The Muhly–Renault–Williams Theorem for Lie Groupoids and its Classical Counterpart

N. P. LANDSMAN*
Korteweg–de Vries Institute for Mathematics, University of Amsterdam, Plantage Muidergracht 24, NL-1018 TV Amsterdam, The Netherlands; e-mail: npl@science.uva.nl

Abstract. A theorem of Muhly–Renault–Williams states that if two locally compact groupoids with Haar system are Morita equivalent, then their associated convolution C^*-algebras are strongly Morita equivalent. We give a new proof of this theorem for Lie groupoids. Subsequently, we prove a counterpart of this theorem in Poisson geometry: If two Morita equivalent Lie groupoids are s-connected and s-simply connected, then their associated Poisson manifolds (viz. the dual bundles to their Lie algebroids) are Morita equivalent in the sense of P. Xu.

1. Introduction

There are two interesting constructions relating groupoids to C^*-algebras. Firstly, a locally compact groupoid G with Haar system λ defines an associated convolution C^*-algebra $C^*(G, \lambda)$ [1]. Secondly, a Lie groupoid G is intrinsically associated with a convolution C^*-algebra $C^*(G)$ [2].

For example, for a Lie group G the C^*-algebra $C^*(G)$ is isomorphic to the usual convolution algebra of G. For a manifold $G_1 = G_0 = M$, one has $C^*(M) \simeq C_0(M)$, and for a pair groupoid over a manifold M, one obtains the C^*-algebra of compact operators on $L^2(M)$.

Involving operator algebras, the above constructions could be said to be of a 'quantum' nature. From that perspective, the Lie case has a 'classical' counterpart involving Poisson manifolds. Namely, a Lie groupoid G canonically defines a Poisson manifold $A^*(G)$ [3–5], which is the dual vector bundle to the Lie algebroid $A(G)$ associated with G [6, 7]. Our interpretation of the passage $G \mapsto A^*(G)$ as the classical analogue of $G \mapsto C^*(G)$ has been justified by an analysis showing that $C^*(G)$ is a deformation quantization (in the sense of Rieffel) of the Poisson manifold $A^*(G)$ [8–11].

For all four cases of locally compact groupoids, Lie groupoids, C^*-algebras, and Poisson manifolds there exists a notion of Morita equivalence; see [11–14], respectively. A remarkable theorem of Muhly–Renault–Williams [11] states that if two locally compact groupoids with Haar system are Morita equivalent, then so are their associated convolution C^*-algebras.

The fact that any Lie groupoid possesses a Haar system [8, 15] establishes the corresponding result for Lie groupoids. Nonetheless, we give a new proof of the Muhly–Renault–Williams theorem for Lie groupoids, which provides considerable insight into the situation. Our proof is not quite independent of the one in [11], for in the technical step of taking completions of various pre-Banach spaces we rely on certain 'hard' results in the locally compact case [1, 11, 16].

Subsequently, we prove a counterpart of this theorem in Poisson geometry: If two Morita equivalent Lie groupoids are s-connected and s-simply connected, then their associated Poisson manifolds (viz. the dual bundles to their Lie algebroids) are Morita equivalent. The essential technical difficulty in the proof of this theorem, namely the completeness of certain Poisson maps, is overcome by constructing the pullback of the action of a Lie groupoid G on a manifold M; this is an action of the symplectic groupoid T^*G on the cotangent bundle T^*M. This construction also clarifies the definition of T^*G itself.

2. The Muhly–Renault–Williams Theorem for Lie Groupoids

2.1. STATEMENT OF DEFINITIONS AND THEOREM

Our generic notation for groupoids is that G_0 is the base space of a groupoid G, with source and target maps $s, t: G_1 \to G_0$, multiplication $m: G_2 \to G_1$ (where $G_2 = G_1 *_{G_0}^{s,t} G_1$), inversion $I: G_1 \to G_1$, and object inclusion $\iota: G_0 \hookrightarrow G_1$ (this inclusion map will often be taken for granted, in that G_0 is seen as a subspace of G_1).

A Lie groupoid is a groupoid for which G_1 and G_0 are manifolds, s and t are surjective submersions, and m and I are smooth. It follows that ι is an immersion, that I is a diffeomorphism, that G_2 is a closed submanifold of $G_1 \times G_1$, and that for each $q \in G_0$, the fibers $s^{-1}(q)$ and $t^{-1}(q)$ are submanifolds of G_1. References on Lie groupoids that are relevant to the themes in this Letter include [4, 7, 8, 17, 18].

Since they play a central role in Morita theory for Lie groupoids, we now define actions and bimodules of Lie groupoids (these notions occur in a large number of papers, and probably go back to Ehresmann, Skandalis and Moerdijk, respectively).

DEFINITION 2.1. (1) Let G be a Lie groupoid and let $M \xrightarrow{\tau} G_0$ be smooth. A left G-action on M (more precisely, on τ) is a smooth map $(x, m) \mapsto xm$ from

$G *_{G_0}^{s,\tau} M$ to M (i.e., one has $s(x) = \tau(m)$), such that $\tau(xm) = t(x)$, $xm = m$ for all $x \in G_0$, and $x(ym) = (xy)m$ whenever $s(y) = \tau(m)$ and $t(y) = s(x)$.

(2) A right action of a Lie groupoid H on $M \xrightarrow{\sigma} H_0$ is a smooth map $(m, h) \mapsto mh$ from $M *_{H_0}^{t,\tau} H$ to M that satisfies $\sigma(mh) = s(h)$, $mh = m$ for all $h \in H_0$, and $(mh)k = m(hk)$ whenever $\sigma(m) = t(h)$ and $t(k) = s(h)$.

(3) A G-H bibundle M carries a left G action as well as a right H-action that commute. That is, one has $\tau(mh) = \tau(m)$, $\sigma(xm) = \sigma(m)$, and $(xm)h = x(mh)$ for all $(m, h) \in M *_{H_0} H$ and $(x, m) \in G *_{G_0} M$. On occasion, we simply write $G \rightarrowtail M \leftarrowtail H$.

The maps τ and σ will sometimes be called the base maps of the given actions.

(4) A left action of a Lie groupoid G on $M \xrightarrow{\tau} G_0$ is called principal when τ is a surjective submersion, and the action is free (in that $xm = m$ iff $x \in G_0$) and proper (that is, the map $(x, m) \mapsto (xm, m)$ from $G *_{G_0} M$ to $M \times M$ is proper).

A similar definition applies to right actions.

We now recall the definition of Morita equivalence of groupoids used in [11], adapted to the smooth (Lie) case [12].

DEFINITION 2.2. A G-H bibundle M between Lie groupoids is called an equivalence bibundle when

(1) M is left and right principal;
(2)   One has $M/H \simeq G_0$ via τ and $G \backslash M \simeq H_0$ via σ.

Two Lie groupoids related by an equivalence bibundle are called Morita equivalent.

This concept of Morita equivalent will be related to that for C^*-algebras [13]. Since various equivalent definitions are possible [19], we recall the one that will be used. For the notion of a Hilbert C^* module that occurs, see [8, 19].

DEFINITION 2.3. (1) An $\mathfrak{A}$-$\mathfrak{B}$ Hilbert bimodule, where $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras, is a Hilbert C^* module $\mathcal{E}$ over $\mathfrak{B}$, along with a nondegenerate $*$-homomorphism of $\mathfrak{A}$ into the C^*-algebra of adjointable operators $\mathcal{L}_{\mathfrak{B}}(\mathcal{E})$.

(2) An equivalence Hilbert bimodule between two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ is an $\mathfrak{A}$-$\mathfrak{B}$ Hilbert bimodule $\mathcal{M}$ that, in addition, is a left Hilbert C^* module over $\mathfrak{A}$, such that

(a)   The range of $\langle \, , \, \rangle_{\mathfrak{B}}$ is dense in $\mathfrak{B}$;
(b)   The range of $_{\mathfrak{A}}\langle \, , \, \rangle$ is dense in $\mathfrak{A}$;
(c)   The $\mathfrak{A}$-valued inner product is related to the $\mathfrak{B}$-valued one by

$$_{\mathfrak{A}}\langle \psi, \varphi \rangle \zeta = \psi \langle \varphi, \zeta \rangle_{\mathfrak{B}}, \tag{2.1}$$

for all $\psi, \varphi, \zeta \in \mathcal{M}$.

(3) Two C^*-algebras are called (strongly) Morita equivalent when there exists an equivalence Hilbert bimodule between them.

The Muhly–Renault–Williams theorem for Lie groupoids then reads

THEOREM 2.4. *If G and H are Morita equivalent as Lie groupoids, then their associated C^*-algebras $C^*(G)$ and $C^*(H)$ are Morita equivalent as C^*-algebras.*

As stated in the Introduction, this theorem follows from the corresponding result for locally compact groupoids with Haar system [11]. The proof in [11] consists of two steps.

In the first step, one sets up a pre-equivalence Hilbert bimodule between $C^*(G, \lambda)$ and $C^*(H, \mu)$, given a G-H equivalence bibundle M. Here a pre-equivalence Hilbert bimodule for C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ is defined as in Definition 2.3, with the difference that $\mathfrak{A}$ and $\mathfrak{B}$ are replaced by dense subalgebras $\mathfrak{A}_0$ and $\mathfrak{B}_0$, respectively, and the Hilbert C^*-module $\mathcal{E}_0$ over $\mathfrak{B}_0$ is not required to be complete. In the case at hand, one has $\mathfrak{A}_0 = C_c(G, \lambda)$, $B_0 = C_c(H, \mu)$, and $\mathcal{E}_0 = C_c(M)$.

For the second step, see Section 2.6 below, and [20]. In the Lie case, we have been able to replace the first step of the proof of the locally compact case in [11] by purely differential geometric arguments. This requires some preparation.

2.2. HALF-DENSITIES ON LIE GROUPOIDS

Following [2], we use the well-known formalism of half-densities, for which we need to establish some notation. Let E be a vector bundle over a manifold M with n-dimensional typical fiber E_m. The bundle $\mathsf{A}(E)$ is defined as $\wedge^n E$ minus the zero section. This is a principal $\mathbb{C}^*$-bundle over M, whose fiber at m is the n-fold antisymmetric tensor product of E_x, with 0 omitted (here $\mathbb{C}^*$ is $\mathbb{C} \backslash \{0\}$, seen as a multiplicative group). For $\alpha \neq 0$, the bundle of α-densities $|\Lambda|^\alpha(E)$ is the line bundle over M associated to $\mathsf{A}(E)$ by the representation $z \mapsto |z|^{-\alpha}$ of $\mathbb{C}^*$ on $\mathbb{C}$. Hence, sections of $|\Lambda|^\alpha(E)$ may be seen as maps $\varphi: \mathsf{A}(E) \to \mathbb{C}$ satisfying $\varphi(zv) = |z|^\alpha \varphi(v)$. One has natural (and obvious) isomorphisms

$$|\Lambda|^\alpha(E) \otimes |\Lambda|^\beta(E) \simeq |\Lambda|^{\alpha+\beta}(E); \tag{2.2}$$

$$|\Lambda|^\alpha(E \oplus F) \simeq |\Lambda|^\alpha(E) \otimes |\Lambda|^\alpha(F). \tag{2.3}$$

The point of this formalism is already evident in the simplest case, where $E = TM$ and $\alpha = 1$; for one may integrate sections of $C_c^\infty(M, |\Lambda|^1(TM))$ over M without choosing a measure (even when M is nonorientable). Similarly, using (2.2), $\int_M fg$ makes sense for $f, g \in C_c^\infty(M, |\Lambda|^{1/2}(TM))$. Generalizing this case, let $M \overset{\tau}{\to} X$ be a fibration for which τ is a surjective submersion, and let $T^\tau M$ be the subbundle of TM whose fibers are tangent to the fibers of τ. One may then integrate $f \in C_c^\infty(M, |\Lambda|^1(T^\tau M))$, or fg, where $f, g \in C_c^\infty(M, |\Lambda|^{1/2}(T^\tau M))$, over any fiber of τ.

2.3. THE CATEGORY OF PRINCIPAL G BUNDLES

Recall the definition of a principal G action (Definition 2.1). The collection of all such actions (or bundles) can be made into a category, with unexpected choice of arrows. This category greatly clarifies both the definition of a Lie groupoid C^*-algebra $C^*(G)$ and the proof of Theorem 2.4. The construction of this category may be found in [15], which contains further details.

Let G be a Lie groupoid, and let $M \xrightarrow{\tau} G_0$ be a principal left G-space. The G-action pulls back to a G-action on $|\Lambda|^{1/2}(T^\tau M)$, which thereby becomes a principal left G-space as well, and one has the isomorphism

$$C^\infty_{c/G}(M, |\Lambda|^{1/2}(T^\tau M))^G \simeq C^\infty_c(G\backslash M, G\backslash|\Lambda|^{1/2}(T^\tau M)). \tag{2.4}$$

Here the left-hand side consists of G-equivariant sections (that is, $\varphi(xm) = x\varphi(m)$) with compact support up to G-translations. As to the right-hand side, note that if E is a vector bundle over X such that E and X are principal left G-manifolds compatible with the bundle projection, then $G\backslash E$ is naturally a vector bundle over $G\backslash X$.

In addition, let $N \xrightarrow{\sigma} G_0$ be a principal left G-space. Then the fiber product $M *_{G_0} N$ is a principal left G-space under the obvious action $x: (m, n) \mapsto (xm, xn)$. We now define the complex vector space

$$(M, N)_G = C^\infty_{c/G}(M *_{G_0} N, |\Lambda|^{1/2}(T^\tau M) \otimes |\Lambda|^{1/2}(T^\sigma N))^G. \tag{2.5}$$

In view of (2.3) and the obvious fact

$$T^{\tau=\sigma}_{(m,n)}(M *_{G_0} N) = T^\tau_m M \oplus T^\sigma_n N \tag{2.6}$$

for $(m, n) \in M *_{G_0} N$, one has the natural isomorphism

$$(M, N)_G \simeq C^\infty_{c/G}(M *_{G_0} N, |\Lambda|^{1/2}(T^{\tau=\sigma}(M *_{G_0} N)))^G, \tag{2.7}$$

which may clarify the meaning of $(M, N)_G$.

The point is now that, given a third principal left G-space $Q \xrightarrow{\rho} G_0$, one has a pairing $(M, N)_G \times (N, Q)_G \to (M, Q)_G$, given by

$$f * g(m, q) = \int_{\sigma^{-1}(\tau(m))} f(m, \cdot) \otimes g(\cdot, q). \tag{2.8}$$

This is well defined in view of (2.2) and subsequent paragraph; note that $\tau(m) = \rho(q)$ by definition of $M * Q$. Furthermore, one has a map $*: (M, N)_G \to (N, M)_G$, given by $f^*(n, m) = \overline{\mathrm{flip}[f(m, n)]}$, where flip: $V \otimes W \to W \otimes V$ is given by $\mathrm{flip}(v \otimes w) = w \otimes v$. This map is involutive, in being antilinear and satisfying $(f * g)^* = g^* * f^*$. It follows that the principal left G-manifolds are the objects of a $*$-category whose arrows are the spaces $(M, N)_G$.

2.4. THE C^*-ALGEBRA OF A LIE GROUPOID

To define $C^*(G)$, note that $G \xrightarrow{t} G_0$ is itself a principal left G-manifold. Hence, the vector space $(G, G)_G$ becomes a $*$-algebra under the above multiplication $(G, G)_G \times (G, G)_G \to (G, G)_G$ and involution $(G, G)_G \to (G, G)_G$. Equipped with a suitable norm, $(G, G)_G$ is a pre-C^*-algebra whose completion is the groupoid C^*-algebra $C^*(G)$. One has the natural isomorphisms (cf. (2.4) and (2.14) below)

$$
\begin{aligned}
(G, G)_G &\simeq C_{c/G}^\infty(G *_{G_0}^{t,t} G, |\Lambda|^{1/2}(T^{t=t}G *_{G_0}^{t,t} G)^G \\
&\simeq C_c^\infty(G/(G *_{G_0}^{t,t} G), G/|\Lambda|^{1/2}(T^{t=t}(G *_{G_0}^{t,t} G))) \\
&\simeq C_c^\infty(G, |\Lambda|^{1/2}(T^s G) \otimes |\Lambda|^{1/2}(T^t G)),
\end{aligned}
\tag{2.9}
$$

so that $(G, G)_G$ is isomorphic with the convolution $*$-algebra defined by Connes [2]. The Lie groupoid C^*-algebra $C^*(G)$ is then the completion of $(G, G)_G$ in the norm $\|f\| = \sup\{\|\pi(f)\|\}$, where the supremum is taken over all representations (on Hilbert spaces) of $(G, G)_G$ (as a $*$-algebra) that are continuous with respect to the inductive limit topology on $(G, G)_G$. The existence of the supremum follows from results in the locally compact case, namely Proposition 4.2 in [16] and Proposition II.1.7 in [1]. Here, as in the second step of the proof of the theorem at hand, it seems that taking completions necessarily involves the theory of locally compact groupoids with Haar system.

The second isomorphism in (2.9) follows from the following, more general case. For a principal left G-manifold M, one has the diffeomorphism $G/(G *_{G_0}^{t,\tau} M) \simeq M$ under the map

$$
[x, m]_G \mapsto x^{-1}m;
\tag{2.10}
$$

this is well defined since $t(x) = \tau(m)$ by the definition of $G *_{G_0}^{t,\tau} M$, so that $(x^{-1}, m) \in G *_{G_0}^{s,\tau} M$. As we have seen in (2.6), one has $T_{(x,m)}G *_{G_0}^{t,\tau} M = T_x^t G \oplus T_m^\tau M$; the derivative of (2.10) maps $T_x^t G$ into $T_{x^{-1}m}^G M$ and maps $T_m^\tau M$ into $T_{x^{-1}m}^\tau M$. Here the vertical tangent space $T_m^G M$ consists of all vectors that are tangent to G orbits. With (2.4) and (2.2) this yields the isomorphism

$$
\begin{aligned}
(G, M)_G &\simeq C_{c/G}^\infty(G *_{G_0}^{t,\tau} M, |\Lambda|^{1/2}(T^{t=\tau}G *_{G_0}^{t,\tau} M))^G \\
&\simeq C_c^\infty(M, |\Lambda|^{1/2}(T^G M) \otimes |\Lambda|^{1/2}(T^\tau M)).
\end{aligned}
\tag{2.11}
$$

The isomorphism (2.9) is evidently a special case of this.

2.5. CONSTRUCTION OF THE PRE-EQUIVALENCE HILBERT BIMODULE

Analogous considerations for right actions lead to a right version of (2.11), viz.

$$
\begin{aligned}
(M, H)_H &\simeq C_{c/H}^\infty(M *_{H_0}^{\sigma,s} H, |\Lambda|^{1/2}(T^{\sigma=s}M *_{H_0}^{\sigma,s} H))^H \\
&\simeq C_c^\infty(M, |\Lambda|^{1/2}(T^\sigma M) \otimes |\Lambda|^{1/2}(T^H M)).
\end{aligned}
\tag{2.12}
$$

For an equivalence bibundle, condition 2 in Definition 2.2 implies

$$T^\sigma M = T^G M; \qquad T^\tau M = T^H M. \tag{2.13}$$

By pullback, we obtain the isomorphism

$$(G, M)_G \simeq (M, H)_H. \tag{2.14}$$

This gives us a pre-equivalence Hilbert bimodule $\mathcal{M}_0$ between $(G, G)_G$ and $(H, H)_H$, as follows:

- Identifying $\mathcal{M}_0$ with $(M, H)_H$, one obtains a right $(H, H)_H$ representation on $\mathcal{M}_0 = (M, H)_H$ from the pairing $(M, H)_H \times (H, H)_H \to (M, H)_H$; that is, for $\psi \in \mathcal{M}_0$ and $B \in (H, H)_H$ one puts $\psi B = \psi * B$.
- Similarly, the map $\langle \psi, \varphi \rangle_{(H,H)_H} = \psi^* * \varphi$ maps from $(M, H)_H^* \times (M, H)_H = (H, M)_H \times (M, H)_H$ into $(H, H)_H \subset C^*(H)$, providing an $(H, H)_H$-valued inner product on $\mathcal{M}_0$.
- On the other hand, identifying $\mathcal{M}_0$ with $(G, M)_G$, one obtains a representation of $(G, G)_G$ on $\mathcal{M}_0$ from the pairing $(G, G)_G \times (G, M)_G \to (G, M)_G$; for $\psi \in \mathcal{M}_0$ and $A \in (G, G)_G$ one puts $A\psi = A * \psi$.
- On the same identification, $_{(G,G)_G}\langle \psi, \varphi \rangle = \psi * \varphi^*$ maps from $(G, M)_G \times (G, M)_G^* = (G, M)_G \times (M, G)_G \to (G, G)_G$, defining a $(G, G)_G$ valued inner product on $\mathcal{M}_0$.

The required algebraic properties, including (2.1), are trivial consequences of the associativity of the $*$-product, and of the involutivity of $*$. Positivity of the inner products and density of their images is also easily established using the method of P. Green [21] (Section 2), as in the locally compact case. Indeed, the Lie analogue of Prop. 2.10 in [11] may be directly proved for Lie groupoids in the same way as for locally compact groupoids. See Lemmas 4.18–4.20 in [22].

2.6. TAKING COMPLETIONS

One now has to show that our pre-equivalence Hilbert bimodule can be completed. As is well known [19, 23], a sufficient condition for this to be possible is that for all $\psi \in \mathcal{M}_0$ one has the bounds $\langle A\psi, A\psi \rangle_{\mathfrak{B}_0} \leqslant \|A\|^2 \langle \psi, \psi \rangle_{\mathfrak{B}_0}$ for all $A \in \mathfrak{A}_0$ and $_\mathfrak{A}\langle \psi B, \psi B \rangle \leqslant \|B\|^2 {}_\mathfrak{A}\langle \psi, \psi \rangle$ for all $B \in \mathfrak{B}_0$. That these bounds are satisfied in our groupoid situation follows from two deep results of Renault, viz. Proposition 4.2 in [16] and Proposition II.1.7 in [1]. See also [20].

Thus we have been unable to modify the final stage of the proof of [11] by specific Lie groupoid arguments, but given the fact that taking completions necessarily abandons the smooth setting, it seems doubtful that such arguments exist.

3. A Classical Analogue of the Muhly–Renault–Williams Theorem for Lie Groupoids

3.1. STATEMENT OF DEFINITIONS AND THEOREM

We recall the passage from a Lie groupoid to its Lie algebroid [6, 7].

Remark 3.1. A Lie groupoid G defines a Lie algebroid $A(G)$ over G_0, as follows:

(1) The vector bundle $A(G)$ over G_0 is the kernel of Tt (the derivative of the target projection $t: G \rightarrow G_0$) restricted (or pulled back) to G_0; hence

$$A(G) = \ker(Tt)_{|G_0}. \tag{3.15}$$

Accordingly, the bundle projection is given by s or t (which coincide on G_0).

(2) The anchor is given by $a = Ts$ (restricted to $A(G)$).
(3) Identifying a section of $A(G)$ with a left-invariant vector field on G_1, the Lie bracket $[\,,\,]_{A(G)}$ is given by the commutator of vector fields on G_1.

For example, TQ is the Lie algebroid of the pair groupoid $Q \times Q$, and the Lie algebra $\mathfrak{g}$ of a Lie group is its Lie algebroid.

Note that, since $\ker(Tt)_{|G_0}$ is a complement to $T(\iota(G_0))$, the Lie algebroid $A(G)$ is isomorphic to the normal bundle $\tilde{A}(G)$ of the embedding $\iota: G_0 \hookrightarrow G$. This isomorphism endows $\tilde{A}(G)$ with the structure of a Lie algebroid as well, isomorphic to $A(G)$, and this alternative version is often called the Lie algebroid of G, too (cf., e.g., [4]).

One part of the connection between Lie algebroids and Poisson manifolds is laid out by the following result [3–5].

PROPOSITION 3.2. *The dual vector bundle E^* to a Lie algebroid E has a canonical Poisson structure that is linear. Conversely, any vector bundle with a linear Poisson structure is dual to a Lie algebroid. This establishes a categorical equivalence between linear Poisson structures on vector bundles and Lie algebroids.*

In particular, the dual vector bundle $A^*(G)$ of the Lie algebroid $A(G)$ of a Lie groupoid G, as well as the dual bundle $\tilde{A}^*(G)$ of $\tilde{A}(G)$ (which is isomorphic to $A^*(G)$) accordingly become Poisson manifolds.

Here linearity means that the Poisson bracket of two linear functions is linear; a function on E^* is, in turn, called linear when it is linear on each fiber. Each section σ of E defines such a function $\tilde{\sigma}$ in the obvious way. Also, each $f \in C^\infty(Q)$ (where Q is the base of E) trivially defines $\tilde{f} \in C^\infty(E^*)$. The Poisson bracket on E^* is then

determined by the following special cases:

$$\{\tilde{f}, \tilde{g}\} = 0; \tag{3.16}$$

$$\{\tilde{\sigma}, \tilde{f}\} = (\widetilde{a_* \sigma})f; \tag{3.17}$$

$$\{\tilde{\sigma}_1, \tilde{\sigma}_2\} = \widetilde{[\sigma_1, \sigma_2]}_E. \tag{3.18}$$

These formulae show quite clearly how the data of a Lie algebroid determine the Poisson structure, which is of a special kind. For example, for a Lie group G the Poisson manifold $A^*(G)$ is just the dual of the Lie algebra of G, equipped with the usual Lie–Poisson structure. For a manifold $G_1 = G_0 = M$ one finds $A^*(G) = M$ with zero Poisson bracket, and for a pair groupoid $G_1 = M \times M$ one obtains $A^*(G) = T^*M$ with the canonical (symplectic) Poisson structure.

Remark 3.3. Note that $\tilde{A}^*(G)$ is the subbundle of T^*G consisting of 1-forms over G_0 that annihilate $TG_0 \subset TG_{|G_0}$. The isomorphism $\tilde{A}^*(G) \simeq A^*(G)$ arises as follows: for each $q \in G_0$ one has a decomposition

$$T_q G = A_q G \oplus T_q G_0; \tag{3.19}$$

cf. (3.15). Hence, $\alpha_q \in A_q^*(G)$ defines $\tilde{\alpha}_q \in \tilde{A}_q^*(G) \subset T_q^*G$ by putting $\tilde{\alpha}_q = \alpha_q$ on $A_q(G)$ and $\tilde{\alpha}_q = 0$ on $T_q G_0$. Conversely, $\tilde{\alpha}_q \in \tilde{A}_q^*(G)$ defines $\alpha_q \in A_q^*(G)$ by restricting it to $A_q(G) \subset T_q G$.

The theory of Morita equivalence of Poisson manifolds was initiated by Xu [14], who gave the following definition.

DEFINITION 3.4. (1) A symplectic bimodule $Q \xleftarrow{q} S \xrightarrow{p} P$ for two Poisson manifolds P, Q consists of a symplectic space S with complete Poisson maps $p: S \to P^-$ and $q: S \to Q$, such that $\{p^*f, q^*g\} = 0$ for all $f \in C^\infty(P)$ and $g \in C^\infty(Q)$.

(2) A symplectic bimodule $Q \leftarrow S \to P$ is called an equivalence symplectic bimodule when:

(a) The maps $p: S \to P$ and $q: S \to Q$ are surjective submersions;
(b) The level sets of p and q are connected and simply connected;
(c) The foliations of S defined by the levels of p and q are mutually symplectically orthogonal (in that the tangent bundles to these foliations are each other's symplectic orthogonal complement).

(3) Two Poisson manifolds are called Morita equivalent when there exists an equivalence symplectic bimodule between them.

Our 'classical' analogue of Theorem 2.4 is now as follows.

THEOREM 3.5. *Let G and H be s-connected and s-simply connected Lie groupoids, with associated Poisson manifolds $A^*(G)$ and $A^*(H)$. If G and H are Morita equivalent as Lie groupoids, then $A^*(G)$ and $A^*(H)$ are Morita equivalent as Poisson manifolds; cf. Definition 3.4.*

The outline of the proof is as follows. Given a G-H bibundle M implementing the Morita equivalence of G and H (see Definition 2.2), we equip $S = T^*M$ with the structure of an $A^*(G)$-$A^*(H)$ symplectic bimodule that satisfies all conditions in Definition 3.4. This involves two constructions that are interesting in their own right, which are the subject of Sections 3.2 and 3.4.

3.2. THE MOMENTUM MAP FOR LIE GROUPOID ACTIONS

The basic construction is valid in more generality than our situation needs.

PROPOSITION 3.6. *A left action of a Lie groupoid G on a manifold M defines a complete Poisson map $J_L : T^*M^- \to A^*(G)$ (called the momentum map of the G action). Here $A^*(G)$ and $T^*M = A^*(M \times M)$ carry the Poisson structure defined in Proposition 3.2 (which induces the canonical one on T^*M).*

Similarly, a right action of a Lie groupoid H on M defines a complete Poisson map $J_R : T^*M \to A^*(H)$.

Except for the completeness of J_L, J_R, the proof is a straightforward generalization of the case where G and H are Lie groups. The G-action leads to a map $\xi^L : A(G) \to TM$, $X \mapsto \xi^L_X$, for which $\tau_{M \to G_0} \circ \tau_{TM \to M}(\xi^L_X) = \tau_{A(G) \to G_0}(X)$. With

$$X = \frac{d\gamma(\lambda)}{d\lambda}_{|\lambda=0} \in \pi^{-1}(q), \tag{3.20}$$

$q \in G_0$, where, by definition of the Lie algebroid $A(G)$, one has

$$t(\gamma(\lambda)) = t(\gamma(0)) = q \tag{3.21}$$

for all λ, this map is given by [8]

$$\xi^L_X(m) = -\frac{d}{d\lambda} \gamma(\lambda)^{-1} m_{|\lambda=0}. \tag{3.22}$$

Here $\tau(m) = q$. Note that $\xi^L_X \in T_mM$, since $\gamma(0) \in G_0$ by definition of the Lie algebroid, and $\gamma(0)m = m$ by definition of a groupoid action. This yields our momentum map by

$$\langle J_L(\theta), X \rangle = \langle \theta, \xi^L_X \rangle. \tag{3.23}$$

One then checks that $J_L : T^*M \to A^*(G)$ is an anti-Poisson map, so that $J_L : T^*M^- \to A^*(G)$ is a Poisson map, as follows. As before, we write $\tau = \tau_{M \to G_0}$.

For $f \in C^\infty(G_0)$ one has $J_L^{\tilde{*}} f = \hat{f}$, where $\hat{f} = f \circ \tau \circ \tau_{T^*M \to M}$, so that

$$\{J_L^{\tilde{*}} f, J_L^* \tilde{g}\}_{T^*M} = \{\hat{f}, \hat{g}\}_{T^*M} = 0 = J_L^*\{f, \tilde{g}\}_{A^*(G)}$$

by (3.16).

For a section σ of $A(G)$, which we take to be of the form $\sigma(q) = X(q)$, as in (3.25), with q-dependent curves $\gamma_q(\lambda)$, one obtains a vector field ξ_σ^L, in terms of which $J_L^* \tilde{\sigma} = \text{sym}(\xi_\sigma^L)$. Here $\text{sym}(\xi) \in C^\infty(T^*M)$ denotes the symbol of a vector field ξ on M. The canonical Poisson bracket on T^*M satisfies

$$\{\text{sym}(\xi), h\}_{T^*M}(\theta_m) = \xi h(m) \tag{3.24}$$

for $h \in C^\infty(M)$, so that

$$\begin{aligned}
&\{J_L^* \tilde{\sigma}, J_L^* \tilde{f}\}_{T^*M}(\theta_m) \\
&= \xi_\sigma^L \hat{f}(m) = -\frac{\mathrm{d}}{\mathrm{d}\lambda} f(\tau[\gamma_q(\lambda)^{-1} m])_{|0} = -\frac{\mathrm{d}}{\mathrm{d}\lambda} f(s(\gamma_q(\lambda)))_{|0} \\
&= -(Ts)(X(q))f(q) = -(a_*\sigma)f(q) = -J_L^*\{\tilde{\sigma}, \tilde{f}\}_{A^*(G)}(\theta_m),
\end{aligned}$$

where $q = \tau(m)$. Here we used (3.17) and Remark 3.1(2).

Finally, using Remark 3.1(3), the property

$$\{\text{sym}(\xi), \text{sym}(\eta)\}_{T^*M} = \text{sym}([\xi, \eta]), \tag{3.25}$$

and (3.18), one proves that

$$\{J_L^* \tilde{\sigma}_1, J_L^* \tilde{\sigma}_2\}_{T^*M} = -J_L^*\{\tilde{\sigma}_1, \tilde{\sigma}_2\}_{A^*(G)}.$$

Since the differentials of the functions in question span $T^*(A^*(G))$, this proves that $J_L: T^*M^- \to A^*(G)$ is a Poisson map.

For the right H-action we define $J_R: T^*M \to A^*(H)$ by

$$\left\langle J_R(\theta_m), \frac{\mathrm{d}h(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle = \left\langle \theta_m, \frac{\mathrm{d}mh(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle, \tag{3.26}$$

where $h(\lambda) \in t_H^{-1}(\sigma(m))$, so that its tangent vector at 0 lies in $A_{\sigma(m)}H$, and the expression $mh(\lambda)$ is defined. This may be shown to be a Poisson map by essentially the same computations as for J_L.

The completeness of J_L and J_R will be proved in Section 3.4. $\qquad \square$

The corresponding momentum maps $\tilde{J}_L: T^*M^- \to \tilde{A}^*(G)$ and $\tilde{J}_R: T^*M \to \tilde{A}^*(H)$ (cf. Remark 3.3) arise in the obvious way, by extending the given expression by 0 on TG_0. However, it is instructive to rewrite $\tilde{J}_L$. Instead of (3.19), we now use the decomposition $TG_{|G_0} = \ker(Ts)_{|G_0} \oplus TG_0$. Relative to this, a vector $dx/d\lambda_{|0} \in \ker(Tt)$, with $\gamma(0) = q \in G_0$, decomposes as

$$\frac{\mathrm{d}\gamma(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} = -\frac{\mathrm{d}\gamma(\lambda)^{-1}}{\mathrm{d}\lambda}_{|\lambda=0} + \frac{\mathrm{d}s(\gamma(\lambda))}{\mathrm{d}\lambda}_{|\lambda=0}. \tag{3.27}$$

Hence, on $\ker(Ts)_{|G_0} \subset TG_{|G_0}$, we simply have

$$\left\langle \tilde{J}_L(\theta_m), \frac{\mathrm{d}z(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle = \left\langle \theta_m, \frac{\mathrm{d}z(\lambda)m}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle. \qquad (3.28)$$

Here $z(\lambda)$ lies in the s-fiber above $\tau(m) \in G_0$, so that the right-hand side is defined. Compare this with (3.23), which may be written as

$$\left\langle J_L(\theta_m), \frac{\mathrm{d}\gamma(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle = -\left\langle \theta_m, \frac{\mathrm{d}\gamma(\lambda)^{-1}m}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle, \qquad (3.29)$$

where $\gamma(\lambda)$ lies in the t-fiber above $\tau(m)$.

COROLLARY 3.7. *Let G and H be Lie groupoids, and let M be a G-H bibundle. Then there exist maps J_L, J_R for which*

$$A^*(G) \xleftarrow{J_L} T^*M^- \xrightarrow{J_R} A^*(H) \qquad (3.30)$$

is a symplectic bimodule.

The definition of a groupoid bibundle easily implies that the last condition in Definition 3.4.1 is met: Firstly,

$$\{J_L^{*\sim}f, J_R^*\tilde{g}\}_{T^*M} = \{\hat{f}, \check{g}\}_{T^*M} = 0,$$

where $\check{g} = g \circ \sigma \circ \tau_{T^*M \to M}$. Secondly, using (3.24), one has

$$\{J_L^*\tilde{\sigma}, J_R^*\tilde{g}\}_{T^*M} = \xi_\sigma^L \check{g} = 0,$$

since $\sigma: M \to H_0$ is G-invariant. Similarly,

$$\{J_L^{*\sim}f, J_R^*\tilde{\sigma}\}_{T^*M} = -\xi_\sigma^R \hat{f} = 0,$$

since $\tau: M \to G_0$ is H-invariant. Finally, using (3.25) and the fact that the G and H actions on M commute, one computes

$$\{J_L^*\tilde{\sigma_1}, J_R^*\tilde{\sigma_2}\}_{T^*M} = \mathrm{sym}([\xi_{\sigma_1}^L, \xi_{\sigma_2}^R]) = \mathrm{sym}(0) = 0.$$

Checking Poisson commutativity for the given functions suffices.   □

3.3. THE COTANGENT BUNDLE OF A LIE GROUPOID

In order to prove completeness of the maps J_L and J_R, we will need the cotangent bundle of a Lie groupoid [4]. We here reinterpret their source and target maps in terms of the momentum maps J_L and J_R of the preceding section.

PROPOSITION 3.8. *The cotangent bundle T^*G^- of a Lie groupoid G becomes a symplectic groupoid over $\tilde{A}^*(G)$ in the following way (we here work with $\tilde{A}^*(G)$ rather*

than $A^*(G)$ in order to facilitate the use of [4]). Consider G as a G-G bibundle in the obvious way. The source map $\tilde{s}: T^*G \to A^*(G)$ is given by $\tilde{s} = \tilde{J}_R$, the target is $\tilde{t} = \tilde{J}_L$, the object inclusion map is $\tilde{A}^*(G) \hookrightarrow T^*G$, inversion is $\tilde{I} = -I^*$, and multiplication is defined as follows.

First note that, by definition of $\tilde{J}_L$ and $\tilde{J}_R$, one has $\tilde{s}(\alpha_x) \in \tilde{A}^*_{s(x)}(G)$ and $\tilde{t}(\beta_y) \in \tilde{A}^*_{t(x)}(G)$. Hence, the condition $(\alpha_x, \beta_y) \in T^*G_2$ implies $(x, y) \in G_2$. As in [4], one shows that the former condition implies that there exists a (necessarily unique) $\gamma_{xy} \in T^*_{xy}G$ such that

$$\alpha_x(X) + \beta_y(Y) = \gamma_{xy}(T_{(x,y)}m(X, Y)) \tag{3.31}$$

for all $(X, Y) \in T_{(x,y)}G_2$, and this γ_{xy} in fact lies in $\tilde{A}^*_{xy}(G)$. The multiplication $\tilde{\mp}$ in T^*G is then given by

$$\alpha_x \tilde{\mp} \beta_y = \gamma_{xy}. \tag{3.32}$$

3.4. THE PULLBACK OF A LIE GROUPOID ACTION

We will prove that J_L and J_R are complete by constructing symplectic actions of the symplectic groupoids T^*G and T^*H (cf. Proposition 3.8) on T^*M with base maps J_L and J_R, respectively, Completeness then follows from Thm. 3.1 in [14], stating that the base map of a symplectic groupoid action is automatically complete.

The following theorem covers the general situation. It generalizes Ex. 3.9 in [17] from groups to groupoids, and its corollary of completeness generalizes Lemma 3.1 in [24].

THEOREM 3.9. *Let G be a Lie groupoid acting on a manifold M, with associated momentum map $J_L: T^*M^- \to A^*(G)$ (cf. Proposition 3.6).*

There exists a symplectic action of T^*G^- (cf. Proposition 3.8) on T^*M^- with base map J_L. In particular, J_L is complete.

Take $\alpha_x \in T^*_xG$ and $\theta_m \in T^*_mM$ such that $\tilde{s}(\alpha_x) = J_L(\theta_m)$. According to (3.29) and Proposition 3.8, using (3.26) applied to the case $M = G$, this condition implies $s(x) = \tau(m)$, and otherwise reads

$$\left\langle \alpha_x, \frac{\mathrm{d}x\gamma(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle = -\left\langle \theta_m, \frac{\mathrm{d}\gamma(\lambda)^{-1}m}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle. \tag{3.33}$$

Here $\gamma(\lambda) \in t^{-1}(s(x))$. We now define $\alpha_x \cdot \theta_m \in T^*_{xm}M$ as follows. Given $dn/d\lambda_{|0} \in T_{xm}M$, one picks a t-cover $g(\cdot)$ in G of the curve $\tau(n(\cdot))$ in G_0; that is, one has $g(0) = x$ and $t(g(\lambda)) = \tau(n(\lambda))$. We then put

$$\left\langle \alpha_x \cdot \theta_m, \frac{\mathrm{d}n(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle = \left\langle \theta_m, \frac{\mathrm{d}g(\lambda)^{-1}n(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle + \left\langle \alpha_x, \frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle. \tag{3.34}$$

The arbitrariness in the choice of $g(\cdot)$ is immaterial because of (3.33). To see this, one replaces $g(\lambda)$ by a curve $g(\lambda)h(\lambda)$ with the same properties, finding that h drops out of (3.34). Equivalently, we may write (3.34) as

$$\langle \alpha_x \cdot \theta_m, \xi_{xm} \rangle = \langle \theta_m, T_{(x^{-1},xm)}\varphi(T_x I(\eta_x) + \xi_{xm}) \rangle + \langle \alpha_x, \eta_x \rangle. \tag{3.35}$$

Here $\varphi: G *_{G_0}^{s,\tau} M \to M$ is the given G-action, and $\eta_x \in T_x$ covers $T\tau(\xi_{xm})$ under t, i.e., $T_x t(\eta_x) = T_{xm}\tau(\xi_{xm})$. The arbitrariness in η_x is a vector in $\ker(Tt)$, which drops out of (3.35) because of (3.33) and the fact that $\ker(Tt)$ is spanned by vectors of the form occurring on the left-hand side of that equation.

We now check that $J_L(\alpha_x \cdot \theta_m) = \tilde{t}(\alpha_x)$. Evaluating both sides on a vector $\mathrm{d}\gamma/\mathrm{d}\lambda_{|0}$, this condition may be rewritten as

$$\left\langle \alpha_x \cdot \theta_m, \frac{\mathrm{d}\gamma(\lambda)^{-1}xm}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle = \left\langle \alpha_x, \frac{\mathrm{d}\gamma(\lambda)^{-1}x}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle. \tag{3.36}$$

To compute the left-hand side, we take $n(\lambda) = \gamma(\lambda)^{-1}xm$ and $g(\lambda) = \gamma(\lambda)^{-1}x$ in (3.34). The first term on the right-hand side of (3.34) then vanishes, and the second term equals the right-hand side of (3.36).

Next, we verify that

$$\alpha_x \cdot (\beta_y \cdot \theta_m) = (\alpha_x \tilde{+} \beta_y) \cdot \theta_m, \tag{3.37}$$

whenever defined. We compute the left-hand side from (3.34) as

$$\left\langle \alpha_x(\beta_y \cdot \theta_m), \frac{\mathrm{d}n(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle$$

$$= \left\langle \beta_y \cdot \theta_m, \frac{\mathrm{d}g(\lambda)^{-1}n(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle + \left\langle \alpha_x, \frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle$$

$$= \left\langle \theta_m, \frac{\mathrm{d}h(\lambda)^{-1}g(\lambda)^{-1}n(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle$$

$$+ \left\langle \beta_y, \frac{\mathrm{d}h(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle + \left\langle \alpha_x, \frac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda}_{|\lambda=0} \right\rangle.$$

Here g is as specified after (3.33), and h is such that

$$h(0) = y \quad \text{and} \quad t(h(\lambda)) = \tau(g(\lambda)^{-1}n(\lambda)) = s(g(\lambda)).$$

The right-hand side of (3.37) is computed as follows: as the t-cover $\tilde{g}(\cdot)$ of $n(\cdot)$ satisfying $\tilde{g}(0) = xy$ and $t(\tilde{g}(\lambda)) = \tau(n(\lambda))$ we may use $\tilde{g}(\lambda) = g(\lambda)h(\lambda)$. Equation (3.37) is then immediate from (3.31) and (3.32).

Finally, we show that elements of $(T^*G)_0$ act trivially on T^*M. According to the definition of the groupoid structure of T^*G, a unit $\alpha_x \in T_x^*G$ satisfies $x \in G_0$ and $\alpha_x|T_xG_0 = 0$. The former condition implies that in (3.34) we may take $g(\lambda) = \tau(n(\lambda))$, so that $g(\cdot) \subset G_0$. The second condition then implies that the second

term on the right-hand side of (3.34) vanishes, whereas the first term is $\langle \theta_m, dn/d\lambda_{|0} \rangle$; this is because $g(\lambda)^{-1} n(\lambda) = n(\lambda)$, since $g(\lambda)^{-1} \in G_0$.

It is routine to check that the T^*G action on T^*M is smooth. That it is symplectic may be verified from a local computation showing that the graph of $(\alpha_x, \theta_m) \mapsto \alpha_x \cdot \theta_m$ is coisotropic in $T^*G^- \times T^*M^- \times T^*M$. An easy dimensional count then implies that it is Lagrangian.

For the final claim in Theorem 3.9, see the beginning of this section. $\quad\square$

When G is a Lie group, we may choose $\eta_x = 0$ in (3.35) to compute

$$\langle \alpha_x \cdot \theta_m, \xi_{xm} \rangle = \langle \theta_m, T_{(x^{-1}, xm)} \varphi(\xi_{xm}) \rangle = \langle \varphi^*_{x^{-1}} \theta_m, \xi_{xm} \rangle,$$

where $\varphi_x : m \mapsto xm$ is the G action on M. Hence $\alpha_x \cdot \theta_m = \varphi^*_{x^{-1}} \theta_m$, and our T^*G action on T^*M is just the pullback of the G action on M. Also see [17].

3.5. PROOF OF THEOREM 3.5

Let us now specialize Corollary 3.7 to the situation of Theorem 3.5, where the bibundle M satisfies the conditions in Definition 2.2. Condition 1 in the latter easily implies that the maps $p = J_R$ and $q = J_L$ satisfy condition 1 in Definition 3.4. To prove condition 2 (for q to be concrete), we first note that the set $J_L^{-1}(\alpha)$ by construction is a subvectorbundle of the restriction of T^*M to $M_\alpha = \tau^{-1}(\pi^{(*)}(\alpha)) \subset M$ (where $\pi^{(*)} : A^*(G) \to G_0$ is the bundle projection of $A^*(G)$ dual to $\pi : A(G) \to G_0$; cf. Definition 3.1). By property 2 in Definition 2.2, the latter set is an H-orbit, and by property 1 in Definition 2.2 (for H) and Definition 2.1.2 this orbit is diffeomorphic to $t_H^{-1}(\sigma(m))$, where $m \in M_\alpha$ (for a different choice $m' \in M_\alpha$ one has $m' = mh$ for some $h \in t_H^{-1}(\sigma(m))$, and then $t_H^{-1}(\sigma(m'))$ is diffeomorphic with $t_H^{-1}(\sigma(m))$ through $k \mapsto hk$). Hence, by assumption in Theorem 3.5, condition 2 in Definition 3.4 holds for q. An isomorphic argument with G and H interchanged proves this condition for p.

Finally, to prove condition 3 in Definition 3.4, proceed as follows. First compute the tangent spaces TJ_L^{-1} to the fibers of J_L: one has $X \in T_\alpha J_L^{-1}$ iff $X(J_L^* f) = 0$ for all $f \in C^\infty(A^*(G))$. Splitting f into the types $\tilde{f}$ and $\tilde{\sigma}$ discussed earlier, and assuming $X = X_g$ is a Hamiltonian vector field (allowed, as T^*M is symplectic), this implies $g \in C^\infty(M)^G$ or $g = \text{sym}(\xi)$, where $\xi \in \Gamma(T^\tau M)^G$. Similarly, $Y_h \in T_\alpha J_R^{-1}$ when $h \in C^\infty(M)^H$ or $h = \text{sym}(\eta)$, where $\eta \in \Gamma(T^\sigma M)^H$. The inclusion $T_\alpha J_R^{-1} \subseteq (T_\alpha L^{-1})^\perp$ now follows as in the proof of Corollary 3.7 (using the basic fact that $\omega(X_f, X_g) = \{f, g\}$). The opposite inclusion follows from the crucial information (2.13). $\quad\square$

Let us note that the above proof has the following reinterpretation. By a remarkable theorem of Dazord [25] and Xu [14], if P is an integrable Poisson manifold with s-connected and s-simply connected symplectic groupoid $\Gamma(P)$, any complete Poisson map $J : S \to P$ defines a symplectic action of $\Gamma(P)$ on S,

and vice versa. Another theorem of Xu [14] states that two Poisson manifolds P and Q are Morita equivalent iff their associated s-connected and s-simply connected symplectic groupoids $\Gamma(P)$ and $\Gamma(Q)$ are Morita equivalent. Applied to the case at hand, we have

$$P = A^*(G), \ Q = A^*(H), \qquad \Gamma(P) = T^*G^-, \quad \text{and} \quad \Gamma(Q) = T^*H^-.$$

Our proof shows that T^*M^- is a symplectic equivalence bimodule between T^*G^- and T^*H^-, establishing their Morita equivalence as symplectic groupoids. Hence their associated Poisson manifolds $A^*(G)$ and $A^*(H)$ are Morita equivalent as well.

Finally, we wish to mention a few open problems, suggested by (the referee of) this paper. Firstly, if E is a Lie algebroid whose dual bundle E^* is integrable as a Poisson manifold [4], is E integrable as a Lie algebroid? (See [26] for an extensive discussion of the latter integrability problem.) Secondly, can one turn Theorem 3.5 around? Thirdly, is there a functorial notion of strict deformation quantization in the present class of examples, extending the quantization of $A^*(G)$ by $C^*(G)$ developed in [8–10] to a quantization of dual pairs by Hilbert C^* bimodules?

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