Nuclear Physics B365 (1991) 121–160
North-Holland

NUCLEAR
PHYSICS B

THE GEOMETRY OF INEQUIVALENT QUANTIZATIONS

N.P. LANDSMAN* and N. LINDEN

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

Received 2 April 1991
(Revised 24 May 1991)

We describe how inequivalent quantizations (superselection sectors) arise within two related algebraic approaches to quantum mechanics (viz. quantization by canonical groups and by C*-algebras). By construction of the quantum hamiltonian and the path integral of a particle moving on a coset space, we show that the inequivalent quantizations manifest themselves as the particle coupling to a certain fictitious external gauge field, in a representation depending on the superselection sector; various well-known topologically non-trivial Yang–Mills field configurations emerge in this way. The general theory is illustrated by taking the coset space to be a circle and a sphere, which puts θ-angles (hence the Aharonov–Bohm effect) and the Dirac charge quantization condition, respectively, in a new light.

1. Introduction and preliminaries

In this paper we discuss topological quantum effects. The motivation was to understand how the inequivalent quantizations, which, in certain operatorial quantization schemes, may be seen to arise from abstract representation theory, manifest themselves in the physics of the systems under consideration.

Specifically, we analyse the example of the quantum mechanics of a particle moving on a topologically non-trivial configuration space – the quotient space G/H. In this section we describe and contrast two related algebraic approaches to the quantization of these systems: one in which the observables are related to a so-called canonical group [1, 2], and one encoding the observables into a transformation group C*-algebra [3, 4]. In both cases the usual replacement of a sub-alge-

* Supported by SERC.

bra of the Poisson algebra of functions on phase space by operators leads to the possibility of inequivalent quantizations, which may be identified with superselection sectors of the system. This situation is in stark contrast to the case of quantum mechanics on the real line, where the Stone–von Neumann theorem on the representations of the Heisenberg–Weyl group states that there is only one quantization up to unitary equivalence.

Our starting point, then, is the algebra of quantum observables. The inequivalent quantizations arise as unitarily inequivalent representations, which may be found by the induced representation theory of Mackey. This theory has an analytic as well as a geometric side, which we review in subsect. 1.1, as the relevant constructions and results are spread over the literature. The representations appear as Hilbert spaces of sections of certain vector bundles over the configuration space (the case of complex functions on the configuration space being but one example). In sect. 2 we use this representation theory to derive an expression, involving an integral over the subgroup H, for the propagator in any of these quantizations. This allows us to explain the origin of previous formulae for the propagator for particles moving on homogeneous spaces [5–7] which may be seen as a special case (the "trivially induced" case) of our work. We then evaluate the propagator for short times, which later on (in subsect. 2.3) allows us to construct the path integral via the Trotter product formula.

In sect. 3, we derive the same expression from a very different point of view – a generalised heat-kernel expansion. We show how the different quantizations may be viewed as the particle coupling to a specific external gauge field (the so-called H-connection) via a term $P\exp(-\int A)$ in the propagator (the gauge field is A and P denotes path-ordering). This term is slightly awkward since it does not arise simply as a term in the action (due to the path-ordering) but we show how it may be re-written (using auxiliary variables) so that the propagator is a conventional path-integral. A similar path-ordered exponential (as well as its absorption into the action) occurs in the context of the so-called Polyakov spin factor [8, 9], which, indeed, entirely fits into the framework of our formalism.

We end by describing specific examples of the general scheme: a particle moving on a 2-sphere, which is related to magnetic monopoles, and on a circle; here the relevance of our formalism to anyon statistics and its applications to high-T_c superconductivity and the quantum Hall effect should be mentioned.

Apart from a reformulation and explicit interrelation of the two quantization schemes mentioned, the key results of this paper are expression (3.6) for the hamiltonian in an arbitrary superselection sector ("quantization"), the corresponding expression (2.16) for the propagator, and the resulting path integral (3.28). We use several approaches and points of view, and include a number of consistency checks and corollaries. Since we employ a variety of (known) geometric and analytic techniques, we have included some background discussion on the mathematics used, in order to make the paper as readable as possible.

1.1. HOMOGENEOUS SPACES AND INDUCED REPRESENTATIONS

Notation. The physical system whose quantization we study in this paper is a particle moving on a homogeneous configuration space $Q = G/H$, alternatively called a coset space. We assume G to be a finite-dimensional Lie-group with Lie algebra $\mathfrak{g}$, and H a compact subgroup with Lie algebra $\mathfrak{h}$, such that Q is a smooth (i.e. C^∞) manifold. Convenient summaries of the relevant mathematics of such spaces are refs. [10–12], where further references to the literature may be found. Below we review the minimal amount of material necessary to understand what follows.

For notational simplicity we assume G to be unimodular (an assumption automatically satisfied by H due to its compactness), in which case both G and H possess a left- and right-invariant Haar measure, which we will simply denote by $\mathrm{d}x$ and $\mathrm{d}h$, respectively (elements of G will generically be called x or y, whereas h stands for an element of H). In that case Q has a unique measure (up to a multiplicative constant) $\mathrm{d}q$, which is invariant under the natural (left-) action of G on Q, under which $x \in G$ sends a coset $q = \{yH\}$ to $xq = \{xyH\}$ (chapter 4 of ref. [13]). Given a normalization of the Haar measures on G and H, the invariant measure is uniquely determined by requiring

$$\int_G \mathrm{d}x f(x) = \int_Q \mathrm{d}q \int_H \mathrm{d}h f(s(q)h) \tag{1.1}$$

for all $f \in C_c(G)$ (where $C_c(X)$ is the space of all continuous functions with compact support on a space X) and an arbitrary measurable section s: $Q \to G$, i.e. $p \circ s = \mathrm{id}$, where p: $G \to Q$ is the canonical projection $px = \{xH\}$. Denoting the coset $\{H\}$ by q_0, this means that $s(q)q_0 = q$. Our assumption of unimodularity is satisfied in practically all cases of interest (and certainly by the examples we shall consider), but may easily be lifted at the expense of introducing the appropriate Radon–Nikodym derivatives in the relevant formulae (cf. ref. [13]).

Since H is compact, it follows that the pair G, H is reductive [11], which means that the Lie algebra $\mathfrak{g}$ has a decomposition as a vector space $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, with $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$. We label a basis of generators of $\mathfrak{g}$ by T_a, $a = 1, \dots, d_G$, those of $\mathfrak{q}$ by T_α, $\alpha = 1, \dots, d_Q$, and those of $\mathfrak{h}$ by T_i, $i = 1, \dots, d_H$. The structure constants appearing in $[T_a, T_b] = C_{ab}{}^c T_c$ are real, so that in unitary representations of G the T_a are represented by skew-adjoint operators. Representations will be denoted by the symbol π.

Induced representations. An induced representation of a group G generalizes the left-regular representation on $L^2(G/H)$ (for the general functional-analytic theory see ref. [13]; for the geometric realization cf. ref. [14]; our formulation in terms of local sections follows ref. [4], also cf. ref. [1]. The general theory of fibre bundles employed in the geometric construction may be found in refs. [10, 15].)

Firstly, note that G is the total space of a principal fiber bundle $P=(G, Q, p, H)$ with base manifold $Q=G/H$ and group H, the projection $p: G \to Q$ being the canonical one defined above. The action of the "gauge group" H on G is simply given by right-multiplication. Now choose a unitary representation π_χ of H on a Hilbert space $\mathscr{H}_\chi$, which we assume to be irreducible (although this is not strictly necessary for the following construction). By compactness of H, $\mathscr{H}_\chi$ is of finite dimension d_χ. The symbol χ is regarded as an element of the dual $\hat{H}$, which, by definition, is the set of equivalence classes of irreducible unitary representations of H, and π_χ is a particular (and arbitrary) choice of a representative of the class χ. For example, the dual of SO(3) is $\mathbb{N}$, and χ may be identified with the spin j; the matrix $\pi_\chi(x)$ would be called $D^j(R)$ $(x \equiv R \in SO(3))$. One then forms the vector bundle E^χ associated to P; its base space is Q, and its fibers are isomorphic to $\mathscr{H}_\chi$, which is identified with the fiber $p^{-1}(q_0)$. Points of E^χ are equivalence classes $[x, \psi_\chi]$ $(\psi_\chi \in \mathscr{H}_\chi)$ under the equivalence relation $(x, \psi_\chi) \equiv (xh^{-1}, \pi_\chi(h)\psi_\chi)$ for all $h \in H$. Then G acts on E^χ by $y[x, \psi_\chi] = [yx, \psi_\chi]$, and therefore it also acts on Γ^χ (the space of smooth cross sections of E^χ with compact support) by means of

$$(\pi^\chi(x)\Psi^\chi)(q) = x\Psi^\chi(x^{-1}q). \tag{1.2}$$

The fibers being isomorphic to the Hilbert space $\mathscr{H}_\chi$, there is a natural hermitian structure $\langle \cdot, \cdot \rangle_{\mathscr{H}_\chi}$ in each of them, in terms of which one can define an inner product in Γ^χ by

$$(\Psi_1^\chi, \Psi_2^\chi) = \int_Q dq \langle \Psi_1^\chi(q), \Psi_2^\chi(q) \rangle_{\mathscr{H}_\chi}. \tag{1.3}$$

The Hilbert space $\mathscr{H}^\chi$ is the closure of Γ^χ in this inner product, with respect to which the representation π^χ is unitary.

The realization of an induced representation constructed above is useful for abstract, coordinate free geometric considerations. In practice it is convenient to realize the sections Ψ^χ in terms of local trivializations of the bundle. We cover Q with open sets $\{U_\alpha\}_{\alpha \in I}$ for some index set I, so that the U_α are homeomorphic to a euclidean space, and do this in such a way that we can locally define smooth sections s_α: $U_\alpha \to G$ (that is, $p \circ s_\alpha$ is the identity map). For $q \in U_{\alpha\beta} \equiv U_\alpha \cap U_\beta$, the sections s_α and s_β are then related by the gauge transformation

$$s_\beta(q) = s_\alpha(q) h_{\alpha\beta}(q), \tag{1.4}$$

where now $h_{\alpha\beta}$: $Q \to H$ is required to be smooth, and to be such that it satisfies the consistency (cocycle) condition $h_{\alpha\beta}(q) h_{\beta\gamma}(q) = h_{\alpha\gamma}(q)$ on $q \in U_\alpha \cap U_\beta \cap U_\gamma$.

Relative to such a local trivialization, a section Ψ^χ is represented by a collection of functions $\psi^\chi(\cdot, \alpha)$: $U_\alpha \to \mathscr{H}_\chi$, $\alpha \in I$, defined by $\psi^\chi(q, \alpha) = s_\alpha(q)^{-1}\Psi^\chi(q)$, where the right-hand side, which is an element of the fiber $p^{-1}(q_0)$, is identified with a vector in $\mathscr{H}_\chi$ by the rule $[e, \psi_\chi] \equiv \psi_\chi$ (so that $[h, \psi_\chi] \equiv \pi_\chi(h)\psi_\chi$). This implies that on an overlap region $U_{\alpha\beta}$, sections in different "gauges" α and β are related by the gauge transformation

$$\psi^\chi(q, \beta) = \pi_\chi\left(h_{\alpha\beta}(q)^{-1}\right)\psi^\chi(q, \alpha). \tag{1.5}$$

Conversely, the section Ψ^χ may be recovered from its local representatives by $\Psi^\chi(q) = [s_\alpha(q), \psi^\chi(q, \alpha)]$, which by the definition of the equivalence class $[\cdot, \cdot]$ and the above relation between sections in different gauges is independent of the choice of α.

We denote the whole collection of local sections $\{\psi^\chi(\cdot, \alpha)\}_{\alpha \in I}$ simply by ψ^χ. The definition of ψ^χ in terms of Ψ^χ easily implies that the group representation π^χ acts on the local sections according to

$$(\pi^\chi(x)\psi^\chi)(q, \alpha) = \pi_\chi\left((s_\alpha(q))^{-1} x s_\beta(x^{-1}q)\right)\psi^\chi(x^{-1}q, \beta). \tag{1.6}$$

Here it has been supposed that $q \in U_\alpha$ and $x^{-1}q \in U_\beta$; which particular U_β is selected in overlap regions is immaterial due to the compatibility conditions given above. (Strictly speaking, we should use a different symbol from the π^χ in (1.2) here.) The inner product (1.3) can be written as

$$(\psi_1^\chi, \psi_2^\chi) = \sum_{\alpha \in I} \int_{U_\alpha} \mathrm{d}q\, \rho_\alpha(q)\left(\psi_1^\chi(q, \alpha), \psi_2^\chi(q, \alpha)\right)_\chi < \infty, \tag{1.7}$$

in terms of a partition of unity* ρ.

We remark that when H is abelian the vector bundles E^χ are all line bundles, so that it should be clear that the construction above generalizes the usual bundle-theoretic approach to magnetic monopoles, cf. subsect. 5.1 below (also see ref. [16]).

Semi-direct products. For reasons to become clear shortly, we now apply this construction to a particular semi-direct product extension of G, namely

$$G_c = G \ltimes V. \tag{1.8}$$

Here V is assumed to be a finite-dimensional vector space, on which G acts ($x \in G$ sending a point $v \in V$ to $xv \in V$). Denoting elements of G_c by pairs (x, v), $x \in G$, $v \in V$, the multiplication law in G_c is given by $(x, v) \cdot (y, w) = (xy, v + xw)$.

* This is a collection of functions ρ_α: $Q \to \mathbb{R}$ such that the support of each ρ_α is in U_α, and $\sum_{\alpha \in I} \rho_\alpha(q) = 1$ for all q. The value of the inner product is not affected by the precise choice of ρ in view of the compatibility between different trivializations of ψ^χ in overlap regions, and the unitarity of π_χ.

The general representation theory of semi-direct products [13] teaches us that (at least in the so-called regular case) all irreducible representations of G_c may be found by inducing from subgroups $H_c \subset G_c$ given by $H_c = H \ltimes V$, where H is the stability group (little group) of an arbitrary point q_0 in a G-orbit Q in V^* (the dual of V, i.e. the set of continuous linear functionals on V). The action of G on V^* is given by the rule $(xw, v) = (w, x^{-1}v)$, where (w, v) is the value of the functional $w \in V^*$ on $v \in V$; if we identify V with V^* via the natural inner product in $V \simeq \mathbb{R}^n$ then (w, v) is evidently just the inner product of w and v. Hence $Q = \{xq_0 | x \in G\} \simeq G/H = G_c/H_c$. (Note that the action of G_c on Q is given by the action of G, i.e. V acts trivially.)

One then induces from a particular class of representations of H_c, namely those given by

$$\pi_{q_0, \chi}((h, v)) = e^{i(q_0, v)} \pi_\chi(h), \tag{1.9}$$

where π_χ is an irreducible representation of H, as above; recall that $q_0 \in Q \subset V^*$. Hence the carrier space of $\pi_{q_0, \chi}$ is simply $\mathscr{H}_\chi \otimes \mathbb{C} \simeq \mathscr{H}_\chi$.

The corresponding representation of G_c is then given by the previous construction, with the replacements $G \to G_c$, $H \to H_c$, and $\chi \to q_0, \chi$. By letting Q run over all G-orbits in V, and χ over $\hat{H}$, one thus obtains all irreducible representations of G_c. We now assume that Q and q_0 have been fixed, and replace the label q_0, χ simply by χ. The induced representation of G_c is accordingly simply called π^χ. To find a convenient realization in a Hilbert space of local sections, we use the family of sections s_α: $Q \to G$ employed before, and trivially extend it to a collection of sections s'_α: $Q \to G_c$ by $s'_\alpha(q) = (s_\alpha(q), 0)$.

With this choice, the Hilbert space carrying the representation $\pi^{q_0, \chi} \equiv \pi^\chi$ of G_c *coincides* with the space $\mathscr{H}^\chi$ carrying π^χ of G, and a short computation using eq. (1.6) shows that the representation itself is given by

$$(\pi^\chi((x, v))\psi^\chi)(q, \alpha) = e^{i(q, v)} \pi_\chi\Big((s_\alpha(q))^{-1} x s_\beta(x^{-1}q)\Big)\psi^\chi(x^{-1}q, \beta). \tag{1.10}$$

It then emerges that the representation $\pi^\chi(G)$ given before is just the restriction of $\pi^\chi(G_c)$ to G (identifying $x \in G$ with $(x, 0) \in G_c$). Whereas π^χ is irreducible as a representation of G_c, it is reducible as a representation of G.

This completes our account of induced representations. We will now use this formalism to quantize a particle whose configuration space is Q.

1.2. QUANTIZATION ON HOMOGENEOUS SPACES

The operatorial quantization of a particle moving on $Q = G/H$ goes back to Mackey [17], whose work was subsequently extended by various authors (cf. refs. [1,3,4,18,19] and references therein). Below we will motivate, examine and interrelate two similar quantization techniques, whose results form the basis of our later derivation of path integrals.

Quantization and reduction of the observables. We adhere to the conventional quantization programme, which is to replace (smooth) functions f, g on a phase space M ($=T^*Q$ in our case) by operators $\pi(f), \pi(g)$ in such a way that Poisson brackets are replaced by (i times) commutators, i.e. $\pi(\{f,g\}) = i[\pi(f), \pi(g)]$. Our convention here is that $\{f,g\} = -\omega(X_f, X_g)$, with $\omega = -\mathrm{d}\theta$ the canonical symplectic form on T^*Q expressed in terms of the Liouville form θ, and X_f the hamiltonian vector field defined by a function f [1,15].

It is well known (e.g. ref. [2]) that this cannot be done for all $f, g \in C^\infty(M)$ if the algebra $\pi(C^\infty(M))$ is to act irreducibly on $\mathscr{H}$, so that the next step in the quantization programme is to identify a subset $\mathscr{F} \in C^\infty(M)$ that can actually be "quantized". We follow refs. [1,2,20] in choosing the elementary functions that can be quantized *a priori* to be the (smooth) functions on configuration space $C^\infty(Q) \subset C^\infty(T^*Q)$ (identifying a function on Q with its pull-back to T^*Q), and the symbols of vector fields Y on Q (this choice can be justified by localization arguments). The symbol f_Y of Y is defined by its value at a point $m \in M$

$$f_Y(m) = \iota_Y m(\pi(m)) = \langle m, Y \rangle_{\pi(m)}, \tag{1.11}$$

that is, the value of the 1-form m on the vector $Y(\pi(m))$ (π is the projection of $M = T^*Q$ onto Q). In local coordinates, $f_Y(p,q) = Y^i(q)p_i$ if $Y = Y^i(q)\partial/\partial q^i$. It can be shown [1,2] that the hamiltonian vector field X_{f_Y} on M canonically related to f_Y (see above) generates diffeomorphisms on M which are the pullback of those on Q generated by Y, and in that sense the association of f_Y with Y given in eq. (1.11) is the natural one. The Poisson algebra generated by these observables is [1,2]

$$\begin{aligned} \{f_X, g\} &= Xg, \\ \{f_X, f_Y\} &= f_{[X,Y]}, \\ \{f, g\} &= 0, \end{aligned} \tag{1.12}$$

with $f, g \in C^\infty(Q)$, and f_X, f_Y as in (1.11). Here $[X,Y]$ is the commutator of the two vector fields on Q.

We now set out to quantize the subalgebra of the Poisson algebra $C^\infty(M)$ which is generated by the two types of functions g and f_Y specified above (since f_X is linear in p this is similar to choosing the vertical polarization in geometric quantization [2]). Denoting the quantum operator corresponding to g by $\pi(g)$, etc., the quantum algebra to be satisfied is obtained by replacing { , } in (1.12) by i[,], i.e.

$$i[\pi(f_X),\pi(g)] = \pi(Xg),$$

$$i[\pi(f_X),\pi(f_Y)] = \pi(f_{[X,Y]}),$$

$$[\pi(f),\pi(g)] = 0. \tag{1.13}$$

Beware of the different meaning of the commutator in the left-hand side (commutator of operators) and the right-hand side (Lie bracket or commutator of vector fields). By identifying the set of complete vector fields on Q with the Lie algebra of the diffeomorphism group Diff(Q) (up to a sign, cf. eq. (1.14) below), and the vector space $C^\infty(Q)$, regarded as an additive abelian group, with its Lie algebra, one sees [1,2] that (1.3) is the Lie algebra of the semi-direct product Diff(Q) $\ltimes$ $C^\infty(Q)$, with Diff(Q) acting on $C^\infty(Q)$ by $(\varphi f)(q) = f(\varphi^{-1}(q))$.

We now look at (1.13) as abstract algebraic relations between the observables f_X, f_Y, f, g, and at the quantization $\pi(\dots)$ as a particular representation of these abstract relations. The problem of finding all such representations can be completely solved if we cut down the number of basic observables to be quantized even further. This can be done satisfactorily if $Q = G/H$, and we accordingly specialize to this case in what follows.

We can exploit the fact that an element X of the Lie algebra $\mathfrak{g}$ of G, defines a complete vector field (called $\tilde{X}$) on Q defined by

$$(\tilde{X}f)(q) = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{e}^{-tX}q)\big|_{t=0}. \tag{1.14}$$

Since G acts transitively on Q, the vector fields corresponding to the entire set of generators T_a form a (generally over-complete) basis at any point of Q, so that one loses no essential information in restricting the system (1.13) to such vector fields. We write $f_X \equiv f_{\tilde{X}}$ in what follows.

To illustrate this procedure, and to show that (1.13) are essentially "canonical" commutation relations, one may take the example $G = Q = \mathbb{R}^n$. Choose X to be the generator $X_i = \partial/\partial q^i$ and f the coordinate function f^j (i.e. $f(q) = q^j$); then, in the notation $P_i = \pi(f_{X_i})$ and $Q^j = \pi(f^j)$, (1.13) says $i[P_i, Q^j] = \delta_i^j$, etc.

As we will make explicit for $n = 3$, there is a much more interesting way to write $Q = \mathbb{R}^n$ as a homogeneous space. This is to take $Q = \mathbb{R}^3 \simeq E(3)/SO(3) \equiv G/H$, where $E(3) = SO(3) \ltimes \mathbb{R}^3$ is the euclidean group. The commutation relations (1.13) then reproduce the usual commutators of position, momentum and angular momentum.

We now may identify an abstract algebraic structure corresponding to these commutation relations in two essentially different ways.

Quantization by transformation group C-algebras.* The first approach [3,4] seeks to retain the whole algebra $C^\infty(Q)$ (as well as the vector fields T_a), and succeeds in doing so by taking the abstract algebraic structure encoding the commutation relations (1.13) to be an operator algebra $\mathscr{A}$, i.e. an algebra of objects that in all respects behave like operators on a Hilbert space, but are not necessarily concretely represented as such. More specifically, the mathematical analysis is much simplified by taking $\mathscr{A}$ to be a C*-algebra, that is, an operator algebra with involution which is isomorphic to a norm-closed algebra of bounded operators on some Hilbert space (cf., e.g. the introductory lectures by Roberts [21], or a mathematical textbook like ref. [22]). In the spirit of quantum field theory [21] we call $\mathscr{A}$ the algebra of observables of the given system.

Once $\mathscr{A}$ has been identified, one can study the inequivalent representations of this algebra by (bounded) operators on concretely given Hilbert spaces. The inequivalent representations may alternatively be identified with the superselection sectors of the system, or with "inequivalent quantizations".

In the specific case of a particle moving on $Q = G/H$, we follow refs. [3,4] in taking $\mathscr{A}$ to be the so-called transformation group C*-algebra $C^*(G, Q)$. The point of this choice is that this algebra encodes the commutation relations (1.13) in the following sense: there is a bijective correspondence between (non-degenerate) representations $\pi(C^*(G,Q))$ and (integrable) representations π of the system (1.13) (supplemented by the anti-commutator of f, g), in which f_X corresponds to an element X of $\mathfrak{g}$ as explained above, and $f, g \in C^\infty(Q)$. (This result easily follows by specializing the results in ref. [3] to the smooth case, i.e. G and H as specified in the beginning of this section. In this context, the representation π is called integrable if the representation of the Lie algebra $\mathfrak{g}$ can be exponentiated to a unitary representation of G, cf. ref. [13].)

As detailed in ref. [3], the representation theory of the algebra $\mathscr{A} = C^*(G, Q)$, and therefore of the commutation relations (1.13) follows from Mackey's imprimitivity theorem [13,17], and is completely known. One finds that there is a one-to-one correspondence between (equivalence classes of) irreducible representations of $C^*(G,Q)$ (hence superselection sectors or inequivalent quantizations of the particle on Q) and elements of the dual $\hat{H}$ defined before. We call the representation of $C^*(G,Q)$ associated to $\chi \in \hat{H}$ by the name π^χ, and its carrier Hilbert space $\mathscr{H}^\chi$.

This notation is justified by the fact that the representation $\pi^\chi(\mathfrak{g})$ associated to $\pi^\chi(\mathscr{A})$ is obtained by taking the "derivative" of a unitary representation $\pi^\chi(G)$, which is precisely the induced representation, realized on the Hilbert space of sections $\mathscr{H}^\chi$, constructed in the previous subsection. More precisely, it can be shown [23] that the operators $\pi^\chi(f_X)$, $X \in \mathfrak{g}$, can be sensibly defined on the domain $\Gamma^\chi \subset \mathscr{H}^\chi$ of smooth compactly supported sections, and are given by

$$\pi^\chi(f_X) = -i\frac{\mathrm{d}}{\mathrm{d}t}\pi^\chi(\exp(tX))|_{t=0} \equiv -i\pi^\chi(X). \tag{1.15}$$

The commutation relations hold on this domain. Moreover, the functions in $C^\infty(Q)$ are simply represented by multiplication operators, i.e.

$$(\pi^\chi(f)\Psi^\chi)(q) = f(q)\Psi^\chi(q), \tag{1.16}$$

and similarly with Ψ^χ replaced by a local section ψ^χ_α.

The representation theory may be illustrated by the example $Q = \mathbb{R}^3$, realized as the coset E(3)/SO(3). As we have seen before, the dual of SO(3) is $\mathbb{N}$, so that the different quantizations of a particle moving in $\mathbb{R}^3$ are labeled by spin (to get half-integral spins in this context one has to replace E(3) by its covering group $SU(2) \ltimes \mathbb{R}^3$). The realization of π^j on a Hilbert space of cross-sections constructed in subsect. 1.1 is particularly straightforward here, as the base space $Q = \mathbb{R}^3$ is contractible, so that the bundle $P = (E(3), \mathbb{R}^3, p, SO(3))$ is trivial and admits a global smooth cross section s. The sections Ψ^j are accordingly simply functions on $\mathbb{R}^3$ with values in the spin j representation space $\mathscr{H}_j = \mathbb{C}^{2j+1}$, and $\mathscr{H}^j = L^2(\mathbb{R}^3, d^3q, \mathscr{H}_j)$. The simplest choice for s is $s(q) = (1, q)$, upon which (1.6) simplifies to

$$(\pi^j((R,a))\psi^j)_m(q) = D^j(R)^l_m \psi^j_l(R^{-1}(q-a)). \tag{1.17}$$

Canonical group quantization. A method (which will turn out to be closely related to the preceding one) of relating the commutation relations (1.13) (with the vector fields restricted to $\mathfrak{g}$) to a more conventional mathematical structure than a C*-algebra, has been formulated by Isham [1] (for applications cf. refs. [24–26]). Here the abstract algebraic object encoding the commutation relations is taken to be a group G_c (and the relevant representations are the (weakly) continuous unitary ones, see below), and in the case under consideration ($Q = G/H$) there is sufficient symmetry to choose G_c to be a finite-dimensional Lie group (in which case one actually may equivalently take the algebraic object to be a C*-algebra, namely the group algebra $C^*(G_c)$!; the (weakly) continuous unitary representations of a locally compact group G are in bijective correspondence with the non-degenerate representations of $C^*(G)$ [22]).

The first step in the canonical group quantization of a configuration space Q is to find a group $\tilde{G}$ which acts transitively on Q; in the case $Q = G/H$ we take $\tilde{G} = G$ (this choice is not mandatory; in fact the representation of a space Q as a coset G/H is non-unique, cf. ref. [3] for a discussion of this point). The problem, then, is to find a finite number of functions, which generate the algebra $C^\infty(Q)$, and which are chosen in such a way that the algebra (1.13) (or, equivalently, the Poisson brackets (1.12)) closes, for in that case the commutation relations define a finite-dimensional Lie algebra, which can be associated with the group G_c. This problem is equivalent to finding a finite generating set $\{g^i\} \subset C^\infty(Q)$ for which $\tilde{X}g^i$ (cf. (1.14)) is a linear combination of the other g's for all i and all $X \in \mathfrak{g}$.

This problem can be solved for a large class of homogeneous configuration spaces $Q = G/H$, which turns out to include many examples of interest (including vector spaces, which in ref. [1] are dealt with using quite different techniques). The idea [1, 2, 18] is to find a representation π_v of G on a vector space V such that there exists a G-orbit on V* (cf. text below (1.8)) which is diffeomorphic to Q. In detail, G acts on V* by the dual representation π_v^* defined by $(\pi_v^*(x)w, v) = (w, \pi_v(x^{-1})v)$, $w \in V^*$, $v \in V$, and one seeks a G-orbit $Q \subset V^*$ for which the action $\pi_v^*(x)q$ coincides with the action xq defined at the beginning of this section.

For example, the n-sphere S^n can be realized as the homogeneous space $S^n = SO(n+1)/SO(n)$, taking $V = \mathbb{R}^{n+1}$, with π_v the defining (fundamental) representation of $G = SO(n+1)$: the dual action on $V^* \simeq \mathbb{R}^{n+1}$ can be identified with the action of G on V, whose orbits are all n-spheres (except for the point 0).

Now choose a basis $\{e^i\}$, $i = 1, \dots, d_V$, in V, relative to which the generators T_a of $\mathfrak{g}$ have matrix elements $\pi_v(T_a)_i^{\ j} \equiv C_{ai}^{\ j}$. One can define the required functions g^i by $g^i(q) = (q, e^i)$ (remember that $q \in Q \subset V^*$). Thus g^i is the i-coordinate of the embedding of Q in V*. (Note that the g^i are not the coordinates of Q in a certain local coordinate system (such as the angle φ for $Q = S^1$), but the cartesian coordinates of the embedding of Q in V*, which are continuous functions on Q.) Defining $\tilde{T}_a$ by (1.14), it then easily follows that $\tilde{T}_a g^i = C_{aj}^{\ i} g^j$. With this choice, the algebra (1.12) becomes (writing $f_a \equiv f_{T_a}$)

$$\{f_a, g^i\} = C_{aj}^{\ i} g^j, \qquad \{f_a, f_b\} = C_{ab}^{\ c} f_c, \qquad \{g^i, g^j\} = 0, \tag{1.18}$$

where the $C_{ab}^{\ c}$ are the structure constants of G. This is isomorphic to the Lie algebra of the semidirect product (1.8), with respect to the action π_v of G on V. This G_c is taken to be the "canonical group", which encodes the commutation relations (1.18), which are a special case of the fundamental ones (1.13), in the following way (cf. the situation in the previous subsection for quantization using C*-algebras!): there is a bijective correspondence between the weakly continuous unitary representations of G_c, and the integrable representations of the system (1.18).

The simplest non-trivial example is the unit circle, $Q = S^1$. The canonical group is $G_c = SO(2) \ltimes \mathbb{R}^2$, with SO(2) acting on $\mathbb{R}^2$ in the usual way by rotation around the origin. The generator of SO(2) then corresponds to the angular momentum, whereas the generators of $\mathbb{R}^2$ are the position coordinates $x = \cos\varphi$, $y = \sin\varphi$ (expressed in polar coordinates).

The representation theory of G_c has already been discussed in subsect. 1.1. According to the general theory, irreducible representations of G_c correspond to G-orbits in V*. By assumption, one of these orbits "is" Q, and the representations of G_c based on this orbit are in turn labeled by the dual $\hat{H}$. Those representations $\pi^\chi(G_c)$ are realized on a Hilbert space $\mathscr{H}^\chi$ of sections of some vector bundle over Q, constructed in subsect. 1.1. Hence a particularly natural class of representations of the Poisson algebra (1.18) is obtained by setting

$$\pi^\chi(f_a) = -i\pi^\chi(T_a), \tag{1.19}$$

as in eq. (1.15), with $\pi^\chi(g^i)$ given by a multiplication operator, cf. eq. (1.16). These operators, then, satisfy the algebra (1.18), with { , } replaced by i[,], as required.

It is illuminating to see how a particle moving on $\mathrm{Q} = \mathbb{R}$ is quantized in this formalism: one takes $\mathrm{G} = \mathbb{R}$ and $\mathrm{V} = \mathbb{R}^2$, with the $\mathbb{R}$-action on $\mathbb{R}^2$ given by

$$a(p_1, p_2) = (p_1, ap_1 + p_2), \tag{1.20}$$

where $a \in \mathbb{R}$, and p_i are the usual cartesian coordinates in $\mathbb{R}^2$. The canonical group is accordingly $\mathrm{G_c} = \mathbb{R} \ltimes \mathbb{R}^2$, which turns out to be isomorphic to the Heisenberg–Weyl group of $\mathbb{R}$. The dual action on $\mathrm{V}^* \simeq \mathbb{R}^2$ (identified with extended configuration space) is then

$$a(x_1, x_2) = (x_1 - ax_2, x_2). \tag{1.21}$$

The G-orbits in V^* which are diffeomorphic to Q are given by $x_2 = \text{const.} \equiv c$, and the corresponding induced representations in the realization given in (1.10) are the usual Schrödinger representation of the canonical commutation relations with $c = \hbar$.

This result may be extended to $\mathrm{Q} = \mathbb{R}^n$ for any n; one takes $\mathrm{G} = \mathbb{R}^n$ and $\mathrm{V} = \mathbb{R}^{n+1}$, with a G-action and its dual given by setting $R = 1$ in the formulae below. One then finds a one-parameter family of orbits in V^* which are diffeomorphic to $\mathbb{R}^n$, and the parameter once again emerges as Planck's constant assuming different values in different (and inequivalent!) representations.

In the transformation group C*-algebra approach we have seen that spin arises on choosing G to be the euclidean group ($\mathrm{G} = \mathrm{E}(3)$ for $n = 3$); a similar construction can be performed here (and trivially generalizes to any n). For $\mathrm{Q} = \mathbb{R}^3$ we now choose $\mathrm{G} = \mathrm{E}(3)$ and $\mathrm{V} = \mathbb{R}^4$. Denoting elements of G by (R, a), $R \in \mathrm{SO}(3)$, $a \in \mathbb{R}^3$, the G-action on V is given by ($p \in \mathbb{R}^3$)

$$(R, a)(p, p_4) = (Rp, (a, Rp) + p_4), \tag{1.22}$$

which clearly generalizes eq. (1.20). The dual action on V^* follows as

$$(R, a)(x, x_4) = (Rx - ax_4, x_4). \tag{1.23}$$

There are three orbit-types, among which the family $x_4 = c$ ($c \neq 0$) consists of orbits diffemorphic (in face equal) to $\mathbb{R}^3$, which implies $\mathrm{H} = \mathrm{SO}(3)$. If one takes $q_0 = (0, c)$ and $c = 1$, the realization (1.10) of $\pi^j(\mathrm{G_c})$ restricted to $\mathrm{G} \subset \mathrm{G_c}$ precisely reproduces (1.17) (and taking $c \neq 1$ corresponds to setting $\hbar \neq 1$ in the relevant formulae). Also, the generators of $\mathbb{R}^3 \subset V \subset G_c$ are the usual cartesian position coordinates; the fourth generator of V equals the constant c.

Comparison of the two methods. Recalling the representation theory of the algebra of observables $\mathscr{A} = C^*(G, Q)$ discussed before, and comparing it with the one of the canonical group G_c, the following should now be clear: *The (equivalence classes of) irreducible representations of* $C^*(G, Q)$ *are identical to those representations of the canonical group* G_c *which correspond to* one *suitably chosen* G*-orbit in* V^*, *which is diffeomorphic to* Q. By "suitably chosen" we mean that the (dual) action of G on that particular orbit should coincide with the action of G on Q used in the construction of the C*-algebra $C^*(G, Q)$.

In those representations, the operators $\pi(g^i) \equiv q^i$ belong to the algebra $C^\infty(Q)$ (in its representation by multiplication operators on $\mathscr{H}^\chi$), and in fact generate it (in a suitable topological sense). Of course, the Lie algebra of G_c holds in any representation of G_c, but the close connection with the algebraic structure of $C^\infty(Q)$ and $C^*(G, Q)$ is only valid in those representations of G_c which are based on the suitably chosen orbit Q.

We see, then, that we may quantize a particle moving on $Q = G/H$ in two alternative ways. In the transformation group C*-algebra approach one declares the algebra of observables to be $\mathscr{A} = C^*(G, Q)$, and finds its inequivalent irreducible representations, identified with inequivalent quantizations, labeled by elements χ of the dual group $\hat{H}$, and realized on Hilbert spaces $\mathscr{H}^\chi$ of sections of certain vector bundles over Q; in the canonical group approach one starts with the canonical group G_c given by (1.8), and studies the inequivalent irreducible representations of this group. A number of such representations can, for all physical intents and purposes, be identified with the representations π^χ of $C^*(G, Q)$ on the Hilbert spaces $\mathscr{H}^\chi$, and may, accordingly, be identified with inequivalent quantizations of the system. It is these representations that we will investigate further in this paper.

The other representations of G_c arise because the canonical group method cuts down the function space $C^\infty(Q)$ in a rather arbitrary way, tailored to get a closed Poisson algebra, so that one loses information about the manifold Q (which is fully contained in the total space $C^\infty(Q)$). The potential relevance of the "other" representations is discussed in ref. [1].

2. Heat kernel and short-time propagator

2.1. TIME-EVOLUTION

Quantization of the hamiltonian. The conclusion of our discussion of quantization on a homogeneous space Q is that we end up with a Hilbert space $\mathscr{H}^\chi$ of sections of a certain vector bundle over Q; in what follows we shall use the explicit realization of this space in terms of the local sections ψ^{χ}, as explained below the definition (1.3). On this Hilbert space, then, the Poisson algebra (1.12) is represented by the operators given in (1.15) and (1.16) (which are essentially self-adjoint on the domain of compactly supported smooth sections Γ^{χ} [23]). In this way, we have quantized arbitrary (smooth) functions of q, and functions which are linear in p.

To quantize the hamiltonian we need to deal with functions that are quadratic in the momenta, which presents an infamous difficulty. One cannot, in general, put $\pi^{\chi}(f_X f_Y) = \pi^{\chi}(f_X)\pi^{\chi}(f_Y)$, since the left-hand side does not depend on the ordering of the f's, whereas the right-hand side does. To avoid this problem we specialize to hamiltonians of the type

$$H = \sum_a (f_{X_a})^2 + V(q), \tag{2.1}$$

where $\Sigma_a X_a^2$ is a Casimir operator of G (which is not necessarily a sum over a complete basis in $\mathfrak{g}$, cf. $\Sigma_{i=1}^3 P_i^2$ for $\mathrm{G} = \mathrm{E}(3)$). To apply our quantization method the potential V is *a priori* restricted to be smooth, but the method can presumably be extended to deal with arbitrary measurable potentials (barring problems involving self-adjointness). We can quantize H by putting

$$H^{\chi} \equiv \pi^{\chi}(H) = -\sum_a \pi^{\chi}(X_a)^2 + \pi^{\chi}(V), \tag{2.2}$$

which is ambiguous only up to a constant, as befits the energy. Note that for $\mathrm{Q} = \mathrm{G} = \mathbb{R}^3$, (2.2), with appropriate X_a, leads to the usual Schrödinger equation. The above discussion may be made more precise by involving the symbol calculus for differential operators on vector bundles, but we will not need such generality and abstraction. (Another approach to (2.2) is to start from a *-automorphism group on $\mathscr{A} = C^*(\mathrm{G}, \mathrm{Q})$, representing the time-evolution of the system, and showing that the hamiltonian H^{χ} actually implements this automorphism in the irreducible representation $\pi^{\chi}(\mathscr{A})$ [4, 16]. The hamiltonian is naturally defined up to a constant in that case, too.)

Heat kernel. We wish to find the imaginary-time propagator (heat kernel) for the hamiltonian H^{χ}. The heat kernel K^{χ} is defined by the propagation property

$$(\mathrm{e}^{-tH^{\chi}}\psi^{\chi})(q,\alpha) = \sum_{\beta \in \mathrm{I}} \int_{\mathrm{U}_\beta} \mathrm{d}q' \rho_\beta(q') K^{\chi}(q,\alpha;q',\beta;t)\psi^{\chi}(q',\beta), \tag{2.3}$$

which is to hold for all $\psi^{\chi} \in \mathscr{H}^{\chi}$. The notation is as follows: the labels α, β attached to q, q' refer to the sections s_α: U_α: $\mathrm{Q} \to \mathrm{G}$ (etc.) with respect to which the respective ψ^{χ}'s are defined, that is they specify the gauge in which K^{χ} is defined, cf. (1.4) etc.; also cf. (2.5) below (it is obviously assumed that $q \in \mathrm{U}_\alpha$ and

$q' \in \mathrm{U}_\beta$). Also, ρ is a partition of unity, cf. (1.7). Since $\psi^\chi(q,\alpha) \in \mathscr{H}_\chi$, $K^\chi(q,\alpha;q',\beta;t)$ is a $d_\chi \times d_\chi$ matrix; we suppress all matrix and vector indices; one may put $\psi^\chi = (\psi^\chi)^i e_i$, where the e_i form a basis of $\mathscr{H}_\chi$. In Dirac notation, the above definition would read (sum over j implicit)

$$\langle q,\alpha;i|\mathrm{e}^{-tH^\chi}|\psi^\chi\rangle = \sum_{\beta\in \mathrm{I}} \int_{\mathrm{U}_\beta} \mathrm{d}q' \rho_\beta(q') \langle q,\alpha,i|\mathrm{e}^{-tH^\chi}|q',\beta;j\rangle \langle q',\beta;j|\psi^\chi\rangle, \tag{2.4}$$

so that the heat kernel $(K^\chi(q,\alpha;q',\beta;t))_{ij} = \langle q,\alpha;i|\mathrm{e}^{-tH^\chi}|q',\beta;j\rangle$ is indeed the usual (imaginary-time) propagator.

Under a gauge transformation (1.4) the wave-function ψ^χ transforms according to eq. (1.5), which implies that

$$K^\chi(q,\gamma;q',\delta;t) = \pi_\chi\big(h_{\gamma\alpha}(q)\big) K^\chi(q,\alpha;q',\beta;t) \pi_\chi\big(h_{\beta\delta}(q')\big), \tag{2.5}$$

with $h_{\gamma\alpha} = h_{\alpha\gamma}^{-1}$, and the right-hand side is obviously a product of matrices.

Many analytic properties are known in case that H^χ is an elliptic partial differential operator, e.g. refs. [27,28]. For example, for Q compact K^χ is C^∞ in both q and q' for $t > 0$. In general, K^χ satisfies a parabolic PDE, which follows from the fact that the left-hand side of eq. (2.3) satisfies the PDE

$$(\partial_t + H^\chi)\psi^\chi(q,t) = 0$$

if ψ^χ is in the domain of H^χ (which is the case if $\psi^\chi \in \Gamma^\chi$). This implies

$$\big(\partial_t + H^\chi(\partial_q)\big) K^\chi(q,\alpha;q',\beta;t) = 0 \tag{2.6}$$

for $t > 0$; as indicated by the notation, H^χ acts on the first variable. The boundary condition for $t = 0$ follows from the definition (2.3):

$$K^\chi(q,q',0) = \delta(q,q'), \tag{2.7}$$

where the right-hand side is the invariant δ-function on Q, defined with respect to the measure $\mathrm{d}q$. The symmetry of H^χ implies

$$\overline{K^\chi(q,\alpha;q',\beta;t)} = K^\chi(q',\beta;q,\alpha;t). \tag{2.8}$$

Finally, let us assume that H^χ is elliptic; in that case it can be shown by elliptic regularity theorems that K^χ must be a smooth cross-section of E^χ as a function of q, with q' and $t > 0$ fixed [23].

2.2. MOTION ON $Q = G/H$ VS MOTION ON G

General formula. A key step in our approach is to relate the heat kernel on Q for the hamiltonian (2.2) to the heat kernel on G with respect to a very similar hamiltonian H_G, cf. eq. (2.12) below. Motion on a group manifold G is a special case of our formalism, contained by setting $Q = G$, hence $H = \{e\}$*. This gives only one representation of the quantum algebra, namely on the Hilbert space $\mathscr{H} = L^2(G)$ (defined with respect to a given left-invariant Haar measure on G). The (trivially) induced representation $\pi(G)$ is just the left-regular representation, defined by $(\pi(y)\psi)(x) = \psi(y^{-1}x)$, which is a special case of (1.6). The representation π playing the role of π^χ, the momenta and position variables are then quantized by eqs. (1.15) and (1.16).

The motivation for looking at $\mathscr{H}$ (which in the present context may be regarded as the Hilbert space completion of the space of smooth sections of the trivial line bundle over G) comes from the observation [4] that each $\mathscr{H}^\chi$ is unitarily equivalent to a subspace of $\mathscr{H}$. To see this, we first define the operator P_χ^j: $\mathscr{H} \to \mathscr{H}$ by

$$\left(P_\chi^j\psi\right)(x) = d_\chi \int_H \mathrm{d}h\, \pi_\chi(h)^{jj}\psi(xh) \tag{2.9}$$

(no sum over j), which is a projector, as follows from the orthogonality relations for compact groups [13]. Subsequently, define an operator V_χ^j: $\mathscr{H} \to \mathscr{H}^\chi$ by

$$\left(V_\chi^j\psi\right)(q, \alpha)^i = \sqrt{d_\chi} \int_H \mathrm{d}h\, \pi_\chi(h)^{ij}\psi(s_\alpha(q)h). \tag{2.10}$$

It can be checked (from the same orthogonality relations) that V_χ^j annihilates the orthogonal complement of $P_\chi^j\mathscr{H}$, and is unitary on $P_\chi^j\mathscr{H}$ itself; accordingly V_χ^j is a partial isometry. Its main virtue is that it intertwines the left-regular representation π on $\mathscr{H}$ and the induced representation π^χ on $\mathscr{H}^\chi$, as follows from (1.6) and (2.10):

$$V_\chi^j \circ \pi = \pi^\chi \circ V_\chi^j. \tag{2.11}$$

Note that V_χ^j indeed maps any $\psi \in \mathscr{H}$ into a well-defined section, that is, the image $V_\chi^j\psi$ automatically has the correct gauge transformation property (1.5). Since this works for any fixed $j = 1, \ldots, d_\chi$, we see that $\mathscr{H}$ contains d_χ copies of $\mathscr{H}^\chi$ (up to unitary equivalence).

* One may alternatively put $G = (G \times G)/H$, with H the diagonal subgroup of $G \times G$, but such generality is not needed here.

Now define, analogously to (2.2), the hamiltonian

$$H_{\mathrm{G}} = -\sum_a \pi(X_a)^2 + \tilde{V} \tag{2.12}$$

on $\mathscr{H}$, where $\tilde{V}$ is a multiplication operator given by $\tilde{V}(x) = V(xq_0)$. This definition is motivated by the property (following from (2.2), (2.11), and (2.12)) that

$$V_\chi^j \circ H_{\mathrm{G}} = H^\chi \circ V_\chi^j . \tag{2.13}$$

Let K_{G} be the heat kernel on G for the hamiltonian (2.12), i.e.

$$(\mathrm{e}^{-tH_{\mathrm{G}}}\psi)(x) = \int_{\mathrm{G}} \mathrm{d}x'\, K_{\mathrm{G}}(x, x', t)\psi(x') . \tag{2.14}$$

It easily follows from the fact that $\Sigma_a X_a^2$ is a Casimir operator, and from the definition of $\tilde{V}$, that H_{G} commutes with the right-regular representation π' of H on $\mathscr{H}$ (defined by $(\pi'(h)\psi)(x) = \psi(xh)$), which implies the invariance property

$$K_{\mathrm{G}}(xh, xh', t) = K_{\mathrm{G}}(x, x', t) \tag{2.15}$$

for all $h \in \mathrm{H}$.

We now use (2.13) to write $(\exp(-tH^\chi)V_\chi^j\psi)(q)$ in two different ways, one involving (2.3) and the other one using (2.14). Using (2.10) and (2.15), and the arbitrariness of $\psi \in \mathscr{H}$ we are led to the central result of this section, which relates the propagator in any superselection sector χ to the one on G:

$$K^\chi(q, \alpha; q', \beta; t) = \int_{\mathrm{H}} \mathrm{d}h\, \pi_\chi(h) K_{\mathrm{G}}\big(s_\alpha(q)h, s_\beta(q'), t\big), \tag{2.16}$$

where we assume (as usual) that $q \in \mathrm{U}_\alpha$ and $q' \in \mathrm{U}_\beta$. The correct gauge transformation property (2.5) is ensured by (2.15). This formula derives and generalizes an equation postulated by Dowker [5] for the trivially induced case $\chi = \mathrm{id}$. It is useful to rewrite eq. (2.16) as

$$K^\chi(q, \alpha; q', \beta; t) = (d_\chi)^{-1/2}\big(V_\chi^j K_{\mathrm{G}}\big(\cdot, s_\beta(q'), t\big)\big)(q), \tag{2.17}$$

where K_{G}, with its second argument and $t > 0$ fixed, is regarded as an element of $\mathscr{H}$. It then follows immediately from (2.13) that K^χ as given by eqs. (2.16) or (2.17) indeed satisfies the heat equation (2.6).

Riemann normal coordinates. Our next goal is to explicitly compute the H-integral in eq. (2.16) in the short-time limit of K_{G}, and to interpret the result geometrically. To do so, we exploit the existence of G-invariant metrics g, g^{Q}, and

g^{H} (and associated invariant measures dx, dq, and dh) on G, Q, and H, respectively [10, 29]. The structure of the heat kernel on a riemannian manifold is intimately related to the latter's geodesics, and this motivates us to attempt to compute (2.16) in Riemann normal coordinates (RNC) (cf., e.g. ref. [30]). The RNC on (a geodesically convex neighbourhood of the identity of) G are constructed via the exponential map of $\mathfrak{g}$ into G [11, 29]. This map is defined by $e^X = \gamma(1)$, where $\gamma(t)$ is the unique one-parameter group tangent to $X \in (\mathrm{TG})_e \simeq \mathfrak{g}$; γ is, in fact, a geodesic. The RNC of a point $\exp(x^a T_a) \in \mathrm{G}$ are simply x^a. Similarly, the RNC of a point $q = \exp(q^\alpha T_\alpha) q_0 \in \mathrm{Q}$ are just q^α, and those of $\exp(h^i T_i) \in \mathrm{H}$ are h^i. This construction exploits the fact that a curve $\exp(tX) q_0 \subset \mathrm{Q}$ is a geodesic if $X \in \mathfrak{q}$ [11, 12].

We now need to express the metrics g, g^{Q}, and g^{H} in terms of RNC. The main difficulty is in g^{Q}. For a given section s: $\mathrm{Q} \to \mathrm{G}$ we can construct a vielbein on Q by [12] $e_\alpha(q) = s(q)' T_\alpha$, where $T_\alpha \in \mathfrak{q}$ is regarded as an element of $(\mathrm{TQ})_{q_0}$, and x' is the derivative of the map x: $\mathrm{Q} \to \mathrm{Q}$ defined by the left-action of G on Q. If $g^{\mathrm{Q}}(T_\alpha, T_\beta) = \delta_{\alpha\beta}$ then by definition of g^{Q} we have that $g^{\mathrm{Q}}(e_\alpha, e_\beta)(q) = \delta_{\alpha\beta}$. To be concrete, in what follows we take s to be the canonical section s_c, defined in a neighbourhood of q_0 by

$$s_c\left(e^{q^\alpha T_\alpha} q_0\right) = e^{q^\alpha T_\alpha}, \tag{2.18}$$

where the left-hand side may be read as $s_c(q^\alpha)$ in RNC. The Baker–Campbell–Hausdorff-formula allows us to express the vectors $\partial/\partial q^\alpha$ in RNC in terms of the vielbein:

$$\frac{\partial}{\partial q^\alpha} = e_\alpha + \tfrac{1}{2} C_{\alpha\beta\gamma} q^\beta e_\gamma - \tfrac{1}{6} C_{\alpha\beta c} C_{\delta c\gamma} q^\beta q^\delta e_\gamma + \mathrm{O}(q^3). \tag{2.19}$$

Here Greek and Latin indices which are early in the alphabet are summed from 1 to d_{Q}, and 1 to d_{G}, respectively; below, we also use the convention that "middle" Latin indices are summed from 1 to d_{H} (in accordance with the conventions stated in subsect. 1.1). Also, we have lowered the third index on C with the euclidean metric g, with respect to which the generators T_a defining the C's are orthonormal. We can now compute $g^{\mathrm{Q}}_{\alpha\beta}(q)$, and therefore the determinant $g^{\mathrm{Q}}(q)$ in RNC, with the result

$$g^{\mathrm{Q}}(q) = 1 + \left(\tfrac{1}{4} C_{\alpha\gamma\epsilon} C_{\alpha\delta\epsilon} - \tfrac{1}{3} C_{\alpha\gamma c} C_{\delta c\alpha}\right) q^\gamma q^\delta + \mathrm{O}(q^3). \tag{2.20}$$

One similarly (and more easily) finds

$$g(x) = 1 + \left(\tfrac{1}{4} C_{adc} + \tfrac{1}{3} C_{cda}\right) C_{abc} x^b x^d + \mathrm{O}(x^3), \tag{2.21}$$

and

$$g^{\mathrm{H}}(h) = 1 - \tfrac{1}{12} C_{ikl} C_{jkl} h^i h^j + \mathrm{O}(h^3) . \tag{2.22}$$

2.3. SHORT-TIME HEAT KERNEL

Propagator on G. Although the above considerations, and in particular the expression (2.16), are valid for any $\mathrm{Q} = \mathrm{G}/\mathrm{H}$ for which H is compact, we can only compute K_{G} (following the literature) if the hamiltonian H_{G} reduces to (minus) the Laplace–Beltrami operator Δ_{LB} on G. To match this with the hamiltonian (2.12) we firstly assume that $\tilde{V} = 0$ (it is easy to restore the potential later on in the path integral by a Feynman–Kac-like formula, see below), and secondly take G to be compact*, with the Casimir operator defining (2.1) and (2.12) given by the usual $C_2(\mathrm{G}) = -\Sigma_a T_a^2$. Then it is well known that [12,29]

$$H_{\mathrm{G}} = \pi(C_2(\mathrm{G})) = -\Delta_{\mathrm{LB}} , \tag{2.23}$$

where Δ_{LB} is constructed from any left-invariant metric g on G (for concreteness' sake we may take the unique $\mathrm{G} \times \mathrm{G}$ invariant metric, which exists by compactness of G).

In general, the heat equation on a d-dimensional manifold M

$$(\partial_t - \Delta_{\mathrm{LB}}(\partial_x)) K(x, x', t) , \tag{2.24}$$

with boundary conditions analogous to (2.7, 2.8), can be solved by the asymptotic Hadamard–Minakshisundaram–DeWitt (HMD) expansion (cf. refs. [12, 28, 30] and references therein)

$$K(x, x', t) = (4\pi t)^{-d/2} \Delta^{1/2}(x, x') \mathrm{e}^{-\sigma(x, x')/2t} \sum_{n=0}^{\infty} a_n(x, x') t^n . \tag{2.25}$$

Here $\sigma(x, x')$ is one-half times the geodesic distance squared between x and x' (which are assumed to be joined by a unique geodesic, i.e. the validity of the solution (2.25) is restricted to the case in which x' lies in a geodesically convex neighbourhood of x). A simple reparametrization argument allows one to express this as

$$\sigma(x, x') = \tfrac{1}{2} \int_0^1 \mathrm{d}s \, g_{ab}(x(s)) \frac{\mathrm{d}x^a}{\mathrm{d}s} \frac{\mathrm{d}x^b}{\mathrm{d}s} , \tag{2.26}$$

where $x(s)$ is an affinely parametrized geodesic with $x(0) = x$ and $x(1) = x'$.

* This assumption may be relaxed, as long as the hamiltonians (2.23) and (3.1) below are elliptic, and have a reductive decomposition.

The second factor in the expansion (2.25) is the square-root of the so-called Van Vleck–Morette determinant, defined by

$$\Delta(x,x') = g(x)^{-1/2}\det\left[\frac{\partial\sigma(x,x')}{\partial x^a \partial x'^b}\right] g(x')^{-1/2}. \tag{2.27}$$

These expressions greatly simplify in RNC based at x' [30], in which $\sigma(x,x') = \frac{1}{2}x^a x^a$ and $\Delta(x,x') = (g(x)g(x'))^{-1/2}$.

We are now in a position to calculate the right-hand side of eq. (2.16). The hamiltonian H^χ defining the heat kernel K^χ is, by (2.1) (or (2.13) and (2.23)) just $H^\chi = \pi^\chi(C_2(\mathrm{G}))$. Since this operator is G-invariant (that is, it commutes with $\pi^\chi(\mathrm{G})$) it follows that for all $x \in \mathrm{G}$

$$K^\chi(xq,\alpha;xq',\beta;t) = K^\chi(q,\alpha;q',\beta;t) \tag{2.28}$$

(appropriately modified if xq and q do not lie in the same patch U_α, etc.). We accordingly need only compute K^χ for fixed $q' = q_0$. The situation further simplifies by working in the canonical section s_c defined in (2.18); note that $s_c(q_0) = e$. We also assume that q lies in the RNC patch of q_0 (which entails no loss of generality, as eventually we will be interested only in the case where q and q' are infinitesimally close), so that we compute K^χ in a single gauge $s_\alpha = s_\beta = s_c$.

As our motivation is mainly to derive a path-integral representation of the full heat kernel by "infinitely" many insertions of the short-time one, we only need K_G and K^χ in the short-time and short-distance limit. More precisely, it is sufficient to know the kernels up to $\mathrm{O}(t)$, where $\mathrm{O}((x-x')^2)$ is counted as $\mathrm{O}(t)$; hence in $\sigma(x,x')/2t$ we need σ to quartic accuracy in $x - x'$ [30, 31]. Analogous statements hold on Q. The exact expression for K_G is known for, e.g. semi-simple Lie groups [12] (and typically is gaussian), but we will not exploit this fact.

Computation. To do the computation, we first express

$$s_c(q)h = \exp(q^\alpha T_\alpha)\exp(h^i T_i)$$

as a single exponential $\exp(x^a(q,h)T_a)$ using the BCH formula, so that $\sigma(s_c(q)h, e) = \frac{1}{2}x^a x^a$, which we need to quartic accuracy in products of q^α and h^i. We then insert (2.25) and (2.27), and use the known [28, 30] result

$$a_0(x,x') + a_1(x,x')t = \mathrm{e}^{R(x)t/6} + \dots, \tag{2.29}$$

up to the desired accuracy, with the Ricci scalar on G being given by the constant $R = \frac{1}{4}C_{abc}C_{abc}$. Subsequently, eqs. (2.21) and (2.22) are used to approximate the Van Vleck–Morette determinant, and the measure $\mathrm{d}h$, respectively.

The only remaining complication is that because H is compact the H-integral is of the type $\int_{-a}^{a}\prod_i \mathrm{d}h^i \ldots$, where the a's are constants of order 1, and the dots include a gaussian weighting factor. However, elementary asymptotic analysis (for example, using Mellin transform techniques) reveals that $\int_1^\infty \mathrm{d}h \exp(-h^2/t) \leqslant ct^n$ for all $n \in \mathbb{N}$ and some constant c, so that to our desired accuracy we are justified in replacing the existing bounds on the h-integration interval by $\pm\infty$, that is, we may ignore the compactness of H, as well as the fact that the RNC patch may not coincide with all of H.

Doing the gaussian integral than leads to a result which, after some re-arrangement, can be expressed as

$$K^\chi(q,c;q_0,c;t) = (4\pi t)^{-d_{\mathrm{Q}}/2}\Delta^{1/2}(q,q_0)\exp\left(-\frac{\sigma(q,q_0)}{2t} + \frac{t}{6}R_{\mathrm{Q}} - tC_2^\chi\right) + \ldots, \tag{2.30}$$

where as explained before, the neglected terms are of $O(t^2)$ and $O(q^3)$. Here σ and Δ are defined as in (2.26) and (2.27); in fact, in RNC one simply has $\sigma(q,q_0) = \frac{1}{2}q^\alpha q^\alpha$. The Ricci scalar is defined with respect to the metric g^{Q} and is given by the constant

$$R_{\mathrm{Q}} = C_{i\beta\gamma}C_{\beta\gamma i}, \tag{2.31}$$

cf. ref. [12] (note that the f's explicitly depend on the metric, for they are defined with respect to an orthonormal set of generators). Finally, the constant C_2^χ is the Casimir operator for $\pi_\chi(\mathrm{H})$, that is,

$$C_2^\chi = -\sum_i \pi_\chi(T_i)^2, \tag{2.32}$$

where the $d_\chi \times d_\chi$ unit matrix is implicit in the left-hand side. The term involving C_2^χ in eq. (2.30) obviously comes from π_χ in eq. (2.16).

The result (2.30) has been derived in a special gauge and for a special point $q' = q_0$, but the general expression for $K^\chi(q,\alpha;q',\beta;t)$ can be derived from (2.28) and (2.5) with (1.4) and (2.18).

3. Path integrals and external Yang–Mills fields

3.1. HAMILTONIAN

H-connection. After the explicit calculation in sect. 2, we will now further our understanding of the heat kernel (2.30), and derive a more general expression for it that in principle allows one to compute the higher-order corrections in t. We continue to assume that G is compact. Our starting point is the following hamiltonian on $\mathscr{H}^\chi$ (cf. the text preceding eq. (2.28))

$$H^\chi = \pi^\chi(C_2(\mathrm{G})) = -\sum_a \left(\pi^\chi(T_a)\right)^2, \tag{3.1}$$

which is an elliptic PDO, which can in principle be computed from eqs. (1.6) and (1.15). That is, $\pi^\chi(T_a)^2 = (\mathrm{d}/\mathrm{d}t)(\mathrm{d}/\mathrm{d}s)\pi^\chi(\exp tT_a \exp sT_a)|_{t=s=0}$ on the domain $\Gamma^\chi \in \mathscr{H}^\chi$ of smooth sections of E^χ. Fortunately, we can give an explicit expression for H^χ as a Laplace–Beltrami operator in a certain background gauge field of a known type, namely the H-connection A^{H} (sect. II.11 of ref. [10]). This is the natural connection on the principal fiber bundle $\mathrm{P} = (\mathrm{G}, \mathrm{Q}, p, \mathrm{H})$ described in subsect. 1.1 (induced representations). A^{H} is just the part of the Maurer–Cartan form θ_{MC} on G which takes values in $\mathfrak{h}$, i.e. $A^{\mathrm{H}} = \theta^i T_i$, where θ_i are canonical left-invariant one-forms on G dual to the Lie algebra of $\mathrm{H} \subset \mathrm{G}$ [10, 12, 15].

It is remarkable that A^{H} is a solution of the Yang–Mills equation [32], which fact has been the main motivation for its appearance in theoretical physics so far (e.g. refs. [33–35]). In fact, some of the well known solutions such as the Dirac monopole field (for $\mathrm{Q} = \mathrm{S}^2 = \mathrm{SO}(3)/\mathrm{U}(1)$; cf. subsect. 4.1) and the BPST instanton (for $\mathrm{Q} = \mathrm{S}^4 = \mathrm{SO}(5)/\mathrm{SO}(4)$) are examples of the H-connection, cf. ref. [35].

Any section s_α of P defines a gauge field $A^{\mathrm{H}}_\alpha \equiv s^*_\alpha A^{\mathrm{H}}$ on Q with respect to the "gauge group" H. A remarkable property of this gauge field, which we record for later use, is

$$\int_{q_0}^{q} A^{\mathrm{H}}_{\mathrm{c}} = 0, \tag{3.2}$$

where $A^{\mathrm{H}}_{\mathrm{c}}$ is the H-connection in the canonical section (2.18); the line integral from q_0 to q is taken along a geodesic. This follows from the fact that geodesics through q_0 have the form $\gamma(t) = \exp(tq^\alpha T_\alpha)q_0$, so that $\langle s^*_{\mathrm{c}}A^{\mathrm{H}}, \mathrm{d}\gamma/\mathrm{d}t\rangle_{\exp(tq^\alpha T_\alpha)q_0} = \langle A^{\mathrm{H}}, q^\alpha L_\alpha\rangle_{\exp tq^\alpha T_\alpha} = 0$. In general, the integral of A^{H}_α along arbitrary geodesics on Q (that is, those not passing through q_0) does not vanish.

Explicit form of the hamiltonian. Employing the H-connection, now regarded as a $\pi_\chi(\mathfrak{h})$-valued connection $A^\chi \equiv \pi_\chi(A^{\mathrm{H}})$ on the vector bundle E^χ, and the metric g^{Q} on Q (cf. subsect. 2.2), we can form the gauge-covariant Laplace–Beltrami operator

$$\begin{aligned} \Delta_{\mathrm{LB}}(A^\chi) &= (\mathrm{d} + A^\chi)^*(\mathrm{d} + A^\chi) \\ &= \left(g^{\mathrm{Q}}\right)^{\mu\nu}\left(\nabla_\mu + A^\chi_\mu\right)\left(\partial_\nu + A^\chi_\nu\right), \end{aligned} \tag{3.3}$$

which acts on smooth sections of the bundle E^χ, and extends to an operator on $\mathscr{H}^\chi$ which is essentially self-adjoint precisely on the domain Γ^χ of smooth sections of E^χ [23]. In the first equation in (3.3), the covariant derivative $d + A^\chi$ is understood as a map from the space of sections of E^χ to the space of section-valued one-forms on Q, and $(d + A^\chi)^*$ (mapping the section-valued one-forms into the sections) is the adjoint of this map with respect to the inner product (derived from the metric g^Q) on the space of p-forms ($p = 0, 1$ in this case), cf. refs. [15, 36]; in the second form ∇ denotes the covariant derivative with respect to the metric connection. We omit the "gauge-dependence" index α on A^χ; by definition, the action of $\Delta_{LB}(A^\chi)$ on a local section is $\Delta_{LB}(A^\chi)\psi^\chi(q, \alpha) = \Delta_{LB}(A^\chi_\alpha(q))\psi^\chi(q, \alpha)$.

For compact Q, the operator $\Delta_{LB}(A^\chi)$ has discrete spectrum [36], so that $\mathscr{H}^\chi$ has a complete basis of eigenfunctions. It can be shown by elementary harmonic analysis (appendix B of ref. [12]) that the eigenfunctions $(\psi^\chi)^I_\gamma$ are labeled by $\gamma \in \hat{G}$ (the dual of G, i.e. the set of equivalence classes of unitary irreducible representations of G), and by a further degeneracy index I (by Frobenius reciprocity, only those γ occur for which $\pi_\gamma(H)$ contains $\pi_\chi(H)$). One then has from eq. B.71 of ref. [12]

$$\Delta_{LB}(A^\chi)(\psi^\chi)^I_\gamma = \left(-C_2^\gamma(G) + C_2^\chi\right)(\psi^\chi)^I_\gamma, \tag{3.4}$$

where $C_2^\gamma(G)$ is the value of the Casimir operator $C_2(G)$ in the irreducible representation $\pi_\gamma(G)$, and C_2^χ has already been defined in (2.32) as $C_2(H)$ in π_χ. Now use the fact that $(\psi^\chi)^I_\gamma$ transforms as an irreducible multiplet under $\pi^\chi(G)$, i.e.

$$\pi^\chi(x)(\psi^\chi)^I_\gamma = \pi_\gamma(x)(\psi^\chi)^I_\gamma \tag{3.5}$$

for all $x \in G$. Combining (3.1), (3.4) and (3.5) we thus find

$$H^\chi = -\Delta_{LB}(A^\chi) + C_2^\chi. \tag{3.6}$$

We see that one of the effects of being in a non-trivially induced representation (sector) is the appearance of a certain background Yang–Mills field in the hamiltonian.

3.2. GAUGE-COVARIANT HEAT EQUATION

Generalized HMD expansion. We now return to the heat equation (2.6), with H^χ given by (3.6) of subsect. 3.1. The ansatz (2.25) can be generalized to (cf. ref. [28])

$$K^\chi(q, \alpha; q', \beta; t) = \frac{\Delta^{1/2}(q, q')}{(4\pi t)^{d_Q/2}} e^{-\sigma(q,q')/2t} e^{-tC_2^\chi} I^\chi(q, \alpha; q', \beta) \sum_{n=0}^{\infty} \tilde{a}_n(q, q') t^n, \tag{3.7}$$

where the $d_\chi \times d_\chi$ matrix $I^\chi(q,\alpha;q',\beta)$ as well as the $\tilde{a}_n$ depend on A^χ. Inserting (3.7) into (2.6) with the hamiltonian (3.6), and use of the identities [28]

$$\tfrac{1}{2}\sigma^\mu\sigma_\mu = \sigma\,, \qquad \sigma^\mu\partial_\mu\Delta^{1/2} = \tfrac{1}{2}\big(d_{\mathrm{Q}} - \nabla^\mu\partial_\mu\sigma\big)\Delta^{1/2} \tag{3.8}$$

(with $\sigma_\mu \equiv \sigma_\mu(q,q') \equiv \partial\sigma(q,q')/\partial q^\mu$; we now use μ,ν rather than α,β to avoid confusion with the α on A^{H} denoting its gauge dependence, and to stress that we do not necessarily work in the RNC q^α on Q) shows that I^χ must satisfy

$$\sigma^\mu D_\mu I^\chi(q,\alpha;q',\beta) = 0\,, \tag{3.9}$$

where the gauge-covariant derivative is defined by its action

$$\big(D_\mu\psi^\chi\big)(q,\alpha) = \big(\partial_\mu + A^\chi_\alpha(q)_\mu\big)\psi^\chi(q,\alpha) \tag{3.10}$$

on any object in a gauge α. The initial condition

$$I^\chi(q,\alpha;q,\alpha) = 1 \tag{3.11}$$

is required in order to satisfy the boundary condition (2.7).

Since the vector $\sigma^\mu\partial_\mu \in (\mathrm{TQ})_q$ is tangent to the geodesic from q' to q, eq. (3.9) with (3.11) is solved by (cf. eq. (3.24) below for a more detailed argument)

$$I^\chi(q,\alpha;q',\alpha) = P\exp - \int_{q'}^{q} A^\chi_\alpha\,, \tag{3.12}$$

where, to start with, we have assumed that $q,q' \in \mathrm{U}_\alpha$. The integral is over the geodesic (assumed unique) from q' to q, and the symbol P path-orders the matrix-valued gauge field along the geodesic, with "later" gauge fields (whose argument is closer to the latest point q) standing to the left of "earlier" ones (those closer to the earliest point q'). If $q \in \mathrm{U}_\alpha$ and $q' \in \mathrm{U}_\beta$ then solution (3.12) has to be replaced by

$$\begin{aligned} I^\chi(q,\alpha;q',\beta) &= P\exp - \int_{q'}^{q} A^\chi_{\alpha\leftarrow\beta} \\ &\equiv P\left\{\left(\exp - \int_{q_1}^{q} A^\chi_\alpha\right)\prod_{i=1}^{n}\left[\pi_\chi\big(h_{\gamma_{i-1}\gamma_i}(q_i)\big)\exp - \int_{q_{i+1}}^{q_i} A^\chi_{\gamma_i}\right]\right\}, \end{aligned} \tag{3.13}$$

where $\gamma_0 = \alpha$, $\gamma_n = \beta$, and $q_{n+1} = q'$. The geodesic from q' to q is supposed to pass through the patches $\mathrm{U}_0 \equiv \mathrm{U}_\alpha, \mathrm{U}_1, \ldots, \mathrm{U}_n \equiv \mathrm{U}_\beta$, such that $q_i \in \mathrm{U}_{i-1} \cap \mathrm{U}_i$. The transformation property of A^χ under a gauge transformation [15] yields

$$P\exp-\int_{q_{i+1}}^{q_i} A^\chi_\beta = \pi_\chi\big(h_{\beta\alpha}(q_i)\big)P\left(\exp-\int_{q_{i+1}}^{q_i} A^\chi_\alpha\right)\pi_\chi\big(h_{\alpha\beta}(q_{i+1})\big), \quad (3.14)$$

which assures firstly that (3.13) is independent of the choice of the intermediate points q_i, and secondly that I^χ transforms exactly as K^χ, cf. eq. (2.5). Therefore, I^χ carries the full gauge dependence of K^χ, so that we conclude that the $\tilde{a}_n$ in (3.7) are gauge-invariant.

Consistency of the ansatz. One can covariantly differentiate eq. (3.9) an arbitrary number of times with respect to q, and then put $q=q'$. Use of the initial condition (3.11) and the known coincidence limits of the covariant derivatives of $\sigma(q,q')$ [28] then yields

$$(D_\mu I^\chi)(q,\alpha;q,\alpha)=0, \quad (3.15)$$

$$\big(\{\nabla_\mu+(A^\chi_\alpha)_\mu,\nabla_\nu+(A^\chi_\alpha)_\nu\}I^\chi\big)(q,\alpha;q,\alpha)=0, \quad (3.16)$$

etc.; here { , } is the anti-commutator. The second equation is equivalent to

$$(D_\mu D_\nu I^\chi)(q,\alpha;q,\alpha)=\tfrac{1}{2}(F^\chi_\alpha)_{\mu\nu}I^\chi(q,\alpha), \quad (3.17)$$

since

$$\left[\nabla_\mu+A^\chi_\mu,\nabla_\nu+A^\chi_\nu\right]=\left[D_\mu,D_\nu\right]=F^\chi_{\mu\nu}, \quad (3.18)$$

with D_μ given by (3.10).

It is not at all obvious that (3.12) satisfies (3.15) and (3.17), so we find it rather instructive to explicitly check this. We assume $q,q'\in \mathrm{U}_\alpha$ and write $A^\chi_\alpha=A$ in the following (which, in any case, is independent of the explicit form of the gauge field). By definition

$$\int_{q'}^{q}A=\int_0^1 \mathrm{d}\lambda\left\langle A,\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}\right\rangle_{\gamma(\lambda)}=\int_0^1\mathrm{d}\lambda\, A_\rho(\gamma(\lambda))\frac{\mathrm{d}\gamma^\rho}{\mathrm{d}\lambda}(\lambda). \quad (3.19)$$

Here γ: $[0,1]\to \mathrm{Q}$ is a geodesic with $\gamma(0)=q'$ and $\gamma(1)=q$. Since γ depends on its terminal points, we write $\gamma(q,q',\lambda)$. One has

$$\frac{\partial}{\partial q^\nu}\gamma^\rho(q,q',\lambda)=B^\rho_\nu(q,q',\lambda), \quad (3.20)$$

where $B(q,q',\lambda)$ is a Jacobi field, that is, a solution of the equation of geodesic deviation with boundary conditions

$$B^\rho_\nu(q,q',0)=0,\qquad B^\rho_\nu(q,q',1)=\delta^\rho_\nu, \tag{3.21}$$

cf. ref. [37] for a review of this notion (in particular, for a coordinate free version of (3.20) and (3.21)); also see ref. [15]. Hence

$$\frac{\partial}{\partial q^\nu}\left(A_\rho(\gamma(\lambda))\frac{\mathrm{d}\gamma^\rho}{\mathrm{d}\lambda}(\lambda)\right)=\partial_\sigma A_\rho(\gamma(\lambda))B^\sigma_\nu(\lambda)\frac{\mathrm{d}\gamma^\rho}{\mathrm{d}\lambda}(\lambda)+A_\rho(\gamma(\lambda))\frac{\mathrm{d}}{\mathrm{d}\lambda}B^\rho_\nu(\lambda), \tag{3.22}$$

where we have suppressed the q,q' dependence of B and γ.

Now use the formula (see ref. [38], p. 340, in a rather different notation)

$$\delta\left(P\exp-\int_{q'}^q A\right)=-\int_0^1\mathrm{d}\lambda\,P\exp\left(-\int_\lambda^1\mathrm{d}\lambda'\left\langle A,\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda'}\right\rangle_{\gamma(\lambda')}\right)$$
$$\times\,\delta\left(\left\langle A,\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda}\right\rangle_{\gamma(\lambda)}\right)P\exp\left(-\int_0^\lambda\mathrm{d}\lambda'\left\langle A,\frac{\mathrm{d}\gamma}{\mathrm{d}\lambda'}\right\rangle_{\gamma(\lambda')}\right) \tag{3.23}$$

for any variation δ. We take $\delta=\partial/\partial q^\nu$, use (3.19),(3.22), partially integrate the $\mathrm{d}/\mathrm{d}\lambda$ in the 2nd term in eq. (3.22), and use (3.21) to find

$$\left(\frac{\partial}{\partial q^\nu}+A_\nu(q)\right)P\exp\left(-\int_{q'}^q A\right)=-P\left\{\int_{q'}^q\iota_{B_\nu}F\exp\left(-\int_{q'}^q A\right)\right\}, \tag{3.24}$$

where F is the curvature (3.18), and B_ν is the vector field $B_\nu(\gamma(\lambda))=B^\rho_\nu\partial/\partial\gamma^\rho(\lambda)$, cf. eq. (3.20); our convention is $\iota_B F=B^\rho F_{\rho\sigma}\,\mathrm{d}q^\sigma$.

Since $\sigma^\nu B_\nu$ solves the geodesic deviation equation with initial value zero, and final value $\sigma^\nu\partial_\nu$, i.e. being tangent to the geodesic, so that there is no geodesic deviation, it must vanish. This shows, on contracting eq. (3.24) with σ_μ, that (3.12) indeed solves eq. (3.9). Also, (3.24) and (3.12) trivially imply (3.15).

To find $D_\mu D_\nu I^\chi$ we need the covariant derivative of the right-hand side of eq. (3.24). The only term that contributes in the limit $q'\to q$ is the one in which ∂_μ acts on $\iota_B F$, that is, in the coincidence limit we may ignore the exponential, as well as the A_μ in D_μ. We thus have to evaluate

$$\frac{\partial}{\partial q^\mu}\int_{q'}^q\iota_{B_\nu}F=\frac{\partial}{\partial q^\mu}\int_0^1\mathrm{d}\lambda\frac{\mathrm{d}\gamma^\rho}{\mathrm{d}\lambda}(q,q',\lambda)B^\sigma_\nu(q,q',\lambda)F_{\sigma\rho}(\gamma(\lambda)). \tag{3.25}$$

Because $\mathrm{d}\gamma(q,q,\lambda)/\mathrm{d}\lambda=0$ we only get a contribution in the coincidence limit if

∂_μ acts on the first term in the integrand. Clearly

$$\frac{\partial}{\partial q^\mu}\frac{\mathrm{d}\gamma^\rho}{\mathrm{d}\lambda}(q,q',\lambda)=\frac{\mathrm{d}}{\mathrm{d}\lambda}B^\rho_\mu(q,q',\lambda).$$

It can be shown directly from the geodesic deviation equation that [37] $B^\rho_\mu(q,q,\lambda)=\lambda\delta^\rho_\mu$. Using these equations in (3.25), the desired result (3.17) follows.

Expansion coefficients. Let us return to the expansion (3.7). To derive a path integral we need $\tilde{a}_0(q,q')$ to order $\sigma(q,q')$, whereas $\tilde{a}_1$, as it is in an $\mathrm{O}(t)$ term, is only needed in the coincidence limit, cf. ref. [31] and subsect. 2.3. The desired quantities can be found from the recursion relations satisfied by the $\tilde{a}_n$, which follow by inserting (3.7) with (3.12) (or (3.13)) in the heat equation (2.6), and by taking the coincidence limit in these relations, cf. refs. [28, 39]. The result is, that to the desired accuracy there is no A^χ dependence, so that one has

$$\tilde{a}_0+\tilde{a}_1 t=\mathrm{e}^{R_{\mathrm{Q}}t/6}+\dots, \tag{3.26}$$

cf. (2.29), with R_{Q} given in (2.31).

The results (3.7), (3.13) and (3.26) give the heat kernel in the short-time approximation, which we had already computed explicitly in subsect. 2.3 as (2.30) (in a special gauge, and for the special value $q'=q_0$). The two methods, totally different in nature, produce the same answer, as follows from (3.2).

3.3. PATH INTEGRAL

Time-slice path integral. We return to the full hamiltonian (2.2), which we write as $H^\chi=H_0^\chi+V^\chi$, with H_0^χ given by (3.1), rewritten as (3.6) with (3.3) (this H_0^χ was previously called H^χ, in the absence of a potential). The following time-slicing derivation of the path integral is valid in euclidean time t (the one used so far) if we assume that V is bounded from below (like H_0^χ), or in real time $t_{\mathrm{R}}=-it$ if $H_0^\chi+V^\chi$ is essentially self-adjoint on the domain of smooth sections Γ^χ. The last-mentioned condition is satisfied if V (and therefore V^χ, which is just V times the $d_\chi\times d_\chi$ unit matrix) is real-valued, and the sum of a square-integrable function (i.e. an element of $\mathscr{H}^\chi$) and a function which is essentially bounded with respect to the measure $\mathrm{d}q$ on Q (cf. theorem X.15 of ref. [40], which can be generalized to the present situation). Note that our $H_0^\chi=-\Delta_{\mathrm{LB}}(A^\chi)$, which includes a Yang–Mills field, is not the usual free hamiltonian $-\Delta_{\mathrm{LB}}$; the latter cannot be used in the Trotter formula (eq. (3.27)) since it fails to be essentially self-adjoint on Γ^χ (or any other reasonable domain).

We derive the path integral in the standard fashion* from the Trotter formula [44]

$$e^{-tH^\chi} = \operatorname*{s-lim}_{n\to\infty} \left(e^{-tH_0^\chi/n}\, e^{-tV^\chi/n}\right)^n \tag{3.27}$$

for $t \geqslant 0$; here s-lim indicates that the limit is to be understood in the strong operator topology. Using the heat kernel K^χ to $O(t) \simeq O((q-q')^2)$ [31], that is, substituting the expansion (3.7) with (2.26), (3.13) and (3.26) we obtain a path-integral representation for the heat kernel itself (in euclidean time) or the transition amplitude from q' to q in time $t_R = -it$ (in real time)

$$K^\chi(q,\alpha;q',\beta;t) = \int_{q_{q\leftarrow q'}} \mathscr{D}q\, e^{-S[q]}\, P\, e^{-\int_q A^\chi_{\alpha\leftarrow\beta}}, \tag{3.28}$$

where the integral is over all paths q for which $q(0)=q'$ and $q(t)=q$. The action is given by

$$S[q] = \int_0^t ds \left[\tfrac{1}{4} g^Q_{\mu\nu}(q(s)) \frac{dq^\mu}{ds}\frac{dq^\nu}{ds} + V(q(s)) - \tfrac{1}{6}R_Q + C_2^\chi\right], \tag{3.29}$$

which strictly speaking should include the $d_\chi \times d_\chi$ unit matrix (we have written q^μ for the μ-coordinate of $q(s)$). The full amplitude (3.28) is a $d_\chi \times d_\chi$ matrix, so that our path integral is not of the usual form, where the integrand is a single exponential of a scalar functional. Expressions of the type (3.28) have previously been used in refs. [45,46]. As we will explain later on, it is possible to get rid of the path-ordering by introducing extra variables.

The precise definition of the functional integral used here is

$$K^\chi(q,\alpha;q',\beta;t) = \operatorname*{s-lim}_{n\to\infty} \int_Q \prod_{i=1}^n \frac{dq_i\, \Delta^{1/2}(q_i,q_{i-1})}{(4\pi t/n)^{d_Q/2}}\, e^{-S[q_{i,i-1}]}\, P\, e^{-\int A^\chi_{\gamma_i\leftarrow\gamma_{i-1}}}, \tag{3.30}$$

where $q_0 = q'$, $q_n = q$, dq is the invariant measure on Q (cf. (1.1) and subsect. 2.2), $q_{i,i-1}$ is a geodesic from q_{i-1} to q_i, and $S[q_{i,i-1}]$ is defined as in (3.29), with 0 and t replaced by t_{i-1} and t_i, (if we assume that $q_{i,i-1}$ is affinely parametrized, with $q_{i,i-1}(t_{i-1}) = q_{i-1}$ and $q_{i,i-1}(t_i) = q_i$; here $t_i - t_{i-1} = t/n$). Furthermore, $A^\chi_{\gamma_i\leftarrow\gamma_{i-1}}$ is defined as in (3.13), assuming that $q_i \in U_i$, etc. The way the potential V appears in (3.28) is compatible with (3.27) if one notes that, to $O(t/n)$, one has $(t/n)\, V(q_{i-1}) = \int_{t_{i-1}}^{t_i} V(q(s))$ (at least for continuous V; more general potentials are handled as in the proof of theorem X.68 of ref. [40]). As a technical point, we

* For an overview of path-integral techniques cf. refs. [41–43], the last two being mathematically rigorous expositions.

remark that the limit $n \to \infty$ in (3.30) is in the strong operator topology, that is, we regard K^{χ} as an integral operator on $\mathscr{H}^{\chi}$ and take the limit on any fixed vector in $\mathscr{H}^{\chi}$.

One of the features of the definition (3.30) of the path integral is that the short-time propagator is given by the action evaluated on a "classical" path (a geodesic) connecting the endpoints (rather than by some mid- or end-point evaluation of the lagrangian). This procedure is further explained in ref. [47] (where only the trivially induced case $\chi = \mathrm{id}$ is considered).

Finally, as explained in ref. [30] (also cf. ref. [31]), one can omit the Van Vleck–Morette determinant $\Delta^{1/2}$ in the measure in (3.30) if one replaces $R_Q/6$ in the action (2.29) by $R_Q/3$; in fact, one may use an arbitrary power of Δ in the measure if the coefficient of the Ricci scalar is appropriately modified.

Wiener measure approach. If we stick to euclidean time t, it is possible to pursue a different approach to the functional integral, which is not based on time-slicing. One can exploit the fact that $H_0^{\mathrm{id}} = -\Delta_{\mathrm{LB}}$ defines a (conditional) Wiener measure $\mu_W^{(c)}$ on the space $\mathscr{P}_Q = C(\mathbb{R}^+, Q)$ of continuous functions on $\mathbb{R}^+$ with values in Q (that is, continuous paths in Q with a given starting (and ending) point), see, e.g. refs. [42, 50]. In terms of the Wiener measure, one can write a generalized (from $\chi = \mathrm{id}$ to general χ) Feynman–Kac formula

$$K^{\chi}(q, \alpha; q', \beta; t) = \int_{\mathscr{P}_Q} \mathrm{d}\mu_W^c(q) \mathrm{e}^{-\int_0^t \mathrm{d}s\, V(q(s))}\, P\, \mathrm{e}^{-\int_q A^{\chi}_{\alpha \leftarrow \beta}}\, \mathrm{e}^{(R_Q/6 - C_2^{\chi})t}, \quad (3.31)$$

where μ_W^c is the conditional Wiener measure concentrated on paths starting at q' and ending at q. In this case, the line integral of A^{χ} is a stochastic integral of a type investigated in refs. [51, 52]; the gauge field appears as a "vertical drift" affecting free diffusion on Q. The precise definition of the stochastic line integral of the gauge field is the following obvious generalization of the flat-space expression given in ref. [51]

$$P\, \mathrm{e}^{-\int_q A} = \underset{n \to \infty}{\text{s-lim}}\, \bar{\Pi}_{i=1}^{n}\, \mathrm{e}^{-\frac{1}{2}[A_\mu(q_i)\sigma^\mu(q_i, q_{i-1}) + A_\mu(q_{i-1})\hat{\sigma}^\mu(q_i, q_{i-1})]}, \quad (3.32)$$

where, in the interest of notational clarity, we have omitted the gauge dependence of $A \equiv A^{\chi}$; if q and q' lie in different patches then expression (3.32) has to be appropriately modified in the manner of (3.13). The expression above is a functional of the brownian path q, in which we have selected points $q_i = q(t_i)$, where $t_i = it/n$ (do not confuse this i with $\sqrt{-1}$). The product $\bar{\Pi}$ is ordered along the path q, i.e. the terms are in the order $i = n, n-1, \dots, i = 1$ from left to right. The vector σ_μ has been defined below (3.8), and [28] $\hat{\sigma}_\mu(q, q') = -\partial\sigma(q, q')/\partial(q')^\mu$ (which equals $\sigma^\mu(q, q')$ in RNC, or in cartesian coordinates in flat space-time).

This approach provides an interesting example of brownian motion on vector bundles [50]; moreover, it has the advantage that (3.31) is a bona fide integral (as opposed to (3.30), which is the limit of a sequence of integrals, but not itself an integral). On the other hand, one no longer recognizes the action in (3.31).

Gauge-invariant formalism. The heat kernel K^χ is explicitly gauge-dependent, cf. eq. (2.5). We can, however, perform a unitary transformation on $\mathscr{H}^\chi$ which leads to state vectors and a modified heat kernel which are invariant under all "based" gauge transformations. A based gauge transformations is one which maps a normalized section (i.e. $s(q_0)=e$) into a normalized section; this implies that the transformation function must satisfy $h_{\alpha\beta}(q_0)=e$, cf. eq. (1.4). The unitary transformation in question is given by

$$(U\psi^\chi)(q,\alpha) \equiv \psi^\chi_{\mathrm{GI}}(q) = \left(P\,\mathrm{e}^{-\int_{q_0}^{q} A^\chi_{\alpha\leftarrow\alpha_0}}\right)^{-1}\psi^\chi(q,\alpha)\,, \tag{3.33}$$

where the integral is along a geodesic (assumed unique) joining $q_0 \in \mathrm{U}_{\alpha_0}$ and q. It follows from (1.5) and (3.14) that ψ^χ_{GI} is independent of the choice of α, and is indeed invariant under based gauge transformations (these Mandelstam-type wave functions have, of course, often been used in the past, cf. ref. [46] and references therein). (This construction does not, in fact, depend on the use of geodesics, so that it is easy to amend the formalism in case that q and q_0 do not lie in a single RNC patch; one just has to give a unique prescription for a path joining q_0 to any point in Q.)

Let U map $\mathscr{H}^\chi$ into $\mathscr{H}^\chi_{\mathrm{GI}}$. One may define a propagation kernel K^χ_{GI} in $\mathscr{H}^\chi_{\mathrm{GI}}$ analogously to (2.3), and clearly $K^\chi_{\mathrm{GI}} = UK^\chi U^{-1}$ (where K^χ is regarded as an integral operator on $\mathscr{H}^\chi$). Then, going through the same derivation as before (or performing the unitary transformation above), it follows that K^χ_{GI} has a path-integral representation similar to (3.28); the only difference is that $P\exp -\int_q A^\chi_{\alpha\leftarrow\beta}$ has to be replaced by $P_{q_0}\exp -\int_\Delta A^\chi_{\alpha_0\leftarrow\alpha\leftarrow\beta\leftarrow\alpha_0}$, where Δ is an ordered triangular path $q_0 \to q' \to q \to q_0$, and the suffix q_0 on P indicates that the ordering along this path starts at q_0. In the regularized expression (3.30) one accordingly has to replace the geodesic path $q_{i,i-1}$, along which A^χ is integrated, by an ordered triangle $q_0 \to q_{i-1} \to q_i \to q_0$, all sides being geodesics.

It follows from the transformation properties (3.14) that this factor, and thereby the whole path integral, is indeed invariant under based gauge transformations; this fact can be made more explicit by invoking the non-abelian Stokes theorem [53], which converts $P_{q_0}\exp -\int_\Delta A$ into a rather complicated ordered surface integral of the curvature F.

In this approach, gauge dependence is traded for path (and q_0) dependence, which in our application seems reasonable because the point q_0 and the set of geodesics on Q are indeed preferred objects.

Removal of the path ordering. It is possible to get rid of the path ordering in the functional integral (3.28), at the expense of introducing extra, unphysical variables. The starting observation is that one can introduce d_χ creation and annihilation operators $a_i^{(*)}$, either fermionic or bosonic, with vacuum $|0\rangle$, in terms of which

$$(P\,\mathrm{e}^{-\int_{q'}^{q} A})_{ij} = \lim_{\varepsilon \downarrow 0} \langle 0|T\Big[a_i(t)\mathrm{e}^{-\int_0^t \mathrm{d}s\, a_k^*(s)\langle A_{kl}(q(s)), \mathrm{d}q/\mathrm{d}s\rangle a_l(s)}\, a_j^*(0)\Big]|0\rangle, \quad (3.34)$$

where $A_{kl} = (A^\chi)^a \pi_\chi(T_a)_{kl}$ (we omit reference to the gauge dependence of A for simplicity), which is assumed to be traceless. The time-evolution of the oscillator operators is governed by the hamiltonian $H = \varepsilon a_k^* a_k$; we need to go through this regularization, for without any time dependence of the operators the time-ordering T is ambiguous. To prove (3.34), which is a generalization of a similar expression for a Wilson loop given by Polyakov [8] (also cf. ref. [48]), expand the exponential and use Wick's theorem. The propagator is $\lim_{\varepsilon \downarrow 0}\langle 0|T[a_k(s_1)a_l^*(s_2)]|0\rangle = \delta_{kl}\theta(s_1 - s_2)$; many contractions will vanish because they produce traces of A, or else conflicting step-functions, and the surviving ones precisely yield the correct terms occurring in the expansion of the left-hand side.

Using a coherent-state formalism [49], the right-hand side of (3.34) can be written as a path integral, yielding

$$(P\,\mathrm{e}^{-\int_{q'}^{q} A})_{ij} = \lim_{\varepsilon \downarrow 0} \frac{\delta}{\delta \bar{J}_i(t)} \frac{\delta}{\delta J_j(0)} \int_{z_0 = 0} \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2\pi i} \mathrm{e}^{-\bar{S}[z,\bar{z},A,J,\bar{J}]}\Big|_{J=\bar{J}=0}, \quad (3.35)$$

with the action

$$\bar{S}[z,\bar{z},A,J,\bar{J}] =$$

$$\int_0^t \mathrm{d}s \left\{ -\frac{1}{2}\left[\frac{\mathrm{d}\bar{z}_k}{\mathrm{d}s} z_k - \bar{z}_k \frac{\mathrm{d}z_k}{\mathrm{d}s}\right] + \bar{z}_k \left\langle A_{kl}(q(s)), \frac{\mathrm{d}q}{\mathrm{d}s} \right\rangle z_l \right.$$

$$\left. + \varepsilon \bar{z}_k z_k - \bar{J}_k(s) z_k - \bar{z}_k J_k(s) \right\}, \quad (3.36)$$

with $z_k = z_k(s)$, etc. The boundary conditions $z_k(0) = \bar{z}_k(0) = z_k(t) = \bar{z}_k(t) = 0$ on the paths over which the functional integral extends follow immediately in the coherent-state formalism [49], where the ground state is just the coherent state $|0\rangle = |z = 0\rangle$. This works for both bosons and fermions (for which the factor $2\pi i$ in the measure is to be omitted). Note that the $J, \bar{J}$ differentiation cannot be interchanged with the path integration: if one insists on differentiating and subsequently putting the sources to zero first, an additional regularization is necessary, in which the factors $z(t)$ and $\bar{z}(0)$ are to be replaced by $\lim_{\delta \to 0} z(t - \delta)$

and $\lim_{\delta\to 0}\bar{z}(\delta)$, respectively, the limit to be taken after performing the path integral.

Using (3.35) and (3.36) we can now write (3.28) as a path integral with a single action

$$K^{\chi}(q,\alpha;q',\beta;t)_{ij}=\lim_{\epsilon\downarrow 0}\frac{\delta}{\delta\bar{J}_i(t)}\frac{\delta}{\delta J_j(0)}$$

$$\times\int_{q_q\leftarrow q'}\mathscr{D}q\int_{z_0\leftarrow 0}\frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{2\pi i}\,\mathrm{e}^{-S[q]-\tilde{S}[z,\bar{z},A,J,\bar{J}]}\Big|_{J=\bar{J}=0}, \quad (3.37)$$

where S is the action (3.29).

For the above manipulations to be justified it is necessary to adopt the time-slicing definition (3.30) of the path integral. A similar procedure may be followed for the gauge-invariant path integral. It is not clear to us, however, how the Wiener integral (3.31) with (3.32) can be cast into the un-path-ordered framework: the continuum path integral (3.35) seems useless here, whereas a discretized functional integral analogous to the one given in ref. [48] would rely on the approximation $\exp(T_{i,i-1})\simeq 1+T_{i,i-1}$ (where $T_{i,i-1}$ stands for the generic argument of the exponentials in (3.32)), which does not hold with respect to the Wiener measure.

4. Examples

4.1. PARTICLE ON A SPHERE

Set-up. To illustrate our abstract considerations in a concrete example, we now consider a particle moving on the two-sphere $\mathrm{Q}=\mathrm{S}^2$; as we shall see, the superselection sectors of this system can be interpreted in terms of a fictitious Dirac magnetic monopole, in whose field the particle moves [16]. The well-known description of a monopole in terms of line bundles [54, 55] then follows straightforwardly from our general formalism.

We write $\mathrm{S}^2=\mathrm{SO}(3)/\mathrm{SO}(2)$, i.e. we take $\mathrm{G}=\mathrm{SO}(3)$ and $\mathrm{H}=\mathrm{SO}(2)=\mathrm{U}(1)$. The Lie algebra of SO(3) is written, as usual, as $[T_a,T_b]=\epsilon_{abc}T_c$. In the operator-algebraic description, the algebra of observables is $\mathscr{A}=\mathrm{C}^*(\mathrm{SO}(3),\mathrm{S}^2)$; as explained in subsect. 1.2, the irreducible representations π^n of this algebra are labeled by $\hat{\mathrm{H}}=\widehat{\mathrm{SO}(2)}=\mathbb{Z}$ (the dual of SO(2) is $\mathbb{Z}$, because the irreducible unitary representations of SO(2) are all one-dimensional and given by $\pi_n(\theta)=\exp(in\theta)$, $\theta\in[0,2\pi)$, $n\in\mathbb{Z}$. Accordingly, $\mathscr{H}_n=\mathbb{C}$).

In the canonical group method, we embed S^2 in $\mathbb{R}^3$ in the usual way as the unit sphere, so that $\mathrm{G}_{\mathrm{c}}=\mathrm{SO}(3)\ltimes\mathbb{R}^3\equiv\mathrm{E}(3)$ [1]. The functions $g^i(q)$ in (1.18) are accordingly just the three cartesian coordinates of a point $q \in S^2$, that is, $g^1(\varphi, \theta) = \sin\theta\cos\varphi$, etc., where φ and θ are spherical coordinates on S^2. The irreducible unitary representations of E(3) correspond to SO(3)-orbits in $\mathbb{R}^3$, which are of two different types: one is the origin, all the other ones are two-spheres with radius $r > 0$. Among the latter we select the one with $r = 1$ and obtain a family of representations labeled by the unitary irreducible representations of its stability group H = SO(2). As detailed in sect. 1, these representations π^n exactly correspond to those of the algebra of observables $\mathscr{A}$ defined previously.

Following the prescription in subsect. 1.1, we can give an explicit realization of the representation π^n of $\mathscr{A}$ or E(3) on a Hilbert space of sections (also cf. refs. [1,16]). To start with, the principal bundle $P = (SO(3), S^2, p, SO(2))$ is defined by the usual topology on the total space SO(3) and the base space S^2, and the projection p: $SO(3) \to S^2$. The latter is defined by choosing the special point $q_0 \in S^2$ to be $(0,0,1)$ (the embedding of S^2 in $\mathbb{R}^3$ being understood), i.e. $px = xq_0$, $x \in SO(3)$. Hence H = SO(2) is identified with the subgroup of rotations around the z-axis. In terms of the parametrization of SO(3) by Euler angles [56] the projection is explicitly given by $pR(\alpha, \beta, \gamma) = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta)$. Thus the bundle P is just the Hopf fibration of S^3 over S^2 (up to a discrete factor $\mathbb{Z}_2$), which already suggests a connection with magnetic monopole theory [33].

The associated vector bundles $E^\chi \equiv E^n$ over S^2 have one-dimensional fibers $\mathscr{H}_n = \mathbb{C}$, and are therefore line bundles. Their structure is well known in physics [54,55], and the realization of the induced representation π^n on the Hilbert space $\mathscr{H}^n$ of L^2-sections of E^n has been detailed in ref. [16]. Suffice it to remark here that the representatives $i\pi^n(T_a) \equiv J_a^n$ of the generators of SO(3) are exactly the angular momentum operators of a charged particle moving in the field of a U(1) magnetic monopole of (automatically quantized) charge $eg = n$, which are essentially self-adjoint on the domain Γ^n of smooth sections of E^n.

H-connection. We will now demonstrate by straightforward computation that the H-connection on the bundle P is indeed the magnetic monopole field (as was already shown in a different way in refs. [34,35]). We work in the canonical section s_c (2.18). This is found by noting that

$$(\varphi, \theta) = e^{-\theta\sin\varphi T_1 + \theta\cos\varphi T_2} q_0, \tag{4.1}$$

so that the RNC on S^2 are given by $q^1(\varphi, \theta) = -\theta\sin\varphi$ and $q^2(\varphi, \theta) = \theta\cos\varphi$. By eqs. (2.18) and (4.1), the canonical section is given by

$$s_c(\varphi, \theta) = e^{-\theta\sin\varphi T_1 + \theta\cos\varphi T_2} = e^{\varphi T_3} e^{\theta T_2} e^{-\varphi T_3}; \tag{4.2}$$

the second equality expresses s_c in Euler angles, and shows that s_c coincides with the section (gauge) s_+ used in refs. [16,55]. In RNC on SO(3) we obviously have $s_c(\varphi, \theta) = (-\theta\sin\varphi, \theta\cos\varphi, 0)$.

To find the H-connection A^{H} on P we will first give a few formulae which are valid for any coset space G/H. One may express the vector field $\partial/\partial x^a$, where x^a are RNC on G, in terms of the basis of left-invariant vector fields L_a [10, 12, 15] at any point x (which lies in the RNC patch of e). It follows from problem III.4 of ref. [15] that

$$\frac{\partial}{\partial x^a} = M(x)^b{}_a L_b(x), \tag{4.3}$$

with

$$M(x)^b{}_a = \left(\frac{1 - \mathrm{e}^{-\pi_{\mathrm{ad}}(x)}}{\pi_{\mathrm{ad}}(x)}\right)^b{}_a, \tag{4.4}$$

where π_{ad} is the adjoint representation of G on its Lie algebra $\mathfrak{g}$.

Specializing to G = SO(3), the matrix M is known to be (cf. problem III.5(8) of ref. [15])

$$M(x)^b{}_a = \delta^b_a \frac{\sin x}{x} + \frac{x^b x_a}{x^2}\left(1 - \frac{\sin x}{x}\right) - \frac{\epsilon_{ac}^{\ \ b} x^c}{x^2}(1 - \cos x), \tag{4.5}$$

where $\epsilon_{ac}^{\ \ b} = \epsilon_{acb}$, $x = \sqrt{x^a x^a}$. The H-connection $\theta^3 \mathrm{T}_3$ on $P = (SO(3), S^2, p, SO(2))$ then follows in the RNC x as

$$A^{\mathrm{H}}(x) = T_3 M^3{}_a \,\mathrm{d}x^a. \tag{4.6}$$

In the gauge s_{c} we thus have, using eqs. (4.2), (4.5) and (4.6)

$$A^{\mathrm{H}}_{\mathrm{c}}(\varphi, \theta) \equiv \left(s^*_{\mathrm{c}} A^{\mathrm{H}}\right)(\varphi, \theta) = T_3(1 - \cos\theta)\,\mathrm{d}\varphi, \tag{4.7}$$

which is indeed the usual Dirac monopole field (in this gauge); in the literature one finds the expression $A_\varphi = (1 - \cos\theta)/\sin\theta$, which is taken with respect to the orthonormal basis one-form $\omega^\varphi = \sin\theta\,\mathrm{d}\varphi$ [55]. Note that (4.7) indeed satisfies (3.2), as geodesics emerging from $q_0 = (0, 0, 1)$ have $\varphi = \text{const}$. The connection A^{H} on the principal bundle P defines the connections A^n on the associated vector bundles E^n: since $\pi_n(T_3) = in$ we have

$$A^n_c(\varphi, \theta) = \pi_n\left(A^{\mathrm{H}}_{\mathrm{c}}\right) = in(1 - \cos\theta)\,\mathrm{d}\varphi, \tag{4.8}$$

which is a monopole field with charge $eg = n$; this charge is automatically quantized as a consequence of the representation theory of the algebra of observables.

Path integral. Since $C_2^n = n^2$, the hamiltonian (3.6) in the representation (superselection sector) π^n and the gauge s_c is

$$H^n = -(\nabla + A_c^n)^2 + n^2, \tag{4.9}$$

which is the hamiltonian of a charged particle moving in a monopole field with $eg = n$ [55].

The path integral (3.28) can be written down after the following (trivial) remarks. The Van Vleck–Morette determinant Δ (cf. (2.27)) being non-trivial already for S^2, it is fruitful to follow the procedure sketched after (3.30), i.e. to omit $\Delta^{1/2}$ in the measure in (3.30), and replace 1/6 by 1/3 in the action (3.28) [30]. Note that by eq. (2.19) the Ricci scalar on $Q = S^2$ is $R_Q = 2$ (which is indeed correct for the unit sphere!). Finally, the canonical metric g^Q on S^2 is the usual one (that is, $ds^2 = d\theta^2 + (\sin\theta)^2 d\varphi^2$), and the path integral (3.28) follows. Note that the path-ordering symbol can be omitted, as the gauge field is abelian; accordingly, the term with the gauge field can be included in the lagrangian. The fact that this term is as involved as (3.13) (specialized to the abelian case) was previously noted in ref. [57], also cf. ref. [58].

In the trivially induced case $n = 0$, which is ordinary quantum mechanics on the two-sphere, the path integral (now without an external field) can be evaluated exactly [7, 41]; it is a challenge to extend this result (which holds for motion on any split-rank symmetric space, and any semi-simple Lie group [12]) to the non-trivially induced case $n \neq 0$ (or, generally, $\chi \neq 0$).

4.2. PARTICLE ON A CIRCLE

Representations of the algebra of observables. The quantum mechanics of a particle moving on a circle has extensively been studied in the literature (cf. refs. [1, 41] and references therein), and is often cited as a prime example of a system admitting inequivalent quantizations, whose emergence is usually related to the multiple connectedness of its configuration space $Q = S^1 \equiv S$. In our approach (also cf. refs. [1, 16]) this topological feature is buried in the group- and representation-theoretic aspects of the situation, although both the traditional approach and our method eventually relate the various quantizations to the representations of $\mathbb{Z}$, which arises as $\pi_1(S)$ and H, respectively.

We take $G = \mathbb{R}$ and $H = \mathbb{Z}$ (so that $S = \mathbb{R}/\mathbb{Z}$); accordingly, the action of $\mathbb{R}$ on S is given by $\varphi \to \varphi + 2\pi x \bmod 2\pi$, $\varphi \in [0, 2\pi) \equiv S$, $x \in \mathbb{R}$. In the C*-algebraic approach (cf. subsect. 1.2) we quantize the system by taking the algebra of observables to be $\mathscr{A} = C^*(\mathbb{R}, S)$. In the canonical group method we embed S in $\mathbb{R}^2 \equiv \mathbb{C}$ as the unit circle, and take $G_c = \mathbb{R} \ltimes \mathbb{C}$, where $x \in \mathbb{R}$ acts on $z \in \mathbb{C}$ by mapping it to $\exp(2\pi i x)z$.

As we have seen in subsect. 1.1, the irreducible representations of $C^*(G,H)$ are classified by $\hat{H}$, which in the present case is $\hat{\mathbb{Z}} = SO(2)$ (namely, $\theta \in SO(2)$ defines the unitary irreducible representation π_θ of $\mathbb{Z}$ by $\pi_\theta(n) = \exp(in\theta)$). The unitary irreducible representations of G_c fall into two classes: the first consisting of only one representation, based on the $\mathbb{R}$-orbit $0 \in \mathbb{C}$, and the second consisting of the representations based on the circles S_r with radius $r > 0$. We take $r = 1$ (the relevance of the other S_r is discussed in ref. [1]), and by the theory reviewed in sect. 1 the representations π^θ of G_c based on this orbit are essentially the same as those of the C*-algebra $\mathscr{A}$.

By the procedure explained in subsect. 1.1 we thus get a collection of representations π^θ of $\mathscr{A}$ and G_c, $\theta \in [0, 2\pi)$, realized on the Hilbert space $\mathscr{H}^\theta$ of sections of a line bundle E^θ over S. As explained in ref. [16], the momentum operator $p^\theta = -i\pi^\theta(T)$ (cf. (1.15)), where T is the generator of $\mathbb{R}$, is given by $-i\,\mathrm{d}/\mathrm{d}\varphi$, which is defined and essentially self-adjoint on the domain Γ^θ of smooth sections of E^θ (which, if regarded as a subspace of the Hilbert space $L^2[0, 2\pi]$, consists of the C^∞ functions ψ^θ satisfying the twisted boundary condition

$$\psi^\theta(2\pi) = \exp(-i\theta)\psi^\theta(0)).$$

However, we are not really in the geometric situation of subsect. 3.1, as the little group $H = \mathbb{Z}$ is discrete, so that there are no vertical and horizontal vector fields, that is, connections, on the bundles E^θ. Fortunately enough, we can use almost the entire theory developed in sect. 3, in particular the expression (3.6) for the hamiltonian, if, following ref. [16], we perform a unitary transformation, which maps $\mathscr{H}^\theta$ to $L^2[0, 2\pi]$, and bijectively maps the domain $\Gamma^\theta \subset \mathscr{H}^\theta$ into $C^\infty(S)$, which consists of those functions in $C^\infty([0, 2\pi])$ which, together with all their derivatives, are periodic. In this realization one has

$$H^\theta = \left(-i\frac{\mathrm{d}}{\mathrm{d}\varphi} - \vartheta\right)^2, \tag{4.10}$$

(with $\vartheta = \theta/2\pi$) which is essentially self-adjoint on $C^\infty(S)$ [16]. This is indeed a special case of (3.6), with the gauge field

$$A^\theta = -i\vartheta\,\mathrm{d}\varphi = \pi_\theta\left(\frac{-T_1}{2\pi}\,\mathrm{d}\varphi\right), \tag{4.11}$$

where $T_1 = i$ is, in a formal sense, the (anti-hermitian) generator of $\mathbb{Z}$ (so that $\pi_\theta(n) = \exp(n\pi_\theta T_1)$). This gauge field is precisely the one leading to the Aharonov–Bohm effect; it is also associated with anyon statistics, which is conjectured to play a role in high-T_c superconductivity and the fractional quantum Hall effect [59]. We wish to point out that the alternative description of a particle with anyon statistics as a particle with normal statistics moving in an external Aharonov–Bohm gauge field is an immediate consequence of our formalism.

Heat kernel. The heat equation (2.6), with $H^{\chi} = H^{\theta}$ given by (4.10) is solved exactly by

$$K^{\theta}(\varphi, \varphi', t) = \frac{e^{-t\vartheta^2}}{2\pi}\theta_3\left(\frac{\varphi - \varphi' - 2i\vartheta t}{2\pi}, \frac{it}{\pi}\right), \tag{4.12}$$

in terms of the Jacobi theta function

$$\theta_3(z, \tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n z + \pi i \tau n^2}, \tag{4.13}$$

defined for $\operatorname{Im} \tau > 0$. Using the Poisson summation formula, this can be rewritten as (cf. e.g. ref. [41])

$$K^{\theta}(\varphi, \varphi', t) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} e^{-(\varphi - \varphi' + 2\pi n)^2/4t} e^{i\vartheta(\varphi - \varphi' + 2\pi n)}, \tag{4.14}$$

which is manifestly periodic in φ and φ', and therefore satisfies the boundary condition discussed below eq. (2.8), which in the present case requires that K^{θ} is in $C^{\infty}(S) \times C^{\infty}(S)$ for $t > 0$ (recall that we have "untwisted" the bundle E^{θ} by the unitary transformation mentioned earlier in this section).

The only complication in recognizing that (4.14) for $t \to 0$ coincides with the asymptotic expansion (3.7) (with (3.12) and (4.11)) comes from the fact that the coordinates φ and φ' used in (4.14) are discontinuous at $0, 2\pi$. Therefore, points in S which are actually nearby may be described by coordinates φ and φ' which differ by $\pm 2\pi$. Accordingly, for $t \to 0$ the only contribution in the sum comes from the $n = 0$ term if $\varphi - \varphi' = O(\sqrt{t})$, and from the $n = \pm 1$ term if $\varphi - \varphi' = O(\sqrt{t}) \mp 2\pi$. In any case, the surviving term is just $\sigma(\varphi, \varphi')/2t$ in the first exponential (cf. (2.26) and preceding text for the definition of σ), and $i\vartheta\, d(\varphi, \varphi')$ in the second one, where the signed distance d is obtained by taking the value of n in $d_n(\varphi, \varphi') = \varphi - \varphi' + 2\pi n$ for which $\sigma(\varphi, \varphi') = (\varphi - \varphi' + 2\pi n)^2/2$. This is well defined whenever φ and φ' can be joined by a unique geodesic; the set of points for which this is not the case is of measure zero in the regularized path integral (3.30), and may be ignored.

Path integral. We conclude that we are exactly in the situation described in subsect. 3.2, so that we can write down the path integral (3.28), with the gauge field A given by (4.11), the path-ordering being absent, and the constants R_{Q} and C_2^{χ}

both vanishing. We can absorb the gauge field into the action, and thus obtain

$$K^{\theta}(\varphi,\varphi',t)=\int_{\Phi_{\varphi\leftarrow\varphi'}}\mathscr{D}\Phi\,\mathrm{e}^{-S[\Phi]},\tag{4.15}$$

with the action

$$S[\Phi]=\int_0^t \mathrm{d}s\left[\tfrac{1}{4}(\mathrm{d}\varphi/\mathrm{d}s)^2+V(\Phi(s))+i\vartheta\,\mathrm{d}\varphi/\mathrm{d}s\right],\tag{4.16}$$

where for clarity $\mathrm{d}\varphi/\mathrm{d}s\equiv\mathrm{d}\Phi(s)/\mathrm{d}s$.

This result, in which ϑ labels a superselection sector, or, (in this context) equivalently, an induced representation, may be compared with the conventional way the theta angle is introduced into the path integral [41]. To do so, we make the standard observation that $\int_0^t \mathrm{d}s\,(\mathrm{d}\phi/\mathrm{d}s)=d(\varphi,\varphi')+2\pi n$, where n is the winding number of the path, i.e. the number of times it wraps around the circle. This fact is not completely trivial, as the path Φ is generically non-differentiable, but is easily seen to be true in the time-slice definition (3.30) (by virtue of the specific definition of a stochastic integral, it even holds in the Wiener measure definition of the path integral (3.31)). We may, therefore, rewrite (4.15) as

$$K^{\theta}(\varphi,\varphi',t)=\mathrm{e}^{i\vartheta\,d(\varphi,\varphi')}\sum_{n=-\infty}^{\infty}\mathrm{e}^{in\theta}\int_{(\Phi_{\varphi\leftarrow\varphi'})_n}\mathscr{D}\Phi\,\mathrm{e}^{-S_0[\Phi]},\tag{4.17}$$

where, as indicated by our notation, the path integral occurring in the nth term in the sum is over paths with winding number n only. The action S_0 is just (4.16) but without the ϑ-term.

The form (4.17) of the path integral is exactly the one found in the literature [41], where it is derived in a very different way. Our derivation shows that (4.17) is a special case of (3.28): the θ-dependence of the path integral ultimately arises because the irreducible representations of the algebra of observables of a particle moving on a circle are labeled by a theta-angle [16]. The conventional approach obtains the θ-dependence from the representation theory of the first homotopy group $\pi_1(\mathrm{S})=\mathbb{Z}$, and in the present case this avenue leads to the same result as ours. However, it has recently been claimed [60] (and seems to be folklore anyway), on the basis of an extrapolation of the conventional approach to more general configuration spaces than the circle, that *only* multiply connected configuration spaces admit inequivalent quantizations. The results of refs. [1, 4, 16] as well as of this paper show that such a statement is false: we suggest that inequivalent quantizations correspond to inequivalent representations of the algebra of observables of the system (whether this algebra is a "C*-algebra", or the universal enveloping algebra of a "canonical group", or some other mathematical object is a secondary matter), and there are countless examples of simply connected configuration spaces for which such inequivalent representations exist (the two-sphere

The authors wish to thank S. Albeverio, A. Dancer, C. Isham, K. Fredenhagen, H. Osborn; and P. Tuckey for discussions.

References

[1] C.J. Isham, *in* Relativity, groups and topology 2, Les Houches, 1983, ed. B.S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984)
[2] V. Guillemin and S. Sternberg, Symplectic techniques in physics (Cambridge Univ. Press, Cambridge, 1984)
[3] N.P. Landsman, Lett. Math. Phys. 20 (1990) 11
[4] N.P. Landsman, Rev. Math. Phys. 2 (1991) 45
[5] J.S. Dowker, J. Phys. A5 (1972) 936
[6] M. Böhm and G. Junker, J. Math. Phys. 28 (1987) 1978; 30 (1989) 1195
[7] C. Grosche and F. Steiner, Z. Phys. C36 (1987) 699 and references therein
[8] A.M. Polyakov, *in* Fields, strings and critical phenomena, Les Houches, 1988, ed. E. Brézin and J. Zinn-Justin (North-Holland, Amsterdam, 1990)
[9] P. Batenburg and A.M.J. Schakel, Mod. Phys. Lett. A4 (1989) 1355;
J. Grundberg, T.H. Hansson and A. Karlhede, Nucl. Phys. B347 (1990) 420
[10] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. 1 (Wiley, New York, 1963)
[11] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Vol. 2 (Wiley, New York, 1969)
[12] R. Camporesi, Phys. Rep. 196 (1990) 1
[13] A.O. Barut and R. Raczka, Theory of group representations and applications (PWN, Warsaw, 1977)
[14] N. Wallach, Harmonic analysis on homogeneous spaces (Dekker, New York, 1973)
[15] Y. Choquet-Bruhat, C. DeWitt-Morette and M. Deillard-Bleick, Analysis, manifolds and physics, 2nd edition (North-Holland, Amsterdam, 1982)
[16] N.P. Landsman, Rev. Math. Phys. 2 (1991) 73
[17] G. Mackey, Induced representations (Benjamin, New York, 1968)
[18] H. Doebner and J. Tolar, J. Math. Phys. 16 (1975) 975
[19] G. Giavarini and E. Onorfi, Int. J. Mod. Phys. A5 (1990) 4311
[20] A. Ashtekar, Commun. Math. Phys. 71 (1980) 59
[21] D. Kastler, ed., The algebraic theory of superselection sectors: Introduction and recent examples (World Scientific, Singapore, 1990)
[22] G. Pedersen, C*-algebras and their automorphism groups (Academic Press, New York, 1979)
[23] N.P. Landsman, in preparation
[24] C.J. Isham and N. Linden, Class. Quant. Grav. 4 (1987) 1333
[25] R. Loll, Nucl. Phys. B350 (1991) 831
[26] N. Linden and M.J. Perry, Nucl. Phys. B357 (1991) 289
[27] E.B. Davies, Heat kernels and spectral theory (Cambridge Univ. Press, Cambridge, 1989)
[28] S.A. Fulling, Aspects of quantum field theory in curved spacetime (Cambridge Univ. Press, Cambridge, 1989)
[29] S. Helgason, Groups and geometric analysis (Academic Press, Orlando, 1984)
[30] L. Parker, *in* Recent developments in gravitation, ed. M. Lévy and S. Deser (Plenum, New York, 1979)
[31] D.W. McLaughlin and L.S. Schulman, J. Math. Phys. 12 (1971) 2520
[32] H.T. Laquer, *in* Group actions on manifolds, Contemp. Math. 36 (AMS, Providence, 1983)
[33] A. Trautman, Int. J. Theor. Phys. 16 (1977) 561

[34] Z. Horváth and L. Palla, Nucl. Phys. B142 (1978) 327
[35] F.A. Bais and P. Batenburg, Nucl. Phys. B269 (1986) 363
[36] R. Kuwabara, J. Math. Tokushima Univ. 16 (1982) 1
[37] F.H. Molzahn, T.A. Osborn and S.A. Fulling, Ann. Phys. (N.Y.) 204 (1990) 64
[38] J.A. Schouten, Ricci-calculus (Springer, Berlin, 1954)
[39] D. McAvity and H. Osborn, Class. Quant. Grav. 8 (1991) 603
[40] M. Reed and B. Simon, Fourier analysis, self-adjointness (Academic Press, New York, 1975)
[41] L.S. Schulman, Techniques and applications of path integration (Wiley, New York, 1981)
[42] K.D. Elworthy, *in* Mathematical aspects of superspace, ed. H.-J. Siefert et al. (Reidel, Dordrecht, 1989)
[43] S. Albeverio, S. Paycha and S. Scarlatti, *in* Functional integration, geometry, and strings, ed. Z. Haba and J. Sobczyk (Birkhäuser, Basel, 1989)
[44] M. Reed and B. Simon, Functional analysis (Academic Press, New York, 1972)
[45] J.F. Hamilton and L.S. Schulman, J. Math. Phys. 12 (1971) 161
[46] J.S. Dowker, *in* Functional integration and its applications, ed. A.M. Arthurs (Clarendon, Oxford, 1975)
[47] C. DeWitt-Morette, Ann. Inst. Henri Poincaré (N.S.) 11 (1969) 152
[48] S. Samuel, J. Math. Phys. 19 (1978) 1438
[49] J. Klauder and B-S. Skagerstam, Coherent states (World Scientific, Singapore, 1985)
[50] Ya.I. Belopolskaya and Yu.L. Dalecky, Stochastic equations and differential geometry (Kluwer, Dordrecht, 1990)
[51] J. Pothoff, J. Math. Phys. 25 (1984) 52
[52] S. Albeverio, R. Høegh-Krohn, H. Holden and T. Kolsrud, *in* Springer Lecture Notes in Mathematics 1390 (Springer, New York, 1989)
[53] L. Schlesinger, Math. Ann. 99 (1928) 413
[54] W. Greub and H-R. Petry, J. Math. Phys. 16 (1975) 1397
[55] L.C. Biedenharn and J.D. Louck, The Racah–Wigner algebra in quantum physics, Encycl. Math. Appl. Vol. 9 (Addison–Wesley, Reading, MA, 1981)
[56] L.C. Biedenharn and J.D. Louck, Angular momentum in quantum physics, Encycl. Math. Appl. Vol. 8 (Addison–Wesley, Reading, MA, 1981)
[57] T.T. Wu and C.N. Yang, Phys. Rev. D14 (1977) 437
[58] O. Alvarez, Commun. Math. Phys. 100 (1983) 279
[59] F. Wilczek, Fractional statistics and anyon superconductivity (World Scientific, Singapore, 1990)
[60] E.C.G. Sudarshan, T.D. Imbo and C.S. Imbo, Ann. Inst. Henri Poincaré (P.T.) 49 (1988) 387