity of the argument.

The proof system consists of an axiom scheme plus a set of *introduction* and *elimination* rules. In Figure 1 the system is displayed. Here, we write $\Gamma, \phi \vdash \phi$ instead of $\Gamma \cup \{\phi\} \vdash \phi$ as a simplification.

$$\Gamma, \phi \vdash \phi \text{ (Ax)}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi} (\to I) \qquad \frac{\Gamma \vdash \phi \to \psi \quad \Gamma \vdash \phi}{\Gamma \vdash \psi} (\to E)$$

$$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi} (\wedge I) \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi} (\wedge E) \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi} (\wedge E)$$

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} (\vee I) \qquad \frac{\Gamma \vdash \phi}{\Gamma \vdash \psi \vee \phi} (\vee I) \qquad \frac{\Gamma, \phi \vdash \upsilon \quad \Gamma, \psi \vdash \upsilon \quad \Gamma \vdash \psi \vee \phi}{\Gamma \vdash \upsilon} (\vee E)$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \phi} (\bot E)$$

Figure 1: Intuitionistic Natural Deduction (NJ)

This formalizes *intuitionistic propositional calculus* (IPC). By adding for example the axiom of *the law of the excluded middle* to IPC one obtains a system that is equivalent to *classical propositional calculus* **(CPC)** [17]. This is also the case if we would add the *double negation elimination* rule, which allows us to replace any formula of the form $\neg\neg\phi$ with ϕ. The reason for this is that in presence of the other axioms, assuming the law of the excluded middle is equivalent to assuming the rule of double negation elimination. Either way, this shows that IPC is a proper subsystem of CPC. One may also consider the extended IPC with the quantifiers $\forall$ ("for all") and $\exists$ ("there exists"), which is called *intuitionistic predicate calculus* **(IQC)**. While the meaning of these quantifiers is important and interesting in intuitionism because of their constructive interpretation, we only work with IPC in this thesis.

We can interpret intuitionistic propositional language algebraically using Heyting algebras. For this we analyze the algebraic properties of formulas with respect to their provability. It is easy to see that implication ($\to$) as defined for IPC behaves in a reflexive and transitive way. In other words, for every Γ we have:

- $\Gamma \vdash \phi \to \phi$;
- If $\Gamma \vdash \phi \to \psi$ and $\Gamma \vdash \psi \to \nu$ then $\Gamma \vdash \phi \to \nu$.

For implication to define a partial order relation on the propositional formulas it also needs to be anti-symmetric. Right now this is not the case. However, we can achieve this by identifying equivalent formulas: we define an order relation $\sim_\Gamma$ on a fixed set of propositional formulas Γ as follows:

$$\phi \sim_\Gamma \psi \text{ iff } \Gamma \vdash \phi \to \psi \text{ and } \Gamma \vdash \psi \to \phi.$$

This defines an equivalence relation on Γ:

- (Reflexivity) For all $\phi \in \Gamma$ we have $\phi \sim_\Gamma \phi$ since $\Gamma \vdash \phi \to \phi$.
- (Symmetry) For all $\phi, \psi \in \Gamma$ we have $\phi \sim_\Gamma \psi$ if and only if $\psi \sim_\Gamma \phi$ by definition of $\sim_\Gamma$.
- (Transitivity) Because of the transitivity of implication, we clearly have that $\phi \sim_\Gamma \psi$ and $\psi \sim_\Gamma \nu$ imply $\phi \sim_\Gamma \nu$ for all $\phi, \psi, \nu \in \Gamma$.

Now, let $\mathcal{L}_{\sim_\Gamma} = \Phi / \sim_\Gamma = \{ [\phi]_{\sim_\Gamma} \mid \phi \in \Phi\}$, where $[\phi]_{\sim_\Gamma} = \{\psi \mid \phi \sim_\Gamma \psi\}$. We often call $\mathcal{L}_{\sim_\Gamma}$ the *Lindenbaum algebra*. We create a partial order $\leq_{\sim_\Gamma}$ on $\mathcal{L}_{\sim_\Gamma}$ by defining

$$[\phi]_{\sim_\Gamma} \leq_{\sim_\Gamma} [\psi]_{\sim_\Gamma} \text{ iff } \Gamma \vdash \phi \to \psi.$$

We show that $\leq_{\sim_\Gamma}$ is well-defined and that it indeed defines a partial order on $\mathcal{L}_{\sim_\Gamma}$:

- Assume we have $[\phi_1]_{\sim_\Gamma} \leq_{\sim_\Gamma} [\psi_1]_{\sim_\Gamma}$ and $\phi_1 \sim_\Gamma \phi_2$ and $\psi_1 \sim_\Gamma \psi_2$. We know that $\Gamma \vdash \phi_1 \to \psi_1$. By definition of $\sim_\Gamma$ we have $\Gamma \vdash \phi_2 \to \phi_1$ and $\Gamma \vdash \psi_1 \to \psi_2$, so we use the transitivity of the implication to conclude $\Gamma \vdash \phi_2 \to \psi_2$, so $[\phi_2]_{\sim_\Gamma} \leq_{\sim_\Gamma} [\psi_2]_{\sim_\Gamma}$. This means that the relation does not depend on the choice of the representative, so it is well-defined.
- We have already concluded that the implication relation between formulas is reflexive and transitive. This directly implies the reflexivity and transitivity of $\leq_{\sim_\Gamma}$. For the anti-symmetry, assume $[\phi]_{\sim_\Gamma} \leq_{\sim_\Gamma} [\psi]_{\sim_\Gamma}$ and $[\psi]_{\sim_\Gamma} \leq_{\sim_\Gamma} [\phi]_{\sim_\Gamma}$, so $\Gamma \vdash \phi \to \psi$ and $\Gamma \vdash \psi \to \phi$. By definition of $\sim_\Gamma$ we now have $\phi \sim_\gamma \psi$, so $[\phi]_{\sim_\Gamma} = [\psi]_{\sim_\Gamma}$. This means that we have proven the anti-symmetry of $\leq_{\sim_\Gamma}$, so $\leq_{\sim_\Gamma}$ is a partial order.

Now, we define

$$[\phi]_{\sim_\Gamma} \cup [\psi]_{\sim_\Gamma} = [\phi \vee \psi]_{\sim_\Gamma};$$
$$[\phi]_{\sim_\Gamma} \cap [\psi]_{\sim_\Gamma} = [\phi \wedge \psi]_{\sim_\Gamma}.$$

It is easy to check that this is well-defined and that these operations, the join and meet, make the partial order into a lattice. It is even a distributive lattice: the distributivity follows from the distributivity of $\vee$ and $\wedge$. By setting

$$[\bot]_{\sim_\Gamma} = \{\phi \mid \Gamma \vdash \neg\phi\} \text{ and } [\top]_{\sim_\Gamma} = \{\phi \mid \Gamma \vdash \phi\}$$

we have found our bottom element $0 = [\bot]_{\sim_\Gamma}$ and top element $1 = [\top]_{\sim_\Gamma}$, so the lattice is bounded. We define the implication

$$[\phi]_{\sim_\Gamma} \to [\psi]_{\sim_\Gamma} = [\phi \to \psi]_{\sim_\Gamma}$$

to see that the Lindenbaum algebra is in fact a Heyting algebra. We only have to check that this indeed defines an implication operation in the sense of a Heyting algebra. First, notice that

$$[(\phi \to \psi) \wedge \phi]_{\sim_\Gamma} = [\phi \to \psi]_{\sim_\Gamma} \cap [\phi]_{\sim_\Gamma} \leq_{\sim_\Gamma} [\psi]_{\sim_\Gamma}$$

for all propositional formulas ϕ, ψ. Furthermore, if we have ν such that

$$[\nu \wedge \phi]_{\sim_\Gamma} = [\nu]_{\sim_\Gamma} \cap [\phi]_{\sim_\Gamma} \leq_{\sim_\Gamma} [\psi]_{\sim_\Gamma}$$

then $\Gamma \vdash (\nu \wedge \phi) \to \psi$ which means $\Gamma, (\nu \wedge \phi) \vdash \psi$, so $\Gamma, \nu, \phi \vdash \psi$ and therefore $\Gamma, \nu \vdash (\phi \to \psi)$. This means $\Gamma \vdash \nu \to (\phi \to \psi)$, so $[\nu]_{\sim_\Gamma} \leq_{\sim_\Gamma} [\phi \to \psi]_{\sim_\Gamma}$. So this means that the implication as defined above is indeed a Heyting algebra implication and therefore transforms the Lindenbaum algebra into a Heyting algebra.

The algebraic interpretation of IPC works as follows: we define a *valuation* in a Heyting algebra $\mathcal{H}$, that is a function that sends propositional variables PV to elements of the Heyting algebra $\mathcal{H}$. By interpreting $\bot$ as 0 and the logical connectives $\wedge, \vee, \to$ as the operations $\wedge, \vee, \to$ in the Heyting algebra, we extend the function to formulas. This results in a unique Heyting algebra homomorphism from $\mathcal{L}_{\sim_\Gamma}$ to H. The valuation is described more precisely in the following definition.

Definition 2.3.3. *Let $\mathcal{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$ be a Heyting algebra. A valuation in $\mathcal{H}$ is a map $v : PV \rightarrow H$, where PV is the set of propositional variables. Given a valuation v in $\mathcal{H}$, we recursively define the value $[\![\phi]\!]_v$ of a formula ϕ with respect to v:*

$$\begin{aligned} &[\![p]\!]_v = v(p) \textit{ for } p \in PV; \\ &[\![\bot]\!]_v = 0; \\ &[\![\phi \wedge \psi]\!]_v = [\![\phi]\!]_v \wedge [\![\psi]\!]_v; \\ &[\![\phi \vee \psi]\!]_v = [\![\phi]\!]_v \vee [\![\psi]\!]_v; \\ &[\![\phi \rightarrow \psi]\!]_v = [\![\phi]\!]_v \rightarrow [\![\psi]\!]_v. \end{aligned}$$

We write $\mathcal{H}, v \vDash \phi$ for $[\![\phi]\!]_v = 1$ and $\mathcal{H}, v \vDash \Gamma$ when $\mathcal{H}, v \vDash \phi$ for all $\phi \in \Gamma$. Then, $\Gamma \vDash \phi$ means that $\mathcal{H}, v \vDash \Gamma$ implies $\mathcal{H}, v \vDash \phi$ for all $\mathcal{H}$ and v and $\vDash \phi$ is an abbreviation for $\varnothing \vDash \phi$. If $\vDash \phi$ holds, we say that ϕ is *intuitionistically valid.*

We can relate the notion of *provability* to the notion of *intuitionistic validity.* We think of a formula which is assigned a 1 as "true" and a formula which is assigned a 0 as "false". The other values of the Heyting algebras that can be assigned to a formula should be thought of as intermediate truth values. We want that a formula is "true" in every Heyting algebra and valuation if and only if it is provable within NJ. It turns out that this is the case. This is proven in the following theorem.

Theorem 2.3.4 (Soundness and Completeness)**.** *The following are equivalent:*

(i) $\Gamma \vdash \phi$*;*

(ii) $\Gamma \vDash \phi$*.*

Proof. We prove both statements separately:

1. (i) $\implies$ (ii): Let $\Gamma = \{\nu_1, ..., \nu_n\}$. If v is a valuation in a Heyting algebra $\mathcal{H}$, then $[\![\Gamma]\!]_v$ stands for $[\![\nu_1]\!]_v \wedge ... \wedge [\![\nu_n]\!]_v$. This equals 1 when $\Gamma = \varnothing$. By induction with respect to derivations we prove that, for all v:

   $$\text{If } \Gamma \vdash \phi \text{ then } [\![\Gamma]\!]_v \leq [\![\phi]\!]_v.$$

   Then, the statement of the theorem follows from the special case where $]\!]\Gamma[\![_v = 1$.

   For all formulas ϕ we have $[\![\phi]\!]_v \leq [\![\phi]\!]_v$ so the basis of our induction holds. Now, we proceed by cases depending on the last rule used in the proof. As an example, we consider the ($\vee$E)-rule. We have to prove that $[\![\Gamma]\!]_v \leq [\![\nu]\!]_v$ using the induction hypothesis

   $$[\![\Gamma]\!]_v \wedge [\![\phi]\!]_v \leq [\![\nu]\!]_v \ , \ [\![\Gamma]\!]_v \wedge [\![\psi]\!]_v \leq [\![\nu]\!]_v \text{ and } [\![\Gamma]\!]_v \leq [\![\phi \vee \psi]\!]_v.$$

   We have that $[\![\Gamma]\!]_v \leq [\![\phi \vee \psi]\!]_v$ implies that

   $$[\![\Gamma]\!]_v = [\![\Gamma]\!]_v \wedge [\![\phi \vee \psi]\!]_v = ([\![\Gamma]\!]_v \wedge [\![\phi]\!]_v) \vee ([\![\Gamma]\!]_v \wedge [\![\psi]\!]_v).$$

   Since both components are less than or equal to $[\![\nu]\!]_v$, so is $[\![\Gamma]\!]_v$. The demonstrations for the other rules go in a similar way.

2. (ii) $\implies$ (i): For this part of the proof, we rely on our construction of the Lindenbaum algebra. Suppose we have $\Gamma \vDash \phi$ but $\Gamma \nvdash \phi$. Then $\phi \not\sim_\Gamma \top$, i.e. $[\phi]_\sim \neq 1$ in $\mathcal{L}_\Gamma$. Define a valuation v in $\mathcal{L}_\Gamma$ by $v(p) = [p]_\sim$. By construction, we now have that $[\![\psi]\!]_v = [\psi]_\sim$ for all ψ. This leads to $[\![\phi]\!]_v \neq 1$, which is a contradiction. Therefore, it must be the case that $\Gamma \vDash \phi \implies \Gamma \vdash \phi$. $\square$

By using a valuation in a Heyting algebra and its relation to a locale of opens, we can see that the law of the excluded middle does not hold. Recall that for a topological space we have $\neg A = (A^c)^\circ$. Consider the topological space $\mathbb{R}$ with its standard topology, generated by open intervals of the form (a, b). The locale of opens $\Omega(\mathbb{R})$ is a complete Heyting algebra by Proposition 2.2.5. Let A be a formula. Then, by taking $v(A) = (0, \infty)$ we have $[\![A \vee \neg A]\!]_v = [\![A]\!]_v \vee [\![\neg A]\!]_v = (0, \infty) \cup (-\infty, 0]^\circ = \mathbb{R} \setminus \{0\}$. The value of the formula is thus not equal to $\mathbb{R}$. This means that $A \vee \neg A$ is not intuitionistically valid, since we have found an example of a valuation v and a Heyting algebra such that $[\![A \vee \neg A]\!]_v \neq 1$. By using Theorem 2.3.4 we can now also conclude that $\nvdash A \vee \neg A$.

For the algebraic interpretation of CPC, we can consider a different kind of lattices: Boolean algebras. To introduce these, we first have to define a stronger version of the pseudo-complement.

Definition 2.3.5. *Let L be a bounded lattice with top element 1 and bottom element 0. Let $a \in L$. We say that b is a* ***complement*** *of a iff $a \vee b = 1$ and $a \wedge b = 0$.*

In a distributive lattice, complements are unique when they exist. This is shown in the following proposition.

Proposition 2.3.6. *Let L be a distributive lattice and let $a, b, c \in L$. Then there exists at most one $x \in L$ such that $x \wedge a = b$ and $x \vee a = c$.*

Proof. Suppose x and y both satisfy the conditions in the proposition, so assume

$$x \wedge a = b \text{ and } x \vee a = c$$

and

$$y \wedge a = b \text{ and } y \vee a = c.$$

Then

$$\begin{aligned} x &= x \wedge (x \vee a) = x \wedge c = x \wedge (y \vee a) \\ &= (x \wedge y) \vee (x \wedge a) \quad \textit{(by distributivity)} \\ &= (x \wedge y) \vee b = x \wedge y, \end{aligned}$$

since $b = x \wedge a = y \wedge a$ is a lower bound for $\{x, y\}$. Similarly, we have $y = x \wedge y$, so $x = y$. $\square$

This means that in the case that L is distributive we can speak of *the* complement of an element $a \in L$. Earlier, we said that the complement defines a stronger version of the pseudo-complement $\neg a$, which is defined for a Heyting algebra as $\neg a := a \to 0$. By definition, we have $x \leq \neg a = (a \to 0)$ iff $x \wedge a = 0$. Now, notice that when it exists, the complement b of a is the greatest element of L such that $b \wedge a = a \wedge b = 0$: suppose $a \wedge c = 0$, then $c \leq b$ since

$$c = 1 \wedge c = (a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) = 0 \vee (b \wedge c) = b \wedge c.$$

Using the characterization of Remark 2.2.4, this shows that any complement b of a is also the pseudo-complement of a. We say *the* pseudo-complement, since it is unique by definition.

Now we can define Boolean algebras.

Definition 2.3.7. *A* ***Boolean*** *algebra is a bounded, distributive lattice B such that every element $a \in B$ has a complement, denoted by $\neg a$.*

An example of a Boolean algebra is the power set lattice $(\mathcal{P}(X), \subseteq)$ as defined in Example 2.1.40. The complement of a set $A \in \mathcal{P}(X)$ is defined as the set-theoretic complement, $A^c = X \setminus A$. It is easy to see that this turns $\mathcal{P}(X)$ into a Boolean algebra.

In classical propositional calculus (CPC), there are traditionally two truth values: 0 and 1. Typically, the semantics of classical propositional formulas are defined such that each logical connective is viewed as an operation acting on the set $\{0, 1\}$ of truth values. A valuation in a Boolean algebra can be defined similarly to Definition 2.3.4, thereby generalizing the ordinary two-valued semantics. For further details, we refer to Sørensen and Urzyczyn [20]. A *tautology* is the classical equivalent of an intuitionistically valid formula. The theorem for Soundness and Completeness can be easily adapted to classical logic, with the Boolean algebra arising from $\{0, 1\}$ being sufficient to demonstrate completeness. Again, we refer to Sørensen and Urzyczyn for more details. However, no finite Heyting algebra serves the same role for IPC as the two-valued Boolean algebra does for IPC. Gödel demonstrated this in 1932 in his work *Zum intuitionistischen Aussagenkalkül* [8], indicating a significant difference between classical and intuitionistic logic.

It becomes clear very quickly that Boolean algebras are a special kind of Heyting algebras. For a Boolean algebra B, implication is defined as $a \to b = \neg a \vee b$ for $a, b \in B$. We show that this also defines an implication in the sense of a Heyting algebra, by showing that $\neg a \vee b$ is the greatest element such that $(\neg a \vee b) \wedge a \leq b$:

- $(\neg a \vee b) \wedge a = (\neg a \wedge a) \vee (b \wedge a) = 0 \vee (b \wedge a) = b \wedge a \leq b.$
- Assume we have $c \in B$ such that $c \wedge a \leq b$. Then

$$c = c \wedge 1 = c \wedge (\neg a \vee a) = (c \wedge \neg a) \vee (c \wedge a) \leq (c \wedge \neg a) \vee b = (c \vee b) \wedge (\neg a \vee b) \leq \neg a \vee b.$$

So each Boolean algebra with implication defined as above is a Heyting algebra. From our definition we can easily see that the following proposition holds:

Proposition 2.3.8. *A Heyting algebra H is a Boolean algebra if and only if for every $x \in H$ we have $x \vee \neg x = 1$.*

Proof. Saying that for every $x \in H$ we have is $x \vee \neg x = 1$ is equivalent to saying that for every $x \in H$ the complement $\neg x$ exists. Here, we use that $x \wedge \neg x = 0$ holds for all $x \in H$, which is true since $x \wedge \neg x = \neg x \wedge x \leq 0$ if and only if $\neg x \leq (x \to 0) = \neg x$. Since a Boolean algebra is defined as a bounded, distributive lattice such that for every element the complement exists, this now proves the proposition. □

The fact that $x \vee \neg x = 1$ has to hold in order to make a Heyting algebra a Boolean algebra matches our idea of classical logic being a proper subsystem of intuitionistic logic, where the law of the excluded middle has to be added.

We can equivalently describe Boolean algebras in a different way, by using the *regular elements.* The **regular** elements of a Heyting algebra H are the elements x of H such that $\neg\neg x = x$. Just as the addition of the law of the excluded middle or double negation elimination is equivalent, so is the classification of Boolean algebras with the complemented or regular elements of a Heyting algebra. Here, it is important that the regular elements of a Heyting algebra H in fact constitute a Boolean algebra. For this, and for the equivalence between regular and complemented elements, see Borceux [3].

We can relate all this to our topological space $\mathbb{R}$. We may wonder: what are the regular elements of $\Omega(\mathbb{R})$? As established at the end of Section 2.2, for $U \in \Omega(\mathbb{R})$ we have that $\neg\neg U$ equals the interior of the closure of U. This means that an open subset U of $\mathbb{R}$ is a *regular* element of $\Omega(\mathbb{R})$ if and only if it is equal to the interior of its closure. The open intervals of $\mathbb{R}$ are examples of this. In topology, we know the regular elements as the *regular open subsets* of the topology. This means that the regular open subsets of $\mathbb{R}$ constitute a Boolean algebra.

This gives us a little bit of insight on how the continuum can be used to model either classical or intuitionistic logic. We will come back to this in the last chapter. First, we explore the relation between the pointfree description of spaces and the spaces itself.

2.4 Locales and spaces

In this section, we expand on the relation between locales and spaces. For the next part, we use Picado and Pultr [19] and Mac Lane and Moerdijk [15] as sources.

We start with a definition.

Definition 2.4.1. *Let X be a topological space. An element W of $\Omega(X)$ is called a* ***meet-irreducible open set*** *if the following property holds for any open $U, V \subseteq X$:*

$$\text{if } U \cap V \subseteq W, \text{ then either } U \subseteq W \text{ or } V \subseteq W.$$

A space X is said to be ***sober*** *if the only meet-irreducible sets $W \neq X$ are of the form $X \smallsetminus \overline{\{x\}}$, where $\overline{\{x\}}$ is the closure of $\{x\}$.*

We can also define sobriety in terms of closed subsets. A closed subset $Y \subseteq X$ is said to be **join-irreducible** if it cannot be written as the union of two smaller closed subsets, that is, whenever Y_1 and Y_2 are closed sets such that $Y = Y_1 \cup Y_2$, then $Y = Y_1$ or $Y = Y_2$. We know that for any $x \in X$ that $\overline{\{x\}}$ is a join-irreducible closed set. Then, X is sober if and only if every non-empty join-irreducible closed set is the closure of a unique point.

In the following lemma we prove that both definitions are equivalent.

Lemma 2.4.2. *Let $U \subsetneq X$ be open. Then U is a meet-irreducible open set if and only if $X \smallsetminus U$ is a join-irreducible closed set.*

Proof. We prove the statement in both directions.

- First notice that for U open we have $X \smallsetminus U$ closed. Suppose U is meet-irreducible and suppose $X \smallsetminus U = Y_1 \cup Y_2$ with Y_1, Y_2 closed. Then there exist U_1, U_2 open such that $Y_1 = X \smallsetminus U_1$ and $Y_2 = X \smallsetminus U_2$ By taking the complement we see that

  $$U = (X \smallsetminus U)^c = (Y_1 \cup Y_2)^c = Y_1^c \cap Y_2^c = U_1 \cap U_2.$$

  By the meet-irreducibility of U we have $U_1 \subseteq U$ or $U_2 \subseteq U$, so

  $$X \smallsetminus U = U^c \subseteq U_1^c = Y_1 \text{ or } X \smallsetminus U = U^c \subseteq U_2^c = Y_2.$$

  This shows that $X \smallsetminus U = Y_1$ or $X \smallsetminus U = Y2$, so $X \smallsetminus U$ is a join-irreducible closed set.

- Suppose now that $X \setminus U$ is a join-irreducible closed set. By definition, U is open. Suppose there exist $U_1, U_2 \subseteq X$ open such that $U_1 \cap U_2 \subseteq U$. We have

  $$X \setminus U = X \setminus (U \cup (U_1 \cap U_2)) = X \setminus ((U \cup U_1) \cap (U \cup U_2)) = (X \setminus (U \cup U_1)) \cup (X \setminus (U \cup U_2)).$$

  Since $U, U_1, U_2 \subseteq X$ open we have $X \setminus (U \cup U_1)$ and $X \setminus (U \cup U_2)$ closed. By the join-irreducibility of $X \setminus U$ this implies that

  $$X \setminus U = X \setminus (U \cup U_1) \text{ or } X \setminus U = X \setminus (U \cup U_2).$$

  This means that we either have that $U_1 \subseteq U$ or $U_2 \subseteq U$, so U is a meet-irreducible open set. □

Example 2.4.3. An example of a non-sober space is the topology of the natural numbers $\mathbb{N}$ with the cofinite topology, which is given by $\tau_{\text{cofinite}} = \{U \subseteq \mathbb{N} \mid U = \varnothing \text{ or } U^c \text{ is finite}\}$. We can see that this topological space is not sober by noticing that $\mathbb{N}$ itself is a join-irreducible closed subset, since the closed subsets of $\mathbb{N}$ are the finite subsets of $\mathbb{N}$ and $\mathbb{N}$ itself. If we have closed subsets F_1, F_2 of $\mathbb{N}$ such that $\mathbb{N} = F_1 \cup F_2$, then it cannot be the case that F_1 and F_2 are both finite. Since the only infinite closed subset is $\mathbb{N}$ itself, we have $\mathbb{N} = F_1$ or $\mathbb{N} = F_2$. Because $\mathbb{N}$ does not equal the closure of a single point, we conclude that this topological space is not sober.

Proposition 2.4.4. *Each Hausdorff space is sober.*

Proof. Suppose $U \neq X$ is meet-irreducible and U open. Let $x_1, x_2 \in X$ and consider disjoint open $U_i \subseteq X$ such that $U_i \ni x_i$ for $i = 1, 2$. We cannot have $x_1, x_2 \notin U$, since $(U \cup U_1) \cap (U \cup U_2) \subseteq U$ would imply that $U \cup U_1 \subseteq U$ or $U \cup U_2 \subseteq U$. This means that either $x_1 \in U$ or $x_2 \in U$. In Hausdorff spaces, $\overline{\{x\}} = \{x\}$, so the only meet-irreducible set $U \neq X$ is of the form $X \setminus \overline{\{x\}}$. □

For example, this means that $\mathbb{R}$ is sober, since it is Hausdorff.

We have seen that the concept of a topological space can be extended and generalized through locales. In the case of *sober* spaces, no information on the topological space X is lost by passing to its locale of opens $\Omega(X)$. We can understand this by noticing that we can view a point of a topological space X as a continuous map $1 \to X$, where 1 is the one-point space [12]. From this we get the idea to define a *point* of a locale L as a continuous map $\Omega(1) \to L$, where $\Omega(1)$ is the one-point topological space. In other words, we can look at the induced frame homomorphisms $p : L \to \Omega(1) \cong \{0, 1\} = 2$. It turns out that for a topological space X we can make a bijection between the frame homomorphisms $f : \Omega(X) \to 2$ and the join-irreducible closed subsets of X, as is shown in the following lemma [18]:

Lemma 2.4.5. *Let X be a topological space. There is a bijection between the join-irreducible closed subsets of X, denoted by IrrClSub(X), and the frame homomorphisms $\Omega(X) \to 2$, given by*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Frame}}(\Omega(X), 2) &\longrightarrow \mathrm{IrrClSub}(X) \\ \varphi &\longmapsto X \setminus U_0(\varphi) \end{aligned}$$

where $U_0(\varphi)$ is the union of all elements $U \in \Omega(X)$ such that $\varphi(U) = 0$:

$$U_0(\varphi) := \bigcup_{\substack{U \in \Omega(X) \\ \varphi(U) = 0}} U.$$

Proof. First we need to show that $X \setminus U_0(\varphi)$ is indeed a join-irreducible closed subset of X. Using Lemma 2.4.2, we therefore show that $U_0(\varphi)$ is a meet-irreducible open subset of X. Since $U_0(\varphi)$ is an union of open sets, it is open. Assume $U_1 \cap U_2 \subseteq U_0(\varphi)$. Since φ is a frame homomorphism, it preserves finite meets, inclusion and arbitrary joins, so we have

$$\varphi(U_1) \cap \varphi(U_2) = \varphi(U_1 \cap U_2) \subseteq \varphi(U_0(\varphi)) = 0.$$

This shows that $\varphi(U_1) = 0$ or $\varphi(U_2) = 0$, so $U_1 \subseteq U_0(\varphi)$ or $U_2 \subseteq U_0(\varphi)$. This shows that $U_0(\varphi)$ is a meet-irreducible open subset, so $X \setminus U_0(\varphi)$ is a join-irreducible closed subset of X.

Conversely, given a join-irreducible closed subset $X \setminus U_0$ of X, define a frame homomorphism $\varphi : \Omega(X) \to 2$ as follows:

$$\varphi : U \mapsto \begin{cases} 0 & \text{if } U \subseteq U_0; \\ 1 & \text{otherwise.} \end{cases}$$

We show that this preserves finite meets and arbitrary joins:

- For the finite meets, take two opens U_1, U_2.
  - Assume we have $\varphi(U_1 \cap U_2) = 0$, then by definition $U_1 \cap U_2 \subseteq U_0$. Since U_0 is a meet-irreducible open set, either $U_1 \subseteq U_0$ or $U_2 \subseteq U_0$, so either $\varphi(U_1) = 0$ or $\varphi(U_2) = 0$. This means $\varphi(U_1) \cap \varphi(U_2) = 0$. Moreover, if $\varphi(U_1) \cap \varphi(U_2) = 0$, then $\varphi(U_1) = 0$ or $\varphi(U_2) = 0$. In both cases, since for $i = 1, 2$ we have $U_1 \cap U_2 \subseteq U_i$, it follows that $\varphi(U_1 \cap U_2) = 0$. So $\varphi(U_1 \cap U_2) = 0$ if and only if $\varphi(U_1) \cap \varphi(U_2) = 0$.
  - Similarly: suppose $\varphi(U_1 \cap U_2) = 1$, then $U_1 \cap U_2 \nsubseteq U_0$ and so $U_1 \nsubseteq U_0$ and $U_1 \nsubseteq U_0$. This means $\varphi(U_1) = 1$ and $\varphi(U_2) = 1$ whence $\varphi(U_1) \cap \varphi(U_2) = 1$. On the other hand, if $\varphi(U_1) \cap \varphi(U_2) = 1$ then $\varphi(U_1) = 1$ and $\varphi(U_2) = 1$ so $U_1 \nsubseteq U_0$ and $U_2 \nsubseteq U_0$. Using the meet-irreducibility of U_0 again, $\varnothing \neq U_1 \cap U_2 \nsubseteq U_0$, so $\varphi(U_1 \cap U_2) = 1$. So $\varphi(U_1 \cap U_2) = 1$ if and only if $\varphi(U_1) \cap \varphi(U_2) = 1$.

  This means that in both cases, finite meets are preserved.

- For arbitrary joins, let $\{U_i\}_i$ be a family of opens.
  - $\varphi(\bigcup_i U_i) = 0$ precisely if $\bigcup_i U_i \subseteq U_0$, which is the case precisely if $U_i \subseteq U_0$ for all i, which means by definition that $\varphi(U_i) = 0$ for all i and this is equivalent to $\bigcup_i \varphi(U_i) = 0$. So $\varphi(\bigcup_i U_i) = 0$ if and only if $\bigcup_i \varphi(U_i) = 0$.
  - $\varphi(\bigcup_i U_i) = 1$ precisely if there is at least one U_i such that $U_i \nsubseteq U_0$, which means that there is at least one U_i such that $\varphi(U_i) = 1$ and this is equivalent to $\bigcup_i \varphi(U_i) = 1$. So $\varphi(\bigcup_i U_i) = 1$ if and only if $\bigcup_i \varphi(U_i) = 1$.

  Now we have also shown that arbitrary joins are preserved in both cases.

This means that $\varphi : \Omega(X) \to 2$ is a frame homomorphism. It is easy to see that the two operations are each other's inverses, so we have our bijection. □

This shows that the points for a sober space correspond exactly to its join-irreducible closed subsets, which in turn coincide with the closure of singletons. This is why we say that no information is lost: the points of the space correspond bijectively to the points of the corresponding locale.

Instead of starting with a space, we can also start with a locale. To make a topological space from a locale, one needs to find a corresponding set of points *and* a suitable topology. We may wonder which types of locales correspond with $\Omega(X)$ for a topological space X. It turns out that this is the case for a special kind of locales: *spatial* ones.

As established before, for any locale L the points of L correspond to the frame homomorphisms $f : L \to 2$. In turn, the frame homomorphisms $f : L \to 2$ correspond bijectively to the so-called *completely prime filters* of L, as is described in Johnstone 1982 [12]. Important to note, also according to Johnstone, is the fact that often the points of a locale are identified with the prime ideals of the corresponding frame. Here, the classically true fact that a filter is completely prime if and only if its complement is a principal prime ideal is used [14]. However, this complementation uses the law of the excluded middle, so we cannot constructively identify our points with the prime ideals.

We also have to mention here that finding prime ideals relies on non-constructive principles. The reason for this is that the statement that asserts the existence of the relevant prime ideals uses Zorn's lemma, which is equivalent to the axiom of choice (AC). To be more precise: the statement turns out to be equivalent to $(AC)_F$, the axiom that states that every family of non-empty finite sets has a choice function. As shown by Gödel and Cohen, (AC) is not derivable within Zermelo-Fraenkel set theory. For the same reasons, $(AC)_F$ cannot be derived in ZF either. Moreover, the axiom itself is not constructively acceptable. This means that to ensure the existence of our prime ideals, which correspond classically to points of our locales, we have to take non-constructive steps. For more details on this, see Chapter 10 of Davey and Priestley [7].

We write pt(L) for the set of points of a locale L as defined above. For $a \in L$, define

$$\phi(a) = \{p \in \mathrm{pt(L)} \mid p(a) = 1\}.$$

This defines a topology on $\mathrm{pt}(L)$. In this way, we can find a set of points and a topology starting from a locale. For all $a \in L$ the map $\phi(a)$ is surjective. To use ϕ to go from a locale L to a space $\Omega(X)$ in a proper way, $\phi(a)$ has to be injective for all $a \in L$. When this is the case, L has *enough points* and is called **spatial**. So, any *spatial* locale L is (frame-)isomorphic to the space $\Omega(\mathrm{pt}(L))$.

As mentioned before, finding the points of a locale uses non-constructive principles. This means that proving that a certain locale L is spatial is often only possible in non-constructive mathematics. In other words: we should be careful with talking about points of a locale. Nevertheless, when we do permit the pt-operator, we can obtain an equivalence between sober spaces and spatial locales.

To be precise: we want

$$\mathrm{pt} : X \cong \mathrm{pt}(\Omega(X))$$

to be a homeomorphism and

$$\phi : L \cong \Omega(\mathrm{pt}(L))$$

to be a frame isomorphism, that is, a bijective frame homomorphism. In the context of category theory, we can think of this as a *duality.* The equivalence is exactly the case for X sober and for L spatial, as is shown in Johnstone 1982 [12]. This means that within classical mathematics working with spatial locales can thus be seen as an alternative way of working with sober topological spaces.

Pointless topology is important in constructive mathematics. There are some results which can be reformulated in the context of locales, in such a way that the localic reformulation is classically equivalent to the original result, and are then provable without the use of non-constructive methods. An example of this is Tychonoff's theorem, which classically states that the product of compact topological spaces is compact with respect to the product topology. In Zermelo–Fraenkel set theory, this theorem is equivalent to the axiom of choice, so it is not constructive. However, in Johnstone 1981 [11], a choice-free proof of a localic equivalent of this theorem is given: it is constructively shown that arbitrary products of compact locales are compact. This proof will not be included in this thesis, but the existence of it shows how the point-free approach can be beneficial to constructive mathematics.

Again, we must emphasize that the difference between locales and topological spaces is crucial in contexts where choice principles are not allowed. An important example of this is the localic real line $L(\mathbb{R})$, as discussed in Vickers [27]. The locale $L(\mathbb{R})$ is generated by basic open sets (p,q) with $p \in \mathbb{Q} \cup \{-\infty\}$ and $q \in \mathbb{Q} \cup \{\infty\}$, which are subject to the following relations:

- $1 = (-\infty, \infty)$;
- $(p,q) \wedge (p_0, q_0) = (\max(p, p_0), \min(q, q_0))$;
- $(p,q) \leq 0$ if $p \geq q$;
- $(p,s) \leq (p,r) \vee (q,s)$ if $p \leq q < r \leq s$;
- $(p,q) \leq \bigvee\{(p_0,q_0) \mid p < p_0 < q_0 < q\}$ if $p < q$.

The reason that this specific locale $L(\mathbb{R})$ is interesting in this discussion is that it can constructively be non-spatial, meaning that we cannot always retrieve the points of the original topological space $\mathbb{R}$. Here we see that in a constructive context, the difference between the space and the locale is relevant. On top of that, the locale $L(\mathbb{R})$ is quite significant, since important mathematical theorems such as the Heine-Borel theorem hold constructively within it, unlike in point-set topology. Working with real numbers in a localic context, without the goal of retrieving a topological space, can therefore lead to a more constructive form of mathematics.

However, the construction of $L(\mathbb{R})$ does not fully adhere to constructive methods. While the basic operations are defined in a natural way, ensuring the locale structure requires less transparent steps. For the details on this, we refer to Chapter IV of Johnstone 1982 [12]. Still, this construction should be contrasted with $\Omega(\mathbb{R})$, the locale of all opens of $\mathbb{R}$, which is in itself an *impredicative construction* as it assumes the *complete* collection of open subsets. Such an assumption is unacceptable to constructivists, who require that each set be built from explicit, finite procedures, rather than from a presupposed totality of sets. The locale $L(\mathbb{R})$ offers a more natural way of viewing open subsets, using rational intervals as a basis. Yet, questions remain: how constructive is this localic real line? And how does this version of the continuum relate to our intuitionistic continuum?

In the next chapter, we reflect on the main question of this thesis. The relation between the intuitionistic and pointless continuum, especially with respect to the primal philosophy of intuitionism, is being discussed and we conclude our findings in this way.

Conclusion

In this conclusion, we reflect on the relationship between intuitionism and pointless topology, particularly as applied to the continuum.

We have seen that we can regard the real line as a locale of its open sets. This turns out to be a Heyting algebra, which means that we can use our pointless continuum to describe intuitionistically valid and provable formulas. Interestingly, this leads to *intuitionistic* logic instead of *classical* logic. This difference arises because we have to take the openness of our sets into account in the case of the elements of a topology. This is not necessary in the power set locale, which turns out to be Boolean and therefore classical. The only difference between these two locales lies in the definition of the meet-operation: for the locale of *open* subsets, it is defined as the *interior* of the intersection, while for the locale of *all* subsets, it is defined as the intersection *itself.* It is clear that because of taking the interior, we find examples in which the law of the excluded middle is no longer valid. The pointless continuum is therefore suitable for capturing intuitionistic logic.

The connection between pointless topology as a complete frame and Heyting algebras shows the deep connection between intuitionism and pointless topology on a structural level. Crucial here is that this correspondence is a result of the *formalization* of intuitionism. Looking into the meaning of this would mean that we give intuitionistic logic and therefore *language* a big role in this discussion, which contradicts the primal philosophy of intuitionism. We must therefore look for the connection in a different sense, connecting pointless topology with intuitionism on a more conceptual level.

What is important to understand in this discussion is the fact that in the development of pointless topology, Brouwer's idea of mathematics as a mental construction is not taken into account at all. Only at one point our human intuition plays a role, namely in seeing the points as abstractions of realistic places. This structure leads to a more constructive continuum. Although the motivation of pointless topology thus lies in constructive mathematics, most of the work has been done using classical logic. Important in this is that set theory, on which it relies, is based on classical logic. On top of that, the constructions used in pointless topology are often not constructive, making them incompatible with the intuitionistic perspective. To use the constructive advantages of pointless topology, one must build up the theory in a constructive manner as well. In the literature used for this thesis, this topic is often overlooked.

Something that we have not gotten into in this thesis for time-related reasons is the precise construction of the localic real line as generated by open intervals with rational endpoints. This representation of the localic real line leads to constructive versions of theorems about the reals, such as a constructive version of Heine-Borel. It would have been interesting to explore how compatible this specific construction of the localic real line is with constructivism and intuitionism. Vickers [27] suggests that the presentation of $L(\mathbb{R})$ using generators and relations, as described at the end of Section 2.4, is at least *predicative*. This approach is appealing to constructivists because it avoids presupposing a collection that already includes the elements being constructed. One could take this presentation as a surrogate for the frame to avoid impredicative constructions. This is done in the field of *formal topology*, which would have been an interesting addition to this thesis.

This locale $L(\mathbb{R})$ also reminds us of our intuitionistic continuum, as it is built up by rational intervals. With these rational intervals, like in the intuitionistic continuum, we can approach our points. A point of the localic real line is determined by the open intervals in which it is contained. Since constructively the localic real line is not spatial, we cannot retrieve the exact points themselves. Therefore, we can only see the points with non-trivial extent, meaning we stay in our approximation and never reach an endpoint. Only by using classical, non-constructive methods could we retrieve the points, which would create the classical point-set continuum. We may therefore conclude that we should treat the real numbers in the pointless continuum in the same way as in the intuitionistic continuum. This could mean that every total real function on the constructive pointless continuum would be continuous as well. Since this result would be specific for the localic real line $L(\mathbb{R})$ within a constructive framework, a closer examination is necessary before drawing definitive conclusions. However, it might be an interesting outcome of combining pointless topology and intuitionism.

The use of open intervals with rational endpoints is something the pointless construction of $L(\mathbb{R})$ and the intuitionistic continuum have in common. The intuitionistic notion of a real number is much closer to the *pointless* one than to the *pointwise*. It is determined by the rational open intervals and should be seen as a forever unfinished process. The fact that this is also the case for pointless topology results from applying the same constructivist philosophy, preventing us from retrieving our original point-set topological space. But since pointless topology finds its motivation in constructive mathematics, one *should* use this philosophy of mathematics here. This viewpoint is also advocated by other mathematicians, leading to more rigorous versions of pointless topology. Some work in this area has been done by Graham Manuell [16] and by developers of formal topology, such as Vickers [27]. For future research, it might be interesting to compare these theories to the intuitionistic continuum, as they are more constructively motivated.

To conclude, to fully make use of the constructive benefits of pointless topology, one should build up this theory in a constructive manner. Only then can we meaningfully compare the pointless continuum with the intuitionistic one. This approach would allow us to assess not only the structural similarities of pointless topology and intuitionism, but also their constructive versions of the continuum. And who knows what else this might lead to.

Reflection

As mentioned in the introduction of this thesis, our view of mathematics can influence how we practice it. In writing this thesis, my perspective on the continuum has shifted from a never-ending line of contiguous points to one where potential infinity and constructive methods play significant roles. This perspective developed mainly through working with choice sequences and exploring intuitionistic philosophy. Although the order of presentation in this thesis may suggest otherwise, I delved into intuitionism only after researching pointless topology. I feel that this experience with intuitionism altered the way I viewed pointless topology afterward, as it was not motivated or developed from the philosophy I had just learned to accept.

Looking back, it might have been interesting to approach pointless topology with a constructive motivation already in mind. In that case, the construction of $L(\mathbb{R})$ would have been given a bigger role, investigating its possibilities in constructive mathematics more closely. After all, my conclusion was that we should define the localic real line in a constructive manner to be able to fully compare it with the intuitionistic continuum as two constructive perspectives of the continuum. In this context, perhaps similar intuitionistic principles might apply to this pointless continuum as well. Of course, this is something that I can only conclude afterwards, since it is the project itself that has created this view of mine. While I feel I cannot yet make a definitive statement about the compatibility of pointless topology and intuitionism, due to my limited exploration of the relevant constructions, writing this thesis has provided fresh insights into the continuum and expanded my mathematical knowledge and perspectives. I will incorporate these insights into future projects.

In the future, I would be interested in further developing pointless topology and exploring its possibilities from a constructive perspective rather than a neutral one. I acknowledge that this viewpoint may be influenced by my current understanding of a constructive continuum, which is mostly based on Brouwer's philosophy. Nonetheless, new mathematical intuitions, even those that are shaped by certain experiences or philosophies, are worth investigating. It is crucial, however, to clarify our starting points in those cases and to define the philosophical framework guiding our work. While much of mathematics is done in a precise and careful way, little to no attention is paid to this in mathematical publications. This oversight may stem from a lack of awareness about our underlying perspectives. The only thing we can do is admit the choices we have made because of certain views or experiences that we are *aware* of. That is why I include this reflection in my thesis: to acknowledge the philosophical basis that is shaping my mathematical approach and to encourage a thoughtful consideration of foundational perspectives in mathematical research.

- Josje van der Laan

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