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The Kadison–Singer Conjecture

In 1959, Richard Kadison and Isadore Singer stated a seemingly technical conjecture in operator theory on Hilbert space. They managed to disprove one already tricky case, but left the second, even more difficult case open. In the subsequent 50 years, their conjecture was shown to be related to various other problems in different areas of mathematics. Using unexpected techniques from linear algebra and probability theory, the full conjecture was proved in 2013 year by Adam Marcus, Daniel Spielman, and Nikhil Srivastava, who received the 2014 Pólya Prize for this remarkable achievement.

In this article, we discuss their conjecture (and its proof) in the light of a more general question that Kadison and Singer had in mind (which was partly inspired by quantum mechanics).

1. Linear algebra and convexity

The Kadison–Singer conjecture is concerned with infinite-dimensional Hilbert spaces H, but the underlying situation is already interesting in finite dimension. Hence we start with the Hilbert space

$$H = \mathbb{C}^n,$$

with standard inner product

$$\langle w, z \rangle = \sum_{i=1}^{n} \overline{w}_i z_i,$$

which we evidently take to be linear in the *second* entry. For the moment we identify operators with matrices [1].

Let $M_n(\mathbb{C})$ be the complex $n \times n$ matrices, regarded as an algebra (which we always assume to be *complex* and *associative*) with *involution*, namely the operation $a \mapsto a^*$ of hermitian conjugation. Abstractly, an **involution** on an algebra A is an anti-linear anti-homomorphism $* : A \to A$, so if we write $*(a) = a^*$, then for all $a, b \in A$ and $\lambda \in \mathbb{C}$ we have

$$
\begin{aligned}
(\lambda a + b)^* &= \overline{\lambda}a^* + b^*; \\
(ab)^* &= b^*a^*.
\end{aligned}
$$

Note that $M_n(\mathbb{C})$ has a unit, viz. the unit matrix 1_n. An algebra with involution (and unit) is called a (unital) *-**algebra**. Beside $M_n(\mathbb{C})$, another unital *-algebra of interest to us is $D_n(\mathbb{C})$, i.e., the subalgebra of $M_n(\mathbb{C})$ consisting of all diagonal matrices, with the involution $*$ inherited from $M_n(\mathbb{C})$.

In connection with the Kadison–Singer conjecture, the following concept is crucial. A **state** on a unital *-algebra A (with unit 1_A) is a linear map

$$\omega : A \to \mathbb{C}$$

that satisfies

$$
\begin{aligned}
\omega(1_A) &= 1; \\
\omega(a^*a) &\geq 0, \quad \text{for all } a \in A.
\end{aligned}
$$

Inspired by quantum mechanics, this concept was introduced by John von Neumann [2], albeit in the special case where A is the unital *-algebra $B(H)$ of all bounded operators on some Hilbert space H (see below). The general notion of a state in the above sense is due to Gelfand & Naimark [3] and Segal [4]. The states on A form a convex set $S(A)$, whose extremal points are called **pure**

states. That is, ω is pure iff any decomposition

$$\omega = t\omega' + (1 - t)\omega''$$

for $\omega', \omega'' \in S(A)$ and $t \in (0, 1)$ is necessarily trivial, in that $\omega' = \omega'' = \omega$. States that are not pure are **mixed**.

Von Neumann also defined a **density matrix** as an hermitian matrix $\rho \in M_n(\mathbb{C})$ whose eigenvalues $\{\lambda_i\}_{i=1}^{n}$, are non-negative and sum to unity, or equivalently, as a positive (semi-definite) matrix (in that $\langle \psi, \rho\psi \rangle \geq 0$ for each $\psi \in \mathbb{C}^n$) with unit trace. The point, then, is that states on $M_n(\mathbb{C})$ bijectively correspond to density matrices through

$$(1) \qquad \omega(a) = \text{Tr}\,(\rho a).$$

Upon the identification (1), pure states correspond to one-dimensional projections [5] $|\psi\rangle\langle\psi|$, i.e., ω is pure iff

$$(2) \qquad \omega(a) = \langle \psi, a\psi \rangle$$

for some unit vector $\psi \in \mathbb{C}^n$.

The states on $A = D_n(\mathbb{C})$ are similarly easy to describe. The positive elements of $D_n(\mathbb{C})$ (i.e. those elements of $D_n(\mathbb{C})$ that can be written as a^*a for some $a \in D_n(\mathbb{C})$) are precisely the matrices with only non-negative real numbers on the diagonal. Since a state

$$\omega : D_n(\mathbb{C}) \to \mathbb{C}$$

is linear, it should take the form

$$\omega(a) = \sum_{i=1}^{n} p_i a_{ii}.$$

Since a state is also positive and unital, we know that

$$p_i \geq 0 \quad \text{for all } i;$$
$$\sum_{i=1}^{n} p_i = 1.$$

In other words, the function

$$p : \{1, \ldots, n\} \rightarrow [0, 1];$$
$$p(i) = p_i,$$

is a probability distribution. Clearly, the map $\omega \mapsto p$ is a bijection between $S(D_n(\mathbb{C}))$ and the set of probability distributions on $\{1, \ldots, n\}$. This map is affine, in the sense that it preserves the convex structure. Hence we only need to determine the extreme points of the convex set of probability distributions to determine the pure states on $D_n(\mathbb{C})$. These extreme points are easily shown to be those probability distributions that satisfy $p_i = 1$ for some $i \in \{1, \ldots, n\}$ and $p_j = 0$ for all other j. Hence the pure states on $D_n(\mathbb{C})$ are of the form

$$\omega_i(a) = a_{ii}.$$

All this may be neatly illustrated for $n = 2$, where the density matrices ρ on $\mathbb{C}^2$ are parametrized by the unit ball $B^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$, in $\mathbb{R}^3$ according to

$$\rho(x, y, z) = \frac{1}{2} \begin{pmatrix} 1 + z & x + iy \\ x - iy & 1 - z \end{pmatrix}.$$

This isomorphism

$$S(M_2(\mathbb{C})) \cong B^3$$

is affine (i.e., it preserves the convex structure), and indeed, the extremal points (x, y, z) in B^3 form the two-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. The corresponding density matrices satisfy $\rho^2 = \rho$ and hence (given that already $\rho^* = \rho$ and $\text{Tr}(\rho) = 1$) they are the one-dimensional projections on $\mathbb{C}^2$.

For the diagonal matrices we have

$$S(D_2(\mathbb{C})) \cong [0, 1],$$

since the pure states on $D_2(\mathbb{C})$ are the two points

$$\omega_i(a) = a_{ii}, \ i = 1, 2.$$

2. The Kadison–Singer Property

Having introduced the basic definitions, let us now streamline the world of the Kadison–Singer conjecture by introducing the Kadison–Singer *property* [6].

Let H be a Hilbert space and denote the *-algebra of all bounded operators on H by $B(H)$, equipped with the adjoint

as an involution, as above. In quantum mechanics one is particularly interested in *abelian* unital *-algebras

$$A \subset B(H),$$

since these define 'classical measurement contexts' in the sense of Bohr [7]. Note that above we discussed the case $A = D_n(\mathbb{C})$, which is indeed abelian.

In Bohr's 'Copenhagen Interpretation' of quantum mechanics, the outcome of any measurement must be recorded in the language of classical physics, which roughly speaking means that such an outcome (assumed sharp, i.e., dispersion-free) defines a pure state on some such A. The question, then, is whether such an outcome also fixes the state of the quantum system as a whole (assuming the latter is pure).

Mathematically, this means the following. Both A and $B(H)$ have states, and states on $B(H)$ obviously *restrict* to states on A. In the reverse direction, we can ask whether states on A *extend* to states on $B(H)$. It turns out that (due to the Hahn–Banach theorem of functional analysis [8]) they always do, but what is at stake is the question whether this extension is *unique*. As suggested above, this question is particularly interesting for *pure* states, and hence we say that A has the **Kadison–Singer property** iff *each* pure state on A extends *uniquely* to a state on $B(H)$. Simple arguments in convexity theory [6, 9] show that if the extension is unique, then it is necessarily pure, so that one might as well say that

A has the Kadison–Singer property iff each pure state on A extends uniquely to a pure state on $B(H)$.

Let us look at this property in a different way, initially for finite-dimensional Hilbert spaces H. Following Dirac, physicists typically write $|\lambda\rangle$ for a (unit) eigenvector of some hermitian operator a with eigenvalue λ. They understand that this fails to identify the corresponding vector state (2) unless a is maximal (in the sense of having non-degenerate spectrum): indeed, if a is not maximal, then it has an eigenvalue λ having at least two orthogonal eigenvectors, which clearly define different vector states on $B(H)$. However, in the maximal case Dirac's notation $|\lambda\rangle$ is used apparently without realizing that even in that case there might be an ambiguity; it was left to Kadison and Singer to note this [9].

Fortunately, in H is finite-dimensional, there is no problem.

Theorem 1. *For each $n \in \mathbb{N}$, the algebra $D_n(\mathbb{C}) \subset M_n(\mathbb{C})$ has the Kadison–Singer property.*

Proof. Consider the pure state ω_i on $D_n(\mathbb{C})$, where $i \in \{1, \ldots, n\}$ is arbitrary. Then writing e_i for the i'th basis vector of $\mathbb{C}^n$, we see that the functional

$$\mu : M_n(\mathbb{C}) \rightarrow \mathbb{C};$$
$$\mu(a) = \langle e_i, ae_i \rangle = a_{ii},$$

is a pure state extension of ω. The only thing that is left to prove is that μ is the *unique* pure state extension of ω. So, suppose that $\mu' : M_n(\mathbb{C}) \rightarrow \mathbb{C}$ is also a pure state extension of ω. Then

$$\mu'(a) = \langle \psi, a\psi \rangle$$

for some unit vector $\psi \in \mathbb{C}^n$. Since μ' extends ω, we know that then

$$|\langle \psi, e_i \rangle|^2 = \mu'(|e_i\rangle\langle e_i|) = \omega(|e_i\rangle\langle e_i|) = 1,$$

whence $\psi = ze_i$ for some $z \in \mathbb{C}$ such that $|z| = 1$. Therefore,

$$\mu'(a) = \langle \psi, a\psi \rangle$$
$$= |z|^2 \langle e_i, ae_i \rangle$$
$$= \mu(a)$$

for each $a \in M_n(C)$, i.e. $\mu' = \mu$ and μ is the unique pure state extension of ω. $\square$

We say that a unital abelian *-algebras $A \subset B(H)$ is **maximal** if there is no abelian unital *-algebra $B \subset B(H)$ that *properly* contains A. If H is finite-dimensional, then the unital *-algebra generated by $a = a^*$ and the unit is maximal abelian iff a is maximal as defined above.

Corollary 2. *Suppose H is a finite-dimensional Hilbert space and suppose that $A \subset B(H)$ is a maximal abelian unital *-algebra. Then A has the Kadison–Singer property.*

Proof. The Kadison–Singer property is stable under unitary equivalence, in that for any unitary u (i.e. $uu^* = 1 = u^*u$), $A \subset B(H)$ has the said property iff $uAu^{-1} \subset B(uH)$ has it. We show that

$$A = uD_n(\mathbb{C})u^*$$

for some unitary matrix u; a unitary change of basis then reduces the argument to the previous case. Since A is maximal abelian, by spectral theory it is generated by n mutually orthogonal one-dimensional projections $f_i = |w_i\rangle\langle w_i|$, $i = 1, \ldots, n$, where the w_i form an orthonormal basis. Putting the latter as columns in a matrix yields u. $\square$

3. Infinite-dimensional Hilbert space

After this warm-up we move to the actual setting of the Kadison–Singer conjecture, viz. an infinite-dimensional *separable* Hilbert space H (i.e., H has a *countable* orthonormal basis). All such spaces are (unitarily, or, equivalently, isometrically) isomorphic. For what follows, two key examples are the space

$$(3) \qquad H = \ell^2(\mathbb{N}),$$

of all functions $\psi : \mathbb{N} \to \mathbb{C}$ for which

$$\sum_n |\psi(n)|^2 < \infty,$$

with inner product

$$\langle \varphi, \psi \rangle = \sum_n \overline{\varphi(n)} \psi(n),$$

and the space

$$(4) \qquad H = L^2(0,1)$$

consisting of all the measurable functions $\psi : (0,1) \to \mathbb{C}$ (up to equivalence with respect to null sets) for which

$$\int_0^1 dx \, |\psi(x)|^2 < \infty,$$

with inner product given by

$$\langle \varphi, \psi \rangle = \int_0^1 dx \, \overline{\varphi(x)} \psi(x).$$

We now look at the unital *-algebra $B(H)$ of all *bounded* operators on H. The infinite-dimensionality of H leads to a number of new phenomena:

- There exist states on $B(H)$ that are not given by (1).
- There exist unitarily *in*equivalent maximal abelian *-algebras in $B(H)$.

In the first point we interpret (1) in the appropriate way, in that we replace density *matrices* by density *operators* [2], that is, positive operators ρ for which

$$\sum_i \langle e_i, \rho e_i \rangle = 1$$

for some (and hence for any) basis (e_i) of H. Von Neumann showed that a state ω on $B(H)$ takes the form (1) iff

$$\omega \left(\sum_n f_n \right) = \sum_n \omega(f_n)$$

for any countable family (f_n) of mutually orthogonal projections (this is similar to the countable additivity condition in the definition of a measure). Such states are called **normal**. The existence of non-normal states is the same as the existence of **singular** states: these are the states that vanish on all one-dimensional projections, and thereby on all compact operators. Trivially, singular states are not normal. In fact, any state is either normal, or singular, or it can be written as a convex combination of a normal and a singular state. This has the immediate corollary that every *pure* state is either normal or singular.

It is however a non-trivial matter to write down states on $B(H)$ that are not normal. Using the Hahn–Banach Theorem, it can be shown that for any $t \in [0,1]$, there exists a (necessarily non-normal) state ω on $B(L^2(0,1))$ such that $\omega(m_x) = t$, where m_x is the position operator of quantum mechanics, i.e., the multiplication operator on (4) given by

$$(5) \qquad m_x \psi(x) = x \psi(x).$$

More generally, if some bounded operator $a \in B(H)$ has $\lambda \in \mathbb{C}$ in its continuous spectrum $\sigma_c(a)$ [10], then there exists a necessarily non-normal state ω on $B(H)$ such that $\omega(a) = \lambda$, see [11, Prop. 4.3.3].

The difference between normal states and singular states is very important for the Kadison–Singer property, so we say a little more about it. Let $A \subset B(H)$ be any unital *-algebra (i.e., A is not necessarily abelian) that satisfies

$$(6) \qquad A'' = A,$$

where the **commutant** S' of any subset $S \subset B(H)$ is defined by

$$S' = \{a \in B(H) \mid ab = ba, b \in S\},$$

and $S'' = (S')'$. By definition, this makes A a **von Neumann algebra**. For example, $B(H)$ is itself a von Neumann algebra, but also, it is easy to see that if A is maximal abelian in $B(H)$, then it is a von Neumann algebra, too: commutativity gives $A \subseteq A'$, whilst maximality pushes this into an equality

$$A = A',$$

which implies (6).

Von Neumann algebras were initially called **rings of operators** by von Neumann himself, and historically their investigation by von Neumann and his assistant Francis Murray [13] launched the (now) vast area of **operator algebras**. Despite the tremendous prestige of von Neumann, initially few mathematicians recognized the importance of this development [16]; among them were Israel Gelfand and Mark Naimark, who created the theory of C*-algebras [3] (which incorporate von Neumann algebras, see also [14, 15]), and also Kadison himself.

The deeper significance of the normality condition, then, was unearthed by Shôichirô Sakai [17], who proved that a unital *-algebra $A \subseteq B(H)$ is a von Neumann algebra iff it is closed in the norm-topology inherited from $B(H)$ (i.e., A is a C*-algebra) *and* is the dual of some Banach space. For example, $B(H)$ is the dual of $B_1(H)$, the space of trace-class operators on H equipped with its own intrinsic norm $\|a\|_1 = \mathrm{Tr}\,|a|$, where $|a| = \sqrt{a^*a}$. This duality property endowes A with a second intrinsic topology, viz. the pertinent weak*-topology, and a state $\omega : A \to \mathbb{C}$ (which is automatically norm-continuous) is normal iff it is weak*-continuous, too [18].

4. Classification of MASA's

We now turn to the second point, i.e., the existence of unitarily inequivalent maximal abelian unital *-algebras $A \subset B(H)$, to be called **MASA**'s from now on.

We start with some examples. First, for the Hilbert space (3) we have the **discrete subalgebra**

$$(7) \qquad A_d = \ell^\infty(\mathbb{N})$$

of all bounded functions $f : \mathbb{N} \to \mathbb{C}$ (with pointwise multiplication), which acts on $\ell^2(\mathbb{N})$ by generalizing (5): $f \in \ell^\infty(\mathbb{N})$ defines a multiplication operator m_f by

$$(8) \qquad m_f \psi(x) = f(x)\psi(x).$$

Second, for the Hilbert space (4) we have the **continuous subalgebra**

$$(9) \qquad A_c = L^\infty(0,1)$$

of all essentially bounded measurable functions $f : (0,1) \to \mathbb{C}$ (with pointwise multiplication), acting as in (8). It is not difficult to show that [8]

$$\ell^\infty(\mathbb{N})' = \ell^\infty(\mathbb{N});$$
$$L^\infty(0,1)' = L^\infty(0,1),$$

so that both A_d and A_c are MASA's. In particular, they are von Neumann algebras. Indeed, in the light of Sakai's result just mentioned it is a standard result in functional analysis that [8]

$$\ell^\infty(\mathbb{N}) \cong \ell^1(\mathbb{N})^*;$$
$$L^\infty(0,1) \cong L^1(0,1)^*.$$

In fact, these are essentially the only examples of MASA's on separable Hilbert spaces. An early result of von Neumann himself was that any abelian von Neumann algebra on a separable Hilbert space is generated by a single self-adjoint operator [19], and this is the key to their classification [11, Thm. 9.4.1]:

Theorem 3. *If H is infinite-dimensional and separable, a maximal abelian *-algebra $A \subset B(H)$ is unitarily equivalent to one of the following:*

- The discrete subalgebra A_d, cf. (7);
- The continuous subalgebra A_c, cf. (9);
- A direct sum $A_c \oplus A_d$;
- A direct sum $A_c \oplus D_n(\mathbb{C})$, where $n \in \mathbb{N}$.

The last two cases (or rather a family of cases), realized on either the Hilbert space $L^2(0,1) \oplus \ell^2(\mathbb{N})$ or $L^2(0,1) \oplus \mathbb{C}^n$, are called **mixed subalgebras**.

The proof of this result relies on the notion of *minimal projections*. A projection p on a Hilbert space H is a linear operator satisfying $p^2 = p^* = p$; it is well known that such operators bijectively correspond to the closed linear subspaces pH of H that form their images. More generally, a projection in a C*-algebra A is an element $p \in A$ that satisfies the same equalities. On the set $P(A)$ consisting of the projections in A, we can define a natural order, which coincides with the notion of positivity for A. For example, in the algebra $\ell^\infty(\mathbb{N})$, the projections are exactly the indicator functions 1_W of subsets $W \subseteq \mathbb{N}$ and $1_W \leq 1_Y$ if and only if $W \subseteq Y$. Of course, the zero-element of A is the minimal element of $P(A)$ with respect to this order, but we say a projection is a **minimal projection** if it is a minimal element of the ordered set $P(A) \setminus \{0\}$. One can easily see that in the case of $\ell^\infty(\mathbb{N})$, the minimal projections are then exactly the indicator functions of single points. Furthermore, the whole algebra is generated by these indicator functions of single points. For the finite dimensional case, i.e. for $D_n(\mathbb{C})$ where $n \in \mathbb{N}$, this is exactly the same.

However, for the continuous subalgebra $L^\infty(0,1)$ the situation is different. Again, the projections are indicator functions, but since for any (measurable) set $A \subseteq [0,1]$ such that $\mu(A) > 0$ there is a $B \subseteq A$ such that $0 < \mu(B) < \mu(A)$, this algebra has no minimal projections and is therefore certainly not generated by them. A mixed subalgebra keeps the middle ground between the discrete and the continous case: it does have minimal projections (coming from the discrete part), but is not generated by them.

Hence we see that the discrete, continuous and mixed cases can be distinguished by considering the number of minimal projections and the question whether the whole algebra is generated by these minimal projections. As it turns out, these two pieces of information classify all maximal abelian unital *-algebras on separable Hilbert spaces: whenever such an algebra has the same properties as one of the three cases we discussed, it is unitarily equivalent to this case; see [6, 11] for details.

5. The Kadison–Singer conjecture

The real goal of the Kadison–Singer conjecture, to which we are now about to turn, is to give a classification of all abelian unital *-algebras $A \subset B(H)$ that have the Kadison–Singer property, where H is a separable Hilbert space. Although we have seen that the finite-dimensional case is misleading as a model for the infinite-dimensional one in at least two ways, one fact remains:

Lemma 4. *Only MASA's can possibly have the Kadison–Singer property.*

Proof. We use some operator algebra theory. It is easy to show that states on unital *-algebras A in $B(H)$ are continuous (i.e., bounded), so we may as well assume that A is closed in the operator norm (in which case it is a so-called C*-algebra). Since A is also abelian, the pure state space $\partial_e S(A)$ coincides with the **Gelfand spectrum** $\Omega(A)$ of A. i.e., the set of all nonzero multiplicative linear functionals on A. This is a compact Hausdorff space too (again in the weak*-topology on the dual space A^*). Gelfand and Naimark proved that A is isomorphic (as a C*-algebra) to the algebra $C(\Omega(A))$ of complex-valued continuous functions on $\Omega(A)$, so that

$$A \cong C(\partial_e S(A))$$

for any abelian C*-algebra A.

Now suppose that $A_1 \subseteq A_2 \subset B(H)$, where A_1 and A_2 are abelian C*-algebras and A_1 has the Kadison–Singer property. Then any pure state ω_1 on A_1 extends uniquely to a pure state ω on $B(H)$, which in turn restricts to a pure state ω_2 on A_2. The map $\omega_1 \mapsto \omega_2$ from $\partial_e S(A_1)$ to $\partial_e S(A_2)$ is then a continuous isomorphism, since its inverse is given by restriction from A_2 to A_1. Hence this isomorphism induces an isomorphism between $C(\partial_e S(A_1))$ and $C(\partial_e S(A_2))$, i.e. between A_1 and A_2, which can easily be shown to be the inclusion function $A_1 \hookrightarrow A_2$. Hence $A_1 = A_2$, so that any C*-subalgebra with the Kadison–Singer property must be maximal. $\qquad\square$

Recall that the Kadison–Singer property is stable under unitary equivalence. In view of the above lemma, in order to complete the classification of abelian *-algebras $A \subset B(H)$ having the Kadison–Singer property (where H is a separable Hilbert space) we only need to answer the question whether the discrete, continuous and mixed subalgebras have the Kadison–Singer property. Note that we have already answered the question positively for the discrete algebra $D_n(\mathbb{C})$ whenever $n \in \mathbb{N}$. However, the other cases, including the infinite discrete case A_d, need a more careful analysis. Indeed, one reason for the difficulty of the subject is that although the unique extension property is hard to prove for an *arbitrary* pure state, we can quite easily prove the following result.

Proposition 5. *Let $A \subset B(H)$ be a MASA (and hence a von Neumann algebra). Then any normal pure state on A has a unique extension to $B(H)$.*

Using density operators, this can be proved as in the finite-dimensional case. It follows that in looking for possible pure states on A without unique extensions to $B(H)$, one necessarily enters the realm of singular states. As we noted, these are hard to grasp, and having already encountered the Hahn–Banach theorem in this context, it may not be surprising that the world of ultrafilters and the like plays a role in the analysis of the Kadison–Singer conjecture. Furthermore, we are not able to treat the singular states on two different MASA's in the same way: each MASA needs a different approach.

Let us start with the continuous case. Kadison and Singer already proved in their original article from 1959 that the continuous subalgebra does not have the Kadison–Singer property. Twenty years later, in 1979, Joel Anderson [20] gave a more straightforward proof of the same fact, and also improved upon it. He proved that there is no pure state on the continuous subalgebra (9) at all that extends in a unique way to a (pure) state on $B(L^2(0,1))$, which is definitely stronger than the negation of having the Kadison–Singer property. Anderson used the Stone–Čech compactification of $\mathbb{N}$ (realized via ultrafilters) in order to be able to describe all pure states on A_c. A careful and tricky argument then gave the desired result (see also [6]).

It is easy to show that if a direct sum of algebras has the Kadison–Singer property, that then all summands must have the Kadison–Singer property too. Hence the fact that the continuous subalgebra does not have the Kadison–Singer property has the immediate corollary that no mixed subalgebra has the Kadison–Singer property. Therefore, if any MASA on a separable Hilbert space has the Kadison–Singer property, it must be unitarily equivalent to the discrete subalgebra A_d (or, if $\dim(H) = n < \infty$, to $D_n(\mathbb{C})$). Thus Kadison and Singer realized that the only open case for the classification of MASA's having the Kadison–Singer property was the discrete algebra (7). They were unable to answer the question for this algebra and left it open.

In the subsequent years and decades, this question became known as the **Kadison–Singer conjecture**:

Any pure state on the abelian von Neumann algebra $\ell^\infty(\mathbb{N})$, realized as multiplication operators on the Hilbert space $\ell^2(\mathbb{N})$, has a unique extension to a (necessarily pure) state on $B(\ell^2(\mathbb{N}))$.

In other words, $\ell^\infty(\mathbb{N}) \subset B(\ell^2(\mathbb{N}))$ *has the Kadison–Singer property.*

6. Proof of the KS-conjecture

In the years that followed, many people worked on this problem. Before the turn of the century, the most notable progress was made by the aforementioned Anderson. He straightened out some of the details in the article by Kadison and Singer and reformulated what later became known as the **paving conjecture**. This is a statement that is equivalent to the Kadison–Singer conjecture and says the following:

For every $\varepsilon > 0$ there is an $l_\varepsilon \in \mathbb{N}$ such that for all $a \in B(\ell^2(\mathbb{N}))$ that satisfy $\mathrm{diag}(a) = 0$, there exists a set of projections $\{p_i\}_{i=1}^{l_\varepsilon} \subseteq \ell^\infty(\mathbb{N})$ such that

$$\sum_{i=1}^{l_\varepsilon} p_i = 1$$

and

$$(10) \qquad \|p_i a p_i\| \leq \varepsilon \|a\|$$

for every $i \in \{1, \ldots, l_\varepsilon\}$.

Here, we have used the function

$$\mathrm{diag}(a) : \mathbb{N} \to \mathbb{C}$$

which is defined by

$$\mathrm{diag}(a)(n) = \langle \delta_n, a\delta_n \rangle.$$

The strength and difficulty in proving this conjecture is contained in the uniformity of l_ε: there is one fixed l_ε that should work for *all* a.

In turn, using Tychonoff's theorem, it can be shown that this paving theorem for operators on $\ell^2(\mathbb{N})$ is equivalent to a paving theorem for matrices. To be more precise, the Kadison–Singer conjecture is equivalent to:

For every $\varepsilon > 0$ there is an $l_\varepsilon \in \mathbb{N}$ such that for all $n \in M_n(\mathbb{C})$ and all $a \in M_n(\mathbb{C})$ such that $\mathrm{diag}(a) = 0$, there is a set of diagonal projections

$$\{p_i\}_{i=1}^{l_\varepsilon} \subseteq D_n(\mathbb{C})$$

such that $\sum_{i=1}^m p_i = 1$ and (10).

This equivalence is quite remarkable, since we can now use tools of linear algebra to draw conclusions about the infinite-dimensional discrete algebra.

In 2004, Nik Weaver [21] formulated a new conjecture, which he showed was equivalent to the paving conjecture. **Weaver's conjecture** was reformulated by Terence Tao [23] as follows:

Suppose $k, m, n \in \mathbb{N}$ and let $C \geq 0$. Furthermore, let $\{A_i\}_{i=1}^k \subseteq M_n(\mathbb{C})$ be a set of positive semi-definite matrices of rank 1, such that

$$\|A_i\| \leq C \text{ for all } 1 \leq i \leq k;$$

$$\sum_{i=1}^k A_i = 1.$$

Then there exists a partition of $\{Z_i\}_{i=1}^m$ of $\{1, \ldots, k\}$ such that for all $j \in \{1, \ldots, m\}$ we have

$$\left\| \sum_{i \in Z_j} A_i \right\| \leq \left(\frac{1}{\sqrt{m}} + \sqrt{C} \right)^2.$$

The true breakthrough came when the theory of random matrices was used. In 2013, computer scientists Adam Marcus, Daniel Spielman and Nikhil Srivastava proved the following theorem:

Theorem 6. *Suppose $\{Y_i\}_{i=1}^n$ is a set of independent random variables taking a finite number of values in the set of positive semi-definite $n \times n$-matrices of rank 1 and let $C > 0$. Furthermore, let*

$$Y = \sum_{i=1}^n Y_i,$$

and suppose that

$$\mathbb{E}Y = 1,$$

and

$$\mathbb{E}\|Y_i\| \leq C,$$

for all $i \in \{1, \ldots, n\}$. Then there is at least one realization $\{A_i\}_{i=1}^n$ of the set $\{Y_i\}_{i=1}^n$ such that

$$\|A\| \leq (1 + \sqrt{C})^2,$$

where

$$A = \sum_{i=1}^n A_i.$$

They proved this theorem considering zeroes of so-called *real stable polynomials*. Using this theorem, the Weaver conjecture can be easily proven (cf. [6]). As a consequence, the Kadison–Singer conjecture was finally proven, 54 years after Kadison and Singer posed their question. Furthermore, it completes the classification of unital abelian C*-subalgebras with the Kadison–Singer property in the case of separable Hilbert spaces:

Theorem 7. *Suppose H is a separable Hilbert space and let $A \subset B(H)$ be an abelian, unital *-algebra. Then A has the Kadison–Singer property if and only if it is unitarily equivalent to the discrete algebra A_d (or, if H is n-dimensional, to the diagonal matrices $D_n(\mathbb{C})$, $n \in \mathbb{N}$).*

Let us close by noting that we have just scratched the surface of the world opened by the Kadison–Singer conjecture and its proof; for further information see [23, 24, 25].

References

[1] Hilbert's student E. Schmidt is reported to have warned von Neumann against using the abstract language of operators: *"Nein! Nein! Sagen Sie nicht Operator, sagen Sie Matrix!"*.

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[5] Throughout this paper a 'projection' p is an *orthogonal* projection ($p^2 = p^* = p$).

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[9] R.V. Kadison & I.M. Singer, Extensions of pure states. *American Journal of Mathematics* 81, 383–400 (1959). The authors actually attribute the general idea of the conjecture to I.E. Segal and I. Kaplansky.

[10] We have $\lambda \in \sigma(a)$, i.e., the full spectrum of a, when $a - \lambda \cdot 1$ is not invertible (Hilbert), or, equivalently, when there exists a sequence (ψ_n) of unit vectors for which $\lim_{n \to \infty} \|(a - \lambda)\psi_n\| = 0$ (Weyl). Then $\lambda \in \sigma_d(a)$ (i.e., the *discrete* spectrum of a) when a has an eigenvector with eigenvalue λ, and $\sigma_c(a) = \sigma(a) \backslash \sigma_d(a)$. In finite dimension $\sigma(a) = \sigma_d(a)$ and hence $\sigma_c(a) = \emptyset$, but on the infinite-dimensional space (4) we have, for example, $\sigma_c(m_x) = [0,1]$ whilst $\sigma_d(m_x) = \emptyset$.

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[16] For example, following a lecture by von Neumann on operator algebras at Harvard sometime in the 1930s, G.H. Hardy is reported to have said to G.D. Birkhoff:
> *"He is quite clearly a brilliant man, but why does he waste his time on this stuff?"*

This anecdote may in fact tell us more about Hardy's own narrow-minded attitudes than about operator algebras, but even von Neumann's close friend and colleague S. Ulam displays a clear lack of appreciation in his autobiography *Adventures of a Mathematician* from 1976.

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[18] Sakai also proved that the predual of a von Neumann algebra is unique (which is not necessarily the case for general Banach spaces).

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