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Dedicated to my parents

Preface

My journey in the world of operator algebras started when I was a student, and became interested in the relationship between quantum physics and classical mechanics, which also forms the motivation behind the subject of this thesis. The search for more information about this relationship eventually lead me to an issued version of the inaugural lecture of Klaas Landsman about John von Neumann and operator algebras. Little did I know at that moment that he would become my doctoral supervisor. Much did I learn from him as a PhD student, not only about operator algebras, but also about the scientific society. Most importantly, he has been very supportive during all these years; I really got the feeling that I could always rely on him. In other words, I could not wish for a better supervisor. Klaas, thank you very much!

I also would like to thank my co-author Chris Heunen. It was and is a pleasure to work together with you. Thank you for your all your insights, and for your help in finding my way into the scientific world. I am looking forward to continue our research!

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The companionship of many friends in and outside academia was of great value. I feel honored to have Sietse Ringers and Sander Wolters as paranymphs. I also want to thank all my friends in several orchestras for enjoyable musical times. Finally, I want to thank my family, who were an invaluable support, in particular my parents to whom I dedicate this work.

Contents

1 Introduction

The main subject of this thesis is the relationship between C*-algebras and the order-theoretic structures defined by their commutative C*-subalgebras ordered by inclusion. A C*-algebra is a normed *unital* algebra A over $\mathbb{C}$ such that

$$\|ab\| \leq \|a\|\|b\|,$$

for each $a, b \in A$, complete in the topology induced by its norm $\|\cdot\|$, and equipped with an involution, i.e., an anti-linear map $A \to A$, $a \mapsto a^*$, satisfying

$$(a^*)^* = a,$$

and

$$(ab)^* = b^* a^*,$$

for each $a, b \in A$, and such that

$$\|a^* a\| = \|a\|^2,$$

for each $a \in A$. We say that a subset B of A is a C*-subalgebra if it is a topologically closed subalgebra of A that is closed under the involution, and that contains the identity element of A. We refer to Appendix C for more details on C*-algebras. We denote the set of all commutative C*-subalgebras of A by $\mathcal{C}(A)$, which is a partially ordered set (poset) if, as already mentioned, we order it by inclusion. Since *-isomorphic C*-algebras have order-isomorphic posets of commutative C*-subalgebras, $\mathcal{C}(A)$ becomes an invariant for C*-algebras. Moreover, for both mathematics and physics, $\mathcal{C}(A)$ turns out to be a 'natural' invariant of C*-algebras, and there might as well be interesting applications to computer science.

1.1 Mathematical motivation

In many fields of mathematics, one of the most important problems is the classification of the relevant structures up to isomorphism, which is often done by means of invariants. For instance, Dynkin diagrams form a complete invariant for finite-dimensional semi-simple Lie algebras, orientable compact surfaces are classified by their genus, and Hilbert spaces are uniquely determined by their dimension. In the same spirit, one of the major problems in the theory of C*-algebras is finding and investigating invariants in order to classify as many C*-algebras as possible. We list a few invariants for C*-algebras. The best known invariant is probably *K-theory*, a generalization of topological K-theory, which has been used by Elliott and others to classify several classes of C*-algebras, such as the separable approximately finite-dimensional C*-algebras [30]. The latter class can also be classified by means of *Bratteli diagrams*. The most important subclass of C*-algebras is formed by the von Neumann algebras; *dimension functions* play an important role in the classification of von Neumann factors, i.e., von Neumann algebras with trivial center (such functions are actually K-theoretic in nature).

Perhaps the most important theorem in operator algebras, and in particular for this thesis, is the famous theorem by Gelfand and Naimark that completely characterizes commutative C*-algebras. Given a compact Hausdorff space X, the algebra $C(X)$ of continuous functions $X \to \mathbb{C}$ with pointwisely defined algebraic operations becomes a commutative C*-algebra in the supremum norm. The Gelfand–Naimark Theorem states that the converse holds as well: for each commutative C*-algebra A there exists a compact Hausdorff space X, called the *Gelfand spectrum* of A, such that A is *-isomorphic to A. This theorem is extremely powerful, for it can be extended to a duality, called *Gelfand duality*, between the category of commutative C*-algebras and the category of compact Hausdorff spaces. Moreover, it has led to various perspectives on Gelfand duality for non-commutative C*-algebras.

Firstly, one can try to *extend* Gelfand duality to a duality between all C*-algebras and some other category containing the compact Hausdorff spaces as a subcategory. The first ideas in this direction were developed by Akemann [1] and by Giles and Kummer [40]. Based on their work, Mulvey introduced the notion of a *quantale* [90], which was further developed by Borceux, Rosický, and others [13, 76].

Secondly, one can regard the category of all C*-algebras as the dual of an undefined category of 'non-commutative topological spaces' and try to translate topological and geometrical notions to the language of commutative C*-algebras in order to *generalize* these notions to non-commutative C*-algebras. As an example, the *real rank* of a C*-algebra is a non-commutative generalization of the Lebesgue covering dimension of a topological space: the real rank of a commutative C*-algebra equals the Lebesgue covering dimension of its Gelfand spectrum [17]. A more advanced example is the notion of a *spectral triple*, which turns out to be the non-commutative generalization of a Riemannian spin manifold [23].

In this thesis, we offer a third perspective on the relation between Gelfand duality and non-commutative C*-algebras: we intend to *exploit* it in order to learn more about the structure of non-commutative C*-algebras. Our approach has some similarities with algebraic quantum field theory [44]. Here one considers a C*-algebra A, and assigns to each region O of spacetime a C*-subalgebra $A(O)$ of A, such that $A(O_1) \subseteq A(O_2)$ if $O_1 \subseteq O_2$.

In a similar way, we start with a C*-algebra A and cover A as much as possible[1] by its commutative C*-subalgebras. The idea is to learn about the structure of A by looking at the structure of its commutative C*-subalgebras and their mutual relations, expressed by their ordering the C*-subalgebras by set-theoretic inclusion. We already denoted this partially ordered set of commutative C*-subalgebras of A by $\mathcal{C}(A)$. Gelfand duality plays an implicit but crucial role here, as it is an essential ingredient of the proof of Hamhalter's Theorem (cf. §4.7), which allows us to reconstruct elements of $\mathcal{C}(A)$ as C*-algebras from the order structure of $\mathcal{C}(A)$. Therefore, we can regard $\mathcal{C}(A)$ purely as a partially ordered set, obtaining a powerful new invariant for C*-algebras, which is based on exploiting Gelfand duality as much as possible.

1.2 Motivation from physics

The advantage of the use of C*-algebras in physics lies in the fact that they can be used for the description of both quantum and classical systems. Since observables and symmetries of a quantum system are usually identified with selfadjoint and unitary operators on some Hilbert space, respectively, C*-algebras can be used for the description of quantum systems. Since classical systems are usually described by phase spaces, namely through the corresponding function spaces, this allows us to describe classical systems by C*-algebras, too. Consequently, C*-algebras form the perfect mathematical framework for the description of the interplay between classical and quantum systems. We refer for instance to [80, 81], which discuss quantization as well as the classical limit in terms of C*-algebras.

Physically, the only way to access a quantum system is by means of measurements, which yield classical information. The so-called *doctrine of classical*

[1] Complete coverage is only possible if A is already commutative. Nevertheless, the set A_{sa} of all selfadjoint elements (which spans A) is always completely covered by the set of commutative C*-subalgebras of A.

concepts of Niels Bohr states that all observations on quantum systems, such as the readout stage of a quantum computation, must be expressed in terms of classical physics [12]. Using the correspondence between commutative C*-algebras and classical systems, one can identify every commutative C*-subalgebra of a C*-algebra A with a piece of classical information stored in the quantum system corresponding to A. Hence the elements of $\mathcal{C}(A)$ represent all accessible information about the quantum system corresponding to A, making it a natural invariant for C*-algebras from the point of view of physics, too.

1.3 Relevance for quantum logic and topos theory

Given a C*-algebra A, the set $\mathrm{Proj}(A)$ of its projections becomes an *orthomodular poset* (cf. Appendix B.4) if we order it by $p \leq q$ if and only if $pq = p$. Moreover, $\mathrm{Proj}(A)$ is a Boolean algebra if A is commutative[2], and since Boolean algebras model classical logic, it is plausible to see orthomodular posets as models for a new logic relevant for quantum physics, interpreting the meet operation as logical conjunction, the join operator as a logical disjunction, and the orthocomplementation as a logical negation. For this reason, Birkhoff and von Neumann interpreted orthomodular posets as the order-theoretic structures corresponding to a logic they called *quantum logic*. We will see in Chapter 6 that $\mathrm{Proj}(A)$ is encoded in $\mathcal{C}(A)$ and can be decoded.

Quantum logic, however, has its drawbacks. Firstly, due to the non-distribu -tivity of orthomodular posets, there is no implication operator (in the sense of a right adjoint of the meet operator). Secondly, the law of the excluded middle in the form $x \vee x^{\perp} = 1$ holds, which does not match the fact that there are statements in quantum mechanics, like 'Schrödinger's cat is dead', which are typically neither true nor false.

Therefore, *intuitionistic logic*, which is distributive, but lacks the law of the excluded middle, seems to be a better choice for a logic describing reasoning about quantum systems. This is one of the motivations for the development of *quantum toposophy*, also known as the *topos approach to quantum mechanics*, where the origins of the object $\mathcal{C}(A)$ can be found. We refer to [19, 20, 59, 111] for more information about the topos approach to quantum mechanics, which is still an active subject, see for instance [56].

[2]The converse does not hold, since there exist non-commutative C*-algebras without non-trivial projections [11].

1.4 Computer science

Domain theory (cf. Chapter B.6) can be seen as the study of the approximation of information. The setting is a poset P whose elements represent bits of information. The steps of a calculation are represented by elements of a directed set; the supremum of this set represents the outcome of the calculation. For this reason it is desirable to assume that P has the property that the supremum of any directed subset of P exists: P is a directed complete poset (dcpo). There is also a notion of finite information, represented by elements p of P, called *compact elements*, for which any calculation of p already contains p as a calculation step. If every element of P can be calculated by compact elements, we say that P is a *domain*.

The importance of domains lies in the fact that they can be used as semantics for computer programming languages and for formal languages like the lambda calculus; solving so-called *domain equations* is one of the methods of finding suitable domains as models for some language. Without giving all details, a domain equation is an equation of the form $FX \cong X$, where F is a suitable endofunctor or bifunctor on some suitable category, usually the category of dcpo's with a suitable choice of morphisms. We refer to [39, Chapter IV] for the meaning of the word 'suitable' in this context. A solution of this equation is a domain L satisfying $FL \cong L$. The most famous example of a domain equation might be $X \cong [X, X]$ in the category of dcpo's with a least element and Scott continuous maps as morphisms. Here $[X, X]$ denotes the dcpo of all Scott continuous maps from X to itself. Since in (untyped) lambda calculus arguments can be seen as functions and *vice versa*, solutions of this equation are models for lambda calculus.

For any C*-algebra A, the poset $\mathcal{C}(A)$ is a dcpo, and if A is a so-called *scattered* C*-algebra A (cf. Chapter 2.3), it turns out that $\mathcal{C}(A)$ is even a domain. Although it is not clear (yet) of which language $\mathcal{C}(A)$ is an interpretation, domains of the form $\mathcal{C}(A)$ for some scattered C*-algebra A might be relevant for computer science.

1.5 A complete invariant?

In [22], Connes proved the existence of a C*-algebra N that is not *-isomorphic to its opposite algebra N^{op}, i.e., the C*-algebra with the same underlying vector space as N, but with multiplication $(a, b) \mapsto ba$. Since N^{op} clearly has the same commutative subalgebras as N, it follows that $\mathcal{C}(N) = \mathcal{C}(N^{\mathrm{op}})$, which shows that $\mathcal{C}(A)$ is not a complete invariant of C*-algebras. Since the algebra N is a von Neumann factor, it follows that the poset $\mathcal{V}(M)$ of commutative von Neumann subalgebras of a von Neumann algebra M is not a complete invariant for von Neumann algebras either.

One could try to add extra structure to $\mathcal{C}(A)$ in order to obtain a complete invariant, i.e., such that the $\mathcal{C}$-functor can be 'extended' to a full and faithful functor. For *AW*-algebras*, an algebraic generalization of von Neumann algebras (cf. Chapter 2.4), such an invariant, called an *active lattice*, was found by Heunen and Reyes [61]. For arbitrary C*-algebras, a first step in the direction of a complete invariant is showing that an order isomorphism between $\mathcal{C}(A)$ and $\mathcal{C}(B)$ implies that the *Jordan structures* of A and B are isomorphic, i.e., that there is a linear and involution-preserving bijection $\varphi : A \to B$ such that $\varphi(a^2) = \varphi(a)^2$ for each $a \in A$. Such a map is called a *Jordan isomorphism*. Clearly, a *-isomorphism is a Jordan isomorphism. Conversely, a Jordan isomorphism between A and B can be regarded as a 'mixture' of a *-isomorphism and a anti-*-isomorphism between A and B (cf. Corollaries 5.75 and 5.76 in [2]). Hence proving the existence of a Jordan isomorphism between A and B is about 'half' the work of proving the existence of a *-isomorphism between A and B. As shown by Döring and Harding in [25], and Hamhalter in [47, 48, 50] (the latter reference together with Turilova), an order isomorphism between posets of commutative subalgebras of algebras A and B is uniquely implemented by a Jordan isomorphism between A and B. Their main results are listed in §9.1.

At least in the present thesis, we decided not to worry about the explicit construction of a *-isomorphism between A and B and hence to treat $\mathcal{C}(A)$ as an (albeit incomplete) invariant of C*-algebras. Therefore, we aim to investigate which classes of C*-algebras can be singled out by $\mathcal{C}(A)$ and which C*-algebras are completely determined by $\mathcal{C}(A)$ up to *-isomorphism, by looking how C*-algebraic properties of A are reflected by order-theoretic properties of $\mathcal{C}(A)$. A (minor) advantage of this strategy is that we no longer have to exclude the case that A has a type I_2 summand.

1.6 Conventions and notations

- We use the Kadison-Ringrose definition of a C*-algebra [67, 68]. Hence we assume that all C*-algebras are unital and all *-homomorphisms preserve the identity element, unless indicated otherwise.

There are two reasons for this convention. Firstly, in the unital case, it was shown by J. Hamhalter [47] (see also Chapter 4) that $\mathcal{C}(A)$ completely determines any commutative C*-algebra A up to isomorphism. Given a noncommutative C*-algebra A, it follows that every element of $\mathcal{C}(A)$ can be reconstructed as a C*-algebra from the order structure of $\mathcal{C}(A)$. This fact is very useful, and will often be exploited. In the non-unital case, however, this reconstruction of elements of the poset of commutative C*-subalgebras seems not always possible, which bereaves us from a very useful technique. Secondly, as noted above, the origins of $\mathcal{C}(A)$ lie in the topos approach to quantum mechanics, where the existence of an internal version of Gelfand duality is essential. For a long time, only a unital version of internal Gelfand duality was known; for this reason it was natural to work with unital C*-algebras. It should be noted, however, that Henry recently proved an internal version of non-unital Gelfand duality [55]. Nevertheless, it is not clear whether the constructions in the topos approach to quantum mechanics can completely be transfered to a non-unital setting.

- Given two sets A and B, we write $A \subseteq B$ to denote that A is a subset of B, and $A \subset B$ to indicate that A is a *proper* subset of B, i.e., $A \subseteq B$ and $A \neq B$.

The reason is that we will often use the inclusion as a partial order, and it is desirable in order theory to distinguish inequalities from strict inequalities.

- Unless indicated otherwise, topological spaces are always denoted by upper case letters at the end of the alphabet: U, V, W, X, Y, Z.
- C*-algebras are denoted by upper case letters at the beginning of the alphabet: A, B, C, D, E, F.
- We use lower case letters te denote elements of topological spaces and C*-algebras. Functions between topological spaces are usually denoted by f, g, h, or k.
- We can consider collections of certain subsets of some given set ordered by inclusion. Partially ordered sets obtained in this way are denoted by calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$.
- $\mathcal{C}(A)$ denotes the poset of commutative C*-subalgebras of a C*-algebra A, ordered by inclusion.

- $\mathcal{C}_{\text{fin}}(A)$ denotes the poset of finite-dimensional commutative C*-subalgebras of a C*-algebra A, ordered by inclusion.
- $\mathcal{C}_{\text{AF}}(A)$ denotes the subposet of $\mathcal{C}(A)$ consisting of all elements in $\mathcal{C}(A)$ that are approximately finite-dimensional. For each $C \in \mathcal{C}(A)$, we have $C \in \mathcal{C}_{\text{AF}}(A)$ if and only if C has real rank zero if and only if C is generated by its projections (cf. Theorem 2.2.3).
- $\mathcal{B}(P)$ denotes the poset of Boolean subalgebras of an orthomodular poset P, ordered by inclusion.
- $\mathcal{A}(A)$ denotes the poset of commutative AW*-subalgebras of an AW*-algebra A, ordered by inclusion.
- $\mathcal{V}(M)$ denotes the poset of commutative von Neumann-subalgebras of a von Neumann algebra M, ordered by inclusion, i.e., if M is a von Neumann algebra in $B(H)$, then $C \in \mathcal{V}(M)$ if and only if $C \subseteq B(H)$ is commutative such that $C'' = C$ and $C \subseteq M$. The poset $\mathcal{V}(M)$ equals $\mathcal{A}(M)$ (cf. Theorem 8.1.2).
- $B(X)$ denotes the Boolean algebra of clopen subsets of a topological space X.
- $C(X)$ denotes the space of all continuous functions $X \to \mathbb{C}$, where X is some compact Hausdorff space.
- C_K denotes the space of all continuous functions $X \to \mathbb{C}$ that are constant on some closed subset $K \subseteq X$.
- $\sim_B$ denotes the equivalence relation given by $x \sim_B y$ if and only if $f(x) = f(y)$ for each $f \in B$, where $B \subseteq C(X)$.
- Both C_k and $C(k)$ are used to denote the map $C(Y) \to C(X)$, $f \mapsto f \circ k$ if $k : X \to Y$ is a continuous map between topological spaces X and Y.
- $[x]_B$ denotes the equivalence class of the element $x \in X$ under the equivalence relation $\sim_B$.
- $C^*(S)$ denotes the C*-subalgebra of a C*-algebra A generated by a subset $S \subseteq A$. By our conventions, $C^*(S)$ contains the identity element of A.
- $p\,C q$ denotes that two elements p and q in an orthomodular poset commute.

- $\langle S \rangle$ denotes the Boolean subalgebra of an orthomodular poset P generated by a non-empty subset S of P consisting of mutually commuting elements.
- $S^{\perp}$ denotes the set $\{s^{\perp} : s \in S\}$ for each subset S of an orthoposet.
- $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ denotes an inner product on a Hilbert space H. We adapt the convention that inner products are always linear in the second variable and anti linear in the first variable.
- $\mathcal{D}(X)$ denotes depending on the context either the set of u.s.c. decompositions of a compact Hausdorff space or the set of down-sets of a poset X.
- $\Downarrow Y$ denotes the set $\downarrow Y \cap X$ if X is a fixed subset of a poset P and Y an arbitrary subset of X.

We consider the following categories:

- **Sets**, the category of sets with functions.
- **Poset**, the category of posets with order morphisms.
- **OMP**, the category of orthomodular posets with orthomodular morphisms.
- **DCPO**, the category of dcpo's with Scott continuous maps.
- **CptHd**, the category of compact Hausdorff spaces with continuous maps.
- **Stone**, the category of Stone spaces with continuous maps.
- **Bool**, the category of Boolean algebras with Boolean morphisms.
- **CStar**, the category of C*-algebras with *-homomorphisms.
- **CCStar**, the category of commutative C*-algebras with *-homomorphisms.
- **AWStar**, the category of AW*-algebras with AW*-homomorphisms.

1.7 Outline and results

The prerequisites for reading this thesis are the basics on operator algebras, order theory, Boolean algebras and domain theory. All notions and theorems in these fields that we use are introduced in the appendices.

Chapter 2 gives an overview of several classes of C*-algebras of interest to us, namely finite-dimensional C*-algebras; approximately finite-dimensional C*-algebras (AF-algebras), i.e., C*-algebras that are obtained as an inductive limit of finite-dimensional C*-algebras; locally AF algebras, which are generalizations of AF-algebras; scattered C*-algebras, which form a subclass of locally AF-algebras, which will play an important role in Chapter 7, and which can be seen as non-commutative generalization of so-called scattered topological spaces; Rickart C*-algebras; and finally AW*-algebras, which are algebraic generalizations of von Neumann algebras. All these classes turn out to be C*-algebras of real rank zero. We give the definitions of all these classes, and state some results about commutative C*-subalgebras of algebras in these classes.

Chapter 3 introduces the poset $\mathcal{C}(A)$ of commutative C*-subalgebras of a C*-algebra A. We discuss special elements of $\mathcal{C}(A)$, such as the least element and the maximal elements. Moreover, we give the expressions of suprema and infima of subsets of $\mathcal{C}(A)$. Finally, we discuss how $\mathcal{C}(A)$ can be extended to a functor from C*-algebras to posets.

Chapter 4 treats $\mathcal{C}(A)$ in the special case that A itself is a commutative C*-algebra. Firstly, we give order-theoretic conditions on $\mathcal{C}(A)$ that precisely correspond with A being a commutative C*-algebra. We continue with the introduction of so-called ideal subalgebras of A, which are relatively simple elements that can be seen as building blocks of $\mathcal{C}(A)$. We proceed by discussing posets that are isomorphic to $\mathcal{C}(A)$, but which are defined in terms of X, the Gelfand spectrum of A. These are the poset $\mathcal{D}(X)$ of all u.s.c. decompositions of a compact Hausdorff space X and the poset $\mathcal{Q}(X)$ of all compact Hausdorff quotients of X. These posets are useful for a better understanding of the structure of $\mathcal{C}(A)$, and are used in order to give an order-theoretic characterization of the ideal algebras of A as elements of $\mathcal{C}(A)$. Finally, the characterization of ideal algebras is used for Hamhalter's proof that an order isomorphism $\mathcal{C}(A) \to \mathcal{C}(B)$ for two commutative C*-algebras A and B is induced by a *-isomorphism between A and B.

Chapter 5 discusses finite-dimensional C*-algebras in terms of commutative C*-subalgebras. We show that there are several order-theoretic properties of $\mathcal{C}(A)$ corresponding to A being finite-dimensional. Furthermore, given a finite-dimensional C*-algebra A and an arbitrary C*-algebra B, we use the Artin-Wedderburn Theorem in order to show that an order isomorphism between $\mathcal{C}(A)$ and $\mathcal{C}(B)$ implies the existence of a *-isomorphism between A and B. The results in this chapter have been publised in [84].

Chapter 6 discusses two ways of reconstructing the projections in a C*-algebra from $\mathcal{C}(A)$. Firstly, we discuss the poset $\mathcal{B}(P)$ of Boolean subalgebras of an orthomodular poset P. Given a C*-algebra A, we prove that $\mathcal{B}(\mathrm{Proj}(A))$ is order isomorphic to $\mathcal{C}_{\mathrm{AF}}(A)$, i.e., the poset of commutative C*-subalgebras of A that are also AF-algebras. Then, given another C*-algebra B such that $\mathcal{C}(A)$ and $\mathcal{C}(B)$ are order isomorphic, we use the Harding–Navara Theorem [52] to show that there is an orthomodular isomorphism between $\mathrm{Proj}(A)$ and $\mathrm{Proj}(B)$, the projection posets of A and B, respectively. In the penultimate section, we assume that A is commutative and construct a poset in terms of elements of $\mathcal{C}(A)$ that is orthomodular isomorphic to $\mathrm{Proj}(A)$. In the last section, we construct a poset in terms of $\mathcal{C}(A)$ for any C*-algebra A whose center is at least three-dimensional.

Chapter 7 is based on joint work with Heunen [60]. Here we investigate the domain theory of $\mathcal{C}(A)$, which means that we investigate to which class of C*-algebras a C*-algebra A belongs if A is an algebraic domain, a continuous domain, a quasialgebraic domain, or a quasicontinuous domain. It turns out that A is a scattered C*-algebra in all these cases.

Chapter 8 discusses the poset $\mathcal{A}(A)$ of commutative AW*-subalgebras of an AW*-algebra A. We show that $\mathcal{A}(A)$ has similar properties as $\mathcal{C}(A)$. We explore the relation between $\mathcal{C}(A)$ and $\mathcal{A}(A)$, discuss the domain theory of $\mathcal{A}(A)$, and show that the projection lattice of an AW*-algebra A can be reconstructed from $\mathcal{A}(A)$. Furthermore, we prove that any commutative AW*-algebra A is determined by $\mathcal{A}(A)$ up to isomorphism. Finally, we show that $\mathcal{A}(A)$ (and hence $\mathcal{C}(A)$) determines each type I AW*-algebra (cf. Definition 8.6.8) A up to isomorphism, which generalizes Theorem 5.3.1.

Chapter 9 gives an overview of the results of Döring and Harding in [25] and Hamhalter in [47, 48] about the relation between the Jordan structure of a C*-algebra and its posets of commutative subalgebras. The main result in the last reference is that if A is an AW*-algebra without a type I_2 summand, and B any other AW*-algebra, then each order isomorphism $\mathcal{C}(A) \to \mathcal{C}(B)$ is induced by a Jordan isomorphism between A and B. We extend the theorem to all AW*-algebras, but the price we have to pay is that it is no longer assured that the order isomorphism between the posets of commutative subalgebras is induced by a Jordan isomorphism between A and B. As an application, we show that if A is a W*-algebra, then the existence of an order isomorphism between $\mathcal{C}(A)$ and $\mathcal{C}(B)$ implies that B is a W*-algebra, too.

Chapter 10 is extracted from [83] and discusses Grothendieck topologies on a given poset P, which can be used in order to define sheaves on P. We show the existence of a class of Grothendieck topologies that are induced by subsets of P, and prove that P satisfies the ascending chain condition if and only if all Grothendieck topologies on P are induced by some subset of P, which leads to another order-theoretic property that $\mathcal{C}(A)$ satisfies if and only if A is finite-dimensional. We also discuss morphisms of sites, i.e., posets equipped with a Grothendieck topology, in particular in the case that the associated Grothendieck topology is induced by a subset of the poset.

We summarize the most important results about $\mathcal{C}(A)$. Some of these results are not included in this thesis, in which case we give a reference.

Theorem 1.7.1. Let A be a C*-algebra. Then:

(1) A is commutative if and only if $\mathcal{C}(A)$ is a lattice;

(2) A is finite-dimensional if and only if there is an $M \in \max \mathcal{C}(A)$ that is finite-dimensional;

(3) A is finite-dimensional if and only if $\mathcal{C}(A)$ is Noetherian if and only if $\mathcal{C}(A)$ is Artinian if and only if $\mathcal{C}(A)$ is order scattered;

(4) A is scattered if and only if each $C \in \mathcal{C}(A)$ is an AF-algebra if and only if each $M \in \max \mathcal{C}(A)$ is scattered;

(5) A is scattered if and only if $\mathcal{C}(A)$ is an algebraic domain if and only if $\mathcal{C}(A)$ is a continuous domain if and only if $\mathcal{C}(A)$ is a quasialgebraic domain if and only if $\mathcal{C}(A)$ is a quasicontinuous domain;

(6) A is an AW*-algebra if each $M \in \max \mathcal{C}(A)$ is an AW*-algebra;

(7) A is a Rickart C*-algebra if each $M \in \max \mathcal{C}(A)$ is a Rickart C*-algebra.

Proof.

(1) [7, Proposition 14], included in this thesis as Proposition 4.1.1;

(2) A standard result in the theory of C*-algebras, see for instance Exercise 4.12 in [67];

(3) Theorem 5.1.2;

(4) Kusuda's Theorem [78, 79], included in this thesis as Theorem 2.3.4;

(5) Theorem 7.4.1.

Statements (6) and (7) are both proven by Saito and Wright in [99]. □

Theorem 1.7.2. Let A and B be C*-algebras and let $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$ be an order isomorphism.

(1) If A is a commutative C*-algebra, then there is a *-isomorphism $\varphi : A \to B$ such that $\mathcal{C}(\varphi) = \Phi$, which is unique if $\dim A \neq 2$;

(2) If A is finite-dimensional, then A and B are *-isomorphic;

(3) There is an orthomodular isomorphism $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ such that $\Phi(C) = C^*(\varphi[\mathrm{Proj}(C)])$ for each $C \in \mathcal{C}(A)$ that is generated by projections;

(4) If A is an AW*-algebra, then so is B. Moreover, given the unique decomposition
$$A = A_{\mathrm{I}} \oplus A_{\mathrm{II}_1} \oplus A_{\mathrm{II}_\infty} \oplus A_{\mathrm{III}},$$
where A_ν is an AW*-algebra of type ν ($\nu = \mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$), then there exist AW*-algebras B_ν of type ν ($\nu = \mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$) such that
$$B \cong B_{\mathrm{I}} \oplus B_{\mathrm{II}_1} \oplus B_{\mathrm{II}_\infty} \oplus B_{\mathrm{III}},$$
such that A_{I} and B_{I} are *-isomorphic, and such that A_ν and B_ν are Jordan *-isomorphic (cf. Definition 9.1.4) for $\nu = \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$;

(5) If A is a W*-algebra, then so is B;

(6) If $A = B(H)$ for some Hilbert space H, then A and B are *-isomorphic.

Proof.

(1) Hamhalter's Theorem [47], included in this thesis as Theorem 4.7.5;

(2) Theorem 5.3.1;

(3) Theorem 6.4.4;

(4) Proposition 8.4.1 assures that B is an AW*-algebra, and that $\mathcal{A}(A) \cong \mathcal{A}(B)$. It follows from Theorem 8.6.23 that B is the C*-sum of AW*-algebras B_ν ($\nu = \mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$) such that $\mathcal{A}(A_\nu) \cong \mathcal{A}(B_\nu)$. The same theorem implies that B_ν is of type ν, and that $A_{\mathrm{I}} \cong B_{\mathrm{I}}$. Finally, it follows from Theorem 9.2.8 that A_ν and B_ν are Jordan *-isomorphic for $\nu = \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}$;

(5) Corollary 9.3.9;

(6) Corollary 8.6.24.

□

All results without references are due to the author, except (5) of Theorem 1.7.1, which is due to Heunen and the author.

2 Classes of C*-algebras

In this chapter we discuss several classes of C*-algebras that occur in the rest of the thesis. The structure of finite-dimensional C*-algebras (i.e., C*-algebras whose underlying vector spaces are finite-dimensional) is well known, but most textbooks do not discuss the structure of commutative C*-subalgebras of finite-dimensional C*-algebras.

Finite-dimensional C*-algebras are generated by their projections. We will mainly be interested in classes of C*-algebras with plenty of projections; several of these classes can be seen as non-commutative generalizations of commutative C*-algebras that are generated by their projections. A commutative C*-algebra A has zero-dimensional Gelfand spectrum if and only if A is generated by its projections. The non-commutative generalization of the Lebesgue covering dimension of topological spaces is the *real rank* of a C*-algebra, which was introduced by Brown and Pedersen [17]. Hence the non-commutative generalization of C*-algebras with zero-dimensional Gelfand spectra are the C*-algebras of *real rank zero*. We will see that indeed a commutative C*-algebra has real rank zero if and only if its spectrum is zero-dimensional (cf. Definition A.2.2).

The class of C*-algebras of real rank zero includes all other classes discussed in this thesis. One of these is formed by the *approximately finite-dimensional C*-algebras* (AF-algebras), i.e., C*-algebras that can be obtained as an inductive limit of finite-dimensional C*-subalgebras. We will see that in the commutative case all these classes coincide: A commutative C*-algebra A is AF if and only if its real rank is zero if and only if it is generated by its projections (i.e., elements $p \in A$ such that $p = p^2 = p^*$) (cf. Theorem 2.2.3). Moreover, we shall prove that the projections determine commutative algebras of real rank zero up to *-isomorphism (cf. Theorem 2.2.5), which is not true for non-commutative C*-algebras of real rank zero.

Scattered topological spaces form a subclass of zero-dimensional spaces (cf. Definition A.2.2); it is no surprise that their non-commutative counterparts are called *scattered* C*-algebras. It turns out that it is easy to decide from the order structure of $\mathcal{C}(A)$ whether a C*-algebra is scattered or not (which follows from Theorem 2.3.4). This fact will be used in Chapter 7 for the description of the domain theory of $\mathcal{C}(A)$, in which scattered C*-algebras play the lead role.

The last subclass of C*-algebras of real rank zero that we consider are the *AW*-algebras*, which can be seen as an algebraic generalization of von Neumann algebras (which therefore have real rank zero, too, but do not play a distinguished role in this thesis, despite their popularity and historical significance for operator algebras as a whole). We can see from the order structure of $\mathcal{C}(A)$ whether or not A is an AW*-algebra (cf. Proposition 8.4.1), which is essential in order to see whether or not A is even a W*-algebra (cf. Theorem 9.3.8). Hence investigating W*-algebras by means of $\mathcal{C}(A)$ requires the analysis of $\mathcal{C}(A)$ when A is an AW*-algebra. Thus AW*-algebras form a natural class to investigate through their posets of commutative subalgebras. Initially, it was hoped that the class of von Neumann algebras coincided with the class of AW*-algebras, but in fact it turns out there is an equivalence relation on the collection of AW*-algebras with 2^c equivalence classes, where c denotes the cardinality of $\mathbb{R}$, and such that the class of von Neumann algebras merely coincides with a single equivalence class [98].

2.1 Finite-dimensional C*-algebras

We collect the most important results about commutative C*-subalgebras of finite-dimensional C*-algebras. We omit the proofs, since these are either simple, or easy to find in the literature.

Theorem 2.1.1 (Artin-Wedderburn)**.** Let A be a finite-dimensional C*-algebra with projections $\mathrm{Proj}(A)$. Then there are numbers $k, n_1, \ldots, n_k \in \mathbb{N}$ such that

$$A \cong \bigoplus_{i=1}^{k} \mathrm{M}_{n_i}(\mathbb{C}). \tag{1}$$

Here, the number k is unique, whereas the numbers $n_1, \ldots, n_k$ are unique up to permutation.

Proof. [106, Theorem I.11.2]. □

Proposition 2.1.2. Let A be a commutative C*-algebra with projections $\mathrm{Proj}(A)$ (cf. §C.3). Then the following conditions are equivalent:

(a) A is n-dimensional;

(b) $A \cong \mathbb{C}^n$, with pointwise algebraic operations;

(c) $A \cong C(X)$, where X is a discrete space of n points;

(d) A has exactly n minimal projections $p_1, \ldots, p_n$, which are orthogonal (i.e., $p_i p_j = \delta_{ij} p_i$) and form a basis for A such that $\sum_{i=1}^n p_i = 1_A$;

(e) A is spanned by n orthogonal non-zero projections;

(f) $\mathrm{Proj}(A)$ is a finite Boolean algebra with exactly n atoms that generates A;

(g) A is generated by a finite Boolean subalgebra of $\mathrm{Proj}(A)$ with exactly n atoms.

It turns out that the maximal commutative C*-subalgebras determine whether a C*-algebra is finite-dimensional or not. The proof of the next proposition can be found in [69] as the solution of [67, Exercise 4.6.12].

Proposition 2.1.3. Let A be a C*-algebra and M a maximal commutative C*-subalgebra. If M is finite-dimensional, then A must be finite-dimensional as well.

The following proposition implies that all maximal commutative C*-subalgebras of matrix algebras are *-isomorphic. It will be used in order to proof that all commutative C*-subalgebras of any fixed finite-dimensional C*-algebra are *-isomorphic.

Proposition 2.1.4. Let $A = \mathrm{M}_n(\mathbb{C})$ and $M \in \max \mathcal{C}(A)$. Then M is n-dimensional and there is some $u \in \mathrm{U}(n)$ such that $M = \{udu^* : d \in D_n\}$, where D_n is the commutative C*-subalgebra of A consisting of all diagonal matrices.

2.2 AF-algebras and C*-algebras of real rank zero

In this section we introduce C*-algebras that can be approximated by finite-dimensional C*-algebras (AF-algebras), and C*-algebras of real rank zero. The latter can be seen as the non-commutative generalizations of zero-dimensional compact Hausdorff spaces (also called Stone spaces, cf. Proposition A.2.5 and the remark directly preceding it).

Definition 2.2.1. Let A be a C*-algebra. Then we say that A is:

- of *real rank zero* if the set of self-adjoint elements of A with finite spectrum is dense in the set A_{sa} of self-adjoint elements of A;

- *approximately finite-dimensional* or *AF* if there exists a directed set $\mathcal{D}$ of finite-dimensional C*-subalgebras D of A such that $A = \overline{\bigcup \mathcal{D}}$.
- *locally approximately finite-dimensional* or *locally AF* if for each finite subset $\{x_1, \ldots, x_n\} \subseteq A$ and each $\epsilon > 0$ there is a finite-dimensional C*-subalgebra D having elements $y_1, \ldots, y_n \in D$ such that $\|x_i - y_i\| < \epsilon$ for each $i = 1, \ldots, n$.

We note that we do not require AF-algebras to be separable (as is usually done), in which case there exists a classification in terms of Bratteli diagrams [14] or K-theory [30]. It also turns out that in the definition of separable AF-algebras one can replace the directed set of finite-dimensional C*-subalgebras by an increasing sequence of finite-dimensional C*-subalgebras.

The proof that every locally AF-algebra has real rank zero in the next lemma is taken from [79, Theorem 2.3].

Lemma 2.2.2. Every AF-algebra is locally AF. Every locally AF-algebra has real rank zero.

Proof. Let A be AF, and let $\mathcal{D}$ be a directed collection of finite-dimensional C*-subalgebras of A such that $A = \overline{\bigcup \mathcal{D}}$. Let $x_1, \ldots, x_n \in A$ and $\epsilon > 0$. Then $x_i \in \overline{\bigcup \mathcal{D}}$ for each $i = 1, \ldots, n$. Since $\bigcup \mathcal{D}$ is dense in A, there is a $y_i \in \bigcup \mathcal{D}$ such that $\|x_i - y_i\| < \epsilon$. Hence for each $i = 1, \ldots, n$, there is a $D_i \in \mathcal{D}$ such that $y_i \in D_i$. Since $\mathcal{D}$ is directed, it follows that there is a $D \in \mathcal{D}$, which by definition of $\mathcal{D}$ is finite-dimensional, such that $D_1, \ldots, D_n \subseteq D$. Hence $y_1, \ldots, y_n \in \mathcal{D}$ and $\|x_i - y_i\| < \epsilon$ for each $i = 1, \ldots, n$. We conclude that A is locally AF.

Now assume that A is locally AF. Let $a \in A_{\text{sa}}$ and let $\epsilon > 0$. Then there is a finite-dimensional C*-subalgebra B of A and an $b \in B$ such that $\|a - b\| < \epsilon$. Let $b_1 = \frac{b+b^*}{2}$. Then $b_1 \in B_{\text{sa}}$, and $\|a - b_1\| = \|a - b\|$. Since b_1 is an element of a finite-dimensional algebra, it follows from Lemma C.1.25 that its spectrum is finite. We conclude that A has real rank zero. □

It was shown by Bratteli [14, Theorem 2.2] that the classes of separable AF algebras and locally AF-algebras actually coincide, a result which is generalized in [35, Theorem 1.5] to C*-algebras with dense subset of cardinality $\kappa \leq \aleph_1$. However, the same theorem states that if $\kappa > \aleph_1$, one always can find a locally AF-algebra with dense subset of cardinality κ that is not AF.

On the other hand, for commutative C*-algebras, we shall prove below that the classes of AF-algebras, locally AF-algebras and real rank zero algebras coincide regardless of the size of κ.

Theorem 2.2.3. Let A be a commutative C*-algebra with Gelfand spectrum X. Then the following statements are equivalent:

(1) A is generated by its projections;

(2) The linear span of $\mathrm{Proj}(A)$, the collection of projections in A, is a dense *-subalgebra of A;

(3) A is AF;

(4) A is locally AF;

(5) A has real rank zero;

(6) For each $a \in A$ and $\epsilon > 0$, there is a projection $p \in A$ and some $b \in A$ such that $p = ab$ and $\|a - ap\| < \epsilon$;

(7) X can be separated by $\mathrm{Proj}(A)$;

(8) X is a Stone space.

The implication (1) $\implies$ (6) is stated in [71, Lemma 2.1]. A more general statement, which implies the equivalence between (5) and (8), originally proven in [17, Proposition 1.1].

Proof. The implications (3) $\implies$ (4) $\implies$ (5) are proven in Lemma 2.2.2. We proceed by showing (8) $\implies$ (6) $\implies$ (7) $\implies$ (2) $\implies$ (8), and (2) $\implies$ (1) $\implies$ (3), and end by proving (5) $\implies$ (7).

For (8) $\implies$ (6), assume that X is zero-dimensional, let $a \in C(X)$ and $\epsilon > 0$. If $\|a\| < \epsilon$, then we can choose $p = b = 0$. Otherwise, let

$$K = \{x \in X : |a(x)| \geq \epsilon\}, \quad V = \{x \in X : |a(x)| > \epsilon/2\}.$$

Notice that K is non-empty and is contained in V. Moreover, V is open, whereas K is closed, hence compact, for X is compact Hausdorff. Since K and $X \setminus V$ are disjoint closed subsets, normality of X allows us to find, for each $x \in K$, an open neighborhood U_x of x and an open set W_x such that $X \setminus V \subseteq W_x$ and $U_x \cap W_x = \emptyset$. By zero-dimensionality of X, we can assume that U_x is clopen. It follows that $U_x \cap X \setminus V = \emptyset$ for each $x \in K$. The U_x form an open cover of K, hence by compactness of K, we can find a finite subcover, whose union U is necessarily clopen and contains K. Moreover, we have $U \cap X \setminus V = \emptyset$, hence

$U \subseteq V$. Since U is clopen, the characteristic function of U, which we denote by p, is a projection in $C(X)$. Let $x \in X$ such that $x \notin U$. Then $x \notin K$, hence

$$|a(x) - p(x)a(x)| = |a(x)| < \epsilon.$$

If $x \in U$, then $|a(x) - p(x)a(x)| = 0$, hence we find that $\|a - pa\| < \epsilon$. Notice that a is non-zero on U, hence the function $b : X \to \mathbb{C}$ defined by $b(x) = \frac{1}{a(x)}$ if $x \in U$ and $b(x) = 0$ otherwise is continuous. By construction, $ab = p$.

Assume that (6) holds. In order to show that (7) follows, let $x, y \in X$ be distinct points. By Urysohn's Lemma, there is a function $f' \in C(X)$ such that $f'(x) \neq f'(y)$. Let $f = f' - f'(y)$. Since $1_X \in C(X)$, it follows that $f \in C(X)$, and we have $f(x) \neq 0 = f(y)$. Let $\epsilon = |f(x)|$. Then we can find a projection $p \in C(X)$ such that $p = fg$ for some $g \in C(X)$ and such that $\|f - pf\| < \epsilon$. In particular, this implies $|f(x) - p(x)f(x)| < |f(x)|$ and since $f(x) \neq 0$, it follows that $p(x) \neq 0$. On the other hand, $p(y) = 0$ for $f(y) = 0$, and we conclude that the projections separate the points of X.

For (7) $\Longrightarrow$ (2), assume that the projections in A separate the points of X. The linear span of the projections is then a *-subalgebra that separates the points of X, hence the Stone-Weierstrass Theorem assures that (2) holds.

Assume that (2) holds and let $x, y \in X$ be distinct points. By Urysohn's Lemma, there is a function $f \in C(X)$ such that $f(x) \neq f(y)$. Now assume that $p(x) = p(y)$ for each projection p. Since f can be approximated by linear combinations of projections, we obtain a contradiction with $f(x) = f(y)$. Hence there must be a projection p such that $p(x) \neq p(y)$. Since the image of p lies in $\{0, 1\}$, we can assume without loss of generality that $p(x) = 0$ and $p(y) = 1$. Then $x \in C$, where $C = p^{-1}[\{0\}]$ is clopen, whereas $y \notin C$. Thus (8) holds.

Assume again that (2) holds. Since linear combinations of projections are clearly contained in $C^*(\mathrm{Proj}(A))$, the C*-subalgebra of A generated by $\mathrm{Proj}(A)$ (cf. Definition C.1.19), we find that

$$\mathrm{Span}(\mathrm{Proj}(A)) \subseteq C^*(\mathrm{Proj}(A)).$$

Since the latter algebra is closed, this implies

$$\overline{\mathrm{Span}(\mathrm{Proj}(A))} \subseteq C^*(\mathrm{Proj}(A)).$$

Thus (1) follows. Conversely, assume that (1) holds and let

$$\mathcal{D} = \{C^*(p_1, \ldots, p_n) : n \in \mathbb{Z}_+, p_1, \ldots, p_n \in \mathrm{Proj}(A)\}.$$

Then $\mathcal{D}$ is directed, for if $C^*(p_1, \ldots, p_n)$ and $C^*(q_1, \ldots, q_m)$ are elements of $\mathcal{D}$, then they are clearly contained in $C^*(p_1, \ldots, p_n, q_1, \ldots, q_m)$, which is also an element of $\mathcal{D}$. Let $D = C^*(p_1, \ldots, p_n)$ be an element of $\mathcal{D}$. We note that $\mathrm{Proj}(A)$ is a Boolean algebra by Proposition C.3.2. By Proposition B.4.25, the Boolean subalgebra B generated by $p_1, \ldots, p_n$ is finite. Hence $D \subseteq C^*(B)$, and since the latter is finite-dimensional by Proposition 2.1.2, it follows that D is finite-dimensional as well. We conclude that $\mathcal{D}$ consists of finite-dimensional C*-subalgebras. If $p \in \mathrm{Proj}(A)$, then clearly $p \in D$ for some $D \in \mathcal{D}$, hence $\mathrm{Proj}(A) \subseteq \bigcup \mathcal{D}$. It now follows from (1) that

$$A = C^*(\mathrm{Proj}(A)) \subseteq \overline{\bigcup \mathcal{D}},$$

whence $A = \overline{\bigcup \mathcal{D}}$. Thus A is AF.

Finally, assume that (5) holds, and let $x, y \in X$ be distinct points. By Urysohn's Lemma, there is an $f \in C(X)$ such that $f(x) = 1$ and $f(y) = 0$. We can assume that f is self adjoint, otherwise replace f by $\frac{f+f^*}{2}$, then still $f(x) = 1$ and $f(y) = 0$. Since $C(X)$ has real rank zero, there is a $g \in C(X)_{\mathrm{sa}}$ with $\sigma(g)$ finite, i.e., $g[X]$ is a finite set, such that $\|f - g\| < \frac{1}{2}$. Hence

$$|1 - g(x)| = |f(x) - g(x)| \leq \sup_{z \in X}\{|f(z) - g(z)|\} = \|f - g\| < \frac{1}{2},$$

and

$$|g(y)| = |f(y) - g(y)| < \frac{1}{2}.$$

This implies that $\frac{1}{2} < g(x) < 1\frac{1}{2}$ and $-\frac{1}{2} < g(y) < \frac{1}{2}$, hence $g(x) \neq g(y)$. Since $g[X]$ is finite, it follows that $\{g(x)\}$ is open in the image of g, hence $U = g^{-1}[\{g(x)\}]$ is clopen. Let p be the characteristic function on U, whence $p(x) = 1$. Then p is a projection, and since $U \cap g^{-1}[\{g(y)\}] = \emptyset$, it follows that $y \notin U$, so $p(y) = 0$. We conclude that (7) holds. □

Example 2.2.4. The Cantor space $2^{\mathbb{N}}$ is a totally disconnected space (cf. Examples A.2.4), hence a Stone space. It follows that $C(2^{\mathbb{N}})$ satisfies one and hence all of the conditions in Theorem 2.2.3.

The proof of the following theorem relies on the combination of Stone duality with Gelfand duality.

Theorem 2.2.5. Let A and B be commutative AF-algebras and $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ a Boolean isomorphism (see Definition B.4.26). Then there is a unique *-isomorphism $\widetilde{\varphi} : A \to B$ such that $\widetilde{\varphi}|_{\mathrm{Proj}(A)} = \varphi$.

Proof. By Theorem 2.2.3, there are Stone spaces X and Y and *-isomorphisms $\psi_A : A \to C(X)$ and $\psi_B : B \to C(Y)$. By Proposition C.3.4, the restrictions

$$\psi_A : \mathrm{Proj}(A) \to \mathrm{Proj}(C(X)), \qquad \psi_B : \mathrm{Proj}(B) \to \mathrm{Proj}(C(Y))$$

are orthomodular isomorphisms. Since all C*-algebras we are considering here are commutative, all projections posets are Boolean algebras (cf. Proposition C.3.2), hence the restrictions ψ_A and ψ_B are Boolean isomorphisms (see Lemma B.4.27). By Proposition C.3.6, there is a Boolean isomorphism

$$\rho_X : \mathrm{Proj}(C(X)) \to B(X),$$

where $B(X)$ denotes the Boolean algebra of clopen subsets of X. Explicitly, ρ_X is defined by $\rho_X(p) = p^{-1}[\{0\}]$, and its inverse is defined by $\rho_X^{-1}(U) = \chi_U$ for each $U \in B(X)$, where χ_U is the characteristic function of U. *Mutatis mutandis*, we find a Boolean isomorphism

$$\rho_Y : \mathrm{Proj}(C(Y)) \to B(Y),$$

hence $\tau : B(X) \to B(Y)$ defined by

$$\tau = \rho_Y \circ \psi_B \circ \varphi \circ \psi_A^{-1} \circ \rho_X^{-1} \tag{2}$$

is a Boolean isomorphism. By Stone duality (Theorem B.5.8), there is a unique homeomorphism $h : Y \to X$ such that $B(h) = \tau$. It follows from Proposition C.2.5 that $C(h) : C(X) \to C(Y)$ is a *-isomorphism, hence

$$\widetilde{\varphi} = \psi_B^{-1} \circ C(h) \circ \psi_A \tag{3}$$

is a *-isomorphism $A \to B$. Moreover, by $\tau = B(h)$, we obtain

$$\varphi = \psi_B^{-1} \circ \rho_Y^{-1} \circ B(h) \circ \rho_X \circ \psi_A.$$

Let $p \in \mathrm{Proj}(A)$. Then

$$\begin{aligned}
\varphi(p) &= \psi_B^{-1} \circ \rho_Y^{-1} \circ B(h) \circ \rho_X \circ \psi_A(p) \\
&= \psi_B^{-1} \circ \rho_Y^{-1} \circ B(h)\big(\psi_A(p)^{-1}[\{0\}]\big) \\
&= \psi_B^{-1} \circ \rho_Y^{-1} \Big(h^{-1}\big(\psi_A(p)^{-1}[\{0\}]\big)\Big) \\
&= \psi_B^{-1} \circ \rho_Y^{-1} \Big(\big(\psi_A(p) \circ h\big)^{-1}[\{0\}]\Big) \\
&= \psi_B^{-1} \circ \rho_Y^{-1} \Big(\big(C(h) \circ \psi_A(p)\big)^{-1}[\{0\}]\Big) \\
&= \psi_B^{-1} \circ \rho_Y^{-1} \circ \rho_Y\big(C(h) \circ \psi_A(p)\big) \\
&= \psi_B^{-1} \circ C(h) \circ \psi_A(p). \\
&= \widetilde{\varphi}(p).
\end{aligned}$$

Hence φ is the restriction of $\widetilde{\varphi}$. We show that $\widetilde{\varphi}$ is the unique extension of φ. So assume that $\theta : A \to B$ is a *-isomorphism such that $\theta|_{\mathrm{Proj}(A)} = \varphi$. Then

$$\psi_B \circ \theta \circ \psi_A^{-1} : C(X) \to C(Y)$$

is a *-homomorphism, hence there is a unique homeomorphism $k : Y \to X$ such that

$$C(k) = \psi_B \circ \theta \circ \psi_A^{-1}. \tag{4}$$

Since

$$C(k)|_{\mathrm{Proj}(C(X))} = \psi_B \circ \theta \circ \psi_A^{-1}|_{\mathrm{Proj}(C(X))} = \psi_B \circ \varphi \circ \psi_A^{-1}|_{\mathrm{Proj}(C(X))},$$

it follows from (2) that

$$\tau = \rho_Y \circ C(k)|_{\mathrm{Proj}(C(X))} \circ \rho_X^{-1}.$$

Let $U \subseteq X$ be clopen. It now follows that

$$\tau(U) = \rho_Y \circ C(k)(\chi_U) = \rho_Y(\chi_U \circ k) = \rho_Y(\chi_{k^{-1}[U]}) = k^{-1}[U] = B(k)(U).$$

Since h is the unique homeomorphism such that $\tau = B(h)$, we find $h = k$, hence combining (3) and (4) yields $\theta = \widetilde{\varphi}$. □

Lemma 2.2.6. Let A be a C*-algebra for which the linear span of its projections is dense. Then the projections in the center of A form exactly the center of the projections in A, i.e., we have

$$\text{Proj}(Z(A)) = C(\text{Proj}(A)).$$

Proof. Let $p \in \text{Proj}(Z(A))$. Since $p \in Z(A)$, we have $pa = ap$ for each $a \in A$. In particular, we have $pq = qp$ for each $q \in \text{Proj}(A)$. By Proposition C.3.2, we have $p\,C q$ for each $q \in \text{Proj}(A)$. Thus, $p \in C(\text{Proj}(A))$.

Conversely, let $p \in C(\text{Proj}(A))$. Then $p\,C q$ for each $q \in \text{Proj}(A)$, and Proposition C.3.2 implies that $pq = qp$ for each $q \in \text{Proj}(A)$. Now let $a \in \text{Span}(\text{Proj}(A))$, i.e., there are $q_1, \ldots, q_n \in \text{Proj}(A)$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that $a = \lambda_1 q_1 + \ldots + \lambda_n q_n$, and it follows that $pa = ap$. Let $a \in A$. Since the linear span of $\text{Proj}(A)$ is dense in A, there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\text{Span}(\text{Proj}(A))$ such that for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $\|a - a_n\| < \frac{\epsilon}{2}$ for each $n \geq N$. Let $\epsilon > 0$. Then

$$\begin{aligned} \|ap - pa\| &= \|ap - pa_n + pa_n - pa\| = \|ap - a_n p + pa_n - pa\| \\ &\leq \|(a - a_n)p\| + \|p(a - a_n)\| \leq 2\|p\|\|a - a_n\| < \epsilon, \end{aligned}$$

for $\|p\| \leq 1$. Since we can choose ϵ arbitrarily small, we find $\|ap - pa\| = 0$, i.e., $ap = pa$. We conclude that $p \in Z(A)$, so $p \in \text{Proj}(Z(A))$. ☐

The first two statements in the next proposition are well known, cf. [10]. The third statement is a special case of [47, Lemma 3.2].

Proposition 2.2.7. Let A be a C*-algebra such that all its maximal commutative C*-subalgebras are generated by their projections. Then:

(1) A has real rank zero;

(2) the linear span of the projections in A is dense in A;

(3) if M is a maximal commutative C*-subalgebra of A, then any non-zero projection p contained in M is minimal with respect to M if and only if it is minimal with respect to A.

Proof. For (1), let $a \in A_{\text{sa}}$. Then Proposition C.1.15 assures the existence of some maximal commutative C*-subalgebra M such that $a \in M$. Since M is generated by its projections, it follows from Theorem 2.2.3 that M has real rank zero. Hence for each $\epsilon > 0$, there is some $b \in M_{\text{sa}}$ with finite spectrum such that $\|a - b\| < \epsilon$. Since $b \in A_{\text{sa}}$, we conclude that A has real rank zero.

For (2), let $a \in A$ be arbitrary. Then we can write $a = a_1 + ia_2$, where $a_1 = \frac{a+a^*}{2}$ and $a_2 = \frac{a-a^*}{2i}$ are self adjoint. For each $i = 1, 2$, Proposition C.1.15 assures that a_i is contained in some maximal commutative C*-subalgebra M_i of A. Hence there is a sequence r_n consisting of linear combinations of projections in M_1 converging in the norm topology to a_1, and similarly, there is a sequence s_n consisting of linear combinations of projections in M_2 converging ito a_2. Then $r_n + is_n$ is a linear combination of projections in A converging to a. It follows that every element of A can be approximated in norm by linear combinations of projections.

Finally, we prove (3). Assume $p \in A$ is a minimal projection. Then $0 < q \leq p$ implies $q = p$ for each projection $q \in A$. If M is a maximal commutative C*-subalgebra of A containing p, then clearly $0 < q \leq p$ implies $q = p$ for each projection $q \in M$, so p is minimal with respect to M. Conversely, assume that p is a projection in a maximal commutative C*-subalgebra M such that p is minimal with respect tot M. Let q be a projection in A such that $0 < q \leq p$. Thus $pq = qp = q$. Now let r be a projection in M. Since $p, r \in M$ and M is commutative, we have $pr \in M$ and $pr \leq p$. By minimality of p, we obtain $pr = 0$ or $pr = p$. If $pr = 0$, then $qr = qpr = 0$. If $pr = p$, then $qr = qpr = qp = q$. In both cases, q commutes with r. Since M is generated by its projections, q commutes with all elements of M. By maximal commutativity of M, it follows that $q \in M$. By minimality of p with respect to M, we obtain $q = p$, so p is minimal with respect to A. □

2.3 Scattered C*-algebras

We introduce the class of scattered C*-algebras. These algebras are interesting for us, since they can be completely characterized by their commutative C*-subalgebras.

Definition 2.3.1. A C*-algebra A is called *scattered* if every self-adjoint element of A has countable spectrum.

Examples 2.3.2.

- Let X be the one-point compactification of the natural numbers. Let $f \in C(X)$, then $\sigma(f)$ is countable, for X is countable and $\sigma(f) = f[X]$. Hence $C(X)$ is a scattered C*-algebra.

- Let $K(H)$ denote the *compact operators* on a Hilbert space H. Let

$$A = \{a + \lambda 1_H : a \in K(H), \lambda \in \mathbb{C}\},$$

i.e., A is the *unitization* of $K(H)$. It follows from [24, Theorem VII.7.1] that $\sigma(a)$ is countable for each $a \in K(H)$. Since

$$\sigma(a + \lambda 1_H) = \{\mu + \lambda : \mu \in \sigma(a)\},$$

it follows that the spectrum of every element of A is countable. Thus, A is a (non-commutative) scattered C*-algebra.

It follows from Kusuda's Theorem [79] that scattered C*-algebras can be characterized in terms of commutative C*-subalgebras. Since we adjusted the content of the theorem to make it more suitable for our purposes, we include a proof. However, we emphasize that the proof is essentially the same as Kusuda's; we do not use other techniques. For the proof, we need one standard result from point-set topology, the Cantor–Bendixson Theorem. Recall that a topological space is called *second countable* if it has a countable basis.

Theorem 2.3.3 (Cantor–Bendixson)**.** *Let X be a second countable space. Then X can be written as a disjoint union of a countable set C and a closed set P without isolated points.*

Theorem 2.3.4 (Kusuda)**.** *Let A be a C*-algebra. Then the following statements are equivalent:*

(a) *A is scattered;*

(b) *Every maximal commutative C*-subalgebra of A is scattered;*

(c) *Every maximal commutative C*-subalgebra of A has scattered spectrum (cf. Definition A.2.2);*

(d) *Every commutative C*-subalgebra of A is scattered;*

(e) *Every commutative C*-subalgebra of A has scattered spectrum;*

(f) *Every commutative C*-subalgebra A is an AF-algebra;*

(g) *Every commutative C*-subalgebra of A has a totally disconnected spectrum;*

(h) $\mathcal{C}(A)$ does not contain an element *-isomorphic to $C([0,1])$.

Proof. We show that (a) implies (h) by contraposition. Assume that A contains a commutative C*-subalgebra *-isomorphic to $C([0,1])$, and let $g : [0,1] \to [0,1]$ be given by $g(x) = x$. Then g is an element of $C([0,1])$, and g is clearly self adjoint. Since $\sigma(g) = g\big[[0,1]\big]$, we have $\sigma(g) = [0,1]$. Since $C([0,1])$ is *-isomorphic to some commutative C*-subalgebra of A, it follows that g can be identified with a self-adjoint element a of A such that $\sigma(a) = [0,1]$, which is clearly uncountable. We conclude that A cannot be scattered. For the next implications, we will apply the following results multiple times: Proposition C.1.15, which assures that every commutative C*-subalgebra of A is always contained in some maximal commutative C*-subalgebra; Theorem A.2.7, which characterizes scattered spaces X in terms of continuous surjections with domain X; and finally Proposition C.2.5, from which we can derive the following: if C is a commutative C*-subalgebra of A with spectrum X, then the existence of a continuous surjection $X \to Y$ onto some compact Hausdorff space Y implies the existence of some C*-subalgebra of C with spectrum Y, and conversely, given a C*-subalgebra of C with spectrum Y, there must exist a continuous surjection $X \to Y$.

The implication (h) $\implies$ (c) is also proven by contraposition. Assume that there is some maximal commutative C*-subalgebra M with spectrum X that is not scattered. Hence there is some continuous surjection $f : X \to [0,1]$, and it follows that M, hence A, has a commutative C*-subalgebra with spectrum homeomorphic to $[0,1]$.

We show that (c) implies (g), so assume that all maximal commutative C*-subalgebras of A have scattered spectrum and let C be a commutative C*-subalgebra of A with spectrum Y. Since C can be embedded into some maximal commutative C*-subalgebra M with scattered spectrum X, it follows that there exists a continuous surjection $f : X \to Y$, hence Y has totally disconnected spectrum. For (g) $\implies$ (e), let C be a commutative C*-subalgebra of A with spectrum X. Then all C*-subalgebras of C have totally disconnected spectrum. This implies that if $f : X \to Y$ is a continuous surjection onto the compact Hausdorff space Y, then Y is totally disconnected. We conclude that X is scattered.

In order to show that (e) implies (a), let $a \in A$ be self adjoint. Then $\sigma(a) \subseteq \mathbb{R}$, so second countable. Theorem 2.3.3 now implies that $\sigma(a)$ can be written as a disjoint union of a countable set and a closed set P without isolated points. By the functional calculus, $C(\sigma(a))$ is a commutative C*-algebra *-isomorphic to $C^*(a)$, which is a commutative C*-subalgebra of A. Hence $\sigma(a)$ is scattered. Since P is a closed subset of $\sigma(a)$ without isolated points, scatteredness of $\sigma(a)$ implies that P must be empty. It follows that $\sigma(a)$ must be countable, so A is scattered. It follows that in particular (a) and (e) are equivalent, which can be used to prove the equivalence between (a) and (b). Let A be scattered and M a maximal commutative C*-subalgebra. Then all commutative C*-subalgebras of A have scattered spectrum, so in particular all commutative C*-subalgebras of M have scattered spectrum, so M is scattered. Conversely, if all maximal commutative C*-subalgebras of A are scattered and C is some commutative C*-subalgebra of A, then C can be embedded in some maximal commutative C*-subalgebra of A. Hence C has scattered spectrum.

Clearly (d) implies (b). For the converse, we will again use the equivalence between (a) and (e). Assume that every maximal commutative C*-algebra of A is scattered and let C be a commutative C*-subalgebra of A. Then C can be embedded in some maximal commutative C*-subalgebra M of A, and since M is scattered, all its commutative C*-subalgebras have scattered spectrum. This implies that in particular all commutative C*-subalgebras of C have scattered spectrum, hence C is scattered. We conclude that (b) and (d) are equivalent. Finally, the equivalence between (f) and (g) follows directly from Theorem 2.2.3. □

Theorem 2.3.4 is a reformulation of [79, Theorem 2.3], which contains more statements than explicitly listed in Theorem 2.3.4. We did not include those statements for the reason that we do not need them for the content of this thesis. One of these statements is that A is scattered if and only if every (not necessarily commutative) C*-subalgebra is a locally AF algebra. It follows that scattered C*-algebras are locally AF (which is also stated in [82, Lemma 5.1]), hence scattered C*-algebras have real rank zero by Lemma 2.2.2. We note that it also follows from combining Theorem 2.3.4 and Proposition 2.2.7 that scattered C*-algebras have real rank zero.

Corollary 2.3.5. A commutative C*-algebra is scattered if and only if its Gelfand spectrum is scattered.

Corollary 2.3.6. Any finite-dimensional C*-algebra is scattered.

Proof. Let A be finite-dimensional, then all its commutative C*-subalgebras are finite-dimensional, so certainly AF-algebras. It now follows directly from Theorem 2.3.4 that A is scattered. $\square$

Corollary 2.3.7. Let A be a scattered C*-algebra. Then A has a minimal projection. If A is infinite-dimensional, then A has infinitely many minimal projections.

Proof. Let M be a maximal commutative C*-subalgebra of A. Then the Gelfand spectrum X of M is scattered by Theorem 2.3.4, hence it must have an isolated point x. Hence the set S of isolated points of M is non-empty. If A is infinite-dimensional, then it follows from Proposition 2.1.3 that M is infinite-dimensional, too. Hence X has infinitely many points. Assume that S is finite, then S is clopen, hence $X \setminus S$ is clopen. Since $X \setminus S$ is closed, it has an isolated point $\{x\}$. Since $X \setminus S$ is open, it follows that $\{x\}$ is open in X, hence an isolated point not in S. We conclude that S cannot be finite.

It follows from Proposition C.3.6 that the characteristic function of $\{x\}$ with $x \in X$ isolated is a minimal projection in $C(X)$. Hence M has at least one minimal projection, and infinitely many if A is infinite-dimensional. It follows from Theorem 2.3.4 that each maximal commutative C*-subalgebra of A is an AF-algebra, hence M is generated by its projections (cf. Theorem 2.2.3). By Proposition 2.2.7, every minimal projection in M is a minimal projection in A, hence we conclude that A has at least one minimal projection, and infinitely many if A is infinite-dimensional. $\square$

2.4 AW*-algebras

We now describe *AW*-algebras*, a class of C*-algebras that was introduced by Kaplansky in order to axiomatize the algebraic aspects of W*-algebras. AW*-algebras can be regarded as 'abstract W*-algebras', which explains the terminology. The main references for this section are [6, 71, 72]. It turns out that every W*-algebra is an AW*-algebra, but the converse is not true [29].

Definition 2.4.1. Let A be a C*-algebra and $S \subseteq A$ a nonempty subset. Then we define the *right-annihilator* of S as the set

$$R(S) = \{a \in A : sa = 0 \text{ for all } s \in S\}.$$

Similarly, the *left-annihilator* of S is defined by

$$L(S) = \{a \in A : as = 0 \text{ for all } s \in S\}.$$

If we want to emphasize the ambient algebra A, we write $R_A(S)$ instead of $R(S)$.

Lemma 2.4.2. [6, §1.3] Let A be a C*-subalgebra and and denote $S^* = \{s^* : s \in S\}$ for each $S \subseteq A$. If $S \subseteq A$ is nonempty, then $L(S) = R(S^*)^*$ and $R(S) = L(S^*)^*$. Moreover, if $p \in \mathrm{Proj}(A)$, then $R(S) = pA$ if and only if $L(S^*) = Ap$.

Definition 2.4.3. Let A be a C*-algebra. Then A is called a *Rickart C*-algebra* if for each $a \in A$, there is a projection $p \in A$ such that $R(\{a\}) = pA$. If for each nonempty $S \subseteq A$ there is a projection $p \in A$ such that $R(S) = pA$, we say that A is an *AW*-algebra*. If A is an AW*-algebra with one-dimensional center, we call A an *AW*-factor*.

Notice that if $R(S) = pA$ for some projection p, Proposition C.3.2 implies that p is unique. Moreover, Lemma 2.4.2 implies that for each x in a Rickart algebra there is some $p \in \mathrm{Proj}(A)$ such that $L(\{x\}) = Ap$, and similarly, for each nonempty subset S of an AW*-algebra A, there is a projection $p \in \mathrm{Proj}(A)$ such that $L(S) = Ap$.

The following lemma gives a method of constructing new AW*-algebras from a given AW*-algebra.

Lemma 2.4.4. [6, Proposition 4.8] Let A be an AW*-algebra and $p \in A$ a non-zero projection. Then pAp is an AW*-algebra with identity element p.

There are several equivalent definitions of AW*-algebras. The most important ones are listed in the following theorem:

Theorem 2.4.5. Let A be a C*-algebra. Then the following statements are equivalent:

(1) A is an AW*-algebra;

(2) every maximal commutative C*-subalgebra of A is generated by its projections and $\mathrm{Proj}(A)$ is a complete orthomodular lattice;

(3) every maximal commutative C*-subalgebra of A is generated by its projections and the supremum of every family of orthogonal projections exists.

(4) A is a Rickart C*-algebra such that $\mathrm{Proj}(A)$ is a complete orthomodular lattice;

(5) A is a Rickart C*-algebra and the supremum of every family of orthogonal projections exists;

(6) every maximal commutative C*-subalgebra of A has extremally disconnected spectrum (cf. Definition A.2.2).

The equivalences between (1), (4) and (5) in the next theorem are proven in [6, Proposition 4.1]. The equivalence between (1) and (3) is proven in [71]. The equivalence between (3) and (6) is proven by Saito and Wright in [99].

Corollary 2.4.6. [6, Theorem 7.1] Let A be a commutative C*-algebra. Then A is an AW*-algebra if and only if its Gelfand spectrum is extremally disconnected.

Corollary 2.4.7. Let A be a C*-algebra. Then A is an AW*-algebra if and only if all its maximal commutative C*-subalgebras are AW*-algebras.

We note that [99] also contains a proof of the statement that a C*-algebra A is a Rickart C*-algebra if and only if all its maximal commutative C*-subalgebras are Rickart C*-algebras. One could ask to which class of C*-algebras A belongs if one requires that *all* its commutative C*-subalgebras are AW*-algebras instead of only the maximal ones. This question is answered in Corollary 2.4.31 below.

Definition 2.4.8. Let A be a Rickart C*-algebra and $a \in A$. Then we define the *right projection* $RP(a)$ and *left projection* $LP(a)$ of a by $RP(a) = p$, $LP(a) = q$ where p and q are the unique projections satisfying $R(\{a\}) = (1-p)A$ and $L(\{a\}) = A(1-q)$. If we want to emphasize the ambient algebra A, we write $RP_A(a)$ and $LP_A(a)$ instead of $RP(a)$ and $LP(a)$, respectively.

Lemma 2.4.9. [6, Proposition 3.3] Let A be a Rickart C*-algebra and $a \in A$. Then

(1a) $aRP(a) = a$, and $ar = a$ implies $RP(a) \leq r$ for each $r \in \mathrm{Proj}(A)$;

(1b) $LP(a)a = a$, and $ra = a$ implies $LP(a) \leq r$ for each $r \in \mathrm{Proj}(A)$;

(2a) $ab = 0$ if and only if $RP(a)b = 0$ for each $b \in A$;

(2b) $ba = 0$ if and only if $bLP(b) = 0$ for each $b \in A$.

Proposition 2.4.10. [6, Proposition 3.6] Let A be a Rickart C*-algebra, $a \in A$ and $\{p_i\}_{i\in I}$ a collection of projections that have a supremum p. Then $ap = 0$ if and only if $ap_i = 0$ for each $i \in I$. Similarly, we have $pa = 0$ if and only if $p_i a = 0$ for each $i \in I$.

We now give the definition of an AW*-subalgebra. We have seen that the projections of an AW*-algebra form a complete lattice. For this reason, we do not only require an AW*-subalgebra B of an AW*-algebra A to be an AW*-algebra on its own, but also that $\mathrm{Proj}(B)$ be a sublattice of $\mathrm{Proj}(A)$.

Definition 2.4.11. Let A be an AW*-algebra. Then a C*-subalgebra B of A is called an *AW*-subalgebra* of A if the following two conditions hold:

(1) B is an AW*-algebra;

(2) If $\{p_i\}_{i\in I}$ is a collection of projections in B, then their supremum calculated in A is an element of B.

Proposition 2.4.12. [6, Proposition 4.7] Let B be a C*-subalgebra of an AW*-algebra A such that $B = B''$. Then B is an AW*-subalgebra of A.

Lemma 2.4.13. Let A be an AW*-algebra. Then the maximal commutative AW*-subalgebras of A are precisely the maximal commutative C*-subalgebras of A.

Proof. Since a maximal commutative AW*-subalgebra is in particular a commutative C*-subalgebra, which can be embedded in some maximal commutative C*-subalgebra by Proposition C.1.15, it is sufficient to show that all maximal commutative C*-subalgebras are AW*-subalgebras. Thus, let M be a maximal commutative C*-subalgebra of A, so that $M \subseteq M'$. Since M is maximal abelian, we have $M = M'$. Write $x = x_1 + ix_2$, where $x_1 = \frac{x+x^*}{2}$ and $x_2 = \frac{x-x^*}{2i}$ are self-adjoint. Then $x_i y = yx_i$ for each $y \in M$ and each $i = 1, 2$. It follows from Lemma C.1.22 that for each $i = 1, 2$, the C*-algebra $C^*(M \cup \{x_i\})$ is a commutative C*-subalgebra of A containing M and x. By maximality of M, $M = C^*(M \cup \{x_i\})$, hence $x_i \in M$ for $i = 1, 2$. We conclude that $x = x_1 + ix_2$ is an element of M, and we conclude that $M = M'$. In particular we have $M = M''$, hence M is an AW*-subalgebra of A. ☐

Corollary 2.4.14. Each AW*-algebra A has a real rank zero. Moreover, the span of $\mathrm{Proj}(A)$ is dense in A.

Proof. Since the maximal commutative C*-subalgebras of A are generated by their projections, the statement follows from Proposition 2.2.7. $\square$

The statement that each AW*-algebra has real rank zero is known, see for instance [46, p. 35] or [10]. In the latter reference, it is also stated that the span of the projections in any C*-algebra of real rank zero is dense.

In the proof of the next lemma, we use techniques from the proof of [71, Theorem 5.2].

Lemma 2.4.15. Let A be an AW*-algebra. Then

$$RP(x) = \bigvee \{p \in \mathrm{Proj}(A) : p = ax^*x \text{ for some } a \in A\}.$$

Proof. If $x = 0$, then there is only one projection that is a multiple of x^*x, namely $p = 0$, which is clearly the right projection of 0. Assume that $x \neq 0$ and let $y = x^*x$ We notice that $R(\{x\}) = R(\{y\})$, since we have $ya = 0$ if and only if $(xa)^*(xa) = a^*x^*xa = 0$ if and only if $xa = 0$. It now follows from the definition of the right projection that $RP(x) = RP(y)$.

Let S be the set of all projections $p \in A$ such that $p = ay$ for some $a \in A$ and let $q = \bigvee S$. We aim to show that $q = RP(y)$. Since y is self adjoint, Proposition C.1.15 assures that we can find a maximal abelian C*-subalgebra M of A such that $y \in M$. By Theorem 2.4.5, M is generated by its projections. It follows from Theorem 2.2.3 that for each $\epsilon > 0$, we can find a $p \in M$ such that $p = ay$ for some $a \in M$, and such that $\|y - yp\| < \epsilon$. This implies that $p \in S$, so we can also conclude that S is nonempty.

By Lemma 2.4.9, we have $yRP(y) = y$, so $y\big(1 - RP(y)\big) = 0$. Let $p \in S$. Then there is an $a \in A$ such that $p = ay$, whence $p\big(1 - RP(y)\big) = 0$. Thus $p = pRP(y)$, so $p \leq RP(y)$. Since q is the supremum of S, we conclude that $q \leq RP(y)$.

Let $\epsilon > 0$. We observed that we can find a $p \in M \cap S$ such that $\|y - yp\| < \epsilon$. Since $p \leq q$, we have $pq = p$, so

$$(1 - p)(1 - q) = 1 - p - q + pq = 1 - q.$$

Hence

$$\|y(1 - q)\| = \|y(1 - p)(1 - q)\| \leq \|y(1 - p)\|\|1 - q\| < \epsilon\|1 - q\| = \epsilon.$$

Since we can take ϵ arbitrarily small, we find that $y(1-q) = 0$. Hence $y = yq$ and by Lemma 2.4.9, we find that $RP(y) \leq q$. We conclude that $RP(y) = q$, which is the statement we wanted to prove. □

Proposition 2.4.16. Let H be a Hilbert space. Then $B(H)$ is an AW*-algebra. Moreover, the von Neumann algebras on H coincide with the AW*-subalgebras of $B(H)$ (hence every von Neumann algebras is an AW*-algebra).

This follows from [6, Proposition 4.3, Proposition 4.9, Exercise 4.24].

The next lemma is obvious. Nevertheless, in view of the delicacies involving AW*-subalgebras we provide a proof.

Lemma 2.4.17. Let A be an AW*-algebra and $B \subseteq A$ an AW*-subalgebra of A. Furthermore, let $C \subseteq B$ be a C*-subalgebra. Then C is an AW*-subalgebra of B if and only if it is an AW*-subalgebra of A.

Proof. Assume that C is an AW*-subalgebra of B. In particular, C is an AW*-algebra. Moreover, if $S \subseteq \mathrm{Proj}(C)$, then the same proposition assures that the supremum of S calculated in C equals the supremum of S calculated in B, which equals the supremum of S calculated in A. Hence C is an AW*-subalgebra of A.

Conversely, assume that C is an AW*-subalgebra of A such that $C \subseteq B$. By definition C is an AW*-algebra. Let $S \subseteq \mathrm{Proj}(C)$. Then $S \subseteq B$ and since C and B are AW*-subalgebras of A, it follows that the supremum of S calculated in C equals the supremum of S calculated in A, which equals the supremum of S calculated in B. Hence C is an AW*-subalgebra of B. □

Lemma 2.4.18. [6, Proposition 4.8] Let A be an AW*-algebra. Then:

- the center $Z(A)$ is a commutative AW*-subalgebra of A;
- if $\{C_i\}_{i\in I}$ is a collection of AW*-subalgebras of A, then $C = \bigcap_{i\in I} C_i$ is an AW*-subalgebra of A. Moreover, C is commutative if C_i is commutative for some $i \in I$.

By Lemma 2.4.18, the intersection of AW*-subalgebras of an AW*-algebra A is an AW*-subalgebra of A. Let $S \subseteq A$ be a set, then the set of AW*-subalgebras of A containing S is non-empty, since it contains A itself. Hence the intersection of all AW*-subalgebras containing S is the smallest AW*-subalgebra of A containing S, so the following definition makes sense.

Definition 2.4.19. Let A be an AW*-algebra and $S \subseteq A$. Then we define the AW*-subalgebra $AW^*(S)$ of A *generated* by S as the smallest AW*-subalgebra of A containing S. If S is finite, say $S = \{s_1, \ldots, s_n\}$, we rather write $AW^*(s_1, \ldots, s_n)$ instead of $AW^*(\{s_1, \ldots, s_n\})$.

Lemma 2.4.20. Let A be an AW*-algebra and S a *-closed subset of A. Then S is commutative if and only if $AW^*(S)$ is a commutative AW*-subalgebra of A.

Proof. Clearly S is commutative if $AW^*(S)$ is commutative. If S is commutative, let $C = S''$. It follows from Proposition C.1.15 that C is a commutative C*-algebra containing S such that $C = C''$. Proposition 2.4.12 assures that C is an AW*-subalgebra of A, hence by definition of $AW^*(S)$ and by Lemma 2.4.18, it $AW^*(S)$ must be commutative. □

Lemma 2.4.21. Let A be an AW*-subalgebra and $S \subseteq A$ such that $C^*(S)$ is an AW*-subalgebra of A. Then $C^*(S) = AW^*(S)$.

Proof. The collection of C*-subalgebras of A containing S contains the collection of AW*-subalgebras of A containing S. Therefore $C^*(S) \subseteq AW^*(S)$. On the other hand, since $C^*(S)$ is an AW*-subalgebra of A and it contains S, we must have $AW^*(S) \subseteq C^*(S)$. Hence we have equality. □

Definition 2.4.22. Let $\varphi : A \to B$ be a *-homomorphism between AW*-algebras A and B. Then we say that φ is an *AW*-homomorphism* if $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ preserves arbitrary suprema. We denote the category of AW*-algebras with AW*-homomorphisms by **AWStar**. If φ is a bijective AW*-homomorphism, then φ is called an *AW*-isomorphism*.

The following results mainly deal with AW*-homomorphisms, and are well known to AW*-experts. Since the main references [6, 71, 72, 73] do not treat AW*-homomorphisms, we include the proofs.

Lemma 2.4.23. Let A be an AW*-algebra, B a C*-algebra, and $\varphi : A \to B$ a *-isomorphism. Then B is an AW*-algebra and φ is an AW*-isomorphism.

Proof. Let $S \subseteq B$ be non-empty. Then $T = \varphi^{-1}[S]$ is a non-empty subset of A, hence $R(T) = pA$ for some projection $p \in A$. Then $q = \varphi(p)$ is a projection in B. Let $a \in R(T)$, and $s \in S$. Then $s = \varphi(x)$ for some $x \in T$, hence

$$s\varphi(a) = \varphi(x)\varphi(a) = \varphi(ax) = \varphi(0) = 0,$$

so $\varphi(a) \in R(S)$. Conversely, if $b \in R(S)$, then $b = \varphi(a)$ for some $a \in A$. For each $x \in T$, we have $\varphi(x) \in S$, hence

$$0 = \varphi(x)b = \varphi(x)\varphi(a) = \varphi(xa),$$

hence $xa = 0$, so $a \in R(T)$. Thus, $\varphi[R(T)] = R(S)$. Moreover,

$$\varphi[pA] = \{\varphi(pa) : a \in A\} = \{\varphi(p)\varphi(a) : a \in A\} = \{qb : b \in B\} = qB,$$

and since $R(T) = pA$, we find $R(S) = qB$, so B is an AW*-algebra. In order to show that φ is an AW*-isomorphism, note that Proposition C.3.4 assures that $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ is an orthomodular isomorphism, which is an order isomorphism by Lemma B.4.9. Hence $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ preserves all suprema, so $\varphi : A \to B$ is indeed an AW*-isomorphism. □

Lemma 2.4.24. Let $\varphi : A \to B$ be an AW*-homomorphism between AW*-algebras A and B. Then we have for each $x \in A$,

$$RP(\varphi(x)) = \varphi(RP(x)).$$

Proof. Let $x \in A$. By Lemma 2.4.9, we have $xRP(x) = x$, hence $\varphi(x)\varphi(RP(x)) = \varphi(x)$. It follows from the same lemma that $RP(\varphi(x)) \leq \varphi(RP(x))$. Using the fact that φ preserves suprema of projections, Lemma 2.4.15 yields

$$\varphi(RP(x)) = \bigvee\{\varphi(p) \in \mathrm{Proj}(A) : p = ax^*x \text{ for some } a \in A\}.$$

Now for any $p \in \mathrm{Proj}(A)$, if $p = ax^*x$ for some $a \in A$, then $\varphi(p) = b\varphi(x)^*\varphi(x)$ for some $b \in B$, namely $b = \varphi(a)$. Hence

$$\{\varphi(p) \in \mathrm{Proj}(A) : p = ax^*x \text{ for some } a \in A\}$$

is a subset of

$$\{q \in \mathrm{Proj}(B) : q = b\varphi(x)^*\varphi(x) \text{ for some } b \in B\}.$$

By Lemma 2.4.15, we obtain

$$\bigvee\{q \in \mathrm{Proj}(B) : q = b\varphi(x)^*\varphi(x) \text{ for some } b \in B\} = RP(\varphi(x)),$$

hence we obtain $\varphi(RP(x)) \leq RP(\varphi(x))$. □

Proposition 2.4.25. Let $\varphi : A \to B$ be an AW*-homomorphism between AW*-algebras A and B. Then

- $\varphi[C]$ is an AW*-subalgebra of B for each AW*-subalgebra C of A;
- $\varphi^{-1}[D]$ is an AW*-subalgebra of A for each AW*-subalgebra D of A.

Proof. Let C be an AW*-subalgebra of A and D an AW*-subalgebra of B. By Theorem C.1.6, we know that $\varphi[C]$ is a C*-subalgebra of B and $\varphi^{-1}[D]$ a C*-subalgebra of A. We first show that $\varphi[C]$ is an AW*-subalgebra of A. Let $y \in \varphi[C]$. Then $y = \varphi(x)$ for some $x \in C$. We notice that $RP(x) \in C$, for C is an AW*-subalgebra of A. Moreover, by Lemma 2.4.24 we have

$$RP(y) = RP(\varphi(x)) = \varphi(RP(x)),$$

so $RP(y) \in \varphi[C]$. Let $\{q_i\}_{i\in I}$ a family of projections in $\varphi[C]$. Then there are $x_i \in C$ such that $\varphi(x_i) = q_i$ for each $i \in I$. Let $p_i = RP(x_i)$, then we have $p_i \in C$ for each $i \in I$, since C is an AW*-subalgebra. By definition of the order on $\mathrm{Proj}(B)$ we have $q_i q = q_i$ if and only if $q_i \leq q$. Hence by Lemma 2.4.9, we find

$$q_i = RP(q_i) = RP(\varphi(x_i)) = \varphi(RP(x_i)) = \varphi(p_i),$$

where we used Lemma 2.4.24 in the third equality. The p_i are projections in C, which is an AW*-subalgebra, hence $\bigvee_{i\in I} p_i \in C$. Since φ preserves suprema of projections in A, this implies

$$\bigvee_{i\in I} q_i = \bigvee_{i\in I} \varphi(p_i) = \varphi\left(\bigvee_{i\in I} p_i\right),$$

hence $\bigvee_{i\in I} q_i \in \varphi[C]$. We conclude that $\varphi[C]$ is an AW*-subalgebra of B.

We show that $\varphi^{-1}[D]$ is an AW*-subalgebra of A. Let $x \in \varphi^{-1}[D]$, then $\varphi(x) \in D$, which is an AW*-subalgebra of B, hence $RP(\varphi(x)) \in D$. By Lemma 2.4.24, we find $\varphi(RP(x)) = RP(\varphi(x))$, hence $RP(x) \in \varphi^{-1}[D]$. Let $\{p_i\}_{i\in I}$ be a collection of projections in $\varphi^{-1}[D]$. Then $\{\varphi(p_i)\}_{i\in I}$ is a collection of projections in D, and since D is an AW*-subalgebra of B, we find $\bigvee_{i\in I} \varphi(p_i) \in D$. Now, since φ is an AW*-homomorphism, we find

$$\varphi\left(\bigvee_{i\in I} p_i\right) = \bigvee_{i\in I} \varphi(p_i),$$

hence $\bigvee_{i\in I} p_i \in \varphi^{-1}[D]$. We conclude that $\varphi^{-1}[D]$ is an AW*-subalgebra of A. □

The first claim in the next proposition is taken from [6, Proposition 9.1].

Proposition 2.4.26. Let $\{A_i\}_{i\in I}$ be a collection of AW*-algebras, and let $A = \bigoplus_{i\in I} A_i$. Then A is an AW*-algebra, and the projection map $\pi_i : A \to A_i$ is an AW*-homomorphism for each $i \in I$.

Proof. For each $j \in I$, let $\pi_j : A \to A_i$, $(a_i)_{i\in I} \mapsto a_j$ be the projection on the j-th factor. Clearly π_j is a *-homomorphism. Let $S \subseteq A$ be non-empty. For each $i \in I$, let $S_i = \pi_i[S]$. Let $a = (a_i)_{i\in I} \in A$. We show that

$$a \in R(S) \implies a_i \in R(S_i),$$

where we $R(S_i)$ regard as a subset of A_i. Indeed, if $a \in R(S)$, and $s_i \in S_i$, then $s_i = \pi_i(s)$ for some $s \in S$. Since π_i is a *-homomorphism, we have

$$s_i a_i = \pi_i(s)\pi_i(a) = \pi_i(sa) = 0.$$

So indeed $a_i \in R(S_i)$. Since each A_i is an AW*-algebra, it follows that for each $i \in I$, there is a projection $p_i \in A_i$ such that $R(S_i) = p_i A_i$. Let $p = (p_i)_{i\in I}$. Since $\|p_i\| \in \{0, 1\}$ for each $i \in I$, we find that $\sup_{i\in I} \|p_i\| < \infty$, hence $p \in A$. If $a \in R(S)$, then $a_i \in R(S_i) = p_i A_i$, hence for each $i \in I$, there are $x_i \in A_i$ such that $a_i = p_i x_i$. Then

$$p_i a_i = p_i^2 x_i = p_i x_i = a_i,$$

hence

$$a = (a_i)_{i\in I} = (p_i a_i)_{i\in I} = (p_i)_{i\in I}(a_i)_{i\in I} = pa,$$

so $a \in pA$. Conversely, let $a \in pA$. Then $a = px$ for some $x \in A$. Let $s \in S$, so $s = (s_i)_{i\in I}$ with $s_i = \pi_i(s)$. Thus $s_i \in S_i$, hence

$$sp = (s_i)_{i\in I}(p_i)_{i\in I} = (s_i p_i)_{i\in I} = 0,$$

for $p_iA = R(S_i)$. Hence

$$sa = spx = 0,$$

so $a \in R(S)$. Thus $R(S) = pA$, so A is an AW*-algebra. Let $S \subseteq \mathrm{Proj}(A)$. By Lemma C.3.9, there is an order isomorphism $\varphi : \mathrm{Proj}(A) \to \prod_{i\in I} \mathrm{Proj}(A_i)$ given by $p \mapsto \big(\pi_i(p)\big)_{i\in I}$. By Proposition B.1.15, $\varphi[S] \subseteq \prod_{i\in I} \mathrm{Proj}(A_i)$ must have a supremum, which is equal to $\varphi\left(\bigvee S\right)$. Let $\psi_j : \prod_{i\in I} \mathrm{Proj}(A_i) \to \mathrm{Proj}(A_j)$ denote the canonical projection map on the j-th component. Notice that $\psi_j \circ \varphi = \pi_j$. Now, we can apply Lemma B.1.18 in order to conclude that the existence of a supremum of $\varphi[S]$ implies the existence of a supremum of $\psi_j\left[\varphi[S]\right]$ in $\mathrm{Proj}(A_j)$, which is given by $\psi_j\left(\bigvee \varphi[S]\right)$. Hence

$$\bigvee \pi_j[S] = \bigvee \psi_j\left[\varphi[S]\right] = \psi_j\left(\bigvee \varphi[S]\right) = \psi_j \circ \varphi\left(\bigvee S\right) = \pi_j\left(\bigvee S\right).$$

We conclude that π_j preserves suprema of projections, hence π_j is an AW*-homomorphism. □

Proposition 2.4.27. [6, Proposition 10.2] Let A be an AW*-algebra and $\{p_i\}_{i\in I}$ a collection of mutually orthogonal projections in $Z(A)$ such that $\bigvee_{i\in I} p_i = 1_A$. Then $A_i := p_iA$ is an AW*-algebra with identity element p_i for each $i \in I$, and $\varphi : A \to \bigoplus_{i\in I} A_i$, $a \mapsto (p_ia)_{i\in I}$ is a *-isomorphism.

Conversely, if $A = \bigoplus_{i\in I} A_i$ for some collection $\{A_i\}_{i\in I}$ of AW*-algebras, then A is an AW*-algebra, and there is a collection of orthogonal projections $\{p_i\}_{i\in I}$ in $Z(A)$, where $p_i = (p_{i,j})_{j\in I}$ is given by

$$p_{i,j} = \begin{cases} 1_{A_j}, & j = i, \\ 0_{A_j}, & j \neq i, \end{cases} \tag{5}$$

such that $\bigvee_{i\in I} p_i = 1_A$, and the restriction of the projection map $\pi_i : A \to A_i$ to the domain pA_i is a *-isomorphism $p_iA \to A_i$ for each $i \in I$.

Proposition 2.4.28. Any finite-dimensional C*-algebra is an AW*-algebra.

Proof. Let A be a finite-dimensional C*-algebra. By the Artin-Wedderburn Theorem, $A \cong \bigoplus_{i=1}^k \mathrm{M}_{n_i}(\mathbb{C})$. We note that $\mathrm{M}_{n_i}(\mathbb{C}) = B(\mathbb{C}^{n_i})$, which is an AW*-algebra by Proposition 2.4.16. By Proposition 2.4.26, $\bigoplus_{i=1}^k \mathrm{M}_{n_i}(\mathbb{C})$ is an AW*-algebra, hence A is an AW*-algebra by Lemma 2.4.23. □

Corollary 2.4.29. Let A be an AW*-algebra and C a finite-dimensional C*-subalgebra of A. Then C is an AW*-subalgebra of A.

Proof. By Proposition 2.4.28, C is an AW*-algebra. Let $S \subseteq \text{Proj}(C)$. By Proposition 2.1.2, $\text{Proj}(C)$ is finite, hence S is finite. Any two elements p and q in S commute in A, hence by Proposition C.3.2, their join in $\text{Proj}(A)$ is given by $p + q - pq$, which is an element of C. Since S is finite, this implies $\bigvee S \in C$. We conclude that C is an AW*-subalgebra of A. □

Corollary 2.4.30. The finite-dimensional C*-algebras are precisely the scattered AW*-algebras.

Proof. Let A be a finite-dimensional C*-algebra. Then A is scattered by Corollary 2.3.6 and is an AW*-algebra by Proposition 2.4.28. Conversely, if A is a scattered AW*-algebra, let M be a maximal commutative C*-subalgebra of A. By Theorem 2.3.4, the spectrum of M is scattered. By Theorem 2.4.5, the spectrum of M is extremally disconnected as well. It now follows from Lemma A.2.3 that the spectrum of M is finite, hence M must be finite-dimensional, see for instance Proposition 2.1.2. Hence Proposition 2.1.3 assures that A itself is finite-dimensional. □

Corollary 2.4.31. Let A be a C*-algebra. Then A is finite-dimensional if and only if all its commutative C*-subalgebras are AW*-algebras.

Proof. If A is finite-dimensional, then it follows from Corollary 2.4.30 that its commutative C*-subalgebras (which are necessarily finite-dimensional, too) are AW*-algebras. Conversely, if all commutative C*-subalgebras of A are AW*-algebras, then in particular all maximal commutative C*-subalgebras of A are AW*-algebras. Hence A is an AW*-algebra by Theorem 2.4.5. Moreover, by Corollary 2.4.6, the Gelfand spectrum of any commutative C*-subalgebra of A is extremally disconnected. It now follows from Lemma A.2.3 that all commutative C*-subalgebras of A have a totally disconnected Gelfand spectrum. Theorem 2.3.4 now assures that A is scattered, hence A is finite-dimensional by Corollary 2.4.30. □

3 Posets of commutative C*-subalgebras

In this chapter, we define the poset of commutative C*-subalgebras of a C*-algebra and investigate some of its properties, such as the existence of suprema and infima, and functoriality.

3.1 Definition and general properties

We start by giving a formal definition of posets of commutative C*-subalgebras.

Definition 3.1.1. Let A be a C*-algebra. We denote the set of its commutative C*-subalgebras by $\mathcal{C}(A)$, which becomes a poset if we order it by inclusion[3].

Lemma 3.1.2. Let A be a C*-algebra. Then $Z(A)$ is the intersection of all maximal commutative C*-subalgebras of A.

Proof. Let $x \in \bigcap \max \mathcal{C}(A)$, i.e., $x \in M$ for each maximal commutative C*-subalgebra of A. Let $y \in A$. Then y can be written as a linear combination of two self-adjoint elements a_1, a_2. It now follows from Proposition C.1.15 that there are maximal commutative C*-subalgebras M_1, M_2 of A such that $a_i \in M_i$ for $i = 1, 2$. Since $x \in M_1, M_2$, it follows that x commutes with both a_1 and a_2. Hence x commutes with y, so $x \in Z(A)$. Thus the intersection of all maximal commutative C*-subalgebras is contained in $Z(A)$.

Now assume that $x \in Z(A)$. Since x commutes with all elements of A, it commutes in particular with x^*. Hence x is normal. We have $x^* \in Z(A)$ as well, for $Z(A)$ is a *-subalgebra of A. Let M be a maximal commutative C*-subalgebra of A. Then $M \cup \{x, x^*\}$ is a set of mutually commuting elements, which is *-closed. By Lemma C.1.22, we find that there is a commutative C*-subalgebra $C^*(M \cup \{x, x^*\})$ containing $M \cup \{x, x^*\}$. But since M is maximal, $C^*(M \cup \{x, x^*\})$ must be equal to M. As a consequence, $x \in M$, so we find that x is contained in every maximal commutative C*-subalgebra of A. Hence $Z(A)$ is contained in the intersection of all maximal commutative C*-subalgebras of A. □

Although neither the statement of the lemma above is complicated, nor is its proof difficult, we could not find it in the C*-algebra literature. However, similar

[3] In the literature one sometimes refers to $\mathcal{C}(A)$ as the 'poset of classical contexts', since one can regard a commutative C*-subalgebra of a C*-algebra A as a 'classical context' of the quantum system represented by A.

statements in other fields can be found, for instance Lemma B.4.24, which is in the setting of orthomodular posets. Moreover, it turns out as well that the center of a Lie algebra $\mathfrak{g}$ equals the intersection of the so called maximal abelian $\mathfrak{g}$-reductive subalgebras [62, Proposition 4.20].

Theorem 3.1.3. Let A be a C*-algebra. Then:

(a) The C*-subalgebra $\mathbb{C}1_A$ is the least element of $\mathcal{C}(A)$;

(b) The infimum $\bigwedge \mathcal{S}$ of a non-empty subset $\mathcal{S} \subseteq \mathcal{C}(A)$ is given by $\bigcap \mathcal{S}$;

(c) The supremum $\bigvee \mathcal{S}$ of a subset $\mathcal{S} \subseteq \mathcal{C}(A)$ exists if and only all elements of $\bigcup \mathcal{S}$ commute, in which case $\bigvee \mathcal{S} = C^*(\bigcup \mathcal{S})$;

(d) If a subset $\mathcal{D} \subseteq \mathcal{C}(A)$ is directed, then its supremum $\bigvee \mathcal{D}$ exists, and is given by $\overline{\bigcup \mathcal{D}}$; in case A is finite-dimensional, then $\bigvee \mathcal{D} = \bigcup \mathcal{D}$;

(e) For each $C \in \mathcal{C}(A)$, there is an $M \in \max \mathcal{C}(A)$ such that $C \subseteq M$. In particular, $\max \mathcal{C}(A)$ is non-empty, and its elements are exactly the maximal commutative C*-subalgebras of A;

(f) The center $Z(A)$ is the infimum of $\max \mathcal{C}(A)$.

In particular, $\mathcal{C}(A)$ is a complete semilattice, in particular a dcpo (cf. Definition B.6.1).

Proof. Clearly $\mathbb{C}1_A$ is a commutative C*-algebra, and since $1_A \in C$ for each $C \in \mathcal{C}(A)$, it follows that $\mathbb{C}1_A \subseteq C$, which proves (a). By Lemma C.1.18, the intersection of any non-empty subset $\mathcal{S}$ of $\mathcal{C}(A)$ is a commutative C*-subalgebra of A, hence is the infimum of $\mathcal{S}$. We note that if $\mathcal{S}$ is empty, then $\bigcap \mathcal{S} = A$, which needs not be commutative. For (c), clearly $C^*(\bigcup \mathcal{S})$ is the smallest C*-subalgebra of A containing all elements of $\mathcal{S}$. Moreover, since all elements of $\mathcal{S}$ are *-closed, it follows that $\bigcup \mathcal{S}$ is *-closed. By Lemma C.1.22, $C^*(\bigcup \mathcal{S})$ is commutative if $\bigcup \mathcal{S}$ is commutative. The converse is trivial. We prove (d): Let $S = \bigcup \mathcal{D}$. We show that S is a commutative *-algebra. Let $x, y \in S$ and $\lambda, \mu \in \mathbb{C}$, there are $D_1, D_2 \in \mathcal{D}$ such that $x \in D_1$ and $y \in D_2$. Since $\mathcal{D}$ is directed, there is some $D_3 \in \mathcal{D}$ such that $D_1, D_2 \subseteq D_3$. Hence $x, y \in D_3$, whence $\lambda x + \mu y, x^*, xy \in D_3$, and since D_3 is commutative, it follows that $xy = yx$. Since $D_3 \subseteq S$, and $1_A \in D_3$ it follows that S is a commutative *-subalgebra of A. By Lemma C.1.21, we find that $\overline{S}$ is a commutative C*-subalgebra of A.

If A is finite-dimensional, all subspaces of A are closed, hence $\bigcup \mathcal{D}$ is already closed. Clearly $C \in \max \mathcal{C}(A)$ if and only if C is a maximal commutative C*-subalgebra of A. Then (e) follows directly from Proposition C.1.15. Finally, (f) is exactly Lemma 3.1.2. It now follows from (b) and (d) that $\mathcal{C}(A)$ is a complete semilattice. □

If A is not finite-dimensional, then we do not necessarily have $\bigvee \mathcal{D} = \bigcup \mathcal{D}$ for each directed $\mathcal{D} \subseteq \mathcal{C}(A)$. For instance, let $A = C([0,1])$ and

$$\mathcal{D} = \{C_{[0,1/n]} : n \in \mathbb{N}\}.$$

Then each function in $\bigcup \mathcal{D}$ is constant on some neighborhood of 0. Therefore, if $f : [0,1] \to \mathbb{C}$ is defined by $f(x) = x$, then $f \in C([0,1])$, but clearly $f \notin \bigcup \mathcal{D}$. However, one can easily show that $\bigcup \mathcal{D}$ is a *-subalgebra of $C([0,1])$ that separates points in $[0,1]$, so $\overline{\bigcup \mathcal{D}} = C([0,1])$ by the Stone-Weierstrass Theorem. Thus $\bigcup \mathcal{D} \neq \bigvee \mathcal{D}$.

The statement that $\mathcal{C}(A)$ is a dcpo has been noted before in [27] and [102].

Corollary 3.1.4. Let A be a C*-algebra and $C \in \mathcal{C}(A)$. If $\{S_i\}_{i \in I}$ is some collection of subsets of C, then $C^*(S_i) \in \mathcal{C}(A)$ for each $i \in I$, the supremum of $\{C^*(S_i)\}_{i \in I}$ exists, and is equal to

$$\bigvee_{i \in I} C^*(S_i) = C^*\left(\bigcup_{i \in I} S_i\right). \tag{6}$$

Proof. Since $S_i \subseteq C$ for each $i \in I$, Lemma C.1.22 assures that $C^*(S_i) \in \mathcal{C}(A)$. Moreover, we have $\bigcup_{i \in I} S_i \subseteq C$, hence it follows from Theorem 3.1.3 that

$$\bigvee_{i \in I} C^*(S_i) = C^*\left(\bigcup_{i \in I} C^*(S_i)\right).$$

Now by Lemma C.1.20, $C^* : \mathcal{P}(C) \to {\downarrow} C$ has an upper adjoint. Hence C^* preserves suprema (Lemma B.1.23), which proves (6). □

3.2 Functoriality of the map $A \mapsto \mathcal{C}(A)$

It is known that $\mathcal{C} : \mathbf{CStar} \to \mathbf{Poset}$ is a functor [57, Proposition 5.3.3]. This statement is sharpened in the next theorem, which states several other properties of $\mathcal{C}$ as a functor as well.

Theorem 3.2.1. $\mathcal{C} : \mathbf{CStar} \to \mathbf{DCPO}$ becomes a functor if, for each *-homomorphism $\varphi : A \to B$ between C*-algebras A and B, we define $\mathcal{C}(\varphi) : \mathcal{C}(A) \to \mathcal{C}(B)$ by $C \mapsto \varphi[C]$. Moreover, $\mathcal{C}(\varphi)$ has the following properties:

(a) If $S \subseteq A$ is a commutative *-closed subset, then

$$\mathcal{C}(\varphi)(C^*(S)) = C^*(\varphi[S]).$$

(b) If $\{C_i\}_{i \in I} \subseteq \mathcal{C}(A)$ is a family such that $\bigvee_{i \in I} C_i$ exists, then $\bigvee_{i \in I} \mathcal{C}(\varphi)(C_i)$ exists, and

$$\mathcal{C}(\varphi)\left(\bigvee_{i \in I} C_i\right) = \bigvee_{i \in I} \mathcal{C}(\varphi)(C_i). \tag{7}$$

In particular, $\mathcal{C}(\varphi)$ is Scott continuous (cf. Definition B.6.1).

(c) If φ is injective, or if A is commutative, then $\mathcal{C}(\varphi)$ has an upper adjoint (cf. Definition B.1.20)

$$\mathcal{C}(\varphi)_* : \mathcal{C}(B) \to \mathcal{C}(A), \quad D \mapsto \varphi^{-1}[D],$$

which satisfies

$$\mathcal{C}(\varphi)_*\left(\bigcap_{j \in J} D_j\right) = \bigcap_{j \in J} \mathcal{C}(\varphi)_*(D_j), \tag{8}$$

for each family $\{D_i\}_{i \in I} \subseteq \mathcal{C}(B)$ such that $I \neq \emptyset$.

(d) If φ is injective, then $\mathcal{C}(\varphi)$ is an order embedding such that

$$\mathcal{C}(\varphi)\left(\bigcap_{i \in I} C_i\right) = \bigcap_{i \in I} \mathcal{C}(\varphi)(C_i), \tag{9}$$

for each family $\{C_i\}_{i\in I} \subseteq \mathcal{C}(A)$ such that $I \neq \emptyset$. Moreover, the following identities hold:

$$\begin{aligned} \mathcal{C}(\varphi)_* \circ \mathcal{C}(\varphi) &= 1_{\mathcal{C}(A)}; \\ \mathcal{C}(\varphi) \circ \mathcal{C}(\varphi)_*|_{\mathcal{C}(\varphi)[\mathcal{C}(A)]} &= 1_{\mathcal{C}(B)}\big|_{\mathcal{C}(\varphi)[\mathcal{C}(A)]}, \end{aligned}$$

and

$$\downarrow\mathcal{C}(\varphi)[\mathcal{C}(A)] = \mathcal{C}(\varphi)[\mathcal{C}(A)].$$

(e) If $\mathcal{C}(\varphi)$ is surjective, then φ is surjective.

(f) If φ is a *-isomorphism, then $\mathcal{C}(\varphi)$ is an order isomorphism.

Proof. Let $C \in \mathcal{C}(A)$. Then the restriction of φ to C is a *-homomorphism with codomain B. By Theorem C.1.6 it follows that $\varphi[C]$ is a C*-subalgebra of B. Since φ is multiplicative, it follows that $\varphi[C]$ is commutative, so $\varphi[C] \in \mathcal{C}(B)$. Moreover, we have $\varphi[C] \subseteq \varphi[D]$ if $C \subseteq D$, so $\mathcal{C}(\varphi)$ is an order morphism. If $\varphi : A \to B$ and $\psi : B \to D$ are *-homomorphisms, then

$$\mathcal{C}(\psi \circ \varphi)(C) = \psi \circ \varphi[C] = \psi[\varphi[C]] = \mathcal{C}(\psi) \circ \mathcal{C}(\varphi)(C),$$

for each $C \in \mathcal{C}(A)$, and if $\iota_A : A \to A$ is the identity morphism, then

$$\mathcal{C}(\iota_A)(C) = \iota_A[C] = C = 1_{\mathcal{C}(A)}(C),$$

for each $C \in \mathcal{C}(A)$, so $\mathcal{C}(\iota_A)$ is the identity morphism of $\mathcal{C}(A)$. Thus $\mathcal{C}$ is a functor **CStar** $\to$ **Poset**. It follows from (b) that $\mathcal{C}$ is actually a functor **CStar** $\to$ **DCPO**. We prove properties (a)-(f).

(a) We recall Lemma C.1.22, which assures that $C^*(S) \in \mathcal{C}(A)$, hence $\varphi[C^*(S)]$ is an element of $\mathcal{C}(B)$, because

$$\varphi[C^*(S)] = \mathcal{C}(\varphi)(C^*(S)). \tag{10}$$

We have

$$\varphi[S] \subseteq \varphi[C^*(S)],$$

whence

$$C^*(\varphi[S]) \subseteq \varphi[C^*(S)]. \tag{11}$$

For the inclusion in the other direction, we note that

$$S \subseteq \varphi^{-1}\left[\varphi[S]\right] \subseteq \varphi^{-1}[C^*(\varphi[S])].$$

Since $\varphi^{-1}[C^*(\varphi[S])$ is a C*-subalgebra of A by Theorem C.1.6, we obtain

$$C^*(S) \subseteq \varphi^{-1}[C^*(\varphi[S])],$$

and it follows from Example B.1.21 that

$$\varphi[C^*(S)] \subseteq C^*(\varphi[S]). \tag{12}$$

The statement now follows from combining (10), (11), and (12).

(b) Assume that $\bigvee_{i \in I} C_i$ exists in $\mathcal{C}(A)$, i.e., $\bigcup_{i \in I} C_i$ is a commutative set (using Theorem 3.1.3). By the same theorem and (a), we obtain

$$\begin{aligned} \mathcal{C}(\varphi)\left(\bigvee_{i \in I} C_i\right) &= \varphi\left[C^*\left(\bigcup_{i \in I} C_i\right)\right] &&= C^*\left(\varphi\left[\bigcup_{i \in I} C_i\right]\right) \\ &= C^*\left(\bigcup_{i \in I} \varphi[C_i]\right) &&= C^*\left(\bigcup_{i \in I} \mathcal{C}(\varphi)(C_i)\right) \\ &= \bigvee_{i \in I} \mathcal{C}(\varphi)(C_i). \end{aligned}$$

(c) Assume that φ is injective, or that A is commutative. Let $D \in \mathcal{C}(B)$, then $\varphi^{-1}[D]$ is a C*-subalgebra of A by Theorem C.1.6, which is clearly commutative if A itself is commutative. In the case that φ is injective, let $x, y \in \varphi^{-1}[D]$. Then $\varphi(x), \varphi(y) \in D$, so

$$\varphi(xy - yx) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x) = 0.$$

By the injectivity of φ, it follows that $xy = yx$, so $\varphi^{-1}[D]$ is a commutative C*-subalgebra of A. So $D \mapsto \varphi^{-1}[D]$ is a well-defined map $\mathcal{C}(B) \to \mathcal{C}(A)$. It now follows from Example B.1.21 that $D \mapsto \varphi^{-1}[D]$ is indeed the upper adjoint of $\mathcal{C}(\varphi)$. By Lemma B.1.23, $\mathcal{C}(\varphi)_*$ preserves all existing infima, hence (8) holds.

(d) Assume that φ is injective. Let $\{C_i\}_{i\in I} \subseteq \mathcal{C}(A)$ with I non empty. We always have

$$\varphi\left[\bigcap_{i\in I} C_i\right] \subseteq \bigcap_{i\in I} \varphi[C_i],$$

hence let $x \in \bigcap_{i\in I} \varphi[C_i]$. Then for each $i \in I$, there is an $c_i \in C_i$ such that $x = \varphi(c_i)$. Fix some $j \in I$ and let $c = c_j$. Then for each $i \in I$, we have $\varphi(c_i) = x = \varphi(c)$. By injectivity of φ, it follows that $c_i = c$, so $c \in \bigcap_{i\in I} C_i$. Thus, $x \in \varphi\left[\bigcap_{i\in I} C_i\right]$, and we conclude that

$$\varphi\left[\bigcap_{i\in I} C_i\right] = \bigcap_{i\in I} \varphi[C_i],$$

which is exactly (9).

It follows from (c) that $\mathcal{C}(\varphi)$ has an upper adjoint $\mathcal{C}(\varphi)_*$ defined by $D \mapsto \varphi^{-1}[D]$ for each $D \in \mathcal{C}(B)$. By injectivity of φ, we find

$$\mathcal{C}(\varphi)_* \circ \mathcal{C}(\varphi)(C) = \varphi^{-1}[\varphi[C]] = C,$$

for each $C \in \mathcal{C}(A)$, hence $\mathcal{C}(\varphi)_* \circ \mathcal{C}(\varphi) = 1_{\mathcal{C}(A)}$.

In order to show that $\mathcal{C}(\varphi)$ is an order embedding, let $C_1, C_2 \in \mathcal{C}(A)$. Since $\mathcal{C}(\varphi)$ is an order morphism, $C_1 \subseteq C_2$ implies

$$\mathcal{C}(\varphi)(C_1) \subseteq \mathcal{C}(\varphi)(C_2). \tag{13}$$

Conversely, if (13) holds, it follows from $\mathcal{C}(\varphi)_* \circ \mathcal{C}(\varphi) = 1_{\mathcal{C}(A)}$ that $C_1 \subseteq C_2$. Thus, $\mathcal{C}(\varphi)$ is an order embedding.

Let $D \in \mathcal{C}(\varphi)[\mathcal{C}(A)]$. Then $D = \mathcal{C}(\varphi)(C)$ for some $C \in \mathcal{C}(A)$. By Lemma B.1.22, we find

$$\mathcal{C}(\varphi) \circ \mathcal{C}(\varphi)_*(D) = \mathcal{C}(\varphi) \circ \mathcal{C}(\varphi)_* \circ \mathcal{C}(\varphi)(C) = \mathcal{C}(\varphi)(C) = D.$$

Thus

$$\mathcal{C}(\varphi) \circ \mathcal{C}(\varphi)_*|_{\mathcal{C}(\varphi)[\mathcal{C}(A)]} = 1_{\mathcal{C}(B)}\big|_{\mathcal{C}(\varphi)[\mathcal{C}(A)]}.$$

We show that $\mathcal{C}(\varphi)[\mathcal{C}(A)]$ is a down-set. Let $D \in \downarrow\mathcal{C}(\varphi)[\mathcal{C}(A)]$. Hence there is some $C \in \mathcal{C}(A)$ such that $D \subseteq \mathcal{C}(\varphi)(C)$. Then we find

$$\mathcal{C}(\varphi)_*(D) \subseteq \mathcal{C}(\varphi)_* \circ \mathcal{C}(\varphi)(C) = C.$$

Since

$$D \subseteq \mathcal{C}(\varphi)(C) = \varphi[C] \subseteq \varphi[A],$$

we find

$$\mathcal{C}(\varphi) \circ \mathcal{C}(\varphi)_*(D) = \varphi[\varphi^{-1}[D]] = D \cap \varphi[A] = D.$$

So $D = \mathcal{C}(\varphi)(E)$ with $E = \mathcal{C}(\varphi)_*(D)$, hence $D \in \mathcal{C}(\varphi)[\mathcal{C}(A)]$. We conclude that $\downarrow\mathcal{C}(\varphi)[\mathcal{C}(A)] = \mathcal{C}(\varphi)[\mathcal{C}(A)]$.

(e) Assume that $\mathcal{C}(\varphi)$ is surjective. Let $b \in B$, and let $b_1 = \frac{b+b^*}{2}$ and $b_2 = \frac{b-b^*}{2i}$. Then $b = b_1 + ib_2$, and b_1 and b_2 are self-adjoint elements of B. It follows from Lemma C.1.22 that $C^*(b_i) \in \mathcal{C}(B)$ for each $i = 1, 2$, hence by the surjectivity of $\mathcal{C}(\varphi)$, there are $C_1, C_2 \in \mathcal{C}(A)$ such that $\mathcal{C}(\varphi)(C_i) = C^*(b_i)$. Since $\mathcal{C}(\varphi)(C_i) = \varphi[C_i]$, this means that there are $a_1 \in C_1$ and $a_2 \in C_2$ such that $\varphi(a_i) = b_i$. Let $a = a_1 + ia_2$. Then $\varphi(a) = b$, hence φ is surjective.

(f) This follows directly from the functoriality of $\mathcal{C}$ and the fact that φ has an inverse. □

Corollary 3.2.2. Let $\varphi : A \to B$ be an injective *-homomorphism. Then $\mathcal{C}(\varphi) : \mathcal{C}(A) \to \mathcal{C}(B)$ is Lawson continuous (cf. Definition B.6.1).

It turns out that the condition that φ be injective is essential to assure that $\mathcal{C}(\varphi)$ is Lawson continuous, which follows from the next counterexample by Chris Heunen, whom we thank. Let $A = \mathbb{C}^3$ and $B = \mathbb{C}^2$ and let $\varphi : A \to B$ be given by $(\lambda, \mu, \nu) \mapsto (\lambda, \nu)$. Let $C_1, C_2 \in \mathcal{C}(A)$ be given by

$$C_1 = \{(\lambda, \mu, \mu) : \lambda, \mu \in \mathbb{C}\},$$

$$C_2 = \{(\lambda, \lambda, \mu) : \lambda, \mu \in \mathbb{C}\}.$$

Then

$$\varphi[C_1 \cap C_2] = \varphi[\{(\lambda, \lambda, \lambda) : \lambda \in \mathbb{C}\}] = \{(\lambda, \lambda) : \lambda \in \mathbb{C}\} \neq B = \varphi[C_1] \cap \varphi[C_2].$$

Hence $\mathcal{C}(\varphi)$ does not preserve infima.

We have seen that if φ is injective, then $\mathcal{C}(\varphi)$ has an upper adjoint $\mathcal{C}(\varphi)_*$, which preserves all existing infima. It might be interesting to know when $\mathcal{C}(\varphi)_*$ is Scott continuous. It turns out that the condition that the domain of $\mathcal{C}(\varphi)_*$ is *meet-continuous* (cf. Definition B.6.1) is sufficient.

Proposition 3.2.3. Let A be a C*-algebra. Then $\mathcal{C}(A)$ is meet-continuous if and only if for each C*-algebra B and each injective *-homomorphism $\varphi : B \to A$, the order morphism $\mathcal{C}(\varphi)_* : \mathcal{C}(A) \to \mathcal{C}(B)$ is Scott continuous.

Proof. Assume that $\mathcal{C}(\varphi) : \mathcal{C}(B) \to \mathcal{C}(A)$ has a Scott continuous upper adjoint for each injective $\varphi : B \to A$. Let $C \in \mathcal{C}(A)$ and $\mathcal{D} \subseteq \mathcal{C}(A)$ be directed. Since $C \subseteq A$, we can consider the embedding $\iota : C \to A$. By the first statement, $\mathcal{C}(\iota)_*$ exists and is Scott continuous. Moreover, if $D \in \mathcal{C}(A)$, we have

$$\mathcal{C}(\iota)_*(D) = \iota^{-1}[D] = \{x \in C : \iota(x) \in D\} = C \cap D.$$

By Scott continuity of $\mathcal{C}(\iota)_*$, we find

$$C \cap \bigvee{}^{\uparrow} \mathcal{D} = \mathcal{C}(\iota)_* \left(\bigvee{}^{\uparrow} \mathcal{D}\right) = \bigvee{}^{\uparrow} \mathcal{C}(\iota)_*[\mathcal{D}] = \bigvee{}^{\uparrow} \{C \cap D : D \in \mathcal{D}\},$$

so $\mathcal{C}(A)$ is meet-continuous.

Assume that $\mathcal{C}(A)$ is meet-continuous and let $\varphi : B \to A$ injective. By Theorem 3.2.1, the upper adjoint $\mathcal{C}(\varphi)_*$ exists and is given by $D \mapsto f^{-1}[D]$. Let $\mathcal{D} \subseteq \mathcal{C}(B)$ be directed. Since $\mathcal{C}(A)$ is a dcpo, $\bigvee \mathcal{D}$ exists, and since $\mathcal{C}(\varphi)_*$ is an order morphism, $\mathcal{C}(\varphi)_*[\mathcal{D}]$ is clearly directed, so $\bigvee \mathcal{C}(\varphi)_*[\mathcal{D}]$ exists as well. Now,

$$\mathcal{C}(\varphi)_*(D) \subseteq \mathcal{C}(\varphi)_* \left(\bigvee \mathcal{D}\right)$$

for each $D \in \mathcal{D}$, hence

$$\begin{aligned} \bigvee \{\varphi^{-1}[D] : D \in \mathcal{D}\} &= \bigvee \{\mathcal{C}(\varphi)_*(D) : D \in \mathcal{D}\} \\ &\subseteq \mathcal{C}(\varphi)_* \left(\bigvee \mathcal{D}\right) \\ &= \varphi^{-1}\left[\overline{\bigcup \mathcal{D}}\right] \\ &= \varphi^{-1}\left[\bigvee \mathcal{D}\right]. \end{aligned}$$

Now, let $x \in \varphi^{-1}[\bigvee \mathcal{D}]$ be self adjoint, so $\varphi(x) \in \bigvee \mathcal{D}$. Since x is self adjoint, it follows that $C^*(x) \in \mathcal{C}(B)$, and by Theorem 3.2.1, we have

$$\varphi[C^*(x)] = \mathcal{C}(\varphi)[C^*(x)] = C^*(\varphi(x)).$$

Since $\varphi(x) \in \bigvee \mathcal{D}$, we have $C^*(\varphi(x)) \subseteq \bigvee \mathcal{D}$, so $\varphi[C^*(x)] \subseteq \bigvee \mathcal{D}$. Write $C = C^*(x)$. By meet-continuity of $\mathcal{C}(A)$, we find

$$\varphi[C] = \varphi[C] \cap \bigvee \mathcal{D} = \bigvee \{\varphi[C] \cap D : D \in \mathcal{D}\}.$$

Since $\varphi[B]$ is a C*-subalgebra of A by Theorem C.1.6, it follows that $\varphi[B]$ is closed in A. Since φ is injective, $\varphi : B \to \varphi[B]$ is a *-isomorphism, and hence a homeomorphism. Let $S \subseteq \varphi[B]$, then $\varphi^{-1}[\overline{S}] = \overline{\varphi^{-1}[S]}$, where $\overline{S}$ is the closure of S in A as well as the closure of S in $\varphi[B]$, since $\varphi[B]$ is closed in A. Then $\varphi[C] \subseteq \varphi[B]$ and the injectivity of φ yields

$$\begin{aligned} C &= \varphi^{-1}[\varphi[C]] = \varphi^{-1}\left[\bigvee \{\varphi[C] \cap D : D \in \mathcal{D}\}\right] \\ &= \varphi^{-1}\left[\overline{\bigcup \{\varphi[C] \cap D : D \in \mathcal{D}\}}\right] = \overline{\varphi^{-1}\left[\bigcup \{\varphi[C] \cap D : D \in \mathcal{D}\}\right]} \\ &= \overline{\bigcup \{\varphi^{-1}[\varphi[C] \cap D] : D \in \mathcal{D}\}} \subseteq \overline{\bigcup \{\varphi^{-1}[D] : D \in \mathcal{D}\}} \\ &= \bigvee \{\varphi^{-1}[D] : D \in \mathcal{D}\}. \end{aligned}$$

Since $x \in C$, it follows that $x \in \bigvee \{\varphi^{-1}[D] : D \in \mathcal{D}\}$.

Now let $x \in \varphi^{-1}[\bigvee \mathcal{D}]$ arbitrary. Then we can write $x = x_1 + ix_2$, where $x_1 = \frac{x+x^*}{2}$ and $x_2 = \frac{x-x^*}{2i}$ are self adjoint. Since $\varphi^{-1}[\bigvee \mathcal{D}]$ is a *-algebra, hence closed under the *-operation and linear combinations, x_1 and x_2 are elements of $\varphi^{-1}[\bigvee \mathcal{D}]$. By the previous analysis, we find that $x_1, x_2 \in \bigvee \{\varphi^{-1}[D] : D \in \mathcal{D}\}$, hence $x = x_1 + ix_2 \in \bigvee \{\varphi^{-1}[D] : D \in \mathcal{D}\}$. We conclude that

$$\varphi^{-1}\left[\bigvee \mathcal{D}\right] = \bigvee \{\varphi^{-1}[D] : D \in \mathcal{D}\},$$

that is

$$\mathcal{C}(\varphi)_*\left(\bigvee \mathcal{D}\right) = \bigvee \{\mathcal{C}(\varphi)_*(D) : D \in \mathcal{D}\}.$$

Thus, $\mathcal{C}(\varphi)_*$ is Scott continuous. $\square$

We will find other criteria for the meet-continuity of $\mathcal{C}(A)$ in Corollary 7.4.2.

4 C*-subalgebras of commutative C*-algebras

In this section, we describe the properties of $\mathcal{C}(A)$ in the case that A is a commutative C*-algebra. By Gelfand duality, A is *-isomorphic to $C(X)$ for some compact Hausdorff space X. It follows that we can describe properties of $\mathcal{C}(A)$ in terms of topological properties. To be more precise, we show that $\mathcal{C}(A)$ is both order isomorphic to the poset $\mathcal{D}(X)$ of so-called *u.s.c. decompositions* of X and to the poset of equivalence classes of surjective continuous maps with domain X. The proof that these order isomorphisms exist is not very difficult, but quite long.

The poset $\mathcal{D}(X)$ has been intensively studied by Firby [36, 37, 38], who succeeded in giving a complete characterization of posets that are order isomorphic to a $\mathcal{D}(X)$ for some compact Hausdorff space X [37, Theorem 3.1.9], and showed that two compact Hausdorff spaces X and Y are homeomorphic if and only if $\mathcal{D}(X)$ and $\mathcal{D}(Y)$ are order isomorphic, which is equivalent to the statement that two commutative C*-algebras A and B are *-isomorphic if and only if $\mathcal{C}(A)$ and $\mathcal{C}(B)$ are order isomorphic. Another proof of this last statement is given by Mendivil in [88, Theorem 11] using the language of ideals.

The last section of this chapter is a review of Hamhalter's Theorem, which states the most general version of the theorem of Firby, in which also the *-isomorphisms between A and B inducing the order isomorphisms between $\mathcal{C}(A)$ and $\mathcal{C}(B)$ are classified. Its implications are huge, since it allows us to reconstruct the underlying commutative C*-algebra for any element of $\mathcal{C}(A)$ for any C*-algebra A.

4.1 Criteria for commutativity

The next proposition describes order-theoretic properties of $\mathcal{C}(A)$ corresponding to A being a commutative C*-algebra.

Proposition 4.1.1. [7, Proposition 14] Let A be a C*-algebra. Then the following statements are equivalent:

(a) A is commutative;

(b) $\mathcal{C}(A)$ has a greatest element;

(c) $\mathcal{C}(A)$ is bounded;

(d) $\mathcal{C}(A)$ is a complete lattice.

Proof. Assume that A is commutative. Then A is clearly the greatest element of $\mathcal{C}(A)$, hence (b) follows. Since Theorem 3.1.3 assures that $\mathcal{C}(A)$ always has a least element, it follows that (b) and (c) are equivalent.

For (b) $\implies$ (d), assume that $\mathcal{C}(A)$ contains a greatest element, then the infimum of an empty subset of $\mathcal{C}(A)$ exists (and equals A). By Theorem 3.1.3, $\mathcal{C}(A)$ contains all non-empty infima. Thus $\mathcal{C}(A)$ contains all infima, hence Lemma B.1.13 assures that $\mathcal{C}(A)$ is a complete lattice.

Finally, assume that $\mathcal{C}(A)$ is a complete lattice. Let $a, b \in A$. Then a can be written as a linear combination of two self-adjoint elements a_1 and a_2. By Lemma C.1.22, $C^*(a_i)$ is a commutative C*-subalgebra of A for each $i \in \{1,2\}$. Let $C_a = C^*(a_1) \vee C^*(a_2)$. Then $C_a \in \mathcal{C}(A)$ and since $C^*(a_1), C^*(a_2) \subseteq C_a$, it follows that $a_1, a_2 \in C_a$. Hence $a \in C_a$. In the same way, there is a $C_b \in \mathcal{C}(A)$ such that C_b. Now, $a, b \in C_a \vee C_b$ for $C_a, C_b \subseteq C_a \vee C_b$, and since $C_a \vee C_b$ is commutative, it follows that $ab = ba$. We conclude that A is commutative. □

4.2 Ideal subalgebras

Since a proper closed ideal I of a commutative C*-algebra A does not contain the identity element of A, we have $I \notin \mathcal{C}(A)$. However, if we add the identity to I, we do obtain an element of $\mathcal{C}(A)$. Let X be the Gelfand spectrum of A. Then we note that there exists a 1-1 correspondence between closed ideals of A and closed subsets of X (c.f. Theorem C.2.2). If $A = C(X)$, let K be the closed subset of X corresponding to I. Then the C*-subalgebra generated by I and 1_A is exactly the algebra of all functions constant on K. This motivates the following definition:

Definition 4.2.1. Let X be compact Hausdorff and let $K \subseteq X$ be closed. Then

$$C_K = \{f \in C(X) : f \text{ is constant on K}\}$$

is a C*-subalgebra of $C(X)$, called the *ideal subalgebra generated by K*.

We check that C_K is a C*-subalgebra of $C(X)$ if $K \subseteq X$ is closed. Clearly $1_X \in C_K$, and if $f, g \in C_K$ and $\lambda, \mu \in \mathbb{C}$, it is also clear that $\lambda f + \mu g$ and fg are constant on K, and hence are elements of C_K. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in C_K converging to $f \in C(X)$. Let $x, y \in K$. Given $\epsilon > 0$, there is some $N \in \mathbb{N}$

such that $\|f - f_n\| < \frac{\epsilon}{2}$ for each $n \geq N$, hence

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f(y)| \\ &= |f(x) - f_n(x)| + |f_n(y) - f(y)| \\ &\leq 2\|f - f_n\| < \epsilon. \end{aligned}$$

Since ϵ can be chosen arbitrarily small, we obtain $f(x) = f(y)$, whence $f \in C_K$.

The ideal subalgebras will be of importance, since they turn out to be the 'building blocks' of $\mathcal{C}(C(X))$ in the sense that every element of $\mathcal{C}(C(X))$ can be written as an infimum of some collection of ideal subalgebras, to the effect that the proofs of almost all theorems in this thesis directly or indirectly involve ideal algebras.

Lemma 4.2.2. Let X be compact and Hausdorff, and let $K, L \subseteq X$ be closed subsets. Then:

(a) $C_K = C(X)$ if and only if K is empty or a singleton;

(b) $K \subseteq L$ implies $C_L \subseteq C_K$;

(c) $K \cap L \neq \emptyset$ implies $C_K \cap C_L = C_{K \cup L}$;

(d) If $\#K \geq 2$, then $C_L \subseteq C_K$ implies $K \subseteq L$;

(e) If $\#K \geq 2$, then $C_L = C_K$ implies $K = L$.

Proof. Let K be empty or a singleton, then clearly each $f \in C(X)$ is constant on K, so $C_K = C(X)$. Conversely, if $\#K > 1$, then there are two distinct points in $K \subseteq X$, which cannot be separated by C_K. Hence by the Stone-Weierstrass Theorem it follows that $C_K \neq C(X)$, which proves (a). Let $K \subseteq L$ and $f \in C_L$. Then f is constant on L, so certainly constant on K. It follows that (b) must hold. For (c), notice that (b) implies $C_{K \cup L} \subseteq C_K \cap C_L$. Now let $f \in C_K \cap C_L$. Let $x, y \in K \cup L$ and choose a $z \in K \cap L$. We have $x \in K$ or $x \in L$. Without loss of generality, assume $x \in K$. Then $f(x) = f(z)$, since f constant on K and $z \in K$. In the same way we find $f(y) = f(z)$, so $f(x) = f(y)$, whence f is constant on $K \cup L$. We conclude that $f \in C_{K \cup L}$.

For (d), notice that $C_L \subseteq C_K$ translates to the statement that each $f \in C(X)$ must be constant on K if f is constant on L. Assume that $K \nsubseteq L$. Then there is a point $x \in K$ such that $x \notin L$. Since K has at least two points, there is a $y \in K$ such that $x \neq y$. Now, $\{x\}$ and $L \cup \{y\}$ are disjoint closed sets, so by Urysohn's

Lemma, we can find $f \in C(X)$ such that $f[\{x\}] = 0$ and $f[L \cup \{y\}] = 1$. But this means that f is constant on L, but not on K contradicting $C_L \subseteq C_K$. So we must have $K \subseteq L$. Let $C_L = C_K$, then from $C_L \subseteq C_K$ we find that $K \subseteq L$, so L must also have cardinality strictly greater than one. Hence $C_K \subseteq C_L$ implies $L \subseteq K$. We conclude that $K = L$. □

Notice that if K and L are both one-point sets, then $C_K = C_L = C(X)$, whilst it is still possible that $K \neq L$. So the condition for (d) and (e) to the effect that K has cardinality of at least two is necessary.

Definition 4.2.3. Let X be a compact Hausdorff space. We denote the set of closed subsets of X containing at least two points by $\mathcal{F}(X)$, which we order by inclusion.

Proposition 4.2.4. The map $\mathcal{F}(X)^{\mathrm{op}} \to \mathcal{C}(C(X))$ given by $K \mapsto C_K$ is an order embedding.

Proof. This follows directly from (b) and (d) of Lemma 4.2.2. □

If B is a C*-subalgebra of $C(X)$ with Gelfand spectrum Y, it is convenient if we have an expression for ideal subalgebras of B in terms of ideal subalgebras of $C(X)$ and *vice versa*.

Proposition 4.2.5. Let X, Y be compact Hausdorff spaces, let $q : X \to Y$ be a surjective continuous map. Let $C_q : C(Y) \to C(X)$ be the map $f \mapsto f \circ q$, and let $B = C_q[C(Y)]$. Then $\mathcal{C}(C_q) : \mathcal{C}(C(Y)) \to \mathcal{C}(B)$ is an order isomorphism such that

$$\mathcal{C}(C_q)^{-1}\left(\bigcap_{i \in I} C_{K_i} \cap B\right) = \bigcap_{i \in I} C_{q[K_i]}$$

for each collection $\{K_i\}_{i \in I}$ of closed subsets of X, and

$$\mathcal{C}(C_q)\left(\bigcap_{j \in J} C_{L_j}\right) = \bigcap_{j \in J} C_{q^{-1}[L_j]} \cap B$$

for each collection $\{L_j\}_{j \in J}$ of closed subsets of Y.

Proof. Since $q : X \to Y$ is surjective, Proposition C.2.5 assures that $C_q : C(Y) \to C(X)$ is an injective *-homomorphism. By Theorem 3.2.1, we find that $\mathcal{C}(C_q) : \mathcal{C}(C(Y)) \to \mathcal{C}(C(X))$ is an order embedding, and by definition of B, its image equals $\downarrow B = \mathcal{C}(B)$. Moreover, by the same theorem, $\mathcal{C}(C_q)$ has an upper adjoint $\mathcal{C}(C_q)_* : \mathcal{C}(C(X)) \to \mathcal{C}(C(Y))$, whose restriction $\mathcal{C}(B) \to \mathcal{C}(C(Y))$ is the inverse of $\mathcal{C}(C_q) : \mathcal{C}(C(Y)) \to \mathcal{C}(B)$. If $K \subseteq X$ is closed, then by definition of $\mathcal{C}(C_q)_*$, we find

$$\begin{aligned}\mathcal{C}(C_q)_*(C_K) = C_q^{-1}[C_K] &= \{f \in C(Y) : C_q(f) \in C_K\} \\ &= \{f \in C(Y) : f \circ q \in C_K\} = C_{q[K]},\end{aligned}$$

for $f \circ q \in C_K$ if and only if $f \in C_{q[K]}$. Notice that $q[K]$ is closed, since Lemma A.1.1 assures that q is a closed map. By definition of B, we have $\mathcal{C}(C_q)_*(B) = C(Y)$. Let $\{K_i\}_{i\in I}$ be a collection of closed subsets of X. Since $\mathcal{C}(C_q)_*$ preserves intersections and $\bigcap_{i\in I} C_{K_i} \cap B \in \mathcal{C}(B)$, we find

$$\begin{aligned}\mathcal{C}(C_q)_*\left(\bigcap_{i\in I} C_{K_i} \cap B\right) &= \mathcal{C}(C_q)_*(B) \cap \bigcap_{i\in I} \mathcal{C}(C_q)_*(C_{K_i}) \\ &= C(Y) \cap \bigcap_{i\in I} C_{q[K_i]} = \bigcap_{i\in I} C_{q[K_i]}.\end{aligned}$$

Now let $\{L_j\}_{j\in J}$ be a collection of closed subsets of Y. Then $\{q^{-1}[L_j]\}_{j\in J}$ is a collection of closed subsets of X, hence the previous result yields

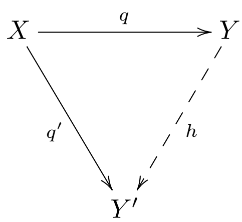
$$\mathcal{C}(C_q)^{-1}\left(\bigcap_{j\in J} C_{q^{-1}[L_j]} \cap B\right) = \bigcap_{j\in J} C_{q[q^{-1}[L_j]]} = \bigcap_{j\in J} C_{L_j},$$

where we used the surjectivity of q in the last equality. □

4.3 Quotient spaces

In this section we consider a compact Hausdorff space X and look at the continuous surjections with domain X and compact Hausdorff codomain. We can define an equivalence class on the set of these surjections. Moreover, the set of equivalence classes can be equipped with an order such that the resulting poset is order isomorphic to $\mathcal{C}(C(X))$.

Definition 4.3.1. Let X be a compact Hausdorff space. We can define an equivalence relation on the set of all continuous surjections with domain X and compact Hausdorff codomain as follows. If $q : X \to Y$ and $q' : X \to Y'$ are continuous surjections onto compact Hausdorff spaces Y and Y', respectively, we say that q is equivalent to q', written $q \sim q'$, if there is a homeomorphism $h : Y \to Y'$ such that $q' = h \circ q$, i.e., if the following diagram commutes:



The equivalence class of q is denoted by $[q]$. The collection of all such equivalence classes is denoted by $\mathcal{Q}(X)$.

The proof of lemma is easy, hence we omit it.

Lemma 4.3.2. Let X be a compact Hausdorff space. For each $[q : X \to Y]$ and $[q' : X \to Y']$ in $\mathcal{Q}(X)$, define $[q'] \leq [q]$ if and only if there is a continuous surjection $h : Y \to Y'$ such that $q' = h \circ q$. Then $\leq$ is well-defined and is a partial order on $\mathcal{Q}(X)$.

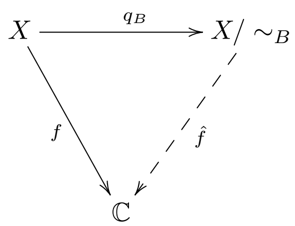
It follows from Lemma A.1.1 that if X and Y are compact Hausdorff spaces, and $q : X \to Y$ is continuous and surjective, then q is closed, so a quotient map. Conversely, every quotient map $X \to Y$ is automatically continuous and surjective. Thus $\mathcal{Q}(X)$ can be seen as the set of quotients of X. Moreover, it turns out that $\mathcal{C}(C(X))$ and $\mathcal{Q}(X)$ are order isomorphic. Before we prove this, we state the following definition.

Definition 4.3.3. Let B be a (not necessarily unital) C*-subalgebra of $C(X)$. Then we define the equivalence relation $\sim_B$ on X by $x \sim_B y$ if and only if $f(x) = f(y)$ for each $f \in B$. We denote the equivalence class of x under this equivalence relation by $[x]_B$. The natural map $X \to X/\sim_B$, $x \mapsto [x]_B$ will be denoted by q_B, where we suppress the index B if no confusion is possible.

Lemma 4.3.4. Let B a C*-subalgebra of $C(X)$. Then $X/\sim_B$ is compact Hausdorff in the quotient topology induced by $q_B : X \to X/\sim_B$.

Proof. By definition of the quotient topology, $q_B : X \to X/\sim_B$ is continuous. Then $X/\sim_B$ is compact since it is the continuous image of a compact space.

Let $[x]_B, [y]_B$ be different points in $X/\sim_B$. Then there is an $f \in B$ such that $f(x) \neq f(y)$. Let $\hat{f} : X/\sim_B \to \mathbb{C}$ be defined by $\hat{f}([z]_B) = f(z)$ for each $z \in X$. Notice that this is well-defined, since $[z]_B = [z']_B$ implies $f(z) = f(z')$ for each $z, z' \in X$. By definition of $\hat{f}$, the following diagram commutes:



(14)

Since $f(x) \neq f(y)$ and $\mathbb{C}$ is Hausdorff, there are open and disjoint $U, V \subseteq \mathbb{C}$ such that $f(x) \in U$ and $f(y) \in V$. Since U and V are disjoint, $(\hat{f})^{-1}[U]$ and $(\hat{f})^{-1}[V]$ are disjoint. Now,

$$q_B^{-1} \circ (\hat{f})^{-1}[U] = (\hat{f} \circ q_B)^{-1}[U] = f^{-1}[U],$$

which is open, for f is continuous. By definition of the quotient topology on $X/\sim_B$, it follows that $(\hat{f})^{-1}[U]$ is open, and in a similar way it follows that $(\hat{f})^{-1}[V]$ is open as well. We conclude that $X/\sim_B$ is Hausdorff. □

Lemma 4.3.5. Let B a C*-subalgebra of $C(X)$ and $f \in B$. Then there is a unique $\hat{f} \in C(X/\sim_B)$ such that $f = \hat{f} \circ q_B$, i.e., such that Diagram 14 commutes.

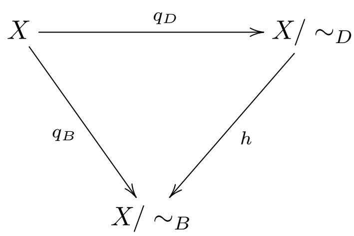
Proof. Define $\hat{f} : X/\sim_B \to \mathbb{C}$ by $\hat{f}([x]_B) = f(x)$. This is well-defined, for f is constant on $[x]_B$ by definition of $\sim_B$. Since Lemma 4.3.4 assures that $X/\sim_B$ is compact Hausdorff, it follows from Lemma A.1.1 that $\hat{f}$ is continuous and the unique function such that the Diagram 14 commutes. □

The next proposition is a reformulation of [108, Proposition 5.1.3].

Proposition 4.3.6. The assignment $\mathcal{C}(C(X)) \to \mathcal{Q}(X)$, $B \mapsto [q_B : X \to X/\sim_B]$ is an order isomorphism with inverse $[q : X \to Y] \mapsto C_q[C(Y)]$.

Proof. By Lemma 4.3.4, it follows that $B \mapsto [q_B]$ is a well-defined map from $\mathcal{C}(C(X))$ to $\mathcal{Q}(X)$. Let $B, D \in \mathcal{C}(C(X))$ such that $B \subseteq D$. We define the map $h : X/\sim_D \to X/\sim_B$ by $h([x]_D) = [x]_B$. This is well defined, since if

$[x]_D = [y]_D$, then $f(x) = f(y)$ for each $f \in D$, so certainly $f(x) = f(y)$ for each $f \in B$, whence $[x]_B = [y]_B$. By definition of h the following diagram commutes:



Since q_B and q_D are continuous surjections onto compact Hausdorff spaces, is follows from Lemma A.1.1 that h is continuous and surjective. We conclude that $[q_B] \leq [q_D]$, so $B \mapsto [q_B]$ is indeed an order morphism $\mathcal{C}(C(X)) \to \mathcal{Q}(X)$. Let $[q : X \to Y] \in \mathcal{Q}(X)$. It follows from Proposition C.2.5 that

$$C_q : C(Y) \to C(X)$$

is an injective *-homomorphism, so $C_q[C(Y)]$ is a C*-subalgebra of $C(X)$. Let $q' : X \to Y'$ be a continuous surjection onto a compact Hausdorff space Y'. Assume that $[q] = [q']$, then there is some homeomorphism $h : Y \to Y'$ such that $q' = h \circ q$. Moreover, Proposition C.2.5 assures that $C_h : C(Y') \to C(Y)$ is a *-isomorphism. By the same Proposition, we find

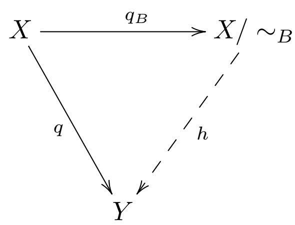
$$C_{q'}[C(Y')] = C_{h \circ q}[C(Y')] = C_q \circ C_h[C(Y')] = C_q[C(Y)].$$

We conclude that the assignment $[q : X \to Y] \mapsto C_q[C(Y)]$ is a well-defined map $\mathcal{Q}(X) \to \mathcal{C}(C(X))$. Now assume that $[q'] \leq [q]$. Then there is a continuous surjection $k : Y \to Y'$ such that $q' = k \circ q$, and Proposition C.2.5 assures that $C_k : C(Y') \to C(Y)$ is an injective *-homomorphism, hence $C_k[C(Y')] \subseteq C(Y)$. It follows by the same lemma that

$$C_{q'}[C(Y')] = C_{k \circ q}[C(Y')] = C_q[C_k[C(Y')]] \subseteq C_q[C(Y)],$$

hence the map $[q : X \to Y] \mapsto C_q[C(Y)]$ is an order morphism $\mathcal{Q}(X) \to \mathcal{C}(C(X))$.

We now prove that the maps $\mathcal{C}(C(X)) \to \mathcal{Q}(X)$ and $\mathcal{Q}(X) \to \mathcal{C}(C(X))$ are each other's inverses. Let $[q : X \to Y] \in \mathcal{Q}(X)$ and let $B = C_q(C(Y))$. We have to show that there is a homeomorphism $h : X/\sim_B \to Y$ such that $q = h \circ q_B$, or, equivalently, such that the following diagram commutes:



Hence define h by $h([x]_B) = q(x)$. We have to show that h is well defined. Let $[x]_B = [y]_B$. This means that $f(x) = f(y)$ for each $f \in B$. Since each $f \in B$ equals $g \circ q$ for some $g \in C(Y)$, we find that $g \circ q(x) = g \circ q(y)$ for each $g \in C(Y)$. By Lemma C.1.10, $C(Y)$ separates all points of Y, so we must have $q(x) = q(y)$. Thus h is well defined. Clearly h is surjective. For injectivity, let $x, y \in X$ such that $[x]_B \neq [y]_B$. Then there is some $f \in B$ such that $f(x) \neq f(y)$, whence $g \circ q(x) \neq g \circ q(y)$ for some $g \in C(Y)$. Thus $q(x) \neq q(y)$. Since q and q_B are continuous surjections and $X/\sim_B$ is compact Hausdorff, it follows from Lemma A.1.1 that h is a surjective continuous and closed map. Since h is also injective, it follows that h must be a homeomorphism. Now let $B \in \mathcal{C}(C(X))$. We have to proof that $B = C_{q_B}[C(X/\sim_B)]$. Firstly, since q_B is surjective, we find that $C_{q_B} : C(X/\sim_B) \to C(X)$ is an injective *-homomorphism. Let $f \in B$. It follows from Lemma 4.3.5 that there is a unique $\hat{f} \in C(X/\sim_B)$ such that $\hat{f} \circ q_B = f$. That is, B lies in the image of C_{q_B} and $C_{q_B}^{-1} : B \to C(X/\sim_B)$, $f \mapsto \hat{f}$ is an injective *-homomorphism. Let $[x]_B, [y]_B \in X/\sim_B$ such that $[x]_B \neq [y]_B$. Then there is an $f \in B$ such that $f(x) \neq f(y)$. Hence

$$C_{q_B}^{-1}(f)\big([x]_B\big) = \hat{f}([x]_B) = \hat{f} \circ q_B(x) = f(x) \neq f(y) = C_{q_B}^{-1}(f)\big([y]_B\big).$$

It follows that $C_{q_B}^{-1}[B]$ separates point of $X/\sim_B$, hence $C_{q_B}^{-1}[B] = C(X/\sim_B)$ by the Stone-Weierstrass Theorem. We conclude that $C_{q_B}[C(X/\sim_B)] = B$. $\square$

4.4 U.s.c. decompositions

If X is a compact Hausdorff space and Y is a quotient of X, then Y is necessarily compact, but not always Hausdorff. However, it turns out that there is a way of describing exactly the Hausdorff quotients by considering the partitions of X induced by quotient maps, which is desirable, since we found in

Proposition 4.3.6 that compact Hausdorff quotients of X correspond with C*-subalgebras of $C(X)$. It turns out that certain conditions on such a partition exactly correspond to Y being Hausdorff. It follows that we obtain a description of C*-subalgebras of $C(X)$ in terms of certain partitions of X, a fact which was first mentioned by Firby in [36].

Definition 4.4.1. Let X be a topological space and $\mathcal{P}$ a partition of X. Then an open set U of X is called *saturated* if U is the union of elements of $\mathcal{P}$. A partition $\mathcal{P}$ of X is called an *upper semicontinuous* (shortly *u.s.c.*) *decomposition* of X if all members of $\mathcal{P}$ are closed subsets of X and if for each $K \in \mathcal{P}$ and open $U \subseteq X$ such that $K \subseteq U$, there is a saturated open set V such that $K \subseteq V \subseteq U$. The set of u.s.c. decompositions of X is denoted by $\mathcal{D}(X)$ and is ordered by $\mathcal{P}_1 \leq \mathcal{P}_2$ if and only if $\mathcal{P}_1$ is a *refinement* of $\mathcal{P}_2$, i.e., for each $K_1 \in \mathcal{P}_1$, there is some $K_2 \in \mathcal{P}_2$ such that $K_1 \subseteq K_2$.

The set of partitions on a set S is usually ordered by refinement [42, Chapter IV.4], hence $\mathcal{D}(X)$ is a subposet of the lattice of partitions of X.

Lemma 4.4.2. Let $\mathcal{P}$ be an u.s.c. decomposition of a topological space X. Then an open set $V \subseteq X$ is saturated if and only if for each $K \in \mathcal{P}$, $K \cap V \neq \emptyset$ implies $K \subseteq V$.

Proof. Assume that U is saturated and let $K \in \mathcal{P}$ such that $K \cap V \neq \emptyset$. Since V is the union of elements of $\mathcal{P}$, all elements of $\mathcal{P}$ are disjoint, and $K \cap V \neq \emptyset$, we must have $K \subseteq V$. Conversely, assume that $K \cap V \neq \emptyset$ implies $K \subseteq V$ for each $K \in \mathcal{P}$. For each $x \in V$, we can find an $K_x \in \mathcal{P}$, and since $x \in K_x \cap V$, we have $K_x \subseteq V$. Hence $V = \bigcup_{x \in V} K_x$. □

Example 4.4.3. Let X be finite. Equipped with the discrete topology, X is compact Hausdorff, and every partition of X is an u.s.c. decomposition.

Example 4.4.4. Let X be a compact Hausdorff space and let $K_1, \ldots, K_n$ be non-empty closed disjoint subsets of X. Then

$$\mathcal{P} = \{K_1, \ldots, K_n\} \cup \left\{ \{x\} : x \notin \bigcup_{i=1}^{n} K_i \right\}$$

is a decomposition into closed subsets of X. Let $K \in \mathcal{P}$ and U open such that $K \subseteq U$. If $K = K_i$ for some $j \in \{1, \ldots, n\}$, let $V = U \cap X \setminus (\bigcup_{i \neq j} K_i)$.

If $K = \{x\}$, let $V = U \cap X \setminus (\bigcup_{i=1}^{n} K_i)$. In both cases, V is saturated and $K \subseteq V \subseteq U$. So $\mathcal{P}$ is an u.s.c. decomposition of X.

Lemma 4.4.5. Let $\mathcal{P}$ be an u.s.c. decomposition of a compact Hausdorff space X. If we equip $\mathcal{P}$ with the quotient topology with respect to the natural map $q_{\mathcal{P}} : X \to \mathcal{P}$, then $\mathcal{P}$ is compact Hausdorff.

Proof. Since X is compact and $q_{\mathcal{P}}$ is continuous by definition, we find that $\mathcal{P}$ is compact. Let $K_1, K_2 \in \mathcal{P}$ such that $K_1 \neq K_2$. Hence K_1 and K_2 are disjoint closed sets of X, and since the latter is compact Hausdorff, there are disjoint open sets U_1 and U_2 such that $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$. Since $\mathcal{P}$ is an u.s.c. decomposition of X and $K_1, K_2 \in \mathcal{P}$, we can find saturated sets $V_1, V_2 \subseteq X$ such that $K_i \subseteq V_i \subseteq U_i$ for $i = 1, 2$. We necessarily have $V_1 \cap V_2 = \emptyset$. We always have $V_i \subseteq q_{\mathcal{P}}^{-1}\big[q_{\mathcal{P}}[V_i]\big]$. Let $v \in V_i$. Then

$$v \in q_{\mathcal{P}}^{-1}\big[q_{\mathcal{P}}[\{v\}]\big] \cap V_i,$$

and since V_i is saturated and $q_{\mathcal{P}}^{-1}\big[q_{\mathcal{P}}[\{v\}]\big] \in \mathcal{P}$, it follows from Lemma 4.4.2 that $q_{\mathcal{P}}^{-1}[q_{\mathcal{P}}[\{v\}]] \subseteq V_i$. We conclude that

$$V_i = \bigcup_{v \in V_i} q_{\mathcal{P}}^{-1}\big[q_{\mathcal{P}}[\{v\}]\big] = q_{\mathcal{P}}^{-1}\big[q_{\mathcal{P}}[V_i]\big],$$

so $q_{\mathcal{P}}[V_i]$ is open in $\mathcal{P}$ by definition of the quotient topology. Furthermore, since $K_i \subseteq V_i$, we have $\{y_i\} = q_{\mathcal{P}}[K_i] \subseteq q_{\mathcal{P}}[V_i]$. Finally, $q_{\mathcal{P}}[V_1] \cap q_{\mathcal{P}}[V_2] = \emptyset$, since otherwise we have

$$V_1 \cap V_2 = q_{\mathcal{P}}^{-1}\big[q_{\mathcal{P}}[V_1]\big] \cap q_{\mathcal{P}}^{-1}\big[q_{\mathcal{P}}[V_2]\big] = q_{\mathcal{P}}^{-1}\big[q_{\mathcal{P}}[V_1] \cap q_{\mathcal{P}}[V_2]\big] \neq \emptyset,$$

a contradiction. Thus $\mathcal{P}$ is Hausdorff. □

Lemma 4.4.6. Let X and Y be compact Hausdorff spaces and $f : X \to Y$ continuous. Then $\mathcal{D}_f : \mathcal{D}(Y) \to \mathcal{D}(X)$ defined by

$$\mathcal{P} \mapsto \{f^{-1}[K] : K \in \mathcal{P}\}$$

is a well-defined order morphism. Moreover, if $g : Y \to Z$ is another continuous function between compact Hausdorff spaces, then

$$\mathcal{D}_{g \circ f} = \mathcal{D}_f \circ \mathcal{D}_g,$$

so that $\mathcal{D} : \mathbf{CptHd}^{\mathrm{op}} \to \mathbf{Poset}$ is a functor.

Proof. Let $\mathcal{P} \in \mathcal{D}(Y)$. We have to show that $\mathcal{D}_f(\mathcal{P})$ is an u.s.c. decomposition of X. Let $K \in \mathcal{P}$, then K is closed, hence $f^{-1}[K] \subseteq X$ is closed. If $x \in X$, then $x \in f^{-1}[K]$ for some $K \in \mathcal{P}$ for $\mathcal{P}$ is a partition of Y. Moreover, if $K, K' \in \mathcal{P}$ such that $K \neq K'$, then $K \cap K' = \emptyset$, so

$$f^{-1}[K] \cap f^{-1}[K'] = f^{-1}[K \cap K'] = \emptyset,$$

whence $\mathcal{D}_f(\mathcal{P})$ is a partition of X consisting of closed subsets. Let $f^{-1}[K]$ be an element of $\mathcal{D}_f(\mathcal{P})$ and $U \subseteq X$ open such that $f^{-1}[K] \subseteq U$. If $x \notin U$, then $f(x) \notin K$, so $f[X \setminus U] \subseteq Y \setminus K$. We find that $K \subseteq Y \setminus f[X \setminus U]$, which is open for f is a closed function by Lemma A.1.1. Since $K \in \mathcal{P}$, there is a saturated open set $V \subseteq Y$ such that

$$K \subseteq V \subseteq Y \setminus f[X \setminus U].$$

Clearly $f^{-1}[K] \subseteq f^{-1}[V]$, which is open for f is continuous. Let x be an element of $f^{-1}[V]$. Then $f(x) \in Y \setminus f[X \setminus U]$, so $x \notin X \setminus U$, and we conclude that $f^{-1}[V] \subseteq U$. Since V is saturated, there is a subset $\mathcal{S} \subseteq \mathcal{P}$ such that $V = \bigcup \mathcal{S}$. But then

$$f^{-1}[V] = f^{-1}\left[\bigcup \mathcal{S}\right] = \bigcup_{K' \in \mathcal{S}} f^{-1}[K'],$$

to the effect that V is the union of elements of $\mathcal{D}_f(\mathcal{P})$. Hence V is saturated, proving that $\mathcal{D}_f(\mathcal{P})$ is an u.s.c. decomposition of X. Let $\mathcal{P}$ and $\mathcal{P}'$ in $\mathcal{D}(Y)$ such that $\mathcal{P} \leq \mathcal{P}'$. Let $f^{-1}[K] \in \mathcal{D}_f(\mathcal{P})$, then $\mathcal{P} \leq \mathcal{P}'$ implies the existence of some $K' \in \mathcal{P}'$ such that $K \subseteq K'$. Hence $f^{-1}[K] \subseteq f^{-1}[K']$, so $\mathcal{D}_f(\mathcal{P}) \leq \mathcal{D}_f(\mathcal{P}')$. Thus $\mathcal{D}_f$ is an order morphism. Finally, let Z be compact Hausdorff and $g : Y \to Z$ continuous. Let $\mathcal{P} \in \mathcal{D}(Z)$. Then

$$\begin{aligned} \mathcal{D}_f \circ \mathcal{D}_g(\mathcal{P}) &= \left\{f^{-1}[L] : L \in \mathcal{D}_g(\mathcal{P})\right\} \\ &= \left\{f^{-1}\left[g^{-1}[K]\right] : K \in \mathcal{P}\right\} \\ &= \left\{(g \circ f)^{-1}[K] : K \in \mathcal{P}\right\} \\ &= \mathcal{D}_{g \circ f}(\mathcal{P}), \end{aligned}$$

hence $\mathcal{D}_{g \circ f} = \mathcal{D}_f \circ \mathcal{D}_g$. □

Proposition 4.4.7. For each compact Hausdorff space X, define the map $\alpha_X : \mathcal{C}(C(X)) \to \mathcal{D}(X)^{\mathrm{op}}$ by $B \mapsto \{[x]_B : x \in X\}$. Then α_X is an order isomorphism with inverse $\mathcal{P} \mapsto \bigcap_{K \in \mathcal{P}} C_K$. Moreover, $\alpha : \mathcal{C}(C(\cdot)) \to \mathcal{D}(\cdot)^{\mathrm{op}}$ is a natural isomorphism, i.e., the following diagram commutes if X and X' are compact Hausdorff spaces and $f : X \to Y$ is continuous:

$$\begin{array}{ccc} \mathcal{C}(C(Y)) & \xrightarrow{\alpha_Y} & \mathcal{D}(Y)^{\mathrm{op}} \\ {\scriptstyle \mathcal{C}(C_f)}\downarrow & & \downarrow{\scriptstyle \mathcal{D}_f} \\ \mathcal{C}(C(X)) & \xrightarrow[\alpha_X]{} & \mathcal{D}(X)^{\mathrm{op}} \end{array}$$

Proof. Let X be a compact Hausdorff space and let B be a C*-subalgebra of $C(X)$. By Lemma 4.3.4, $q_B : X \to X/\sim_B$ is a continuous surjection onto a compact Hausdorff space. Since $\big\{\{[x]_B\} : x \in X\big\}$ is an u.s.c. decomposition of $X/\sim_B$ and q_B is continuous, Lemma 4.4.6 assures that

$$\alpha_X(B) = \{[x]_B : x \in X\} = \{q_B^{-1}[\{[x]_B\}] : [x]_B \in X/\sim_B\}$$

is a u.s.c. decomposition of X. Let D be another C*-subalgebra of $C(X)$ such that $B \subseteq D$. Then $f(x) = f(y)$ for each $f \in D$ implies $f(x) = f(y)$ for each $f \in B$, hence $[x]_D \subseteq [x]_B$. It follows that

$$\{[x]_D : x \in X\} \leq \{[x]_B : x \in X\},$$

so $\alpha_X : \mathcal{C}(C(X)) \to \mathcal{D}(X)^{\mathrm{op}}$ is indeed an order morphism. Denote the map $\mathcal{D}(X)^{\mathrm{op}} \to \mathcal{C}(C(X))$, $\mathcal{P} \mapsto \bigcap_{K \in \mathcal{P}} C_K$ by η_X. Clearly η_X is a well-defined map. We show that it is an order morphism, and that it is the inverse of α_X. If $\mathcal{P}, \mathcal{P}' \in \mathcal{D}(X)$ such that $\mathcal{P} \leq \mathcal{P}'$, then for each $K \in \mathcal{P}$, there is some $K' \in \mathcal{P}'$ with $K \subseteq K'$. Let $f \in \bigcap_{K' \in \mathcal{P}'} C_{K'}$. Thus f is constant on each $K' \in \mathcal{P}'$, so certainly constant on each $K \in \mathcal{P}$. We conclude that

$$\bigcap_{K' \in \mathcal{P}'} C_{K'} \subseteq \bigcap_{K \in \mathcal{P}} C_K,$$

hence η_X is indeed an order morphism $\mathcal{D}(X)^{\mathrm{op}} \to \mathcal{C}(C(X))$.

Let $B \in \mathcal{C}(C(X))$. We have to show that $B = \eta_X \circ \alpha_X(B)$, that is, $B = \bigcap_{x\in X} C_{[x]_B}$. If $f \in B$, then f is constant on $[x]_B$ for each $x \in X$, hence $f \in \bigcap_{x\in X} C_{[x]_B}$. Now, let $f \in \bigcap_{x\in X} C_{[x]_B}$. Define $\hat{f} : X/\sim_B \to \mathbb{C}$ by $\hat{f}([x]_B) = f(x)$. Since f is constant on $[x]_B$ for each $x \in X$, $\hat{f}$ is well-defined, and by definition, Diagram 14 commutes. Lemma A.1.1 assures now that $\hat{f}$ is continuous, so $\hat{f} \in C(X/\sim_B)$. By Proposition 4.3.6, we have $B = C_{q_B}[C(X/\sim_B)]$ and since $f = \hat{f} \circ q_B$, it follows that $f \in B$. Hence $B = \bigcap_{x\in X} C_{[x]_B}$. Now, let $\mathcal{P} \in \mathcal{D}(X)$. We have to show that $\mathcal{P} = \alpha_X \circ \eta_X(\mathcal{P})$. That is, if $B = \bigcap_{K\in\mathcal{P}} C_K$, then we must have $\mathcal{P} = \{[x]_B : x \in X\}$. Let $K \in \mathcal{P}$ and $x \in K$. If $y \in K$, then $f(x) = f(y)$ for each $f \in B$, then $y \in [x]_B$. Thus $K \subseteq [x]_B$. Now assume that $y \notin K$. Hence $y \in L$ for some $L \in \mathcal{P}$, and $K \cap L = \emptyset$. By Lemma 4.4.5, $\mathcal{P}$ is a compact Hausdorff space and the quotient map $q_{\mathcal{P}} : X \to \mathcal{P}$ is continuous. Since $C(\mathcal{P})$ separates points of $\mathcal{P}$, we find that there is a continuous function $h : \mathcal{P} \to \mathbb{C}$ such that $h(K) \neq h(L)$. It follows that $f = h \circ q_{\mathcal{P}}$ is an element of B, and $f(x) \neq f(y)$. We conclude that $K = [x]_B$, hence we indeed have $\mathcal{P} = \{[x]_B : x \in X\}$, so $\eta_X = \alpha_X^{-1}$.

Finally, we show that α is a natural isomorphism. Since α_X is an order isomorphism for each compact Hausdorff space X, it is enough to show that α is a natural transformation. So if $f : X \to Y$ is a continuous function between compact Hausdorff spaces, we have to show that

$$\alpha_X \circ \mathcal{C}(C_f) = \mathcal{D}_f \circ \alpha_Y \tag{15}$$

Let $B \in \mathcal{C}(C(Y))$. Then

$$\mathcal{D}_f \circ \alpha_Y(B) = \mathcal{D}_f(\{[y]_B : y \in Y\}) = \big\{f^{-1}\big[[y]_B\big] : y \in Y\big\}.$$

Assume that $f^{-1}\big[[y]_B\big] \neq \emptyset$. Then there is some $x \in X$ such that $f(x) \in [y]_B$, that is $[y]_B = [f(x)]_B$. Hence

$$\mathcal{D}_f \circ \alpha_Y(B) = \big\{f^{-1}\big[[f(x)]_B\big] : x \in X\big\}.$$

Now,

$$\alpha_X \circ \mathcal{C}(C_f)(B) = \alpha_X(C_f[B]) = \{[x]_{C_f[B]} : x \in X\}.$$

Then $x' \in [x]_{C_f[B]}$ if and only if $k(x) = k(x')$ for each $k \in C_f[B]$ if and only if $g \circ f(x) = g \circ f(x')$ for each $g \in B$ if and only if $f(x') \in [f(x)]_B$ if and only if

$x' \in f^{-1}\big[[f(x)]_B\big]$. Hence we find that

$$\alpha_X \circ \mathcal{C}(C_f)(B) = \mathcal{D}_f \circ \alpha_Y(B),$$

whence α is indeed a natural transformation. Since α_X is an order isomorphism for each compact Hausdorff space, it follows that α is a natural isomorphism. □

Corollary 4.4.8. Let $f : X \to Y$ be a continuous function between compact Hausdorff spaces, then

$$C_f\left[\bigcap_{K\in\mathcal{P}} C_K\right] = \bigcap_{K\in\mathcal{P}} C_{f^{-1}[K]}$$

for each $\mathcal{P} \in \mathcal{D}(Y)$.

Proof. This follows from

$$\mathcal{C}(C_f) \circ \alpha_Y^{-1} = \alpha_X^{-1} \circ (\alpha_X \circ \mathcal{C}(C_f)) \circ \alpha_Y^{-1} = \alpha_X^{-1} \circ (\mathcal{D}_f \circ \alpha_Y) \circ \alpha_Y^{-1} = \alpha_X^{-1} \circ \mathcal{D}_f,$$

where (15) is used in the second equality. □

Corollary 4.4.9. Let $f : X \to Y$ be a continuous function between compact Hausdorff spaces, and B be a C*-subalgebra of $C(Y)$. Then

$$C_f\,[B] = \bigcap_{y\in Y} C_{f^{-1}\big[[y]_B\big]},$$

where $[y]_B$ is interpreted as a subset of Y (rather than as a point in $Y/\sim_B$).

Proof. This follows from the previous corollary, since Proposition 4.4.7 assures that $\{[y]_B : y \in Y\}$ is an u.s.c. decomposition of Y. □

Corollary 4.4.10. [47, Proposition 2.2] Let X be a compact Hausdorff space and B a C*-subalgebra of $C(X)$. Then

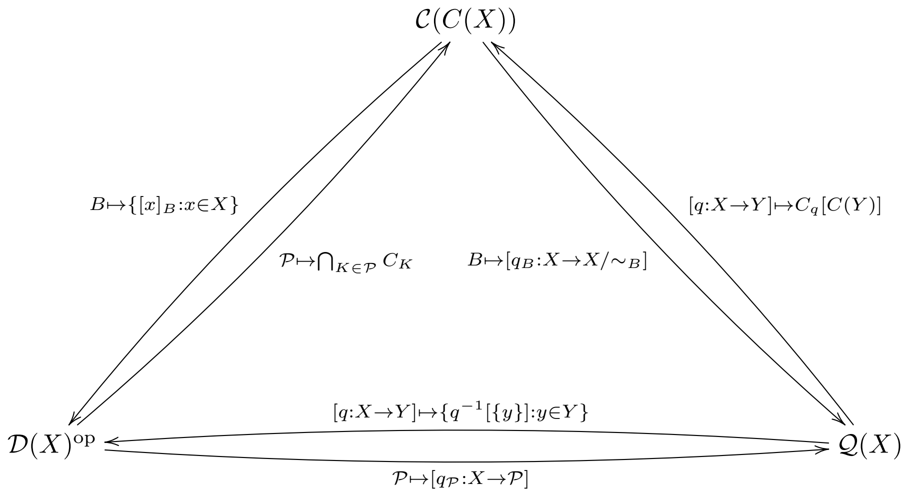
$$B = \bigcap_{x\in X} C_{[x]_B}.$$

Proof. Let $X = Y$ and let $f : X \to Y$ be the identity. Then the statement follows from the previous corollary. □

Corollary 4.4.11. Let X be a compact Hausdorff space, and let B and C be C*-subalgebras of $C(X)$ such that $B \subseteq C$. If for each $x, y \in X$ such that $[x]_C \neq [y]_C$ there is an $f \in B$ such that $f(x) \neq f(y)$, then $B = C$.

Proof. Since $B \subseteq C$, we have $[x]_C \subseteq [x]_B$ for $f(x) = f(y)$ for each $f \in C$ implies $f(x) = f(y)$ for each $f \in B$. Now, assume that $[x]_C \neq [y]_C$. By assumption, there is some $f \in B$ such that $f(x) \neq f(y)$, so $[x]_B \neq [y]_B$. By contraposition, it follows that $[x]_B = [y]_B$ implies $[x]_C = [y]_C$. We conclude that $[x]_C = [x]_B$ for each $x \in X$, hence $C = B$ by Corollary 4.4.10. $\square$

Theorem 4.4.12. Let X be a compact Hausdorff space. Then the following diagram commutes, and consists only of order isomorphisms:



Proof. By Proposition 4.3.6 and Proposition 4.4.7, we only have to show that the maps of the bottom side of the triangle are compositions of maps of the other sides. Then the triangle commutes, and since the maps of the other sides are order isomorphisms, it follows that the maps of the bottom side are order isomorphisms as well. Let $\mathcal{P}$ be an u.s.c. decomposition of X. Then $\mathcal{P}$ corresponds to $B = \bigcap_{K \in \mathcal{P}} C_K$ in $\mathcal{C}(C(X))$. Let $q_B : X \to X/\sim_B$ be the quotient map. By Proposition 4.4.7, $X/\sim_B = \mathcal{P}$ and $q_B = q_{\mathcal{P}}$, which shows that the map $\mathcal{D}(X)^{\mathrm{op}} \to \mathcal{Q}(X)$ in the triangle is the composition of the map $\mathcal{D}(X)^{\mathrm{op}} \to \mathcal{C}(C(X))$ and the map $\mathcal{C}(C(X)) \to \mathcal{Q}(X)$. Now let $q : X \to Y$

be a continuous surjection onto a compact Hausdorff space. Then $[q] \in \mathcal{Q}(X)$ corresponds to $B = C_q[C(Y)]$ in $\mathcal{C}(C(X))$. Let $x \in X$ and $y = q(x)$. Then $x' \in [x]_B$ if and only if $f \circ q(x) = f \circ q(x')$ for each $f \in C(Y)$. Since $C(Y)$ separates points of Y, it follows that $x' \in [x]_B$ if and only if $q(x') = q(x)$. Thus $[x]_B = \{q^{-1}[\{q(x)\}]\}$, so

$$\{[x]_B : x \in X\} = \{q^{-1}[\{q(x)\}] : x \in X\} = \{q^{-1}[\{y\}] : y \in Y\},$$

where we used the surjectivity of q in the last equality. It follows that the assignment $\mathcal{Q}(X) \to \mathcal{D}(X)^{\mathrm{op}}$ is exactly the composition of the map $\mathcal{Q}(X) \to \mathcal{C}(C(X))$ and the map $\mathcal{C}(C(X)) \to \mathcal{D}(X)^{\mathrm{op}}$. □

4.5 The covering relation on $\mathcal{C}(C(X))$

In this section we describe the covering relation (cf. Definition B.1.7) on $\mathcal{C}(C(X))$, where X is some compact Hausdorff space. The covering relation expresses whether or not one can find an element between two comparable elements in a poset. The covering relation reveals much information about the structure of a poset, for instance we shall see in the next chapter that it plays an important role in deciding whether a C*-algebra A is finite-dimensional or not.

Lemma 4.5.1. Let X be a compact Hausdorff space and $\mathcal{P} = \{K_i\}_{i \in I}$ an u.s.c. decomposition of X. Then

$$\mathcal{P}_{s,t} = \{K_i\}_{i \neq s,t} \cup \{K_s \cup K_t\}$$

is an u.s.c. decomposition of X, which covers $\mathcal{P}$ in $\mathcal{D}(X)$ for any two distinct points $s, t \in I$.

Proof. Clearly $\mathcal{P}_{s,t}$ is a partition of closed subsets of X. Let $i \neq s, t$ and $U \subseteq X$ open such that $K_i \subseteq U$. Since $\mathcal{P}$ is an u.s.c. decomposition of X, there is an open V, saturated with respect to $\mathcal{P}$, such that $K_i \subseteq V \subseteq U$. Let

$$V' = V \setminus (K_s \cup K_t).$$

Then V' is open, and saturated with respect to $\mathcal{P}_{s,t}$. Moreover, we clearly have $K_i \subseteq V' \subseteq U$. Now, let U open in X such that $K_s \cup K_t \subseteq U$. Then $K_s \subseteq U$, so there is an open V_s, saturated with respect to $\mathcal{P}$, such that $K_s \subseteq V_s \subseteq U$. Similarly, there is an open V_t, saturated with respect to $\mathcal{P}$, such that $K_t \subseteq V_t \subseteq$

U. Let $V = V_s \cup V_t$. Then V is saturated with respect to $\mathcal{P}_{s,t}$ and

$$K_s \cup K_t \subseteq V \subseteq U.$$

Thus $\mathcal{P}_{s,t}$ is an u.s.c. decomposition of X. We have $\mathcal{P} \leq \mathcal{P}_{s,t}$ without equality, since clearly each $K_i \in \mathcal{P}$ is contained in some element of $\mathcal{P}_{s,t}$ and $K_s \neq K_s \cup K_t$. Let $\mathcal{P}' \in \mathcal{D}(X)$ such that

$$\mathcal{P} \leq \mathcal{P}' \leq \mathcal{P}_{s,t}$$

and assume that $\mathcal{P} \neq \mathcal{P}'$. Let $K_i \in \mathcal{P}$ with $i \neq s, t$. Then $K_i \in \mathcal{P}_{s,t}$, and since $\mathcal{P}_{s,t} \leq \mathcal{P}'$, there is a $K \in \mathcal{P}'$ such that $K_i \subseteq K$. Moreover, since $\mathcal{P}' \leq \mathcal{P}$, there is a $K_j \in \mathcal{P}$ such that $K \subseteq K_j$. For any $x \in K_i$, we must have $x \in K_j$, and since $\mathcal{P}$ forms a partition of X, it follows that $i = j$. Hence $K = K_i$. Now, consider K_s. There must be some $K_s' \in \mathcal{P}'$ such that $K_s \subseteq K_s'$. Moreover, there must be some $K \in \mathcal{P}_{s,t}$ such that $K_s' \subseteq K$, which implies that $K_s \subseteq K$. By definition of $\mathcal{P}_{s,t}$, we must have $K = K_s \cup K_t$, hence

$$K_s \subseteq K_s' \subseteq K_s \cup K_t.$$

Similarly, there is a $K_t' \in \mathcal{P}'$ such that

$$K_t \subseteq K_t' \subseteq K_s \cup K_t,$$

and it follows that $K_s' \cup K_t' = K_s \cup K_t$. Since $\mathcal{P}' \neq \mathcal{P}$, we must have either $K_s \neq K_s'$ or $K_t \neq K_t'$. Without loss of generality, assume that $K_s \neq K_s'$. Then there is some $x \in K_s'$ such that $x \notin K_s$. It follows that $x \in K_t$, so $x \in K_t'$. Hence $K_s' \cap K_t' \neq \emptyset$, so $K_s' = K_t' = K_s \cup K_t$. It follows that $\mathcal{P}' = \mathcal{P}_{s,t}$, so $\mathcal{P}_{s,t}$ covers $\mathcal{P}$. □

Proposition 4.5.2. Let X be a compact Hausdorff space, and let B and D in $\mathcal{C}(C(X))$. Then the following statements are equivalent:

(a) B is covered by D;

(b) There are distinct points $y, z \in X$ such that $B = D \cap C_{\{y,z\}}$ and $D \nsubseteq C_{\{y,z\}}$;

(c) There are distinct points $y, z \in X$ such that $[y]_D \neq [z]_D$ and

$$B = \bigcap_{x \notin [y]_D \cup [z]_D} C_{[x]_D} \cap C_{[y]_D \cup [z]_D}; \tag{16}$$

(d) There is an $u \in X$ such that $[u]_B$ has a non-trivial separation $\{K, L\}$, and

$$D = \bigcap_{x \notin [u]_B} C_{[x]_B} \cap C_K \cap C_L. \tag{17}$$

In particular, the co-atoms of $\mathcal{C}(C(X))$ are exactly the ideal subalgebras $C_{\{y,z\}}$ for $y, z \in X$ such that $y \neq z$, and each element $B \in \mathcal{C}(C(X))$ such that $B \neq C(X)$ is contained in some co-atom.

Proof. (a) $\Longrightarrow$ (b): Let D be a cover of B. Then $B \subseteq D$, but $B \neq D$. Hence there is a $g \in D$ such that $g \notin B$. By Corollary 4.4.10, there must be an $x' \in X$ such that g is not constant on $[x']_B$, otherwise $g \in B$. So let $y, z \in [x']_B$ be distinct points such that $g(y) \neq g(z)$. Hence $g \notin C_{\{y,z\}}$, so D is not contained in $C_{\{y,z\}}$. It follows that $D \cap C_{\{y,z\}} \subset D$, but $D \cap C_{\{y,z\}}$. On the other hand, since $\{y, z\} \subseteq [x']_B$, Lemma 4.2.2 assures that $C_{[x']_B} \subseteq C_{\{y,z\}}$, so

$$B = \bigcap_{x \in X} C_{[x]_B} \subseteq \bigcap_{x \in X} C_{[x]_B} \cap C_{\{y,z\}} = B \cap C_{\{y,z\}} \subseteq D \cap C_{\{y,z\}}.$$

Thus

$$B \subseteq D \cap C_{\{y,z\}} \subset D,$$

where the last inclusion is proper. Since D covers B, it follows that $B = D \cap C_{\{y,z\}}$.

(b) $\Longrightarrow$ (c): Since $B = D \cap C_{\{y,z\}}$, we have $B \subseteq D$. We also have $B \neq D$, since otherwise $D = D \cap C_{\{y,z\}}$, contradicting $D \nsubseteq C_{\{y,z\}}$. The last expression also implies that there is some $f \in D$ that is not constant on $\{y, z\}$. In other words, $f(y) \neq f(z)$, hence $[y]_D \neq [z]_D$. Corollary 4.4.10 assures that

$$D = \bigcap_{x \in X} C_{[x]_D}.$$

By Lemma 4.2.2, we find that

$$C_{[y]_D} \cap C_{[z]_D} \cap C_{\{y,z\}} = C_{[y]_D \cup [z]_D},$$

hence

$$B = D \cap C_{\{y,z\}} = \bigcap_{x \notin [y]_D \cup [z]_D} C_{[x]_D} \cap C_{[y]_D \cup [z]_D}.$$

(c) $\Longrightarrow$ (d): Let $K = [y]_D$ and $L = [z]_D$. Notice that K and L are both non-empty and closed, hence they form a non-trivial separation of $K \cup L$. By Lemma 4.5.1,

$$\big\{[x]_D : x \notin K \cup L\big\} \cup \{K \cup L\}$$

is an u.s.c. decomposition of X. Since B satisfies (16), it follows from Theorem 4.4.12 that

$$\{[x]_B : x \in X\} = \{[x]_D : x \notin K \cup L\} \cup \{K \cup L\}.$$

Since $x \in [x]_B \cap [x]_D$ for each $x \in X$, we find that if $x \notin K \cup L$, then $[x]_D = [x]_B$. If $x \in K \cup L$, then $[x]_B = K \cup L$. Choose $u \in K \cup L$, then it follows that K and L form a non-trivial separation of $[u]_B$, and

$$D = \bigcap_{x \in X} C_{[x]_D} = \bigcap_{x \notin K \cup L} C_{[x]_B} \cap C_{[y]_D} \cap C_{[z]_D} = \bigcap_{x \notin [u]_B} C_{[x]_B} \cap C_K \cap C_L.$$

(d) $\Longrightarrow$ (a): By Theorem 4.4.12, B is covered by D in $\mathcal{C}(C(X))$ if and only if the u.s.c. decomposition of X corresponding to D is covered by the u.s.c. decomposition of X corresponding to B in $\mathcal{D}(X)$. The statement now follows from Lemma 4.5.1. Finally, B is a co-atom in $\mathcal{C}(C(X))$ if and only if B is covered by $C(X)$, that is, if and only if there are $y, z \in X$ such that $B = C(X) \cap C_{\{y,z\}}$ and $B \neq C(X)$. It follows that B is a co-atom if and only if $B = C_{\{y,z\}}$ for some $y, z \in X$ such that $B \neq C(X)$. If $y = z$, then $C_{\{y,z\}} = C_{\{y\}} = C(X)$, so y and z must be distinct. Then $C_{\{y,z\}}$ fails to separate y and z, hence the Stone-Weierstrass Theorem implies that $C_{\{y,z\}} \neq C(X)$. Hence the co-atoms of $\mathcal{C}(C(X))$ are exactly the ideal subalgebras $C_{\{y,z\}}$ with $y \neq z$. If $B \in \mathcal{C}(C(X))$ such that $B \neq C(X)$, then B fails to separate some $y, z \in X$ such that $y \neq z$. Hence all functions in B are constant on $\{y, z\}$, so $B \subseteq C_{\{y,z\}}$. □

Corollary 4.5.3. Let A be a commutative C*-algebra and let $B, D \in \mathcal{C}(A)$. Then D covers B if and only if there is a co-atom C of $\mathcal{C}(A)$ such that $D \nsubseteq C$ and $B = C \cap D$.

Proof. By the commutative Gelfand–Naimark Theorem, there is a *-isomorphism $\varphi : A \to C(X)$ for some compact Hausdorff space X. By Theorem 3.2.1, $\mathcal{C}(\varphi) : \mathcal{C}(A) \to \mathcal{C}(C(X))$ is an order isomorphism, which preserves covering relation by Proposition B.1.15. The statement follows directly from the previous proposition. □

4.6 Order-theoretic characterizations of ideal subalgebras

It follows from Theorem 4.4.12 that the ideal subalgebras of $C(X)$ are *meet-dense* in $\mathcal{C}(C(X))$, i.e., every element of $\mathcal{C}(C(X))$ can be written as the infimum of a certain collection of ideal subalgebras. This formalizes the heuristic statement that ideal subalgebras 'form the building blocks of $\mathcal{C}(C(X))$'. In this section, we give characterizations of ideal subalgebras in terms of the order on $\mathcal{C}(C(X))$.

Definition 4.6.1. Let A be a commutative C*-algebra. Then we say that $C \in \mathcal{C}(A)$ is a *co-bounding element* if at least one of the following conditions hold:

(i) $C = A$ or C is a co-atom;

(ii) C is covered by exactly three co-atoms;

(iii) For any two co-atoms C_1, C_2 such that $C \subseteq C_1, C_2$, there exists a co-atom C_3 such that $C \subseteq C_3$, and $C_1 \cap C_3$ and $C_2 \cap C_3$ are both covered by exactly three co-atoms.

The collection of co-bounding elements of $\mathcal{C}(A)$ is denoted by $\mathcal{I}(A)$, which becomes a poset with the order inherited from $\mathcal{C}(A)$.

The terminology 'co-bounding element' is derived from [37], where the so called *bounding elements* of $\mathcal{D}(X)$ are introduced. These elements correspond to co-bounding elements of $\mathcal{C}(C(X))$ under the duality between $\mathcal{D}(X)$ and $\mathcal{C}(C(X))$ described in Theorem 4.4.12. The next proposition, which is the reason why we denote the set of co-bounding elements of $\mathcal{C}(A)$ by $\mathcal{I}(A)$, is a translation of [37, Lemma 3.1.1] in terms of $\mathcal{C}(C(X))$ instead of u.s.c. decompositions.

Proposition 4.6.2. Let X be a compact Hausdorff space. Then the collection $\mathcal{I}(C(X))$ of co-bounding elements of $\mathcal{C}(C(X))$ is exactly the collection of ideal subalgebras of $C(X)$.

Proof. We have $C(X) = C_\emptyset$, but also $C(X) = C_{\{x\}}$ for each $x \in X$. Hence the ideal subalgebras C_K with K empty or a singleton are all equal to $C(X)$. By Proposition 4.5.2, the co-atoms of $\mathcal{C}(C(X))$ are exactly the ideal subalgebras C_K with K a two-point set. Thus the elements in $\mathcal{C}(C(X))$ satisfying (i) of Definition 4.6.1 are exactly $C(X)$ and the co-atoms of $\mathcal{C}(C(X))$.

Let D be a co-atom, so $D = C_{\{y,z\}}$ for some distinct points $y, z \in X$. By Theorem 4.4.12, $[x]_D = \{x\}$ if $x \neq y, z$ and $[y]_D = [z]_D = \{y, z\}$. It now follows from Proposition 4.5.2 that the elements in $\mathcal{C}(A)$ covered by D are exactly the algebras of the form $C_{\{x,y,z\}}$ for any $x \in X \setminus \{y, z\}$ and the algebras of the form $C_{\{y,z\}} \cap C_{\{x,w\}}$ for any distinct $x, w \in X \setminus \{y, z\}$. Clearly, the algebras of the form $C_{\{y,z\}} \cap C_{\{x,w\}}$ are covered by only two co-atoms: $C_{\{y,z\}}$ and $C_{\{x,w\}}$, whereas the algebras of the form $C_{\{x,y,z\}}$ are covered by exactly three co-atoms: $C_{\{y,z\}}$, $C_{\{x,y\}}$ and $C_{\{x,z\}}$. It follows that the elements of $\mathcal{C}(A)$ satisfying (ii) of Definition 4.6.1 are exactly the ideal subalgebras C_K with K a three-point set.

Now let $C = C_K$, where K is closed with $\#K \geq 3$ and let C_1, C_2 be co-atoms such that $C_K \subseteq C_1, C_2$. Hence $C_1 = C_{\{x,y\}}$ and $C_2 = C_{\{u,v\}}$ for distinct $x, y \in X$ and distinct u, v in X. If $\{x, y\} = \{u, v\}$, then there is some other point $z \in K \setminus \{x, y\}$. Hence if we choose $C_3 = C_{\{x,z\}}$, we find by Lemma 4.2.2 that

$$C_1 \cap C_3 = C_2 \cap C_3 = C_{\{x,y,z\}},$$

which is exactly covered by three co-atoms. If $\{x, y\} \neq \{u, v\}$, assume without loss of generality that $x \neq u \neq y$ and let $C_3 = C_{\{x,u\}}$. Then $C_1 \cap C_3 = C_{\{x,y,u\}}$ and $C_2 \cap C_3 = C_{\{x,u,v\}}$, both covered by exactly three co-atoms, so C satisfies (iii).

Finally, we show that any $C \in \mathcal{C}(C(X))$ satisfying (iii) but not (i) of Definition 4.6.1 must be of the form C_K for some closed K with $\#K \geq 3$. Let $\{K_i\}_{i \in I}$ be the u.s.c. decomposition corresponding to C, so $C = \bigcap_{i \in I} C_{K_i}$. Since C does not satisfy (i), there is either at least one $i \in I$ such that $\#K_i \geq 3$, or there are at least distinct $i, j \in I$ such that $\#K_i, \#K_j \geq 2$. Let $i, j \in I$ such that $\#K_i, \#K_j \geq 2$. Notice that we do not assume that $i \neq j$; we actually aim to show that $K_i = K_j$. Choose distinct $x, y \in K_i$ and distinct $u, v \in K_j$. Then $C_1 = C_{\{x,y\}}$ and $C_2 = C_{\{u,v\}}$ are co-atoms such that $C \subseteq C_1, C_2$. Hence there is some co-atom C_3 such that $C \subseteq C_3$, and $C_1 \cap C_3$ and $C_2 \cap C_3$ are both covered by exactly three co-atoms. Write $C_3 = \{w, z\}$ for some distinct points $w, z \in X$. Then $C_{\{x,y\}} \cap C_{\{z,w\}} = C_{L_1}$ and $C_{\{u,v\}} \cap C_{\{z,w\}} = C_{L_2}$ for some three-point subsets $L_1, L_2 \subseteq X$. It follows that $C_{L_1} \subseteq C_{\{x,y\}}$ and by Lemma 4.2.2, we find that $\{x, y\} \subseteq L_1$. Similarly, $\{z, w\} \subseteq L_1$, so $\{x, y, z, w\} \subseteq L_1$. Now, L_1 is a three-point set, so $\{x, y\} \cap \{z, w\} \neq \emptyset$. In a similar way, we find that $\{u, v, z, w\} \subseteq L_2$, and so $\{u, v\} \cap \{z, w\} \neq \emptyset$. Now, by Lemma 4.2.2, we find

$$C_1 \cap C_2 \cap C_2 = C_{\{x,y\}} \cap C_{\{u,v\}} \cap C_{\{w,z\}} = C_{\{u,v,w,x,y,z\}}.$$

Notice that $\{u, v, w, x, y, z\}$ is not necessarily a six-point set, but it is at least a two-point set. We denote the u.s.c. decomposition of X corresponding to $C_{\{u,v,w,x,y,z\}}$ by $\mathcal{P}$. Thus

$$\mathcal{P} = \big\{\{u, v, w, x, y, z\}\big\} \cup \big\{\{s\} : s \notin \{u, v, w, x, y, z\}\big\},$$

and since $C \subseteq C_{\{u,v,w,x,y,z\}}$, Proposition 4.4.7 assures that $\mathcal{P} \leq \{K_s\}_{s \in I}$. Thus $\{u, v, w, x, y, z\} \subseteq K_s$ for some $s \in I$, which is necessarily unique as the $\{K_t\}_{t\in I}$ is a partition of X. Now, $x \in K_i$, so $s = i$, but $u \in K_j$, so also $s = j$. We conclude that $K_i = K_j$ for each $i, j \in I$ such that $\#K_i, \#K_j \geq 2$. Hence there is exactly one $i \in I$ such that $\#K_i \geq 2$, so

$$C = \bigcap_{j \in I} C_{K_j} = C_{K_i},$$

since $C_{K_j} = A$ if K_j is a singleton. So C is an ideal subalgebra, and since we assumed that it is not of the form (i), K_i must have at least three points. We conclude that $C = C_K$ for some closed set K if and only C satisfies (i), (ii) or (iii) of Definition 4.6.1. □

Corollary 4.6.3. Let A and B be commutative C*-algebras such that $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$ is an order isomorphism. Then Φ restricts to an order isomorphism $\mathcal{I}(A) \to \mathcal{I}(B)$.

Proof. Since $\mathcal{I}(A)$ and $\mathcal{I}(B)$ are completely defined in terms of covering relations in $\mathcal{C}(A)$ and $\mathcal{C}(B)$, respectively, it follows from Proposition B.1.15 that Φ restricts to an order isomorphism $\mathcal{I}(A) \to \mathcal{I}(B)$. □

Corollary 4.6.4. Let A be a commutative C*-algebra. Then $\mathbb{C}1_A \in \mathcal{I}(A)$.

Proof. By the commutative Gelfand–Naimark Theorem, there is a *-isomorphism $\psi : A \to C(X)$ for some compact Hausdorff space X. By Theorem 3.2.1, $\mathcal{C}(\psi) : \mathcal{C}(A) \to \mathcal{C}(C(X))$ is an order isomorphism, which maps $\mathbb{C}1_A$ to $\mathbb{C}1_{C(X)}$. Since $\mathbb{C}1_{C(X)} = C_X$, it follows from Proposition 4.6.2 that $\mathbb{C}1_{C(X)} \in \mathcal{I}(C(X))$. Now, Corollary 4.6.3 assures that $\mathbb{C}1_A \in \mathcal{I}(A)$. □

Proposition 4.6.5. Let X be a compact Hausdorff space. If $\{K_i\}_{i\in I}$ is a collection of closed subspaces of X, then

$$\bigvee_{i \in I} C_{K_i} = C_{\bigcap_{i\in I} K_i} \tag{18}$$

in $\mathcal{C}(C(X))$. In particular, the subposet $\mathcal{I}(C(X))$ of $\mathcal{C}(C(X))$ consisting of all ideal subalgebras of $C(X)$ forms a complete lattice. Its infimum operation is given by

$$\bigwedge_{i\in I} C_{K_i} = C_{\overline{\bigcup_{j\in J} K_j}}, \quad J = \{i \in I : \#K_i \geq 2\}. \tag{19}$$

Proof. Let $\{K_i\}_{i\in I}$ be a collection of closed subspaces of X. Then for each $j \in I$, we have $\bigcap_{i\in I} K_i \subseteq K_j$. Let $K = \bigcap_{i\in I} K_i$. Then K is closed, and $K \subseteq K_j$ implies

$$C_{K_j} \subseteq C_K,$$

whence

$$\bigvee_{i\in I} C_{K_i} \subseteq C_K.$$

Now, let $x, y \in X$ such that $[x]_{C_K} \neq [y]_{C_K}$. Hence there is some $f \in C(X)$ constant on K such that $f(x) \neq f(y)$. It cannot happen that both x and y are elements of K, so without loss of generality, assume that $x \notin K$. Then there is a $j \in I$ such that $x \notin K_j$. By Urysohn's Lemma, there is a $g \in C(X)$ such that $g[K_j \cup \{y\}] = \{1\}$ and $g(x) = 0$. Then $g \in C_{K_j}$, so $g \in \bigvee_{i\in I} C_{K_i}$. Since $g(x) \neq g(y)$, Corollary 4.4.11 assures that (18) holds. It follows that the subposet of all ideal subalgebras is closed under arbitrary suprema, hence Lemma B.1.13 assures that this subposet is a complete lattice.

In order to see that the infimum operation in $\mathcal{I}(C(X))$ is given by (19), let $K = \overline{\bigcup_{j\in J} K_j}$ with $J = \{i \in I : \#K_j \geq 2\}$. By Lemma 4.2.2, $C_{K_i} = C(X)$ for $i \in I \setminus J$. If $j \in J$, then $K_j \subseteq K$, whence

$$C_K \subseteq C_{K_i}$$

for each $i \in I$. Now assume that $L \subseteq X$ is closed such that $C_L \subseteq C_{K_i}$ for each $i \in I$. By Lemma 4.2.2, we have $K_j \subseteq L$ for each $j \in J$, so $\bigcup_{j\in J} K_j \subseteq L$. Since L is closed, it follows that $K \subseteq L$, whence $C_L \subseteq C_K$. It follows that $C_K = \bigwedge_{i\in I} C_{K_i}$. □

4.7 Hamhalter's Theorem

Given a compact Hausdorff space X, recall that $\mathcal{F}(X)$ is the poset of all closed subsets of X containing at least two points ordered by inclusion (Definition

4.2.3). The next lemma describes the covering relation in $\mathcal{F}(X)$. Its proof is easy, hence we omit it.

Lemma 4.7.1. Let X be a compact Hausdorff space. If $F, G \in \mathcal{F}(X)$, then F covers G if and only if $G = F \setminus \{x\}$ for some isolated point $x \in F$. Moreover, if X contains more than two points, then a point $x \in X$ is isolated if and only if $X \setminus \{x\} \in \mathcal{F}(X)$ if and only if $X \setminus \{x\}$ is a co-atom in $\mathcal{F}(X)$.

Also the proof of the next lemma is easy.

Lemma 4.7.2. Let X be a completely regular space. Then for each closed subset F of X we have $F = \bigcap\{\overline{O} : O \text{ open}, F \subseteq O\}$.

Lemma 4.7.3. Let X and Y be compact Hausdorff spaces with $2 \leq \#X < \infty$. Let $\Psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ be an order isomorphism. Then there is a homeomorphism $h : X \to Y$ such that $h[F] = \Psi(F)$ for each $F \in \mathcal{F}(X)$.

Proof. Since X is finite, compact and Hausdorff, X is discrete. Hence

$$\#\mathcal{F}(X) = \#\mathcal{P}(X) - \#X - 1,$$

where the 1 arises because of the empty set. If we denote $n = \#X$, then $\#\mathcal{F}(X)$ is finite, and equal to $2^n - n - 1$. Now, $n \geq 2$, so $\#\mathcal{F}(X) \geq 1$. If $\#Y = 0$ or $\#Y = 1$, then $\mathcal{F}(Y) = \emptyset$, so $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ cannot be order isomorphic. If $\#Y = \infty$, then clearly $\#F(Y) = \infty$, which is not possible if $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ are order isomorphic. Since $\#\mathcal{F}(Y) = 2^m - m - 1$ if $m = \#Y$, which is clearly strictly increasing in m if $m \geq 2$, we should have $n = m$ if $\#F(X) = \#F(Y)$, which is the case if $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ are order isomorphic. If $n = 2$, then $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. Then $h(x_i) = y_i$ for $i = 1, 2$ is clearly a homeomorphism, such that $\Psi(F) = h[F]$ for $F \in \mathcal{F}(X) = \{X\}$.

Let $n > 2$, Then for each $x \in X$, $X \setminus \{x\}$ is a closed set of X, and contains at least two elements, so $X \setminus \{x\} \in \mathcal{F}(X)$. Moreover, by Lemma 4.7.1, $X \setminus \{x\}$ is a co-atom in $\mathcal{F}(X)$. By Proposition B.1.15, $\Psi(X \setminus \{x\})$ should also be a co-atom in $\mathcal{F}(Y)$, hence of the form $Y \setminus \{y\}$ for some $y \in Y$. We define $h : X \to Y$ by the equation $\{h(x)\} = Y \setminus \Psi(X \setminus \{x\})$. Then h is injective, since $h(x) = h(y)$ for $x, y \in X$ imply

$$Y \setminus \Psi(X \setminus \{x\}) = Y \setminus \Psi(X \setminus \{y\}),$$

so $\Psi(X \setminus \{x\}) = \Psi(X \setminus \{y\})$. Now, Ψ is a bijection, so $X \setminus \{x\} = X \setminus \{y\}$, which implies $x = y$. To show surjectivity of h, let $y \in Y$. Then $Y \setminus \{y\} \in \mathcal{F}(Y)$, so there is some $F \in \mathcal{F}(X)$ such that $\Psi(F) = Y \setminus \{y\}$. Since $Y \setminus \{y\}$ is a co-atom in $\mathcal{F}(Y)$, F must be a co-atom in $\mathcal{F}(X)$, so $F = X \setminus \{x\}$ for some $x \in X$. We conclude that

$$\{h(x)\} = Y \setminus \Psi(F) = \{y\},$$

so h is indeed surjective. As a bijection between discrete spaces, h must be a homeomorphism.

Finally, we show that $h[F] = \Psi(F)$ for each $F \in \mathcal{F}(X)$. First notice the following: if the infimum of a family $\{F_i\}_{i\in I}$ in $\mathcal{F}(X)$ exists, then it must be equal to $\bigcap_{i\in I} F_i$. Indeed, let F be the infimum in $\mathcal{F}(X)$. Then $F \subseteq F_i$ for each $i \in I$, hence $F \subseteq \bigcap_{i\in I} F_i$. Since $\mathcal{F}(X) \subseteq \mathcal{P}(X)$ and $\bigcap_{i\in I} F_i$ is the infinum of $\{F_i\}_{i\in I}$ in $\mathcal{P}(X)$, we obtain $\bigcap_{i\in I} F_i \subseteq F$. We conclude that $F = \bigcap_{i\in I} F_i$. Now let $F \in \mathcal{F}(X)$. Then

$$\bigcap_{x\notin F} X \setminus \{x\} = X \setminus \bigcup_{x\notin F} \{x\} = F.$$

So $F \in \mathcal{F}(X)$ is the infimum of the family $\{X \setminus \{x\}\}_{x\notin F}$ in $\mathcal{F}(X)$. Now, Ψ is an order isomorphism, and so Ψ preserves infima if they exists. Hence

$$\begin{aligned} \Psi(F) &= \Psi\left(\bigcap_{x\notin F} X \setminus \{x\}\right) = \bigcap_{x\notin F} \Psi(X \setminus \{x\}) = \bigcap_{x\notin F} \{Y \setminus h(x)\} \\ &= Y \setminus \bigcup_{x\notin F} \{h(x)\} = Y \setminus h[X \setminus F]. \end{aligned}$$

Since h is a bijection, we must have $h[X \setminus F] = Y \setminus h[F]$, hence we indeed find $\Psi(F) = h[F]$. □

Proposition 4.7.4. [47, Theorem 2.3] Let X and Y be compact Hausdorff spaces with $\#X \geq 2$. Let $\Psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ an order isomorphism. Then there is a homeomorphism $h : X \to Y$ such that $h[F] = \Psi(F)$ for each $F \in \mathcal{F}(X)$.

Proof. The case that X has finite cardinality is covered by Lemma 4.7.3, hence assume that X has infinite cardinality and let $x \in X$. If x is not isolated, we define $h(x)$ as follows. Let $\mathcal{O}(x)$ denote the set of all open neighborhoods of x. Since x is not isolated, each $O \in \mathcal{O}(x)$ contains at least another element, so $\overline{O} \in \mathcal{F}(X)$. Moreover, finite intersections of elements of $\{\overline{O} : O \in \mathcal{O}(x)\}$ are still in $\mathcal{F}(X)$. Indeed, if $O_1, \ldots, O_n \in \mathcal{O}(x)$, then $O_1 \cap \ldots \cap O_n$ is an open set containing x, and since $\overline{O_1 \cap \ldots \cap O_n} \subseteq \overline{O_1} \cap \ldots \cap \overline{O_n}$, it follows that $\overline{O_1} \cap \ldots \cap \overline{O_n} \in \mathcal{F}(X)$. Since Ψ is an order isomorphism, we find that finite intersections of $\{\Psi(\overline{O}) : O \in \mathcal{O}(x)\}$ are contained in $\mathcal{F}(Y)$. This implies that $\{\Psi(\overline{O}) : O \in \mathcal{O}(x)\}$ satisfies the finite intersection property. As Y is compact, it follows that $I_x = \bigcap_{O \in \mathcal{O}(x)} \Psi(\overline{O})$ is non-empty. We can say more: it turns out that I_x contains exactly one element. Indeed, assume that there are two different point $y_1, y_2 \in I_x$. Then $\{y_1, y_2\} \in \mathcal{F}(Y)$, so $\Psi^{-1}(\{y_1, y_2\}) \in \mathcal{F}(X)$. Since $\{y_1, y_2\} \in \Psi(\overline{O})$ for each $O \in \mathcal{O}(x)$, we also find that $\Psi^{-1}(\{y_1, y_2\}) \subseteq \overline{O}$ for each $O \in \mathcal{O}(x)$. This implies that

$$\Psi^{-1}(\{y_1, y_2\}) \subseteq \bigcap_{O \in \mathcal{O}(x)} \overline{O} = \{x\},$$

where the last equality holds by the normality of X. But this is a contradiction with $\Psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ being a bijection. So I_x contains exactly one point. We define $h(x)$ such that $\{h(x)\} = I_x$. Notice that $h(x)$ cannot be isolated in Y. If $h(x)$ would be isolated, then Lemma 4.7.1 implies that $Y \setminus \{h(x)\}$ would be a co-atom in $\mathcal{F}(Y)$. By Proposition B.1.15, $\Psi^{-1}(Y \setminus \{h(x)\}$ is a co-atom in $\mathcal{F}(X)$, which must be of the form $X \setminus \{z\}$ for some isolated $z \in X$. Since x is not isolated, we cannot have $x = z$, so $X \setminus \{z\}$ is an open neighborhood of x, which is even clopen since z is isolated. By definition of $h(x)$, we must have $h(x) \in \Psi(X \setminus \{z\}$, but this is exactly $Y \setminus \{h(x)\}$. We found a contradiction, hence $h(x)$ cannot be isolated. Now assume that x is an isolated point. By Lemma 4.7.1, $X \setminus \{x\}$ is a co-atom in $\mathcal{F}(X)$, so Proposition B.1.15 guarantees that $\Psi(X \setminus \{x\})$ should be a co-atom in $\mathcal{F}(Y)$ as well. By Lemma 4.7.1 again, this means that $\Psi(X \setminus \{x\}) = Y \setminus \{y\}$ for some unique $y \in Y$, which must be isolated, since $Y \setminus \{y\}$ is closed. We define $h(x) = y$.

In an analogous way, Ψ^{-1} induces a map $\sigma : Y \to X$. So if $y \in Y$ is isolated, we $\sigma(y)$ is defined as the unique point such that $X \setminus \{\sigma(y)\} = Y \setminus \{y\}$, and if y is not isolated, $\sigma(y)$ is defined as the unique point such that $\{y\} = \bigcap_{O \in \mathcal{O}(y)} \Psi^{-1}(\overline{O})$. We shall show that h and σ are each other's inverses. Let $x \in X$ be isolated. Then $h(x)$ is isolated as well, and is defined by the equation $\Psi(X \setminus \{x\}) = Y \setminus \{h(x)\}$. Since Ψ is an order isomorphism, we have $X \setminus \{x\} = \Psi^{-1}(Y \setminus \{h(x)\})$. Since $h(x)$ is isolated, we find by definition of σ that $\sigma(h(x)) = x$. In a similar way we find that $h(\sigma(y)) = y$ for each isolated $y \in Y$. Now assume that x is not isolated and let $F \in \mathcal{F}(X)$ such that $x \in F$.

Then

$$\begin{aligned}
\{h(x)\} &= \bigcap_{O\in\mathcal{O}(x)} \Psi(\overline{O}) \\
&\subseteq \bigcap\{\Psi(\overline{O}) : O \text{ open}, F\subseteq O\} \\
&= \Psi\left(\overline{O} : O \text{ open}, F\subseteq O\}\right) \\
&= F.
\end{aligned}$$

Here the last equality follows by Lemma 4.7.2, which can be used since X is compact and Hausdorff, so it is certainly completely regular. The second last equality follows from the fact that $\bigcap\{\overline{O} : O \text{ open}, F \subseteq O\}$ is closed since it is the intersection of closed sets. Moreover, the intersection contains more than one point, since F contains two or more points and $F \subseteq \overline{O}$ for each O. Hence $\bigcap\{\overline{O} : O \text{ open}, F \subseteq O\} \in \mathcal{F}(X)$, and since Ψ is an order isomorphism, it preserves infima, which justifies the second last equality. Hence $h(x) \in \Psi(F)$ for each $F \in \mathcal{F}(X)$ containing x. Since x is not isolated, $h(x)$ is not isolated as well. Hence in a similar way, we find that $\sigma(h(x)) \in \Psi^{-1}(G)$ for each $G \in \mathcal{F}(Y)$ containing $h(x)$. Let $z = \sigma(h(x)$. Combining both statements, we find that $z \in F$ for each $F \in \mathcal{F}(X)$ such that $x \in F$. In other words, $z \in \bigcap\{F \in \mathcal{F}(X) : x \in F\}$. Since x is not isolated, we each $O \in \mathcal{O}(x)$ contains at least two points. Hence

$$\bigcap\{F \in \mathcal{F}(X) : x \in F\} \subseteq \bigcap\{\overline{O} : O \in \mathcal{O}(x)\} = \{x\},$$

where we used Lemma 4.7.2 in the last equality. We conclude that $z = x$, so $\sigma(h(x)) = x$. In a similar way, we find that $h(\sigma(y)) = y$ for each non-isolated $y \in Y$. We conclude that h is a bijection with $h^{-1} = \sigma$.

We have to show that if $F \in \mathcal{F}(X)$, then $h[F] = \Psi(F)$. Let $x \in F$. When we proved that h is a bijection, we already noticed that $h(x) \in \Psi(F)$ if x is not isolated. If x is isolated in X, then we first assume that F has at least three points. Since $\{x\}$ is open, $G = F \setminus \{x\}$ is closed. Since F contains at least three points, $G \in \mathcal{F}(X)$. So G is covered by F in $\mathcal{F}(X)$, so $\Psi(F)$ covers $\Psi(G)$. By Lemma 4.7.1, there must be an element $y_G \in Y \setminus \Psi(G)$ such that

$$\Psi(F) = \Psi(G \cup \{x\}) = \Psi(G) \cup \{y_G\}.$$

We have $G \cup \{x\}, X \setminus \{x\} \in \mathcal{F}(X)$, so

$$\begin{aligned}\Psi(G) &= \Psi(G \cup \{x\} \cap X \setminus \{x\}) = \Psi(G \cup \{x\}) \cap \Psi(X \setminus \{x\}) \\ &= (\Psi(G) \cup \{y_G\}) \cap (Y \setminus \{h(x)\}),\end{aligned}$$

where $\Psi(X \setminus \{x\}) = Y \setminus \{h(x)\}$ by definition of values of h at isolated points. Since $x \notin G$ and Ψ preserves inclusions, this latter equation also implies $\Psi(G)$ is a subset of $Y \setminus \{h(x)\}$. Hence we find

$$\Psi(G) = (\Psi(G) \cup \{y_G\}) \cap (Y \setminus \{h(x)\}) = \Psi(G) \cup (\{y_G\} \cap Y \setminus \{h(x)\}).$$

Thus we obtain $\{y_G\} \cap Y \setminus \{h(x)\} \subseteq \Psi(G)$, but since $y_G \notin \Psi(G)$, we must have $h(x) = y_G$. As a consequence, we obtain $\Psi(F) = \Psi(G) \cup \{h(x)\}$, so $h(x) \in \Psi(F)$.

Summarizing, if F has at least three points, we found that $h(x) \in \Psi(F)$ for $x \in F$ regardless whether x is isolated or not. So $h[F] \subseteq \Psi(F)$ for each $F \in \mathcal{F}(X)$ such that F has at least three points. Let $F \in \mathcal{F}(X)$ have exactly two points. Then there are $F_1, F_2 \in \mathcal{F}(X)$ with exactly three points such that $F = F_1 \cap F_2$. Then since h as a bijection and Ψ as an order isomorphism both preserve intersections in $\mathcal{F}(X)$, we find

$$h[F] = h[F_1 \cap F_2] = h[F_1] \cap h[F_2] \subseteq \Psi(F_1) \cap \Psi(F_2) = \Psi(F_1 \cap F_2) = \Psi(F).$$

So $h[F] \subseteq \Psi(F)$ for each $F \in \mathcal{F}(X)$. In a similar way, we find $h^{-1}[G] \subseteq \Psi^{-1}[G]$ for each $G \in \mathcal{F}(Y)$. So if we substitute $G = \Psi(F)$, we obtain $h^{-1}[\Psi(F)] \subseteq F$. Since h is a bijection, it follows that $\Psi(F) = h[F]$ for each $F \in \mathcal{F}(X)$. As a consequence, h induces a one-one correspondence between closed subsets of X and closed subsets of Y. Hence h is a homeomorphism. □

Theorem 4.7.5. [47, Theorem 2.4] Let A and B be commutative C*-algebras. Given an order isomorphism $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$, there is a *-isomorphism $\varphi : A \to B$ such that $\Phi = \mathcal{C}(\varphi)$. Moreover, if $\dim A \neq 2$, φ is the unique *-isomorphism inducing Φ in this way.

Proof. We first assume that $A = C(X)$ and $B = C(Y)$ for some compact Hausdorff spaces X and Y. If A is one-dimensional, then $\mathcal{C}(A) = 1$, and clearly B must be one-dimensional as well. Moreover, clearly there is only one *-isomorphism inducing Φ. If A is two-dimensional, $\mathcal{C}(A)$ and so $\mathcal{C}(B)$ are two-point posets. Clearly B must be isomorphic to $\mathbb{C}^2$. Since $\Phi = \mathcal{C}(1_{\mathbb{C}^2}) = \mathcal{C}(\varphi)$, where $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ is given by $(a,b) \mapsto (b,a)$, it follows that Φ is not uniquely determined by a *-isomorphism. Assume that $\dim A > 2$. We can use Proposition 4.6.2 for an order-theoretic description of the elements of $\mathcal{C}(A)$ that are ideal subalgebras. As a result, we find that Φ restricts to an order isomorphism between the ideal subalgebras in $\mathcal{C}(A)$ and the ideal subalgebras in $\mathcal{C}(B)$. It now follows from Lemma 4.2.2 that there is an order isomorphism $\rho : \mathcal{F}(X) \to \mathcal{F}(Y)$ such that $\Phi(C_F) = C_{\rho(F)}$. By Proposition 4.7.4, we find that there is a homeomorphism $h : X \to Y$ such that $h[F] = \rho(F)$ for each $F \in \mathcal{F}(X)$. Let $\varphi = C_{h^{-1}} : C(X) \to C(Y)$ be the corresponding *-isomorphism. That is, $\varphi(f) = f \circ h^{-1}$. If $C_F \in \mathcal{C}(A)$ is an ideal subalgebra, we find

$$\varphi[C_F] = C_{h^{-1}}[C_F] = C_{h[F]} = C_{\rho(F)} = \Phi(C_F),$$

where we used Corollary 4.4.8 in the second equality. By Corollary 4.4.10, we find that each $D \in \mathcal{C}(A)$ can be written as $D = \bigcap_{x \in X} C_{[x]_D}$. If $[x]_D$ is a singleton, then $C_{[x]_D} = C(X)$. Hence if S is the set of all $x \in X$ such that $[x]_D$ is not a singleton, we find $D = \bigcap_{x \in S} C_{[x]_C}$. This also works if S is empty. This means that $D = C(X)$, and the empty intersection gives $C(X)$ as well. Since $C_{[x]_D}$ is an ideal subalgebra in $\mathcal{C}(A)$ if $[x]_D$ is not a singleton, we find that each $D \in \mathcal{C}(A)$ can be written as an intersection of ideal subalgebras. As a consequence, we find that $\varphi[D] = \Phi(D)$ for each $D \in \mathcal{C}(A)$. In other words, $\Phi = \mathcal{C}(\varphi)$. For uniqueness of φ, assume that φ_1 and φ_2 both induce Φ. Then $\mathcal{C}(\varphi_1 \circ \varphi_2^{-1})$ is the identity order isomorphism on $\mathcal{C}(A)$. Let $\varphi = \varphi_1 \circ \varphi_2^{-1}$ and let $h : X \to X$ be a homeomorphism such that $C_{h^{-1}} = \varphi$, where we recall that $C_{h^{-1}}(f) = f \circ h^{-1}$. Then for each $F \in \mathcal{F}(X)$, we find

$$C_F = \mathcal{C}(\varphi)(C_F) = \varphi[C_F] = C_{h^{-1}}[C_F] = C_{h[F]},$$

where in the last equality we used Corollary 4.4.8. Using Lemma 4.2.2 we now obtain $h[F] = F$. For arbitrary $x \in X$, we can find $y, z \in X$ such that $\{x, y, z\}$ is a three-point set, since A is at least three-dimensional. Since $\{x, y\}$ and $\{x, z\}$ are elements of $\mathcal{F}(X)$, and $h[\{x, y\}] = \{x, y\}$ and $h[\{x, z\}] = \{x, z\}$, we find that $h(x) = x$. Hence h is the identity, whence $\varphi = C_{h^{-1}}$ is the identity on $C(X)$. We conclude that $\varphi_1 = \varphi_2$.

If A and B are commutative C*-algebras in abstract sense, let X and Y be their Gelfand spectra, respectively, and let $\psi_A : A \to C(X)$ and $\psi_B : B \to C(Y)$ be *-isomorphisms. By Theorem 3.2.1, $\mathcal{C}(\psi_A) : \mathcal{C}(A) \to \mathcal{C}(C(X))$ and

$\mathcal{C}(\psi_B) : \mathcal{C}(B) \to \mathcal{C}(C(Y))$ are order isomorphisms, and

$$\Psi = \mathcal{C}(\psi_B) \circ \Phi \circ \mathcal{C}(\psi_A)^{-1}$$

is an order isomorphism $\mathcal{C}(C(X)) \to \mathcal{C}(C(Y))$, hence there is a *-isomorphism $\psi : C(X) \to C(Y)$ such that $\Psi = \mathcal{C}(\psi)$. If $\dim A \neq 2$, then $\dim C(X) \neq 2$, in which case ψ is the unique *-isomorphism such that $\Psi = \mathcal{C}(\psi)$. Let $\varphi = \psi_B^{-1} \circ \psi \circ \psi_A$. By functoriality of $\mathcal{C}$ (see Theorem 3.2.1), we find

$$\mathcal{C}(\varphi) = \mathcal{C}(\psi_B)^{-1} \circ \mathcal{C}(\psi) \circ \mathcal{C}(\psi_A) = \mathcal{C}(\psi_B)^{-1} \circ \Psi \circ \mathcal{C}(\psi_A) = \Phi.$$

If $\dim A \neq 2$, and $\theta : A \to B$ is a *-isomorphism such that $\mathcal{C}(\theta) = \Phi$, then

$$\mathcal{C}(\psi_B \circ \theta \circ \psi_A^{-1}) = \mathcal{C}(\psi_B) \circ \mathcal{C}(\theta) \circ \mathcal{C}(\psi_A)^{-1} = \mathcal{C}(\psi_B) \circ \Phi \circ \mathcal{C}(\psi_A)^{-1} = \Psi,$$

and by uniqueness of ψ inducing Ψ, we find $\psi = \psi_B \circ \theta \circ \psi_A^{-1}$, whence

$$\theta = \psi_B^{-1} \circ \psi \circ \psi_A = \varphi.$$ □

Corollary 4.7.6. Let A be a (not necessarily commutative) C*-algebra and let $C \in \mathcal{C}(A)$. If B is a commutative C*-algebra such that $\mathcal{C}(B)$ and $\downarrow C \subseteq \mathcal{C}(A)$ are order isomorphic, then B and C are isomorphic as C*-algebras.

Proof. If $C \in \mathcal{C}(A)$, then $\downarrow C$ is a subposet of $\mathcal{C}(A)$, which is clearly order isomorphic to $\mathcal{C}(C)$. Hence if B is any commutative C*-algebra such that $\downarrow C \cong \mathcal{C}(B)$, we find $\mathcal{C}(C) \cong \mathcal{C}(B)$, hence $B \cong C$ by the previous theorem. □

5 Finite-dimensional C*-algebras

This chapter is devoted to the characterization of finite-dimensional C*-algebras in terms of commutative subalgebras. We start from finding order-theoretic properties of $\mathcal{C}(A)$ that exactly specify that A is finite-dimensional. We proceed by considering the Artin-Wedderburn Theorem, which states that there are numbers $k, n_1, \ldots, n_k \in \mathbb{N}$ such that A is *-isomorphic to $\bigoplus_{i=1}^{k} \mathrm{M}_{n_i}(\mathbb{C})$. Finally, we find a method for retrieving the numbers $k, n_1, \ldots, n_k$ from $\mathcal{C}(A)$.

5.1 Criteria for being finite-dimensional

We first introduce the following subposet of $\mathcal{C}(A)$.

Definition 5.1.1. Let A be a C*-algebra. Then we denote the subposet of $\mathcal{C}(A)$ consisting of all finite-dimensional commutative C*-subalgebras of A by $\mathcal{C}_{\mathrm{fin}}(A)$.

We will find an order-theoretic characterization of $\mathcal{C}_{\mathrm{fin}}(A)$ as the so-called *compact elements* of $\mathcal{C}(A)$ in Chapter 7. Recall the definitions of Artinian and Noetherian posets (see Definition B.2.1) and the definition of an order scattered poset (see Definition B.1.8). We now state the main theorem of this section.

Theorem 5.1.2. Let A be a C*-algebra. Then the following statements are equivalent:

(1) A is finite-dimensional;

(2) $\mathcal{C}(A) = \mathcal{C}_{\mathrm{fin}}(A)$;

(3) $\mathcal{C}(A)$ is Artinian;

(4) $\mathcal{C}(A)$ is Noetherian;

(5) $\mathcal{C}(A)$ is order scattered.

We divide the proof of Theorem 5.1.2 into several lemmas[4].

Lemma 5.1.3. Let A be a C*-algebra. Then:

[4]We thank Michael Mislove for bringing the notion of order scattered posets to our attention, and for providing the crucial element in the proof of Lemma 5.1.10, namely the construction of an order dense chain of closed subsets in an infinite scattered compact Hausdorff space.

- $\mathbb{C}1_A$ is the least element of $\mathcal{C}_{\text{fin}}(A)$;
- $\mathcal{C}_{\text{fin}}(A)$ is a down-set when regarded as a subset of $\mathcal{C}(A)$;
- $\mathcal{C}_{\text{fin}}(A) = \mathcal{C}(A)$ if A is finite-dimensional.

Proof. By Theorem 3.1.3, $\mathcal{C}(A)$ has a least element $\mathbb{C}1_A$, which is clearly one-dimensional. Hence $\mathbb{C}1_A \in \mathcal{C}_{\text{fin}}(A)$. If $C \in \mathcal{C}_{\text{fin}}(A)$ and $D \in \mathcal{C}(A)$ such that $D \subseteq C$, then D must be finite-dimensional, for C is finite-dimensional. Thus, $D \in \mathcal{C}_{\text{fin}}(A)$, and we conclude that $\mathcal{C}_{\text{fin}}(A)$ is a down-set as a subposet of $\mathcal{C}(A)$. Finally, it is trivial that $\mathcal{C}_{\text{fin}}(A) = \mathcal{C}(A)$ if A is finite-dimensional. $\square$

Lemma 5.1.4. Let A be a C*-algebra and $B \in \mathcal{C}(A)$ finite-dimensional. Then $C \in \mathcal{C}(A)$ is a cover of B if and only if $B \subseteq C$ and $\dim(B) + 1 = \dim(C)$.

Proof. Assume that B is finite-dimensional. Then clearly $C \in \mathcal{C}(A)$ is a cover of B if $B \subseteq C$ and $\dim(B) + 1 = \dim(C)$. Conversely, assume that C is a cover of B. By definition of a cover, we have $B \subsetneq C$. Let $n = \dim(C)$ and let X be the spectrum of C. By Theorem 4.4.12, the spectrum of B is homeomorphic to $X/\sim_B$, must consist of n points. Hence there are n closed subsets $K_1, \ldots, K_n$ forming a partition of X such that

$$\{[x]_B : x \in X\} = \{K_1, \ldots, K_n\}.$$

By Corollary 4.4.10, we have $B = \bigcap_{x \in X} C_{[x]_B} = \bigcap_{i=1}^{n} C_{K_i}$. Now, Proposition 4.5.2 implies that there is a $j \in \{1, \ldots, n\}$ such that $K_j = L_1 \cup L_2$ with L_1, L_2 disjoint non-empty closed subsets, and such that

$$C = C_{K_1} \cap \ldots \cap C_{K_{j-1}} \cap C_{L_1} \cap C_{L_2} \cap C_{K_{j+1}} \cap \ldots \cap C_{K_n}.$$

Since $\{K_1, \ldots, K_{j-1}, L_1, L_2, K_{j+1}, \ldots, K_n\}$ is a partition of X, it follows from Theorem 4.4.12 that $[x]_C = K_i$ if $x \in K_i$ with $i \neq j$, and $[x]_C = L_i$ if $x \in L_i$ with $i = 1, 2$. However, since X is the spectrum of C, $C(X)$ separates all points of X, so $[x]_C \neq [y]_C$ if $x \neq y$. Hence $\#X = n+1$, so $\dim(C) = \dim(B) + 1$. $\square$

Lemma 5.1.5. Let A be a C*-algebra. Then $\mathcal{C}_{\text{fin}}(A)$ is graded with a rank function

$$\dim : \mathcal{C}_{\text{fin}}(A) \to \mathbb{N}$$

assigning to each element $C \in \mathcal{C}_{\text{fin}}(A)$ its dimension. Moreover, if A is finite-dimensional, then the range of dim is bounded from above by the dimension of A.

Proof. By Lemma 5.1.3, $\mathbb{C}1_A$, which is clearly one-dimensional, is the least element of $\mathcal{C}_{\text{fin}}(A)$. Let $C_1, C_2 \in \mathcal{C}_{\text{fin}}(A)$. It is clear that $C_1 \subset C_2$ (so $C_1 \neq C_2$) implies $\dim(C_1) < \dim(C_2)$. Since Lemma 5.1.3 assures that $\mathcal{C}_{\text{fin}}(A)$ is a down-set in $\mathcal{C}(A)$, it follows that C_1 is covered by C_2 in $\mathcal{C}_{\text{fin}}(A)$ if and only if C_1 is covered by C_2 in $\mathcal{C}(A)$. It now follows from Lemma 5.1.4 for each $C_1, C_2 \in \mathcal{C}(A)$ that C_2 covers C_1 if and only if $C_1 \subsetneq C_2$ and $\dim(C_2) = \dim(C_1) + 1$. Hence $\dim : \mathcal{C}_{\text{fin}}(A) \to \mathbb{N}$ is indeed a rank function. It is trivial that the dimension of A is an upper bound for the dimension of its subalgebras if A is finite-dimensional. ☐

Definition 5.1.6. Let X be a topological space with topology $\mathcal{O}(X)$. Then X is called *Noetherian* if the poset $\mathcal{O}(X)$ ordered by inclusion is Noetherian.

The next lemma is an easy exercise in [53] (Exercise I.1.7).

Lemma 5.1.7. Let X be a topological space. Then X is Noetherian if and only if every subset of X is compact. Moreover, if X is Noetherian and Hausdorff, then X must be finite.

Lemma 5.1.8. Let A be a finite-dimensional C*-algebra. Then $\mathcal{C}(A)$ is order scattered.

Proof. Since A is finite-dimensional, we have $\mathcal{C}(A) = \mathcal{C}_{\text{fin}}(A)$, hence it follows from Lemma 5.1.5 that there exists a unique rank function $d : \mathcal{C}(A) \to \mathbb{N}$. Since $d(C_1) < d(C_2)$ if $C_1 \subset C_2$ for each $C_1, C_2 \in \mathcal{C}(A)$, it follows that d is an order morphism. Let $\mathcal{C} \subseteq \mathcal{C}(A)$ be a chain. Then the restriction $d : \mathcal{C} \to \mathbb{N}$ is an order embedding. Indeed, let $C_1, C_2 \in \mathcal{C}$ such that $d(C_1) \leq d(C_2)$. Since $\mathcal{C}$ is a chain, we have either $C_1 \subset C_2$ or $C_2 \subset C_1$ or $C_1 = C_2$. If $C_2 \subset C_1$, then $d(C_2) < d(C_1)$ contradicting $d(C_1) \leq d(C_2)$. Hence we must have $C_1 \subseteq C_2$.

Now, if $\mathcal{C}$ is order dense chain of at least two points, it follows that $d[\mathcal{C}]$ is an order dense chain in $\mathbb{N}$, which is impossible since $\mathbb{N}$ is clearly order scattered. Hence $\mathcal{C}(A)$ cannot have order dense chains of at least two points, so $\mathcal{C}(A)$ is order scattered. ☐

Lemma 5.1.9. Let A be a C*-algebra such that A is not scattered. Then $\mathcal{C}(A)$ is not order scattered.

Proof. Assume that A is not scattered. By Theorem 2.3.4, there is some $C \in \mathcal{C}(A)$ that is *-isomorphic to $C([0,1])$. We first show that $\mathcal{C}(C([0,1]))$ contains an infinite order dense chain. Consider the subset

$$\mathcal{L} = \left\{ C_{[x,1]} : x \in [0,1) \right\}.$$

Notice that $[x,1]$ has at least two points for each $x \in [0,1)$. Hence it follows from Proposition 4.2.4 that for each $x, y \in [0,1)$ we have

$$x \leq y \iff [y,1] \subseteq [x,1] \iff C_{[x,1]} \subseteq C_{[y,1]}.$$

Thus, $x \mapsto C_{[x,1]}$ is an order embedding $[0,1) \to \mathcal{L}$, which is clearly surjective, hence an order isomorphism. Since $[0,1)$ is clearly order dense, it follows that $\mathcal{L}$ is order dense as well.

By Theorem 3.2.1, the *-isomorphism between C and $C([0,1])$ induces an order isomorphism between $\mathcal{C}(C([0,1]))$ and $\mathcal{C}(C) = \downarrow C$. Hence under this order isomorphism $\mathcal{L}$ corresponds to an order dense chain of $\downarrow C$, hence of $\mathcal{C}(A)$ for $\downarrow C$ is a down-set of $\mathcal{C}(A)$. We conclude that $\mathcal{C}(A)$ cannot be order scattered. □

Lemma 5.1.10. Let X be an infinite scattered compact Hausdorff space and let $A = C(X)$. Then $\mathcal{C}(A)$ is not ordered scattered.

Proof. We first note that X must contain an infinite number of isolated points. Indeed, if X has only finite isolated points, say $x_1, \ldots, x_n$, then $X \setminus \{x_1, \ldots, x_n\}$ is clopen, Since $X \setminus \{x_1, \ldots, x_n\}$ is closed, it contains an isolated point x_{n+1}, which is an isolated point of X, since $X \setminus \{x_1, \ldots, x_n\}$ is open, which gives a contradiction. Now we can choose a countable set Y of isolated points of X. Notice that Y is open, but Y cannot be closed, otherwise we obtain a contradiction with the compactness of X. Let Z be the topological boundary of Y, i.e., $Z = \overline{Y} \setminus Y$. Since Y is open, it follows that Z is closed. Moreover, if $S \subseteq Y$, then $Z \cup S$ is closed. Indeed, $Z \cup S = (\overline{Y} \setminus Y) \cup S = \overline{S} \setminus (Y \setminus S)$, and since $Y \setminus S$ consists only of isolated points, it is open, hence $Z \cup S$ is closed. Since Y is countable, we can label its elements by $\mathbb{Q}$. Hence, write $Y = \{x_q\}_{q \in \mathbb{Q}}$. For each $q \in \mathbb{Q}$, let

$$K_q = Z \cup \{x_r : r \in \downarrow q\},$$

and notice that K_q is closed and infinite. Since $q \mapsto \downarrow q$ is an order embedding $\mathbb{Q} \mapsto \mathcal{P}(\mathbb{Q})$ where $\mathbb{Q}$ is equipped with the usual order, it follows that the map $q \mapsto K_q$ is an order embedding of $\mathbb{Q}$ into $\mathcal{F}(X)$, the poset of all closed subsets of X of at least two points ordered by inclusion. Now Proposition 4.2.4 states that the map $K \mapsto C_K$ is an order embedding of $\mathcal{F}(X)^{\mathrm{op}}$ into $\mathcal{C}(A)$. Hence, we obtain an order embedding $\iota : \mathbb{Q}^{\mathrm{op}} \to \mathcal{C}(A)$. Since $\mathbb{Q}$ (and hence also $\mathbb{Q}^{\mathrm{op}}$) is clearly order dense, it follows that the image of ι is an order dense chain in $\mathcal{C}(A)$. We conclude that $\mathcal{C}(A)$ is not order scattered. □

Proof of Theorem 5.1.2. Assume that A is finite-dimensional. By Lemma 5.1.3, we have $\mathcal{C}(A) = \mathcal{C}_{\mathrm{fin}}(A)$. It follows from Lemma 5.1.5 that $\mathcal{C}(A)$ has a rank function whose range is bounded from above. By Lemma B.2.6, $\mathcal{C}(A)$ is both Artinian and Noetherian.

Assume that A is not finite-dimensional. By Proposition C.1.15, A has some maximal commutative C*-subalgebra M, which cannot be finite-dimensional by Proposition 2.1.3, it follows that M cannot be finite-dimensional. Since M is a commutative C*-algebra, the commutative Gelfand–Naimark Theorem assures that $M = C(X)$ for some compact Hausdorff space X, which must have an infinite number of points by Proposition 2.1.2.

We construct a descending chain in $\mathcal{C}(A)$ as follows. By the Axiom of Dependent Choice, we can find $\{x_1, x_2, x_3, \ldots\} \subseteq X$. Let

$$C_n = \{f \in C(X) : f(x_1) = \ldots = f(x_n)\}$$

for each $n \in \mathbb{N}$, i.e., $C_n = C_{\{x_1,\ldots,x_n\}}$ regarded as an ideal subalgebra. Clearly, we have $C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$. Assume that $i < j$. Then $\{x_1, \ldots, x_i\}$ and $\{x_j\}$ are disjoint closed sets, hence Urysohn's Lemma assures the existence of some $f \in C(X)$ such that $f[\{x_1, \ldots, x_i\}] = \{1\}$ and $f(x_j) = 0$. Clearly, $f \in C_i$, but $f \notin C_j$. This shows that $C_i \neq C_j$, so the chain is descending, but it never stabilizes.

We construct an ascending chain in $\mathcal{C}(A)$ as follows. First we notice that since X is infinite and Hausdorff, Lemma 5.1.7 implies that X is not Noetherian. So there is an ascending chain $O_1 \subseteq O_2 \subseteq \ldots$ of open subsets of X that does not stabilize. For each $i \in \mathbb{N}$, let $F_i = X \setminus O_i$. Then $F_1 \supseteq F_2 \supseteq \ldots$ is a descending chain of closed subsets of X, which does not stabilize. For each $i \in I$ let $C_i = C_{F_i}$. Then C_i is a C*-subalgebra of $C(X)$ and if $i \leq j$, we have $F_i \supseteq F_j$, so $C_i \subseteq C_j$. Moreover, if $i < j$ and $F_i \neq F_j$, then there is some $x \in F_i$ such that $x \notin F_j$. By Urysohn's Lemma, there is an $f \in C(X)$ such that $f(x) = 0$ and $f(y) = 1$ for each $y \in F_j$. Hence $f \in C_j$, but $f \notin C_i$. It follows that $C_i \neq C_j$, so $C_1 \subseteq C_2 \subseteq \ldots$ is an ascending chain that does not stabilize.

Thus $\mathcal{C}(A)$ contains an ascending chain as well as a descending chain, neither of which stabilizes, so $\mathcal{C}(A)$ can be neither Noetherian nor Artinian. Hence both the Noetherian and the Artinian condition on $\mathcal{C}(A)$ imply that A must be finite-dimensional.

Finally, we prove that A is finite-dimensional if and only if $\mathcal{C}(A)$ is order scattered. Firstly, it is proven in Lemma 5.1.8 that $\mathcal{C}(A)$ is order scattered if A is finite-dimensional. Assume that A is infinite-dimensional. If A is not scattered, it follows from Lemma 5.1.9 that $\mathcal{C}(A)$ is not order scattered. If A is scattered, let M be a maximal commutative C*-subalgebra of A, which has scattered spectrum X by Theorem 2.3.4. Moreover, since A is infinite-dimensional, it follows from Proposition 2.1.3 that M is infinite-dimensional, hence X cannot be finite. Since $M \cong C(X)$, it now follows from Lemma 5.1.10 that $\mathcal{C}(M)$ is not order scattered. Since $\mathcal{C}(M) = \downarrow M$ as subposet of $\mathcal{C}(A)$, it follows that neither $\mathcal{C}(A)$ is order scattered. $\square$

5.2 Decomposition of factors

Given C*-algebras $A_1, \ldots, A_n$ we consider the C*-sum $A = A_1 \oplus \ldots \oplus A_n$, which is, categorically speaking, the product of the A_i. We aim to investigate how to relate the structure of $\mathcal{C}(A)$ to the structures of $\mathcal{C}(A_1), \ldots, \mathcal{C}(A_n)$. It turns out that $\mathcal{C}$ does not preserve products. Nevertheless, if each A_i has trivial center, we will see that we can identify a subposet of $\mathcal{C}(A)$ that is order isomorphic to the product of the $\mathcal{C}(A_i)$.

Lemma 5.2.1. Let $A_1, \ldots, A_n$ be C*-algebras. Let $A = \bigoplus_{i=1}^n A_i$ and $C \in \mathcal{C}(A)$ such that $Z(A) \subseteq C$. Then there are $C_i \in \mathcal{C}(A_i)$ such that $Z(A_i) \subseteq C_i$ and $C = \bigoplus_{i=1}^n C_i$.

Proof. Let $\pi_i : A \to A_i$ be the projection on the i-th factor. Then we obtain an order morphism $\mathcal{C}(\pi_i) : \mathcal{C}(A) \to \mathcal{C}(A_i)$. Let $C_i = \mathcal{C}(\pi_i)(C)$, or equivalently, $C_i = \pi_i[C]$. Then

$$Z(A_i) = \pi_i \left[\bigoplus_{i=1}^n Z(A_i) \right] = \pi_i[Z(A)] \subseteq \pi_i[C] = C_i$$

for each $i \in I$. Let $c \in C$, then $\pi_i(c) \in C_i$ for each $i = 1, \ldots, n$, so $c = \pi_1(c) \oplus \ldots \oplus \pi_n(c)$. Thus $c \in \bigoplus_{i=1}^n C_i$, hence $C \subseteq \bigoplus_{i=1}^n C_i$. Let $c_1 \oplus \ldots \oplus c_n \in \bigoplus_{i=1}^n C_i$. This means that for each $i = 1, \ldots, n$ there is a $d^i \in C$ such that $\pi_i(d^i) = c_i$.

Here $d^i = d^i_1 \oplus \ldots \oplus d^i_n$, with $d^i_j \in A_j$, and in particular we have $d^i_i = c_i$. For each $j = 1, \ldots, n$, let $e^j \in A$ be the element $e^j_1 \oplus \ldots \oplus e^j_n$, with

$$e^j_i = \begin{cases} 1_{A_i} & i = j; \\ 0_{A_i} & i \neq j. \end{cases}$$

Here 1_{A_i} and 0_{A_i} denote the unit and the zero of A_i, respectively. Since $1_{A_i}, 0_{A_i} \in Z(A_i)$, Lemma C.1.28 assures that $e^j \in Z(A)$. Since $Z(A) \subseteq C$, we find that $e^j \in C$. It follows that $f^j = e^j d^j \in C$. Here $f^j = f^j_1 \oplus \ldots \oplus f^j_n$ with

$$f^j_i = \begin{cases} c_i & i = j; \\ 0_{A_i} & i \neq j. \end{cases}$$

Now,

$$f^1 + \ldots + f^n = c_1 \oplus \ldots \oplus c_n = c,$$

and since $f^j \in C$, it follows that $c \in C$. So $C = \bigoplus_{i=1}^n C_i$. □

Proposition 5.2.2. Let $A_1, \ldots, A_n$ be C*-algebras and let $A = \bigoplus_{i=1}^n A_i$. Let

$$\Gamma : \prod_{i=1}^n \mathcal{C}(A_i) \to \mathcal{C}(A)$$

be given by $(C_1, \ldots, C_n) \mapsto C_1 \oplus \ldots \oplus C_n$, and let

$$\Delta : \mathcal{C}(A) \mapsto \prod_{i=1}^n \mathcal{C}(A_i)$$

be the map $\mathcal{C}(\pi_1) \times \ldots \times \mathcal{C}(\pi_n)$, where $\pi_i : A \to A_i$ denotes the projection on the i-th factor. Thus Δ maps $C \in \mathcal{C}(A)$ to $(\pi_1[C], \ldots, \pi_n[C])$. Then:

(i) Γ is an embedding of posets;

(ii) Δ is surjective;

(iii) $\Delta \circ \Gamma = 1_{\prod_{i=1}^n \mathcal{C}(A_i)}$ and $1_{\mathcal{C}(A)} \leq \Gamma \circ \Delta$;

(iv) the restriction of Γ to a map $\prod_{i=1}^n \uparrow Z(A_i) \to \uparrow Z(A)$ is an order isomorphism with inverse Δ.

Proof. For each $i = 1, \ldots, n$, let $C_i \in \mathcal{C}(A_i)$. Then $C_1 \oplus \ldots \oplus C_n$ is clearly a commutative C*-subalgebra of A, where $1_{A_1} \oplus \ldots \oplus 1_{A_n} = 1_A$. Hence the image of Γ lies in $\mathcal{C}(A)$, so Γ is well defined. Furthermore, $\mathcal{C}(\pi_i) : \mathcal{C}(A) \to \mathcal{C}(A_i)$ is an order morphism by Theorem 3.2.1.

(i) Let $(C_1, \ldots, C_n)$ and $(D_1, \ldots, D_n)$ be elements of $\prod_{i=1}^n \mathcal{C}(A_i)$. Then $(C_1, \ldots, C_n) \leq (D_1, \ldots, D_n)$ implies $C_i \subseteq D_i$ for each $i = 1, \ldots, n$. Hence $C_1 \oplus \ldots \oplus C_n \subseteq D_1 \oplus \ldots \oplus D_n$, which says that

$$\Gamma((C_1, \ldots, C_n)) \subseteq \Gamma((D_1, \ldots, D_n)).$$

Conversely, if $\Gamma((C_1, \ldots, C_n)) \subseteq \Gamma((D_1, \ldots, D_n))$, we have

$$C_1 \oplus \ldots \oplus C_n \subseteq D_1 \oplus \ldots \oplus D_n.$$

If we let act $\mathcal{C}(\pi_i)$ on this inclusion, we obtain $C_i \subseteq D_i$ for each i in $\{1, \ldots, n\}$. Hence $(C_1, \ldots, C_n) \leq (D_1, \ldots, D_n)$. Thus Γ is an embedding of posets.

(ii) Let $(C_1, \ldots, C_n) \in \prod_{i=1}^n \mathcal{C}(A_i)$. If $C = C_1 \oplus \ldots \oplus C_n$, then $C \in \mathcal{C}(A)$ and

$$\Delta(C) = (\mathcal{C}(\pi_1)(C), \ldots, \mathcal{C}(\pi_n)(C)) = (\pi_1[C], \ldots, \pi_n[C]) = (C_1, \ldots, C_n).$$

(iii) Let $(C_1, \ldots, C_n) \in \prod_{i=1}^n \mathcal{C}(A_i)$. Let $C = \Gamma((C_1, \ldots, C_n))$. Then $C = C_1 \oplus \ldots \oplus C_n$, and by the calculation in (ii), we obtain $\Delta(C) = (C_1, \ldots, C_n)$. Hence $\Delta \circ \Gamma = 1_{\prod_{i=1}^n \mathcal{C}(A_i)}$.

Let $C \in \mathcal{C}(A)$. Then

$$\Gamma \circ \Delta(C) = \Gamma((\pi_1[C], \ldots, \pi_n[C])) = \pi_1[C] \oplus \ldots \oplus \pi_n[C].$$

Let $c \in C$. Since $C \subseteq A$, and $A = \bigoplus_{i=1}^n A_i$, we have $c = c_1 \oplus \ldots \oplus c_n$, with $c_i \in A_i$ for each $i = 1, \ldots, n$. Hence $c_i = \pi_i(c)$, and we find $c = \pi_1(c) \oplus \ldots \oplus \pi_n(c)$, so $c \in \pi_1[C] \oplus \ldots \oplus \pi_n[C]$. But

$$\pi_1[C] \oplus \ldots \oplus \pi_n[C] = \Gamma((\pi_1[C], \ldots, \pi_n[C])) = \Gamma \circ \Delta(C).$$

Hence $c \in \Gamma \circ \Delta(C)$, so $C \subseteq \Gamma \circ \Delta(C)$. We conclude that $1_{\mathcal{C}(A)} \leq \Gamma \circ \Delta$.

(iv) In order to show that Γ restricts to an order isomorphism

$$\prod_{i=1}^{n} \uparrow Z(A_i) \to \uparrow Z(A)$$

with inverse Δ, it is enough to show that $\Gamma \circ \Delta(C) = C$ for each C in $\uparrow Z(A)$. Then the statement follows directly from the equality in (iii). So let $C \in \mathcal{C}(A)$ such that $Z(A) \subseteq C$. Then Lemma 5.2.1 assures that there are $C_i \in \mathcal{C}(A_i)$ for each $i = 1, \ldots, n$ such that $C = C_1 \oplus \ldots \oplus C_n$ and $Z(A_i) \subseteq C_i$ for each $i = 1, \ldots, n$. Then $\pi_i[C] = C_i$, hence

$$\Gamma \circ \Delta(C) = \Gamma((\pi_1[C], \ldots, \pi_n[C])) = \Gamma((C_1, \ldots, C_n)) = C_1 \oplus \ldots \oplus C_n = C.$$

□

Proposition 5.2.3. Let $A = \bigoplus_{i=1}^{k} \mathrm{M}_{n_i}(\mathbb{C})$, where $k, n_1, \ldots, n_k \in \mathbb{N}$. Then $[Z(A), M] \cong \prod_{i=1}^{k} \mathcal{C}(\mathbb{C}^{n_i})$ for each $M \in \max \mathcal{C}(A)$.

Proof. By Proposition 5.2.2, there is an order morphism

$$\Delta : \mathcal{C}(A) \to \prod_{i=1}^{j} \mathcal{C}(\mathrm{M}_{n_i}(\mathbb{C}))$$

whose restriction to $\uparrow Z(A)$ is an order isomorphism with inverse Γ. Let M in $\max \mathcal{C}(A)$. By Theorem 3.1.3, it follows that $Z(A) \subseteq M$, so $M \in \uparrow Z(A)$, hence $M \in \max \uparrow Z(A)$. Proposition B.1.15 assures that $\Delta(M)$ is a maximal element of $\prod_{i=1}^{k} \mathcal{C}(\mathrm{M}_{n_i}(\mathbb{C}))$. By Lemma B.1.19 we find that there are M_{n_i} in $\max \mathcal{C}(\mathrm{M}_{n_i}(\mathbb{C}))$ for each $i = 1, \ldots, k$ such that $\Delta(M) = M_{n_1} \times \ldots \times M_{n_k}$ and $\downarrow \Delta(M) = \downarrow M_{n_1} \times \ldots \times \downarrow M_{n_k}$. Since Γ is the inverse of Δ and has codomain $\uparrow Z(A)$, we find that

$$\Gamma[\downarrow \Delta(M)] = \downarrow M \cap \uparrow Z(A) = [Z(A), M].$$

Hence the restriction $\Gamma : \downarrow M_{n_1} \times \ldots \times \downarrow M_{n_k} \to [Z(A), M]$ is an order isomorphism. Notice that all maximal elements of $\mathcal{C}(\mathrm{M}_{n_i}(\mathbb{C}))$ are *-isomorphic by Proposition 2.1.4. More specifically, $M_{n_i} \in \max \mathcal{C}(\mathrm{M}_{m_i}(\mathbb{C}))$ is *-isomorphic to D_{n_i}. Since

$$D_{n_i} = \{\mathrm{diag}(\lambda_1, \ldots, \lambda_{n_i})\},$$

we find that M_{n_i} is *-isomorphic to $\mathbb{C}^{n_i}$. Hence there is an embedding $f : \mathbb{C}^{n_i} \to \mathrm{M}_{n_i}(\mathbb{C})$ such that $f[C^{n_i}] = M_{n_i}$. By Theorem 3.2.1, we find that

$$\mathcal{C}(f) : \mathcal{C}(\mathbb{C}^{n_i}) \to \mathcal{C}(\mathrm{M}_{n_i}(\mathbb{C}))$$

is an order embedding with image $\downarrow M_{n_i}$. We conclude that $\mathcal{C}(\mathbb{C}^{n_i}) \cong \downarrow M_{n_i}$ in $\mathcal{C}(\mathrm{M}_{n_i}(\mathbb{C}))$. Hence $[Z(A), M] \cong \prod_{i=1}^{k} \mathcal{C}(\mathbb{C}^{n_i})$. ☐

Proposition 5.2.4. Let A be a commutative finite-dimensional C*-algebra. If $\dim A = 1$, then $\mathcal{C}(A)$ is the one-point lattice. If $\dim A \geq 2$, then $\mathcal{C}(A)$ is a directly indecomposable lattice (cf. Definition B.3.1).

Proof. By Proposition 4.1.1, $\mathcal{C}(A)$ is a bounded lattice. Let X be the spectrum of A. If $\dim A = 1$, then $\mathcal{C}(A) = \{\mathbb{C}1_A\}$, so $\mathcal{C}(A) = \mathbf{1}$, the one-point lattice. If $\dim A = 2$, then $\mathcal{C}(A) = \{A, \mathbb{C}1_A\}$. So $\mathcal{C}(A)$ contains no other elements than a greatest and a least one, and is therefore certainly directly indecomposable. Let $\dim A \geq 3$ and let X be the Gelfand spectrum of A. Then X has at least three points (see for instance Proposition 2.1.2). Let $B \in \mathcal{C}(A)$, assumed not equal to $\mathbb{C}1_A$ or A. By Corollary 4.4.10, we have

$$B = \bigcap_{x \in X} C_{[x]_B}.$$

Since X is finite, it follows that $X/\sim_B$ is finite as well. Notice that we cannot have $[x]_B = \{x\}$ for all $x \in X$, otherwise $B = C(X) = A$. Neither can $X/\sim_B$ be a singleton set, since otherwise $B = \mathbb{C}1_A$. For each element $[x]_B$ in $X/\sim_B$, choose a representative x. Let K be the set of representatives. Notice that K is not a singleton set, since $X/\sim_B$ contains at least two elements. Also notice that K is not unique, since there is at least one $[x]_B \in X/\sim_B$ containing two or more points. Since X is discrete, it follows that K is closed. Let $f \in B \cap C_K$ and let $x, y \in X$ be points such that $x \neq y$. If $[x]_B = [y]_B$, then $f(x) = f(y)$. If $[x]_B \neq [y]_B$, then there are $x', y' \in K$ such that $x' \in [x]_B$ and $y' \in [y]_B$. Since $f \in C_K$, we find that $f(x') = f(y')$. Since $f \in B$, we obtain $f(x) = f(x')$ and $f(y) = f(y')$. Combining all equalities gives $f(x) = f(y)$. So in all cases, $f(x) = f(y)$. So f must be constant, and we conclude that $B \cap C_K = \mathbb{C}1_A$.

Since $\mathcal{C}(A)$ is a lattice, $B \vee C_K$ exists. Let $f \in C(X)$. Define $g : X \to \mathbb{C}$ by $g(x) = f(k)$ if $x \in [k]_B$, where $k \in K$. Notice that g is well defined, since K is a collection of representatives. Moreover, since X is discrete, g is continuous, so $g \in C(X)$. By definition, we have $g \in \bigcap_{x \in X} C_{[x]_B}$, so $g \in B$. Let $h = f - g$.

Then $h \in C(X)$, and if $k \in K$, we find $h(k) = f(k) - g(k) = 0$, so h is constant on K. We conclude that $f = g + h$ with $g \in B$ and $h \in C_K$. Hence $A = C(X) = B \vee C_K$. We find that C_K is a complement of B. However, K is not unique, and therefore neither is C_K. We conclude that A and $\mathbb{C}1_A$ are the only elements with a unique complement, so $\mathcal{C}(A)$ is indirectly indecomposable. □

The proof of this proposition is based on the proof of the directly indecomposability of partition lattices in [96]. More can be said about $\mathcal{C}(A)$ when A is a commutative C*-algebra of dimension n, namely that $\mathcal{C}(A)$ is order isomorphic to the lattice of partitions of the set $\{1, \dots, n\}$. We refer to [58] for a complete characterization of $\mathcal{C}(A)$ when A is a commutative finite-dimensional C*-algebra.

5.3 Commutative C*-subalgebras determine finite-dimensional C*-algebras

We arrive at the main result in this chapter, which states that $\mathcal{C}(A)$ determines each finite-dimensional C*-algebra A up to isomorphism.

Theorem 5.3.1. Let A be a finite-dimensional C*-algebra and B any C*-algebra such that $\mathcal{C}(A) \cong \mathcal{C}(B)$. Then $A \cong B$.

Proof. Let A be a finite-dimensional C*-algebra, and B a C*-algebra. Let $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$ an order isomorphism. By Theorem 5.1.2, $\mathcal{C}(A)$ is Noetherian. Let $\mathcal{D} \subseteq \mathcal{C}(B)$ be a non-empty subset. Then $\Phi^{-1}[\mathcal{D}] \subseteq \mathcal{C}(A)$ is non-empty, and must have a maximal element M by the Noetherian property. By Proposition B.1.15, it follows that $\Phi(M)$ is a maximal element of $\mathcal{D}$, so $\mathcal{C}(B)$ is Noetherian as well. Hence Theorem 5.1.2 implies that B is finite-dimensional. It follows from Lemma 5.1.5 that both $\mathcal{C}(A)$ and $\mathcal{C}(B)$ have a rank function assigning to each element its dimension. By Lemma B.2.7 the rank function is unique, hence it follows from Lemma B.2.8 that $\dim(\Phi(C)) = \dim(C)$ for each $C \in \mathcal{C}(A)$. Therefore, we can reconstruct the dimensions of elements of $\mathcal{C}(A)$ and $\mathcal{C}(B)$, and the dimension is preverved by Φ. By the Artin-Wedderburn Theorem, there are unique $k, k' \in \mathbb{N}$ and unique $\{n_i\}_{i=1}^{n}, \{n'_i\}_{i=1}^{k'}$ with $n_i, n'_i \in \mathbb{N}$ such that

$$\begin{aligned} A &\cong \bigoplus_{i=1}^{k} \mathrm{M}_{n_i}(\mathbb{C}); \\ B &\cong \bigoplus_{i=1}^{k'} \mathrm{M}_{n'_i}(\mathbb{C}). \end{aligned}$$

Without loss of generality, we may assume that the n_i and n'_i form an descending (but not necessarily strictly descending) finite sequence.

By Theorem 3.1.3, we have $Z(A) = \bigcap \max \mathcal{C}(A)$ and $\bigcap \max \mathcal{C}(B) = Z(B)$. Since the intersection is the infimum operation in $\mathcal{C}(A)$ and $\mathcal{C}(B)$, and order isomorphisms preserve both infima and maximal elements, we find that $\Phi(Z(A)) = Z(B)$, so $\dim(Z(A)) = \dim(Z(B))$. Using Lemma C.1.28, we find that

$$Z(A) = \bigoplus_{i=1}^{n} Z(\mathrm{M}_{n_i}(\mathbb{C})),$$

and since the dimension of the center of a matrix algebra is 1, we find that $\dim Z(A) = k$. In the same way, we find that $\dim Z(B) = k'$, so we must have $k = k'$. Let $M \in \max \mathcal{C}(A)$. By Proposition B.1.15 it follows that $\Phi(M)$ is a maximal element of $\mathcal{C}(B)$, and since $\Phi(Z(A)) = Z(B)$, we find that Φ restricts to an order isomorphism $[Z(A), M] \to [Z(B), \Phi(M)]$. By Proposition 5.2.3, we obtain an order isomorphism

$$\prod_{i=1}^{k} \mathcal{C}(\mathbb{C}^{n_i}) \cong \prod_{i=1}^{k} \mathcal{C}(\mathbb{C}^{n'_i}).$$

It is possible that for some i we have $n_i = 1$, in which case it follows from Proposition 5.2.4 that $\mathcal{C}(\mathbb{C}^{n_i}) = \mathbf{1}$. Since we assumed that $\{n_i\}_{i=1}^{n}$ is a descending sequence, there is a greatest number r below k such that $n_r \neq 1$. Likewise, let s be the greatest number such that $n'_s \neq 1$. Then we obtain an order isomorphism

$$\prod_{i=1}^{r} \mathcal{C}(\mathbb{C}^{n_i}) \cong \prod_{i=1}^{s} \mathcal{C}(\mathbb{C}^{n'_i}).$$

By Proposition 5.2.4 and Corollary B.3.3, we now find $r = s$, and there is a permutation $\pi : \{1, \ldots, r\} \to \{1, \ldots, r\}$ such that $\mathcal{C}(\mathbb{C}^{n_i}) \cong \mathcal{C}(\mathbb{C}^{n'_{\pi(i)}})$ for each $i \in \{1, \ldots, r\}$. Let $\Psi_i : \mathcal{C}(\mathbb{C}^{n_i}) \to \mathcal{C}(\mathbb{C}^{n'_{\pi(i)}})$ be the accompanying order isomorphism. Lemma 5.1.5 assures that the function assigning to each element of $\mathcal{C}(\mathbb{C}^{n_i})$ its dimension, is a rank function, and similarly the dimension function is a rank function for $\mathcal{C}(\mathbb{C}^{n'_{\pi(i)}})$. By Lemma B.2.8, we find that $\dim(C) = \dim(\Psi_i(C))$ for each $C \in \mathcal{C}(\mathbb{C}^{n_i})$. Hence

$$n_i = \dim(\mathbb{C}^{n_i}) = \dim(\Psi_i(\mathbb{C}^{n_i})) = \dim(\mathbb{C}^{n'_{\pi(i)}}) = n'_{\pi(i)},$$

where we used Proposition B.1.15 in the third equality. By definition of r, we must have $n_i = n'_i = 1$ for all $i \geq r$. Hence we can extend π to a permutation $\{1, \ldots, k\} \to \{1, \ldots, k\}$ by setting $\pi(i) = i$ for each $i \geq r$. Hence $k = k'$ and $\{n_1, \ldots, n_k\}$ and $\{n'_1, \ldots, n'_k\}$ are the same sets up to permutation. We conclude that A and B must be *-isomorphic. □

If A is a finite-dimensional C*-algebra and B is a C*-algebra such that there is an order isomorphism $\Psi : \mathcal{C}(A) \to \mathcal{C}(B)$, then there it might be the case that even though A and B are *-isomorphic, we have $\Psi \neq \mathcal{C}(f)$ for each *-isomorphism $f : A \to B$. For instance, let $A = B = \mathrm{M}_2(\mathbb{C})$. Then

$$\mathcal{C}(\mathrm{M}_2(\mathbb{C})) = \{\mathbb{C}1_{\mathrm{M}_2(\mathbb{C})}\} \cup \{uD_2u^* : u \in \mathrm{U}(2)\},$$

where $D_2 = \{\mathrm{diag}(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 \in \mathbb{C}\}$. Furthermore, it follows from [2, Theorem 4.27] that for each *-isomorphism $\varphi : \mathrm{M}_2(\mathbb{C}) \to \mathrm{M}_2(\mathbb{C})$, there is some $u \in \mathrm{U}(2)$ such that $\varphi(x) = uxu^*$ for each $x \in \mathrm{M}_2(\mathbb{C})$. Hence

$$\mathcal{C}(\varphi) : \mathcal{C}(\mathrm{M}_2(\mathbb{C})) \to \mathcal{C}(\mathrm{M}_2(\mathbb{C}))$$

is given by $C \mapsto uCu^*$.

Choose $v \in \mathrm{U}(2)$ such that $D_2 \neq vD_2v^*$ and define $\Psi : \mathcal{C}(\mathrm{M}_2(\mathbb{C})) \to \mathcal{C}(\mathrm{M}_2(\mathbb{C}))$ by $\Psi(D_2) = vD_2v^*$, $\Psi(vD_2v^*) = D_2$ and $\Psi(C) = C$ for all other $C \in \mathcal{C}(\mathrm{M}_2(\mathbb{C}))$. Then Ψ is clearly an order isomorphism. However, $\Psi \neq \mathcal{C}(\psi)$ for any *-isomorphism $\psi : \mathrm{M}_2(\mathbb{C}) \to \mathrm{M}_2(\mathbb{C})$.

6 Recovering orthomodular posets of projections

In this chapter we give two ways of reconstructing the (orthomodular) poset $\text{Proj}(A)$ of projections in a C*-algebra A from $\mathcal{C}(A)$. In the first case, we consider an order isomorphism $\mathcal{C}(A) \to \mathcal{C}(B)$ and show that it induces an orthomodular isomorphism $\text{Proj}(A) \to \text{Proj}(B)$. In order to obtain this result, in the first section we introduce the poset $\mathcal{B}(P)$ of all Boolean subalgebras of an orthomodular poset P. In many ways, this poset is similar to $\mathcal{C}(A)$, for instance it is a complete semilattice, but it has the advantage of determining P completely, as follows from the Harding–Navara Theorem, which we will state in §6.3. In §6.2, we consider the poset $\mathcal{C}_{\text{AF}}(A)$ of commutative AF-subalgebras of A. Furthermore, we show that it is order isomorphic to $\mathcal{B}(\text{Proj}(A))$. In §6.4, we use this order isomorphism and the Harding–Navara Theorem in order to show the existence of an orthomodular isomorphism $\text{Proj}(A) \to \text{Proj}(B)$.

The last two sections are devoted to a direct reconstruction of $\text{Proj}(A)$ from $\mathcal{C}(A)$. In order words, we construct an orthomodular poset from $\mathcal{C}(A)$ that is isomorphic to $\text{Proj}(A)$ as an orthomodular poset. A slight drawback of this method is that we have to put constraints on A, namely we have to assume that its center is at least three-dimensional. §6.5 deals with the commutative case, which is used in §6.6 for the non-commutative case.

6.1 Posets of Boolean subalgebras

We start by defining the poset of Boolean subalgebras of a given orthomodular poset. We refer to Appendix B.4 for the preliminaries on orthomodular posets and Boolean algebras.

Definition 6.1.1. Let P be an orthomodular poset. We denote the poset of all its Boolean subalgebras of P, ordered by inclusion, by $\mathcal{B}(P)$.

The next theorem can be seen as an analogue of Theorem 3.1.3 for $\mathcal{B}(P)$.

Proposition 6.1.2. Let P be an orthomodular poset. Then:

(a) $\mathcal{B}(P)$ has a least element $\{0,1\}$;

(b) $\mathcal{B}(P)$ is atomistic (cf. Definition B.1.8), and the atoms of $\mathcal{B}(P)$ are exactly the Boolean subalgebras generated by elements in $P \setminus \{0,1\}$, i.e., the algebras $\langle p \rangle = \{0, p, p^\perp, 1\}$, where $p \in P \setminus \{0,1\}$;

(c) The supremum $\bigvee \mathcal{S}$ of a subset $\mathcal{S} \subseteq \mathcal{B}(P)$ exists if and only all elements of $\bigcup \mathcal{S}$ commute, in which case $\bigvee \mathcal{S} = \langle \bigcup \mathcal{S} \rangle$;

(d) $\mathcal{B}(P)$ is an algebraic complete semilattice (cf. Definition B.6.1 and Definition B.6.4), in which:

- $\bigwedge \mathcal{S} = \bigcap \mathcal{S}$ for each non-empty $\mathcal{S} \subseteq \mathcal{B}(P)$;
- $\bigvee \mathcal{D} = \bigcup \mathcal{D}$ for each directed $\mathcal{D} \subseteq \mathcal{B}(P)$;
- the compact elements are exactly the finite Boolean subalgebras of P.

Proof. For a proof of (a) and (b), we refer to [52, Proposition 2.3]. For (c), let $\{B_i\}_{i \in I}$ be a collection of Boolean subalgebras of P such that $\bigcup_{i \in I} B_i$ consists of mutually commuting elements. It follows from Lemma B.4.23 that $\langle \bigcup_{i \in I} B_i \rangle$, which is clearly the supremum of the B_i, exists. Conversely, if $\bigvee_{i \in I} B_i$ exists in $\mathcal{B}(P)$, then it contains B_i for each $i \in I$, hence $\bigcup_{i \in I} B_i \subseteq \bigvee_{i \in I} B_i$. Since the right-hand side of this inclusion consists of mutually commuting elements, so does the left-hand side.

Finally, we prove (d). It follows from Lemma B.4.22 that $\mathcal{B}(P)$ has all non-empty infima given by the intersection operator. Let $\mathcal{D} \subseteq \mathcal{B}(P)$ be directed. Let $p, q \in \bigcup \mathcal{D}$. Then $p \in D_1$ and $q \in D_2$ for some $D_1, D_2 \in \mathcal{D}$, hence there must be some $D \in \mathcal{D}$ such that $p, q \in D$. Since D is a Boolean subalgebra of P, it follows that p and q commute, that their meet and join exist and are contained in D, and that $p^\perp \in D$. It follows that that $\bigcup \mathcal{D}$ is a Boolean subalgebra of P, which is clearly the supremum of $\mathcal{D}$.

Let B be any Boolean subalgebra of P, and let $\mathcal{D}$ be the set of all finite Boolean subalgebras of B. Then $\mathcal{D}$ is directed, since if $D_1, D_2 \in \mathcal{D}$, then $D_1 \cup D_2$ is finite and consists of mutually commuting elements, hence Proposition B.4.25 assures that the Boolean subalgebra $\langle D_1 \cup D_2 \rangle$ of B generated by $D_1 \cup D_2$ is finite. Thus $D_1, D_2 \subseteq \langle D_1 \cup D_2 \rangle$ and $\langle D_1 \cup D_2 \rangle \in \mathcal{D}$. By the same proposition, $\langle p \rangle$ is a finite Boolean subalgebra of B for each $p \in P$, so $\langle p \rangle \in \mathcal{D}$, whence $B = \bigcup \mathcal{D} = \bigvee \mathcal{D}$. Hence each element B of $\mathcal{B}(P)$ is de directed supremum of all finite Boolean subalgebras of B.

In particular, if B is compact, it follows that B must be contained in some finite Boolean subalgebra of B, which clearly implies that B must be a finite Boolean subalgebra of P. Conversely, if B is a finite Boolean subalgebra of P, let $\mathcal{D} \subseteq \mathcal{B}(P)$ be directed such that $B \subseteq \bigvee \mathcal{D}$. Write $B = \{b_1, \ldots, b_n\}$. Since $\bigvee \mathcal{D} = \bigcup \mathcal{D}$, we can find a $D_i \in \mathcal{D}$ such that $b_i \in D_i$ for each $i \in \{1, \ldots, n\}$.

Since $\mathcal{D}$ is directed, there is some $D \in \mathcal{D}$ such that $D_1, \ldots, D_n \subseteq D$. Hence $B \subseteq D$, so B is compact.

We conclude that the compact elements are exactly the finite Boolean subalgebras of P, and that every element of $\mathcal{B}(P)$ can be written as a directed supremum of finite Boolean subalgebras of P. Hence $\mathcal{B}(P)$ is algebraic. □

Algebraicity of $\mathcal{B}(P)$ is no surprise. If P is a Boolean algebra, then $\mathcal{B}(P)$ becomes the lattice of Boolean subalgebras of P, whose algebraicity is proven in [43]. The orthomodular case is only a slight generalization of this Boolean case.

Corollary 6.1.3. Let P be an orthomodular poset. Then $\mathcal{B}(P)$ is a zero-dimensional compact Hausdorff space in the Lawson topology (cf. Definition B.6.9).

Proof. This follows directly from Proposition 6.1.2 and Theorem B.6.10. □

Corollary 6.1.4. Let P be an orthomodular poset and let B be the collection of all Lawson clopen subsets of $\mathcal{B}(P)$ ordered by inclusion. Then B is a Boolean algebra.

Proof. This follows from the standard fact that the clopen sets of any topological space from a Boolean algebra. □

The next lemma implicitly uses [70, Lemma 1.3.3].

Lemma 6.1.5. Let P and Q be orthomodular posets and $\varphi : P \to Q$ an orthomodular morphism. If $B \subseteq Q$ is a Boolean subalgebra, then:

- $\varphi^{-1}[B]$ is non-empty, and is closed under orthocomplementation, existing binary joins, and existing binary meets;
- If P is a Boolean algebra, or if φ is injective and P an orthomodular lattice, then $\varphi^{-1}[B]$ is a Boolean subalgebra of P.

Proof. We have $\varphi(0) = 0$, hence $0 \in \varphi^{-1}[B]$. Let $x, y \in \varphi^{-1}[B]$. By Proposition B.4.8, φ preserves existing binary meets and existing binary joins. Hence if $x \vee y$ exists, we have $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$, and since B is closed under binary joins, we obtain $x \vee y \in \varphi^{-1}[B]$. In a similar way, we find that the existence of $x \wedge y$ implies that $x \wedge y \in \varphi^{-1}[B]$. It follows from $\varphi(x^\perp) = \varphi(x)^\perp$ and the fact that B is closed under orthocomplementation that $x^\perp \in \varphi^{-1}[B]$.

If P is a Boolean algebra, then $\varphi^{-1}[B]$ consists of mutually commuting elements. We claim that $\varphi^{-1}[B]$ also consists of mutually commuting elements if P is an orthomodular lattice and φ is injective. So assume that P is an orthomodular lattice and φ is injective. Let $x, y \in \varphi^{-1}[B]$. Then their meet and join exists[5], hence

$$\varphi\big(x \wedge (x^\perp \vee y)\big) = \varphi(x) \wedge \big(\varphi(x)^\perp \vee \varphi(y)\big) = \big(\varphi(x) \wedge \varphi(p)^\perp\big) \vee \big(\varphi(x) \wedge \varphi(y) = \varphi(x \wedge y),$$

where we used Proposition B.4.8 in the first and last equalities, and the distributivity of B in the second equality.

It now follows from the injectivity of φ that

$$x \wedge (x^\perp \vee y) = x \wedge y.$$

Let $p = x$, $q = y^\perp$ and $r = x \wedge y^\perp$. Then $r \leq p, q$ and

$$p \wedge r^\perp = x \wedge (x \wedge y^\perp)^\perp = x \wedge (x^\perp \vee y) = x \wedge \leq y = q^\perp,$$

hence Lemma B.4.14 implies that $p = x$ and $q = y^\perp$ commute. It now follows from Lemma B.4.5 that p and q commute as well. We conclude in both cases that $\varphi^{-1}[B]$ is a Boolean subalgebra, since it is a subset of P closed under orthocomplementation, existing binary meets, and existing binary joins, and consists of mutually commuting elements. □

The next theorem is an almost complete analogue of Theorem 3.2.1 for $\mathcal{B}(P)$, except statements (c) and (d), where in some cases we need to assume that P is an orthomodular lattice.

Theorem 6.1.6. $\mathcal{B}$: **OMP** $\to$ **DCPO** becomes a functor if, for each orthomodular morphism $\varphi : P \to Q$ between orthomodular posets P and Q, we define $\mathcal{B}(\varphi) : \mathcal{B}(P) \to \mathcal{B}(Q)$ by $B \mapsto \varphi[B]$. Moreover, $\mathcal{B}(\varphi)$ has the following properties:

(a) If $S \subseteq P$ is a set of mutually commuting elements, then

$$\mathcal{B}(\varphi)(\langle S \rangle) = \langle \varphi[S] \rangle.$$

[5] For this reason, we assumed P to be an orthomodular lattice rather than an orthomodular poset.

(b) If $\{B_i\}_{i\in I} \subseteq \mathcal{B}(P)$ is a family such that $\bigvee_{i\in I} B_i$ exists, then $\bigvee_{i\in I} \mathcal{B}(\varphi)(B_i)$ exists, and

$$\mathcal{B}(\varphi)\left(\bigvee_{i\in I} B_i\right) = \bigvee_{i\in I} \mathcal{B}(\varphi)(B_i). \tag{20}$$

In particular, $\mathcal{B}(\varphi)$ is Scott continuous.

(c) If P is a Boolean algebra, or if φ is injective and P is an orthomodular lattice, then $\mathcal{B}(\varphi)$ has an upper adjoint $\mathcal{B}(\varphi)_* : \mathcal{B}(Q) \to \mathcal{B}(P)$, $D \mapsto \varphi^{-1}[D]$, which satisfies

$$\mathcal{B}(\varphi)_*\left(\bigcap_{j\in J} D_j\right) = \bigcap_{j\in J} \mathcal{B}(\varphi)_*(D_j) \tag{21}$$

for each family $\{D_i\}_{i\in I} \subseteq \mathcal{B}(Q)$ such that $I \neq \emptyset$.

(d) If φ is injective, in particular if φ is an order embedding, then $\mathcal{B}(\varphi)$ is an order embedding such that

$$\downarrow \mathcal{B}(\varphi)[\mathcal{B}(P)] = \mathcal{B}(\varphi)[\mathcal{B}(P)],$$

and

$$\mathcal{B}(\varphi)\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} \mathcal{B}(\varphi)(B_i) \tag{22}$$

for each family $\{B_i\}_{i\in I} \subseteq \mathcal{B}(P)$ such that $I \neq \emptyset$. Moreover, if P is an orthomodular lattice, the following identities hold:

$$\begin{aligned} \mathcal{B}(\varphi)_* \circ \mathcal{B}(\varphi) &= 1_{\mathcal{B}(P)}; \\ \mathcal{B}(\varphi) \circ \mathcal{B}(\varphi)_*|_{\mathcal{B}(\varphi)[\mathcal{B}(P)]} &= 1_{\mathcal{B}(Q)}\big|_{\mathcal{B}(\varphi)[\mathcal{B}(P)]}. \end{aligned}$$

(e) If $\mathcal{B}(\varphi)$ is surjective, then φ is surjective.

(f) If φ is an orthomodular isomorphism, then $\mathcal{B}(\varphi)$ is an order isomorphism.

Proof. Let $\varphi : P \to Q$ be an orthomodular morphism and let B a Boolean subalgebra of P. Then $0 \in \varphi[B]$. Moreover, Proposition B.4.8 assures that $\varphi[B]$ is closed under orthocomplementation, consists of mutually commuting elements, and contains 1. By Lemma B.4.5, all binary meets and joins exist in $\varphi[B]$. Since φ preserves binary meets and binary joins, again by Proposition B.4.8, it follows that $\varphi[B]$ is closed under binary meets and binary joins. By Definition B.4.17, $\varphi[B]$ is a Boolean subalgebra of Q. Clearly $B_1 \subseteq B_2$ in $\mathcal{B}(P)$ implies $\varphi[B_1] \subseteq \varphi[B_2]$, hence $\mathcal{B}(\varphi)$ is an order morphism. It remains to prove that $\mathcal{B}(\varphi)$ is Scott continuous, which follows from (b). We prove properties (a)-(f):

(a) Since $\varphi[S] \subseteq \varphi[\langle S \rangle] = \mathcal{B}(\varphi)(\langle S \rangle)$, we have $\langle \varphi[S] \rangle \subseteq \mathcal{B}(\varphi)(\langle S \rangle)$. Let $B = \langle \varphi[S] \rangle$. Since $S \subseteq \varphi^{-1}[\varphi[S]]$ and $\varphi[S] \subseteq B$, we find $S \subseteq \varphi^{-1}[B]$. Let D be a Boolean subalgebra of P such that $S \subseteq D$. Then any pair of elements in D commute, hence Lemma 6.1.5 assures that $D \cap \varphi^{-1}[B]$ is a Boolean subalgebra of P containing S, whence $\langle S \rangle \subseteq \varphi^{-1}[B]$. By Example B.1.21, we obtain $\varphi[\langle S \rangle] \subseteq B$, i.e., $\mathcal{B}(\varphi)[\langle S \rangle] \subseteq \langle \varphi[S] \rangle$.

(b) Similar to the proof of Theorem 3.2.1(b), but by using Proposition 6.1.2 instead of Theorem 3.1.3.

(c) Assume that P is a Boolean algebra, or that φ is injective and P is an orthomodular lattice. Let $D \in \mathcal{B}(Q)$, then Lemma 6.1.5 assures that $\varphi^{-1}[D]$ is a Boolean subalgebra of P. So $D \mapsto \varphi^{-1}[D]$ is a well-defined map $\mathcal{B}(Q) \to \mathcal{B}(P)$. It now follows from Example B.1.21 that $D \mapsto \varphi^{-1}[D]$ is indeed the upper adjoint of $\mathcal{B}(\varphi)$. By Lemma B.1.23, $\mathcal{B}(\varphi)_*$ preserves all existing infima, hence (21) holds.

(d) Assume that φ is injective. Let $\{B_i\}_{i \in I} \subseteq \mathcal{B}(P)$ with I non empty. We always have

$$\varphi\left[\bigcap_{i \in I} B_i\right] \subseteq \bigcap_{i \in I} \varphi[B_i],$$

hence let $q \in \bigcap_{i \in I} \varphi[B_i]$. Then for each $i \in I$, there is an $p_i \in B_i$ such that $q = \varphi(p_i)$. Fix some $j \in I$ and let $p = p_j$. Then for each $i \in I$, we have $\varphi(p_i) = q = \varphi(p)$. By injectivity of φ, it follows that $p_i = p$, so $p \in \bigcap_{i \in I} C_i$. Thus, $q \in \varphi\left[\bigcap_{i \in I} B_i\right]$, and we conclude that

$$\varphi\left[\bigcap_{i \in I} B_i\right] = \bigcap_{i \in I} \varphi[B_i],$$

which is exactly (22).

In order to show that $\mathcal{B}(\varphi)$ is an order embedding, let $B_1, B_2 \in \mathcal{B}(P)$. Since $\mathcal{B}(\varphi)$ is an order morphism, $B_1 \subseteq B_2$ implies

$$\mathcal{B}(\varphi)(B_1) \subseteq \mathcal{B}(\varphi)(B_2). \tag{23}$$

Conversely, if (23) holds, it follows from the injectivity of φ that

$$B_1 = \varphi^{-1}[\varphi[B_1]] = \varphi^{-1}[\mathcal{B}(B_1)] \subseteq \varphi^{-1}[\mathcal{B}(B_2)] = \varphi^{-1}[\varphi[B_2]] = B_2,$$

hence $\mathcal{B}(\varphi)$ is an order embedding. We show that $\mathcal{B}(\varphi)[\mathcal{B}(P)]$ is a down-set. Let $D \in \downarrow\mathcal{B}(\varphi)[\mathcal{B}(P)]$. Hence there is some $B \in \mathcal{B}(P)$ such that $D \subseteq \mathcal{B}(\varphi)(B)$. Let $E = \varphi^{-1}[D] \cap B$. By Lemma 6.1.5, $\varphi^{-1}[D]$ is non-empty and is closed under orthocomplementation, existing binary joins and existing binary meets. Hence it contains some element p, hence 0, for $0 = p \wedge p\perp$. Since B is a Boolean subalgebra of P, it follows from Proposition B.4.16 that $0 \in B$. Hence $0 \in E$. Thus, E is non-empty, closed under orthocomplementation, existing binary joins and existing binary meets, and consists of mutually commuting elements, hence E is a Boolean subalgebra. Now $\varphi[E] = D$. Indeed, if $q \in \varphi[E]$, then $q = \varphi(p)$ for some $p \in E$, hence $p \in \varphi^{-1}[D]$, so $q \in D$. Conversely, let $q \in D$. Since $D \subseteq \mathcal{B}(\varphi)(B) = \varphi[B]$, we find that $q \in \varphi[B]$. Hence there is some $p \in B$ such that $\varphi(p) = q$. Since $q \in D$, we find that $p \in \varphi^{-1}[D]$, so $p \in E$. We conclude that $q \in \varphi[E]$. It follows that $\mathcal{B}(\varphi)(E) = \varphi[E] = D$, hence $D \in \mathcal{B}(\varphi)[\mathcal{B}(P)]$. We conclude that $\downarrow\mathcal{B}(\varphi)[\mathcal{B}(P)] = \mathcal{B}(\varphi)[\mathcal{B}(P)]$.

Assume that P is an orthomodular lattice. Then it follows from (c) that $\mathcal{B}(\varphi)$ has an upper adjoint $\mathcal{B}(\varphi)_*$ defined by $D \mapsto \varphi^{-1}[D]$ for each $D \in \mathcal{B}(Q)$. By injectivity of φ, we find

$$\mathcal{B}(\varphi)_* \circ \mathcal{B}(\varphi)(B) = \varphi^{-1}[\varphi[B]] = B$$

for each $B \in \mathcal{B}(P)$, hence $\mathcal{B}(\varphi)_* \circ \mathcal{B}(\varphi) = 1_{\mathcal{B}(P)}$. Let $D \in \mathcal{B}(\varphi)[\mathcal{B}(P)]$. Then $D = \mathcal{B}(\varphi)(B)$ for some $B \in \mathcal{B}(P)$. By Lemma B.1.22, we find

$$\mathcal{B}(\varphi) \circ \mathcal{B}(\varphi)_*(D) = \mathcal{B}(\varphi) \circ \mathcal{B}(\varphi)_* \circ \mathcal{B}(\varphi)(B) = \mathcal{B}(\varphi)(B) = D.$$

Thus

$$\mathcal{B}(\varphi) \circ \mathcal{B}(\varphi)_*|_{\mathcal{B}(\varphi)[\mathcal{B}(P)]} = 1_{\mathcal{B}(Q)}\big|_{\mathcal{B}(\varphi)[\mathcal{B}(P)]}.$$

(e) Assume that $\mathcal{B}(\varphi)$ is surjective. Let $q \in Q$, then Lemma B.4.23 assures that q generates a Boolean subalgebra $\langle q \rangle$. By surjectivity of $\mathcal{B}(\varphi)$, there is a $B \in \mathcal{B}(A)$ such that $\mathcal{B}(\varphi)(B) = \langle q \rangle$. Since $\mathcal{B}(\varphi)(B) = \varphi[B]$, this means that there is a $p \in B$ such that $\varphi(p) = q$. Thus, φ is surjective.

(f) This follows directly from functoriality of $\mathcal{B}$ and the fact that φ has an inverse. □

6.2 Posets of commutative AF-subalgebras

Combining Stone duality with Gelfand duality gives an equivalence between Stone spaces and commutative AF-algebras. We are going to exploit this equivalence in order to identify a certain subposet of $\mathcal{C}(A)$ that is order isomorphic to $\mathcal{B}(\mathrm{Proj}(A))$.

Definition 6.2.1. Let A be a C*-algebra. Then we denote the subposet of $\mathcal{C}(A)$ consisting of all commutative C*-subalgebras of A that are also AF-algebras by $\mathcal{C}_{\mathrm{AF}}(A)$.

Lemma 6.2.2. Let A be a C*-algebra. If $\mathcal{D} \subseteq \mathcal{C}_{\mathrm{AF}}(A)$ is directed, then $\bigvee \mathcal{D}$ as calculated in $\mathcal{C}(A)$ is an element of $\mathcal{C}_{\mathrm{AF}}(A)$, hence $\bigvee S$ equals the supremum of $\mathcal{D}$ as calculated in $\mathcal{C}_{\mathrm{AF}}(A)$.

Proof. Let $\mathcal{D} \subseteq \mathcal{C}_{\mathrm{AF}}(A)$ be directed, let $\bigvee \mathcal{D}$ be the supremum of $\mathcal{D}$ in $\mathcal{C}(A)$. By Theorem 2.2.3, we have $D = C^*(\mathrm{Proj}(D))$ for each $D \in \mathcal{D}$. Since $\mathrm{Proj}(D) \subseteq \bigvee \mathcal{D}$ for each $D \in \mathcal{D}$, using Corollary 3.1.4 we obtain:

$$\bigvee \mathcal{D} = \bigvee_{D \in \mathcal{D}} C^*(\mathrm{Proj}(D)) = C^*\left(\bigcup_{D \in \mathcal{D}} \mathrm{Proj}(D)\right).$$

In other words, $\bigvee \mathcal{D}$ is generated by a subset of $\mathrm{Proj}(\bigvee \mathcal{D})$, hence it is generated by $\mathrm{Proj}(\bigvee \mathcal{D})$ itself. Using Theorem 2.2.3 again, it follows that $\bigvee \mathcal{D}$ is an AF-algebra, hence $\bigvee \mathcal{D} \in \mathcal{C}_{\mathrm{AF}}(A)$. □

Proposition 6.2.3. Let A be a C*-algebra. Then

$$\Psi_A : \mathcal{C}(A) \to B(\mathrm{Proj}(A)), \quad C \mapsto \mathrm{Proj}(C),$$

is an order morphism with lower adjoint (cf. Definition B.1.20)

$$\Phi_A : \mathcal{B}(\mathrm{Proj}(A)) \to \mathcal{C}(A), \quad B \mapsto C^*(B),$$

i.e., we have

$$C^*(B) \subseteq C \iff B \subseteq \mathrm{Proj}(C),$$

for each $B \in \mathcal{B}(\mathrm{Proj}(A))$ and $C \in \mathcal{C}(A)$.

Proof. It follows from Proposition C.3.2 that $\mathrm{Proj}(C)$ is a Boolean subalgebra of $\mathrm{Proj}(A)$ for each commutative C*-subalgebra C of A. Let $B \in \mathcal{B}(\mathrm{Proj}(A))$. By Lemma B.4.13, B consists of elements that mutually commute in an order theoretic sense. Hence all elements of B commute in an operator algebraical sense by Proposition C.3.2. Since projections are self adjoint, it follows that B is *-closed. Hence Lemma C.1.22 assures that $C^*(B)$ is a commutative C*-subalgebra of A. Thus $\Psi_A : \mathcal{C}(A) \to \mathcal{B}(\mathrm{Proj}(A))$ and $\Phi_A : \mathcal{B}(\mathrm{Proj}(A)) \to \mathcal{C}(A)$ are well-defined maps, which are order morphisms, since they clearly preserve inclusions. Let $B \in \mathcal{B}(\mathrm{Proj}(A))$ and $C \in \mathcal{C}(A)$. Assume that $B \subseteq \mathrm{Proj}(C)$. Then

$$C^*(B) \subseteq C^*(\mathrm{Proj}(C)) \subseteq C^*(C) = C.$$

Assume that $C^*(B) \subseteq C$. Since B consists of projections, and $B \subseteq C^*(B)$, we obtain

$$B \subseteq \mathrm{Proj}(C^*(B)) \subseteq \mathrm{Proj}(C),$$

hence Ψ_A and Φ_A form a Galois connection. □

Theorem 6.2.4. For any *-homomorphism $\varphi : A \to A'$ between C*-algebras A and A', define

$$\mathcal{C}_{\mathrm{AF}}(\varphi) : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(A'), \quad C \mapsto \varphi[C].$$

Then $\mathcal{C}_{\mathrm{AF}}$ becomes a well-defined functor **CStar** → **DCPO**. Furthermore, if A is a C*-algebra, then

$$\Psi_A : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{B}(\mathrm{Proj}(A)), \quad C \mapsto \mathrm{Proj}(C)$$

is an order isomorphism, and $\Psi : \mathcal{C}_{\mathrm{AF}} \to \mathcal{B} \circ \mathrm{Proj}$ is a natural isomorphism[6], i.e., for each *-homomorphism $\varphi : A \to A'$ between C*-algebras A and A' the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}_{\mathrm{AF}}(A) & \xrightarrow{\Psi_A} & \mathcal{B}(\mathrm{Proj}(A)) \\ {\scriptstyle \mathcal{C}_{\mathrm{AF}}(\varphi)}\downarrow & & \downarrow{\scriptstyle \mathcal{B}(\mathrm{Proj}(\varphi))} \\ \mathcal{C}_{\mathrm{AF}}(A') & \xrightarrow[\Psi_{A'}]{} & \mathcal{B}(\mathrm{Proj}(A')). \end{array}$$

Proof. Let A be a C*-algebra. Since Ψ_A is the restriction of the map $\Psi_A : \mathcal{C}(A) \to \mathcal{B}(\mathrm{Proj}(A))$ in Proposition 6.2.3 to $\mathcal{C}_{\mathrm{AF}}(A)$, it follows that Ψ_A is an order morphism. If we consider the map

$$\Phi_A : \mathcal{B}(\mathrm{Proj}(A)) \to \mathcal{C}(A), \quad B \mapsto C^*(B)$$

in the same proposition, it follows from Theorem 2.2.3 that $\Phi_A(B) \in \mathcal{C}_{\mathrm{AF}}(A)$ for each $B \in \mathcal{B}(\mathrm{Proj}(A))$. Hence we can consider its corestriction $\Phi_A : \mathcal{B}(\mathrm{Proj}(A)) \to \mathcal{C}_{\mathrm{AF}}(A)$ to the codomain $\mathcal{C}_{\mathrm{AF}}(A)$, which is an order morphism, since the original map is an order morphism. If $C \in \mathcal{C}_{\mathrm{AF}}(A)$, we know by Theorem 2.2.3 that C is generated by its projections, hence

$$C^*(\mathrm{Proj}(C)) = C, \tag{24}$$

or, equivalently,

$$\Phi_A \circ \Psi_A = 1_{\mathcal{C}_{\mathrm{AF}}(A)}.$$

We aim to prove that

$$\Psi_A \circ \Phi_A = 1_{\mathcal{B}(\mathrm{Proj}(A))},$$

which amounts to showing that

$$\mathrm{Proj}(C^*(B)) = B, \tag{25}$$

[6] Recall Lemma B.6.2, which states that order isomorphisms are exactly the isomorphisms in **DCPO**.

for each Boolean subalgebra $B \in \mathcal{B}(\mathrm{Proj}(A))$. By Stone duality, there is a Stone space X such that $B \cong B(X)$, where $B(X)$ is the Boolean algebra of clopen subsets of X. By Proposition C.3.6, there is a Boolean isomorphism $B(X) \cong \mathrm{Proj}(C(X))$. Hence we can assume that $B = \mathrm{Proj}(C(X))$ for some Stone space X. By (24) we now find:

$$C^*(B) = C^*(\mathrm{Proj}(C(X))) = C(X),$$

whence

$$\mathrm{Proj}(C^*(B)) = \mathrm{Proj}(C(X)) = B.$$

We conclude that $\Psi_A : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{B}(\mathrm{Proj}(A))$ is an order isomorphism with inverse Φ_A.

We now prove that $\mathcal{C}_{\mathrm{AF}}$ is a functor **CStar** $\to$ **DCPO**. Firstly, by Lemma 6.2.2, $\mathcal{C}_{\mathrm{AF}}(A)$ is a dcpo, which also follows from the order isomorphism between $\mathcal{C}_{\mathrm{AF}}(A)$ and $\mathcal{B}(\mathrm{Proj}(A))$, where the latter is a dcpo by Proposition 6.1.2. Let $\varphi : A \to A'$ be a *-homomorphism. Let $C \in \mathcal{C}_{\mathrm{AF}}(A)$ and $E = \varphi[C]$. We have to show that E is an AF-algebra as well. Since C is an AF-algebra, there is some directed set $\mathcal{D} \subseteq \downarrow C$ such that $\bigvee \mathcal{D} = C$. Since $E = \mathcal{C}(\varphi)(C)$, and Theorem 3.2.1 assures that $\mathcal{C}(\varphi)$ is Scott continuous, it follows that

$$\mathcal{D}' = \{\mathcal{C}(\varphi)(D) : D \in \mathcal{D}\}$$

is directed and

$$E = \mathcal{C}(\varphi)(C) = \mathcal{C}(\varphi)\left(\bigvee \mathcal{D}\right) = \bigvee_{D \in \mathcal{D}} \mathcal{C}(\varphi)(D) = \bigvee \mathcal{D}'.$$

Now by definition of $\mathcal{C}(\varphi)$, we have $\mathcal{D}' = \{\varphi[D] : D \in \mathcal{D}\}$, and since the image of a finite-dimensional C*-algebra is finite-dimensional as well, we find that E is the supremum of a directed set of finite-dimensional C*-subalgebras. Hence E is an AF-algebra. Thus $\mathcal{C}_{\mathrm{AF}}(\varphi)(C) \in \mathcal{C}_{\mathrm{AF}}(A')$ for each $C \in \mathcal{C}_{\mathrm{AF}}(A)$, and since $\mathcal{C}(\varphi)$ is Scott continuous, it follows that its restriction $\mathcal{C}_{\mathrm{AF}}(\varphi)$ is Scott continuous as well. Hence $\mathcal{C}_{\mathrm{AF}}$ is indeed a functor **CStar** $\to$ **DCPO**.

It remains to check that $\Psi_A : \mathcal{C}_{\mathrm{AF}} \to \mathcal{B} \circ \mathrm{Proj}$ is a natural transformation. Let $C \in \mathcal{C}_{\mathrm{AF}}(A)$. Then

$$\begin{aligned}
\Psi_{A'} \circ \mathcal{C}_{\mathrm{AF}}(\varphi)(C) &= \Psi_{A'} \circ \mathcal{C}(\varphi)(C^*(\mathrm{Proj}(C))) \\
&= \Psi_{A'}(C^*(\varphi[\mathrm{Proj}(C)])) \\
&= \mathrm{Proj}(C^*(\mathrm{Proj}(\varphi)[\mathrm{Proj}(C)])) \\
&= \mathrm{Proj}(C^*(\mathcal{B}(\mathrm{Proj}(\varphi))(\mathrm{Proj}(C))) \\
&= \mathcal{B}(\mathrm{Proj}(\varphi))(\mathrm{Proj}(C)) \\
&= \mathcal{B}(\mathrm{Proj}(\varphi)) \circ \Psi_A(C),
\end{aligned}$$

where the first equality follows by the definitions of $\mathcal{C}_{\mathrm{AF}}(\varphi)$ and $\mathcal{C}(\varphi)$, and by (24). The second equality follows from (a) in Theorem 3.2.1, the third by definition of the action of Proj on *-homomorphisms, and the penultimate equality follows from (25). □

Corollary 6.2.5. Let A be a C*-algebra. Then $\mathcal{C}_{\mathrm{AF}}(A)$ is an algebraic complete semilattice, which becomes the Gelfand spectrum of some commutative AF-algebra when equipped with the Lawson topology.

Proof. This follows directly from Theorem 6.2.4, Proposition 6.1.2, Corollary 6.1.3, and Theorem 2.2.3. □

Corollary 6.2.6. Let A be a C*-algebra such that all its commutative C*-subalgebras are AF-algebras. Then $\mathcal{C}(A)$ is an algebraic complete semilattice, which becomes the Gelfand spectrum of some commutative AF-algebra when equipped with the Lawson topology.

We will explore further domain-theoretic properties of $\mathcal{C}(A)$ in §7. It is shown in [47, Proposition 3.3] that $\mathbb{C}^2$ and $\mathrm{M}_2(\mathbb{C})$ are the only C*-algebras with two-dimensional maximal commutative C*-subalgebras. For our purposes it is enough to show that this statement holds for C*-algebras with maximal commutative C*-subalgebras generated by their projections.

We recall that maximal Boolean subalgebras of orthomodular posets are also called *blocks* (cf. Definition B.4.18).

Corollary 6.2.7. Let A be a C*-algebra whose maximal commutative C*-subalgebras are generated by projections. Then

$$M \mapsto \mathrm{Proj}(M)$$

is a bijection $\max \mathcal{C}(A) \to \max \mathcal{B}(\mathrm{Proj}(A))$ with inverse

$$B \mapsto C^*(B).$$

Moreover, if A is neither *-isomorphic to $\mathbb{C}^2$ nor to $\mathrm{M}_2(\mathbb{C})$, then:

- A does not have two-dimensional maximal commutative C*-subalgebras;
- $\mathrm{Proj}(A)$ does not have blocks of precisely four elements.

Proof. If $M \in \max \mathcal{C}(A)$, it follows from Theorem 2.2.3 that $M \in \max \mathcal{C}_{\mathrm{AF}}(A)$. Conversely, if $M \in \max \mathcal{C}_{\mathrm{AF}}(A)$, then Proposition C.1.15 assures that $M \subseteq N$ for some maximal commutative C*-subalgebra N of A. Since $N \in \max \mathcal{C}_{\mathrm{AF}}(A)$, it follows that $M = N$. Hence $\max \mathcal{C}(A) = \max \mathcal{C}_{\mathrm{AF}}(A)$. It now follows from Theorem 6.2.4 that $M \mapsto \mathrm{Proj}(M)$ is a bijection with inverse $B \mapsto C^*(B)$. The same theorem assures that $M = C^*(\mathrm{Proj}(M))$ for each $M \in \max \mathcal{C}(A)$.

First assume that A is infinite-dimensional. By Proposition 2.1.3, each maximal commutative C*-subalgebras M of A must be infinite-dimensional, too. Since $M = C^*(\mathrm{Proj}(M))$, it follows from Proposition 2.1.2 that $\mathrm{Proj}(M)$ must be infinite. Hence each block of $\mathrm{Proj}(A)$ must be infinite. Hence if A is infinite-dimensional, A cannot have two-dimensional maximal commutative C*-subalgebras and $\mathrm{Proj}(A)$ cannot have blocks of precisely four elements.

Now assume that A is finite-dimensional and let $M \in \max \mathcal{C}(A)$. By Proposition 2.1.2, we find that $\mathrm{Proj}(M)$ is a finite Boolean algebra with n atoms if and only if M is n-dimensional. By Proposition 6.1.2, $\mathrm{Proj}(M)$ contains exactly four elements if M is two-dimensional. By the Artin-Wedderburn Theorem, we have $A \cong \bigoplus_{i=1}^{k} \mathrm{M}_{n_i}(\mathbb{C})$ for some $k, n_1, \ldots, n_k \in \mathbb{N}$. Let $M \in \max \mathcal{C}(A)$. By Lemma 5.2.1 we must have $M \cong \bigoplus_{i=1}^{n} M_i$ for some $M_i \in \max \mathcal{C}(\mathrm{M}_{n_i}(\mathbb{C}))$ for each $i \in \{1, \ldots, k\}$. By Proposition 2.1.4, we find that $M \cong \mathbb{C}^{n_1 + \cdots + n_k}$. We find that M is two-dimensional precisely in the cases $k = 1$, $n_1 = 2$ and $k = 2$, $n_1 = n_2 = 1$, i.e., if $A \cong \mathbb{C}^2$ or $A \cong \mathrm{M}_2(\mathbb{C})$. □

6.3 The theorems of Sachs and Harding–Navara

In this section, we state Sachs's Theorem and its generalization by Harding and Navara to orthomodular posets, which allows us to reconstruct Boolean algebras and orthomodular posets, respectively, from their posets of Boolean subalgebras.

Theorem 6.3.1 (Sachs)**.** Let P and Q be Boolean algebras and let

$$\Phi : \mathcal{B}(P) \to \mathcal{B}(Q)$$

be an order isomorphism. Then there is a Boolean isomorphism $\varphi : P \to Q$ such that $\Phi = \mathcal{B}(\varphi)$, which is the unique Boolean isomorphism inducing Φ in this way as long as P does not consist of precisely four elements.

We sketch a proof of Sachs's Theorem that makes use of methods of Hamhalter's Theorem. The proof uses Stone duality, hence we assume that X and Y are Stone spaces such that $P = B(X)$ and $Q = B(Y)$, i.e., the Boolean algebras of the clopen subsets of X and Y, respectively. We first need the following definition.

Definition 6.3.2. Let P be a Boolean algebra. A Boolean subalgebra of the form $I \cup I^{\perp}$ for some ideal (cf. Definition B.5.2) is called a *dual subalgebra*.

We shall see that dual subalgebras are the Boolean analogues of ideal subalgebras.

Let I and J be ideals of P. By Lemma B.5.3 we have $I \cup I^{\perp} = P$ if and only if $I = P$ or I is a maximal ideal. Assume that both I and J are neither equal to P nor maximal ideals. Then $I \cup I^{\perp} \subseteq J \cup J^{\perp}$ implies $I \subseteq J$. Indeed, if $p \in I$, then $\downarrow p \subseteq I$, and $p \in J \cup J^{\perp}$. Assume that $p \in J^{\perp}$, then $\uparrow p \subseteq J^{\perp}$, hence $\downarrow p \cup \uparrow p \subseteq J \cup J^{\perp}$. If $q \in P$, we have $q = (p \wedge q) \vee (p^{\perp} \wedge q)$. Since $p \wedge q \in \downarrow p$, $p \vee q^{\perp} \in \uparrow p$, and $J \cup J^{\perp}$ is a Boolean subalgebra of P, it follows that $p^{\perp} \wedge q = (p \vee q^{\perp})^{\perp}$, hence q is an element of $J \cup J^{\perp}$. Hence $J \cup J^{\perp} = P$, contradicting our assumption, so $p \in J$. Clearly, $I \subseteq J$ implies $I \cup I^{\perp} \subseteq J \cup J^{\perp}$, hence the assignment $I \mapsto I \cup I^{\perp}$ is an order isomorphism between the set of proper ideals that are not maximal, and the set of dual subalgebras not equal to P. Now we use the following proposition that gives an order-theoretic characterization of dual subalgebras of P:

Proposition 6.3.3. [52, Proposition 3.3] Let P be a Boolean algebra and B a Boolean subalgebra of P. Then the following statements are equivalent:

(1) B is a dual subalgebra;

(2) B is covered by $B \vee A$ for each atom A of $\mathcal{B}(P)$ that is incomparable to B.

It follows that the order isomorphism $\Phi : \mathcal{B}(P) \to \mathcal{B}(Q)$ restricts to an order isomorphism between the posets of dual subalgebras of P and Q. Since P and Q are the greatest elements of these posets, we can further restrict this isomorphism to an order isomorphism between the poset of dual subalgebras of P not equal to P, and the poset of dual subalgebras of Q not equal to Q. Hence we obtain an order isomorphism between the poset of non-maximal proper ideals of P, and the poset of non-maximal proper ideals of Q.

Proposition B.5.4 states the existence of an order isomorphism between the open subsets of X and the ideals of $B(X)$, both ordered by inclusion. This order isomorphism clearly restricts to an order isomorphism between the poset of non-maximal proper ideals of P and the poset of non-maximal open proper subsets of X. Clearly $X \setminus \{x\}$ for any $x \in X$ is a maximal open proper subset. Let $U \subseteq X$ be a open proper subset. Then there is some $x \in X$ such that $x \notin U$, hence $U \subseteq X \setminus \{x\}$, and since U is maximal, it follows that $U = X \setminus \{x\}$. Since we have an order reversing isomorphism between the poset of open subsets of X ordered by inclusion and the poset of closed subsets of X ordered by inclusion, we find that the order isomorphism between non-maximal open proper subsets of X and non-maximal open proper subsets of Y induces an order isomorphism between $\mathcal{F}(X)$ and $\mathcal{F}(Y)$, where $\mathcal{F}(X)$ denotes the poset of all closed subsets of X with at least two points. Now, we can use Proposition 4.7.4 in order to obtain a homeomorphism between X and Y, which induces a Boolean isomorphism $\varphi : P \to Q$. We need one more lemma, which follows from [97, Theorem 5] and the remark preceding [97, Lemma 7].

Lemma 6.3.4. Let P be a Boolean algebra and B a Boolean subalgebra. Then B is the intersection of dual subalgebras.

Hence dual subalgebras of a Boolean algebra play a role similar to ideal subalgebras of a commutative C*-algebra. It now follows precisely in the same way as in the proof of Theorem 4.7.5 that φ induces Φ, and is the unique Boolean isomorphism inducing Φ if $\#X \neq 2$, i.e., if P has other than four elements.

Corollary 6.3.5. Let A and D be commutative AF-algebras and

$$\Phi : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(D)$$

an order isomorphism. Then there is a *-isomorphism $\varphi : A \to D$ such that $\mathcal{C}_{\mathrm{AF}}(\varphi) = \Phi$. Moreover, φ is the unique *-isomorphism inducing Φ in this way as long as $\dim A \neq 2$.

Proof. By Theorem 6.2.4, there is a order isomorphism $\Psi : \mathcal{B}(\mathrm{Proj}(A)) \to \mathcal{B}(\mathrm{Proj}(D))$ defined by $B \mapsto \mathrm{Proj}(\Phi(C^*(B)))$. It now follows from Theorem 6.3.1 that there is a Boolean isomorphism $\psi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ such that $\mathcal{B}(\psi) = \Psi$. This Boolean isomorphism is unique as long as $\mathrm{Proj}(A)$ does not have four elements, i.e., $\dim A \neq 2$. It now follows from Theorem 2.2.5 that there is a unique *-isomorphism $\varphi : A \to B$ whose restriction to $\mathrm{Proj}(A)$ equals ψ. Let $C \in \mathcal{C}_{\mathrm{AF}}(A)$. By Theorem 6.2.4, we have $C^*(\mathrm{Proj}(C)) = C$ and $\Phi(C) = C^*(\Psi(\mathrm{Proj}(C)))$. Hence

$$\begin{aligned}\Phi(C) &= C^*(\mathcal{B}(\psi))(\mathrm{Proj}(C)) = C^*(\psi[\mathrm{Proj}(C)]) = C^*(\varphi[\mathrm{Proj}(C)]) \\ &= \mathcal{C}(\varphi)(C^*(\mathrm{Proj}(C))) = \mathcal{C}(\varphi)(C) = \mathcal{C}_{\mathrm{AF}}(\varphi)(C),\end{aligned}$$

where we used Theorem 3.2.1 in the one but last equality. □

Harding and Navara proved the following theorem, which extends Sachs's Theorem to orthomodular posets.

Theorem 6.3.6 (Harding–Navara)**.** Let P and Q be orthomodular posets and let $\Phi : \mathcal{B}(P) \to \mathcal{B}(Q)$ be an order isomorphism. Then there is an orthomodular isomorphism $\varphi : P \to Q$ such that $\Phi = \mathcal{B}(\varphi)$, which is the unique orthomodular isomorphism inducing Φ in this way if P has no blocks of exactly four elements.

Roughly speaking, the idea behind the proof of the Harding–Navara Theorem is the following. First one assumes that P has no blocks of exactly four elements. Each element $p \in P$ is contained in some block B of $\mathcal{B}(P)$ (cf. Proposition B.4.20). Φ restricts to an order isomorphism $\downarrow B \to \downarrow \Phi(B)$, and since $\downarrow B = \mathcal{B}(B)$, Sachs's Theorem assures the existence of a Boolean isomorphism $\varphi_B : B \to \Phi(B)$ such that $\mathcal{B}(\varphi_B) = \Phi|_{\downarrow B}$. Given another block D containing p, one has to check that $\varphi_B(p) = \varphi_D(p)$. It follows that one can define a unique orthomodular isomorphism $\varphi : P \to Q$ defined by $\varphi(p) = \varphi_B(p)$, where B is a block containing p. If P contains blocks of four elements, one considers the poset P_0 obtained by removing the elements of P that are both atoms and coatoms in P. Then $\mathcal{B}(P_0)$ equals $\mathcal{B}(P)$ with the blocks of $\mathcal{B}(P)$ removed. The general case then follows from the case of P having no blocks of four elements. We refer to [52] for the details.

6.4 Induced orthomodular isomorphisms of projection posets

Given two C*-algebras A and B, we aim to show that if $\mathcal{C}(A)$ and $\mathcal{C}(B)$ are order isomorphic, then $\mathrm{Proj}(A)$ and $\mathrm{Proj}(B)$ are orthomodular isomorphic. We start with the following lemma, which is originally from [47, Lemma 3.1]. We give an alternative proof.

Lemma 6.4.1. Let A be a C*-algebra and let $C \in \mathcal{C}(A)$. Then the following statements are equivalent:

(1) C is an atom of $\mathcal{C}(A)$;

(2) C is two-dimensional;

(3) there is some non-trivial projection $p \in A$ such that $C = C^*(p)$;

(4) there is some non-trivial projection $p \in A$ such that $C = \mathrm{Span}\{p, 1_A - p\}$.

Proof. The equivalence between (1) and (2) follows from Lemma 5.1.4 and the fact that $\mathbb{C}1_A$ is the least element of $\mathcal{C}(A)$. For the equivalence between (2) and (4), we first note that Proposition 2.1.2 assures that C two-dimensional if and only if it is spanned by two minimal projections p and q (minimal with respect to C) such that $p + q = 1_A$. Hence p is non-trivial and

$$C = \mathrm{Span}\{p, 1_A - p\}. \tag{26}$$

Conversely, if C satisfies (26) for some non-trivial projection p, then p and $1_A - p$ are non-zero projections that are clearly orthogonal. By Proposition 2.1.2 it follows that C is two-dimensional. Finally, we note that

$$\mathrm{Span}\{p, 1_A\} = \mathrm{Span}\{p, 1_A - p\}$$

is the smallest C*-subalgebra of A containing p. Hence

$$\mathrm{Span}\{p, 1_A - p\} = C^*(p),$$

which gives the equivalence between (3) and (4). □

Proposition 6.4.2. Let A be a C*-algebra and $C \in \mathcal{C}(A)$ such that $C \neq \mathbb{C}1_A$. Then C is the supremum of some collection of atoms of $\mathcal{C}(A)$ if and only if C is generated by its projections.

Proof. Let $C = C^*(\mathrm{Proj}(C))$. Notice that $\mathbb{C}1_A$ is the smallest C*-subalgebra of A containing 0_A and 1_A. Hence

$$C^*(0_A) = C^*(1_A) = C^*(0_A, 1_A) = \mathbb{C}1_A,$$

and it follows that $\mathrm{Proj}(C)$ must have a non-trivial projection, otherwise we obtain a contradiction with $C \neq \mathbb{C}1_A$. Since $\mathrm{Proj}(C) \subseteq C$, we have $C^*(p) \in {\downarrow} C$ for each $p \in P$. Since $P = \bigcup_{p \in P}\{p\}$, it follows from Corollary 3.1.4 that

$$C = C^*(\mathrm{Proj}(C)) = \bigvee \big\{C^*(p) : p \in \mathrm{Proj}(C)\big\}.$$

Notice that if $p \in \{0, 1\}$, then $C^*(p) = \mathbb{C}1_A$. Hence

$$C = \bigvee \big\{C^*(p) : p \in \mathrm{Proj}(C) \setminus \{0, 1\}\big\},$$

and it follows from Lemma 6.4.1 that C is the supremum of a collection of atoms in $\mathcal{C}(A)$. Conversely, if $C = \bigvee \mathcal{D}$, with $\mathcal{D}$ a collection of atoms in $\mathcal{C}(A)$, we must have

$$\mathcal{D} = \{C^*(p) : p \in P\}$$

for some collection $P \subseteq C$ consisting of projections. Hence

$$C = \bigvee_{p \in P} C^*(p) = C^*\left(\bigcup_{p \in P}\{p\}\right) = C^*(P),$$

where we used Corollary 3.1.4 in the second equality. Since

$$P \subseteq \mathrm{Proj}(C) \subseteq C,$$

it follows that $C = C^*(\mathrm{Proj}(C))$. □

Corollary 6.4.3. Let A be a C*-algebra. Then $\mathcal{C}(A)$ is atomistic if and only if A is scattered.

Proof. By Proposition 6.4.2, and element $C \in \mathcal{C}(A)$ is the supremum of some collection of atoms of $\mathcal{C}(A)$ if and only if it is generated by its projections. By Theorem 2.2.3, this is equivalent with C being an AF-algebra. The statement now follows from Theorem 2.3.4. □

Theorem 6.4.4. Let A and B be C*-algebras.

(1) Any order isomorphism $\mathcal{C}(A) \to \mathcal{C}(B)$ restricts to an order isomorphism $\mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$;

(2) If $\Phi : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$ is an order isomorphism, then there exists an orthomodular isomorphism $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ such that

$$\Phi(C) = C^*(\varphi[\mathrm{Proj}(C)])$$

for each $C \in \mathcal{C}_{\mathrm{AF}}(A)$. Moreover, if $\mathrm{Proj}(A)$ does not contain any block with four elements, then φ is the unique orthomodular isomorphism inducing Φ in this way.

Proof. For (1), let $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$ be an order isomorphism. By Theorem 2.2.3, $\mathcal{C}_{\mathrm{AF}}(A)$ equals the subposet of $\mathcal{C}(A)$ of commutative C*-subalgebras of A that are generated by their projections. Thus if $C \in \mathcal{C}_{\mathrm{AF}}(A)$, then Proposition 6.4.2 assures that C is the supremum of some collection of atoms of $\mathcal{C}(A)$. Since Φ is an order isomorphism, it follows that $\Phi(C)$ is also the supremum of some collection of atoms of $\mathcal{C}(B)$. Again by Proposition 6.4.2, we find that $\Phi(C) \in \mathcal{C}_{\mathrm{AF}}(B)$. It follows that Φ restricts to an order isomorphism $\mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$.

We prove (2). Let $\Phi : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$ be an order isomorphism. By Theorem 6.2.4, we obtain an unique order isomorphism $\Psi : \mathcal{B}(\mathrm{Proj}(A)) \to \mathcal{B}(\mathrm{Proj}(B))$ such that $\Phi(C) = C^*(\Psi(\mathrm{Proj}(C)))$ for each $C \in \mathcal{C}_{\mathrm{AF}}(A)$. It now follows from Theorem 6.3.6 that there exists an orthomodular isomorphism between $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$, which is unique as long as $\mathrm{Proj}(A)$ does not have any blocks consisting of four elements, such that $\mathcal{B}(\varphi) = \Psi$. Hence, if $C \in \mathcal{C}_{\mathrm{AF}}(A)$, we obtain

$$\Phi(C) = C^*(\mathcal{B}(\varphi)(\mathrm{Proj}(C)) = C^*(\varphi[\mathrm{Proj}(C)]).$$

□

Corollary 6.4.5. Let A be a scattered C*-algebra and B an arbitrary C*-algebra. Let $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$ be an order isomorphism. Then B is scattered and there exists an orthomodular isomorphism $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ such that

$$\Phi(C) = C^*(\varphi[\mathrm{Proj}(C)]),$$

for each $C \in \mathcal{C}(A)$. Moreover, if A is not *-isomorphic to either $\mathbb{C}^2$ or $\mathrm{M}_2(\mathbb{C})$, then φ is the unique orthomodular isomorphism inducing Φ in this way.

Proof. By Theorem 2.3.4, the maximal commutative C*-subalgebras of A are scattered. Let $M \in \max \mathcal{C}(A)$. Then $\downarrow M = \mathcal{C}(M)$ and $\downarrow \Phi(M) = \mathcal{C}(\Phi(M))$, hence Φ restricts to an order isomorphism $\mathcal{C}(M) \to \mathcal{C}(\Phi(M))$. By Theorem 4.7.5, it follows that M and $\Phi(M)$ are *-isomorphic, hence $\Phi(M)$ is scattered. Hence B is scattered by Theorem 2.3.4, which also assures that $\mathcal{C}(A) = \mathcal{C}_{\mathrm{AF}}(A)$ and $\mathcal{C}(B) = \mathcal{C}_{\mathrm{AF}}(B)$. We can now apply Theorem 6.4.4 in order to obtain an orthomodular isomorphism $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ such that

$$\Phi(C) = C^*(\varphi[\mathrm{Proj}(C)]),$$

for each $C \in \mathcal{C}(A)$. Moreover, if $\mathrm{Proj}(A)$ does not have blocks of four elements, then φ is the unique orthomodular isomorphism inducing Φ in this way. Assume that A is neither *-isomorphic to $\mathbb{C}^2$ nor to $\mathrm{M}_2(\mathbb{C})$. Since $\mathcal{C}(A) = \mathcal{C}_{\mathrm{AF}}(A)$, it follows from Theorem 2.2.3 that the maximal commutative C*-subalgebras of A are generated by projections. Hence we can apply Corollary 6.2.7 to conclude that $\mathrm{Proj}(A)$ does not have any blocks of precisely four elements. It follows that φ is the unique orthomodular isomorphism inducing Φ. □

6.5 Reconstructing projections in commutative C*-algebras

Theorem 6.4.4 does not give a direct reconstruction of $\mathrm{Proj}(A)$ from $\mathcal{C}(A)$. The rest of this chapter is devoted to describing an orthomodular poset that is orthomodular isomorphic to $\mathrm{Proj}(A)$ completely in terms of $\mathcal{C}(A)$. In this section we consider the case when A is commutative.

Lemma 6.5.1. Let X be a compact Hausdorff space, and let $C \in \mathcal{C}(C(X))$. Then C is an atom of $\mathcal{C}(C(X))$ if and only if there is a non-trivial separation (K_0, K_1) of X such that $C = C_{K_0} \cap C_{K_1}$.

Proof. Recall Lemma 6.4.1, which states that an element $C \in \mathcal{C}(C(X))$ is an atom if and only if there is some non-trivial projection p such that

$$C = \mathrm{Span}\{p, 1 - p\}.$$

By Corollary C.3.7, p corresponds to non-trivial separation (K_0, K_1) with $K_0 = p^{-1}[\{0\}]$ and $K_1 = p^{-1}[\{1\}]$. It follows immediately that

$$C_{K_0} \cap C_{K_1} = \mathrm{Span}\{p, 1-p\},$$

hence $C_{K_0} \cap C_{K_1}$ is an atom of $\mathcal{C}(C(X))$. □

Lemma 6.5.2. Let X be a compact Hausdorff space and let $K_0, K_1 \subseteq X$ be non-empty closed subsets such that $C_{K_0} \cap C_{K_1}$ is an atom of $\mathcal{C}(C(X))$. Then $C_{K_0} \cap C_{K_1}$ is not an ideal subalgebra if and only if K_0 and K_1 both contain two or more points and form a (non-trivial) separation of X, in which case $C_{K_0} \vee C_{K_1} = C(X)$.

Proof. Assume that $C_{K_0} \cap C_{K_1}$ is not an ideal subalgebra. If K_0 is empty or a singleton, then $C_{K_0} = C(X)$, contradicting that $C_{K_0} \cap C_{K_1}$ is not an ideal subalgebra. Similarly, K_1 can neither be empty nor a singleton. If $K_0 \cap K_1 \neq \emptyset$, it follows from Lemma 4.2.2 that $C_{K_0} \cap C_{K_1} = C_{K_0 \cup K_1}$ contradicting that $C_{K_0} \cap C_{K_1}$ is not an ideal subalgebra. Hence $K_0 \cap K_1 = \emptyset$, and we obtain from Proposition 4.6.5 that $C_{K_0} \vee C_{K_1} = C(X)$. Since K_0 and K_1 are disjoint closed subsets of X, it follows directly from Example 4.4.4 and Theorem 4.4.12 that

$$\{K_0, K_1\} \cup \big\{\{x\} : x \notin K_0 \cup K_1\big\}$$

is an u.s.c decomposition of X corresponding to $C_{K_0} \cap C_{K_1}$. Now $C_{K_0} \cap C_{K_1}$ is an atom of $\mathcal{C}(C(X))$, hence by Lemma 6.5.1 there is a non-trivial separation (K_0', K_1') of X such that

$$C_{K_0} \cap C_{K_1} = C_{K_0'} \cap C_{K_1'}.$$

It follows from Theorem 4.4.12 that $\{K_0', K_1'\}$ is the u.s.c. decomposition of X corresponding to $C_{K_0} \cap C_{K_1}$. By the same theorem, every C*-subalgebra of $C(X)$ has a unique u.s.c. decomposition, so $X \setminus (K_0 \cup K_1) = \emptyset$, and we either have $K_0 = K_0'$ and $K_1 = K_1'$ or $K_0 = K_1'$ and $K_1 = K_0'$. In any case, K_0 and K_1 form a separation of X. Finally, assume that K_0 and K_1 contain both two or more points and form a separation of X. Since the intersection operation is the infimum operation in $\mathcal{C}(A)$, $C_{K_0} \cap C_{K_1}$ is the greatest C*-subalgebra in $\mathcal{C}(C(X))$ that is contained in both C_{K_0} and C_{K_1}. The greatest ideal subalgebra contained in C_{K_0} and C_{K_1} is $C_{K_0 \cup K_1}$ according to Proposition 4.6.5, hence if

$C_{K_0} \cap C_{K_1}$ is an ideal subalgebra, then

$$C_{K_0} \cap C_{K_1} = C_{K_0 \cup C_K} = C_X = \mathbb{C}1_A,$$

contradicting the assumption that $C_{K_0} \cap C_{K_1}$ is an atom. Thus $C_{K_0} \cap C_{K_1}$ cannot be an ideal subalgebra. □

Lemma 6.5.3. Let X be a compact Hausdorff space with at least two points and let $K \subseteq X$ be a closed subset. Then C_K is an atom of $\mathcal{C}(C(X))$ if and only if $K = X \setminus \{x\}$ for some isolated point $x \in X$.

Proof. Let $K = X \setminus \{x\}$ for some isolated $x \in X$. Then $C_{\{x\}} = C(X)$, hence it follows from Lemma 6.5.1 that C_K is an atom of $\mathcal{C}(C(X))$.

Conversely, assume that C_K is an atom of $\mathcal{C}(C(X))$. Then $C_K \neq \mathbb{C}1_A$, so $K \neq X$. By Example 4.4.4 and Theorem 4.4.12, the unique u.s.c. decomposition of C_K is

$$\{K\} \cup \big\{\{x\} : x \notin K\big\},$$

and by Lemma 6.5.1, there is a non-trivial separation (K_0, K_1) of X such that $C_K = C_{K_0} \cap C_{K_1}$ with u.s.c. decomposition $\{K_0, K_1\}$. Hence there is exactly one point $x \in X$ that is not contained in K and either $K_0 = K$ and $K_1 = \{x\}$ or $K_0 = \{x\}$ and $K_1 = K$. In any case, $K \cup \{x\} = X$, and x is isolated for K_0 and K_1 are clopen. □

Let A be a commutative C*-algebra with Gelfand spectrum X. We aim to reconstruct the projections in A from $\mathcal{C}(A)$. In order to do so, we first note that the projections of A are in 1-1 correspondence with the clopen subsets of X. Hence it is sufficient to reconstruct the clopen subsets of X from $\mathcal{C}(A)$. Recall the set $\mathcal{I}(A)$ of co-bounding elements of $\mathcal{C}(A)$ (cf. Definition 4.6.1), which is also equal to the set of ideal subalgebras of A (cf. Proposition 4.6.2). Since the ideal subalgebras of A are almost in 1-1 correspondence with closed subsets of X, we have to axiomatize when an ideal subalgebra corresponds to a clopen subset of X. However, the correspondence between $\mathcal{I}(A)$ and closed subsets is not completely bijective, in that both singletons and the empty set are mapped to $A \in \mathcal{I}(A)$. Hence if there is an isolated point $x \in X$, there is no way to recover it from $\mathcal{I}(A)$. However, $X \setminus \{x\}$ is closed as well, and if X has at least three points, then $X \setminus \{x\}$ can be recovered. Consequently, $X \setminus \{x\}$ plays two roles: firstly as a clopen set, and secondly as the complement of an isolated point. In order to deal with this two-fold character of $X \setminus \{x\}$, we have to consider pairs of ideal subalgebras instead of single ideal subalgebras, and characterize those pairs whose corresponding closed subsets are each other's complements. The position of an ideal subalgebra in such a pair decides then whether we have to consider its corresponding closed subset or the complement of this closed subset. We denote the set of these pairs by $\mathcal{P}(A)$ and we show that we can equip it with an order and an orthocomplementation such that $\mathcal{P}(A)$ becomes an Boolean algebra that is isomorphic to $\mathrm{Proj}(A)$.

Definition 6.5.4. Let A be a commutative C*-algebra and let $\mathcal{P}(A)$ be the subset of $\mathcal{I}(A) \times \mathcal{I}(A)$ consisting of the pairs (D_0, D_1) that satisfy at least one of the following conditions:

(P0) $D_0 = \mathbb{C}1_A$ and $D_1 = A$;

(P1) $D_0 = A$ and $D_1 = \mathbb{C}1_A$;

(P2) D_0 is an atom of $\mathcal{C}(A)$, and $D_1 = A$;

(P3) $D_0 = A$, and D_1 is an atom of $\mathcal{C}(A)$;

(P4) $D_0 \cap D_1$ is an atom of $\mathcal{C}(A)$, and $D_0 \cap D_1 \notin \mathcal{I}(A)$.

We will sometimes write $\overline{D}$ instead of (D_0, D_1) when it improves the readability of the expressions,

Since Corollary 4.6.4 assures that $\mathbb{C}1_A \in \mathcal{I}(A)$ for each commutative C*-algebra A, it follows that $\mathcal{P}(A)$ is well defined and nonempty. We proceed by showing that $\mathcal{P}(A)$ can be equipped by an order and an orthocomplementation.

Proposition 6.5.5. Let A be a commutative C*-algebra such that $\dim A \geq 3$. Then $\mathcal{P}(A)$ becomes an orthoposet with least element and a greatest element defined by (P0) and (P1), respectively, if we order it by $(D_0, D_1) \leq (E_0, E_1)$ if and only if all of the following conditions hold:

(O1) $D_0 \subseteq E_0$;

(O2) $E_1 \subseteq D_1$;

(O3) If (D_0, D_1) satisfies (P2) and (E_0, E_1) satisfies (P3), then $D_0 \neq E_1$.

The orthocomplementation on $\mathcal{P}(A)$ is defined for each $(D_0, D_1) \in \mathcal{P}(A)$ by

$$(D_0, D_1)^\perp = (D_1, D_0). \qquad (27)$$

Proof. Notice that A is not an atom of $\mathcal{C}(A)$ since $\dim A \geq 3$ (cf. Lemma 5.1.4). Hence an element of $\mathcal{P}(A)$ cannot satisfy both (P2) and (P3). Let (D_0, D_1), (E_0, E_1), and (F_0, F_1) elements of $\mathcal{P}(A)$. Then $(D_0, D_1) \leq (D_0, D_1)$, since (O3) does not apply because (D_0, D_1) cannot satisfy both (P2) and (P3). Thus $\leq$ is reflexive. For transitivity, assume that

$$(D_0, D_1) \leq (E_0, E_1) \leq (D_0, D_1).$$

Then (O1) and (O2) imply that

$$D_0 \subseteq E_0 \subseteq D_0, \quad E_1 \subseteq D_1 \subseteq E_1,$$

so $(D_0, D_1) = (E_0, E_1)$. To prove that $\leq$ is antisymmetric, assume that

$$(D_0, D_1) \leq (E_0, E_1), \tag{28}$$

$$(E_0, E_1) \leq (F_0, F_1). \tag{29}$$

Then we have to show that

$$(D_0, D_1) \leq (F_0, F_1). \tag{30}$$

Clearly (O1) and (O2) are satisfied, so assume that (D_0, D_1) and (F_0, F_1) satisfy (P2) and (P3), respectively. Hence D_0 and F_1 are atoms of $\mathcal{C}(A)$, and

$$D_1 = F_0 = A.$$

Notice that

$$(E_0, E_1) \neq (\mathbb{C}1_A, A),$$

otherwise it follows from (28) that $D_0 \subseteq \mathbb{C}1_A$, which is impossible since D_0 is an atom. So (E_0, E_1) cannot satisfy (P0), and in a similar way, (29) prohibits that (E_0, E_1) satisfies (P1). Assume that (E_0, E_1) satisfies (P2), so E_0 is an atom and $E_1 = A$. Then (28) implies that $D_0 \subseteq E_0$. Since D_0 and E_0 are both atoms, it follows that $D_0 = E_0$. By (29), it follows that $E_0 \neq F_1$, so $D_0 \neq F_1$. In a similar way, it follows that $D_0 \neq F_1$ if (E_0, E_1) satisfies (P3). The remaining case is (E_0, E_1) satisfying (P4). From (28) and 29, we obtain $D_0 \subseteq E_0$ and $F_1 \subseteq E_1$, respectively. Hence

$$D_0 \cap F_1 \subseteq E_0 \cap E_1.$$

If $D_0 = F_1$, it follows that

$$D_0 = F_1 = E_0 \cap E_1,$$

for $D_0 = F_1$ and $E_0 \cap E_1$ are both atoms of $\mathcal{C}(A)$. But since $E_0 \cap E_1 \notin \mathcal{I}(A)$, it follows that $D_0, F_1 \notin \mathcal{I}(A)$ as well, which is impossible. So $D_0 \neq F_1$, hence (30) indeed holds.

It is obvious that $(\mathbb{C}1_A, A)$ and $(A, \mathbb{C}1_A)$ are the least and the greatest element of $\mathcal{P}(A)$, respectively. We show that $\mathcal{P}(A)$ admits an orthocomplementation defined by (27). First define a map $\mathcal{I}(A) \times \mathcal{I}(A) \to \mathcal{I}(A) \times \mathcal{I}(A)$, $(D_0, D_1) \mapsto (D_0, D_1)^\perp$, where

$$(D_0, D_1)^\perp = (D_1, D_0).$$

We check that this restricts to a map on $\mathcal{P}(A)$. Let $(D_0, D_1) \in \mathcal{I}(A) \times \mathcal{I}(A)$. Notice that (D_0, D_1) satisfies (P0) if and only if (D_1, D_0) satisfies (P1), (D_0, D_1) satisfies (P2) if and only if (D_1, D_0) satisfies (P3) and (D_0, D_1) satisfies (P4) if and only if (D_1, D_0) satisfies (P4). Hence $(D_0, D_1)^\perp \in \mathcal{P}(A)$ if $(D_0, D_1) \in \mathcal{P}(A)$. Moreover, clearly $(D_0, D_1)^{\perp\perp} = (D_0, D_1)$. Let (D_0, D_1) and (E_0, E_1) be elements of $\mathcal{P}(A)$ such that

$$(D_0, D_1) \leq (E_0, E_1).$$

Then $E_1 \subseteq D_1$ and $D_0 \subseteq D_1$, If (E_1, E_0) satisfies (P2) and (D_1, D_0) satisfies (P3), then (E_0, E_1) satisfies (P3) and (D_0, D_1) satisfies (P2), whence $E_1 \neq D_0$, for A is not an atom of $\mathcal{C}(A)$. It follows that

$$(E_1, E_0) \leq (D_1, D_0),$$

so

$$(E_0, E_1)^\perp \leq (D_0, D_1)^\perp.$$

Let $(E_0, E_1) \in \mathcal{P}(A)$ such that

$$(D_0, D_1), (D_1, D_0) \leq (E_0, E_1).$$

Then it follows from (O1) and (O2) that $E_1 \subseteq D_0, D_1 \subseteq E_0$, and clearly $(E_0, E_1) = (A, \mathbb{C}1_A)$ if (D_0, D_1) satisfies (P0) or (P1). Assume that (D_0, D_1) satisfies (P2). It follows immediately that $E_0 = A$. Moreover, E_1 is contained in

D_0, which is an atom of $\mathcal{C}(A)$, but cannot be equal to D_0, otherwise we obtain a contradiction with (O3) with respect to the inequality

$$(D_0, D_1) \leq (E_0, E_1).$$

We conclude that $E_1 = \mathbb{C}1_A$. Assume that (D_0, D_1) satisfies (P3), then (D_1, D_0) satisfies (P2), and we find in exactly the same way that $E_0 = A$ and $E_1 = \mathbb{C}1_A$. Assume that (D_0, D_1) satisfies (P4). Then $E_1 \subseteq D_0 \cap D_1$, which is an atom of $\mathcal{C}(A)$ but not an element of $\mathcal{I}(A)$. Since $E_1 \in \mathcal{I}(A)$, we must have $E_1 = \mathbb{C}1_A$. Let X be the spectrum of A. Then $C(X)$ and A are *-isomorphic, hence by Theorem 3.2.1 and Corollary 4.6.3, we have $\mathcal{I}(A) \cong \mathcal{I}(C(X))$. By Proposition 4.6.2, $\mathcal{I}(C(X))$ is the collection of ideal subalgebras of $C(X)$. It now follows from Lemma 6.5.2 that $D_0 \vee D_1 = A$. Since $D_0, D_1 \subseteq E_0$, this yields $E_0 = A$. Thus in all cases, we find $(E_0, E_1) = (A, \mathbb{C}1_A)$, which is the greatest element of $\mathcal{P}(A)$. Hence $(D_0, D_1) \vee (D_0, D_1)^\perp$ exists and is equal to the greatest element of $\mathcal{P}(A)$. Now let $(E_0, E_1) \in \mathcal{P}(A)$ such that

$$(E_0, E_1) \leq (D_0, D_1), (D_1, D_0).$$

Then

$$(D_0, D_1)^\perp, (D_1, D_0)^\perp \leq (E_0, E_1)^\perp,$$

hence

$$(D_1, D_0), (D_0, D_1) \leq (E_1, E_0).$$

The analysis above with (E_0, E_1) replaced by (E_1, E_0) gives $(E_1, E_1) = (A, \mathbb{C}1_A)$, hence $(E_0, E_1) = (\mathbb{C}1_A, A)$, the least element of $\mathcal{P}(A)$. We conclude that $(D_0, D_1) \wedge (D_0, D_1)^\perp$ exists and is equal to the least element of $\mathcal{P}(A)$. We conclude that $\mathcal{P}(A)$ is an orthoposet. □

Remark 6.5.6. We note that $\mathcal{P}(A)$ defined by (P0)-(P4) and ordered by (O1)-(O3) can also be defined for each C*-algebra A with $\dim A \leq 2$. However, in Proposition 6.5.5 we assumed that A is at least three-dimensional for the reason that we eventually want to prove that $\mathcal{P}(A)$ is isomorphic to $\mathrm{Proj}(A)$, which is certainly not possible if $\dim A \leq 2$. Indeed, if $\dim A = 1$, then $\mathcal{P}(A)$ consists of only one element, namely (A, A) for $A = \mathbb{C}1_A$. However, $\mathrm{Proj}(A)$ has two elements. If $\dim A = 2$, then $\mathcal{C}(A) = \{\mathbb{C}1_A, A\}$, whence the only elements of $\mathcal{P}(A)$ are $(A, \mathbb{C}1_A)$ and $(\mathbb{C}1_A, A)$, whereas $\mathrm{Proj}(A)$ has four elements.

Lemma 6.5.7. Let A and B be commutative C*-algebras that are at least three-dimensional and let $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$ an order isomorphism. Then $\widehat{\Phi} : \mathcal{P}(A) \to \mathcal{P}(B)$ defined by

$$(D_0, D_1) \mapsto (\Phi(D_0), \Phi(D_1))$$

is an order isomorphism such that

$$\widehat{\Phi}\big((D_0, D_1)^{\perp}\big) = \widehat{\Phi}\big((D_0, D_1)\big)^{\perp} \tag{31}$$

for each $(D_0, D_1) \in \mathcal{P}(A)$.

Proof. By Corollary 4.6.3, Φ restricts to an order isomorphism $\mathcal{I}(A) \to \mathcal{I}(B)$. It follows that

$$\widehat{\Phi} : \mathcal{I}(A) \times \mathcal{I}(A) \to \mathcal{I}(B) \times \mathcal{I}(B), \quad (D_0, D_1) \mapsto (\Phi(D_0), \Phi(D_1))$$

is a bijection. Moreover, for each $D \in \mathcal{C}(A)$, we have

$$D = A \iff \Phi(D) = B,$$

$$D = \mathbb{C}1_A \iff \Phi(D) = \mathbb{C}1_B,$$

$$D \text{ is an atom of } \mathcal{C}(A) \iff \Phi(D) \text{ is an atom of } \mathcal{C}(B).$$

Hence for each $(D_0, D_1) \in \mathcal{I}(A) \times \mathcal{I}(A)$, we have

$$(D_0, D_1) \in \mathcal{P}(A) \iff (\Phi(D_0), \Phi(D_1)) \in \mathcal{P}(B),$$

so $\widehat{\Phi}$ restricts to a bijection $\mathcal{P}(A) \to \mathcal{P}(B)$. Let (D_0, D_1) and (E_0, E_1) be elements of $\mathcal{P}(A)$. Since Φ is an order isomorphism, we have $D_0 \subseteq E_0$ and $E_1 \subseteq D_1$ if and only if $\Phi(D_0) \subseteq \Phi(E_0)$ and $\Phi(E_1) \subseteq \Phi(D_1)$. Moreover, we have $D_0 \neq E_1$ if and only if $\Phi(D_0) \neq \Phi(E_1)$, so $\widehat{\Phi} : \mathcal{P}(A) \to \mathcal{P}(B)$ is indeed an order isomorphism. From the definition of $(D_0, D_1)^{\perp}$ and $\widehat{\Phi}$, it easily follows that (31) holds. □

Theorem 6.5.8. Let A be a commutative C*-algebra with $\dim A \geq 3$. Then:

(1) $\mathcal{P}(A)$ is a Boolean algebra, and there exists a Boolean isomorphism $\Psi_A : \mathrm{Proj}(A) \to \mathcal{P}(A)$ definedby

$$\Psi_A(p) = \left(\psi^{-1}\left[C_{\psi(p)^{-1}[\{0\}]}\right], \psi^{-1}\left[C_{\psi(p)^{-1}[\{1\}]}\right]\right),$$

for each $p \in \mathrm{Proj}(A)$, where X is the Gelfand spectrum of A and $\psi : A \to C(X)$ is a *-isomorphism.

(2) Ψ_A does not depend on the choice of X and ψ in following sense: given another compact Hausdorff space Y and a *-isomorphism $\varphi : A \to C(Y)$, then

$$\Psi_A(p) = \left(\varphi^{-1}\left[C_{\varphi(p)^{-1}[\{0\}]}\right], \varphi^{-1}\left[C_{\varphi(p)^{-1}[\{1\}]}\right]\right)$$

for each $p \in \mathrm{Proj}(A)$.

(3) For each $C \in \mathcal{C}(A)$ and $p \in \mathrm{Proj}(A)$, we have $p \in \mathrm{Proj}(C)$ if and only if $D_0 \cap D_1 \subseteq C$, where $(D_0, D_1) = \Psi_A(p)$.

(4) If $C \in \mathcal{C}(A)$ with $\dim C \geq 3$, then there is an order embedding $\Psi_{A,C} : \mathcal{P}(C) \to \mathcal{P}(A)$ that preserves the orthocomplementation and has image

$$\mathrm{Im}(\Psi_{A,C}) = \{(D_0, D_1) \in \mathcal{P}(A) : D_0 \cap D_1 \subseteq C\}$$

such that $\Psi_{A,C}(C \cap D_0, C \cap D_1) = (D_0, D_1)$ for each $(D_0, D_1) \in \mathrm{Im}(\Psi_{A,C})$ and the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Proj}(C) & \hookrightarrow & \mathrm{Proj}(A) \\ \downarrow \Psi_C & & \downarrow \Psi_A \\ \mathcal{P}(C) & \xhookrightarrow[\Psi_{A,C}]{} & \mathcal{P}(A) \end{array} \tag{32}$$

Moreover, if $B \in \mathcal{C}(A)$ such that $C \subseteq B$, then

$$\Psi_{A,B} \circ \Psi_{B,C} = \Psi_{A,C}. \tag{33}$$

Proof. For (1), we first consider the case $A = C(X)$ for some compact Hausdorff space X. Then we can take ψ to be the identity, whence for each $p \in \mathrm{Proj}(C(X))$ we have:

$$\Psi_{C(X)}(p) = \left(C_{p^{-1}[\{0\}]}, C_{p^{-1}[\{1\}]}\right).$$

Claim: $\mathcal{P}(C(X))$ is the image of $\Psi_{C(X)}$.

By Proposition 4.6.2, we can identify elements of $\mathcal{I}(C(X))$ with ideal subalgebras. By Corollary C.3.7, we have a bijective correspondence between projections of $C(X)$ and separations of X via

$$p \mapsto (p^{-1}[\{0\}], p^{-1}[\{1\}]),$$

so we have to show that a pair of closed subsets (K_0, K_1) is a separation of X if and only if (C_{K_0}, C_{K_1}) satisfies (P0)-(P4) of Proposition 6.5.5. If $K_0 = X$ and $K_1 = \emptyset$, or $K_0 = \emptyset$ and $K_1 = X$, then (C_{K_0}, C_{K_1}) are exactly the points in $\mathcal{I}(C(X)) \times \mathcal{I}(C(X))$ satisfying (P0) and (P1). If $K_0 \neq X \neq K_1$, assume first that $K_0 = X \setminus \{x\}$ for some isolated point $x \in X$. If $X = \{x\}$, then (C_{K_0}, C_{K_1}) satisfies both (P0) and (P1). It now follows from Lemma 6.5.3 that (C_{K_0}, C_{K_1}) satisfies (P2). Conversely, the last lemma implies that if D_0 and D_1 are ideal subalgebras such that (D_0, D_1) satisfies (P2), then there is some isolated point $x \in X$ such that

$$K_0 = \{x\}, \quad K_1 = X \setminus \{x\},$$

$$D_0 = C_{K_0}, \quad D_1 = C_{K_1}.$$

In a similar way, we find that the pairs in $\mathcal{I}(C(X)) \times \mathcal{I}(C(X))$ satisfying (P3) are exactly the pairs (C_{K_0}, C_{K_1}) such that $K_0 = \{x\}$ for some isolated point $x \in X$ and $K_1 = X \setminus \{x\}$. The only remaining option is that both K_0 and K_1 both contain more than two points. By Lemma 6.5.2, it follows that (C_{K_0}, C_{K_1}) in $\mathcal{I}(C(X)) \times \mathcal{I}(C(X))$ satisfies (P4) if and only if (K_0, K_1) is a separation of X such that both K_0 and K_1 both contain two or more points.

Claim: $\Psi_{C(X)}$ is an order morphism.

Let $p, q \in \mathrm{Proj}(C(X))$ and assume that $q \leq p$. By Lemma C.3.5, this is equivalent with $p(x) = 0$ implies $q(x) = 0$ for each $x \in X$, which is equivalent with

$q(x) = 1$ implies $p(x) = 1$ for each $x \in X$. Hence

$$p^{-1}[\{0\}] \subseteq q^{-1}[\{0\}], \quad q^{-1}[\{1\}] \subseteq p^{-1}[\{1\}].$$

It follows that

$$C_{q^{-1}[\{0\}]} \subseteq C_{p^{-1}[\{0\}]}, \quad C_{p^{-1}[\{1\}]} \subseteq C_{q^{-1}[\{1\}]}.$$

Thus $\Psi_{C(X)}(q) \leq \Psi_{C(X)}(p)$, so Ψ is an order morphism.

Claim: $\Psi_{C(X)}$ is an orthocomplementation-preserving order isomorphism.

We first have to show that $\Psi_{C(X)}$ is an order embedding. Then the surjectivity of $\Psi_{C(X)}$, which follows from the first claim, guarantees that $\Psi_{C(X)}$ is an order isomorphism. So let $p, q \in \mathrm{Proj}(C(X))$ such that $\Psi_{C(X)}(q) \leq \Psi_{C(X)}(p)$ and let

$$K_0 = p^{-1}[\{0\}], \quad K_1 = p^{-1}[\{1\}],$$
$$L_0 = q^{-1}[\{0\}], \quad L_1 = q^{-1}[\{1\}].$$

Then

$$C_{L_0} \subseteq C_{K_0}, \quad C_{K_1} \subseteq C_{L_1}.$$

First assume that $\#K_0 \geq 2$. By Lemma 4.2.2, we obtain $K_0 \subseteq L_0$, so $p(x) = 0$ implies $q(x) = 0$ for each $x \in X$. It follows that $q \leq p$. Now assume that $\#K_0 \leq 1$, but $\#L_1 \geq 2$. Again by Lemma 4.2.2, we find $L_1 \subseteq K_1$, so $q(x) = 1$ implies $p(x) = 1$ for each $x \in X$, and we conclude that $q \leq p$. Assume that $\#K_0, \#L_1 \leq 1$. Then

$$C_{K_0} = C_{L_1} = C(X),$$

hence

$$\Psi_{C(X)}(p) = (C_{K_0}, C_{K_1})$$

satisfies either (P1) or (P3), and

$$\Psi_{C(X)}(q) = (C_{L_0}, C_{L_1})$$

satisfies (P0) or (P2). If $\Psi_{C(X)}(p)$ satisfies (P1), then $C_{K_1} = \mathbb{C}1_X$, which is only possible if $K_1 = X$. Thus $p(x) = 1$ for each $x \in X$, so $p = 1_X$, whence $q \leq p$. Similarly, if $\Psi_{C(X)}(q)$ satisfies (P0), then $q = 0_X$, so $q \leq p$. Finally, if $\Psi_{C(X)}(p)$ satisfies (P3) and $\Psi_{C(X)}(q)$ satisfies (P2), we find by Lemma 6.5.3 that

$$K_1 = X \setminus \{y\}, \quad L_0 = X \setminus \{z\}$$

for some isolated points $y, z \in X$. Moreover, $\Psi_{C(X)}(q) \leq \Psi_{C(X)}(p)$ implies that $C_{L_0} \neq C_{K_1}$, so $y \neq z$. Hence $p(x) = 1$ for all $x \in X$ except $x = y$ and $q(x) = 1$ only for $x = z$. Since $y \neq z$, we find that $q(x) = 1$ implies $p(x) = 1$, so $q \leq p$. We conclude that $\Psi_{C(X)}(q) \leq \Psi_{C(X)}(p)$ if and only if $q \leq p$, so $\Psi_{C(X)}$ is an order isomorphism. If $p \in C(X)$, then

$$p^{\perp}[\{0\}] = \{x \in X : (1_X - p)(x) = 0\} = \{x \in X : p(x) = 1\} = p^{-1}[\{1\}].$$

Similarly, we find

$$p^{\perp}[\{1\}] = p^{-1}[\{0\}].$$

Hence

$$\Psi_{C(X)}\left(p^{\perp}\right) = (C_{(p^{\perp})^{-1}[\{0\}]}, C_{(p^{\perp})^{-1}[\{1\}]}) = (C_{p^{-1}[\{1\}]}, C_{p^{-1}[\{0\}]}) = \Psi_{C(X)}(p)^{\perp},$$

so $\Psi_{C(X)} : \mathrm{Proj}(C(X)) \to \mathcal{P}(C(X))$ preserves the orthocomplementation.

We can now prove statement (1). Let A be an arbitrary commutative C*-algebra such that $\dim A \geq 3$. Let X be the Gelfand spectrum of A and $\psi : A \to C(X)$ a *-isomorphism. We first note that $\mathrm{Proj}(A)$ and $\mathrm{Proj}(C(X))$ are Boolean algebras (see also Proposition C.3.2). Since ψ is a *-isomorphism, it follows from Proposition C.3.4 and Lemma B.4.9 that $\mathrm{Proj}(A) \to \mathrm{Proj}(C(X))$ is an order isomorphism that preserves the orthocomplementation. Since ψ^{-1} is a *-isomorphism, it follows from Theorem 3.2.1 that $\mathcal{C}(\psi^{-1}) : \mathcal{C}(C(X)) \to \mathcal{C}(A)$ is an order isomorphism. Hence Lemma 6.5.7 assures that $\widehat{\mathcal{C}(\psi^{-1})}$ is an order isomorphism preserving the orthocomplementation. Now

$$\begin{aligned}
\Psi_A(p) &= \Big(\psi^{-1}\big[C_{\psi(p)^{-1}[\{0\}]}\big], \psi^{-1}\big[C_{\psi(p)^{-1}[\{1\}]}\big]\Big) \\
&= \Big(\mathcal{C}(\psi^{-1})\big(C_{\psi(p)^{-1}[\{0\}]}\big), \mathcal{C}(\psi^{-1})\big(C_{\psi(p)^{-1}[\{1\}]}\big)\Big) \\
&= \widehat{\mathcal{C}(\psi^{-1})}\Big(\big(C_{\psi(p)^{-1}[\{0\}]}, C_{\psi(p)^{-1}[\{1\}]}\big)\Big) \\
&= \widehat{\mathcal{C}(\psi^{-1})} \circ \Psi_{C(X)}\big(\psi(p)\big).
\end{aligned}$$

Hence we obtain

$$\Psi_A = \widehat{\mathcal{C}(\psi^{-1})} \circ \Psi_{C(X)} \circ \psi.$$

Since $\widehat{\mathcal{C}(\psi^{-1})}$, $\Psi_{C(X)}$, and ψ are each order isomorphisms preserving the orthocomplentation, it follows that Ψ_A is an orthocomplentation-preserving order isomorphism, too. Since $\mathrm{Proj}(A)$ is a Boolean algebra, this implies that $\mathcal{P}(A)$ is a distributive lattice. Since $\mathcal{P}(A)$ is also an orthoposet, it follows that $\mathcal{P}(A)$ is a Boolean algebra and, using Lemma B.4.27, that Ψ_A is a Boolean isomorphism.

To prove (2), let Y be another compact Hausdorff space and $\varphi : A \to C(Y)$ a *-isomorphism. Then $\psi^{-1} \circ \varphi : C(Y) \to C(X)$ is a *-isomorphism. By Proposition C.2.5, there is a homeomorphism $h : X \to Y$ such that $\psi = C_h \circ \varphi$. We now apply Proposition 4.2.5, where we take $q = h$. Since C_h is a *-isomorphism, we find that $B = C_h[C(Y)] = C(X)$. Hence

$$C_h^{-1}[C_K] = \mathcal{C}(C_h^{-1})(C_K) = \mathcal{C}(C_h^{-1})\big(C_K \cap C(X)\big) = C_{h[K]}$$

for each closed $K \subseteq X$. Now since $\psi = C_h \circ \varphi$, we find

$$\begin{aligned}
\Psi_A(p) &= \Big((C_h \circ \varphi)^{-1}\big[C_{C_h\circ\varphi(p)^{-1}[\{0\}]}\big], (C_h \circ \varphi)^{-1}\big[C_{C_h\circ\varphi(p)^{-1}[\{1\}]}\big]\Big) \\
&= \Big(\varphi^{-1} \circ C_h^{-1}\big[C_{(\varphi(p)\circ h)^{-1}[\{0\}]}\big], \varphi^{-1} \circ C_h^{-1}\big[C_{(\varphi(p)\circ h)^{-1}[\{1\}]}\big]\Big) \\
&= \Big(\varphi^{-1} \circ C_h^{-1}\big[C_{h^{-1}\circ\varphi(p)^{-1}[\{0\}]}\big], \varphi^{-1} \circ C_h^{-1}\big[C_{h^{-1}\circ\varphi(p)^{-1}[\{1\}]}\big]\Big) \\
&= \Big(\varphi^{-1}\big[C_{\varphi(p)^{-1}[\{0\}]}\big], \varphi^{-1}\big[C_{\varphi(p)^{-1}[\{1\}]}\big]\Big).
\end{aligned}$$

In order to prove (3), let $C \subseteq A$ be a C*-subalgebra. Let $p \in \mathrm{Proj}(A)$ and $\Psi_A(p) = (D_0, D_1)$. We want to show that $p \in \mathrm{Proj}(C)$ if and only if $D_0 \cap D_1 \subseteq C$. If p equals either 0_A or 1_A, then $p \in C$ and $D_0 \cap D_1 = \mathbb{C}1_A$. Clearly the latter subalgebra is contained in C, so the statement is true if p is a trivial projection. Assume that p is non-trivial. Then (D_0, D_1) can only satisfy (P2), (P3), or (P4) in Proposition 6.5.5, hence $D_0 \cap D_1$ is an atom of $\mathcal{C}(A)$. Let X be the Gelfand spectrum of A and $\psi : A \to C(X)$ the corresponding *-isomorphism. It follows from the definition of Ψ_A that

$$(\psi[D_0], \psi[D_1]) = (C_{\psi(p)^{-1}[\{0\}]}, C_{\psi(p)^{-1}[\{1\}]}).$$

Since $\psi(p)$ is constant on $\psi(p)^{-1}[\{0\}]$ and $\psi(p)^{-1}[\{1\}]$, it follows that $\psi(p)$ is an element of $\psi[D_0]\cap\psi[D_1]$. Since ψ is a *-isomorphism, it follows from Theorem 3.2.1 that $\mathcal{C}(\psi):\mathcal{C}(A)\to\mathcal{C}(C(X))$ is an order isomorphism sending C to $\psi[C]$. Moreover,

$$\mathcal{C}(\psi)(D_0\cap D_1)=\mathcal{C}(\psi)(D_0)\cap\mathcal{C}(\psi)(D_1)=\psi[D_0]\cap\psi[D_1],$$

hence $\psi[D_0]\cap[D_1]$ is an atom of $\mathcal{C}(C(X))$. Lemma 6.4.1 now assures both that $\psi[D_0]\cap[D_1]$ is two-dimensional and the existence of some non-trivial projection $q\in C(X)$ such that

$$\psi[D_0]\cap\psi[D_1]=C^*(q)=C^*(1-q).$$

Since two-dimensional C*-algebras have two non-trivial projections, and $\psi(p)\in\psi[D_0]\cap\psi[D_1]$ is non-trivial, we either have $\psi(p)=q$ or $\psi(p)=1-q$. It follows that

$$\psi[D_0]\cap\psi[D_1]=C^*(\psi(p)).$$

Applying Theorem 3.2.1, we obtain

$$\begin{aligned}D_0\cap D_1&=\mathcal{C}(\psi^{-1})\circ\mathcal{C}(\psi)(D_0\cap D_1)\\&=\mathcal{C}(\psi^{-1})(\psi[D_0]\cap\psi[D_1])\\&=\mathcal{C}(\psi^{-1})\big(C^*(\psi(p))\big)\\&=C^*(p).\end{aligned}$$

Thus $D_0\cap D_1=\mathrm{Span}\{p,1_A-p\}$ (cf. Lemma 6.4.1), whence $p\in C$ if and only if $D_0\cap D_1\subseteq C$.

Finally, for (4), we start by describing $\Psi_{A,C}:\mathcal{P}(C)\to\mathcal{P}(A)$. Since $\psi:A\to C(X)$ is a *-isomorphism, we have $\psi[C]\subseteq C(X)$, hence by Theorem 4.4.12 we can find a compact Hausdorff space Y and a continuous surjection $k:X\to Y$ such that $C_k[C(Y)]=\psi[C]$. By Proposition C.2.5, C_k is an injective *-homomorphism, hence $C_k^{-1}\circ\psi|_C:C\to C(Y)$ is a *-isomorphism. It now follows from (2) that for each $p\in\mathrm{Proj}(C)$ we have $\Psi_C(p)=(E_0,E_1)$

with, for each $i = 0, 1$,

$$
\begin{aligned}
E_i &= \left(C_k^{-1} \circ \psi|_C\right)^{-1} \left[C_{(C_k^{-1} \circ \psi|_C)(p)^{-1}[\{i\}]}\right] \\
&= \psi|_C^{-1} \circ C_k \left[C_{(C_k^{-1} \circ \psi|_C)(p)^{-1}[\{i\}]}\right] \\
&= \psi|_C^{-1} \left[C_{k^{-1}\left[(C_k^{-1} \circ \psi|_C)(p)^{-1}[\{i\}]\right]} \cap \psi[C]\right],
\end{aligned}
$$

where we used Proposition 4.2.5 in the last equality. Fortunately, this ugly expression can be simplified:

$$
\begin{aligned}
x \in k^{-1}\left[(C_k^{-1} \circ \psi|_C)(p)^{-1}[\{i\}]\right] &\iff k(x) \in (C_k^{-1} \circ \psi|_C)(p)^{-1}[\{i\}] \\
&\iff C_k^{-1} \circ \psi|_C\,(p)(k(x)) = i \\
&\iff C_k\big((C_k^{-1} \circ \psi|_C)(p)\big)(x) = i \\
&\iff \psi(p)(x) = i \\
&\iff x \in \psi(p)^{-1}[\{i\}].
\end{aligned}
$$

Hence

$$
\begin{aligned}
E_i &= \psi|_C^{-1} \left[C_{\psi(p)^{-1}[\{i\}]} \cap \psi[C]\right] \\
&= \psi^{-1} \left[C_{\psi(p)^{-1}[\{i\}]} \cap \psi[C]\right] \cap C \\
&= \psi^{-1} \left[C_{\psi(p)^{-1}[\{i\}]}\right] \cap C,
\end{aligned}
$$

and we see that

$$
\Psi_C(p) = (E_0, E_1) = (C \cap D_0, C \cap D_1),
$$

where $(D_0, D_1) = \Psi_A(p)$. Let $\dim C \geq 3$, then $\Psi_C : \mathrm{Proj}(C) \to \mathcal{P}(C)$ is an order isomorphism. Moreover, by (3) we find that Ψ_A restricts to an order isomorphism $\mathrm{Proj}(C) \to \mathcal{D}$, where

$$
\mathcal{D} = \{(D_0, D_1) \in \mathcal{P}(A) : D_0 \cap D_1 \subseteq C\}.
$$

Hence $\Phi = \Psi_C \circ \Psi_A|_{\mathrm{Proj}(C)}^{-1}$ is an order isomorphism $\mathcal{D} \to \mathcal{P}(C)$, and

$$
\Phi((D_0, D_1)) = (C \cap D_0, C \cap D_1)
$$

for each $(D_0, D_1) \in \mathcal{D}$. Let $\Psi_{A,C} = \Phi^{-1}$. It follows that

$$\Psi_{A,C} : \mathcal{P}(C) \to \mathcal{P}(A)$$

is an order embedding with image $\mathcal{D}$. Moreover, since both Ψ_C and Ψ_A preserve the orthocomplementation, so does $\Psi_{A,C}$. Finally, if $B \in \mathcal{C}(A)$ such that $C \subseteq B$, then $\dim B \geq 3$. Let $p \in \mathrm{Proj}(C)$, then certainly $p \in \mathrm{Proj}(B)$. Let $(D_0, D_1) = \Psi_A(p)$. Then

$$\begin{aligned}(\Psi_{A,B} \circ \Psi_{B,C})^{-1}((D_0, D_1)) &= \Psi_{B,C}^{-1} \circ \Psi_{A,B}^{-1}((D_0, D_1)) \\ &= \Psi_{B,C}((B \cap D_0, B \cap D_1)) \\ &= (C \cap D_0, C \cap D_1) \\ &= \Psi_{A,C}^{-1}((D_0, D_1)).\end{aligned}$$

□

6.6 Reconstructing projections in arbitrary C*-algebras

In this section, we extend the results of the previous section to non-commutative C*-algebras. For the next lemma, recall that for a commutative C*-algebra A, we introduced the abbreviation $\overline{D}$ for $(D_0, D_1) \in \mathcal{P}(A)$.

Lemma 6.6.1. Let A be a commutative C*-algebra, and $B, C \in \mathcal{C}(A)$ such that $B \cap C$ is at least three-dimensional. Let $\overline{D} \in \mathcal{P}(B)$ and $\overline{E} \in \mathcal{P}(C)$. Then

$$\Psi_{A,B}(\overline{D}) = \Psi_{A,C}(\overline{E}) \tag{34}$$

if and only if there is a $\overline{F} \in \mathcal{P}(B \cap C)$ such that

$$\Psi_{B,B\cap C}(\overline{F}) = \overline{D}, \tag{35}$$

and

$$\Psi_{C,B\cap C}(\overline{F}) = \overline{F}. \tag{36}$$

Proof. Assume that (34) holds. Since $B \cap C$ is at least three-dimensional, it follows from Theorem 6.5.8 that the maps Ψ_A, Ψ_B and Ψ_C are order isomorphisms, and $\Psi_{B\cap C}$, $\Psi_{A,B}$, $\Psi_{A,C}$ and $\Psi_{A,B\cap C}$ are order embeddings. It follows that there are unique projections $p_B \in \mathrm{Proj}(B)$ and $p_C \in \mathrm{Proj}(C)$ such that

$\Psi_B(p_B) = \overline{D}$ and $\Psi_C(p_C) = \overline{E}$. By (32) of Theorem 6.5.8, we find

$$\begin{aligned}\Psi_A(p_B) &= \Psi_{A,B} \circ \Psi_B(p_B)\\ &= \Psi_{A,B}(\overline{D})\\ &= \Psi_{A,C}(\overline{E})\\ &= \Psi_{A,C} \circ \Psi_C(p_C)\\ &= \Psi_A(p_C),\end{aligned}$$

and since Ψ_A is an order isomorphism, we obtain $p_B = p_C$. If we write p instead of p_B or p_C, this means that $p \in \mathrm{Proj}(B \cap C)$. Let $\overline{F} = \Psi_{B\cap C}(p)$. Then

$$\Psi_{B,B\cap C}(\overline{F}) = \Psi_{B,B\cap C} \circ \Psi_{B\cap C}(p) = \Psi_B(p) = (\overline{D}),$$

where we used (32) of Theorem 6.5.8 in the second equality. Hence (35) holds, and (36) follows in a similar way. Conversely, if (35) and (36) holds for some $\overline{F} \in \mathcal{P}(B \cap C)$, then

$$\begin{aligned}\Psi_{A,B}(\overline{D}) &= \Psi_{A,B} \circ \Psi_{B,B\cap C}(\overline{F})\\ &= \Psi_{A,B\cap C}(\overline{F})\\ &= \Psi_{A,C} \circ \Psi_{C,B\cap C}(\overline{F})\\ &= \Psi_{A,C}(\overline{E}),\end{aligned}$$

where we applied (33) of Theorem 6.5.8 in the second and third equality. □

Theorem 6.6.2. Let A be a C*-algebra such that $Z(A)$ is at least three-dimensional, or equivalently, $Z(A)$ is neither the least element nor an atom of $\mathcal{C}(A)$. Let

$$\mathcal{P}(A) = \bigcup_{M\in\max\mathcal{C}(A)} \mathcal{P}(M) \Big/ \sim,$$

where $\sim$ is an equivalence relation on $\bigcup_{M\in\max\mathcal{C}(A)} \mathcal{P}(M)$ defined by $\overline{D} \sim \overline{E}$ for $\overline{D} \in \mathcal{P}(M)$, $\overline{E} \in \mathcal{P}(N)$, $M, N \in \max\mathcal{C}(A)$ if and only if there is a $B \in [Z(A), M \cap N]$ and a $\overline{F} \in \mathcal{P}(B)$ such that

$$\Psi_{M,B}(\overline{F}) = \overline{D}$$

and

$$\Psi_{N,B}(\overline{F}) = \overline{E}.$$

Then $\mathcal{P}(A)$ is an orthomodular poset that is orthomodularly isomorphic to $\mathrm{Proj}(A)$ if we define the order and orthocomplementation on $\mathcal{P}(A)$ as follows: if $\left[\overline{D}\right]$ denotes the equivalence class of $\overline{D}$, then the order on $\mathcal{P}(A)$ is defined by $\left[\overline{D}\right] \leq \left[\overline{E}\right]$ if and only if there are $\overline{D'} \in \left[\overline{D}\right]$ and $\overline{E'} \in \left[\overline{E}\right]$ such that $\overline{D'} \leq \overline{E'}$. The orthocomplementation on $\mathcal{P}(A)$ is defined by

$$\left[\overline{D}\right]^{\perp} = \left[\overline{D}^{\perp}\right].$$

Proof. Since the least element of $\mathcal{C}(A)$ corresponds to the only one-dimensional commutative C*-subalgebra of A, and the atoms of $\mathcal{C}(A)$ correspond exactly to the two-dimensional commutative C*-subalgebras of A (cf. Lemma 6.4.1), the condition that $Z(A)$ is neither the least element nor an atom of $\mathcal{C}(A)$ translates to the condition that $Z(A)$ is at least three-dimensional.

Claim: $\sim$ is an is an equivalence relation on $\bigcup_{M \in \max \mathcal{C}(A)} \mathcal{P}(M)$.

Let $M, N, K \in \max \mathcal{C}(A)$, $\overline{D} \in \mathcal{P}(M)$, $\overline{E} \in \mathcal{P}(N)$ and $\overline{F} \in \mathcal{P}(K)$. Clearly $\overline{D} \sim \overline{D}$ and $\overline{D} \sim \overline{E}$ if and only if $\overline{E} \sim \overline{D}$. Assume that $\overline{D} \sim \overline{E}$ and $\overline{E} \sim \overline{F}$. Hence there is are $B \in [Z(A), M \cap N]$, $\overline{G} \in \mathcal{P}(B)$, $C \in [Z(A), N \cap K]$ and $\overline{H} \in \mathcal{P}(C)$ such that

$$\begin{aligned} \Psi_{M,B}(\overline{G}) &= \overline{D}; \\ \Psi_{N,B}(\overline{G}) &= \overline{E}; \\ \Psi_{N,C}(\overline{H}) &= \overline{E}; \\ \Psi_{K,C}(\overline{H}) &= \overline{F}. \end{aligned}$$

By Lemma 6.6.1, we find that the middle two equations imply the existence of some $\overline{K} \in \mathcal{P}(B \cap C)$ such that

$$\Psi_{B,B\cap C}(\overline{K}) = \overline{G},$$

and

$$\Psi_{C,B\cap C}(\overline{K}) = \overline{H}.$$

Then it follows from the remaining equations and (33) of Theorem 6.5.8 that

$$\Psi_{M,B\cap C}(\overline{K}) = \Psi_{M,B} \circ \Psi_{B,B\cap C}(\overline{K}) = \Psi_{M,B}(\overline{G}) = \overline{D},$$

and

$$\Psi_{N,B\cap C}(\overline{K}) = \Psi_{N,C} \circ \Psi_{C,B\cap C}(\overline{K}) = \Psi_{N,C}(\overline{H}) = \overline{F}.$$

Hence $\overline{D} \sim \overline{F}$.

Claim: $\leq$ is a well-defined relation on $\mathcal{P}(A)$.

It follows easily from the definition of $\leq$ that $\left[\overline{D}\right] \leq \left[\overline{E}\right]$, $\overline{D} \sim \overline{D'}$, and $\overline{E} \sim \overline{E'}$ implies $\left[\overline{D'}\right] \leq \left[\overline{E'}\right]$. Hence $\leq$ is indeed a well-defined relation.

Claim: The operation

$$\left[\overline{D}\right] \mapsto \left[\overline{D}\right]^{\perp} = \left[\overline{D}^{\perp}\right]$$

is well defined.

Let $\overline{D} \sim \overline{E}$, where $\overline{D} \in \mathcal{P}(M)$ and $\overline{E} \in \mathcal{P}(N)$, with $M, N \in \max \mathcal{C}(A)$. Hence there is a $B \in [Z(A), M \cap N]$ and an $\overline{F} \in \mathcal{P}(B)$ such that

$$\Psi_{M,B}(\overline{F}) = \overline{D}, \quad \Psi_{N,B}(\overline{F}) = \overline{E}.$$

Since B is at least three-dimensional, Theorem 6.5.8 assures that $\Psi_{M,B}$ and $\Psi_{N,B}$ preserve orthocomplements. It follows that

$$\Psi_{M,B}\left(\overline{F}^{\perp}\right) = \overline{D}^{\perp}, \quad \Psi_{N,B}\left(\overline{F}^{\perp}\right) = \overline{E}^{\perp}.$$

Thus $\overline{D}^{\perp} \sim \overline{E}^{\perp}$, which shows that $\left[\overline{D}\right] \mapsto \left[\overline{D}\right]^{\perp}$ is a well defined operation.

Claim: Define $\Psi : \mathrm{Proj}(A) \to \mathcal{P}(A)$ by $\Psi(p) = [\Psi_M(p)]$, where M is a maximal commutative C*-subalgebra of A containing p. Then $\Psi : \mathrm{Proj}(A) \to \mathcal{P}(A)$ is a well-defined bijection such that $\Psi(q) \leq \Psi(p)$ if and only if $q \leq p$, and such that $\Psi\left(p^{\perp}\right) = \Psi(p)^{\perp}$.

In order to see that Ψ is well defined, let $p \in \mathrm{Proj}(A)$. We first note that the existence of a maximal commutative C*-subalgebra M of A containing p is assured by Proposition C.1.15. Let N be another maximal commutative C*-subalgebra of A containing p. Since $Z(A)$ is three-dimensional, and equal to the intersection of all maximal commutative C*-subalgebras of A (Lemma 3.1.2), it follows that $M \cap N$ is at least three-dimensional and contains p. By Theorem 6.5.8,

$$\Psi_M(p) = \Psi_{M,M\cap N} \circ \Psi_{M\cap N}(p),$$

and

$$\Psi_N(p) = \Psi_{M,M\cap N} \circ \Psi_{M\cap N}(p),$$

whence $[\Psi_M(p)] = [\Psi_N(p)]$. Since M is at least three-dimensional, Theorem 6.5.8 assures that Ψ_M preserves orthocomplements. Hence

$$\Psi\left(p^\perp\right) = \left[\Psi_M\left(p^\perp\right)\right] = \left[\Psi_M(p)^\perp\right] = [\Psi_M(p)]^\perp = \Psi(p)^\perp.$$

We show that Ψ is injective. Let $p, q \in \mathrm{Proj}(A)$ such that $\Psi(p) = \Psi(q)$. Let M and N be maximal commutative C*-subalgebras of A containing p and q, respectively, such that $\Psi(p) = [\Psi_M(p)]$ and $\Psi(q) = [\Psi_N(q)]$. Hence $\Psi_M(p) \sim \Psi_N(q)$, so there is some $B \in [Z(A), M\cap N]$ and a $\overline{G} \in \mathcal{P}(B)$ such that $\Psi_{M,B}(\overline{G}) = \Psi_M(p)$ and $\Psi_{N,B}(\overline{G}) = \Psi_N(q)$. Now, B is at least thee dimensional for it contains $Z(A)$, hence Theorem 6.5.8 assures that $\Psi_{M,B} : \mathcal{P}(B) \to \mathcal{P}(M)$ and $\Psi_{N,B} : \mathcal{P}(B) \to \mathcal{P}(N)$ are order embeddings and $\Psi_B : \mathrm{Proj}(B) \to \mathcal{P}(B)$ is an order isomorphism. It follows that there is a unique $r \in \mathrm{Proj}(B)$ such that $\Psi_B(r) = \overline{G}$. Hence

$$\Psi_M(p) = \Psi_{M,B} \circ \Psi_B(r) = \Psi_M(r),$$

where we used Theorem 6.5.8 in the last equality. Since the same theorem also assures that Ψ_M is an order isomorphism, we obtain $p = r$. In a similar way, we find that $q = r$, whence $p = q$, so Ψ is injective.

If $[\overline{D}] \in \mathcal{P}(A)$, then $\overline{D} \in \mathcal{P}(M)$ for some maximal commutative C*-subalgebra M of A. Since $\Psi_M : \mathrm{Proj}(M) \to \mathcal{P}(M)$ is an order isomorphism and $\mathrm{Proj}(M) \subseteq \mathrm{Proj}(A)$, it follows that $\overline{D} = \Psi_M(p)$ for some $p \in \mathrm{Proj}(A) \cap M$. Hence $[\overline{D}] = \Psi(p)$, so Ψ is surjective.

Let $q \leq p$ in $\mathrm{Proj}(A)$. Then $pq = q = qp$, hence Proposition C.1.15 assures that p, q can be embedded in a single maximal commutative C*-subalgebra M

of A. It follows that $\Psi_M(q) \leq \Psi_M(p)$, so

$$\Psi(q) = [\Psi_M(q)] \leq [\Psi_M(p)] = \Psi(p).$$

Assume that $q, p \in \text{Proj}(A)$ such that $\Psi(q) \leq \Psi(p)$. We aim to show that $q \leq p$. Let M and N be maximal commutative C*-subalgebras containing p and q, respectively, such that $\Psi(q) = [\Psi_N(q)]$ and $\Psi(p) = [\Psi_M(p)]$, so $[\Psi_N(q)] \leq [\Psi_M(p)]$. It follows that there is a maximal commutative C*-subalgebra K and $\overline{D}, \overline{E} \in \mathcal{P}(K)$ such that

$$\Psi_N(q) \sim \overline{D} \leq \overline{E} \sim \overline{\Psi}_M(p).$$

Since $Z(A) \subseteq K$, it follows from Theorem 6.5.8 that $\Psi_K : \text{Proj}(K) \to \mathcal{P}(K)$ is an order isomorphism, hence there are $r, s \in \text{Proj}(K)$ such that $\Psi_K(r) = \overline{D}$ and $\Psi_K(s) = \overline{E}$. Thus

$$\Psi_N(q) \sim \Psi_K(r) \leq \Psi_K(s) \sim \Psi_M(p).$$

The first equivalence in this expression implies the existence of some B in $[Z(A), N \cap K]$ and some $\overline{G} \in \mathcal{P}(B)$ such that

$$\Psi_N(q) = \Psi_{N,B}(\overline{G}), \qquad \Psi_K(r) = \Psi_{K,B}(\overline{G}).$$

Since B is at least three-dimensional, Theorem 6.5.8 assures that

$$\Psi_B : \text{Proj}(B) \to \mathcal{P}(B)$$

is an order isomorphism, so there is some $t \in \text{Proj}(B)$ such that $\Psi_B(t) = \overline{G}$. By the same theorem, we find

$$\Psi_N(q) = \Psi_{N,B} \circ \Psi_B(t) = \Psi_N(t),$$

and since Ψ_N is an order isomorphism, we obtain $q = t$. In a similar way, we find that $\Psi_K(r) = \Psi_K(t)$, hence $r = t$. It follows that $q = r$, and in a similar way, $\Psi_K(s) \sim \Psi_M(p)$ implies $p = s$. Since $\Psi_K(r) \leq \Psi_K(s)$ and Ψ_K is an order isomorphism, we find that $r \leq s$. Thus $q \leq p$, which concludes the proof of the claim.

Finally, it follows from the last claim that $\leq$ is an order on $\mathcal{P}(A)$, the $\perp$-operation on $\mathcal{P}(A)$ is an orthocomplementation making $\mathcal{P}(A)$ an orthomodular poset, and that Ψ is an order isomorphism preserving the orthocomplementation, i.e., an orthomodular isomorphism (if we apply Lemma B.4.9). $\square$

7 Domains of commutative C*-subalgebras

In this chapter we explore the domain-theoretic aspects of $\mathcal{C}(A)$, and aim to identify classes of C*-algebras A for which $\mathcal{C}(A)$ is a domain. We refer to Appendix B.6 for all domain-theoretical notions that we use. We will see that $\mathcal{C}(A)$ is an algebraic domain if and only if A is scattered. Moreover, it turns out that weaker notions of a domain, such as quasi-continuous domains, are all equivalent in the case of $\mathcal{C}(A)$. We note that the first domain-theoretic properties of $\mathcal{C}(A)$ were investigated by Spitters in [102]. Moreover, the domain theory of the related poset $\mathcal{V}(M)$ is described by Döring and Barbosa in [27]. Their results will be generalized in the setting of AW*-algebras in Chapter 8.

7.1 Algebraicity

In this section we prove that $\mathcal{C}(A)$ is algebraic (cf. Definition B.6.4) if and only if A is a scattered C*-algebra. We first have to characterize the compact elements of $\mathcal{C}(A)$ (cf. Definition B.6.4), for which we need the following lemma.

Lemma 7.1.1. Let A be a C*-algebra, and X a compact Hausdorff space such that $C(X) \in \mathcal{C}(A)$. Let $p_1, \ldots, p_n \in X$, and denote the set of open neighborhoods of p_i by $\mathcal{O}(p_i)$ for each $i \in \{1, \ldots, n\}$. Let

$$\mathcal{D} = \left\{ \bigcap_{i=1}^{n} C_{\overline{U_i}} : U_1 \in \mathcal{O}(p_1), \ldots, U_n \in \mathcal{O}(p_n) \right\}.$$

Then $\mathcal{D}$ is a directed family in $\mathcal{C}(A)$ such that $\bigvee^{\uparrow} \mathcal{D} = C(X)$.

Proof. If $U_i, V_i \in \mathcal{O}(p_i)$, notice that $U_i \cap V_i$ is an open neighborhood of p_i. Moreover, we have

$$\overline{U_i \cap V_i} \subseteq \overline{U_i} \cap \overline{V_i} \subseteq \overline{U_i},$$

so Lemma 4.2.2 assures that $C_{\overline{U_i}} \subseteq C_{\overline{U_i \cap V_i}}$. In a similar way, we find $C_{\overline{V_i}} \subseteq C_{\overline{U_i \cap V_i}}$. Hence if $D_1 = \bigcap_{i=1}^n C_{\overline{U_i}}$ and $D_2 = \bigcap_{i=1}^n C_{\overline{V_i}}$ are elements of $\mathcal{D}$, we find that $D_1, D_2 \subseteq D$, where $D = \bigcap_{i=1}^n C_{\overline{U_i \cap V_i}}$, which is clearly an element of $\mathcal{D}$, so $\mathcal{D}$ is directed.

We use the Stone-Weierstrass Theorem in order to show that $\bigvee^{\uparrow} \mathcal{D} = C(X)$. Firstly, $\bigvee^{\uparrow} \mathcal{D}$ clearly contains 1_X, since $1_X \in D$ for each $D \in \mathcal{D}$. In order to show that $f \in \bigvee^{\uparrow} \mathcal{D}$ implies $f^* \in \bigvee^{\uparrow} \mathcal{D}$, first assume that $f \in D$ where $D = \bigcap_{i=1}^n C_{\overline{U_i}}$. Then f is constant on each $\overline{U_i}$. Since f^* is defined by

$f^*(x) = \overline{f(x)}$ for each $x \in X$, we see that f^* is constant on each $\overline{U_i}$, too, so $f^* \in D$. Now assume that $f \in \bigvee^{\uparrow} \mathcal{D}$. Then for each $n \in \mathbb{N}$, there is a $D \in \mathcal{D}$ and a $f_n \in D$ such that $\|f - f_n\| < \frac{1}{n}$. Hence each $f_n^* \in D$ for some $D \in \mathcal{D}$, since

$$\|f_n^* - f^*\| = \|f_n - f\| < \frac{1}{n},$$

it follows that $f_n^* \to f^*$ if $n \to \infty$. Thus $f^* \in \bigvee^{\uparrow} \mathcal{D}$.

Finally, let $x, y \in X$ such that $x \neq y$. We shall show that there is some $f \in \bigvee^{\uparrow} \mathcal{D}$ such that $f(x) \neq f(y)$. Since $\{p_1, \ldots, p_n\}$ is finite, it is closed, hence

$$K = \{y\} \cup (\{p_1, \ldots, p_n\} \setminus \{x\})$$

is closed. Clearly $x \notin K$, hence there are open disjoint subsets U, V such that $x \in V$ and $K \subseteq U$. Since U and V are disjoint, it follows that $x \notin \overline{U}$. Let $B = C_{\overline{U}}$. If $x \notin \{p_1, \ldots, p_n\}$, then we let $U_i = U$ for each $i \in \{1, \ldots, n\}$. If $x = p_j$, then we let $U_j = X$ and $U_i = U$ for each $i \neq j$. In both cases, $B = \bigcap_{i=1}^n C_{\overline{U_i}}$, so $B \in \mathcal{D}$. By Theorem 4.4.12, we have

$$[x]_B = \{x\} \neq \overline{U} = [y]_B,$$

hence there is some $f \in B$ such that $f(x) \neq f(y)$. We conclude that $\bigvee^{\uparrow} \mathcal{D}$ separates points of X, so by the Stone-Weierstrass Theorem, we find that $\bigvee^{\uparrow} \mathcal{D} = C(X)$. □

Proposition 7.1.2. [102, Proposition 15] Let A be a C*-algebra, then

$$\mathcal{K}(\mathcal{C}(A)) = \mathcal{C}_{\text{fin}}(A),$$

i.e., $C \in \mathcal{C}(A)$ is compact if and only if C is finite-dimensional.

Proof. Assume that C is compact. By the Gelfand–Naimark Theorem, there is some compact Hausdorff space X such that $C = C(X)$. Let $p \in X$ and consider $\mathcal{D} = \{C_{\overline{U}} : U \in \mathcal{O}(p)\}$. By Lemma 7.1.1, it follows that $\mathcal{D}$ is a directed family in $\mathcal{C}(A)$ such that $C(X) = \bigvee^{\uparrow} \mathcal{D}$. By compactness of $C = C(X)$, there is an element $C_{\overline{U}}$ in $\mathcal{D}$ such that $C(X) = C_{\overline{U}}$. Since $C(X)$ separates all points of X, it follows that $C_{\overline{U}}$ must separate all points of X as well. However, each $f \in C_{\overline{U}}$ is constant on $\overline{U}$, so $C_{\overline{U}}$ can only separate all points of X if $\overline{U} = \{x\}$. This implies that $\{x\} = U$, so $\{x\}$ is open. Since $x \in X$ was arbitrary, it follows that X is discrete. By compactness of X, it follows that X must have finite cardinality. Hence C is finite-dimensional..

Conversely, assume that C is finite-dimensional, with dimension n. By Proposition 2.1.2, C is spanned by a finite set P of orthogonal projections. Let $\mathcal{D} \subseteq \mathcal{C}(A)$ a directed family such that $C \subseteq \bigvee^{\uparrow} \mathcal{D}$. Since each projection $p \in P$ is contained in this direct limit, Proposition C.4.1 assures that there is an $D \in \mathcal{D}$ and a projection $q \in D$ such that $\|p - q\| < 1$. Since $\bigvee^{\uparrow} \mathcal{D}$ is commutative, it is *-isomorphic to $C(X)$ for some compact Hausdorff space X. It now follows from Lemma C.3.5 that $p = q$, so $p \in D$. Hence if $P = \{p_1, \ldots, p_n\}$, there are $D_1, \ldots, D_n \in \mathcal{D}$ such that $p_i \in D_i$. Since $\mathcal{D}$ is directed, there must be some $D \in \mathcal{D}$ such that $D_1, \ldots, D_n \subseteq D$. So $P \subseteq D$, which implies that $C \subseteq D$. We conclude that C is compact. □

Proposition 7.1.3. Let A be a C*-algebra. Then $\mathcal{C}(A)$ is algebraic if and only if A is scattered.

Proof. It follows from Corollary 6.2.6 that $\mathcal{C}(A)$ is algebraic if all its elements are AF-algebras. The converse follows directly from Proposition 7.1.2 and the definition of an AF-algebra. The statement now follows from Theorem 2.3.4. □

7.2 Continuity

In the previous section, we have seen that $\mathcal{C}(A)$ is algebraic if and only if A is scattered. Since the notion of a continuous domain relaxes that of an algebraic domain, it was hoped that the class of C*-algebras for which $\mathcal{C}(A)$ is a continuous domain contains the scattered C*-algebras as a proper subclass. However, we shall see that $\mathcal{C}(A)$ is continuous if and only if it is algebraic. First we have to characterize the 'way-below relation' $\ll$ on $\mathcal{C}(A)$ (cf. Definition B.6.4), for which we need the following lemma.

Lemma 7.2.1. Let X a compact Hausdorff space and let B a C*-subalgebra of $C(X)$. Then

(1) B is finite-dimensional if and only if $[x]_B \subseteq X$ is open for each $x \in X$;

(2) if X is connected, then B is the (one-dimensional) subalgebra of all constant functions on X if and only if $[x]_B$ is open for some $x \in X$;

(3) if B is not finite-dimensional, then there is an $x \in X$ and a $p \in [x]_B$ such that $B \nsubseteq C_{\overline{U}}$ for each $U \in \mathcal{O}(p)$. If X is connected, then this statement holds for each $x \in X$.

Proof.

(1) By Theorem 4.4.12, B is *-isomorphic to $C(X/\sim_B)$. Let $q : X \to X/\sim_B$ be the quotient map. By definition of the quotient topology, $V \subseteq X/\sim_B$ is open if and only if $q^{-1}[V]$ is open in X. We can regard $[x]_B$ as a subset of X, whereas we can regard $\{[x]_B\}$ as a singleton set in $X/\sim_B$. Since $[x]_B = q^{-1}[\{[x]_B\}]$, we find that $\{[x]_B\}$ is open in $X/\sim_B$ if and only if $[x]_B$ is open in X. Hence $X/\sim_B$ is discrete if and only if $[x]_B$ is open in X for each $x \in X$. Since $X/\sim_B$ is compact as the continuous image of a compact space, $X/\sim_B$ is discrete if and only if $X/\sim_B$ if finite. Hence $[x]_B$ is open in X for each $x \in X$ if and only if B is finite-dimensional.

(2) Since $\{[x]_B : x \in X\}$ is an u.s.c. decomposition of X, each $[x]_B$ is closed. Hence $[x]_B$ is open implies $[x]_B$ is clopen for each $x \in X$. Assume that X is connected and $[x]_B$ is open for some $x \in X$. Then $X = [x]_B$, so $f(y) = f(x)$ for each $f \in B$ and each $y \in X$. Hence B is the algebra of all constant functions on X, and since this algebra is spanned by 1_X, the function given by $x \mapsto 1$, it follows that B is one-dimensional. Conversely, if B is the one-dimensional subalgebra of all constant function on X, then for each $f \in B$ there is some $\lambda \in \mathbb{C}$ such that $f(x) = \lambda$ for each $x \in X$. Hence $f(x) = f(y)$ for each $x, y \in X$, whence for each $x \in X$ we have $[x]_B = X$, which is clearly open.

(3) Assume that B is not finite-dimensional. By (i) there must be some $x \in X$ such that $[x]_B$ is not open, hence there must be a point $p \in [x]_B$ such that $U \nsubseteq [x]_B$ for each $U \in \mathcal{O}(p)$. If X is connected, then (ii) implies that $[x]_B$ is not open for each $x \in X$, so p can be chosen as an element of $[x]_B$ for each $x \in X$. In both cases, we have $U \nsubseteq [x]_B$ for $U \in \mathcal{O}(p)$, hence there is an $q \in U$ such that $q \notin [x]_B$. We have $y \in [x]_B$ if and only if $f(x) = f(y)$ for each $f \in B$. So $p \in [x]_B$ and $q \notin [x]_B$ implies the existence of some $f \in B$ such that $f(p) \neq f(q)$. That is, there is some $f \in B$ such that f is not constant on U, so f is certainly not constant on $\overline{U}$. We conclude that for each $U \in \mathcal{O}(p)$ there is an $f \in B$ such that $f \notin C_{\overline{U}}$, so $B \nsubseteq C_{\overline{U}}$ for each $U \in \mathcal{O}(p)$. $\square$

Proposition 7.2.2. Let A be a C*-algebra and $B, C \in \mathcal{C}(A)$. Then the following statements are equivalent:

(1) $B \ll C$;

(2) $B \in \mathcal{K}(\mathcal{C}(A))$ and $B \subseteq C$;

(3) B is finite-dimensional and $B \subseteq C$.

Proof. By Proposition 7.1.2, B is finite-dimensional if and only if B is compact, which proves the equivalence between (2) and (3).

(2) $\Longrightarrow$ (1) follows from directly from Lemma B.6.5. For (1) $\Longrightarrow$ (3), assume that $B \subseteq C$ and that B is not finite-dimensional. Let X be the spectrum of C, so $C = C(X)$. By Lemma 7.2.1, we find that there is a $p \in X$ such that $B \nsubseteq C_{\overline{U}}$ for each $U \in \mathcal{O}(p)$. Let $\mathcal{D} = \{C_{\overline{U}} : U \in \mathcal{O}(p)\}$, then Lemma 7.1.1 assures that $\mathcal{D}$ is a directed family in $\mathcal{C}(A)$ such that $\bigvee^{\uparrow} \mathcal{D} = C(X)$. However, $B \nsubseteq D$ for each $D \in \mathcal{D}$, so B cannot be way below $C = C(X)$. □

Proposition 7.2.3. Let A be a C*-algebra. Then $\mathcal{C}(A)$ is algebraic if and only if it is continuous (cf. Definition B.6.4).

Proof. Let $C \in \mathcal{C}(A)$. By Proposition 7.2.2, we have $\twoheaddownarrow C = \mathcal{K}(\mathcal{C}) \cap \downarrow C$, whence $C = \bigvee^{\uparrow} \mathcal{K}(\mathcal{C}) \cap \downarrow C$ if and only if $C = \bigvee^{\uparrow} \twoheaddownarrow C$. □

7.3 Quasidomains

We see that $\mathcal{C}(A)$ is not necessarily continuous for each C*-algebra A. The question rises whether $\mathcal{C}(A)$ satisfies certain weaker conditions than algebraicity and continuity. In this section, we investigate when $\mathcal{C}(A)$ is a quasidomain. We first have to characterize the generalized way-below relation in $\mathcal{C}(A)$ (cf. Definition B.6.12). We refer to Definition B.6.13 for the notions of quasialgebraic and quasicontinuous domains as well as the definition of the sets $\mathrm{Fin}(C)$ and $\mathrm{CompFin}(C)$.

Lemma 7.3.1. Let A be a C*-algebra, $C \in \mathcal{C}(A)$, and $\mathcal{F} \subseteq \mathcal{C}(A)$. Then $\mathcal{F} \in \mathrm{Fin}(C)$ if and only if $\mathcal{F}$ contains finitely many elements and $F \ll C$ for some $F \in \mathcal{F}$.

Proof. Let $\mathcal{F}$ contain finitely many elements and assume that $F \ll C$ for some $F \in \mathcal{F}$. Let $\mathcal{D} \subseteq \mathcal{C}(A)$ be directed such that $C \subseteq \bigvee \mathcal{D}$. Since $F \ll C$, we have $F \subseteq D$ for some $D \in \mathcal{D}$, so $D \in \uparrow \mathcal{F}$. Thus $\mathcal{F} \in \mathrm{Fin}(C)$.

Conversely, $\mathcal{F} \in \mathrm{Fin}(C)$. Then we have $\mathcal{F} \ll C$, and $\mathcal{F}$ is non-empty and finite. Let $\mathcal{D} = \{C\}$. Then $\mathcal{D}$ is directed and $C \subseteq \bigvee^{\uparrow} \mathcal{D}$, so there is some $F \in \mathcal{F}$ such that $F \subseteq C$. Since $\mathcal{F}$ contains a finite number of elements, so does C. Let $\{F_1, \ldots, F_n\}$ be the subset of $\mathcal{F}$ of all elements contained in C and assume that each F_i has infinite dimension. Let X be the spectrum of C, so $C = C(X)$, then Lemma 7.2.1 implies the existence of points $p_1, \ldots, p_n \in X$ such that $F_j \nsubseteq C_{\overline{U_j}}$ for each $U_j \in \mathcal{O}(p_j)$. In particular $F_j \nsubseteq \bigcap_{i=1}^n C_{\overline{U_i}}$ for each $U_i \in \mathcal{O}(p_i)$, where $i \in \{1, \ldots, n\}$.

Let

$$\mathcal{D} = \left\{ \bigcap_{i=1}^{n} C_{\overline{U_i}} : U_i \in \mathcal{O}(p_i), i = 1, \ldots, n \right\}.$$

By Lemma 7.1.1, we have $\bigvee^{\uparrow} \mathcal{D} = C(X)$, but $F_i \nsubseteq D$ for each $D \in \mathcal{D}$. If $F \in \mathcal{F}$ such that $F \nsubseteq C$, we cannot have $F \subseteq D$ for some $D \in \mathcal{D}$, since each D is contained in C by construction of $\mathcal{D}$, hence we obtain a contradiction with $\mathcal{F} \ll C$. We conclude that there must be a finite-dimensional $F \in \mathcal{F}$ such that $F \subseteq C$. By Proposition 7.2.2 it follows that $F \ll C$. □

Lemma 7.3.2. Let A be a C*-algebra and let $C \in \mathcal{C}(A)$. If $F \in\downarrow C$, then $\{F\} \in \mathrm{CompFin}(C)$. If $\mathcal{F} \in \mathrm{Fin}(C)$, then there is an $\mathcal{F}' \in \mathrm{CompFin}(C)$ such that $\mathcal{F} \leq \mathcal{F}'$.

Proof. Let $F \ll C$. By Lemma 7.3.1, we have $\{F\} \in \mathrm{Fin}(C)$. By Lemma 7.2.2, we have $F \ll F$, so $\{F\} \ll \{F\}$, whence $\{F\} \in \mathrm{CompFin}(C)$.

Let $\mathcal{F} \in \mathrm{Fin}(C)$. By Lemma 7.3.1, there is an $F \in \mathcal{F}$ such that $F \ll C$. By the previous, we find that $\{F\} \in \mathrm{CompFin}C$. Since $F \in \mathcal{F}$, we have $F \in \uparrow \mathcal{F}$, so $\uparrow \{F\} \subseteq \uparrow \mathcal{F}$. We conclude that $\mathcal{F} \leq \mathcal{F}'$, where $\mathcal{F}' = \{F\}$. □

Proposition 7.3.3. Let A be a C*-algebra. Then the following are equivalent:

- $\mathcal{C}(A)$ is continuous;
- $\mathcal{C}(A)$ is quasialgebraic;
- $\mathcal{C}(A)$ is quasicontinuous.

Proof. Assume $\mathcal{C}(A)$ is continuous and let $C \in \mathcal{C}(A)$. Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{CompFin}(C)$. Since

$$\mathrm{CompFin}(C) \subseteq \mathrm{Fin}(C),$$

it follows by Lemma 7.3.1 that there are $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ such that $F_1, F_2 \ll C$. Hence $F_1, F_2 \in \downarrow C$, and since $\downarrow C$ is directed by continuity of $\mathcal{C}(A)$, it follows that there is some $F \in \downarrow C$ such that $F_1, F_2 \subseteq F$. Let $\mathcal{F} = \{F\}$. By Lemma 7.3.2, we find that $\mathcal{F} \in \mathrm{CompFin}(C)$. Since $F_1, F_2 \subseteq F$, we obtain $\mathcal{F} = \{F\} \subseteq \uparrow\mathcal{F}_1 \cap \uparrow\mathcal{F}_2$, so $\mathrm{CompFin}(C)$ is directed.

Let $B \in \mathcal{C}(A)$ such that $C \nsubseteq B$. We have to show that $B \notin \uparrow\mathcal{F}$ for some $\mathcal{F} \in \mathrm{CompFin}(C)$. Assume the contrary, so $B \in \uparrow\mathcal{F}$ for each $\mathcal{F} \in \mathrm{CompFin}(C)$. By Lemma 7.3.2 we have $\{F\} \in \mathrm{CompFin}(C)$ for each $F \in \downarrow C$, whence $F \subseteq B$ for each $F \in \downarrow C$. Hence $\bigvee \downarrow C \subseteq B$, and by continuity of $\mathcal{C}(A)$ we have $\bigvee \downarrow C = C$, so $C \subseteq B$. We clearly obtain a contradiction, hence $\mathcal{C}(A)$ must be quasialgebraic.

Assume that $\mathcal{C}(A)$ is quasialgebraic and let $C \in \mathcal{C}(A)$. Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Fin}(C)$. By Lemma 7.3.2, there are $\mathcal{F}_1', \mathcal{F}_2' \in \mathrm{CompFin}(C)$ such that $\mathcal{F}_i \leq \mathcal{F}_i'$. By quasialgebraicity of $\mathcal{C}(A)$, $\mathrm{CompFin}(C)$ is directed, so there is an $\mathcal{F}$ in $\mathrm{CompFin}(C)$ such that $\mathcal{F}_1', \mathcal{F}_2' \leq \mathcal{F}$, so $\mathcal{F}_1, \mathcal{F}_2 \leq \mathcal{F}$. Since

$$\mathrm{CompFin}(C) \subseteq \mathrm{Fin}(C),$$

it follows that $\mathrm{Fin}(C)$ is directed. Let $B \in \mathcal{C}(A)$ such that $C \nsubseteq B$. Assume that there is an $\mathcal{F} \in \mathrm{Fin}(C)$ such that $B \in \uparrow\mathcal{F}$. By Lemma 7.3.2, there is an $\mathcal{F}' \in \mathrm{CompFin}(C)$ such that $\mathcal{F} \leq \mathcal{F}'$. This means that $\uparrow\mathcal{F} \subseteq \uparrow\mathcal{F}'$. Hence we have $B \in \uparrow\mathcal{F}'$, which contradicts the quasialgebraicity of $\mathcal{C}(A)$. Hence we must have $B \notin \uparrow\mathcal{F}$ for each $\mathcal{F} \in \mathrm{Fin}(C)$, so $\mathcal{C}(A)$ is quasicontinuous.

Now assume that $\mathcal{C}(A)$ is quasicontinuous. Let $F_1, F_2 \in \downarrow C$. By Lemma 7.3.1, we have $\{F_1\}, \{F_2\} \in \mathrm{Fin}(C)$, and since $\mathrm{Fin}(C)$ is directed, there is an $\mathcal{F} \in \mathrm{Fin}(C)$ such that $\mathcal{F} \subseteq \uparrow\{F_1\} \cap \uparrow\{F_2\}$. In other words, we have $F_1, F_2 \subseteq F$ for each $F \in \mathcal{F}$, and since $\mathcal{F} \in \mathrm{Fin}(C)$, Lemma 7.3.1 assures the existence of some F such that $F \ll C$, so $\downarrow C$ is directed. Let $B = \bigvee \downarrow C$. By Lemma B.6.5, we have $F \subseteq C$ for each $F \in \downarrow C$, hence $B \subseteq C$. If $B \neq C$, then $C \nsubseteq B$, so by quasicontinuity of $\mathcal{C}(A)$, there must be an $\mathcal{F} \in \mathrm{Fin}(C)$ such that $B \notin \uparrow\mathcal{F}$. Hence $F \nsubseteq B$ for each $F \in \mathcal{F}$, and in particular Lemma 7.3.1 implies the existence of some $F \in \mathcal{F}$ such that $F \ll C$, but $F \nsubseteq B$. By definition of B we have $F \subseteq B$ for each $F \ll C$, hence we obtain a contradiction. Thus $\mathcal{C}(A)$ is continuous. □

7.4 Overview and consequences

We can now collect all our results in this chapter.

Theorem 7.4.1. Let A be a C*-algebra. Then the following statements are equivalent:

(1) A is scattered;

(2) Each $M \in \max \mathcal{C}(A)$ is scattered;

(3) Each $C \in \mathcal{C}(A)$ is scattered;

(4) $\mathcal{C}(A) = \mathcal{C}_{\mathrm{AF}}(A)$, i.e., each $C \in \mathcal{C}(A)$ is an AF-algebra;

(5) $\mathcal{C}(A)$ is atomistic;

(6) $\mathcal{C}(A)$ is algebraic;

(7) $\mathcal{C}(A)$ is continuous;

(8) $\mathcal{C}(A)$ is quasialgebraic;

(9) $\mathcal{C}(A)$ is quasicontinuous.

Proof. The equivalences between (1), (2), (3), and (4) follow from Theorem 2.3.4. It follows from Proposition 7.1.3 and Corollary 6.4.3 that (1), (5) and (6) are equivalent. The equivalence between (6) and (7) is stated in Proposition 7.2.3. The equivalence between (7), (8) and (9) follows from Proposition 7.3.3. □

Corollary 7.4.2. Let A be a C*-algebra.

- If A is scattered, then $\mathcal{C}(A)$ is meet-continuous;
- If A has a commutative C*-subalgebra that is an AF-algebra but not scattered, then $\mathcal{C}(A)$ is not meet-continuous.

Proof. It follows from Theorem 7.4.1 and Proposition B.6.7 that $\mathcal{C}(A)$ is meet-continuous if A is scattered. Assume now that $\mathcal{C}(A)$ contains an element C that is an AF-algebra, but not scattered. Moreover, assume that $\mathcal{C}(A)$ is meet-continuous. By Theorem 7.4.1, $\downarrow C = \mathcal{C}(C)$ contains an element B that is not an AF-algebra. Since $\mathcal{C}(A)$ is meet-continuous, we obtain

$$B = B \cap C(X) = B \cap \bigvee \mathcal{D} = \bigvee \{B \cap D : D \in \mathcal{D}\},$$

and since $B \cap D$ is finite-dimensional for each $D \in \mathcal{D}$ for D is finite-dimensional, it follows that B must be an AF-algebra. We obtain a contradiction, hence $\mathcal{C}(A)$ cannot be meet-continuous. $\square$

We note that we did not exhaust all possibilities: the question whether $\mathcal{C}(A)$ is meet-continuous if A is a C*-algebra that has only commutative C*-subalgebras that have connected Gelfand spectra (or equivalently, $\mathcal{C}(A) \setminus \mathbb{C}1_A$ does not contain elements that are AF-algebras) is still open.

Finally, we can equip $\mathcal{C}(A)$ with the so-called *Lawson topology* (cf. Definition B.6.9). The next proposition states the topological properties of $\mathcal{C}(A)$ when A is scattered.

Proposition 7.4.3. Let A be a scattered C*-algebra. Then the domain $\mathcal{C}(A)$ is a Stone space in the Lawson topology. Moreover, $\mathcal{C}(A)$ is scattered in the Lawson topology if and only if A is finite-dimensional.

Proof. By Theorem 3.1.3, $\mathcal{C}(A)$ is a complete semilattice. Theorem 7.4.1 assures that $\mathcal{C}(A)$ is algebraic if A is scattered. It now follows from Theorem B.6.10 that $\mathcal{C}(A)$ is a zero-dimensional compact Hausdorff space, hence a Stone space, when equipped with the Lawson topology. Recall that a basis for the Lawson topology on $\mathcal{C}(A)$ is given by sets of the form $\mathcal{U} \setminus \uparrow \mathcal{F}$ with $\mathcal{F} \subseteq \mathcal{C}(A)$ finite and $\mathcal{U}$ Scott open. Since $\mathcal{C}(A)$ is algebraic, it follows from Proposition B.6.8 that sets of the form $\Uparrow C$ for C compact form a basis for the Scott topology on $\mathcal{C}(A)$. Thus sets of the form $\uparrow C \setminus \uparrow \mathcal{F}$ with C compact and $\mathcal{F}$ finite form a basis for the Lawson topology. Let A have finite dimension, so it is certainly scattered. Take a nonempty subset $\mathcal{S} \subseteq \mathcal{C}(A)$. Since A is finite-dimensional, $\mathcal{C}(A)$ is Noetherian by Theorem 5.1.2, hence $\mathcal{S}$ contains a maximal element M be a maximal element. Since M must be finite-dimensional too, it is compact by Proposition 7.1.2. Hence $\uparrow M$ is Scott open and therefore Lawson open. Maximality of M in $\mathcal{S}$ now gives $\mathcal{S} \cap \uparrow M = \{M\}$, and since $\uparrow M$ is Lawson open, it follows that M is an isolated point of $\mathcal{S}$. Hence $\mathcal{C}(A)$ is scattered.

Conversely, assume A is infinite-dimensional. Then by Theorem 5.1.2, $\mathcal{C}(A)$ has an infinite-dimensional, hence non-compact element C. Then $\mathcal{S} = \downarrow C$ contains an isolated point if $\mathcal{S} \cap \mathcal{U}$ is a singleton for some basic Lawson open $\mathcal{U}$. Hence $\downarrow C \cap \uparrow K \setminus \uparrow \mathcal{F}$ must be a singleton for some finite set $\mathcal{F} \subseteq \mathcal{C}(A)$ and some compact $K \in \mathcal{C}(A)$. In other words, $[K, C] \setminus \uparrow \mathcal{F}$ is a singleton, where we recall that

$$[K, C] = \{D \in \mathcal{C}(A) \mid K \subseteq D \subseteq C\}.$$

We show that there are infinitely many atoms in $[K, C]$. Firstly, K is finite-dimensional, say n-dimensional, so K is spanned by n orthogonal projections (Proposition 2.1.2) adding up to 1_A. C is infinite-dimensional, and it follows from Theorem 7.4.1 that C is scattered. By Corollary 2.3.7, C has infinitely many minimal projections, and by Corollary 6.4.3, $\downarrow C = \mathcal{C}(C)$ is atomistic. It follows from Lemma 6.4.1 that $\text{Span}\{p, 1_A - p\}$ is an atom of $\mathcal{C}(C)$ for each minimal projection $p \in C$. Since K is finite-dimensional, it contains only finitely many projections. Hence there are infinitely many minimal projections $p \in C$ such that $p \notin K$. Let let $p_1, \ldots, p_n$ the orthogonal projections spanning K and let $p \in C$ be a minimal projection such that $p \notin K$. Then p is linearly independent of $p_1, \ldots, p_n$. Moreover, $p = p1_A = p_1 p + \ldots + p_n p$. Since p is minimal, and commutes with each p_i, we find that $pp_i = p$ or $pp_i = 0$. It follows that there is exactly one $j \in \{1, \ldots, n\}$ such that $pp_j = p$, so $p \leq p_j$; the other p_i are orthogonal to p. Note that $p \notin \text{Span}\{p_1, \ldots, p_n\}$ implies that $p < p_j$. Hence $(1 - p)p_j = p_j - p$ is non-zero. It follows that

$$\{p_1, \ldots, p_{j-1}, p, p_j - p, p_{j+1}, \ldots, p_n\}$$

is an orthogonal collection of projections, which by Proposition 2.1.2 spans a C*-subalgebra D_p of C that clearly contains K. Since $\dim D_p = \dim K + 1$, it follows from Lemma 5.1.4 that D_p covers K. Let q be another minimal projection in C such that $q \notin K$. Then $q \neq p_1, \ldots, p_n, p$, hence it is possible that $D_p = D_q$ only if $p_j - p$ is a minimal projection in C, in which case we necessarily have $q = p_j - p$. Since there are infinitely many minimal projections in C not contained in K, it follows that there are infinitely many different covers D_p of K in $\downarrow C$. In other words, there are infinitely many atoms in $[K, C]$. Hence there is no finite subset $\mathcal{F} \subseteq \mathcal{C}(A)$ making $[K, C] \setminus \uparrow \mathcal{F}$ a singleton. We conclude that $\downarrow C$ has no isolated points, so $\mathcal{C}(A)$ cannot be scattered. □

Thus $B := C(\mathcal{C}(A))$ is a commutative AF-algebra if A is scattered. Moreover, we can consider the Lawson topology on $\mathcal{C}(B)$. Only if B is a scattered C*-algebra it is assured that $\mathcal{C}(B)$ is again a Stone space. Now, B is scattered if and only if its Gelfand spectrum $\mathcal{C}(A)$ is scattered. The previous proposition stated that $\mathcal{C}(A)$ is only scattered if A is finite-dimensional.

8 AW*-algebras

In this chapter we study AW*-algebras by means of their commutative C*-subalgebras. Despite the fact that AW*-algebras are an algebraic generalization of von Neumann algebras, most operator algebraists seem to find the former less are interesting than the latter. However, AW*-algebras seem to be more natural objects of study by means of their commutative subalgebras than von Neumann algebras, for two reasons. Firstly, it is possible to decide whether a C*-algebra A is an AW*-algebra by the order structure of $\mathcal{C}(A)$; such a decision procedure is not known for von Neumann algebras. Secondly, given an AW*-algebra A, we can consider the subposet $\mathcal{A}(A)$ of $\mathcal{C}(A)$ consisting of the AW*-subalgebras of A. In case that A is a von Neumann algebra, we will see that $\mathcal{A}(A)$ is equal to $\mathcal{V}(A)$, the poset of von Neumann subalgebras of A. The latter poset has been studied before [25, 27], but it turns out that most statements of $\mathcal{V}(A)$ can be generalized to $\mathcal{A}(A)$ for any AW*-algebra A.

In §8.1, we define the poset $\mathcal{A}(A)$ and study its most general properties. In §8.2, we show that $\mathcal{A}$ is a functor from the category of AW*-algebras to the category of dcpo's. In §8.4, we show that $\mathcal{A}(A)$ can be identified in order-theoretic terms as a subposet of $\mathcal{C}(A)$. Finite-dimensional AW*-algebras are studied from the point of view of $\mathcal{A}(A)$ in §8.3. As a consequence, we obtain a description of the domain theory of $\mathcal{A}(A)$. In §8.5, the AW*-analogue of Hamhalter's Theorem is proven. Moreover, it is shown that $\mathcal{A}(A)$ also contains all information about the projection lattice of A. In §8.6, we show that $\mathcal{A}(A)$ determines homogeneous AW*-algebras up to *-isomorphism. Finally, in the last section, we outline steps of a possible proof of the statement that $\mathcal{A}(A)$ determines type I AW*-algebras up to *-isomorphism.

8.1 Posets of commutative AW*-subalgebras

In this section, we investigate the first properties of the poset of commutative AW*-subalgebras of an AW*-algebra.

Definition 8.1.1.

- Let A be an AW*-algebra. We denote the set of its commutative AW*-subalgebras by $\mathcal{A}(A)$, which as usual becomes a poset if we order it by inclusion;

- The subposet of $\mathcal{A}(A)$ consisting of the finite-dimensional AW*-subalgebras of A is denoted by $\mathcal{A}_{\text{fin}}(A)$;
- Let M be a von Neumann algebra in $B(H)$ for some Hilbert space H. Then we denote the poset of all commutative C*-subalgebras of M that are also von Neumann algebras in $B(H)$ by $\mathcal{V}(M)$.

Theorem 8.1.2. Let A be an AW*-algebra. Then:

(a) $\mathcal{C}_{\text{fin}}(A) = \mathcal{A}_{\text{fin}}(A) \subseteq \mathcal{A}(A) \subseteq \mathcal{C}_{\text{AF}}(A) \subseteq \mathcal{C}(A)$, where all inclusions becomes equalities if A is finite-dimensional;

(b) The AW*-subalgebra $\mathbb{C}1_A$ is the least element of $\mathcal{A}(A)$;

(c) The infimum $\bigwedge \mathcal{S}$ of a non-empty subset $\mathcal{S} \subseteq \mathcal{A}(A)$ is given by $\bigcap \mathcal{S}$;

(d) The supremum $\bigvee \mathcal{S}$ of a subset $\mathcal{S} \subseteq \mathcal{A}(A)$ exists if and only all elements of $\bigcup \mathcal{S}$ commute, in which case $\bigvee \mathcal{S} = AW^*(\bigcup \mathcal{S})$. In particular $\bigvee \mathcal{S}$ exists if $\mathcal{S}$ is directed;

(e) For each $C \in \mathcal{A}(A)$, there is an $M \in \max \mathcal{A}(A)$ such that $C \subseteq M$. In particular, $\max \mathcal{A}(A)$ is non-empty, and equal to $\max \mathcal{C}(A)$;

(f) The center $Z(A)$ is the infimum of $\max \mathcal{A}(A)$;

(g) $\mathcal{A}(A) = \mathcal{V}(A)$ if A is a von Neumann algebra.

In particular, $\mathcal{A}(A)$ is a complete semilattice.

Proof.

(a) By definition of an AW*-algebra, we have $\mathcal{A}(A) \subseteq \mathcal{C}(A)$. By Corollary 2.4.29, we have $\mathcal{C}_{\text{fin}}(A) = \mathcal{A}_{\text{fin}}(A)$. Moreover, if $C \in \mathcal{A}(A)$, then C has extremally disconnected spectrum, which is certainly totally disconnected. It follows from Theorem 2.2.3 that C is an AF-algebra, hence $C \in \mathcal{C}_{\text{AF}}(A)$.

If A is finite-dimensional, then we clearly have $\mathcal{C}(A) = \mathcal{C}_{\text{fin}}(A)$, hence all inclusions become equalities.

(b) Since $\mathbb{C}1_A$ is the least element of $\mathcal{C}(A)$ and an element of $\mathcal{C}_{\text{fin}}(A)$, this follows from (a).

(c) This follows directly from Lemma 2.4.18.

(d) Clearly $AW^*(\bigcup \mathcal{S})$ is the smallest AW*-subalgebra of A containing all elements of $\mathcal{S}$. Moreover, since all elements of $\mathcal{S}$ are *-closed, it follows that $\bigcup \mathcal{S}$ is *-closed. By Lemma 2.4.20, $AW^*(\bigcup \mathcal{S})$ is commutative if $\bigcup \mathcal{S}$ is commutative. The converse is trivial. If $\mathcal{S}$ is directed, then for each $x, y \in \bigcup \mathcal{S}$, there are $C_1, C_2 \in \mathcal{S}$ such that $x \in C_1$ and $y \in C_2$. Since $\mathcal{S}$ is directed, there is some $C \in \mathcal{S}$ such that $C_1, C_2 \subseteq C$, hence x and y must commute, for they are contained in some commutative AW*-subalgebra. We conclude that $\bigcup \mathcal{S}$ consists of pairwise commuting elements, hence $\bigvee \mathcal{S}$ exists in $\mathcal{A}(A)$.

(e) It follows from Lemma 2.4.13 and (e) of Theorem 3.1.3 that $\max \mathcal{A}(A)$ is nonempty and equal to $\max \mathcal{C}(A)$, and that each $C \in \mathcal{A}(A)$ is contained in some $M \in \max \mathcal{A}(A)$.

(f) This follows from (c), (e), and by (f) of Theorem 3.1.3.

(g) By Proposition 2.4.16, the von Neumann algebras on a Hilbert space H coincide with the AW*-subalgebras of $B(H)$. Hence assume that A is a von Neumann algebra on H. Let $C \in \mathcal{A}(A)$, i.e., C is an AW*-subalgebra of A. By Proposition 2.4.16, A is an AW*-subalgebra of $B(H)$, hence a von Neumann algebra on H, so $C \in \mathcal{V}(A)$. Conversely, if $C \in \mathcal{V}(A)$, then C is a C*-subalgebra of A that is a von Neumann algebra on H. Hence C is an AW*-subalgebra of $B(H)$. Since also A is an AW*-subalgebra of $B(H)$, which moreover contains C as a C*-subalgebra, it follows from Lemma 2.4.17 that C is an AW*-subalgebra of A. Hence $C \in \mathcal{A}(A)$.

It follows from (c) and (d) that $\mathcal{A}(A)$ is a complete semilattice. ☐

8.2 Functoriality of the map $A \mapsto \mathcal{A}(A)$

The next theorem describes the properties of $\mathcal{A}$ as a functor from the category of AW*-algebras to the category of dcpo's.

Theorem 8.2.1. $\mathcal{A}$: **AWStar** $\to$ **DCPO** becomes a functor if, for each AW*-homomorphism $\varphi : A \to B$ between AW*-algebras A and B, we define $\mathcal{A}(\varphi) : \mathcal{A}(A) \to \mathcal{A}(B)$ by $C \mapsto \varphi[C]$. Moreover, $\mathcal{A}(\varphi)$ has the following properties:

(a) If $S \subseteq A$ is a commutative *-closed subset, then

$$\mathcal{A}(\varphi)(AW^*(S)) = AW^*(\varphi[S]).$$

(b) If $\{C_i\}_{i\in I} \subseteq \mathcal{A}(A)$ is a family such that $\bigvee_{i\in I} C_i$ exists, then $\bigvee_{i\in I} \mathcal{A}(\varphi)(C_i)$ exists, and

$$\mathcal{A}(\varphi)\left(\bigvee_{i\in I} C_i\right) = \bigvee_{i\in I} \mathcal{A}(\varphi)(C_i). \tag{37}$$

In particular, $\mathcal{A}(\varphi)$ is Scott continuous.

(c) If φ is injective, or if A is commutative, then $\mathcal{A}(\varphi)$ has an upper adjoint

$$\mathcal{A}(\varphi)_* : \mathcal{A}(B) \to \mathcal{A}(A), \quad D \mapsto \varphi^{-1}[D],$$

which satisfies

$$\mathcal{A}(\varphi)_*\left(\bigcap_{j\in J} D_j\right) = \bigcap_{j\in J} \mathcal{A}(\varphi)_*(D_j), \tag{38}$$

for each family $\{D_i\}_{i\in I} \subseteq \mathcal{C}(B)$ such that $I \neq \emptyset$.

(d) If φ is injective, then $\mathcal{A}(\varphi)$ is an order embedding such that

$$\mathcal{A}(\varphi)\left(\bigcap_{i\in I} C_i\right) = \bigcap_{i\in I} \mathcal{A}(\varphi)(C_i), \tag{39}$$

for each family $\{C_i\}_{i\in I} \subseteq \mathcal{A}(A)$ such that $I \neq \emptyset$. Moreover, the following identities hold:

$$\begin{aligned} \mathcal{A}(\varphi)_* \circ \mathcal{A}(\varphi) &= 1_{\mathcal{A}(A)}; \\ \mathcal{A}(\varphi) \circ \mathcal{A}(\varphi)_*|_{\mathcal{A}(\varphi)[\mathcal{A}(A)]} &= 1_{\mathcal{A}(B)}\big|_{\mathcal{A}(\varphi)[\mathcal{A}(A)]}, \end{aligned}$$

and

$$\downarrow \mathcal{A}(\varphi)[\mathcal{A}(A)] = \mathcal{A}(\varphi)[\mathcal{A}(A)].$$

(e) If $\mathcal{A}(\varphi)$ is surjective, then so is φ.

(f) If φ is a *-isomorphism, then $\mathcal{A}(\varphi)$ is an order isomorphism.

Proof. We note that $\mathcal{A}(A) \subseteq \mathcal{C}(A)$ and $\mathcal{A}(B) \subseteq \mathcal{C}(B)$. Consider $\mathcal{C}(\varphi) : \mathcal{C}(A) \to \mathcal{C}(B)$. If $C \in \mathcal{A}(A)$, then $\mathcal{C}(\varphi)(C) = \varphi[C]$ is an element of $\mathcal{A}(B)$ by Proposition 2.4.25, since φ is an AW*-homomorphism. Hence $\mathcal{C}(\varphi)$ restricts to an order morphism $\mathcal{A}(A) \to \mathcal{A}(B)$, and clearly, this restriction is $\mathcal{A}(\varphi)$. Hence

$\mathcal{A} : \mathbf{AWStar} \to \mathbf{Poset}$ is a functor. It follows from (b) that $\mathcal{A}$ is actually a functor $\mathbf{CStar} \to \mathbf{DCPO}$.

(a) The same as the proof of Theorem 3.2.1(a) except some variations, namely we use Lemma 2.4.20 in order to guarantee that $AW^*(S) \in \mathcal{A}(A)$, and Proposition 2.4.25 for the statement that $\varphi^{-1}[AW^*(\varphi[S])$ is an AW*-subalgebra of A.

(b) The same as the proof of Theorem 3.2.1(b), except that we use Theorem 8.1.2 instead if Theorem 3.1.3.

(c) Assume that φ is injective, or A is commutative. It follows from Theorem 3.2.1 that $\mathcal{C}(\varphi)$ has an upper adjoint $D \mapsto \varphi^{-1}[D]$. Since $\mathcal{A}(\varphi)$ is the restriction of $\mathcal{C}(\varphi)$ to $\mathcal{A}(A)$, it follows that $\mathcal{A}(\varphi)$ is an order embedding. Moreover, if $D \in \mathcal{A}(B)$, then it follows from Proposition 2.4.25 that

$$\mathcal{C}(\varphi)_*(D) = \varphi^{-1}[D]$$

is an element of $\mathcal{A}(A)$. Hence it can be restricted to an order morphism $\mathcal{A}(B) \to \mathcal{A}(A)$, which is the upper adjoint of $\mathcal{A}(\varphi)$. By Lemma B.1.23, $\mathcal{A}(\varphi)_*$ preserves all existing infima, hence (38) holds.

(d) Assume that φ is injective. Since $\mathcal{A}(\varphi)$ is the restriction of $\mathcal{C}(\varphi)$, it follows from Theorem 3.2.1 that $\mathcal{A}(\varphi)$ is an order embedding and that (39) holds. By (c), we know that $\mathcal{A}(\varphi)$ has an upper adjoint $\mathcal{A}(\varphi)_*$, $D \mapsto \varphi^{-1}[D]$. By Theorem 3.2.1, we have

$$\mathcal{C}(\varphi)_* \circ \mathcal{C}(\varphi) = 1_{\mathcal{C}(A)},$$

and

$$\mathcal{C}(\varphi) \circ \mathcal{C}(\varphi)_*|_{\mathcal{C}(\varphi)[\mathcal{C}(A)]} = 1_{\mathcal{C}(B)}\big|_{\mathcal{C}(\varphi)[\mathcal{C}(A)]},$$

and $\mathcal{A}(\varphi)_*$ is the restriction of $\mathcal{C}(\varphi)$ to $\mathcal{A}(B)$. Hence we must have

$$\mathcal{A}(\varphi)_* \circ \mathcal{A}(\varphi) = 1_{\mathcal{A}(A)},$$

and

$$\mathcal{A}(\varphi) \circ \mathcal{A}(\varphi)_*|_{\mathcal{A}(\varphi)[\mathcal{A}(A)]} = 1_{\mathcal{A}(B)}\big|_{\mathcal{A}(\varphi)[\mathcal{A}(A)]}.$$

In order to show that $\mathcal{A}(\varphi)[\mathcal{A}(A)]$ is a down-set in $\mathcal{A}(B)$, we first note that

$$\mathcal{A}(\varphi)[\mathcal{A}(A)] \subseteq \mathcal{C}(\varphi)[\mathcal{C}(A)],$$

where the right-hand side is a down-set in $\mathcal{C}(B)$ by Theorem 3.2.1. Hence if $D \in \mathcal{A}(\varphi)[\mathcal{A}(A)]$ and $E \in \mathcal{A}(B)$ such that $E \subseteq D$, then $E = \mathcal{C}(\varphi)(C)$ for some $C \in \mathcal{C}(A)$. Now,

$$C = \mathcal{C}(\varphi)_*\mathcal{C}(\varphi)(C) = \mathcal{C}(\varphi)_*(E) = \mathcal{A}(\varphi)_*(E),$$

since $E \in \mathcal{A}(B)$, so $C \in \mathcal{A}(A)$. Because

$$\mathcal{A}(\varphi)(C) = \mathcal{C}(\varphi)(C) = E,$$

we find that $E \in \mathcal{A}(\varphi)[\mathcal{A}(A)]$.

(e) The same as the proof of Theorem 3.2.1(e), with the exception that we use Lemma 2.4.20 instead of Lemma C.1.22.

(f) This follows directly from functoriality of $\mathcal{A}$ and the fact that φ has an inverse. □

8.3 Finite-dimensional AW*-algebras

Theorem 5.1.2 specifies order-theoretic conditions on $\mathcal{C}(A)$ that are equivalent with A being finite-dimensional. Theorem 7.4.1 states under which conditions $\mathcal{C}(A)$ is a domain. The next theorem is the AW*-analogue of these two theorems: they both state when $\mathcal{A}(A)$ is a domain and specify conditions on $\mathcal{A}(A)$ corresponding to A being a finite-dimensional AW*-algebra.

Theorem 8.3.1. Let A be an AW*-algebra. Then the following statements are equivalent:

(1) A is finite-dimensional;

(2) $\mathcal{A}(A)$ is Noetherian;

(3) $\mathcal{A}(A)$ is an algebraic domain;

(4) $\mathcal{A}(A)$ is a continuous domain.

Proof. Theorem 8.1.2 assures that $\mathcal{A}(A) = \mathcal{C}(A)$ if A is finite-dimensional. It follows from Theorem 5.3.1 that $\mathcal{A}(A)$ is Noetherian. So (1) $\Longrightarrow$ (2).

The implications (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) follow from Proposition B.6.7. We show (4) $\Longrightarrow$ (1) by contraposition. Thus, assume that A is not finite-dimensional. It follows from Proposition 2.1.3 that A has a maximal commutative C*-subalgebra M that is not finite-dimensional. By Theorem 2.4.5, M is generated by its projections, hence it must have a zero-dimensional spectrum X by Theorem 2.2.3. Since M is infinite-dimensional, X must have infinite elements, and there must be a non-isolated point $x \in X$, otherwise we obtain a contradiction with the compactness of X. Choose any $y_1 \in X$ such that $y_1 \neq x$. Since X is Hausdorff, there are disjoint open V_1 and W_1 such that $x \in V_1$ and $y_1 \in W_1$. Since X is zero-dimensional, there must be a clopen U_1 such that $y_1 \in U_1 \subseteq W_1$. We proceed by induction. Let $U_1, \ldots, U_n$ be a finite collection of mutually disjoint clopen subsets such that $x \notin \bigcup_{i=1}^n U_i$. Since the latter set is clopen, it follows that $X \setminus \bigcup_{i=1}^n U_i$ is an open subset containing x. Since x is not isolated, there must be some $y_{n+1} \in X \setminus \bigcup_{i=1}^n U_i$ such that $x \neq y_{n+1}$. Then $\bigcup_{i=1}^n U_i \cup \{x\}$ and $\{y_{n+1}\}$ are two disjoint closed sets, and since X is normal, there are disjoint open V_{n+1} and W_{n+1} such that $\bigcup_{i=1}^n U_i \cup \{x\} \subseteq V_{n+1}$ and $y_{n+1} \in W_{n+1}$. Since X is zero-dimensional, we can choose a clopen U_{n+1} such that $y_{n+1} \subseteq U_{n+1} \subseteq W_{n+1}$. Since $V_{n+1} \cap W_{n+1} = \emptyset$, it follows that $U_i \cap U_{n+1} = \emptyset$ for each $i \in \{1, \ldots, n\}$. We obtain an infinite sequence $U_1, U_2, \ldots$ of non-empty clopen subsets that are mutually disjoint. Let $\chi_1, \chi_2, \ldots$ be the characteristic functions. Then each χ_i is a non-zero projection in $C(X)$, and since the U_i are mutually disjoint, we find that $\chi_i \chi_j = 0$ if $i \neq j$. Thus, under the *-isomorphism between $C(X)$ and M, we obtain an infinite set P of pairwise orthogonal projections in M.

Let $I \subseteq P$ be infinite such that its complement in P is infinite as well, and let $p = \bigvee I$. Since I contains non-zero projections, we have $p \neq 0$. Choose some non-zero $q \in P \setminus I$, then $rq = 0$ for each $r \in I$, hence $pq = 0$ by Proposition 2.4.10, so $p \neq 1$. It follows from Lemma 6.4.1 that $C^*(p)$ is an atom of $\mathcal{C}(A)$. Consider

$$\mathcal{D} = \{C^*(F) : F \subseteq P \text{ is finite}\}.$$

By Proposition 2.1.2, every element of $\mathcal{D}$ is finite-dimensional. It now follows from Corollary 2.4.29 that all elements of $\mathcal{D}$ and $C^*(p)$ are elements of $\mathcal{A}(A)$. Since $C^*(p)$ is an atom of $\mathcal{C}(A)$, it is an atom of $\mathcal{A}(A)$ as well. Notice that $\mathcal{D}$ is directed: each $C^*(F_1)$ $C^*(F_2)$ in $\mathcal{D}$ are contained in $C^*(F_1 \cup F_2)$, which is also an element of $\mathcal{D}$, for $F_1 \cup F_2$ is finite. By Theorem 3.1.3, $\bigvee \mathcal{D}$ exists in $\mathcal{A}(A)$ and is equal to $AW^*(\bigcup \mathcal{D})$. Notice that $I \subseteq \bigcup \mathcal{D}$, so $AW^*(I) \subseteq \bigvee \mathcal{D}$, hence $p = \bigvee I$ is an element of $\bigvee \mathcal{D}$, whence $C^*(p) \subseteq \bigvee \mathcal{D}$. We show that $p \notin D$ for each $D \in \mathcal{D}$. So, let $D \in \mathcal{D}$ and let $\{p_1, \ldots, p_n\} \subseteq P$ be finite such that $D = C^*(p_1, \ldots, p_n)$. We note that $p_{n+1} := 1_A - \sum_{i=1}^n p_i$ is an element in $C^*(p_1, \ldots, p_n)$, which is orthogonal to p_j for each $j \in \{1, \ldots, n\}$, since

$$p_j p_{n+1} = p_j \left(1_A - \sum_{i=1}^n p_i\right) = p_j - \sum_{i=1}^n p_j p_i = p_j - p_j = 0.$$

Hence $\sum_{i=1}^{n+1} p_i = 1_A$. It follows from Lemma C.3.10 that

$$\begin{aligned}\mathrm{Span}\{p_1, \ldots, p_n, 1_A\} &= \mathrm{Span}\,\{p_1, \ldots, p_n, p_{n+1}\} \\ &= C^*(p_1, \ldots, p_n, p_{n+1}) \\ &= C^*(p_1, \ldots, p_n) = D,\end{aligned}$$

where we used $p_{n+1} \in C^*(p_1, \ldots, p_n)$ in the second equality. If $p \in D$, then $p = \sum_{i=1}^n \lambda_i p_i + \lambda 1_A$ for some $\{\lambda_1, \ldots, \lambda_n, \lambda\} \subseteq \mathbb{C}$. Since I is infinite, there is is a non-zero $q \in I$ such that $q \neq p_i$ for each $i \in \{1, \ldots, n\}$. Since q and the p_i are elements of P, so orthogonal, and $p = \bigvee I$, so $q \leq p$, we find

$$q = qp = q \left(\sum_{i=1}^n \lambda_i p_i + \lambda 1_A\right) = \sum_{i=1}^n \lambda_i q p_i + \lambda q = \lambda q,$$

hence $\lambda = 1$, for q is non-zero. Since $P \setminus I$ is infinite, there is some non-zero $q \notin I$ such that $q \neq p_i$ for each $i \in \{1, \ldots, n\}$. By orthogonality of P, and since $q \notin I$, we have $qr = 0$ for each $r \in I$, hence $qp = 0$ using Proposition 2.4.10. Moreover, since $q \neq p_i$, we have $qp_i = 0$ for each $i \in \{1, \ldots, n\}$. We obtain

$$0 = qp = q \left(\sum_{i=1}^n \lambda_i p_i + 1_A\right) = \sum_{i=1}^n \lambda_i q p_i + q = q,$$

which gives a contradiction, since we assumed that $q \neq 0$. We conclude that p cannot be an element of D. Thus, $C^*(p) \nsubseteq D$ for each $D \in \mathcal{D}$. It follows that $C^*(p)$ is not compact, and since it is an atom of $\mathcal{A}(A)$, we find that the set of all elements way below $C^*(p)$ only contains $\mathbb{C}1_A$. Hence

$$\bigvee \twoheaddownarrow C^*(p) = \bigvee\{\mathbb{C}1_A\} = \mathbb{C}1_A \neq C^*(p),$$

and we conclude that $\mathcal{A}(A)$ is not continuous. □

The proof for the implication (4) $\Longrightarrow$ (1) in the previous proposition is taken from [27], where the equivalence between (1) and (4) is stated for von Neumann algebras. It is remarkable that a C*-algebra A can be infinite-dimensional with $\mathcal{C}(A)$ algebraic, whereas $\mathcal{A}(A)$ is never algebraic if A is an infinite-dimensional AW*-algebra. The reason why the argument in the proof of the previous proposition for the implication (4) $\Longrightarrow$ (1) must fail in the C*-algebra case is quite subtle, namely we have $p \in AW^*(\bigcup \mathcal{D})$, since $AW^*(\bigcup \mathcal{D})$ is an AW*-algebra, so it is closed under suprema of projections. However, we have $p \notin C^*(\bigcup \mathcal{D})$, since the norm closure of an inductive limit of commutative algebras does not yield extra projections. If A is finite-dimensional, then $\mathcal{A}(A) = \mathcal{C}(A)$, hence it follows from Theorem 5.1.2 that $\mathcal{A}(A)$ is Artinian and order-scattered. It is not clear yet whether the converse holds as well.

Corollary 8.3.2. Let A be a finite-dimensional AW*-algebra and B an arbitrary AW*-algebra. Then $\mathcal{A}(A) \cong \mathcal{A}(B)$ implies $A \cong B$.

Proof. Since A is finite-dimensional, it follows from Theorem 8.3.1 that $\mathcal{A}(A)$ is Noetherian. Hence $\mathcal{A}(B)$ is Noetherian, whence B must be finite-dimensional as well. By Theorem 8.1.2, we must have

$$\mathcal{C}(A) = \mathcal{A}(A) \cong \mathcal{A}(B) = \mathcal{C}(B).$$

Theorem 5.3.1 now assures the existence of a *-isomorphism $\varphi : A \to B$, which is automatically an AW*-isomorphism by Lemma 2.4.23. □

We found that $\mathcal{C}(A)$ is a domain if A is scattered. We also learned that $\mathcal{A}(A)$ is only a domain if A is finite-dimensional. The question is whether there exists an infinite-dimensional scattered AW*-algebra A, and if so, why $\mathcal{C}(A)$ is a domain whereas $\mathcal{A}(A)$ is not. Corollary 2.4.30 states that the answer to the first question is negative, whence the second question does not have any meaning.

8.4 The relation between $\mathcal{C}(A)$ and $\mathcal{A}(A)$

In this section we show that if A and B are AW*-algebras, any order isomorphism between $\mathcal{C}(A)$ and $\mathcal{C}(B)$ restricts to an order isomorphism between $\mathcal{A}(A)$ and $\mathcal{A}(B)$. Moreover, we show that each order isomorphism $\mathcal{A}(A) \to \mathcal{A}(B)$ uniquely extends to an order isomorphism $\mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$, and conversely, each order isomorphism $\mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$ restricts to an order isomorphism $\mathcal{A}(A) \to \mathcal{A}(B)$.

Proposition 8.4.1. Let A be an AW*-algebra and B a C*-algebra.

(a) If $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$ is an order isomorphism, then B is an AW*-algebra, and Φ restricts to an order isomorphism $\mathcal{A}(A) \to \mathcal{A}(B)$.

Assume, in addition, that B is an AW*-algebra, too.

(b) If $\Phi : \mathcal{A}(A) \to \mathcal{A}(B)$ is an order isomorphism, then it can uniquely be extended to an order isomorphism $\mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$;

(c) If $\Phi : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$ is an order isomorphism, then it restricts to an order isomorphism $\mathcal{A}(A) \to \mathcal{A}(B)$.

Proof. We first note $\mathcal{A}(A) \subseteq \mathcal{C}_{\mathrm{AF}}(A)$ by Theorem 8.1.2. Moreover, we have

$$\max \mathcal{A}(A) = \max \mathcal{C}_{\mathrm{AF}}(A),$$

since $\max \mathcal{C}(A) = \max \mathcal{A}(A)$ (Theorem 8.1.2) and each maximal commutative C*-subalgebra of an AW*-algebra is generated by its projections (Theorem 2.4.5), hence an AF-algebra (Theorem 2.2.3). In a similar way, we have $\mathcal{A}(B) \subseteq \mathcal{C}_{\mathrm{AF}}(B)$ and $\max \mathcal{A}(B) = \max \mathcal{C}_{\mathrm{AF}}(B)$ if B is an AW*-algebra.

It will be convenient to prove (c) first, followed by (b), and then (a). Hence assume that B is an AW*-algebra and that

$$\Phi : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$$

is an order isomorphism. Let $C \in \mathcal{A}(A)$ and let M be a maximal commutative C*-subalgebra of A such that $C \subseteq M$ (whose existence is assured by Proposition C.1.15). It follows from Lemma 2.4.13 that M is an AW*-subalgebra of A, so $M \in \max \mathcal{A}(A)$. It follows that $M \in \max \mathcal{C}_{\mathrm{AF}}(A)$. Since Φ is an order isomorphism, we find that $\Phi(M) \in \max \mathcal{C}_{\mathrm{AF}}(B)$. Moreover, we have

$$\max \mathcal{A}(B) = \max \mathcal{C}_{\mathrm{AF}}(B),$$

hence $\Phi(M)$ is a maximal commutative AW*-subalgebra of B. Now Lemma 2.4.17 assures that C is an AW*-subalgebra of M. Since $\downarrow M \subseteq \mathcal{C}_{\mathrm{AF}}(A)$ is order isomorphic to $\mathcal{C}_{\mathrm{AF}}(M)$, it follows that Φ restricts to an order isomorphism

$$\Phi_M : \mathcal{C}_{\mathrm{AF}}(M) \to \mathcal{C}_{\mathrm{AF}}(\Phi(M)).$$

By Corollary 6.3.5 it follows that there is a *-isomorphism $\varphi_M : M \to \Phi(M)$ such that $\mathcal{C}_{\mathrm{AF}}(\varphi_M) = \Phi_M$. By Theorem 8.2.1, this implies that

$$\mathcal{A}(\varphi_M) : \mathcal{A}(M) \to \mathcal{A}(\Phi(M))$$

is an order isomorphism. Since $\mathcal{C}_{\mathrm{AF}}(\varphi_M) = \Phi_M$, we find

$$\Phi(C) = \Phi_M(C) = \mathcal{C}_{\mathrm{AF}}(\varphi_M)(C) = \varphi_M[C] = \mathcal{A}(\varphi_M)(C),$$

hence $\Phi(C) \in \mathcal{A}(\Phi(M))$. It follows that $\Phi(C)$ is an AW*-subalgebra of $\Psi(M)$, hence $\Phi(C)$ is an AW*-subalgebra of B by Lemma 2.4.17. We conclude that Φ restricts to an order isomorphism $\mathcal{A}(A) \to \mathcal{A}(B)$.

For (b), let $\Phi : \mathcal{A}(A) \to \mathcal{A}(B)$ be an order isomorphism. Let $C \in \mathcal{A}(A)$ be finite-dimensional. Since $\downarrow C = \mathcal{A}(C)$, it follows from Theorem 8.3.1 that $\downarrow C$ is Noetherian. Since Φ is an order isomorphism, we find $\downarrow C \cong \downarrow \Phi(C)$, and since $\mathcal{A}(\Phi(C)) = \downarrow \Phi(C)$, we find that $\mathcal{A}(\Phi(C))$ is Noetherian. Again using Theorem 8.3.1, we find that $\Phi(C)$ is finite-dimensional. It follows that Φ restricts to an order isomorphism between $\mathcal{A}_{\mathrm{fin}}(A)$ and $\mathcal{A}_{\mathrm{fin}}(B)$. By Theorem 8.1.2, this is an order isomorphism $\Phi : \mathcal{C}_{\mathrm{fin}}(A) \to \mathcal{C}_{\mathrm{fin}}(B)$. By Theorem 6.2.4, we have an order isomorphism

$$\Psi_A : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{B}(\mathrm{Proj}(A))$$

such that $\Phi_A(C) = \mathrm{Proj}(C)$ for each $C \in \mathcal{C}_{\mathrm{AF}}(A)$ and $\Phi_A^{-1}(D) = C^*(D)$ for each $D \in \mathcal{B}(\mathrm{Proj}(A))$. By Proposition 6.1.2, $\mathcal{B}(\mathrm{Proj}(A))$ is an algebraic complete semilattice, and the finite Boolean subalgebras of $\mathrm{Proj}(A)$ are exactly the compact elements of $\mathcal{B}(\mathrm{Proj}(A))$. Now, if D is a finite Boolean subalgebra, then it follows from Proposition 2.1.2 that $\Psi_A^{-1}(D) = C^*(D)$ is finite-dimensional. The same proposition assures that $\Psi_A(C) = \mathrm{Proj}(C)$ is a finite Boolean subalgebra of $\mathrm{Proj}(A)$. Hence Φ_A restricts to an order isomorphism between $\mathcal{C}_{\mathrm{fin}}(A)$ and $\mathcal{K}(\mathcal{B}(\mathrm{Proj}(A)))$, the subposet of compact elements of $\mathcal{B}(\mathrm{Proj}(A))$, and in a similar way, it follows that there is an order isomorphism between $\mathcal{C}_{\mathrm{fin}}(B)$ and $\mathcal{K}(\mathcal{B}(\mathrm{Proj}(B)))$. Hence, $\mathcal{C}_{\mathrm{fin}}(A)$ and $\mathcal{C}_{\mathrm{fin}}(B)$ are exactly the sets of compact elements of the algebraic dcpo's $\mathcal{C}_{\mathrm{AF}}(A)$ and $\mathcal{C}_{\mathrm{AF}}(B)$, respectively. It now follows from Proposition B.6.6 that there is a unique order isomorphism $\Psi : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$ extending $\Phi : \mathcal{C}_{\mathrm{fin}}(A) \to \mathcal{C}_{\mathrm{fin}}(B)$.

By (c), Ψ restricts to an order isomorphism $\mathcal{A}(A) \to \mathcal{A}(B)$. Hence if C is an element of $\mathcal{A}(A)$, then $\Psi(C)$ is an AW*-subalgebra of B. For a moment, we will denote the supremum in $\mathcal{C}(A)$ and $\mathcal{C}(B)$ by $\bigvee$, and the supremum in $\mathcal{A}(A)$ and $\mathcal{A}(B)$ by sup. Lemma 6.2.2 assures that if $\mathcal{D} \subseteq \mathcal{C}_{\mathrm{AF}}(A)$ is directed, its supremum equals $\bigvee \mathcal{D}$. Since $C \in \mathcal{C}_{\mathrm{AF}}(A)$, there is a directed subset $\mathcal{D} \subseteq \mathcal{C}_{\mathrm{fin}}(A)$ such that $C = \bigvee \mathcal{D}$. Since $\Psi : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$ is an order isomorphism, we find $\Psi(C) = \bigvee \Psi[\mathcal{D}]$. By Theorem 3.1.3, we find $C = C^*(\bigcup \mathcal{D})$ and $\Psi(C) = C^*(\bigcup \Psi[\mathcal{D}])$. Since C and $\Psi(C)$ are AW*-subalgebras of A and B, respectively, it follows from Lemma 2.4.21 that $C = AW^*(\bigcup \mathcal{D})$ and $\Psi(C) = AW^*(\bigcup \mathcal{D})$. By Theorem 8.1.2, we obtain $C = \sup \mathcal{D}$ and $\Psi(C) = \sup \Psi[\mathcal{D}]$. It follows that

$$\Psi(C) = \sup \Phi[\mathcal{D}] = \Phi[\sup \mathcal{D}] = \Phi(C),$$

where the first equality holds since Φ and Ψ agree on $\mathcal{C}_{\mathrm{fin}}(A)$, and the second since $\Phi : \mathcal{A}(A) \to \mathcal{A}(B)$ is an order isomorphism. We conclude that Ψ extends Φ. Let $\Delta : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$ be another extension of $\Phi : \mathcal{A}(A) \to \mathcal{A}(B)$. Then Δ extends the restriction of Φ to $\mathcal{C}_{\mathrm{fin}}(A)$, and since Ψ is the unique order isomorphism $\mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$ extending $\Phi : \mathcal{C}_{\mathrm{fin}}(A) \to \mathcal{C}_{\mathrm{fin}}(B)$, it follows that $\Delta = \Psi$.

Finally, we prove (a). We first show that if B is a C*-algebra, then the existence of an order isomorphism $\mathcal{C}(A) \to \mathcal{C}(B)$ implies that B must be an AW*-algebra. By Theorem 6.4.4, we find that $\mathrm{Proj}(A) \cong \mathrm{Proj}(B)$. Hence if $\mathrm{Proj}(A)$ is a complete lattice, it follows that $\mathrm{Proj}(B)$ is complete as well. Since A is an AW*-algebra, it follows from Lemma 2.4.13 that its maximal abelian C*-subalgebras are AW*-algebras. Let $M \in \mathcal{C}(A)$ be maximal. By Proposition B.1.15, there must be an $N \in \max \mathcal{C}(A)$ such that $\Phi(N) = M$. Since A is an AW*-algebra, N must be an AW*-algebra as well. Now, $\downarrow M = \mathcal{C}(M)$ and $\downarrow N = \mathcal{C}(N)$, hence Φ restricts to an order isomorphism $\mathcal{C}(N) \to \mathcal{C}(M)$. Theorem 4.7.5 now implies that N is *-isomorphic to M, hence M is an AW*-algebra. By Corollary 2.4.6, the spectrum of M is extremally disconnected, hence totally disconnected by Lemma A.2.3. It follows from Theorem 2.2.3 that M is generated by its projections. Thus B has a complete projection lattice, and all its maximal abelian C*-subalgebras of B are generated by their projections. It follows that B is an AW*-algebra. It now follow from Theorem 6.4.4 that Φ

restricts to an order isomorphism $\mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$, which restricts by (c) to an order isomorphism $\mathcal{A}(A) \to \mathcal{A}(B)$. □

8.5 Commutative AW*-algebras and projection lattices

If A and B are commutative C*-algebras, it follows from Theorem 4.7.5 that $A \cong B$. The von Neumann algebraic analogue of this statement, i.e., that $\mathcal{V}(M) \cong \mathcal{V}(N)$ implies $M \cong N$ for each pair of commutative von Neumann algebras M and N, is due to Döring and Harding [25]. Moreover, they showed that if M and N are non-commutative von Neumann algebras, an order isomorphism $\Phi : \mathcal{V}(M) \to \mathcal{V}(N)$ is uniquely induced by an Jordan isomorphism (see also §9.1) as long as M does not have a type I_2 summand. In order to prove this statement, they showed that Φ is induced by an orthomodular isomorphism $\varphi : \mathrm{Proj}(M) \to \mathrm{Proj}(N)$.

In this section, we extend Döring and Harding's results in the reconstruction of commutative algebras and in the reconstruction of the projection lattices to the class of AW*-algebras. It should be noted that the proofs for $\mathcal{A}(A)$ are essentally the same as Döring and Harding's proofs for $\mathcal{V}(M)$ in [25]. We start by a proposition that is the AW*-analogue of Proposition 4.1.1. We skip the proof, since it is essentially the same as the proof of Proposition 4.1.1, except that one has to use Theorem 8.1.2 instead of Theorem 3.1.3, and Lemma 2.4.20 instead of Lemma C.1.22.

Proposition 8.5.1. Let A be an AW*-algebra. Then the following statements are equivalent:

(a) A is commutative;

(b) $\mathcal{A}(A)$ has a greatest element;

(c) $\mathcal{A}(A)$ is bounded;

(d) $\mathcal{A}(A)$ is a complete lattice.

Theorem 8.5.2. Let A be a commutative AW*-algebra and B be an AW*-algebra. If $\Phi : \mathcal{A}(A) \to \mathcal{A}(B)$ is an order isomorphism, then there is a *-isomorphism $\varphi : A \to B$ such that $\mathcal{A}(\varphi) = \Phi$. Moreover, if $\dim A \neq 2$, then φ is the unique *-isomorphism inducing Φ in this way.

Proof. By Theorem 8.1.2, we have $\mathcal{A}(A) \subseteq \mathcal{C}_{\mathrm{AF}}(A)$ and $\mathcal{A}(B) \subseteq \mathcal{C}_{\mathrm{AF}}(B)$. Using Proposition 8.5.1, we find that $\mathcal{A}(A)$ is a complete lattice, hence $\mathcal{A}(B)$ is a complete lattice, so B must be a commutative as well. By Proposition 8.4.1, Φ can uniquely be extended to an order isomorphism $\Psi : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$. Since A and B are a commutative AF-algebras, we can apply Corollary 6.3.5, which yields a *-isomorphism $\varphi : A \to B$ such that $\mathcal{C}_{\mathrm{AF}}(\varphi) = \Psi$, and which uniquely induces Ψ if $\dim A \neq 2$. Hence for each $C \in \mathcal{A}(A)$, we find

$$\Phi(C) = \Psi(C) = \mathcal{C}_{\mathrm{AF}}(\varphi)(C) = \varphi[C] = \mathcal{A}(\varphi)(C),$$

so $\Phi = \mathcal{A}(\varphi)$. Assume that $\dim A \neq 2$ and let $\psi : A \to B$ be another *-isomorphism such that $\mathcal{A}(\psi) = \Phi$. By Theorem 3.2.1, $\mathcal{C}(\psi) : \mathcal{C}(A) \to \mathcal{C}(B)$ is an order isomorphism, which restricts by Theorem 6.4.4 to

$$\mathcal{C}_{\mathrm{AF}}(\psi) : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B).$$

By Proposition 8.4.1, $\mathcal{C}_{\mathrm{AF}}(\psi)$ restricts to $\mathcal{A}(\psi) : \mathcal{A}(A) \to \mathcal{A}(B)$, which can be rephrased by the statement that $\mathcal{C}_{\mathrm{AF}}(\psi)$ is the extension of $\mathcal{A}(\psi)$. Since $\mathcal{A}(\psi) = \Phi$ and Ψ is the unique extension of Φ, we obtain $\mathcal{C}_{\mathrm{AF}}(\psi) = \Psi$. Since $\dim A \neq 2$, φ is the unique *-isomorphism inducing Ψ, hence $\varphi = \psi$. ☐

Theorem 8.5.3. Let A and B be AW*-algebras, and let $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ be an orthomodular isomorphism. Then

$$\mathcal{A}(A) \to \mathcal{A}(B), \quad C \mapsto C^*(\varphi[\mathrm{Proj}(C)])$$

is an order isomorphism. Conversely, if $\Phi : \mathcal{A}(A) \to \mathcal{A}(B)$ is an order isomorphism, then there exists an orthomodular isomorphism $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ such that

$$\Phi(C) = C^*(\varphi[\mathrm{Proj}(C)]) = AW^*(\varphi[\mathrm{Proj}(C)]), \tag{40}$$

for each $C \in \mathcal{A}(A)$. Moreover, φ is the unique orthomodular isomorphism inducing Φ in this way if A is not *-isomorphic to either $\mathbb{C}^2$ or $\mathrm{M}_2(\mathbb{C})$.

Proof. Let $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ be an orthomodular isomorphism. Then $\mathcal{B}(\varphi) : \mathcal{B}(\mathrm{Proj}(A)) \to \mathcal{B}(\mathrm{Proj}(B))$, $D \mapsto \varphi[D]$ is an order isomorphism by Theorem 6.1.6, hence $\mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$, $C \mapsto C^*(\varphi[\mathrm{Proj}(C)])$ is an order isomorphism by Theorem 6.2.4, which restricts to an order isomorphism $\mathcal{A}(A) \to \mathcal{A}(B)$ by Proposition 8.4.1.

Conversely, assume that $\Phi : \mathcal{A}(A) \to \mathcal{A}(B)$ is an order isomorphism. By Proposition 8.4.1, there is a unique order isomorphism $\Psi : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$ extending Φ. By Theorem 6.4.4, we find an orthomodular isomorphism $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ such that

$$\Psi(C) = C^*(\varphi[\mathrm{Proj}(C)])$$

for each $C \in \mathcal{C}_{\mathrm{AF}}(A)$, and φ is the unique orthomodular isomorphism inducing Ψ in this way if $\mathrm{Proj}(A)$ does not have blocks of four elements. Let $C \in \mathcal{A}(A)$. Since Ψ extends Φ, we obtain that the first equality in (40). Since $\Phi(C)$ is an AW*-subalgebra of B, it follows from Lemma 2.4.21 that the second equality holds, too.

Assume that A is neither *-isomorphic to $\mathbb{C}^2$ nor to $\mathrm{M}_2(\mathbb{C})$. Since the maximal commutative C*-subalgebras of AW*-algebras are generated by projections (Theorem 2.4.5), we can apply Corollary 6.2.7 to conclude that $\mathrm{Proj}(A)$ does not have any blocks of precisely four elements. Let $\psi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ be an orthomodular isomorphism such that $\Phi(C) = C^*(\psi[\mathrm{Proj}(C)])$ for each $C \in \mathcal{A}(A)$. By Theorem 6.1.6, it follows that $\mathcal{B}(\psi) : \mathcal{B}(\mathrm{Proj}(A)) \to \mathcal{B}(\mathrm{Proj}(B))$ is an order isomorphism. By Theorem 6.2.4, we find that the map $\Delta : \mathcal{C}_{\mathrm{AF}}(A) \to \mathcal{C}_{\mathrm{AF}}(B)$, given for each $C \in \mathcal{C}_{\mathrm{AF}}(A)$ by

$$\Delta(C) = C^*(\mathcal{B}(\psi)(\mathrm{Proj}(C))) = C^*(\psi[\mathrm{Proj}(C)]),$$

is an order isomorphism. The restriction of Δ to $\mathcal{A}(A)$ is Φ. However, Ψ is the unique extension of Φ, hence $\Delta = \Psi$. It follows that $\Psi(C) = C^*(\psi[\mathrm{Proj}(C)])$, for each $C \in \mathcal{C}_{\mathrm{AF}}(A)$, but since $\mathrm{Proj}(A)$ does not have any blocks of precisely four elements, φ is the unique orthomodular isomorphism inducing Ψ, it follows that $\varphi = \psi$. □

8.6 Structure of AW*-algebras

If we recall the Artin-Wedderburn Theorem, we see that finite-dimensional C*-algebras can be classified by decomposing them as a direct sum of C*-algebras with trivial center. The number of factors in this decomposition corresponds to the dimension of the center, and the factors themselves can be classified by their dimension. This give a complete invariant for finite-dimensional C*-algebras. A similar invariant (though not complete) can be constructed for von Neumann algebras, by showing that each von Neumann algebra M can be written as a *direct integral* (generalizing the direct sum) over the center of M such that each factor is a von Neumann algebra. Hence classifying von Neumann algebras comes down to the classification of von Neumann factors. Murray and von Neumann made considerable progress by proving that there are three different types of von Neumann factors. Moreover, they showed that the factors in the direct integral decomposition can be rearranged, to the effect that every von Neumann algebra can be written as a direct sum of at most three von Neumann algebras, each of whose direct integral decomposition consists solely of factors of a single type.

For AW*-algebras, we cannot rely on Hilbert space techniques, in particular we cannot always decompose an AW*-algebra by means of a direct integral. Nevertheless, it turns out that large parts of the type classification of von Neumann algebras can be carried over to a type classification of AW*-algebras, which is consistent with the special case of von Neumann algebras. We give definitions and state some classification theorems, but we omit most proofs, since these rely on techniques going beyond the scope of this thesis.

Definition 8.6.1. An AW*-algebra A is called *finite* if $a^*a = 1_A$ implies $aa^* = 1_A$ for each $a \in A$.

Examples 8.6.2.

- Any finite-dimensional C*-algebra is finite, which follows from [6, Proposition 17.1];
- Any commutative AW*-algebra is finite. It follows that finite AW*-algebras need not to be finite-dimensional.

For the next definition, recall Lemma 2.4.4: pAp is an AW*-algebra if A is an AW*-algebra and $p \in A$ a projection.

Definition 8.6.3. Let A be an AW*-algebra and $p \in A$ a projection. Then

- p is called *finite* if pAp is a finite AW*-algebra;
- p is called *abelian* if pAp is a commutative AW*-algebra;
- p is called *central* if $p \in Z(A)$.

Since commutative AW*-algebras are finite, it follows that an abelian projection is always a finite projection.

There is an alternative, more usual definition of finite projections, for which we need the following definition.

Definition 8.6.4. Let A be a C*-algebra. An element $w \in A$ is called a *partial isometry* if $ww^*w = w$. Two projections $p, q \in A$ are said to be *Murray-von Neumann equivalent*, or just briefly *equivalent*, if $p = w^*w$ and $q = ww^*$ for some partial isometry $w \in A$, in which case we write $p \sim q$.

It is not difficult to show that the Murray-von Neumann equivalence is indeed an equivalence relation. A proof can be found in [6, Proposition 1.7].

Lemma 8.6.5. Let A be an AW*-algebra and $p \in A$ a projection. Then p is finite if and only if $p \sim q \leq p$ implies $p = q$ for each projection $q \in A$.

Thus we can also characterize finite projections in terms of the Murray-von Neumann equivalence relation. Since this relation is not very suitable for our purposes, we do not give a proof of the previous lemma, but refer to [6, Proposition 15.2]. Indeed, we aim to describe as much as possible in terms of commutative subalgebras. However, if p and q are distinct projections such that $p \sim q$, then the partial isometry w such that $p = w^*w$ and $q = ww^*$ is not normal. Hence it cannot be embedded into a commutative C*-subalgebra, which makes it difficult to reconstruct the Murray-von Neumann equivalence relation from $\mathcal{A}(A)$ or $\mathcal{C}(A)$. It turns out that we can circumvent the Murray-von Neumann equivalence relation if we want to decide from $\mathcal{C}(A)$ and $\mathcal{A}(A)$ whether A has finite projections or not (cf. Corollary 8.6.14 below).

Definition 8.6.6. Let A be a C*-algebra and $p \in A$ a projection. If there is a projection $q \in Z(A)$ such that

(1) $p \leq q$;

(2) $p \leq r$ implies $q \leq r$ for each projection $r \in Z(A)$,

then q is called the *central cover* of p. We write $q = C(p)$. If $C(p) = 1_A$, then p is called a *faithful* projection.

Lemma 8.6.7. [6, Proposition 6.3] Let A be an AW*-algebra. Then each projection $p \in A$ has an central cover $C(p)$ satisfying $R(pA) = \big(1 - C(p)\big)A$.

Proof. By Lemma 2.4.18, $Z(A)$ is an AW*-algebra. Hence $\mathrm{Proj}(Z(A))$ is a complete lattice. Since $1_A \in \mathrm{Proj}(Z(A))$, the set

$$S = \{q \in \mathrm{Proj}(Z(A)) : p \leq q\}$$

is non-empty. Since $Z(A)$ is an AW*-algebra, the infimum of S in $\mathrm{Proj}(Z(A))$ exists, which is exactly the central cover $C(p)$ of p.

Since A is an AW*-algebra, there is a unique $q \in \mathrm{Proj}(A)$ such that $R(pA) = qA$. Let $a, x \in A$ and $y \in R(pA)$. Then $pby = 0$ for each $b \in A$, hence taking $b = ax$ gives $paxy = 0$. Since $a \in A$ is arbitrary, it follows that $pAxy = 0$. We conclude that $xy \in R(pA)$ for each $x \in A$ and $y \in R(pA)$, i.e., we have $AR(pA) \subseteq R(pA)$, hence $AqA \subseteq qA$. Since $Aq \subseteq AqA$, this implies $Aq \subseteq qA$.

Let $x \in A$ be self-adjoint. Then $xq \in qA$, hence $xq = qy$ for some $y \in A$. It follows that

$$xq = q^2 y = qxq,$$

so

$$xq = (qxq)^* = (xq)^* = qx.$$

Now let $x \in A$ be arbitrary, and write $x = x_1 + ix_2$ with $x_1 = \frac{x+x^*}{2}$ and $x_2 = \frac{x-x^*}{2i}$. Then x_1 and x_2 are self-adjoint, whence $qx = xq$. We conclude that $q \in Z(A)$, which also implies that $1 - q \in Z(A)$. Now $pq = 0$ since $q \in R(pA)$, hence $p(1-q) = p$, i.e., $p \leq 1-q$. Since $1-q \in Z(A)$, it follows that $C(p) \leq 1-q$. On the other hand, if $a, x \in A$, it follows from $C(p) \in Z(A)$ that

$$pa\big(1 - C(p)\big)x = pax - paC(p)x = pax - pC(p)ax = pax - pax = 0,$$

whence $\big(1 - C(p)\big)x \in R(pA)$ for each $x \in A$. Thus $\big(1 - C(p)\big)A \subseteq R(pA) = qA$, and by Proposition C.3.2, we find that $1 - C(p) \leq q$. Thus

$$1 - C(p) = q\big(1 - C(p)\big) = q - qC(p),$$

whence $1 - q = C(p)(1 - q)$, so $1 - q \leq C(p)$. Thus $1 - q = C(p)$. □

We can now define the type classification for AW*-algebras.

Definition 8.6.8. Let A be an AW*-algebra. Then

- A is called a *type I* AW*-algebra if it contains a faithful abelian projection.
- A is called a *type II* AW*-algebra if it contains a faithful finite projection and 0 is the only abelian projection in A. If A is a type II algebra that is finite (or equivalently, such that 1_A is a finite projection), we say that it is *type II*$_1$. If 0_A is the only finite *central* projection, then we say that A is *type II*$_\infty$;

- A is called a *type III AW*-algebra* AW*-algebra if 0 is the only finite projection in A.

Since $C(0) = 0$, it follows that 0 is never a faithful projection. We already observed that abelian projections are finite. Hence an AW*-algebra cannot be of more than one type. The next theorem is a consequence of Proposition 10.2 and Theorem 15.3 of [6].

Theorem 8.6.9. Let A be an AW*-algebra. Then there exists a unique decomposition

$$A = A_{\mathrm{I}} \oplus A_{\mathrm{II}_1} \oplus A_{\mathrm{II}_\infty} \oplus A_{\mathrm{III}},$$

such that A_{I} is a type I AW*-algebra, A_{II_1} is a type II_1 AW*-algebra, A_{II_∞} is a type II_∞ AW*-algebra, and A_{III} is a type III AW*-algebra.

Definition 8.6.10. Let A be an AW*-algebra. If there exists an orthogonal collection $\{p_j\}_{j\in J}$ of abelian projections in A such that $\bigvee_{j\in J} p_j = 1_A$ and $p_i \sim p_j$ for each $i, j \in J$, we way that A is a *homogeneous* AW*-algebra of order card J.

Lemma 8.6.11. Let A be an AW*-algebra and p and q abelian projections. Then $p \sim q$ if and only if $C(p) = C(q)$.

Proof. We do not give a full proof, but collect some results in [6] whose combination prove the statement. Firstly, [6, Proposition 6.1(v)] shows that $p \sim q$ implies $C(p) = C(q)$ in each ring R with a proper involution, i.e., $x^*x = 0$ implies $x = 0$. Clearly C*-algebras, and in particular AW*-algebras have a proper involution, which proves the 'ony if' direction. The 'if' direction follows from combining Corollary 14.1 and Proposition 18.1 of [6]. □

Lemma 8.6.12. Let A be an AW*-algebra. Then A is a homogeneous AW*-algebra of order κ if and only if there is a collection $\{p_i\}_{i\in I}$ of faithful orthogonal abelian projections with $\bigvee_{i\in I} p_i = 1_A$ and such that κ equals the cardinality of I.

Proof. Assume that A is homogeneous of order κ. Hence there is some collection $\{p_i\}_{i\in I}$ of orthogonal abelian projections such that the cardinality of I equals κ and $\bigvee_{i\in I} p_i = 1$. Now, $p_i \leq C(p_i)$ for each $i \in I$, hence $\bigvee_{i\in I} C(p_i) = 1$. Moreover, since $p_i \sim p_j$ for each $i, j \in I$, it follows from Lemma 8.6.11 that $C(p_i) = C(p_j)$. Hence $\bigvee_{i\in I} C(p_i) = C(p_j)$ for each $j \in I$, whence $C(p_j) = 1$ for each $j \in I$. Hence each p_i is faithful. Conversely, if A has a collection $\{p_i\}_{i\in I}$ of faithful abelian projections with cardinality κ, then $C(p_i) = 1 = C(p_j)$ for each $i, j \in I$. Hence Lemma 8.6.11 assures that $p_i \sim p_j$. Hence A is homogeneous of order κ. □

It follows from the previous lemma that a homogeneous AW*-algebra A is a type I algebra. If A is homogeneous with finite order n, we call A a *type* I_n *algebra*.

The von Neumann algebraic version of the next theorem, for which we recall the notion of a modular lattice (cf. Definition B.1.14), was already proven by von Neumann, and forms the basis for his continuous geometry [87].

Theorem 8.6.13. Let A be an AW*-algebra. Then A is finite if and only if $\mathrm{Proj}(A)$ is a modular lattice.

Proof. It follows by combining Theorem 13.1, Corollary 14.1 and Proposition 34.1 that $\mathrm{Proj}(A)$ is modular if A if finite. The converse is the main theorem of [74]. □

Corollary 8.6.14. Let A be an AW*-algebra and $p \in \mathrm{Proj}(A)$. Then p is finite if and only if $\downarrow p$ (regarded as a subposet of $\mathrm{Proj}(A)$) is a modular lattice.

Proof. By definition p is finite if pAp is finite. Hence p is finite if and only if $\mathrm{Proj}(pAp)$ is a modular lattice. Let $q \in \downarrow p$. Then $q = pqp$, hence $q \in pAp$. Conversely, if $q \in \mathrm{Proj}(pAp)$, then $q = pap$ for some $a \in A$, so clearly $pq = q$. We conclude that $\mathrm{Proj}(pAp) = \downarrow p$. □

Recall that for each projection p in any AW*-algebra A, $C(p)$ is defined as the least projection $q \in Z(A)$ such that $p \leq q$.

Lemma 8.6.15. [6, Proposition 6.1] Let A be an AW*-algebra and $p, q \in A$ projections. If $q \leq p$, then $C(q) \leq C(p)$.

Proof. Let $q, p \in \mathrm{Proj}(A)$ such that $q \leq p$. Then

$$\{r \in \mathrm{Proj}(Z(A)) : p \leq r\} \subseteq \{r \in \mathrm{Proj}(Z(A)) : q \leq r\},$$

whence $C(q) \leq C(p)$. □

The following lemma combines [68, Proposition 6.4.2] for AW*-algebras and [6, Proposition 15.6].

Proposition 8.6.16. Let A be an AW*-algebra and $p \in A$ a projection. Then the following statements are equivalent:

(1) p is abelian;

(2) $q = C(q)p$ for each $q \in \text{Proj}(A)$ such that $q \leq p$;

(3) p is a minimal element of $\{q \in \text{Proj}(A) : C(q) = C(p)\}$.

Proof. (1) $\implies$ (3): Let p be abelian, so pAp is a commutative algebra. Let $q \in \text{Proj}(A)$ such that $C(q) = C(p)$ and $q \leq p$. We aim to show that $p = q$. It follows from $q \leq p$ that $pqp = q$, so $q \in pAp$. Let $a \in A$. Since pAp is commutative, we have $q(pap) = (pap)q$, and $q \leq p$ yields $qap = paq$. The last equality yields

$$qap(1 - q) = paq(1 - q) = 0,$$

whence $qAp(1 - q) = \{0\}$. It follows that $p(1 - q) \in R(qA)$. Now, Lemma 8.6.7 implies that $R(qA) = \big(1 - C(q)\big)A$, hence $C(q)x = 0$ for each $x \in R(qA)$. In particular, we find that

$$0 = C(q)p(1 - q) = C(q)p - C(q)pq.$$

Since $C(q) = C(p)$, we have $C(q)p = p$, whence $p = pq$. It follows that $p \leq q$, and the inequality in the other direction holds by assumption, so $p = q$.

(3) $\implies$ (2): Let $p \in \text{Proj}(A)$ such that $C(p) = C(r)$ and $r \leq p$ imply $p = r$ for each $r \in \text{Proj}(A)$. Let $q \in \text{Proj}(A)$ such that $q \leq p$. We aim to show that $q = C(q)p$. Since $q \leq C(q)$, and $C(q) \in Z(A)$, we have $C(q)p = pC(q)$. So, p and $C(q)$ commute, hence their meet exists and equals $C(q)p$, whence $q \leq C(q)p$. Assume that $q < C(q)p$. Since $C(q)p$ is a projection, $\big(1 - C(q)\big)p$ is a projection, too. Since $q \leq C(q)$, this implies $q\big(1 - C(q)\big)p = 0$, whence

$$r := q \vee \Big(\big(1 - C(q)\big)p\Big) = q + \big(1 - C(q)\big)p$$

is a projection (cf. Proposition C.3.2). Note that $r \leq p$, for $q \leq p$. Hence it follows from Lemma 8.6.15 that $C(r) \leq C(p)$. We must have $r < p$, otherwise

$$p = q + p - C(q)p,$$

whence $q = C(q)p$ contradicting our assumption. Since $q \leq r$, it follows from Lemma 8.6.15 that $q \leq C(q) \leq C(r)$. Therefore,

$$q + p - C(q)p = r = C(r)r = C(r)\big(q + p - C(q)p\big) = q + C(r)p - C(q)p,$$

whence $p = C(r)p$. It follows that $C(p) \leq C(r)$. Since we already found the inequality in the other direction, we have $C(p) = C(r)$. Since $r < p$, we now obtain by assumption $r = p$, which is clearly a contradiction. Hence we must have $q = C(q)p$.

(2) $\Longrightarrow$ (1): Let $q, r \in pAp$. Then $q = pap$ for some $a \in A$, hence $qp = q$. So $q \leq p$, whence $q = C(q)p$, and similarly, we have $r = C(r)p$. It is now clear that $qr = rq$, and we can conclude that all projections in pAp commute. Since pAp is an AW*-subalgebra of A by Lemma 2.4.4, it follows from Corollary 2.4.14 that pAp is commutative. Hence p is an abelian projection. □

Lemma 8.6.17. Let A and B be AW*-algebras and let $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ be an orthomodular isomorphism. Then φ restricts to a Boolean isomorphism $\mathrm{Proj}(Z(A)) \to \mathrm{Proj}(Z(B))$. Moreover, if $p \in \mathrm{Proj}(A)$, then

- $C(\varphi(p)) = \varphi(C(p))$;
- p is faithful if and only if $\varphi(p)$ is faithful;
- p is abelian if and only if $\varphi(p)$ is abelian;
- p is finite if and only if $\varphi(p)$ is finite.

Proof. Since A and B are AW*-algebras, it follows from Corollary 2.4.14 that their linear spans of projections are dense. Moreover, $\mathrm{Proj}(Z(A)) = C(\mathrm{Proj}(A))$ and $\mathrm{Proj}(Z(B)) = C(\mathrm{Proj}(B))$ by Lemma 2.2.6. Now, by definition of commutativity in an orthomodular poset, we have $p \in C(\mathrm{Proj}(A))$ if and only if p commutes with all elements of $\mathrm{Proj}(A)$. Using Proposition B.4.8, it follows that $p \in C(\mathrm{Proj}(A))$ if and only if $\varphi(p) \in C(\mathrm{Proj}(B))$, hence φ restricts to an orthomodular isomorphism between $\mathrm{Proj}(Z(A))$ and $\mathrm{Proj}(Z(B))$. Since $Z(A)$ and $Z(B)$ are commutative, $\mathrm{Proj}(Z(A))$ and $\mathrm{Proj}(Z(B))$ are Boolean algebras, hence Lemma B.4.27 assures that

$$\varphi : \mathrm{Proj}(Z(A)) \to \mathrm{Proj}(Z(B))$$

is a Boolean isomorphism. Recall that Boolean isomorphisms and orthomodular isomorphisms are order isomorphisms. Let $p \in \mathrm{Proj}(A)$ and $q = C(p)$.

Then $q \in Z(\text{Proj}(A))$ such that $p \leq q$ and such that $p \leq r$ implies $q \leq r$ for each $r \in \text{Proj}(Z(A))$. Since φ is an order isomorphism restricting to an order isomorphism between $\text{Proj}(Z(A))$ and $\text{Proj}(Z(B))$, it follows that $\varphi(p) \leq \varphi(q)$ and $\varphi(p) \leq r$ implies $\varphi(q) \leq r$ for each $r \in \text{Proj}(Z(B))$. Hence $\varphi(q) = C(\varphi(p))$. It follows that p is faithful if and only if $C(p) = 1$ if and only if $C(\varphi(p)) = \varphi(1)$ if and only if $C(\varphi(p)) = 1$ if and only if $\varphi(p)$ is faithful. By Proposition 8.6.16, p is abelian if and only if

$$p \in \min\{q \in \text{Proj}(A) : C(p) = C(q)\}.$$

Now $C(p) = C(p)$ if and only if

$$C(\varphi(p)) = \varphi(C(p)) = \varphi(C(q)) = C(\varphi(q),$$

whence p is minimal in the set of projections $q \in \text{Proj}(A)$ such that $C(p) = C(q)$ if and only if $\varphi(p)$ is minimal in the set of projections $r \in \text{Proj}(B)$ such that $C(r) = C(\varphi(p))$. We conclude that p is abelian if and only if $\varphi(p)$ is abelian. By Corollary 8.6.14, p is finite if and only if $\downarrow p$ is modular. Since φ is an order isomorphism, it follows that $\downarrow \varphi(p) = \varphi[\downarrow p]$ is modular if and only if $\downarrow p$ is modular. Hence p is finite if and only if $\varphi(p)$ is finite. □

Lemma 8.6.18. The faithful abelian projections in an AW*-factor A are precisely the minimal projections in A.

Proof. By definition of a factor, we have $Z(A) = \mathbb{C}1_A$. Hence the only central projections are 0 and 1. Let p be a minimal projection. Then $p \neq 0$, and since $p \leq C(p)$, we must have $C(p) = 1$, so p is faithful. Let $q \in \text{Proj}(A)$ be another projection such that $C(q) = 1$ and assume that $q \leq p$. We note that $C(0) = 0$, so $r \neq 0$. By minimality of p, we find $p = q$, hence p is minimal in the collection of projections q such that $C(q) = C(p)$, hence an abelian projection by Proposition 8.6.16. Conversely, let p be a faithful abelian projection. Let $q \in \text{Proj}(A)$ such that $0 < q \leq p$. Since $0 < q \leq C(q)$, and 1 is the only non-zero central projection, we find $C(q) = 1$. Since p is faithful, we have $C(p) = 1$. Since p is also abelian, and $C(q) = C(p)$, we obtain $q = p$. We conclude that p is minimal. □

Lemma 8.6.19. Let H be a Hilbert space. Then $B(H)$ is a homogeneous AW*-algebra of order $\dim H$.

Proof. Let $\{e_i\}_{i\in I}$ be an orthonormal basis of H. Denote the standard inner product on H by $\langle\cdot,\cdot\rangle$, and define $p_i : H \to H$ by $h \mapsto \langle e_i, h\rangle e_i$. It is routine to check that p_i is a projection. If $i \neq j$, then for each $h \in H$, we find

$$p_i p_j h = \langle e_i, p_j h\rangle e_i = \big\langle e_i, \langle e_j, h\rangle e_j\big\rangle e_i = \langle e_j, h\rangle\langle e_i, e_j\rangle e_i = 0,$$

for e_i and e_j are orthogonal. Thus, p_i and p_j are orthogonal. Moreover, p_i is minimal. Let $q \in B(H)$ be another projection such that $0 \neq q \leq p_i$. Let $j \in I$ such that $j \neq i$. Since $q = qp_i$, we find

$$qe_j = qp_ie_j = q\langle e_i, e_j\rangle e_j = 0.$$

If Now, since $q \neq 0$, we cannot have $qe_i = 0$. Hence

$$qe_i = p_iqe_i = \langle e_i, qe_i\rangle e_i = \langle qe_i, qe_i\rangle e_i = \|qe_i\|e_i,$$

so $\|qe_i\|$ is an eigenvalue of q. Since the only possible eigenvalues of projections are 0 and 1, and $\|qe_i\| \neq 0$, we obtain $qe_i = e_i$. Clearly $p_ie_j = \delta_{ij}e_j$, so $p_i = q$ and we conclude that p_i is minimal. Let q be a projection such that $p_i \leq q$ for each $i \in I$. Then $qp_i = p_i$, and since $p_ie_i = e_i$, we find

$$qe_i = qp_ie_i = p_ie_i = e_i$$

for each $i \in I$. So q is the identity on elements of an orthonormal basis, hence $q = 1_H$. We conclude that $\bigvee_{i\in I} p_i = 1_H$. Since $B(H)$ is a factor, it follows from Lemma 8.6.18 that the p_i are faithful abelian projections. Therefore, $B(H)$ is homogeneous with order $\dim H$. □

Let A and B be AW*-algebras such that A is of type I, and such that $\mathcal{A}(A)$ and $\mathcal{A}(B)$ are order isomorphic. Our aim is to show that A and B are *-isomorphic. We thank Masanao Ozawa for pointing out that the projection lattice of a type I AW*-algebra determines the algebra up to isomorphism, a fact which has not been published yet. It then follows from Theorem 8.5.3 that A and B are indeed *-isomorphic. We choose a slightly different route, but we rely heavily on the following proposition, which forms a part of Ozawa's proof. We thank him for his permission to include it here.

Proposition 8.6.20. Let A be an AW*-algebra, and $\{p_i\}_{i\in I}$ an orthogonal collection of central projections in A such that $\bigvee_{i\in I} p_i = 1_A$. Let B be an AW*-

algebra and $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ an orthomodular isomorphism. Then $B \cong \bigoplus_{i\in I} \varphi(p_i)B$, where $\varphi(p_i)B$ is an AW*-algebra for each $i \in I$, and $\mathrm{Proj}(p_iA) \to \mathrm{Proj}(\varphi(p_i)B)$, $q \mapsto \varphi(p_iq)$ is an orthomodular isomorphism.

Proof. Since φ is an orthomodular isomorphism, it follows that $\{\varphi(p_i)\}_{i\in I}$ is an orthogonal collection of projections in B such that $\bigvee_{i\in I} \varphi(p_i) = 1_B$. Furthermore, using Lemma 8.6.17, we have

$$C(\varphi(p_i)) = \varphi(C(p_i)) = \varphi(p_i),$$

hence $\varphi(p_i)$ is a central projection for each $i \in I$. It now follows from Proposition 2.4.27 that $\varphi(p_i)B$ is an AW*-algebra for each $i \in I$, and the map $B \to \bigoplus_{i\in I} \varphi(p_i)B$, $b \mapsto \big(\varphi(p_i)b\big)_{i\in I}$ is a *-isomorphism. If, given a fixed $j \in I$, we compose this map with the projection map $\bigoplus_{i\in I} \varphi(p_i)B \to \varphi(p_j)B$ (which is an AW*-homomorphism by Proposition 2.4.26), then we obtain an AW*-homomorphism $B \to \varphi(p_j)B$, $b \mapsto \varphi(p_j)b$. It follows from Proposition C.3.2 that $\mathrm{Proj}(B) \to \mathrm{Proj}(\varphi(p_j)B)$, $q \mapsto \varphi(p_j)q$ is an orthomodular morphism. Composing this morphism with the orthomodular isomorphism φ yields an orthomodular morphism $\psi : \mathrm{Proj}(A) \to \mathrm{Proj}(\varphi(p_j)B)$, $q \mapsto \varphi(p_j)\varphi(q)$, which is clearly surjective. Notice that both p_j and $\varphi(p_j)$ are central projections, hence it follows from Proposition C.3.2 that

$$\psi(q) = \varphi(p_j)\varphi(q) = \varphi(p_j) \wedge \varphi(q) = \varphi(p_j \wedge q) = \varphi(p_jq),$$

for each $q \in \mathrm{Proj}(A)$, where we used Proposition B.4.8 for the second equality. Since $p_jA \subseteq A$, we have $\mathrm{Proj}(p_jA) \subseteq \mathrm{Proj}(A)$, hence let $\psi_j : \mathrm{Proj}(p_jA) \to \mathrm{Proj}(\varphi(p_j)B)$ be the restriction of ψ to $\mathrm{Proj}(p_jA)$. Let $r \in \mathrm{Proj}(\varphi(p_j)B)$. Since ψ is surjective, there is some $q \in \mathrm{Proj}(A)$ such that $\psi(q) = r$. Since $p_jq \in \mathrm{Proj}(p_jA)$, it follows that

$$\psi_j(p_jq) = \psi(p_jq) = \varphi(p_jp_jq) = \varphi(p_jq) = \psi(q) = r,$$

hence ψ_j is surjective. Let $q_1, q_2 \in \mathrm{Proj}(p_jA)$. If $q_1 \leq q_2$, then $\psi_j(q_1) \leq \psi_j(q_2)$, since ψ_j is an orthomodular morphism, hence it preserves the order (cf. Proposition B.4.8). Conversely, assume that $\psi_j(q_1) \leq \psi_j(q_2)$. Then $\varphi(p_jq_1) \leq \varphi(p_jq_2)$. Since φ is an orthomodular isomorphism, it is an order isomorphism (cf. Lemma B.4.9), hence $p_jq_1 \leq p_jq_2$. Since $q_1 \in p_jA$, there is some $a \in A$

such that $q_1 = p_j a$, hence

$$p_j q_1 = p_j^2 a = p_j a = q_1,$$

and similarly, $p_j q_2 = q_2$. Hence $q_1 \leq q_2$, and we conclude that ψ_j is an order embedding. Hence ψ_j is an order isomorphism, since it is a surjective order embedding, and as an orthomodular morphism, it also preserves the orthocomplementation. It now follows from Lemma B.4.9 that ψ_j is an orthomodular isomorphism. □

We also need the following theorems, which describe the structure of type I AW*-algebras.

Theorem 8.6.21. [72, Corollary of Theorem 1] Let A and B be homogeneous AW*-algebras of the same order such that $Z(A) \cong Z(B)$. Then $A \cong B$.

Theorem 8.6.22. [72, Lemma 18] Let A be a type I AW*-algebra. Then there are homogeneous AW*-algebras $\{A_i\}_{i \in I}$ such that $A \cong \bigoplus_{i \in I} A_i$.

Theorem 8.6.23. Let A and B be AW*-algebras such that $\mathcal{A}(A) \cong \mathcal{A}(B)$. Then:

(1) $Z(A)$ and $Z(B)$ are *-isomorphic;

(2) If there is a collection $\{A_i\}_{i \in I}$ of AW*-algebras such that $A \cong \bigoplus_{i \in I} A_i$, then there is a collection $\{B_i\}_{i \in I}$ of AW*-algebras such that $B \cong \bigoplus_{i \in I} B_i$, and such that there is an order isomorphism $\mathcal{A}(A_i) \to \mathcal{A}(B_i)$ for each $i \in I$;

(3) If A is a type I AW*-algebra, then $A \cong B$;

(4) If A is a type II AW*-algebra, then B is a type II AW*-algebra. More specifically, if A is a type II_1 algebra, then so is B, and if A is a type II_∞ algebra, then so is B;

(5) If A is a type III AW*-algebra, then B is a type III AW*-algebra;

(6) If A is a finite AW*-algebra, then B is a finite AW*-algebra;

(7) If A is a factor, then B is a factor;

(8) If $A \cong B(H)$ for some Hilbert space H, then $B \cong B(H)$.

Proof. In several statements, we will implicitly use Theorem 8.5.3, which assures the existence of an orthomodular isomorphism $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$.

(1) By Theorem 8.1.2, we have $Z(A) = \bigcap \max \mathcal{A}(A)$. Denote the order isomorphism between $\mathcal{A}(A)$ and $\mathcal{A}(B)$ by Φ. Then

$$\Phi(Z(A)) = \bigcap \Phi[\max \mathcal{A}(A)] = \bigcap \max \mathcal{A}(B) = Z(B).$$

Moreover

$$\mathcal{A}(Z(A)) = \downarrow Z(A) \cong \Phi[\downarrow Z(A)] = \downarrow \Phi(Z(A)) = \downarrow Z(B) = \mathcal{A}(Z(B)).$$

It now follows from Theorem 8.5.2 that $Z(A) \cong Z(B)$.

(2) By Proposition 2.4.27 there is a collection $\{p_i\}_{i\in I}$ of mutually orthogonal central projections in A such that $\bigvee_{i\in I} p_i = 1_A$ and such that $p_iA \cong A_i$ for each $i \in I$. By Proposition 8.6.20 the collection $\{\varphi(p_i)\}_{i\in I}$ consists of mutually orthogonal central projections in B such that $B \cong \bigoplus_{i\in I} B_i$, where $B_i = \varphi(p_i)B$, and such that $\mathrm{Proj}(p_iA)$ and $\mathrm{Proj}(B_i)$ are isomorphic as orthomodular lattices for each $i \in I$. Fix $i \in I$. Since $A_i \cong p_iA$, it follows that there is an orthomodular isomorphism $\mathrm{Proj}(A_i) \to \mathrm{Proj}(B_i)$. It now follows from Theorem 8.5.3 that there is an order isomorphism $\mathcal{A}(A_i) \to \mathcal{A}(B_i)$.

(3) First assume that A is a homogeneous algebra. By Lemma 8.6.12, A has an orthogonal collection $\{p_i\}_{i\in I}$ of faithful abelian projections such that $\bigvee_{i\in I} p_i = 1$. Since φ is an orthomodular isomorphism, hence an order isomorphism, $\{\varphi(p_i)\}_{i\in I}$ is a collection of mutually orthogonal projections with supremum 1_B. By Lemma 8.6.17, the $\varphi(p_i)$ are faithful and abelian. Hence B is a homogeneous AW*-algebra with the same order as A. It follows from (1) that we are allowed to apply Theorem 8.6.21 to conclude that $A \cong B$.

Now assume that A is a type I AW*-algebra. By Theorem 8.6.22, there is a collection $\{A_i\}_{i\in I}$ of homogeneous algebras such that $A \cong \bigoplus_{i\in I} A_i$. It now follows from (2) that $B = \bigoplus_{i\in I} B_i$ for some collection of AW*-algebras $\{B_i\}_{i\in I}$ such that $\mathcal{A}(A_i) \cong \mathcal{A}(B_i)$ for each $i \in I$. Since A_i is homogeneous, it follows that $A_i \cong B_i$ for each $i \in I$. Hence $A \cong B$.

(4) If 0_A is the only abelian projection in A, then it follows from Lemma 8.6.17 that $0_B = \varphi(0_A)$ is the only abelian projection in B. Moreover, the same lemma assures that A has a faithful finite projection if and only if B has a faithful finite projection. Thus A is a type II AW*-algebra if and only if B is a type II AW*-algebra. Now, A is a type II_1 AW*-algebra if and only if 1_A is finite. Again by Lemma 8.6.17, this is equivalent with $1_B = \varphi(1_A)$ being finite. Hence A is a type II_1 AW*-algebra if and only if B is a type II_1 AW*-algebra. Now assume that A is a type II_∞ algebra. Then 0_A is the only finite central projection of A. By Lemma 8.6.17 it follows that $0_B = \varphi(0_A)$ is the only finite central projection of B, hence B is a type II_∞ algebra.

(5) If 0_A is the only finite projection of A, it follows from Lemma 8.6.17 that $0_B = \varphi(0_A)$ is the only finite projection of B. Hence A is a type III AW*-algebra if and only if B is a type III AW*-algebra.

(6) If A is finite, Theorem 8.6.13 assures that $\mathrm{Proj}(A)$ is modular. Hence $\mathrm{Proj}(B)$ is modular, so B is finite.

(7) A is a factor if and only if its center is one-dimensional. By Theorem 8.1.2, the center is the infimum of the maximal elements of $\mathcal{A}(A)$, which must be equal to the least element of $\mathcal{A}(A)$, for this is $\mathbb{C}1_A$. Since $\mathcal{A}(B)$ is order isomorphic to $\mathcal{A}(A)$, it follows that its least element equals the infimum of its maximal elements as well. Hence the center of B is one-dimensional, i.e., B is a factor.

(8) By Lemma 8.6.19, $B(H)$, hence A, is homogeneous, hence a type I AW*-algebra. By (3), we find that $B \cong A$. Hence $B \cong B(H)$. □

We note that the use of the characterization of finite projections in Corollary 8.6.14 for determining the type of an AW*-algebra is not new. A similar approach to a type classification of JBW-algebras (which are defined in §9.3) can be found in [51, Chapter 5].

Corollary 8.6.24. Let A and B be C*-algebras such that $\mathcal{C}(A) \cong \mathcal{C}(B)$. Then:

(1) If A is a type I AW*-algebra, then $A \cong B$;

(2) If A is a type II AW*-algebra, then B is a type II AW*-algebra. More specifically, if A is a type II_1 algebra, then so is B, and if A is a type II_∞ algebra, then so is B;

(3) If A is a type III AW*-algebra, then B is a type III AW*-algebra;

(4) If A is a finite AW*-algebra, then B is a finite AW*-algebra;

(5) If A is a type II_1 AW*-factor, then B is a type II_1 AW*-factor;

(6) If $A \cong B(H)$ for some Hilbert space H, then $B \cong B(H)$.

Proof. This follows from Proposition 8.4.1 and Theorem 8.6.23. □

We note that the first statement of the last corollary generalizes Theorem 5.3.1. Special cases of the first statement were already proven by Hamhalter, who showed that if M is a type I von Neumann algebra not isomorphic to $\mathbb{C}^2$ and without type I_2 summand, and if N is an arbitrary von Neumann algebra, then $\mathcal{C}(M) \cong \mathcal{C}(N)$ implies $M \cong N$. His proof is based on his generalization of Dye's Theorem for AW*-algebras ([48, Theorem 4.6], which we include as Theorem 9.2.6 below) and his observation that type I von Neumann algebras are anti-*-isomorphic to themselves. Moreover, he proved a similar statement for finite type I AW*-algebras. Finally, we recall Connes' example of a von Neumann algebra not *-isomorphic to its opposite, as discussed in §1.5, which shows that there is no hope that the either the $\mathcal{C}$-functor or the $\mathcal{A}$-functor completely determines all AW*-algebras. We also recall that in [61] extra structure is added to $\mathcal{A}(A)$, resulting in a notion called *active lattices* that completely determines all AW*-algebras.

9 The Jordan structure of C*-algebras

In case we are dealing with scattered C*-algebras and AW*-algebras, we have found correspondences between isomorphisms of projection posets and isomorphisms of posets of commutative subalgebras in Corollary 6.4.5 and Theorem 8.5.3. It turns out that there is a stronger structure whose isomorphisms correspond to isomorphisms of posets of commutative subalgebras, namely the *Jordan structure* of a C*-algebra. This structure, which has previously been studied by Döring and Harding [25] and Hamhalter [47, 48, 50] (the last reference together with Turilova), is more general in the sense that we do not have to restrict ourselves to scattered C*-algebras or AW*-algebras. The Jordan algebra that is naturally associated to a given C*-algebra A turns out to be a *JB-algebra*, which is roughly speaking both a Jordan algebra and a Banach algebra. In the first section we give the definitions of Jordan algebras and of JB-algebras as well as their morphisms. In the second section, we state the most important theorems related to $\mathcal{C}(A)$. In the last section, we apply the statements in the second section in order to prove that if A is a W*-algebra and B a C*-algebra, then $\mathcal{C}(A) \cong \mathcal{C}(B)$ implies that B is a W*-algebra, too.

9.1 Jordan algebras

We start by defining Jordan algebras and JB-algebras.

Definition 9.1.1.

- A (not necessarily associative) algebra A with identity element 1_A over $\mathbb{R}$ is called a *Jordan algebra* if the multiplication $\circ : A \times A \to A$ is commutative and satisfies

$$(x \circ y) \circ (x \circ x) = x \circ (y \circ (x \circ x)).$$

  We will abbreviate $a \circ a$ by a^2.

- A Jordan algebra A is called a *JB-algebra* if it is equipped with a norm $\|\cdot\|$ such that A is complete in the metric induced by $\|\cdot\|$, and such that for each $a, b \in A$:

$$\|a \circ b\| \leq \|a\|\|b\|;$$
$$\|a^2\| = \|a\|^2;$$
$$\|a^2\| \leq \|a^2 + b^2\|.$$

- An $\mathbb{R}$-linear map $\varphi : A \to B$ between Jordan algebras A and B such that $\varphi(1_A) = 1_B$ is called a *Jordan homomorphism* if it preserves the Jordan product, and a *Jordan isomorphism* if, in addition, φ is bijective.

Recall that *-homomorphisms between C*-algebras are always continuous. In the same way, a Jordan homomorphism between JB-algebras is always continuous. For our purposes, the following proposition is sufficient. We refer to [3, Proposition 1.35] for a proof.

Proposition 9.1.2. Let A and B be JB-algebras, and let $\varphi : A \to B$ be a Jordan isomorphism. Then φ is an isometry.

Lemma 9.1.3. Let A and B be C*-algebras. Then:

(1) the algebra A_{sa} of all self-adjoint elements of A becomes a JB-algebra if we equip it with the multiplication $\circ : A_{\text{sa}} \times A_{\text{sa}} \to A_{\text{sa}}$ defined by

$$a \circ b = \frac{ab + ba}{2}$$

for each $a, b \in A_{\text{sa}}$;

(2) any $\mathbb{R}$-linear map $\psi : A_{\text{sa}} \to B_{\text{sa}}$ is a Jordan homomorphism if and only if

$$\psi(a^2) = \psi(a)^2$$

for each $a \in A_{\text{sa}}$;

(3) for each Jordan homomorphism $\psi : A_{\text{sa}} \to B_{\text{sa}}$ there is a unique $\mathbb{C}$-linear map $\varphi : A \to B$ such that

$$\varphi(a^*) = \varphi(a)^*,$$
$$\varphi(a^2) = \varphi(a)^2,$$

for each $a \in A$, and such that $\varphi|_{A_{\mathrm{sa}}} = \psi$. Explicitly, φ is given by

$$\varphi(a) = \psi\left(\frac{a + a^*}{2}\right) + i\psi\left(\frac{a - a^*}{2i}\right)$$

for each $a \in A$. If A and B are commutative, φ is a *-homomorphism.

We note that the notation $a \circ a = aa$, and hence the abbreviation a^2 for $a \circ a$ on A_{sa}, is consistent with the original multiplication on A.

Definition 9.1.4. Let A and B be C*-algebras. Then a $\mathbb{C}$-linear map $\varphi : A \to B$ is called a *Jordan *-homomorphism* if for each $a \in A$ we have:

$$\varphi(a^*) = \varphi(a)^*,$$
$$\varphi(a^2) = \varphi(a)^2.$$

A bijective Jordan *-homomorphism is called a *Jordan *-isomorphism*.

The third part of the last lemma states that for each pair of C*-algebras A and B there is a bijective correspondence between Jordan homomorphisms $A_{\mathrm{sa}} \to B_{\mathrm{sa}}$ and Jordan *-homomorphism $A \to B$.

9.2 The relation between $\mathcal{C}(A)$ and the Jordan structure of A

The (chronologically) first theorem relating posets of commutative von Neumann subalgebras of a von Neumann algebra M to the Jordan structure of M is due to Döring and Harding:

Theorem 9.2.1. [25, Theorem 3.4] Let M and N be von Neumann algebras, where M does not have a type I_2 summand, and let $\Phi : \mathcal{V}(M) \to \mathcal{V}(N)$ be an order isomorphism. Then there is a Jordan *-isomorphism $\varphi : M \to N$ such that

$$\Phi(C) = \varphi[C],$$

for each $C \in \mathcal{V}(M)$. Moreover, if M is not two-dimensional, then φ is the unique Jordan *-isomorphism that induces Φ in this way.

The main ingredients of the proof of the last theorem are the Harding–Navara Theorem (Theorem 6.3.6), and Dye's Theorem for von Neumann algebras (cf.

Theorem 9.2.7 below). In order to generalize this theorem to arbitrary C*-algebras, one first needs to weaken the notion of Jordan *-homomorphisms:

Definition 9.2.2. If A and B are C*-algebras, then we say that a (not necessarily linear) map $\varphi : A \to B$ is a *quasi-Jordan *-homomorphism* if φ restricts to a Jordan *-homomorphism (or equivalently, to a *-homomorphism) on each commutative C*-subalgebra C of A, and call φ a *quasi-Jordan *-isomorphism* if it is a bijective quasi-Jordan *-homomorphism (in which case its inverse is automatically a quasi-Jordan *-homomorphism, too).

It turns out that quasi-Jordan *-isomorphism always induce an order isomorphism between posets of commutative C*-algebras:

Proposition 9.2.3. [47, Proposition 1.1] Let A and B be C*-algebras and let $\varphi : A \to B$ be a quasi-Jordan *-isomorphism. Then the map $\mathcal{C}(A) \to \mathcal{C}(B)$ given by $C \mapsto \varphi[C]$ is an order isomorphism.

The main result of [47] is that in all but two cases the converse holds as well:

Theorem 9.2.4. [47, Theorem 3.4] Let A and B be C*-algebras and

$$\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$$

an order isomorphism. If A is neither *-isomorphic to $\mathbb{C}^2$ nor to $\mathrm{M}_2(\mathbb{C})$, then there is a unique quasi-Jordan *-isomorphism $\varphi : A \to B$ such that for each $C \in \mathcal{C}(A)$ we have:

$$\Phi(C) = \varphi[C].$$

Assume that A is neither *-isomorphic to $\mathbb{C}^2$ nor to $\mathrm{M}_2(\mathbb{C})$. It follows from Theorem 6.4.4 that each order isomorphism $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$ implies the existence of some orthomodular isomorphism $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$. The same statement follows from combining Theorem 9.2.4 with the next simple lemma.

Lemma 9.2.5. Let A and B be C*-algebras and let $\varphi : A \to B$ be a quasi-Jordan *-isomorphism. Then φ restricts to an orthomodular isomorphism

$$\mathrm{Proj}(A) \to \mathrm{Proj}(B).$$

Proof. Let $p \in \mathrm{Proj}(A)$, and let C be a commutative C*-subalgebra of A containing p. Then φ restricts to a *-isomorphism on C to its image, whence $\varphi(p)$

is a projection. Thus φ restricts to a map $\mathrm{Proj}(A) \to \mathrm{Proj}(B)$, which is clearly bijective. Since $1_A \in C$, we also find

$$\varphi(1_A - p) = \varphi(1_A) - \varphi(p) = 1_B - \varphi(p),$$

so φ preserves the orthocomplementation. Now φ preserves the order, since if $q \in \mathrm{Proj}(A)$ such that $q \leq p$, then p and q commute, and we can find a commutative C*-subalgebra D containing both p and q, so $\varphi(q) \leq \varphi(p)$. Since φ^{-1} is a quasi-Jordan *-isomorphism, too, it follows in a similar way that $\varphi(q) \leq \varphi(p)$ implies $q \leq p$. We conclude that φ is an orthomodular isomorphism. □

The disadvantage of quasi-Jordan *-isomorphisms is their lack of linearity. However, in case of AW*-algebras, Hamhalter succeeded in removing the 'quasi' part in the previous theorem:

Theorem 9.2.6. [48, Theorem 4.6] Let A and B be AW*-algebras and

$$\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$$

an order isomorphism. If A does not have a type I_2 summand, then there is a Jordan *-isomorphism $\varphi : A \to B$ such that for each $C \in \mathcal{C}(A)$ we have:

$$\Phi(C) = \varphi[C].$$

Moreover, if A is not two-dimensional, then φ is the unique Jordan *-isomorphism that induces Φ in this way.

The key to the proof of the last theorem is Hamhalter's generalization of Dye's Theorem to AW*-algebras:

Theorem 9.2.7. [48, Theorem 4.3] Let A and B be AW*-algebras such that A does not have a type I_2 summand, and let $\psi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ be an orthomodular isomorphism. Then there is a Jordan *-isomorphism $\varphi : A \to B$ such that $\varphi|_{\mathrm{Proj}(A)} = \psi$.

Given a von Neumann algebra M, we note that $\mathcal{A}(M) = \mathcal{V}(M)$, whereas $\mathcal{C}(M) = \mathcal{V}(M)$ only if M is finite-dimensional. For this reason, Theorem 9.2.6 is almost a generalization of Theorem 9.2.1, but not completely. This would have been the case if in Theorem 9.2.6 we had replaced $\mathcal{C}(A)$ and $\mathcal{C}(B)$ by $\mathcal{A}(A)$ and

$\mathcal{A}(B)$, respectively. It is up to the reader to decide whether it is more desirable to have a $\mathcal{C}$-generalization or an $\mathcal{A}$-generalization of Theorem 9.2.1. The next theorem, which also relies heavily on Theorem 9.2.6, gives the $\mathcal{A}$-generalization, but also states that an order isomorphism between $\mathcal{A}(A)$ and $\mathcal{A}(B)$ in all cases implies that A and B are Jordan *-isomorphic.

Theorem 9.2.8. Let A and B be AW*-algebras, and let $\Phi : \mathcal{A}(A) \to \mathcal{A}(B)$ be an order isomorphism. Then there is a Jordan *-isomorphism $A \to B$. Moreover, if A does not have a type I_2 summand, then there is a Jordan *-isomorphism $\varphi : A \to B$ such that for each $C \in \mathcal{A}(A)$ we have

$$\Phi(C) = \varphi[C].$$

Moreover, if A is not two-dimensional, then φ is the unique Jordan *-isomorphism that induces Φ in this way.

Proof. First assume that A does not have a type I_2 summand. By Theorem 8.5.3, we obtain a unique orthomodular isomorphism
$\psi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ such that

$$\Phi(C) = C^*(\psi[\mathrm{Proj}(C)])$$

for each $C \in \mathcal{A}(A)$. By Theorem 9.2.7, there is a Jordan *-isomorphism $\varphi : A \to B$ extending ψ. Let $C \in \mathcal{A}(A)$. By Proposition 9.2.3, $\varphi[C]$ is a C*-subalgebra of B. By Lemma 9.1.3, the restriction $\varphi : C \to \varphi[C]$ is a *-isomorphism, hence $\varphi[C]$ is a commutative AW*-algebra, which is therefore generated by its projections (cf. Theorem 2.4.5). Hence

$$\Phi(C) = C^*(\psi[\mathrm{Proj}(C)] = C^*(\varphi[\mathrm{Proj}(C)])) = C^*(\mathrm{Proj}(\varphi[C])) = \varphi[C],$$

so φ induces Φ. Let $\theta : A \to B$ be another Jordan *-isomorphism inducing Φ. Let $a \in A_{\mathrm{sa}}$ and let M be a maximal commutative C*-subalgebra of A containing a. Then φ and θ restrict to *-isomorphisms $M \to \Phi(M)$ and Φ restricts to an order isomorphism $\mathcal{A}(M) \to \mathcal{A}(\Phi(M))$. Since A is an AW*-algebra, all its maximal commutative C*-subalgebras are generated by their projections (cf. Theorem 2.4.5). Hence it follows from the premises and Corollary 6.2.7 that M is not two-dimensional. Hence Theorem 8.5.2 assures that there is only one *-isomorphism $M \to \Phi(M)$ inducing $\Phi : \mathcal{A}(M) \to \mathcal{A}(\Phi(M))$. We find that φ

and θ have to agree on M, whence $\varphi(a) = \theta(a)$. Thus φ and θ agree on A_{sa}, hence by Lemma 9.1.3 also on A.

Now assume that A has a type I_2 summand. Write $A = A_1 \oplus A_2$, where A_1 is a type I_2 AW*-algebra, and A_2 does not have a type I_2 summand. By Theorem 8.6.23, $B \cong B_1 \oplus B_2$ for AW*-algebras B_1 and B_2 such that $\mathcal{A}(A_1) \cong \mathcal{A}(B_1)$ and $\mathcal{A}(A_2) \cong \mathcal{A}(B_2)$. The same theorem assures that A_1 and B_1 are *-isomorphic, hence Jordan *-isomorphic. From the case without a type I_2 summand, we find that A_2 and B_2 are Jordan *-isomorphic. We conclude that A and B are Jordan *-isomorphic. $\square$

Corollary 9.2.9. Let A and B be AW*-algebras. Then the following statements are equivalent:

(1) $\mathcal{C}(A)$ and $\mathcal{C}(B)$ are order isomorphic;

(2) $\mathcal{A}(A)$ and $\mathcal{A}(B)$ are order isomorphic;

(3) $\mathrm{Proj}(A)$ and $\mathrm{Proj}(B)$ are orthomodular isomorphic;

(4) A and B are Jordan *-isomorphic.

Proof. Proposition 8.4.1 assures that (1) $\Longrightarrow$ (2) holds. It follows from Theorem 9.2.8 that (2) $\Longrightarrow$ (4) holds. Furthermore, (4) $\Longrightarrow$ (1) follows from Proposition 9.2.3. Finally, the equivalence between (2) and (3) follows from Theorem 8.5.3. $\square$

9.3 Recognizing W*-algebras

Recovering the Jordan structure of an AW*-algebra A from $\mathcal{A}(A)$ allows us to identify whether or not A is a W*-algebra. The idea is that the set A_{sa} of all self-adjoint elements of a W*-algebra actually has more structure than that of a JB-algebra, and this structure can be axiomatized. We first need to introduce a notion of *positivity* on Jordan algebras.

Definition 9.3.1. Let A be a Jordan algebra. Then we say that an element $a \in A$ is *positive* if $a = b^2$ for some $b \in A$. Furthermore, if $a, b \in A$ such that $b - a$ is positive, we write $a \leq b$.

Since an element a in a C*-algebra A is positive if and only if $a = b^2$ for some $b \in A$ (cf. [67, Theorem 4.2.6]), the positive elements in A coincide with the positive elements in its associated JB-algebra A_{sa}.

Definition 9.3.2.

- Let A be a JB-algebra. Then a linear functional $\omega : A \to \mathbb{R}$ is called a *state* of A if $\omega(1) = 1$ and $\omega(a) \geq 0$ for each positive $a \in A$;
- Let A be a C*-algebra. Then a linear functional $\omega : A \to \mathbb{C}$ is called a *state* of A if $\omega(1) = 1$ and $\omega(a) \geq 0$ for each positive $a \in A$.

If A is a C*-algebra, then for each self-adjoint element $a \in A$ there are positive $a_1, a_2 \in A$ such that $a = a_1 - a_2$ (cf. [67, Proposition 4.2.3]). Hence $\omega[A_{\mathrm{sa}}] \subseteq \mathbb{R}$ for each state ω of A. Consequently, the restriction of a state of a C*-algebra A to A_{sa} is a state of A_{sa}. Conversely, if ω is a (JB-algebraic) state of A_{sa}, we can extend it in a unique way to a (C*-algebraic) state $\widetilde{\omega}$ of A by defining

$$\widetilde{\omega}(a) = \omega\left(\frac{a + a^*}{2}\right) + i\omega\left(\frac{a - a^*}{2i}\right)$$

for each $a \in A$. Hence we obtain a bijective correspondence between states of A and states of A_{sa}.

Definition 9.3.3.

- Let A be a JB-algebra. Then A is called *monotone complete* if each increasing net $\{a_\lambda\}_{\lambda \in \Lambda}$ in A that is bounded from above has a least upper bound;
- Let A be a C*-algebra. Then A is called *monotone complete* if A_{sa} is monotone complete as a JB-algebra.

It can be proven that every monotone complete C*-algebra is an AW*-algebra. The converse is a conjecture; no one has yet found an AW*-algebra that is not monotone complete. Just as von Neumann algebras and, more generally, AW*-algebras, monotone complete C*-algebras have a type classification. For more details on monotone complete C*-algebras, we refer to [100]. In particular, every W*-algebra is monotone complete. Before we can state the exact conditions on monotone complete C*-algebras to be W*-algebras, we first need some more definitions.

Definition 9.3.4.

- Let A be a monotone complete JB-algebra, and let ω be a state of A. Then ω is called *normal* if $\lim_{\lambda\in\Lambda}\omega(a_\lambda) = \omega(a)$ for each increasing net $\{a_\lambda\}_{\lambda\in\Lambda}$ with least upper bound a;

- Let A be a monotone complete C*-algebra, and let ω be a state of A. Then ω is called *normal* if its restriction to (the monotone complete JB-algebra) A_{sa} is a normal state.

Definition 9.3.5.

- Let A be a JB-algebra and let S be a collection of states of A. We say that S is a *separating family of states* of A if for each non-zero $a \in A_{\mathrm{sa}}$ there is some $\omega \in S$ such that $\omega(a) \neq 0$;

- Let A be a C*-algebra and let S be a collection of states of A. Then we say that S is a *separating family of states* of A if $\left\{\omega|_{A_{\mathrm{sa}}} : \omega \in S\right\}$ is a separating family of states for the JB-algebra A_{sa}.

Since the positive elements of a C*-algebra coincide with the positive elements of the JB-algebra A_{sa}, we find that a collection S of states of A is a separating family if and only if $\left\{\omega|_{A_{\mathrm{sa}}} : \omega \in S\right\}$ is a separating family of states of A_{sa}. We are now in the position to give an alternative characterization of W*-algebras in terms of states and monotone completeness. We refer to [66] for a proof.

Theorem 9.3.6. [66, Theorem 1] Let A be a C*-algebra. Then A is a W*-algebra if and only if it is monotone complete and admits a separating family of normal states[7].

[7] It should be noted that the definitions of W*-algebras and of separating families of states in [66] differs from ours. What we call a W*-algebra (following the definition in [106], viz. a C*-algebra *-isomorphic to a von Neumann algebra) is called 'a C*-algebra with a faithful representation as a ring of operators'. To increase the possible confusion, in [66] a monotone complete algebra with a separating family of normal states is called a W*-algebra. Moreover, in [66] a slightly different notion of a separating family of states is given, namely a family S of states on A is separating if for each non-zero *positive* $a \in A$, there is an $\omega \in S$ such that $\omega(a) \neq 0$. This condition is weaker than that in the theorem stated here, but still sufficient. Since a family S of states on A is separating according to the definition in [66] if it is separating according to the definition stated here, and a von Neumann algebra always has a separating family of normal states according to our definition, the theorem as stated here follows from the theorem in [66].

This characterization of W*-algebras led to the following class of JB-algebras, which form the Jordan-algebraic counterpart of W*-algebras.

Definition 9.3.7. A JB-algebra A is called a *JBW-algebra* it is monotone complete and admits a separating family of normal states.

It follows immediately from the definitions and Theorem 9.3.6 that a C*-algebra A is a W*-algebra if and only if its associated JB-algebra A_{sa} is a JBW-algebra. Now we can prove the following theorem:

Theorem 9.3.8. Let A be a W*-algebra, let B be an AW*-algebra, and let $\Phi : \mathcal{A}(A) \to \mathcal{A}(B)$ be an order isomorphism. Then B is a W*-algebra.

Proof. By Theorem 9.2.8 there exists a Jordan *-isomorphism $\varphi : A \to B$, which restricts to a Jordan isomorphism $A_{\mathrm{sa}} \to B_{\mathrm{sa}}$. Let $a_1, a_2 \in A_{\mathrm{sa}}$ be such that $a_1 \leq a_2$ in A_{sa}, and let $b_i \in B_{\mathrm{sa}}$ be given by $b_i = \varphi(a_i)$ for each $i = 1, 2$. Since $a_2 - a_1 \geq 0$, there is some $a \in A_{\mathrm{sa}}$ such that $a_2 - a_1 = a^2$. Since φ is a Jordan isomorphism, we find that

$$b_2 - b_1 = \varphi(b_2) - \varphi(b_1) = \varphi(b_2 - b_1) = \varphi(b^2) = \varphi(b)^2,$$

hence $b_1 \leq b_2$. It follows that φ is an order morphism, and in a similar way we find that φ^{-1} is an order morphism. Thus φ is an order isomorphism. Now let $\{b_\lambda\}_{\lambda \in \Lambda}$ be an increasing net in B_{sa} that is bounded from above. It follows that $\{\varphi^{-1}(b_\lambda)\}_{\lambda \in \Lambda}$ is an increasing net in A_{sa} that is bounded from above. Since A_{sa} is a JBW-algebra, hence monotone complete, it follows that $\{\varphi^{-1}(b_\lambda)\}_{\lambda \in \Lambda}$ has a least upper bound a. Hence $b := \varphi(a)$ is the least upper bound of $\{b_\lambda\}_{\lambda \in \Lambda}$, for φ is an order isomorphism. We conclude that B_{sa} is monotone complete.

Let ω be a normal state on A_{sa}. Then $\omega \circ \varphi^{-1}$ is a normal state on B_{sa}. Indeed, $\omega \circ \varphi^{-1}$ is clearly a linear functional $B_{\mathrm{sa}} \to \mathbb{R}$, and $\omega \circ \varphi^{-1}(1_B) = \omega(1_A) = 1$. Furthermore, $\omega \circ \varphi^{-1}$ is continuous, since φ is a bijective isometry (cf. Proposition 9.1.2). Moreover, if $b \in B$ is positive, so $b = c^2$ for some $c \in B$, hence

$$\omega \circ \varphi^{-1}(b) = \omega \circ \varphi^{-1}(c^2) = \omega\big(\varphi^{-1}(c)^2\big),$$

which is positive, since ω is positive. Let $\{b_\lambda\}_{\lambda \in \Lambda}$ be an increasing net in B_{sa} with least upper bound b. Again since φ is an order isomorphism, $\{\varphi^{-1}(b_\lambda)\}_{\lambda \in \Lambda}$ is an increasing net in A_{sa} with least upper bound $\varphi^{-1}(b)$. Since ω is normal, it now follows that

$$\lim_{\lambda \in \Lambda} \omega \circ \varphi^{-1}(b_\lambda) = \omega \circ \varphi^{-1}(b).$$

Hence $\omega \circ \varphi^{-1}$ indeed is a normal state of B_{sa}.

Now let $b \in B_{\mathrm{sa}}$ be non-zero. Since φ is an isometry, we have $\|\varphi(0)\| = \|0\| = 0$, so $\varphi(0) = 0$. By the bijectivity of φ it follows that $\varphi^{-1}(b) \neq 0$. Since A_{sa} is a JBW-algebra, there is some normal state ω of A_{sa} such that $\omega\big(\varphi^{-1}(b)\big) \neq 0$. It follows that $\rho := \omega \circ \varphi^{-1}$ is a normal state of B_{sa} such that $\rho(b) \neq 0$, so the normal states of B_{sa} form a separating family. We conclude from Definition 9.3.7 that B_{sa} is a JBW-algebra, hence B is a W*-algebra. □

Corollary 9.3.9. Let A be a W*-algebra, let B be a C*-algebra, and let $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$ be an order isomorphism. Then B is a W*-algebra.

Proof. This follows from Proposition 8.4.1 and the previous theorem. □

Finally, we note that Haagerup and Hanche-Olsen developed Tomita-Takesaki Theory for JBW-algebras [45]. It might be interesting to investigate whether the Tomita-Takesaki theory involved in Connes' subclassification only depends on the Jordan structure of type III W*-algebras. If this is the case, and if A is a type III_λ W*-algebra, then $\mathcal{C}(A) \cong \mathcal{C}(B)$ implies that B is a type III_λ W*-algebra, too.

10 Grothendieck topologies

We noted in the introduction that $\mathcal{C}(A)$ is not a complete invariant for C*-algebras. The question is what information should be added in order make it complete. In the previous chapter we have seen that an order isomorphism between $\mathcal{C}(A)$ and $\mathcal{C}(B)$ implies the existence of a quasi-Jordan *-isomorphism between A and B, which in some cases is even a Jordan *-isomorphism. The advantage of the Jordan structure is that under mild assumptions, a Jordan isomorphism $\varphi : A \to B$ between C*-algebras A and B can be written as a sum of a *-isomorphism and a *-anti-isomorphism [2, Corollary 5.75]. Hence a Jordan isomorphism is 'almost' a *-isomorphism; we only need extra structure that fixes whether we are dealing with a *-isomorphism or with a *-anti-isomorphism.

Since $\mathcal{C}(A)$ is a central object in the topos approach to quantum mechanics, it is natural to consider a sheaf-theoretic notion as possible extra structure, and the notion of a *Grothendieck topology* is probably the most prominent one in this direction. A Grothendieck topology is a structure that allows us to generalize the notion of *sheaves*, which were originally defined on topological spaces, to sheaves on arbitrary categories. Initially, it was hoped that the class of Grothendieck topologies falls apart in two classes such that if J and K are Grothendieck topologies on $\mathcal{C}(A)$ and $\mathcal{C}(B)$, respectively, and if $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$ is an isomorphism of posets that also 'preserves' the Grothendieck topology, then Φ is induced by a unique *-isomorphism if J and K both belong to the first class, and Φ is induced by a unique *-anti-isomorphism if J and K both belong to the second class.

In the first section we define Grothendieck topologies on posets, and prove that for each poset P there is an injection from the power set of P into the set of all Grothendieck topologies on P. Moreover, we show that each subset X of some poset P generates a Grothendieck topology J_X on P. We also prove that all Grothendieck topologies on the poset are of the form J_X for some subset X of P if and only if P is Artinian[8]. As a consequence, a C*-algebra A is finite-dimensional if and only if all Grothendieck topologies on $\mathcal{C}(A)$ are induced by some subset of $\mathcal{C}(A)$.

[8] We note that a similar statement is proven in [33], namely P is Artinian if and only if all congruences on $\mathcal{D}(P)$, the lattice of down-sets of P, are induced by some subset of P. This is no coincidence, since there exists a bijection between Grothendieck topologies on P and congruences on $\mathcal{D}(P)$ [83].

Since the power set of $\mathcal{C}(A)$ does not clearly fall apart into two disjoint subsets, for finite-dimensional C*-algebras it is not clear (at least to the author) whether Grothendieck topologies form the missing extra structure. Nevertheless, for a topos-theoretic analysis of C*-algebras, Grothendieck topologies are useful to investigate, if only because there are natural subposets of $\mathcal{C}(A)$, such as $\mathcal{C}(B)$ if $B \subseteq A$, or $\mathcal{A}(A)$ if A is an AW*-algebra, and choosing a suitable Grothendieck topology on $\mathcal{C}(A)$ shows that the presheaf topoi corrresponding to these subposets can be regarded as subtopoi of $\mathbf{Sets}^{\mathcal{C}(A)^{\mathrm{op}}}$.

In the last section, we consider morphisms of *sites*, i.e., pairs (P, J) with P a poset and J a Grothendieck topology on P, and show how these can be described in case that the Grothendieck topologies are generated by subsets of the posets.

10.1 Grothendieck topologies on posets

In this section, we define the notion of a Grothendieck topology on a poset P, construct an injection from the power set of P into the set of all Grothendieck topologies on P (cf. Lemma 10.1.12). Moreover, we show that this injection is a bijection if and only if P is Artinian (cf. Theorem 10.1.13). Consequently, if $P = \mathcal{C}(A)$, then this injection is a bijection if and only if A is finite-dimensional (cf. Corollary 10.1.14).

Definition 10.1.1. Let P be a poset. We denote the set of all down-sets of a subset X of P by $\mathcal{D}(X)$.

We stress that $\mathcal{D}(X)$ is not related to the set of all u.s.c. decompositions of a topological space. However, in this chapter, we only deal with down-sets and not with u.s.c. decompositions, so there will be no confusion.

Definition 10.1.2. Let P be a poset. Given an element $p \in P$, a subset $S \subseteq P$ is called a *sieve* on p if $S \in \mathcal{D}(\downarrow p)$, where $\downarrow p = \{q \in P : q \leq p\}$. Equivalently, S is a sieve on p if $q \leq p$ for each $q \in S$ and $r \in S$ if $r \leq q$ for some $q \in S$. Then a *Grothendieck topology* J on P is a map $p \mapsto J(p)$ that assigns to each element $p \in P$ a collection $J(p)$ of sieves on p such that

(1) the maximal sieve $\downarrow p$ is an element of $J(p)$;

(2) if $S \in J(p)$ and $q \leq p$, then $S \cap \downarrow q \in J(q)$;

(3) if $S \in J(p)$ and R is any sieve on p such that $R \cap \downarrow q \in J(q)$ for each $q \in S$, then $R \in J(p)$.

The second and third axioms are called the *stability axiom* and the *transitivity axiom*, respectively. Given $p \in P$, we refer to elements of $J(p)$ as J-*covers* of p. A pair (P, J) consisting of a poset P and a Grothendieck topology J on P is called a *site*. We denote the set of Grothendieck topologies on P by $\mathcal{G}(P)$. This set can be ordered pointwisely: for any two Grothendieck topologies J and K on P we define $J \leq K$ if $J(p) \subseteq K(p)$ for each $p \in P$.

Notice that J becomes a functor $P^{\mathrm{op}} \to \mathbf{Sets}$ if we define $J(q \leq p)S = S \cap \downarrow q$ for each $S \in J(p)$. Indeed, if $r \leq q \leq p$ and $S \in J(p)$, we have

$$J(r \leq p)S = S \cap \downarrow r = (S \cap \downarrow q) \cap \downarrow r = J(r \leq q)J(q \leq p)S.$$

Thus the stability axiom states that $J \in \mathbf{Sets}^{P^{\mathrm{op}}}$. In fact, it is enough to require that $S \in J(p)$ for some $p \in P$ implies $S \cap \downarrow q \in J(q)$ for each $q < p$. Indeed, if $q = p$, then $S \cap \downarrow q = S$, which was already assumed to be in $J(p)$.

Example 10.1.3. If $P = \mathcal{O}(X)$, the set of open subsets of some topological space X ordered by inclusion, the Grothendieck topology corresponding to the usual notion of covering is given by

$$J(U) = \{S \in \mathcal{D}(\downarrow U) : \bigcup S = U\}.$$

Example 10.1.4. Let P be a poset. Then

- The *indiscrete* Grothendieck topology on P is given by $J_{\mathrm{ind}}(p) = \{\downarrow p\}$.
- The *discrete* Grothendieck topology on P is given by $J_{\mathrm{dis}}(p) = \mathcal{D}(\downarrow p)$.
- The *atomic* Grothendieck topology on P can only be defined if P is filtered, and is given by $J_{\mathrm{atom}}(p) = \mathcal{D}(\downarrow p) \setminus \{\emptyset\}$.

Notice that the existence of (finite) meets is sufficient for a poset to be filtered. An example of a poset that is not filtered and where the stability axiom for J_{atom} fails is as follows. Let $P_3 = \{x, y, z\}$ with $y \leq x$ and $z \leq x$. Then $\downarrow y \in J_{\mathrm{atom}}(x)$ and since $z \leq x$, we should have $\downarrow y \cap \downarrow z \in J(x)$. But this means that $\emptyset \in J_{\mathrm{atom}}(x)$, a contradiction. On the other hand, if P is filtered, the stability axiom always holds. Indeed, let $S \in J_{\mathrm{atom}}(x)$ and $z \leq x$. Even with

$z \notin S$, we have $S \cap {\downarrow} z \neq \emptyset$, so $S \cap {\downarrow} z \in J_{\text{atom}}(z)$, since if we choose an arbitrary $y \in S$, there must be a $w \in P$ such that $w \leq z$ and $w \leq y$. The latter inequality implies $w \in S$, so $w \in S \cap {\downarrow} z$.

The following lemma will be very useful; for arbitrary categories it can be found in [86, pp. 110-111]. We give a direct proof for posets.

Lemma 10.1.5. Let J be a Grothendieck topology on P. Then $J(p)$ is a filter of sieves on p in the sense that:

- $S \in J(p)$ implies $R \in J(p)$ for each sieve R on p containing S;
- $S, R \in J(p)$ implies $S \cap R \in J(p)$.

Proof. Let $S \in J(p)$ and $R \in \mathcal{D}({\downarrow} p)$ such that $S \subseteq R$. Then if $q \in S$, we have $q \in R$, so $R \cap {\downarrow} q = {\downarrow} q \in J(q)$. It now follows from the transitivity axiom that $R \in J(p)$.

If $S, R \in J(p)$, and let $q \in R$. Then

$$(S \cap {\downarrow} R) \cap {\downarrow} q = S \cap (R \cap {\downarrow} q) = S \cap {\downarrow} q \in J(q)$$

by the stability axiom. So $(S \cap R) \cap {\downarrow} q \in J(q)$ for each $q \in R$, hence by the transitivity axiom, it follows that $S \cap R \in J(p)$. □

We see that a Grothendieck topology is pointwise closed under finite intersections. In general, a Grothendieck topology is not closed under arbitrary intersections, which leads to the following definition.

Definition 10.1.6. Let J be a Grothendieck topology on a poset P. We say that J is *complete* if for each $p \in P$ and each family $\{S_i\}_{i \in I}$ in $J(p)$, for some index set I, we have $\bigcap_{i \in I} S_i \in J(p)$.

Lemma 10.1.7. For each $p \in P$, denote $\bigcap\{S \in J(p)\}$ by S_p. Then J is complete if and only if $S_p \in J(p)$ for each $p \in P$.

Proof. If J is complete, it follows directly from the definitions of S_p and of the definition of completeness that $S_p \in J(p)$ for each $p \in P$. Conversely, assume that $S_p \in J(p)$ for each $p \in P$. For an arbitrary $p \in P$ let $\{S_i\}_{i \in I}$ a family of sieves in $J(p)$. Then

$$S_p = \bigcap\{S \in J(p)\} \subseteq \bigcap_{i \in I} S_i,$$

so by Lemma 10.1.5, we find that $\bigcap_{i \in I} S_i \in J(p)$. □

Proposition 10.1.8. Let P be a poset. Then $\mathcal{G}(P)$ is a complete lattice where the infimum $\bigwedge_{i\in I} J_i$ of a each collection $\{J_i\}_{i\in I}$ of Grothendieck topologies on P is defined by pointwise intersection: $\left(\bigwedge_{i\in I} J_i\right)(p) = \bigcap_{i\in I} J_i(p)$ for any collection $\{J_i\}_{i\in I}$ of Grothendieck topologies on P.

Proof. Let $\{J_i\}_{i\in I}$ be a collection of Grothendieck topologies on P. We shall prove that $\bigwedge_{i\in I} J_i$ is a Grothendieck topology on P. Let $p \in P$, then $\downarrow p \in J_i(p)$ for each $i \in I$, so $\downarrow p \in \bigcap_{i\in I} J_i(p)$. For stability, assume $S \in \bigcap_{i\in I} J_i(p)$ and let $q \leq p$. Thus $S \in J_i(p)$ for each $i \in I$, so by the stability axiom for each J_i, we find $S \cap \downarrow q \in J_i(q)$ for each $i \in I$. So $S \cap \downarrow q \in \bigcap_{i\in I} J_i(q)$. Finally, let $S \in \bigcap_{i\in I} J_i(p)$ and $R \in \mathcal{D}(\downarrow p)$ be such that $R \cap \downarrow q \in \bigcap_{i\in I} J_i(q)$ for each $q \in S$. Then for each $i \in I$ we have $S \in J_i(p)$ and $R \cap \downarrow q \in J_i(q)$ for each $q \in S$. So for each $i \in I$, by the transitivity axiom for J_i we find that $R \in J_i(p)$. Thus $R \in \bigcap_{i\in I} J_i(p)$. We conclude that $\bigwedge J_i$ is indeed a Grothendieck topology on P. Now, for each $k \in I$, we have $\bigwedge_{i\in I} J_i \leq J_k$, since $\bigcap_{i\in I} J_i(p) \subseteq J_k(p)$ for each $p \in P$. If K is another Grothendieck topology on P such that $J_k \leq K$ for each $k \in I$, then for each $p \in P$

$$\bigcap_{i\in I} J_i(p) \subseteq J_k(p) \subseteq K(p).$$

Hence $\bigwedge_{i\in I} J_i \leq K$. This shows that pointwise intersection indeed defines an infimum operation on $\mathcal{G}(P)$, and by Lemma B.1.13, it follows that $\mathcal{G}(P)$ is a complete lattice. □

We now describe a special class of Grothendieck topologies on a poset that are generated by subsets of the poset.

Proposition 10.1.9. Let P be a poset and X a subset of P. Then

$$J_X(p) = \{S \in \mathcal{D}(\downarrow p) : X \cap \downarrow p \subseteq S\} \tag{41}$$

is a complete Grothendieck topology on P. Moreover, if $X \subset Y \subseteq P$, then $J_Y \leq J_X$.

Proof. We have $\downarrow p \in J_X(p)$, for $\downarrow p$ contains $X \cap \downarrow p$. If $S \in J_X(p)$, i.e. $X \cap \downarrow p \subseteq S$, and if $q < p$, then $\downarrow q \subset \downarrow p$, so

$$X \cap \downarrow q = X \cap \downarrow p \cap \downarrow q \subseteq S \cap \downarrow q.$$

Hence we find that $S \cap \downarrow q \in J_X(q)$, so stability holds. For transitivity, let $S \in J_X(p)$, so $X \cap \downarrow p \subseteq S$. Let R be a sieve on p such that for each $q \in S$ we have $R \cap \downarrow q \in J_X(q)$, so $X \cap \downarrow q \subseteq R \cap \downarrow q$. Since S is a sieve, we have $S = \downarrow S = \bigcup_{q \in S} \downarrow q$, whence we find

$$X \cap S = X \cap \bigcup_{q \in S} \downarrow q = \bigcup_{q \in S} (X \cap \downarrow q) \subseteq \bigcup_{q \in S} (R \cap \downarrow q) = R \cap \bigcup_{q \in S} \downarrow q = R \cap S,$$

from which

$$X \cap \downarrow p = X \cap (X \cap \downarrow p) \subseteq X \cap S \subseteq R \cap S \subseteq R$$

follows. Thus $R \in J_X(p)$ and the transitivity axiom holds. We next have to show that J_X is complete. So let $p \in P$ and let $\{S_i\}_{i \in I} \subseteq J_X(p)$ be a collection of covers with index set I. This means that $X \cap \downarrow p \subseteq S_i$ for each $i \in I$. But this implies that $X \cap \downarrow p \subseteq \bigcap_{i \in I} S_i$, hence $\bigcap_{i \in I} S_i \in J_X(p)$. Finally, let $Y \subseteq P$ such that $X \subseteq Y$. Let $p \in P$ and $S \in J_Y(p)$. Then $Y \cap \downarrow p \subseteq S$, and since $X \subseteq Y$ this implies $X \cap \downarrow p \subseteq S$. So $S \in J_X(p)$. We conclude that $J_Y \leq J_X$. □

We call J_X the *subset Grothendieck topology* generated by the subset X of P. It is easy to see that the indiscrete Grothendieck topology is exactly J_P, whereas the discrete Grothendieck topology is $J_\emptyset$.

Lemma 10.1.10. Let J be a Grothendieck topology on a poset P and define $X_J \subseteq P$ by

$$X_J = \big\{ p \in P : J(p) = \{\downarrow p\} \big\}. \tag{42}$$

If K is another Grothendieck topology on P such that $K \leq J$, then $X_J \subseteq X_K$.

Proof. Let $p \in X_J$. Then $K(p) \subseteq J(p) = \{\downarrow p\}$, and since $\downarrow p \in K(p)$ by definition of a Grothendieck topology, this implies $K(p) = \{\downarrow p\}$. So $p \in X_K$. □

Lemma 10.1.11. Let P be a poset and J a Grothendieck topology on P. If $p \in P$ such that $J(p) = \{\downarrow p\}$, then for each sieve S on p such that $X_J \cap \downarrow p \subseteq S$, we have $S \in J(p)$.

Proof. By definition of X_J we have $p \in X_J$, so each sieve S on p containing $X_J \cap \downarrow p$ contains p. Since the only sieve on p containing p must be equal to $\downarrow p$, we find that $X_J \cap \downarrow p \subseteq S$ implies $S \in J(p)$. □

Lemma 10.1.12. Let P be a poset. Then

(1) $Y = X_{J_Y}$ for each $Y \subseteq P$;

(2) $J_Y \leq J_Z$ if and only if $Z \subseteq Y$ for each $Y, Z \subseteq P$;

(3) $K \leq J_{X_K}$ for each Grothendieck topology K on P;

(4) If P is Artinian, then the map $\mathcal{P}(P)^{\mathrm{op}} \to \mathcal{G}(P)$ given by $X \mapsto J_X$ is an order isomorphism with inverse $J \mapsto X_J$;

(5) If P is Artinian, then every Grothendieck topology on P is complete.

Proof.

(1) Let $x \in Y$. Then $x \in Y \cap \downarrow x$, so if $S \in J_Y(x)$, that is S is a sieve on x containing $Y \cap \downarrow x$, we must have $x \in S$. The only sieve on x containing x is $\downarrow x$, so we find that $J_Y(x) = \{\downarrow x\}$. By definition of X_{J_Y} we find that $x \in X_{J_Y}$.

Conversely, let $x \in X_{J_Y}$. This means that $J_Y(x) = \{\downarrow x\}$, or equivalently, that the only sieve on x containing $Y \cap \downarrow x$ is $\downarrow x$. Now, assume that $x \notin Y$. Then $x \notin Y \cap \downarrow x$, so $\downarrow x \setminus \{x\}$ (possibly empty) is a sieve on x which clearly contains $X \cap \downarrow x$, but which is clearly not equal to $\downarrow x$ in any case. So we must have $x \in Y$.

(2) Let $Y, Z \subseteq P$. We already found in Proposition 10.1.9 that $Z \subseteq Y$ implies $J_Y \leq J_Z$. So assume that $J_Y \leq J_Z$. By Lemma 10.1.10, this implies that $X_{J_Z} \subseteq X_{J_Y}$. But by the first statement of this proposition, this is exactly $Z \subseteq Y$.

(3) Let $S \in K(p)$ and $q \in X_K \cap \downarrow p$. So $q \leq p$ and $K(q) = \{\downarrow q\}$. Since $S \in K(p)$ and $q \leq p$, we have $\downarrow q \cap S \in K(q)$ by stability. In other words $S \cap \downarrow q = \downarrow q$. Thus $\downarrow q \subseteq S$ and so certainly $q \in S$. We see that $X_K \cap \downarrow p \subseteq S$, hence $S \in J_{X_K}$.

(4) It follows from Proposition 10.1.9 and Lemma 10.1.10 that $X \mapsto J_X$ and $J \mapsto X_J$ are order morphisms. Since the first statement of this proposition is equivalent with $J \mapsto X_J$ being a left inverse of $X \mapsto J_X$, we only have to show that it is also a right inverse. In other words, we have to show that each Grothendieck topology K on P equals J_{X_K}. So let K be a Grothendieck topology on P. We shall show using Artinian induction (see

Lemma B.2.4) that for any $p \in P$, each sieve S on p containing $X_K \cap \downarrow p$ is an element of $K(p)$.

Let p be a minimal element. Then the only possible sieves on p are $\downarrow p = \{p\}$ and $\emptyset$. Hence there are only two options for $K(p)$, namely either $K(p) = \{\downarrow p\}$ or $K(p) = \{\emptyset, \downarrow p\}$. In the first case, we find by Lemma 10.1.11 that $X_K \cap \downarrow p \subseteq S$ implies $S \in K(p)$. If $K(p) = \{\emptyset, \downarrow p\}$, then $K(p)$ contains all possible sieves on p, so we automatically have that $S \in K(p)$ for any sieve S on p such that $X_K \cap \downarrow p \subseteq S$.

For the induction step, assume that $p \in P$ is not minimal and assume that for each $q < p$ the condition $X_K \cap \downarrow q \subseteq R$ implies $R \in K(q)$ for each sieve R on q. If $K(p) = \{\downarrow p\}$, we again apply Lemma 10.1.11 to conclude that $S \in K(p)$ for all sieves S on p such that $X_K \cap \downarrow p \subseteq S$. If $K(p) \neq \{\downarrow p\}$, we must have $\downarrow p \setminus \{p\} \in K(p)$, which is non-empty, since p is not minimal. If S is a sieve on p such that $X_K \cap \downarrow p \subseteq S$, we have for each $q \in \downarrow p \setminus \{p\}$, that is, for each $q < p$, that

$$X_K \cap \downarrow q = X_K \cap \downarrow p \cap \downarrow q \subseteq S \cap \downarrow q. \tag{43}$$

Our induction assumption on $q < p$ implies now that $S \cap \downarrow q \in K(q)$ for each $q \in \downarrow p \setminus \{p\}$, hence by the transitivity axiom we find $S \in K(p)$.

So for all sieves S on p such that $X_K \cap \downarrow p \subseteq S$, we found that $S \in K(p)$, hence we have $J_{X_K} \subseteq K$. The other inclusion follows from the third statement of this proposition.

(5) Since all subset Grothendieck topologies are complete, this follows from the fourth statement of this proposition. □

Theorem 10.1.13. Let P be a poset. Then P is Artinian if and only if all Grothendieck topologies on P are subset Grothendieck topologies.

Proof. The previous lemma states that all Grothendieck topologies on P are subset Grothendieck topologies if P is Artinian. For the other direction, we first introduce another Grothendieck topology. Let P be a poset and $X \subseteq P$, and define L_X for each $p \in P$ by

$$L_X(p) = \{S \in \mathcal{D}(\downarrow p) : x \in X \cap \downarrow p \implies S \cap \downarrow x \cap X \neq \emptyset\}.$$

To see that this is a Grothendieck topology on P, assume that $x \in X \cap \downarrow p$, then clearly $\downarrow p \cap \downarrow x \cap X \neq \emptyset$, so $\downarrow p \in L_X(p)$. If $S \in L_X(p)$ and $q \leq p$, assume that $x \in X \cap \downarrow q$. Then $x \in X \cap \downarrow p$, so $S \cap \downarrow x \cap X \neq \emptyset$. Since $x \leq q$, we have $\downarrow x = \downarrow q \cap \downarrow x$, hence $S \cap \downarrow q \cap \downarrow x \cap X \neq \emptyset$. We conclude that $S \cap \downarrow q \in L_X(q)$. Finally, let $S \in L_X(p)$ and $R \in \mathcal{D}(\downarrow p)$ such that $R \cap \downarrow q \in L_X(q)$ for each $q \in S$. Let $x \in X \cap \downarrow p$. Then $S \cap \downarrow x \cap X \neq \emptyset$, so there is some $q \in S \cap \downarrow x \cap X$. Since $q \in S$, we find $R \cap \downarrow q \in L_X(q)$. Since $q \in X$, we find $q \in X \cap \downarrow q$, so $(R \cap \downarrow q) \cap \downarrow q \cap X \neq \emptyset$. Since $q \leq x$, we find $\downarrow q \subseteq \downarrow x$, whence $R \cap \downarrow x \cap X \neq \emptyset$. So $R \in L_X(p)$.

Now assume that P is non-Artinian. Then P contains a non-empty subset X without a minimal element. We show that $L_X \neq J_Y$ for each $Y \subseteq P$. First take $Y = \emptyset$. Then $\emptyset \in J_Y(p)$ for each $p \in P$. However, since X is assumed to be non-empty, there is some $p \in X$. Then $p \in X \cap \downarrow p$, but $\emptyset \cap \downarrow p \cap X = \emptyset$, so $\emptyset \notin L_X(p)$. We conclude that $L_X \neq J_Y$ if $Y = \emptyset$.

Assume that Y is non-empty. Then there is some $p \in Y$, and $J_Y(p) = \{\downarrow p\}$. Assume that $X \cap \downarrow p = \emptyset$. Then

$$x \in X \cap \downarrow p \implies \emptyset \cap \downarrow x \cap X \neq \emptyset$$

holds, so $\emptyset \in L_X(p)$. Thus $L_X \neq J_Y$ in this case. Assume that $X \cap \downarrow p \neq \emptyset$. Hence there is some $x \in X \cap \downarrow p$. Even if $p \in X$, we can assume that x is strictly smaller than p, since X does not contain a minimal element, so $X \cap \downarrow p \setminus \{p\} \neq \emptyset$. Let $S = \downarrow x$, then $S \cap \downarrow x \cap X \neq \emptyset$, so $\downarrow x \in L_X(p)$. We conclude that $L_X(p) \neq \{\downarrow p\}$, so $L_X \neq J_Y$ in all cases. □

Corollary 10.1.14. Let A be a C*-algebra. Then A is finite-dimensional if and only if all Grothendieck topologies on $\mathcal{C}(A)$ are subset Grothendieck topologies.

Proof. This follows directly from Theorem 10.1.13 and Theorem 5.1.2. □

10.2 Morphisms of sites

In this section we explore two notions of morphisms of sites. It turns out that these notions yield the same isomorphisms. Moreover, we are interested in the conditions that an order isomorphism $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$ has to satisfy in order to become a site isomorphism $(\mathcal{C}(A), J) \to (\mathcal{C}(B), K)$. Not surprisingly, if $\mathcal{X} \subseteq \mathcal{C}(A)$ and $\mathcal{Y} \subseteq \mathcal{C}(B)$, then $\Phi : (\mathcal{C}(A), J_{\mathcal{X}}) \to (\mathcal{C}(B), J_{\mathcal{Y}})$ is a site isomorphism if and only if $\Phi[\mathcal{X}] = \mathcal{Y}$ (cf. Corollary 10.2.7).

Definition 10.2.1. [86, §VII.10] Let (P, J) and (Q, K) be sites. An order morphism $\varphi : P \to Q$ *preserves covers* if $\downarrow \varphi[S] \in K(\varphi(p))$ for each $p \in P$

and each $S \in J(p)$. An order morphism $\pi : Q \to P$ has the *covering lifting property* (abbreviated by "clp") if for each $q \in Q$ and each $S \in J(\pi(q))$ there is an $R \in K(q)$ such that $\pi[R] \subseteq S$.

Lemma 10.2.2. Let $\varphi : (P_1, J_1) \to (P_2, J_2)$ and $\pi : (P_2, J_2) \to (P_3, J_3)$ be order morphisms. Then $\pi \circ \varphi$ has the clp if π and φ have the clp. Moreover, if π and φ preserve covers, then $\pi \circ \varphi$ preserves covers.

Proof. Let $p \in P_1$ and $S_3 \in J_3(\pi \circ \varphi(p))$. Since π has the clp, there must be an $S_2 \in J_2(\varphi(p))$ such that $\pi[S_2] \subseteq S_3$. Since φ has the clp, there must be an $S_1 \in J_1(p)$ such that $\varphi[S_1] \subseteq S_2$, hence $\pi \circ \varphi[S_1] \subseteq \pi[S_2]$. Combining both inclusions, we obtain $\pi \circ \varphi[S_1] \subseteq S_3$, hence $\pi \circ \varphi$ must have the clp.

Now assume that π and φ preserve covers and let $p \in P$ and $S \in J_1(p)$. Then $\downarrow \varphi[S] \in J_2(\varphi(p))$, since φ preserves covers, hence $\downarrow \pi[\downarrow \varphi[S]] \in J_3(\pi \circ \varphi(p))$ for π preserves covers. However, in order to show that $\pi \circ \varphi$ preserves covers, we have to show that $\downarrow \pi \circ \varphi[S] \in J_3(\pi \circ \varphi(p))$. Let $x \in \downarrow \pi[\downarrow \varphi[S]]$. Then there is a $y \in \downarrow \varphi[S]$ such that $x \leq \pi(y)$. Moreover, there must be an $s \in S$ such that $y \leq \varphi(s)$. Since π is an order morphism, we find $\pi(y) \leq \pi \circ \varphi(s)$, so $x \leq \pi \circ \varphi(s)$. We conclude that $\downarrow \pi[\downarrow \varphi[S]] \subseteq \downarrow \pi \circ \varphi[S]$, hence by Lemma 10.1.5 it follows that $\downarrow \pi \circ \varphi[S] \in J_3(\pi \circ \varphi(p))$. □

In order to define a correct and suitable notion of morphisms of sites, it seems like we have to choose between the cover preserving property and the clp. However, both notions are related to each other, as follows from the following lemma.

Lemma 10.2.3. [86, Lemma VII.10.3] Let (P, J) and (Q, K) be sites and $\varphi : P \to Q$ the upper adjoint of $\pi : Q \to P$. Then φ preserves covers if and only if π has the clp.

In order to find a satisfactory notion of site isomorphisms, we first recall that a map $\varphi : (P, J) \to (Q, K)$ that either preserve covers or has the clp always restricts to an order morphism between the poset parts of the sites. Since a site isomorphism must have an inverse that is also a site morphism, it follows that the restriction of a site isomorphism to the poset parts of the site should be an order isomorphism. Furthermore, if the map φ above preserves covers and has an inverse π that preserves covers, too, then it follows from the previous lemma that both φ and π have the clp. Likewise, if φ has the clp, and has an inverse that has the clp, too, then the previous lemma assures that both φ and π preserve covers. This leads to the following definition:

Definition 10.2.4. Let (P, J) and (Q, K) be sites. Then an order isomorphism $\varphi : P \to Q$ is called an *isomorphism of sites* if it satisfies one of the following equivalent conditions:

(1) φ preserves covers and has the clp;

(2) φ and φ^{-1} both preserve covers;

(3) φ and φ^{-1} both have the clp.

Lemma 10.2.5. Let $\varphi : P \to Q$ be an order isomorphism. Let $p \in P$. Then $S \in \mathcal{D}(\downarrow p)$ implies $\varphi[S] \in \mathcal{D}(\downarrow \varphi(p))$.

Proof. For each $s \in S \in \mathcal{D}(\downarrow p)$ we have $s \leq p$, so $\varphi(s) \leq \varphi(p)$. If $x \in \varphi[S]$ and $y \leq x$, then $\varphi^{-1}(x) \in S$ and $\varphi^{-1}(y) \leq \varphi^{-1}(x)$, so $\varphi^{-1}(y) \in S$. Hence $y \in \varphi[S]$, and we conclude that $\varphi[S] \in \mathcal{D}(\downarrow \varphi(p))$. □

Proposition 10.2.6. Let (P, J) and (Q, K) be sites and $\varphi : P \to Q$ an order isomorphism. Then the following statements are equivalent:

- φ is an isomorphism of sites $(P, J) \to (Q, K)$;
- $S \in J(p)$ if and only if $\varphi[S] \in K(\varphi(p))$ for each $p \in P$ and $S \in \mathcal{D}(\downarrow p)$.

Proof. Assume that φ is an isomorphism of sites and let $p \in P$ and $S \in J(p)$. Since φ preserves covers, we have $\downarrow \varphi[S] \in K(\varphi(p))$. By the preceding lemma, we have $\varphi[S] \in \mathcal{D}(\downarrow \varphi(p))$, so $\downarrow \varphi[S] = \varphi[S]$. Hence $\varphi[S] \in K(\varphi(p))$. Conversely, if $S \in \mathcal{P}(P)$ such that $\varphi[S] \in K(\varphi(p))$, then the same argument for φ^{-1} instead of φ yields $\varphi^{-1}[\varphi[S]] \in J(\varphi^{-1} \circ \varphi(p))$, so $S \in J(p)$.

Now assume that for each $p \in P$, we have $S \in J(p)$ if and only if $\varphi[S] \in K(\varphi(p))$. Let $p \in P$ and $S \in J(p)$. Then $\varphi[S] \in K(\varphi(p))$, so certainly $\downarrow \varphi[S] \in K(\varphi(p))$. Let $q \in Q$ and $R \in K(q)$. By Lemma 10.2.5 we find $\varphi^{-1}[R] \in \mathcal{D}(\downarrow \varphi^{-1}(q))$. Since $\varphi[\varphi^{-1}[R]] = R$, we find that

$$\varphi[\varphi^{-1}[R]] \in K(\varphi \circ \varphi^{-1}(q)),$$

so $\varphi^{-1}[R] \in J(\varphi^{-1}(q))$. We conclude that both φ and φ^{-1} preserve covers, so φ is an isomorphism of sites. □

Corollary 10.2.7. Let P and Q posets, and $X \subseteq P$ and $Y \subseteq Q$ subsets. Then an order isomorphism $\varphi : P \to Q$ is an isomorphism of sites $(P, J_X) \to (Q, J_Y)$ if and only if $\varphi[X] = Y$.

Proof. Assume that $\varphi : (P, J_X) \to (Q, J_Y)$ is an isomorphism of sites and let $x \in X$. Then $J(x) = \{\downarrow x\}$, so $J_Y(\varphi(x)) = \{\varphi[\downarrow x]\}$ for φ is an isomorphism of sites. Now $J_Y(\varphi(x))$ contains only one element if and only if $\varphi(x) \in Y$, hence $\varphi[X] \subseteq Y$. Replacing φ by φ^{-1} gives $\varphi^{-1}[Y] \subseteq X$, hence $\varphi[X] = Y$. Conversely assume that $\varphi[X] = Y$. Let $p \in P$ and $S \in J_X(p)$. Then $X \cap \downarrow p \subseteq S$, hence $\varphi[X] \cap \varphi[\downarrow p] \subseteq \varphi[S]$. By Lemma 10.2.5 and Proposition B.1.15, we obtain $\varphi[S] \in \mathcal{D}(\downarrow \varphi(p))$ and $\varphi[\downarrow p] = \downarrow \varphi(p)$, respectively. Moreover, $\varphi[X] = Y$, hence $Y \cap \downarrow \varphi(p) \subseteq \varphi[S]$, and we conclude that $\varphi[S] \in J_Y(\varphi(p))$. Since φ is an order isomorphism, we have $\varphi^{-1}[Y] = X$. Hence applying the same arguments to φ^{-1} gives the implication $\varphi[S] \in J_Y(\varphi(p))$ implies $S \in J_X(p)$. We conclude that φ is an isomorphism of sites. □

Corollary 10.2.8.

- Let A and B be C*-algebras and $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$ an order isomorphism. Then
$$\Phi : (\mathcal{C}(A), J_{\mathcal{C}_{\mathrm{AF}}(A)}) \to (\mathcal{C}(B), J_{\mathcal{C}_{\mathrm{AF}}(B)})$$
is an isomorphism of sites.
- Let A and B be AW*-algebras and $\Phi : \mathcal{C}(A) \to \mathcal{C}(B)$ an order isomorphism. Then
$$\Phi : (\mathcal{C}(A), J_{\mathcal{A}(A)}) \to (\mathcal{C}(B), J_{\mathcal{A}(B)})$$
is an isomorphism of sites.

Proof. This follows from the previous corollary, Theorem 6.4.4, and Proposition 8.4.1. □

If (P, J_X) and (Q, J_Y) are sites, it might be interesting as well to examine how to express the cover preserving property and the clp of an order morphism $\varphi : P \to Q$ in terms of φ, X and Y.

Proposition 10.2.9. Let P and Q posets and X and Y subsets of P and Q, respectively. Let $\varphi : P \to Q$ an order morphism. Then $\varphi : (P, J_X) \to (Q, J_Y)$ has the clp if and only if $\varphi[X] \subseteq Y$.

Proof. Assume that φ has the clp and let $x \in X$. Since φ has the clp and $\downarrow(Y \cap \downarrow \varphi(x)) \in J_Y(\varphi(x))$, there must be an $R \in J_X(x)$ such that
$$\varphi[R] \subseteq \downarrow(Y \cap \downarrow \varphi(x)).$$

Since $x \in X$, it follows that $J_X(x)$ contains only $\downarrow x$, hence

$$\varphi[\downarrow x] \subseteq \downarrow(Y \cap \downarrow \varphi(x)).$$

In particular, we must have $\varphi(x) \in \downarrow(Y \cap \downarrow \varphi(x))$. Now assume that $\varphi(x) \notin Y$. Then for each $y \in Y \cap \downarrow \varphi(x)$ we have $y < \varphi(x)$. If $z \in \downarrow(Y \cap \downarrow \varphi(x))$, we must have $z \leq y$ for some $y \in Y \cap \downarrow \varphi(x)$, so $z < \varphi(x)$ for each $z \in \downarrow(Y \cap \downarrow \varphi(x)$. Thus the choice $z = \varphi(x)$ gives a contradiction, so we must have $\varphi(x) \in Y$. We conclude that $\varphi[X] \subseteq Y$.

Assume that $\varphi[X] \subseteq Y$ and let $p \in P$ and $S \in J_Y(\varphi(p))$. Then $Y \cap \downarrow \varphi(p) \subseteq S$, so $\varphi[X] \cap \downarrow \varphi(p) \subseteq S$. Moreover, since S is a down-set, we obtain $\downarrow(\varphi[X] \cap \downarrow \varphi(p)) \subseteq S$. Let $R = \downarrow(X \cap \downarrow p)$, then $R \in J_X(p)$. Let $y \in R$. Then there is an $x \in X$ such that $y \leq x \leq p$. Since φ is an order morphism, we find $\varphi(y) \leq \varphi(x) \leq \varphi(p)$. So $\varphi(x) \in \varphi[X] \cap \downarrow \varphi(p)$, whence $\varphi(y) \in \downarrow(\varphi[X] \cap \downarrow \varphi(p))$. Thus we find that $\varphi(y) \in S$, hence $\varphi[R] \subseteq S$, and we conclude that φ has the clp. □

For $\varphi : (P, J_X) \to (Q, J_Y)$ preserving covers, we can state a similar statement, although we have add bijectivity of φ as an extra condition.

Proposition 10.2.10. Let P and Q posets and X and Y subsets of P and Q, respectively. Let $\varphi : P \to Q$ an order isomorphism. Then $\varphi : (P, J_X) \to (Q, J_Y)$ preserves covers if and only if $Y \subseteq \varphi[X]$.

Proof. Assume that φ preserves covers. Then $\downarrow \varphi[S] \in J_Y(\varphi(p))$ for each $p \in P$ and $S \in J_X(p)$. Notice that Lemma 10.2.5 assures that $\downarrow \varphi[S] = \varphi[S]$ for φ is assumed to be an order isomorphism. Since $\downarrow(X \cap \downarrow p)$ is the least element of $J_X(p)$, Lemma 10.1.5 assures that $\varphi[S] \in J_Y(\varphi(p))$ for each $S \in J_X(p)$ if and only if $\varphi[\downarrow(X \cap \downarrow p))] \in J_Y(\varphi(p))$, we find that φ preserves covers if and only if for

$$Y \cap \downarrow \varphi(p) \subseteq \varphi[\downarrow(X \cap \downarrow p)] \tag{44}$$

for each $p \in P$. Let $y \in Y$. By surjectivity of φ there is some $p \in P$ such that $\varphi(p) = y$. Since φ preserves covers, we find that $y = \varphi(p) \in \varphi[\downarrow(X \cap \downarrow p)]$. Assume that $p \notin X$. Then for each $x \in X \cap \downarrow p$, we must have $x < p$. Then if $p' \in \downarrow(X \cap \downarrow p)$, there must be an $x \in X \cap \downarrow p$ such that $p' \leq x$, hence $p' < p$. Then $\varphi(p') < \varphi(p)$ for φ is an order isomorphism, so $z < \varphi(p)$ for each $z \in \varphi[\downarrow(X \cap \downarrow p)$. Since the choice $z = y$ gives a contradiction, we must have $\varphi(p) \in X$. We conclude that $Y \subseteq \varphi[X]$. Conversely, assume that $Y \subseteq \varphi[X]$

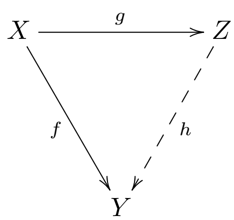
and let $p \in P$. We have to show that (44) holds, so let $y \in Y \cap \downarrow \varphi(p)$. Since $Y \subseteq \varphi[X]$, there is some $x \in X$ such that $\varphi(x) = y$. Hence $\varphi(x) \leq \varphi(p)$, whence $x \leq p$ for φ is an order isomorphism. Hence $y = \varphi(x)$ with $x \in X \cap \downarrow p$, which shows that (44) indeed holds. $\square$

A Topology

A.1 Compact Hausdorff spaces

Compact Hausdorff spaces play a key role in the theory of C*-algebras. The next lemma states conditions when maps between compact and Hausdorff spaces are continuous or closed.

Lemma A.1.1. Let X be compact, Y Hausdorff and $f : X \to Y$ continuous. Then f is closed. Moreover, if Z is compact Hausdorff and $g : X \to Z$ a continuous surjection, then there is at most one map $h : Z \to Y$ such that $f = h \circ g$, i.e. such that the following diagram commutes:



If h exists, then it is continuous and closed. Moreover, h is surjective if f is surjective.

Recall that a collection $\mathcal{C}$ of subsets of some set X has the *finite intersection property* if $\bigcap \mathcal{F} \neq \emptyset$ for each finite subset $\mathcal{F} \subseteq \mathcal{C}$.

Theorem A.1.2. [91, Theorem 26.9] A topological space X is compact if and only if $\bigcap \mathcal{C} \neq \emptyset$ for each collection $\mathcal{C}$ of closed subsets of X with the finite intersection property.

Most spaces in this thesis are compact Hausdorff, hence we will use the following three theorems as lemmas.

Theorem A.1.3. [91, Theorem 32.3] Every compact Hausdorff space is normal.

Theorem A.1.4. [91, Urysohn's Lemma, Theorem 33.1] Let X be a normal space and let A, B be disjoint closed subsets of Y. If $a < b$ in $\mathbb{R}$, then there is a continuous $f : X \to [a, b]$ such that $f(x) = a$ for all $x \in A$ and $f(x) = b$ for each $x \in B$.

Theorem A.1.5. [91, Tietze's Extension Theorem, Theorem 35.1] Let X be a normal space, $K \subseteq X$ closed and $a < b$ in $\mathbb{R}$. Then each continuous map $f : K \to [a, b]$ can be extended to a continuous map $F : X \to [a, b]$.

A.2 Disconnected spaces

Definition A.2.1. Let X be a topological space. A pair (K_0, K_1) of clopen disjoint subsets of X such that $K_0 \cup K_1 = X$ is called a *separation* of X. If both K_0 and K_1 are non-empty, the separation is called non-trivial. X is called *disconnected* if it admits a non-trivial separation.

Definition A.2.2. Let X be a topological space. Then X is called:

- *totally disconnected* if the connected components are precisely the singletons;
- *totally separated* if for each distinct $x, y \in X$, there is a clopen neighborhood C of x not containing y;
- *zero-dimensional* if X has a basis of clopen subsets;
- *scattered* if each non-empty closed subset K of X contains an isolated point, i.e., a point p such that $K \cap U = \{p\}$ for some open set $U \subseteq X$;
- *extremally disconnected* if the closure of each open subset of X is open.

Lemma A.2.3. Let X be a compact Hausdorff space. Then

(1) If X is either extremally disconnected or scattered, then X is totally disconnected;

(2) X is both extremally disconnected and scattered if and only if X is finite;

Proof. The first statement and the 'if' part of the second statement are easy to verify. Assume that X is both scattered and extremally disconnected. Consider the open and discrete set U consisting of all isolated points of X. Assume that $X \setminus U \neq \emptyset$. By scatteredness, $\{x\}$ is open in $X \setminus U$ for some $x \in X \setminus U$. Therefore $X \setminus (U \cup \{x\}) = (X \setminus U) \setminus \{x\}$ is closed in $X \setminus U$ and hence closed in X. Thus both $\{x\}$ and $X \setminus (U \cup \{x\})$ are closed subsets. Since X is compact Hausdorff, there are disjoint open subsets V_1 and V_2 containing x and $X \setminus (U \cup \{x\})$, respectively. We may assume V_1 is closed because X is extremally disconnected.

Observe that V_1 is infinite; otherwise $V_1 \setminus \{x\}$ would be closed and

$$\{x\} = V_1 \setminus (V_1 \setminus \{x\})$$

open, contradicting $x \notin U$. Hence $V_1 \setminus \{x\}$ is infinite, too. Pick two disjoint infinite subsets W_1, W_2 covering $V_1 \setminus \{x\}$. Since W_i is contained U, it must be open. If W_i were closed, then it is compact, contradicting that it both discrete (as a subset of U) and infinite. So $W_i \subsetneq \overline{W_i} \subseteq V_1$. Moreover, $\overline{W_i} \cap W_j = \emptyset$ for $i \neq j$, and $W_1 \cup W_2 = V_1 \setminus \{x\}$, so $\overline{W_i} = W_i \cup \{x\}$ whence $\overline{W_1} \cap \overline{W_2} = \{x\}$. Since X is extremally disconnected, $\overline{W_i}$ is open, whence $\{x\}$ is open, contradicting $x \notin U$. Hence $X = U$, and since U is discrete and compact, it must be finite. □

Examples A.2.4.

- The Cantor space $2^{\mathbb{N}}$ is in the product topology a compact metric space that is totally disconnected [103, Counterexample 57], but neither scattered nor extremally disconnected.

- The space $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ is an example of a compact metric scattered space. In order to see that it is compact, one could note that X is homeomorphic with the one-point compactification of the natural numbers. For a more direct proof, let $\mathcal{U}$ be an open cover of X. Choose a $U \in \mathcal{U}$ such that $0 \in U$. Then there is an $\epsilon > 0$ such that $X \cap (-\epsilon, \epsilon) \subseteq U$. Let $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then $\frac{1}{n} \in U$ for each $n \geq N$. For each $n \in \{1, \ldots, N-1\}$ choose a $U_n \in \mathcal{U}$ such that $\frac{1}{n} \in U_n$. Then we find that $\{U_1, \ldots, U_{N-1}, U\}$ is a finite subcover of $\mathcal{U}$. So X is compact.

  To see that X is scattered, let $n \in \mathbb{N}$. Then $\frac{1}{n}$ is isolated in X, since $X \cap \left(\frac{1}{n+1}, \frac{1}{n-1}\right) = \left\{\frac{1}{n}\right\}$ if $n > 1$ and $X \cap \left(\frac{1}{2}, 2\right) = \left\{\frac{1}{n}\right\}$ if $n = 1$. Let S is a non-empty closed subset of X. If $S = \{0\}$, then 0 is isolated in S. If S contains another point, then that point must be equal to $\frac{1}{n}$ for some $n \in \mathbb{N}$, and since it is isolated in X, it is certainly isolated in S;

- Let X be the Stone-Čech compactification of the natural numbers. Then X is an extremally disconnected compact Hausdorff space. We refer to [103, Counterexample 111] for more details.

Proposition A.2.5. [104, Theorem II.4.2] Let X be a compact Hausdorff space. Then X is totally disconnected if and only if X is totally separated if and only if X is zero dimensional.

Definition A.2.6. Topological spaces satisfying the equivalent conditions of Proposition A.2.5 are also called *Stone spaces*. Extremally disconnected compact Hausdorff spaces are also called *Stonean spaces.* The category of Stone spaces with continuous maps is denoted by **Stone**.

We can now state the theorem that completely characterizes scattered compact Hausdorff spaces.

Theorem A.2.7. Let X be a compact Hausdorff space. Then the following statements are equivalent:

(a) X is scattered;

(b) If $f : X \to Y$ is a continuous surjection onto a compact Hausdorff space Y, then Y is scattered;

(c) If $f : X \to Y$ is a continuous surjection onto a compact Hausdorff space Y, then Y is totally disconnected;

(d) There exists no continuous surjection $f : X \to [0, 1]$.

The proof of the equivalence between (a) and (d) can be found in [101, Theorem 8.5.4]. The proof of (a) $\implies$ (b) can be found in [34, Lemma 12.24]. The remaining implications are easy to verify.

B Order theory

B.1 Basic order-theoretic notions

We recall some definitions in order theory and refer to [28] for a detailed exposition.

Definition B.1.1. Let P be a set and $\leq$ a binary relation on P. Then $\leq$ is called a *preorder* if it satisfies the following axioms:

(P1) $x \leq x$ for each $x \in P$ (reflexivity),

(P2) $x \leq y$ and $y \leq z$ implies $x \leq z$ for each $x, y, z \in P$ (transitivity).

If, furthermore, $\leq$ satisfies

(P3) $x \leq y$ and $y \leq x$ implies $x = y$ for each $x, y \in P$ (antisymmetry), then we call $\leq$ an *partial order*, and P a *partially ordered* set or *poset*. A set can be equipped with more than one order. If we want to emphasize the specific order on a poset P, we rather write $(P, \leq)$ We say that two elements are *comparable* if $x \leq y$ or $y \leq x$, otherwise x and y are called *incomparable*. If all elements in P are comparable, we say that $\leq$ is a *linear order*, and we call P a *linearly ordered set*. A *subposet* X of P is a defined as a subset of $X \subseteq P$ with the order inherited from P. A linearly ordered subposet of a poset is called a *chain*. Given a poset $(P, \leq)$, we define the *opposite poset* P^{op} as the poset with the same underlying set, but with reserved order, i.e., the poset $(P, \sqsubseteq)$ where $x \sqsubseteq y$ if and only if $y \leq x$.

We remark that a poset P defines a so-called *poset category*, i.e., a category $\mathbf{P}$ for which the set of morphisms $\mathbf{P}(p, q)$ between any two objects p and q is either empty or a singleton. Indeed, given a poset P, let $\mathrm{ob}(\mathbf{P}) = P$. Moreover, we define $\mathrm{hom}(\mathbf{P}) = \{f_{q,p} : p, q \in P, p \leq q\}$, with $\mathrm{dom}(f_{q,p}) = p$, $\mathrm{cod}(f_{q,p}) = q$ and composition defined by $f_{r,q} \circ f_{q,p} = f_{r,p}$. It follows that $\mathbf{P}^{\mathrm{op}}$ is exactly the poset category obtained from the poset P^{op}.

Definition B.1.2. Let P be a poset and X a subset. Then a non-empty subset X of P is called:

(i) an *upper set* or *up-set* if $x \leq p$ implies $p \in X$ for each $x \in X$ and $p \in P$;

(ii) an *lower set* or *down-set* if $p \leq x$ implies $p \in X$ for each $x \in X$ and $p \in P$;

(iii) *directed* if for each $x_1, x_2 \in X$ there is a $x \in X$ such that $x_1 \leq x$ and $x_2 \leq x$;

(iv) *filtered* if for each $x_1, x_2 \in X$ there is a $x \in X$ such that $x \leq x_1$ and $x \leq x_2$.

Definition B.1.3. Let P be a poset, $x \in X$ and $X \subseteq P$. Then we define:

- $\uparrow x = \{p \in P : x \leq p\}$, the up-set generated by x;
- $\downarrow x = \{p \in P : y \leq x\}$, the down-set generated by x;
- $\uparrow X = \bigcup_{x \in X} \uparrow x$, the up-set generated by X;
- $\downarrow X = \bigcup_{x \in X} \downarrow x$, the down-set generated by X;

Notice that $\uparrow x = \uparrow\{x\}$ and $\downarrow x = \downarrow\{x\}$. Moreover, X is an up-set if and only if $X = \uparrow X$, and X is a down-set if and only if $X = \downarrow X$.

The following definition is not standard. We will use it to generate up-sets and down-sets in a fixed subposet of some poset.

Definition B.1.4. Let P be a poset and $X \subseteq P$ a fixed non-empty subset. For any $x \in X$ and $Y \subseteq X$, we define:

- $\Uparrow x = X \cap \uparrow x = \{y \in X : x \leq y\}$;
- $\Downarrow x = X \cap \downarrow x = \{y \in X : x \leq y\}$;
- $\Uparrow Y = X \cap \uparrow Y = \bigcup_{y \in Y} \Uparrow y$;
- $\Downarrow Y = X \cap \downarrow Y = \bigcup_{y \in Y} \Downarrow y$.

Definition B.1.5. Let P be a poset and $X \subseteq P$. Then $p \in P$ is called:

- a *maximal* element of X if $\uparrow p \cap X = \{p\}$;
- a *minimal* element of X if $\downarrow p \cap X = \{p\}$;
- an *upper bound* of X if $x \leq p$ for each $x \in X$;
- a *lower bound* of X is $p \leq x$ for each $x \in X$;
- the *greatest* element of X if $p \in X$ and $x \leq p$ for each $x \in X$;
- the *least* element of X if $p \in X$ and $p \leq x$ for each $x \in X$;

A subset X of P can have multiple maximal and minimal elements. The set of all maximal elements of X is denoted by $\max X$, the set of all its minimal elements by $\min X$. Greatest and least elements, however, are always unique. If P itself contains a least and a greatest element, usually denoted by 0 and 1, respectively, we say that P is a *bounded*.

Definition B.1.6. Let P be a poset and $X \subseteq P$. If X has a least upper bound, or a *supremum*, we denote it either by $\bigvee X$ or by $\sup X$. Dually, if X has a greatest lower bound or an *infimum*, we denote it by $\bigwedge X$ or by $\inf X$. If X is finite, we sometimes say that $\bigvee X$ is the join of X. Similarly $\bigwedge X$ is called the meet of X. We write $p_1 \vee p_2$ and $p_1 \wedge p_2$ instead of $\bigvee\{p_1, p_2\}$ and $\bigwedge\{p_1, p_2\}$, respectively.

Definition B.1.7. Let P be a poset and $x, y \in P$. Then:

- $[x, y] = \{z \in P : x \leq z \leq y\}$ is called the *interval* between x and y;
- y is said to *cover* x and x is said to be covered by y if $[x, y] = \{x, y\}$;
- x is called an *atom* if P contains a least element 0 that is covered by x;
- y is called a *co-atom* if P contains a greatest element 1 that covers y.

The covering relation, and in particular the notion of an atom, allow us the define the following kind of posets:

Definition B.1.8. Let P be a poset.

- If P has a least element 0, then P is called *atomic* if for each element $p \in P \setminus \{0\}$ there is some atom $a \in P$ such that $a \leq p$;
- P is called *atomistic* if each element $p \in P$ is the supremum of some collection of atoms in P;
- If P is a chain, then P called *order dense* if no element of P is covered by another element of P, i.e., for each $x, y \in P$ such that $x < y$, there is some $z \in P$ such that $x < z < y$;
- If P is a poset that does not contain any order dense chains of at least two points, then P is called *order scattered*.

Definition B.1.9. Let P be a poset. Then P is called:

(i) a *meet-semilattice* if all binary meets exist;

(ii) a *lattice* if all binary meets and joins exist;

(iii) a *complete lattice* if all suprema and infima exist.

Moreover, if P is a lattice and $X \subseteq P$, then X is called a *sublattice* if $x, y \in X$ implies $x \wedge y \in X$ and $x \vee y \in X$.

Notice that a complete lattice P is automatically bounded, since $\bigvee P$ is its greatest element, and $\bigwedge P$ is its least element.

Definition B.1.10. Let P and Q be posets, and $\varphi : P \to Q$ a map. Then φ is called:

- an *order morphism* if $x \leq y$ implies $\varphi(x) \leq \varphi(y)$ for each $x, y \in P$;
- an *embedding of posets* if $\varphi(x) \leq \varphi(y)$ if and only if $x \leq y$ for each $x, y \in P$;
- an *order isomorphism* if it is an order morphism such that $\varphi \circ \psi = 1_Q$ and $\psi \circ \varphi = 1_P$ for some order morphism $\psi : Q \to P$. Here $1_P : P \to P$ is the identity map $x \mapsto x$;
- a *meet-semilattice morphism* if P and Q are both meet-semilattices and for each $x, y \in P$, we have

$$\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y).$$

- a *lattice morphism* if P and Q are both lattices, and φ is a meet-semilattice morphism such that

$$\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$$

  for each $x, y \in P$. If in addition φ is bijective, then φ is called a *lattice isomorphism*.

Remarks B.1.11.

- Clearly an embedding of posets φ is injective. If $\varphi(x) = \varphi(y)$, then $\varphi(x) \leq \varphi(y)$, so $x \leq y$, and in a similar way, we find $y \leq x$, so $x = y$. The converse does not always hold. Consider for instance the poset $P = \{p_1, p_2, p_3\}$ with $p_1, p_2 < p_3$ and $Q = \{q_1, q_2, q_3\}$ with $q_1 < q_2 < q_3$. Then $\varphi : P \to Q$ defined by $\varphi(p_i) = q_i$ for $i = 1, 2, 3$ is clearly an injective order morphism (it is even bijective), but $p_1 \not\leq p_2$, whereas $\varphi(p_1) \leq \varphi(q_2)$.
- A meet-semilattice morphism $\varphi : P \to Q$ is automatically an order morphism. In order to see this, let $p_1, p_2 \in P$ such that $p_1 \leq p_2$. Then $p_1 = p_1 \wedge p_2$, hence $\varphi(p_1) = \varphi(p_1 \wedge p_2) = \varphi(p_1) \wedge \varphi(p_2)$. It follows that $\varphi(p_1) \leq \varphi(p_2)$.
- A map $\varphi : P_1 \to P_2$ is an order isomorphism if and only if it is a surjective order embedding.

Having introduced order morphisms, we can make the following definition:

Definition B.1.12. We define **Poset** as the category with posets as objects and order morphisms as morphism.

Lemma B.1.13. Let P be a poset. If P has all infima, then it is a complete lattice with supremum defined by

$$\bigvee X = \bigwedge \{p \in P : x \leq p \ \forall x \in X\}$$

for each $X \subseteq P$. Similarly, if P has all suprema, then P is a complete lattice with infimum defined by

$$\bigwedge X = \bigvee \{p \in P : p \leq x \ \forall x \in X\}.$$

Definition B.1.14. Let L be a lattice. Then L is called *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

for each $y, z \in L$. Moreover, L is called *modular* if

$$z \leq x \implies x \wedge (y \vee z) = (x \wedge y) \vee z$$

for each $x, y, z \in L$.

The proof of next proposition is follows from direct verification, hence we leave it.

Proposition B.1.15. Let $\varphi : P \to Q$ be an order isomorphism, $x, y \in P$ and $X \subseteq P$. Then:

(1a) $\varphi[\uparrow X] = \uparrow \varphi[X]$. In particular, if X is an up-set, then so is $\varphi[X]$;

(1b) $\varphi[\downarrow X] = \downarrow \varphi[X]$. In particular, if X is a down-set, then so is $\varphi[X]$;

(2a) If X is directed, then $\varphi[X]$ is directed;

(2b) If X is filtered, then $\varphi[X]$ is filtered;

(3a) If x is maximal in X, then $\varphi(x)$ is maximal in $\varphi[X]$;

(3b) If x is minimal in X, then $\varphi(x)$ is minimal in $\varphi[X]$;

(4a) If x is the greatest element of X, then $\varphi(x)$ is the greatest element of $\varphi[X]$. In particular, if P has a greatest element 1, then so does Q and $\varphi(1) = 1$;

(4b) If x is the least element of X, then $\varphi(x)$ is the least element of $\varphi[X]$. In particular, if P has a least element 0, then so does Q and $\varphi(0) = 0$;

(5a) If $x = \bigvee X$, then $\varphi(x) = \bigvee \varphi[X]$;

(5b) If $x = \bigwedge X$, then $\varphi(x) = \bigwedge \varphi[X]$;

(6) $\varphi\big[[x, y]\big] = [\varphi(x), \varphi(y)]$. In particular, φ preserves covering relations, atoms and co-atoms.

An important theorem in order to find maximal elements in a poset is *Zorn's Lemma*, which is equivalent with the Axiom of Choice.

Theorem B.1.16 (Zorn's Lemma)**.** Let P be a poset such that every non-empty chain of P has an upper bound, then $\max P \neq \emptyset$.

Definition B.1.17. [28, 1.25] Let $\{P_i\}_{i \in I}$ a collection of posets. Then the *cartesian product* of the P_i is defined as the set $\prod_{i \in I} P_i$ equipped by the order defined by $(x_i)_{i \in I} \leq (y_i)_{i \in I}$ if and only if $x_i \leq y_i$ for each $i \in I$. If I has finite cardinality, say n, then we sometimes write $P_1 \times \ldots \times P_n$ instead of $\prod_{i=1}^{n} P_i$.

Lemma B.1.18. Let $\{P_i\}_{i \in I}$ a collection of posets. Let $P = \prod_{i \in I} P_i$ and denote by $\pi_i : P \to P_i$ the projection map. Let $X \subseteq P$. Then its supremum in P exists if and only if the supremum of $\pi_i[X]$ exists in P_i for each $i \in I$, in which case we have

$$\pi_i\left(\bigvee X\right) = \bigvee \pi_i[X]$$

for each $i \in I$. Similarly, the infimum of X exists in P if and only if the infimum of $\pi_i[X]$ exists in P_i for each $i \in I$, in which case we have

$$\pi_i\left(\bigwedge X\right) = \bigwedge \pi_i[X]$$

for each $i \in I$.

Lemma B.1.19. Let P_1, P_2 be posets. Then

$$\max(P_1 \times P_2) = (\max P_1) \times (\max P_2).$$

Moreover, if $p_1 \in P_1$ and $p_2 \in P_2$, we have $\downarrow(p_1, p_2) = \downarrow p_1 \times \downarrow p_2$.

Definition B.1.20. Let P, Q be posets and $\varphi : P \to Q$, $\psi : Q \to P$ order morphism. If

$$\varphi(p) \leq q \iff p \leq \psi(q)$$

for each $p \in P$ and $q \in Q$, then ψ is called the *upper adjoint* of φ and φ is called the *lower adjoint* of ψ, and φ and ψ are said to form a *Galois connection* or an *adjunction*. If the upper adjoint of φ exists, it is sometimes denoted by φ_*. Similarly, the lower adjoint of ψ, if it exists, is sometimes denoted by ψ^*.

If we regard posets as categories, an upper adjoint is exactly a right adjoint.

Example B.1.21. Let X be a set. Then the power set $\mathcal{P}(X)$ becomes a poset if we order it by inclusion. If Y is another set and $f : X \to Y$ a function, then the maps $\mathcal{P}(X) \to \mathcal{P}(Y)$, $A \mapsto f[A]$ and $\mathcal{P}(Y) \to \mathcal{P}(X)$, $B \mapsto f^{-1}[B]$ are order morphisms. Moreover for each $A \subseteq X$ and $B \subseteq Y$, we have

$$f[A] \subseteq B \text{ if and only if } A \subseteq f^{-1}[B],$$

hence the map $B \mapsto f^{-1}[B]$ is the upper adjoint of $A \mapsto f[A]$.

The following lemma can be found in [28, Chapter 7]. Categorical generalizations of these facts are also well-known, but we will not need them here.

Lemma B.1.22. Let P, Q be posets, and $\varphi : P \to Q$, $\psi : Q \to P$ order morphisms. The following three statements are equivalent:

(i) $\psi : Q \to P$ is the upper adjoint of $\varphi : P \to Q$;

(ii) $\psi : Q^{\mathrm{op}} \to P^{\mathrm{op}}$ is the lower adjoint of $\varphi : P^{\mathrm{op}} \to Q^{\mathrm{op}}$;

(iii) Both $1_P \leq \psi \circ \varphi$ and $\varphi \circ \psi \leq 1_Q$.

If these statements hold, then we also have the following:

(1a) $\varphi \circ \psi \circ \varphi = \varphi$;

(1b) $\psi \circ \varphi \circ \psi = \psi$;

(2a) φ is surjective if and only if ψ is injective if and only if $\varphi \circ \psi = 1_Q$.

(2b) φ is injective if and only if ψ is surjective if and only if $\psi \circ \varphi = 1_P$.

(3a) If $\psi' : Q \to P$ is another upper adjoint of φ, then $\psi' = \psi$.

(3b) If $\varphi' : P \to Q$ is another lower adjoint of ψ, then $\varphi' = \psi$.

Lemma B.1.23. Let $\varphi : P \to Q$ be an order morphism with an upper adjoint $\psi : Q \to P$. Then φ preserves all existing suprema, and ψ preserves all existing infima.

Definition B.1.24. Let P be a poset. Then an order morphism $\psi : P \to P$ is called a *closure operator* if for each $p \in P$:

(C1) $p \leq \psi(p)$;

(C2) $\psi \circ \psi(p) = \psi(p)$,

and an *interior operator* or *kernel operator* if for each $p \in P$:

(I1) $\psi(p) \leq p$;

(I2) $\psi \circ \psi(p) = \psi(p)$.

If an order morphism $\varphi : P \to Q$ has an upper adjoint $\psi : Q \to P$, then it follows from Lemma B.1.22 that $\varphi \circ \psi : Q \to Q$ is a closure operator and $\psi \circ \varphi : P \to P$ is an interior operator.

B.2 Chain conditions and graded posets

Definition B.2.1. Let P be a poset. Then P is called *Artinian* if every non-empty subset contains a minimal element. Dually, a poset is called *Noetherian* if every non-empty subset contains maximal element.

Lemma B.2.2. Let P be a poset. Then the following statements are equivalent:

(1) P is Artinian;

(2) All non-empty filtered subsets of P have a least element.

(3) P satisfies the *descending chain condition*: if we have a sequence of elements $p_1 \geq p_2 \geq \ldots$ in P, i.e., a countable descending chain, then the sequence eventually stabilizes, i.e., there is an $n \in \mathbb{N}$ such that $p_k = p_n$ for all $k > n$.

Since P is Noetherian if and only if P^{op} is Artinian, we obtain a similar characterization of Noetherian posets:

Lemma B.2.3. Let P be a poset. Then the following statements are equivalent:

(1) P is Noetherian;

(2) All non-empty directed subsets of P have a greatest element.

(3) P satisfies the *ascending chain condition*: if we have an ascending sequence of elements

$$p_1 \leq p_2 \leq p_3 \leq \dots$$

in P, i.e. a countable ascending chain, then the sequence eventually stabilizes, i.e., there is an $n \in \mathbb{N}$ such that $p_k = p_n$ for all $k > n$.

The following proposition can be found in [65] as Theorem 4.21.

Proposition B.2.4 (Principle of Artinian induction)**.** Let P be an Artinian poset and $\mathcal{P}$ a property such that:

(1) $\mathcal{P}(x)$ is true for each minimal $x \in P$ *(induction basis)*;

(2) $\mathcal{P}(y)$ is true for each $y < x$ implies that $\mathcal{P}(x)$ is true *(induction step)*.

Then $\mathcal{P}(x)$ is true for each $x \in P$.

Definition B.2.5. Let P be a poset. Then P is called *graded* if one can define a function $d : P \to \mathbb{N}$, called a *rank function*, such that:

(i) $d(p) = 1$ for each $p \in \min P$;

(ii) $d(p_1) < d(p_2)$ for each $p_1, p_2 \in P$ such that $p_1 < p_2$;

(iii) p_2 is a cover of p_1 if and only if $p_1 \leq p_2$ and $d(p_2) = d(p_1) + 1$ for each $p_1, p_2 \in P$.

There is no standard definition of a graded poset. For instance, in [95], condition (i) is dropped and $\mathbb{Z}$ is taken as a codomain of rank functions. On the other hand, [77] assumes condition (i), but not condition (ii). For our purposes, it is convenient to combine both definitions. The next three lemmas are easy to verify.

Lemma B.2.6. Let P be a graded poset with rank function $d : P \to \mathbb{N}$. Then P is Artinian. If the range of d is bounded from above, P is Noetherian as well.

Lemma B.2.7. Let P be a graded poset. Then its rank function $d : P \to \mathbb{N}$ is unique.

Lemma B.2.8. Let $\varphi : P \to Q$ be an order isomorphism between graded posets P and Q with rank functions d_P and d_Q, respectively. Then $d_P = d_Q \circ \varphi$.

B.3 Product factorization of posets

Definition B.3.1. [9, III.8] Let P be a poset. Then P is called *directly indecomposable* if $P \cong P_1 \times P_2$ for some posets P_1, P_2 implies that either $P_1 = \mathbf{1}$ and $P_2 = P$ or $P_1 = P$ and $P_2 = \mathbf{1}$, where $\mathbf{1}$ denotes the one-point poset.

The following theorem is originally due to Hashimoto [54], stating that two direct product factorizations of a so-called *connected* poset always have a common refinement. There are several versions of this theorem, for instance, for finite factorizations of lattices it stated in [42, Theorem III.4.2].

Theorem B.3.2. Let P be a meet-semilattice with two direct product factorizations

$$P = \prod_{i \in I} A_i = \prod_{j \in J} B_j.$$

Then there exist a common refinement of these factorizations, i.e., there exist meet-semilattices C_{ij} with $i \in I$ and $j \in J$ such that $A_i \cong \prod_{j \in J} C_{ij}$ for each $i \in I$ and $B_j \cong \prod_{i \in I} C_{ij}$ for each $j \in J$. Moreover, if P and all A_i and B_j are lattices, complete semilattices, or complete lattices, then the C_{ij} can be chosen to be lattices, complete semilattices, or complete lattices, respectively.

Corollary B.3.3. Let $\prod_{i \in I} A_i \cong \prod_{j \in J} B_j$, where the A_i and B_i are directly indecomposable meet-semilattices. Then there exists a bijection $\pi : I \to J$ such that $A_i \cong B_{\pi(i)}$ for each $i \in I$.

Definition B.3.4. Let L be a bounded lattice and $x \in L$. Then $y \in L$ is called a *complement* of x if $x \wedge y = 0$ and $x \vee y = 1$.

The next proposition is an application of [42, Theorem III.4.1].

Proposition B.3.5. Let L be a bounded lattice. If 0 and 1 are the only elements of L with a unique complement (namely each other), then L is directly indecomposable.

B.4 Orthocomplementation

The main references for this section are [5], [32], and [70]. For statements about Boolean algebras, we also used [28] and [41].

Definition B.4.1. Let P be a bounded poset. Then P is called an *orthocomplemented poset* or *orthoposet* if it admits an *orthocomplementation* on P, i.e., a map $P \to P$, $p \mapsto p^\perp$ satisfying for each $p, q \in P$:

(i) $q \leq p$ implies $p^\perp \leq q^\perp$;

(ii) $p^{\perp\perp} = p$;

(iii) $p \wedge p^\perp$ and $p \vee p^\perp$ exist, and are given by

$$p \wedge p^\perp = 0,$$
$$p \vee p^\perp = 1.$$

If $S \subseteq P$, then we define $S^\perp = \{p^\perp : p \in S\}$.

Two elements q and p in an orthoposet P are called *orthogonal* (denoted by $p \perp q$) if $q \leq p^\perp$, or, equivalently, if $p \leq q^\perp$.

Lemma B.4.2. [De Morgan's Laws] Let P be an orthoposet and $\{p_i\}_{i\in I} \subseteq P$. If $\bigvee_{i\in I} p_i$ exists, so does $\bigwedge_{i\in I} p_i^\perp$, which is given by

$$\bigwedge_{i\in I} p_i^\perp = \left(\bigvee_{i\in I} p_i\right)^\perp.$$

Similarly, if $\bigwedge_{i\in I} p_i$ exists, so does $\bigvee_{i\in I} p_i^\perp$, which is given by

$$\bigvee_{i\in I} p_i^\perp = \left(\bigwedge_{i\in I} p_i\right)^\perp.$$

Let P be an orthoposet such that $x \perp y$ implies that $x \vee y$ exists for each $x, y \in P$. Let $p, q \in P$ such that $q \leq p$ holds. Then $q \leq p^{\perp\perp}$, so $q \perp p^\perp$. It follows that $q \vee p^\perp$ exists, hence $q^\perp \wedge p$ exists by De Morgan's Laws. Moreover, we have $q^\perp \wedge p \leq q^\perp$, hence $q^\perp \wedge p \perp q$. Thus condition (ii) in the following definition is well defined.

Definition B.4.3. Let P be an orthoposet. Then P is called an *orthomodular poset* if for each $p, q \in P$:

(i) $p \perp q$ implies that $p \vee q$ exists;

(ii) $q \leq p$ implies $p = q \vee (q^{\perp} \wedge p)$ (the orthomodular law).

If P is also a lattice, we call P an *orthomodular lattice*. If p and q are elements of an orthomodular poset P, then we say that p *commutes* with q (notation $p\,Cq$) if there are pairwise orthogonal elements $e_1, e_2, e_3 \in P$ such that $p = e_1 \vee e_3$ and $q = e_2 \vee e_3$.

It follows easily that a modular lattice is always an orthomodular lattice. We also remark that condition (i) is automatically satisfied if P is an lattice. Furthermore, clearly $p\,Cq$ if and only if $q\,Cp$.

Lemma B.4.4. Let P be an orthomodular poset. Then for each $p, q \in P$:

- $p \perp 0$;
- p commutes with itself;
- $p \perp q$ implies that p and q commute;
- $q \leq p$ implies that p and q commute, and that $p \wedge q^{\perp}$ and q are orthogonal.

Lemma B.4.5. Let p and q be commuting elements in an orthomodular poset P. Then any two elements of the set $\{p, p^{\perp}, q, q^{\perp}\}$ commute and have meets and joins in P.

Lemma B.4.6. Let P be an orthomodular poset and $p, q \in P$. If there are orthogonal e_1, e_2, e_3 such that $p = e_1 \vee e_3$ and $q = e_2 \vee e_3$, then $e_3 = p \wedge q$.

We now define the appropriate notion of morphisms for orthomodular posets [8].

Definition B.4.7. Let $\varphi : P \to Q$ be a map between orthomodular posets. Then φ is called an *orthomodular morphism* if

(i) $\varphi(0) = 0$;

(ii) $\varphi(p^{\perp}) = \varphi(p)^{\perp}$ for each $p \in P$;

(iii) $p \perp q$ implies $\varphi(p \vee q) = \varphi(p) \vee \varphi(q)$ for each $p, q \in P$.

A orthomodular morphism that has an inverse which is an orthomodular morphism as well is called a *orthomodular isomorphism.* We denote the category of orthomodular posets with orthomodular morphisms by **OMP**.

Proposition B.4.8. Let $\varphi : P \to Q$ be an orthomodular morphism between orthomodular posets P and Q. Then φ is an order morphism such that $\varphi(1) = 1$, and that preserves orthogonality and commutativity. Moreover, φ preserves their meets and joins of commuting elements.

Lemma B.4.9. Let $\varphi : P \to Q$ be a map between orthomodular posets P and Q. Then φ is an orthomodular isomorphism if and only if it is an orthocomplement-preserving order isomorphism.

Definition B.4.10. If B be a orthocomplemented lattice that is also distributive, then B is called a *Boolean algebra*. If B is complete as a lattice, it is called a *complete Boolean algebra.*

Lemma B.4.11. The orthocomplementation on a Boolean algebra B is unique.

Example B.4.12. Denote the power set of a set X by $\mathcal{P}(X)$. Then $\mathcal{P}(X)$ is a complete Boolean algebra, where the supremum of a family of subsets is given by the union of elements of the family, the infimum of a family subsets by the intersection of the subsets in the family, and the orthocomplementation by the complement operator.

Lemma B.4.13. Every Boolean algebra is an orthomodular lattice, in which every pair of elements commutes.

If P is an orthomodular poset, and $p, r \in P$ such that $r \leq p$, then r and p commute by Lemma B.4.4, whence $p \wedge r^{\perp}$ exists by Lemma B.4.5. Hence condition (ii) in the next lemma is well defined.

Lemma B.4.14. Let P be an orthomodular poset en $p, q \in P$. Then p commutes with q if and only if there is an $r \in P$ such that

(i) $r \leq p$ and $r \leq q$;

(ii) $p \wedge r^{\perp} \leq q^{\perp}$.

Lemma B.4.15. Let P an orthomodular poset, $p \in P$ and $\{q_i\}_{i\in I} \subseteq P$ such that $p\,C q_i$ for each $i \in I$ and such that the elements $q = \bigvee_{i\in I} q_i$ and $\bigvee_{i\in I}(p\wedge q_i)$ exist. Then p commutes with q, $p \wedge q$ exists and is given by

$$p \wedge q = \bigvee_{i\in I} (p \wedge q_i).$$

Proposition B.4.16. Let P be an orthomodular poset and B a non-empty subposet of P that is closed under orthocomplementation and under all existing binary meets and binary joins. Then $0, 1 \in B$, and the following two statements are equivalent:

- B is a Boolean algebra;
- B consists of pairwise commuting elements.

Definition B.4.17. Let P be an orthomodular poset. We say that a non-empty subposet B is a *Boolean subalgebra* of P if all its elements commute, and if B is closed under orthocomplementation, existing binary meets and existing binary joins.

It follows that if P itself is a Boolean algebra, then $B \subseteq P$ is a Boolean subalgebra if and only if B is closed under binary meets, binary joins and orthocomplementation.

Definition B.4.18. Let P be an orthomodular poset. If M is a maximal Boolean subalgebra of P, i.e., a Boolean subalgebra that is not properly contained in a larger Boolean subalgebra, then we call M a *block*.

Definition B.4.19. Let P be an orthomodular poset. For any non-empty $S \subseteq P$, we define the *commutant* of S as the set

$$C(S) = \{p \in P : p\,Cq \text{ for each } q \in S\}.$$

The commutant $C(P)$ of P itself is called the *center* of P.

The commutant has similar properties as the commutant for C*-algebras. For instance, compare the next proposition with Proposition C.1.15 below. Properties (2) and (3) are taken from [70, Exercise 1.3.11]. The equivalence $S \subseteq C(S)$ if and only if S is contained in some block in property (5) is taken from [70, Lemma I.4.1].

Proposition B.4.20. Let P be an orthomodular poset and $S, T \subseteq P$ non-empty. Then

(1) $C(S)$ is closed under orthocomplementation as well as binary meets and joins providing these exist;

(2) $S \subseteq T$ implies $C(T) \subseteq C(S)$;

(3) $S \subseteq C(C(S))$;

(4) $C(C(C(S))) = C(S)$;

(5) The following statements are equivalent:

- S consists of pairwise commuting elements;
- $S \subseteq C(S)$;
- $C(C(S))$ is a Boolean subalgebra of P;
- S is contained in some block of P.

In particular every element of P and every Boolean subalgebra of P is contained in some block of P, and the set of all blocks of P is not empty;

(6) S is a block of P if and only if $S = C(S)$.

Lemma B.4.21. [70, Lemma 4.2] Let P be an orthomodular poset, $B \subseteq P$ a block, and $p \in B$. Then p is an atom of B if and only if it is an atom of P.

Lemma B.4.22. Let P be an orthomodular poset, and $\{B_i\}_{i\in I}$ a non-empty collection of Boolean subalgebras of P. Then $B = \bigcap_{i\in I} B_i$ a Boolean subalgebra of P.

The next lemma follows from Proposition B.4.20 and Lemma B.4.22.

Lemma B.4.23. Let P be an orthomodular poset and S a subset of P consisting of pairwise commuting elements. Then there exists a smallest Boolean subalgebra of P containing S, denoted by $\langle S\rangle$, and called the *Boolean subalgebra generated by* S, which is given by

$$\langle S\rangle = \bigcap\{B : B \text{ is Boolean subalgebra of } P \text{ such that } S \subseteq B\}. \qquad (45)$$

Lemma B.4.24. [70, Exercise I.4.9] Let P be an orthomodular poset. Then the center $C(P)$ is a Boolean subalgebra of P that equals the intersection of all blocks of P.

Proposition B.4.25. Let P be a Boolean algebra and $S \subseteq P$ a finite non-empty set. Then $\langle S \rangle$ is a finite Boolean subalgebra of P.

Definition B.4.26. A map $\varphi : B_1 \to B_2$ between Boolean algebras B_1 and B_2 is called a *Boolean morphism* if it is a lattice morphism preserving orthocomplementation. A bijective Boolean morphism is called a *Boolean isomorphism*. We denote the category of Boolean algebras with Boolean morphisms by **Bool**.

Lemma B.4.27. Let $\varphi : B_1 \to B_2$ be a map between Boolean algebras B_1 and B_2. Then

(1) φ is an orthomodular morphism if and only if φ is a Boolean morphism;

(2) φ is an order isomorphism if and only if it is a Boolean isomorphism.

The next proposition follows traditionally from Stone's Representation Theorem for finite Boolean algebras. See also [28, Chapter 5].

Proposition B.4.28. Two finite Boolean algebras are order isomorphic if and only if their sets of atoms have the same cardinality. More specific: if φ is a bijection between the sets of atoms between two finite Boolean algebras B and B', then it can uniquely be extended to an order isomorphism, and conversely, if $\varphi : B \to B'$ is an order isomorphism, then it restricts to a bijection between the sets of atoms of B and B'.

B.5 Stone duality

The last proposition of the previous section is a special case of Stone duality, which is sketched in this section.

Lemma B.5.1. Let X be a topological space. Then the poset $B(X)$ of clopen subsets of X ordered by inclusion is a Boolean algebra with Boolean operations defined for each $U, V \in B(X)$ by

$$\begin{aligned} U \vee V &= U \cup V; \\ U \wedge V &= U \cap V; \\ U^{\perp} &= X \setminus U. \end{aligned}$$

Moreover, B becomes a functor from the category of topological spaces with continuous maps to $\mathbf{Bool}^{\mathrm{op}}$, the opposite of the category of Boolean algebras and Boolean morphisms, if we define for each continuous map $f : X \to Y$ between topological spaces X and Y a Boolean morphism $B(f) : B(Y) \to B(X)$ by $U \mapsto f^{-1}[U]$.

Definition B.5.2. Let B be a Boolean algebra and $I \subseteq B$ be a non-empty subset.

- I is called an *ideal* if $a \vee b \in I$ and $p \wedge a \in I$ for each $a, b \in I$ and $p \in B$;
- I is called a *filter* if $a \wedge b \in I$ and $p \vee a \in I$ for each $a, b \in I$ and $p \in B$.

If $I \subseteq B$ is either an ideal or a filter such that $I \neq B$, then I is called *proper*. A filter that is maximal in the set of all proper filters ordered by inclusion is called an *ultrafilter*. Dually, a *maximal ideal* is an ideal that is maximal in the set of all proper ideals ordered by inclusion.

Lemma B.5.3. [41, Lemma 20.1] Let B be a Boolean algebra and $I \subseteq B$ a proper ideal. Then I is a maximal ideal if and only if either $p \in I$ or $p^{\perp} \in I$ for each $p \in B$.

Proposition B.5.4. Let X be a Stone space (cf. Definition A.2.6). Order both the set $\mathcal{I}(B(X))$ of ideals of $B(X)$ and the topology $\mathcal{O}(X)$ of X by inclusion. Then

$$\Phi : \mathcal{O}(X) \to \mathcal{I}(O), \quad O \mapsto \{U \in B(X) : U \subseteq O\}$$

is an order isomorphism with inverse

$$I \mapsto \bigcup_{U \in I} U.$$

We refer to [41, Chapter 35] for a proof.

Definition B.5.5. Let B be a Boolean algebra. Then we define the *Stone spectrum* of B as the topological space $\Sigma(B)$ of all ultrafilters of B whose topology is generated by the basis $\{O_p : p \in B\}$ where $O_p = \{U \in \Sigma(B) : p \in U\}$. If A is another Boolean algebra and $\varphi : A \to B$ a Boolean morphism, then we define the map

$$\Sigma(\varphi) : \Sigma(B) \to \Sigma(A), \quad U \mapsto \varphi^{-1}[U].$$

Theorem B.5.6 (Stone's Representation Theorem for Boolean algebras)**.**

(1) Let B be a Boolean algebra. Then $\Sigma(B)$ is a Stone space such that

$$\varphi : B \to B(\Sigma(B)), \quad p \mapsto O_p = \{U \in \Sigma(B) : p \in U\}$$

is a Boolean isomorphism.

(2) Let X be a Stone space. Then $B(X)$ is a Boolean algebra such that

$$f : X \to \Sigma(B(X)), \quad x \mapsto \{U \in B(X) : x \in U\}$$

is a homeomorphism.

We refer to [18, Theorem IV.4.6] for a proof.

The last theorem can be extended to a duality, called *Stone duality*, between Boolean algebras and Stone spaces. We first need the following lemma, which is Exercise IV.4.9 in [18].

Lemma B.5.7. $\Sigma : \mathbf{Bool} \to \mathbf{Stone}^{\mathrm{op}}$ becomes a functor if we define for each Boolean morphism $\varphi : A \to B$ between Boolean algebras A and B the map $\Sigma(\varphi) : \Sigma(B) \to \Sigma(A)$, which is continuous.

Theorem B.5.8 (Stone duality)**.** The functors $\Sigma : \mathbf{Bool} \to \mathbf{Stone}^{\mathrm{op}}$ and $B : \mathbf{Stone}^{\mathrm{op}} \to \mathbf{Bool}$ form an equivalence of categories.

The theorem follows by proving that

$$B : \mathbf{Bool}(X, Y) \to \mathbf{Stone}\big(B(Y), B(X)\big), \quad f \mapsto B(f)$$

is a bijection. The proof of this latter statement can be found in [41, Chapter 36].

B.6 Domain theory

Definition B.6.1. Let P be a poset. Then we call P:

- a *directed-complete partial order* (dcpo) if for each directed subset D of P (cf. Definition B.1.2) the supremum $\bigvee D$ exists. For each $p \in P$ and $D \subseteq P$, we will use the notation $\bigvee^{\uparrow} D = p$ to express that D is directed with supremum p.

- *meet-continuous* if P is both a meet-semilattice and a dcpo such that

$$p \wedge \bigvee D = \bigvee{}^{\uparrow} \{p \wedge d : d \in D\} \tag{46}$$

  for each $p \in P$ and each directed $D \subseteq P$;

- a *complete semilattice* if P is a dcpo in which all non-empty infima exist.

Let P and Q be dcpo's and $\varphi : P \to Q$ a map. Then we call φ:

- *Scott continuous* if φ is an order morphism that preserves all directed suprema, i.e.,

$$\varphi\left(\bigvee D\right) = \bigvee \varphi[D]$$

  for each directed subset $D \subseteq P$;

- *Lawson continuous* if P and Q are complete semilattices and φ is a Scott continuous map that preserves all non-empty infima, i.e.,

$$\varphi\left(\bigwedge X\right) = \bigwedge \varphi[X]$$

  for each non-empty $X \subseteq P$.

The category of dcpo's with Scott continuous maps is denoted by **DCPO**.

Lemma B.6.2. The isomorphisms in **DCPO** are precisely the order isomorphisms.

Definition B.6.3. Let P be a dcpo. Then the *Scott topology* $\sigma(P)$ is the topology on P consisting of all up-sets U such that $\bigvee^{\uparrow} D \in U$ implies $D \cap U \neq \emptyset$.

An order morphism between dcpo's is Scott continuous if and only if it is continuous with respect to the Scott topologies on the dcpo's, which explains the terminology 'Scott continuous' [39, Proposition II-2.1].

Definition B.6.4. Let P be a dcpo and $p, q \in P$. If for each $D \subseteq P$ the relation $p \leq \bigvee^{\uparrow} D$ implies the existence of some $d \in D$ such that $q \leq d$, then we say that q is *way below* p and write $q \ll p$. For each $p \in P$, we denote the set $\{q \in P : q \ll p\}$ of all elements way below p by $\twoheaddownarrow p$. Furthermore,

- An element $p \in P$ such that $p \ll p$ is called *compact*. The set of all compact elements of P is denoted by $\mathcal{K}(P)$.

- If $p = \bigvee^{\uparrow}(\mathcal{K}(P) \cap \downarrow p)$ for each $p \in P$, we say that P is *algebraic* or is an *algebraic domain*.
- If $p = \bigvee^{\uparrow} \twoheaddownarrow p$ for each $p \in P$, we say that P is *continuous* or is a *continuous domain*.

Lemma B.6.5. [39, Proposition I-1.2] Let P be a dcpo and $p, q, x, y \in P$. If $q \ll p$, then $q \leq p$. If $x \leq q \ll p \leq y$, then $x \ll y$.

The next proposition is a generalization of [25, Lemma 3.3], although the proof is actually not different. Its implication is that we only have to check that the subposets of compact elements are order isomorphic in order to prove that two algebraic dcpo's are order isomorphic. We note that this statement can be generalized to a categorical duality, for which we refer to [39, Corollary IV-1.14].

Proposition B.6.6. Let P and Q be algebraic dcpo's for which there exists an order isomorphism $\varphi : \mathcal{K}(P) \to \mathcal{K}(Q)$. Then there exists a unique order isomorphism $\psi : P \to Q$, given for each $p \in P$ by

$$\psi(p) = \bigvee \varphi[\mathcal{K}(P) \cap \downarrow p], \tag{47}$$

whose restriction to $\mathcal{K}(P)$ is φ.

The statements in the next proposition are taken from Examples I-1.7, Proposition I-4.3, and Proposition I.1.8 of [39].

Proposition B.6.7. Let P be a dcpo.

(1) If P is Noetherian, then it is algebraic;

(2) If P is algebraic, then it is a continuous dcpo such that for each $p, q \in P$ with $q \ll p$, there is a compact $k \in P$ such that $q \leq k \leq p$;

(3) If P is a meet-semilattice that is also a continuous dcpo, then P is meet-continuous.

The statements in the next proposition are taken from Proposition II-1.6 and Corollary II-1.15 in [39].

Proposition B.6.8. Let P be an algebraic dcpo. Then

$$\{\uparrow p : p \in P\},$$

and

$$\{\uparrow k : k \in P \text{ is compact}\}$$

are bases for the Scott topology on P.

Definition B.6.9. Let P be a dcpo. Then subsets of P of the form $U \setminus \uparrow F$, where U is Scott open and F is finite, form a basis for a topology on P, denoted by $\lambda(P)$, and called the *Lawson topology*.

By definition Scott open sets are Lawson open. Similar to Scott continuity, a map $\varphi : P \to Q$ between complete semilattices turns out to be Lawson continuous if and only if it is continuous with respect to the Lawson topologies on P and Q. A proof of this statement can be found in [39, Theorem III-1.8].

The next theorem is a combination of Theorem III-1.9, Theorem III-1.10 and Exercise III-1.14 of [39].

Theorem B.6.10. Let P be an algebraic complete semilattice. Then P equipped with the Lawson topology is zero dimensional, and compact Hausdorff, where

$$\{\uparrow k \setminus \uparrow F : k \in \mathcal{K}(P), F \subseteq \mathcal{K}(P) \text{ finite}\}$$

is a basis consisting of Lawson-clopen subsets.

The notions of algebraic and continuous domains can be weakened. We first need the following definitions.

Definition B.6.11. Let P be a dcpo. Then we define a preorder $\leq$ on the set of non-empty subsets of P given by $X \leq Y$ if $\uparrow Y \subseteq \uparrow X$. A family of non-empty subsets of P is called *directed* if for each F_1, F_2 in the family, there is an F in the family such that $F_1, F_2 \leq F$.

A preorder is no order on P if P has at least two elements p and q such that $q \leq p$. Take $X = \uparrow q$ and let $Y = \{q\}$. Then $X \neq Y$, but $\uparrow X = \uparrow Y$.

Clearly, $\uparrow Y \subseteq \uparrow X$ implies $Y \subseteq \uparrow X$, but the converse holds as well. Let $Y \subseteq \uparrow X$ and $p \in \uparrow Y$. Then there is some $y \in Y$ such that $y \leq p$. Since $y \in Y$ and $Y \subseteq \uparrow X$, there is some $x \in X$ such that $x \leq y$. Hence $x \leq p$, so $p \in \uparrow X$. Hence we find that a family of non-empty subsets of a dcpo P is directed if and only for each F_1, F_2 in the family, there is an F in the family such that $F \subseteq \uparrow F_1 \cap \uparrow F_2$.

Definition B.6.12. Let P be a dcpo and let X and Y be two non-empty subsets of P. Then we say that X is *way below* Y if $\bigvee^{\uparrow} D \in \uparrow Y$ implies that $d \in \uparrow X$ for some $d \in D$, in which case we write $X \ll Y$. For each $p \in P$, we abbreviate $X \ll \{p\}$ by $X \ll p$ and $\{p\} \ll Y$ by $p \ll Y$.

Let $p, q \in P$, then $\{q\} \ll \{p\}$ if and only if $\bigvee^{\uparrow} D \in \uparrow \{p\}$ implies $d \in \uparrow \{q\}$ for some $d \in D$ if and only if $p \ll \bigvee^{\uparrow} D$ implies $q \leq d$ for some $d \in D$ if and only if $q \ll p$. This justifies the abbreviation $q \ll p$ instead of $\{q\} \ll \{p\}$.

Definition B.6.13. A dcpo P is called *quasicontinuous* if for each $p \in P$, the family

$$\mathrm{Fin}(p) = \{F \subseteq P : F \text{ is nonempty, finite, and } F \ll p\}$$

is directed, and if $p \not\leq q$, there is an $F \in \mathrm{Fin}(C)$ such that $q \notin \uparrow F$. A dcpo P is called *quasialgebraic* if for each $p \in P$, the family

$$\mathrm{CompFin}(p) = \{F \in \mathrm{Fin}(p) : F \ll F\}$$

is directed, and if $p \not\leq q$, there is an $F \in \mathrm{CompFin}(p)$ such that $q \notin \uparrow F$.

Notice that $F \ll p$ if $p \leq \bigvee^{\uparrow} D$ implies that there is an $x \in F$ and a $d \in D$ such that $x \leq d$. We refer to [39, §III-3] for more details on quasicontinuous and quasialgebraic dcpo's.

C C*-algebras

C.1 Basic C*-algebra theory

Definition C.1.1. Let A be an algebra over $\mathbb{C}$. Unless mentioned otherwise, we will always assume A has an identity element. Then A is called:

- *commutative* or *abelian* if $ab = ba$ for each $a, b \in A$;
- a **-algebra* if it admits an *involution*, i.e., a map $A \to A$, $a \to a^*$ such that for each $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$:
  - $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$,
  - $(a^*)^* = a$,
  - $(ab)^* = b^* a^*$.

- a *Banach algebra* if its underlying vector space is an Banach space and

  $$\|ab\| \leq \|a\|\|b\|$$

  for each $a, b \in A$.

- a *C*-algebra* if A is a Banach *-algebra such that

  $$\|a^*a\| = \|a\|^2$$

  for each $a \in A$. If A satisfies all axioms of a C*-algebra, except the existence of an identity element, we call A a *non-unital C*-algebra*.

We should also specify the morphisms between C*-algebras.

Definition C.1.2. Let $\varphi : A \to B$ be a linear map between C*-algebras A and B. Then φ is called a **-homomorphism* if for each $a, b \in A$:

- $\varphi(ab) = \varphi(a)\varphi(b)$,
- $\varphi(a^*) = \varphi(a)^*$,
- $\varphi(1_A) = 1_B$.

A bijective *-homomorphism $\varphi : A \to B$ is called a **-isomorphism*, in which case φ has an inverse $\varphi^{-1} : B \to A$, which itself a *-isomorphism. We call A and B **-isomorphic*, abbreviated by $A \cong B$, if there exists a *-isomorphism $A \to B$.

We denote the category consisting of C*-algebras with *-homomorphisms by **CStar**. Its subcategory of commutative C*-algebras is denoted by **CCStar**.

Example C.1.3. Let H be a Hilbert space. A linear operator $a : H \to H$ is called *bounded* if there is some $K > 0$ such that $\|ah\| \leq K\|h\|$ for each $h \in H$. In that case, a is continuous and vice versa. Let $B(H)$ be the set of all bounded linear operators, which is clearly an algebra if we define multiplication by composition. If $a \in B(H)$ there is a unique $a^* \in B(H)$ such that

$$\langle k, ah \rangle = \langle a^*k, h \rangle$$

for each k, h. We can define a norm on $B(H)$ by

$$\|a\| = \sup\{\|ah\| : h \in H, \|h\| = 1\}$$

for each $a \in B(H)$. It follows that $B(H)$ is a C*-algebra. We note that if $H = \mathbb{C}^n$, then $B(H) \cong M_n(\mathbb{C})$, the algebra of $n \times n$-matrices with complex entries.

Example C.1.4. Let X be a compact Hausdorff space and let $C(X)$ be the set of all continuous functions $X \to \mathbb{C}$. Then $C(X)$ becomes a commutative C*-algebra if, for each $f, g \in C(X)$ and $\lambda \in \mathbb{C}$, we define $\lambda f, f+g, fg$ and f^* for each $x \in X$ by

- $(\lambda f)(x) = \lambda(f(x))$,
- $(f+g)(x) = f(x) + g(x)$,
- $(fg)(x) = f(x)g(x)$,
- $(f^*)(x) = \overline{f(x)}$.

We define the norm of $f \in C(X)$ by

$$\|f\| = \sup_{x \in X} |f(x)|,$$

which is well defined by compactness of X.

Definition C.1.5. Let A be a C*-algebra.

- An element $a \in A$ satisfying $a^*a = aa^*$ is called *normal*. If $a^* = a$, then a is called *self adjoint*. If $a = b^*b$ for some other element $b \in A$, then a is called *positive*.
- If S is a subset of A such that $s^* \in S$ for each $s \in S$ is called **-closed*;
- If all elements of a subset S of A commute pairwise, we say that S is a *commutative* subset;
- If B is a *-closed subalgebra of A that is closed in the norm topology of A and that contains 1_A, then we say that B is a *C*-subalgebra* of A;
- If $B \subseteq A$ satisfies all axioms of C*-subalgebra of A, except $1_A \in B$, we call B a *non-unital C*-subalgebra* of A.

Warning. It should be mentioned that according to our terminology a non-unital C*-subalgebra can have an identity element, but it is not equal to 1_A.

Theorem C.1.6. Let A and B be C*-algebras and let $\varphi : A \to B$ be a *-homomorphism. Then φ is continuous with respect to the norm topologies of A and B. If φ is injective, then it is an isometry. Moreover, $\varphi[C]$ is a C*-subalgebra of B for each C*-subalgebra $C \subseteq A$, and $\varphi^{-1}[D]$ is a C*-subalgebra of A for each C*-subalgebra $D \subseteq B$.

Proof. It follows from Propositions I.5.2 and I.5.3 of [106] that φ is continuous and an isometry when injective, respectively. If $D \subseteq B$ is a C*-subalgebra, then it is closed, hence by continuity, $\varphi^{-1}[B] \subseteq A$ is closed as well. Since $1_B \in D$, we have $1_A \in \varphi^{-1}[D]$. Since φ is linear, *-preserving and multiplicative, it follows that $\varphi^{-1}[D]$ is a C*-subalgebra of A. The statement that $\varphi[A]$ is a C*-subalgebra of B follows from [67, Theorem 4.1.9] or from [92, Theorem 3.1.6]. □

Corollary C.1.7. Let A be a C*-algebra, then $B \subseteq A$ is a C*-subalgebra of A if and only if the inclusion $B \to A$ is a *-homomorphism.

The next theorem is the most important theorem about commutative C*-algebras, stating that Example C.1.4 is prototypical.

Theorem C.1.8 (Gelfand–Naimark)**.** Let A be a commutative C*-algebra. Then A is *-isomorphic to $C(X)$ for some compact Hausdorff space X, which is unique up to homeomorphism. The space X is called the *spectrum* of A.

Proof. See [80, Theorem I.1.2.3], or Theorem I.4.4 and Proposition I.4.5 of [106]. □

This theorem, to which we will refer as the *commutative Gelfand–Naimark Theorem*, can be extended to a duality, called *Gelfand duality*, between **CCStar**, the category of commutative C*-algebras, and **CptHd**, the category of compact Hausdorff spaces and continuous functions between them. Using this duality, one can translate statements about commutative C*-algebras into statements about compact Hausdorff spaces.

The proof of commutative Gelfand–Naimark Theorem makes use of the Stone-Weierstrass Theorem, which will be very useful in this thesis in order to prove that subalgebras of commutative C*-algebras are actually C*-algebras. In order to formulate it, we need the following definition:

Definition C.1.9. Let X be a compact Hausdorff space and $S \subseteq C(X)$ as a subset. Then we say that S *separates points* of X if for each distinct $x, y \in X$ there is a function $f \in S$ such that $f(x) \neq f(y)$.

The next Lemma follows directly from Urysohn's Lemma.

Lemma C.1.10. Let X be a compact Hausdorff space. Then $C(X)$ separates points of X.

Theorem C.1.11. [94, Stone-Weierstrass, Theorem 4.3.4] Let X be a compact Hausdorff space and A a *-subalgebra of $C(X)$ (hence containing 1_X) and separating points in X. Then A is dense in $C(X)$ with respect to the norm topology on $C(X)$.

The next theorem is also due to Gelfand and Naimark and shows that Example C.1.3 is prototypical.

Theorem C.1.12 (Gelfand–Naimark)**.** Let A a C*-algebra. Then there is a Hilbert space H such that A is *-isomorphic to some C*-subalgebra of $B(H)$.

Proof. See [106, Theorem I.9.18]. □

We will refer to this theorem as the *Gelfand–Naimark Embedding Theorem*. It explains the name 'C*-algebra', where the 'C' stands for 'closed', since C*-algebras are exactly the *-subalgebras of $B(H)$ that are closed in the norm topology on $B(H)$.

Definition C.1.13. Let A be a C*-algebra. A commutative C*-subalgebra M is called a *maximal commutative C*-subalgebra*, or a *maximal abelian C*-subalgebra* if $M \subseteq C$ implies $M = C$ for each commutative C*-subalgebra $C \subseteq A$.

Definition C.1.14. Let A be a C*-algebra and $S \subseteq A$ a subset. Then we define the *commutant* of S as the set

$$S' = \{x \in A : xy = yx \text{ for each } y \in S\}.$$

The commutant of A itself is called the *center* of A, and is denoted by $Z(A)$. Thus,

$$Z(A) = \{x \in A : xy = yx \text{ for each } y \in A\}.$$

Properties (1)-(4) and first three equivalences in property (5) of the next proposition are taken from [6, Proposition 1.9]. Property (6) is standard.

Proposition C.1.15. Let A be a C*-algebra, and S and T subsets of A. Then:

(1) If S is *-closed, then S' is a C*-subalgebra of A;

(2) $S \subseteq T$ implies $T' \subseteq S'$;

(3) $S \subseteq S''$;

(4) $S''' = S'$;

(5) If S is *-closed, then the following statements are equivalent:

- S is commutative;
- S'' is a commutative C*-subalgebra of A;
- $S \subseteq S'$;
- S is contained in some maximal commutative C*-subalgebra of A.

In particular, every normal element of A and every commutative C*-subalgebra of A can be embedded in some maximal commutative C*-subalgebra, and the set of all maximal commutative C*-subalgebras is not empty.

(6) If S is *-closed, then S is a maximal commutative C*-subalgebra if and only if $S' = S$.

The commutant can be used to define one of the most important subclasses of C*-algebras:

Definition C.1.16. Let H be a Hilbert space. Then a *-subalgebra $M \subseteq B(H)$ is called a *von Neumann algebra* if $M = M''$. A C*-algebra *-isomorphic to a von Neumann algebra is called a *W*-algebra*. Such an algebra does not necessarily act on Hilbert space. A von Neumann algebra or a W*-algebra with one-dimensional center is called a *factor*.

It follows from Proposition C.1.15 that every von Neumann algebra is indeed a C*-algebra. The 'W' in the name 'W*-algebra' is stands for 'weak', since it follow from von Neumann's Double Commutant Theorem [67, Theorem 5.3.1] that von Neumann algebras are precisely the C*-subalgebras of $B(H)$ that are closed in the *weak operator topology*, i.e., the topology on $B(H)$ in which a net $(a_\lambda)_{\lambda\in\Lambda}$ converges to a if and only if $\langle a_\lambda h, k\rangle$ converges to $\langle ah, k\rangle$ for each $h, k \in H$.

Lemma C.1.17. Let A be a C*-algebra. Then the center $Z(A)$ of A is a commutative C*-subalgebra of A.

Lemma C.1.18. Let A be a C*-algebra and $\{C_i\}_{i\in I}$ a collection of C*-subalgebras of A. Then $C = \bigcap_{i\in I} C_i$ is a C*-subalgebra of A. Moreover, if C_i is commutative for some $i \in I$, then C is commutative as well.

By Lemma C.1.18, the intersection of C*-subalgebras of a C*-algebra A is a C*-subalgebra of A. Let $S \subseteq A$ be a set, then the set of C*-subalgebras of A containing S is non-empty, since it contains A itself. Hence the intersection of all C*-subalgebras containing S is the smallest C*-subalgebra of A containing S, so the following definition makes sense.

Definition C.1.19. Let A be a C*-algebra and $S \subseteq A$. Then we define the C*-subalgebra $C^*(S)$ *generated* by S as the smallest C*-subalgebra of A containing S. If S is finite, say $S = \{s_1, \ldots, s_n\}$, we rather write $C^*(s_1, \ldots, s_n)$ instead of $C^*(\{s_1, \ldots, s_n\})$.

Lemma C.1.20. Let A be a C*-algebra, denote by $\mathcal{S}(A)$ the set of all C*-subalgebras of A, and let $\Phi : \mathcal{S}(A) \to \mathcal{P}(A)$ be the order embedding assigning to each C*-subalgebra B of A its underlying set. Then $C^* : \mathcal{P}(A) \to \mathcal{S}(A)$ is the lower adjoint of Φ.

Proof. It easily follows that $C^*(S) \subseteq B$ if and only if $S \subseteq F(B)$ for each C*-subalgebra B of A and each subset $S \subseteq A$. $\square$

Lemma C.1.21. Let A be a C*-algebra, and $C \subseteq A$ a commutative *-subalgebra. Then the closure of C is a commutative C*-subalgebra.

Lemma C.1.22. Let A be a C*-algebra and S a *-closed subset of A. Then S is commutative if and only if $C^*(S)$ is a commutative C*-subalgebra of A.

Definition C.1.23. Let A be a C*-algebra and $a \in A$. The *spectrum of* a is defined as the set

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1_A \text{ is not invertible}\}.$$

Example C.1.24. Let X be a compact Hausdorff space and $f \in C(X)$. Then

$$\sigma(f) = f[X].$$

We state some basis facts about the spectrum of an element in a C*-algebra:

Lemma C.1.25. Let A be a C*-algebra and $a \in A$.

- $\sigma(a)$ is a non-empty compact subset of $\mathbb{C}$;
- If $B \subseteq A$ is a C*-subalgebra containing a, then $\sigma(a)$ calculated in B equals $\sigma(a)$ calculated in A.

Moreover, if a is normal, then

- $\|a\| = \sup_{\lambda \in \sigma(a)} |\lambda|$;
- a is self adjoint if and only if $\sigma(a) \subseteq \mathbb{R}$;
- a is positive if and only if $\sigma(a) \subseteq [0, \infty)$.

We refer to Theorems 3.2.3 and 4.4.5, and Propositions 4.1.1 and 4.1.5 of [67] for a proof.

If a is normal, then it follows from Lemma C.1.22 that $C^*(a)$ is a commutative C*-algebra. By Lemma C.1.25, $\sigma(a)$ is a compact Hausdorff space, $C(\sigma(a))$ is a commutative C*-algebra. It follows that we can connect the spectrum $\sigma(a)$ of a with the Gelfand spectrum of $C(\sigma(a))$.

Theorem C.1.26. [67, Theorem 4.4.5] Let A be a C*-algebra with $a \in A$ normal. Then there exists a unique *-isomorphism

$$\rho : C(\sigma(a)) \to C^*(a) \subseteq A$$

such that:

- $\rho(1_{\sigma(a)}) = 1_A$;
- $\rho(\mathrm{id}_{\sigma(a)}) = a$.

We call ρ the *functional calculus* for the element a and write $f(a)$ instead of $\rho(f)$, for each $f \in C(\sigma(a))$.

Definition C.1.27. Let $\{A_i\}_{i \in I}$ be a family of C*-algebras. Then we define the *C*-sum* of the A_i by

$$\bigoplus_{i \in I} A_i = \left\{ (a_i)_{i \in I} : a_i \in A_i, \sup_{i \in I} \|a_i\| < \infty \right\},$$

which is a C*-algebra with componentwise-defined operations and with norm $\|(a_i)_{i\in I}\| = \sup_{i\in I} \|a_i\|$.

For each $j \in I$, the *projection map* on the j-th factor is defined as the *-homomorphism

$$\pi_j : \bigoplus_{i\in I} A_i \to A_j,\ (a_i)_{i\in I} \mapsto a_j.$$

Lemma C.1.28. Let $\{A_i\}_{i\in I}$ be a collection of C*-algebras and let

$$A = \bigoplus_{i\in I} A_i.$$

Then $Z(A) = \bigoplus_{i\in I} Z(A_i)$.

C.2 Ideals

Definition C.2.1. Let A be a C*-algebra. A (possibly non-unital) subalgebra I of A is called an *ideal* if $x \in I$ and $a \in A$ implies $ax \in I$. If I is an ideal, it is called:

- a **-ideal* if $x \in I$ implies $x^* \in I$;
- *proper* if $I \neq A$;
- *prime* if I is proper and $xy \in I$ implies $x \in I$ or $y \in I$;
- *maximal* if it is maximal in the set of all proper ideals ordered by inclusion;
- *closed* if it is topologically closed in A;
- a *C*-ideal* if it is a closed *-ideal.

Clearly an ideal I equals A if and only if $1_A \in I$. Let I be a proper ideal. Then its closure $\overline{I}$ is a proper ideal as well. To see that $\overline{I}$ is an ideal, let $x \in \overline{I}$. Then for each $\epsilon > 0$, there is some $x_\epsilon \in I$ such that $\|x - x_\epsilon\| < \epsilon$. Let $a \in A$. Then

$$\|ax - ax_\epsilon\| \leq \|a\|\|x - x_\epsilon\| \leq \|a\|\epsilon.$$

Since $ax_\epsilon \in I$, it follows that $ax \in \overline{I}$. So $\overline{I}$ is indeed an ideal. To see that it is proper, assume that $1_A \in \overline{I}$. Then there is some $x \in I$ such that $\|x - 1_A\| < 1$. It is a standard result in Banach algebra theory that this implies that x is invertible, see for instance [67, Lemma 3.1.5]. Hence $1_A \in I$. By contraposition, it follows that $\overline{I}$ is proper when I is proper. As a consequent, the maximal ideals are closed. We shall see below that a C*-ideal is always closed under the *-operation, justifying the terminology 'C*-ideal'.

Since by the commutative Gelfand–Naimark Theorem, every commutative C*-algebra is *-isomorphic to $C(X)$ for some compact Hausdorff, all information about the C*-algebra is stored in X. The next theorem relates closed subsets of X to C*-ideals.

Theorem C.2.2. [67, Theorem 3.4.1] Let X be a compact Hausdorff space and $A = C(X)$. If $K \subseteq X$ is closed, then

$$I_K = \big\{f \in C(X) : f[K] = \{0\}\big\}$$

is closed ideal of $C(X)$. If $I \subseteq C(X)$ is a closed ideal, then

$$K_I = \bigcap_{f \in I} \ker f$$

a closed subset of X. Moreover, the assignments $K \mapsto I_K$ and $I \mapsto K_I$ are each others inverses, and inclusion reversing.

Corollary C.2.3. Let A be a commutative C*-algebra. Then the C*-ideals of A are exactly the closed ideals.

Definition C.2.4. Let $k : X \to Y$ be a continuous map between compact Hausdorff spaces X and Y. Then we denote the map $C(Y) \to C(X)$, $f \mapsto f \circ k$ by $C(k)$. In many cases, we attempt to reduce an excessive use of parenthesis and write C_k instead of $C(k)$.

Proposition C.2.5. Let X and Y be compact Hausdorff spaces.

- If $k : X \to Y$ is a continuous map, then $C(k) : C(Y) \to C(X)$ is a *-homomorphism. Moreover, k is surjective if and only if $C(k)$ is injective (in which case $C(k)$ is an isometry), and k is a homeomorphism if and only if $C(k)$ an *-isomorphism;
- If Z is another compact Hausdorff space and $h : Y \to Z$ continuous, then $C(h \circ k) = C(k) \circ C(h)$;
- If $\varphi : C(Y) \to C(X)$ is a *-homomorphism, then there is a unique continuous map $k : X \to Y$ such that $\varphi = C(k)$.

C.3 Projections

In this section, we describe the order structure of projections in a C*-algebra. Most statements can be found in the standard literature for operator algebras, see for instance [67, 68, 106].

Definition C.3.1. Let A be a C*-algebra. Then $p \in A$ is called a *projection* if $p = p^2 = p^*$. A projection is called *non-trivial* or *proper* if $0 \neq p \neq 1$. The set of projections in A is denoted by $\mathrm{Proj}(A)$. We say that a family of projections $\{p_i\}_{i \in I}$ is *orthogonal* if $p_i p_j = \delta_{ij} p_i$ for each $i, j \in I$.

Properties (1) and (2) of the next proposition are taken from [6, Proposition 1.1].

Proposition C.3.2. Let A be a C*-algebra. If we order $\mathrm{Proj}(A)$ by

$$q \leq p \iff pq = q,$$

and define an orthocomplementation on $\mathrm{Proj}(A)$ by

$$p^\perp = 1_A - p,$$

then $\mathrm{Proj}(A)$ becomes an orthomodular poset, which is a Boolean algebra if A is commutative. The greatest and least element of $\mathrm{Proj}(A)$ are 1_A and 0_A, respectively. Moreover, for each $p, q \in \mathrm{Proj}(A)$ we have

(1) $q \leq p$ if and only if $qp = q$ if and only if $qA \subseteq pA$ if and only if $Aq \subseteq Ap$;

(2) $q = p$ if and only if $qA = pA$ if and only if $Aq = Ap$.

(3) $pq = pq$ if and only if $pq \in \mathrm{Proj}(A)$ if and only if $p\,C q$, i.e., p and q commute in an order-theoretic sense. In all these equivalent cases, the meet and join of p and q exist, and are given by

$$p \wedge q = pq,$$
$$p \vee q = p + q - pq.$$

(4) $pq = 0_A$ if and only if p and q are orthogonal in an order-theoretic sense, in which case

$$p \vee q = p + q.$$

Definition C.3.3. Let A be a C*-algebra. Then the atoms of $\mathrm{Proj}(A)$ are also called *minimal projections*.

The next proposition can be found in a slightly different setting in [7, Lemma 2].

Proposition C.3.4. $\mathrm{Proj} : \mathbf{CStar} \to \mathbf{OMP}$ becomes a functor if, for each *-homomorphism $\varphi : A \to B$ between C*-algebras A and B, we define $\mathrm{Proj}(\varphi) : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ as the restriction of φ to $\mathrm{Proj}(A)$, also denoted by φ if no confusion may arise. Moreover,

- if $\varphi : A \to B$ is injective, then $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ is an order embedding;
- if $\varphi : A \to B$ is a *-isomorphism, then $\varphi : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$ is an orthomodular isomorphism.

The next lemma is standard.

Lemma C.3.5. Let X be a compact Hausdorff space. Then:

(1) each $p \in C(X)$ is a projection if and only if the image of p is a subset of $\{0, 1\}$;

(2) for each $p, q \in \mathrm{Proj}(C(X))$, we have $q \leq p$ if and only if $q(x) = 1$ implies $p(x) = 1$ for each $x \in X$, if and only if $p(x) = 0$ implies $q(x) = 0$ for each $x \in X$;

(3) for each $p, q \in \mathrm{Proj}(C(X))$, we have $\|p - q\| < 1$ if and only if $p = q$.

Proposition C.3.6. Let X be a compact Hausdorff space. Then the assignment $p \mapsto p^{-1}[\{1\}]$ is a Boolean isomorphism $\mathrm{Proj}(C(X)) \to B(X)$, where $B(X)$ denotes the Boolean algebra of clopen subsets of X (cf. Chapter B.5). Its inverse is given by $U \mapsto \chi_U$, where χ_U is the characteristic function on $U \in B(X)$.

Corollary C.3.7. Let X be compact Hausdorff. Then there is a bijection from $\mathrm{Proj}(C(X))$ to the set of ordered pairs (K_0, K_1) of clopen subsets of X such that $K_1 = X \setminus K_0$ given by

$$p \mapsto (p^{-1}[\{0\}], p^{-1}[\{1\}])$$

with inverse $(K_0, K_1) \mapsto p$ with

$$p(x) = \begin{cases} 0, & x \in K_0; \\ 1, & x \in K_1. \end{cases}$$

Under this bijection, p is non-trivial if and only if (K_0, K_1) is a non-trivial separation of X.

Proposition C.3.8. Let H be a Hilbert space and denote the set of closed subspaces of H by $\mathcal{C}(H)$. Then $\mathcal{C}(H)$ becomes a complete orthomodular lattice if we order it by inclusion and define

$$K^\perp = \{h \in H : \langle h, k \rangle = 0 \text{ for each } k \in K\}$$

for each $K \in \mathcal{C}(H)$. Moreover,

$$\mathrm{Proj}(B(H)) \to \mathcal{C}(H), \quad p \mapsto pH = \{ph : h \in H\}$$

is an orthomodular isomorphism.

We refer to [67, §2.5] for a more detailed exposition of the lattice of closed subspaces of a Hilbert space and its relation with projections.

Lemma C.3.9. Let $\{A_i\}_{i \in I}$ be a collection of C*-algebras, and let $A = \bigoplus_{i \in I} A_i$. Let $\pi_i : A \to A_i$ be the projection map on the i-th component. Then $\varphi : \mathrm{Proj}(A) \to \prod_{i \in I} \mathrm{Proj}(A_i)$ defined by $p \mapsto (\pi_i(p))_{i \in I}$ is an orthomodular isomorphism.

Lemma C.3.10. Let A be a commutative C*-algebra and $P \subseteq \mathrm{Proj}(A)$. Then

(a) P is an orthogonal family of projections if all elements of P are minimal projections;

(b) If P is an orthogonal family of non-zero projections, then all elements in P are linear independent;

(c) If P is a finite orthogonal family of projections such that $\sum P = 1_A$, then $C^*(P) = \mathrm{Span}(P)$.

C.4 Direct limits of C*-algebras

Several families of C*-algebras can be defined in terms of directed families belonging to another class of C*-algebras. Categorical speaking, this amounts classes of C*-algebras described in terms of a *direct limit* or *inductive limit*, i.e., a colimit with a directed poset as the index category. The most prominent example of such a class is formed by the approximately finite-dimensional C*-algebras, which are defined in §2.2. Hence it is useful to describe elements (especially projections) of these C*-algebras in terms of elements of the C*-subalgebras in these directed families.

Proposition C.4.1. [110, Proposition L.2.2] Let A be a C*-algebra, and $\mathcal{D}$ a directed set of C*-subalgebras such that $A = \overline{\bigcup \mathcal{D}}$. Let $a \in A$. Then for each $\epsilon > 0$ there is a $D \in \mathcal{D}$ and a $a_D \in D$ such that $\|a - a_D\| < \epsilon$. Moreover, if a is self adjoint, a_D can be chosen to be self adjoint; if a is positive, then a_D can be chosen to be positive; if $0 \leq a \leq 1$, then a_D can be chosen such that $0 \leq a_D \leq 1$; and if a is a projection, then a_D can be chosen to be a projection.

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Samenvatting

Als ik zou willen dat je het begreep, had ik het wel beter uitgelegd. (Johan Cruijff)

Ongetwijfeld zal de titel van dit proefschrift voor de meeste mensen (inclusief veel wiskundigen!) onbegrijpelijk zijn, vandaar een poging tot opheldering. Heel kort samengevat slaat $\mathcal{C}(A)$, wat we uitspreken als 'C van A', op de *geordende* verzameling van alle *commutatieve* C*-deelalgebra's van een *C*-algebra* A. In deze samenvatting zullen we proberen de gecursiveerde woorden te verklaren en te motiveren waarom we in $\mathcal{C}(A)$ geïnteresseerd zijn.

C*-algebra's en commutativiteit

Grof gezegd is een *algebra* een verzameling waarvan de elementen zich als getallen gedragen: we kunnen twee elementen bij elkaar optellen, van elkaar aftrekken, en met elkaar vermenigvuldigen. Veel gebruikelijke rekenregels gelden ook voor algebra's. Zo geldt er bijvoorbeeld

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

voor alle getallen x, y en z. Als we bijvoorbeeld $x = 3$, $y = 2$ en $z = 4$ kiezen, vinden we

$$3 \cdot (2 + 4) = 18 = 3 \cdot 2 + 3 \cdot 4.$$

Nu blijkt deze wet ook te gelden als x, y en z elementen uit een algebra zijn. Een algebra kan je dus in zekere zin zien als een verzameling van gegeneraliseerde getallen.

Nu hebben gewone getallen de eigenschap dat we kunnen bepalen hoe ver ze uit elkaar liggen. De getallen 14 en 18 liggen bijvoorbeeld $18 - 14 = 4$ getallen uit elkaar verwijderd. Aangezien afstand altijd in gewone getallen uitgedrukt wordt en het verschil tussen twee elementen in een algebra geen gewoon getal is, kunnen we in het algemeen niet aangeven hoe ver twee elementen in een algebra uit elkaar liggen. Een *C*-algebra* is een speciaal soort algebra, waarbij er wel een methode is om de afstand tussen twee elementen uit te drukken.

Er is echter één specifieke rekenregel die voor getallen altijd geldig is, maar niet voor alle C*-algebra's. Elke twee gewone getallen x en y *commuteren*, wat wil zeggen dat

$$x \cdot y = y \cdot x.$$

Neem bijvoorbeeld $x = 2$ en $y = 7$, dan hebben we

$$2 \cdot 7 = 14 = 7 \cdot 2.$$

De meeste C*-algebra's daarentegen bevatten minstens twee elementen x en y die onderling niet commuteren:

$$x \cdot y \neq y \cdot x.$$

Echter, er zijn ook voorbeelden te vinden van C*-algebra's waarvan wel alle elementen commuteren. Dergelijke algebra's noemen we *commutatieve* C*-algebra's. Verder is het altijd zo dat elke C*-algebra meerdere commutatieve C*-deelalgebra's heeft, waarbij een C*-deelalgebra grof gezegd een deelverzameling van de oorspronkelijke algebra is die op zichzelf een C*-algebra vormt.

Quantummechanica

Eén van de belangrijkste redenen waarom C*-algebra's bestudeerd worden, is dat ze gebruikt kunnen worden om natuurkundige systemen, zoals een slinger van een klok, een gas in een afgesloten ruimte, of een elektron om een atoomkern, te representeren. Hierbij zijn we geïnteresseerd in de *meetbare grootheden*, zoals de positie en de snelheid van de slinger, maar ook in de *toestand* van het systeem, zoals de bewegingsenergie die de slinger op een bepaald moment heeft, of de temperatuur van het gas in de afgesloten ruimte. Vervolgens willen we kunnen bepalen wat de *verwachtingswaarde* is als we een bepaalde grootheid gaan *meten* gegeven dat het systeem zich in een bepaalde toestand bevindt. Als de slinger een bepaalde bewegingsenergie heeft, willen we bijvoorbeeld kunnen voorspellen wat voor snelheid we zullen meten.

Al deze begrippen en handelingen kunnen in de taal van C*-algebra's weergegeven worden. De meetbare grootheden kunnen als elementen van een C*-algebra gerepresenteerd worden, als een soort gegeneraliseerde getallen dus. Hierbij is het van belang om onderscheid te maken tussen een grootheid en z'n waarde. De waarden die grootheden aannemen worden uitgedrukt door middel van gewone getallen, maar de grootheden zelf zijn geen gewone getallen, maar gedragen zich als gegeneraliseerde getallen. Als het systeem zich in een bepaalde toestand bevindt, en we een bepaalde grootheid willen meten, moet de uitkomst wel altijd een gewoon getal zijn. Snelheid drukken we bijvoorbeeld in gewone getallen uit. Om die reden representeren we een toestand van het systeem met behulp van een functie f die aan elk element a van de C*-algebra (elke meetbare grootheid) een gewoon getal $f(a)$ (de verwachtingswaarde van de meetuitkomst) toekent.

Het voordeel van C*-algebra's is dat dit maatwerk zowel voor de klassieke natuurkunde van Newton als voor de quantummechanica gebruikt kan worden. Quantummechanica is de natuurkundige theorie die het gedrag van materie en energie op (sub)atomaire schaal beschrijft. Deze beschrijving is soms tegen-intuïtief: zo is het bijvoorbeeld onmogelijk om zowel de snelheid en de positie van een deeltje nauwkeurig te meten: als we eerst de positie x bepalen, en dan de snelheid v, dan kunnen we totaal andere uitkomsten krijgen dan als we eerst v bepalen, en dan x. De wiskundige reden hiervoor is omdat x en v in de C*-algebra corresponderend met het quantumsysteem niet commuteren:

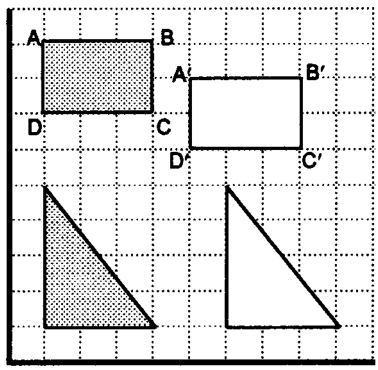
$$vx \neq xv.$$

Aldus zijn de C*-algebra's die quantumsystemen representeren niet noodzakelijk commutatief. In de klassieke natuurkunde daarentegen treden de bovengenoemde effecten niet op. Theoretisch is het bijvoorbeeld wel altijd mogelijk om de snelheid en de positie van een auto op een bepaald tijdstip nauwkeurig bepalen. Er volgt dat de C*-algebra's die met klassieke systemen corresponderen altijd commutatief zijn.

Isomorfismen en invarianten

In de wiskunde zijn we vaak geïnteresseerd in het onderscheiden van bepaalde wiskundige objecten. Dit zullen we illustreren aan de hand van meetkundige figuren, die overigens verder geen rol spelen in dit proefschrift. Hierbij moeten we eerst definiëren wat we überhaupt bedoelen met dat twee objecten 'hetzelfde' zijn. Eerst kijken we naar welke transformaties de structuur waarin we geïnteresseerd zijn behouden. In het geval van meetkunde draait het om de afstand tussen de punten van een meetkundig figuur, dus kijken we naar transformaties die de onderlinge afstand van twee punten bewaren. Als we een driehoek verplaatsen, dan verandert bijvoorbeeld de afstand tussen twee hoekpunten niet. De transformaties die de gewenste structuur behouden noemen we *isomorfismen*. In het geval van meetkunde zijn dit behalve verplaatsingen ook rotaties en spiegelingen. We zeggen nu dat twee objecten 'hetzelfde', oftewel *isomorf*, zijn, als ze er een isomorfisme is dat het ene object op het andere object afbeeldt.

Het woord ‘isomorf’ is overigens afgeleid uit het Grieks en betekent letterlijk ‘gelijkvormig’. De rechthoeken in Figuur 1 (waar kleuren overigens geen rol van betekenis spelen) zijn bijvoorbeeld isomorf:



Figuur 1

Als we de grijze rechthoek vier hokjes naar rechts en één hokje naar beneden verschuiven, wat een verplaatsing is, dan verkrijgen we precies de witte rechthoek, dus de twee rechthoeken zijn inderdaad isomorf. Ook de twee driehoeken zijn isomorf, want we kunnen de grijze driehoek vijf hokjes naar rechts opschuiven, en dan krijgen we precies de witte driehoek.

Dit werkt zo in veel takken van de wiskunde: gegeven een bepaalde structuur waarin we geïnteresseerd zijn, kijken we eerst naar de isomorfismen, dus de transformaties die de gegeven structuur bewaren. Vervolgens zeggen we dat twee objecten isomorf zijn, dus ‘hetzelfde’ zijn, als er een isomorfisme is dat de één in de ander transformeert. Optelling en vermenigvuldiging zijn essentieel voor de structuur van C*-algebra’s, dus isomorfismen tussen twee C*-algebra’s moeten functies f van de ene naar de andere C*-algebra zijn die (onder andere) vermenigvuldiging en de optelling bewaren:

$$f(a+b) = f(a) + f(b),$$
$$f(a \cdot b) = f(a) \cdot f(b).$$

Hoe abstracter de structuur, hoe lastiger het soms is om een concreet isomorfisme te vinden tussen twee objecten die isomorf zijn. Het is soms makkelijker om aan te tonen dat twee objecten *niet* isomorf zijn. Dit doen we door middel van *invarianten*, wat eigenschappen zijn die bewaard blijven onder isomorfismen. Een voorbeeld van een invariant voor meetkundige figuren wordt bijvoorbeeld gevormd door het aantal hoeken van een meetkundig figuur, wat immers niet verandert na een verplaatsing, rotatie of spiegeling. Hieruit kunnen we bijvoorbeeld afleiden dat een driehoek en een rechthoek niet isomorf zijn, omdat een rechthoek meer hoeken heeft dan een driehoek. Een ander voorbeeld van een meetkundige invariant bestaat uit de lengte van de zijden van de meetkundige figuren. Zo zien we dat de rechthoeken in Figuur 1 geen vierkanten zijn, omdat de zijden van de rechthoeken niet allemaal even lang zijn, terwijl dit wel het geval is voor een vierkant. Deze invariant legt driehoeken ook vast: als de zijden van twee driehoeken paarsgewijs even lang zijn, dan kan je bewijzen dat de twee driehoeken isomorf moeten zijn. Dus gegeven twee driehoeken kunnen we altijd bepalen of ze isomorf zijn zonder een concreet isomorfisme te construeren, we hoeven alleen maar de lengte van de zijden te bepalen.

Geordende verzameling

Een andere structuur die relevant voor dit proefschrift is en die tot meer voorbeelden van isomorfismen en invarianten leidt, is de klasse van de *geordende verzamelingen*. Startpunt zijn de natuurlijke getallen

$$1, 2, 3, \dots,$$

die we op natuurlijke wijze kunnen ordenen: als we met $n < m$ aangeven dat het getal n kleiner is dan m, dan is $<$ een relatie die voldoet aan de volgende twee regels:

(1) voor geen enkel getal n geldt $n < n$;

(2) als $n < m$ en $m < k$, dan $n < k$ voor alle getallen n, m en k.

Net zoals een algebra een generalisatie van getallen vormt, kunnen we ook orde generaliseren. We zeggen dat een verzameling een *geordende verzameling* is als er een relatie $<$ tussen de elementen van de verzameling is die voldoet aan de bovenstaande twee regels (waarbij de getallen vervangen worden door de elementen van de verzameling). Een dergelijke relatie noemen we een *orderelatie*.

De natuurlijke getallen vormen dus een geordende verzameling. Een ander voorbeeld van een geordende verzameling is de verzameling van alle mensen die ooit geleefd hebben met de volgende orderelatie: gegeven twee mensen A en B, zeggen we dat $A < B$ als A een nakomeling van B is. Dit is inderdaad een orderelatie, want (1): niemand is nakomeling van zichzelf, en (2): als A een nakomeling van B is, en B een nakomeling van C, dan is A ook een nakomeling van C.

In het laatste voorbeeld is er iets opmerkelijks aan de hand: waar gegeven twee verschillende getallen er altijd één groter is dan de ander, is dit niet het geval voor de verzameling van alle mensen die ooit geleefd hebben. Bijvoorbeeld als Daniël de broer van Thomas is, kan hij noch een nakomeling noch een voorouder van Thomas zijn. We zeggen dat de orde op de verzameling van alle mensen *partieel* is, terwijl de orde op de natuurlijke getallen *totaal* is. In de wiskunde zijn partiële orderelaties overigens vaak interessanter dan totale orderelaties. De geordende verzamelingen in dit proefschrift zijn bijvoorbeeld bijna allemaal partieel. Een ander voorbeeld van een partieel geordende verzameling wordt gevormd door klanten in een supermarkt, die in rijen staan te wachten bij de kassa. Gegeven personen A en B zeggen we dat $A < B$ als A *voor* B in *dezelfde* rij staat. Ook hier geldt weer (1): geen klant staat voor zichzelf in een rij, en (2): als $A < B$ en $B < C$, dan staat A voor B dezelfde rij, en B staat voor C in dezelfde rij. Dus staan A en C in dezelfde rij, en A staat duidelijk voor C, dus $A < C$. Echter, als twee klanten A en B in verschillende rijen staan, dan geldt er noch $A < B$ noch $B < A$, dus de orde is inderdaad partieel.

We kunnen nu ook orde-isomorfismen tussen geordende verzamelingen X en Y definiëren. Dit zijn functies f van X naar Y die de orde moeten behouden:

$$x_1 < x_2 \iff f(x_1) < f(x_2).$$

Geordende verzamelingen zijn dus orde-isomorf als ze dezelfde ordestructuur hebben. Stel bijvoorbeeld dat er in supermarkt 1 op dit moment drie rijen zijn, waarvan twee rijen met elk vijf wachtende klanten en één rij van vier wachtenden, dan zijn de wachtende klanten in supermarkt 1 orde-isomorf met de wachtende klanten van supermarkt 2 als de laatste supermarkt ook op dit moment drie wachtrijen heeft, waarvan twee met elk vijf wachtenden en één rij met vier wachtenden.

Een voorbeeld van een orde-invariant is bijvoorbeeld de eigenschap dat een geordende verzameling totaal geordend is: gegeven twee geordende verzamelingen zodat de één totaal geordend is en de ander partieel, dan kunnen de verzamelingen niet orde-isomorf zijn.

Geordende verzameling vormen een relatief eenvoudige structuur binnen de wiskunde, zeker in vergelijking met C*-algebra's. Daarom kunnen wiskundige problemen vaak versimpeld worden door ze te herleiden tot ordetheoretische problemen. In het bijzonder kunnen we geordende verzamelingen zelf als invariant gebruiken. Bijvoorbeeld de verzameling van alle lijnstukken van een meetkundig figuur geordend op lengte is een geordende verzameling. Dus als we de grijze rechthoek in Figuur 1 bekijken, verkrijgen we een geordende verzameling bestaande uit de lijnstukken AB, BC, CD en AD, zodat

$$AD < AB, \quad AD < CD, \quad BC < AB, \quad BC < CD.$$

Merk op dat deze geordende verzameling partieel is: AB en CD hebben bijvoorbeeld dezelfde lengte, dus de één is niet groter dan de ander. Twee isomorfe meetkundige figuren hebben duidelijk geordende verzamelingen van lijnstukken die orde-isomorf zijn, dus de geordende verzameling van lijnstukken van een meetkundig figuur vormt een meetkundige invariant. We zullen zien dat we op soortgelijke wijze een invariant voor C*-algebra's kunnen vinden.

$\mathcal{C}(A)$ als invariant voor C*-algebra's

Het belangrijkste doel van dit proefschrift is het onderscheiden van verschillende C*-algebra's van elkaar, en daarom zijn we geïnteresseerd in een invariant voor C*-algebra's. Natuurkundig gezien zijn invarianten voor C*-algebra's ook interessant, want deze komen overeen met intrinsieke eigenschappen van quantumsystemen, en leiden tot methoden om quantumsystemen van elkaar te onderscheiden. Een belangrijk criterium voor natuurkundigen is dat deze methoden experimenteel te implementeren zijn, waarvan we gebruik maken in onze keuze voor een invariant.

Zoals reeds gezegd is de quantummechanica soms tegenintuïtief. Als we eerst een bepaalde grootheid a meten, en vervolgens een andere grootheid b die *niet* commuteert met a, wordt de toestand van het quantumsysteem verstoord: het komt in een andere toestand terecht, en welke dat is valt niet met zekerheid te voorspellen. Hierdoor wordt de meetuitkomst van b tot op zekere hoogte onvoorspelbaar (denk bijvoorbeeld aan de snelheid en de positie van een deeltje), waardoor het meetresultaat van b weinig informatie blootgeeft over de toestand van het systeem voor het meten van b. Waarom de natuur zo werkt is voor ons geen punt van onderzoek, maar een startpunt: we kiezen ervoor om alleen grootheden te meten die onderling commuteren. In de taal van C*-algebra's: we moeten kijken naar de verzameling $\mathcal{C}(A)$ van commutatieve C*-deelalgebra's van de C*-algebra A corresponderend met het systeem, want die corresponderen met de experimenteel toegankelijke informatie. Gegeven twee commutatieve C*-deelalgebra's D en E van A zeggen we dat $D < E$ als D een *echte* deelverzameling is van E, waarbij we met 'echt' bedoelen dat D niet gelijk aan E is. Het valt eenvoudig na te gaan dat dit een orderelatie op $\mathcal{C}(A)$ definieert, dus $\mathcal{C}(A)$ is een geordende verzameling. De orderelatie op $\mathcal{C}(A)$ is ook natuurkundig relevant, want het geeft aan dat E meer experimenteel toegankelijke informatie bevat dan D. Het blijkt dat $\mathcal{C}(A)$ en $\mathcal{C}(B)$ orde-isomorf zijn als A en B isomorf als C*-algebra's zijn, dus $\mathcal{C}(A)$ als geordende verzameling is een invariant voor C*-algebra's.

Het is van belang te beseffen dat we $\mathcal{C}(A)$ louter als geordende verzameling beschouwen. We 'vergeten' dus alle informatie die in een commutatieve C*-deelalgebra C van A is opgeslagen, en onthouden alleen dat C minder informatie bevat dan D indien de laatste een andere commutatieve C*-deelalgebra van A is zodat $C < D$. Opmerkelijk genoeg blijkt dat slechts deze kennis van de *hoeveelheid* informatie toch voldoende is om alle informatie die in de C*-deelalgebra's in $\mathcal{C}(A)$ opgeslagen te kunnen reconstrueren. Dit volgt uit de Stelling van Hamhalter die in Hoofdstuk 4 besproken wordt. Feitelijk gooien we dus niets weg als we $\mathcal{C}(A)$ louter als geordende verzameling beschouwen, waardoor het natuurkundig gezien gerechtvaardigd is om naar $\mathcal{C}(A)$ als invariant voor C*-algebra's te kijken.

Vervolgens willen we $\mathcal{C}(A)$ gebruiken om de structuur van C*-algebra's beter te gebruiken: we proberen C*-algebraische eigenschappen van een C*-algebra A te vertalen naar ordetheoretische eigenschappen van $\mathcal{C}(A)$, en we gebruiken dat om uit te vinden voor welke C*-algebra's A er geldt dat ze isomorf zijn met elke willekeurige C*-algebra B waarvoor geldt dat $\mathcal{C}(B)$ orde-isomorf is met $\mathcal{C}(A)$. In andere woorden, we proberen te achterhalen welke C*-algebra's volledig vastgelegd worden door hun commutatieve C*-deelalgebra's.

Curriculum Vitae

Bert Lindenhovius was born on October 20, 1984 in Den Helder, the Netherlands. In 2007 he obtained a Bachelor's degree in physics at the Free University, Amsterdam. In 2010, he received a Bachelor's degree in mathematics at the same university. In 2011, he obtained his Master's degree in mathematical physics at the University of Amsterdam under supervision of Prof. dr. R.H. Dijkgraaf. In the same year, he started his PhD research at Radboud University, Nijmegen under supervision of Prof. dr. N.P. Landsman. In August 2016, he will start as a postdoctoral researcher in the group of Prof. dr. M.W. Mislove at Tulane University, New Orleans.