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Superselection rules from Dirac and BRST quantisation of constrained systems

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Adopting the point of view that inequivalent quantisations correspond to superselection sectors, i.e. unitarily inequivalent representations of the algebra of observables, we show how the superselection sectors of a particle moving on a coset space G/H follow from its quantisation as a constrained system (the unconstrained system being the phase space T*G). Both the Dirac and BRST method are examined; the former works well for compact H, whereas the latter runs into several difficulties. Accordingly, a possible improvement to the BRST procedure is suggested.

1. Introduction

An intriguing feature of the quantum mechanics of non-linear systems is the possibility of inequivalent quantisations [1–4]. Within a C*-algebraic [5,6] or canonical group [3] approach these quantisations arise as unitarily inequivalent representations of the algebra of quantum observables (i.e. superselection sectors). For homogeneous configuration spaces, these are realised on Hilbert spaces of cross sections of vector bundles over the configuration space (for a review see ref. [7]). Within this set of quantisations is the naïve one in which the wave functions are cross sections of a trivial complex line bundle over the configuration space (i.e. complex-valued functions over configuration space). In the case of quantum mechanics on the real line (or any finite-dimensional vector space) the Stone – von Neumann theorem [8] shows that the naïve quantisation is the unique one ⋆, up to unitary equivalence, but this uniqueness is the exception, rather than the rule.

The configuration spaces we shall be concerned with here are of the type G/H where G and H are Lie groups of which G is unimodular (for simplicity) and H compact, and we shall address the question of how the inequivalent quantisations arise when the system is thought of as constrained motion on G (as explained in

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⋆ Unless we allow the assignment of spin as an inequivalent quantisation [1,5,6].

refs. [3,5,6], for a coset space G/H, the inequivalent quantisations are labelled by $\hat{\mathrm{H}}$ – the space of equivalence classes of irreducible representations of H). In sect. 2 we show how the classical constrained phase space $T^*(G/H)$ – the cotangent bundle of G/H – and the space of constrained classical observables $C^\infty(T^*(G/H))$ are derived from the unconstrained system of motion on G. In sect. 3 we treat the quantum mechanics of motion on G/H according to Dirac [9]: we describe a minimal quantisation of motion on G in which the quantised constraints $\pi(p_i)$, $i = 1, \ldots, d_H = \dim H$ are well defined. As we show, the requirement that the $\pi(p_i)$ be among the quantisable functions leads to the treatment of G itself as a coset space, and we demonstrate how the inequivalent quantisations of motion on G/H arise. The Dirac approach works well here as a consequence of the compactness of H; for non-compact H issues of ill-defined inner products and operators with continuous spectra need to be addressed.

In sects. 4 and 5 we extend the system yet further and treat motion on G/H within the BRST formalism [10], first classically, then quantum mechanically. The former analysis has been undertaken by other authors for a general phase space [11–13] and a summary, as it applies to our models, is included for completeness: the result is that the physical observables may be understood as corresponding to the BRST observable cohomology classes at ghost number zero. In sect. 5 we discuss the quantum BRST cohomology. However unlike sects. 2 and 3 where we found that quantisation of the classical constrained system by the Dirac method leads neatly to the quantum constrained system, we see (as has been noted before [14–16,23]) that the BRST quantisation is by no means so satisfactory. We are unable to improve upon previous treatments and thus have to resort to a somewhat ad hoc prescription to regain the correct physical state cohomology; nonetheless, it is illuminating to see how the inequivalent quantisations arise via inequivalent representations of the BRST operator selecting the physical state space. In sect. 6 we present a conclusion and, motivated by ideas from operator algebra theory, propose a framework for BRST state and observable cohomology which emphasises their duality and which may resolve some of the difficulties in sect. 5 in a coherent fashion.

2. Classical reduced phase space

In this section we show how the classical constraints reduce the extended phase space, T^*G – the cotangent bundle of G – to the constrained phase space $T^*(G/H)$: we identify the space $\Gamma_p \subset T^*G$ on which the constraints $p_i = 0$, $i = 1, \ldots, \dim H$ (as defined below) hold, then take the quotient of this space by the gauge transformations generated by the p_i. This is an example of symplectic reduction [4,17].

It will be convenient to use a specific partial coordinate system on T^*G: given a basis $L_a(x)$, $a = 1, \dots, \dim G$, $x \in G$, for the left-invariant vector fields on G, one can define a dual basis $\theta^a(x)$ via $\langle \theta^a, L_b \rangle_x = \delta^a_b$ at each point. Then an arbitrary element $\alpha \in (T^*G)_x$ can be expanded as $\alpha = p_a \theta^a(x)$. We use the p_a as coordinates for $(T^*G)_x$ and these coordinates are globally valid since T^*G is a trivial bundle ($T^*G \simeq G \times \mathfrak{g}^*$ where $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$, the Lie algebra of G). In these coordinates the Liouville form is [17]

$$\theta(p, x) = p_a \theta^a(x), \tag{2.1}$$

so that, using the Maurer–Cartan equations $\mathrm{d}\theta^a = -\frac{1}{2} C_{bc}{}^a \theta^b \wedge \theta^c$, the standard symplectic form is

$$\omega = -\mathrm{d}\theta = -\mathrm{d}p_a \wedge \theta^a + \tfrac{1}{2} C_{bc}{}^a p_a \theta^b \wedge \theta^c. \tag{2.2}$$

The hamiltonian vector field X_f associated to a (C^1) function $f(x, p)$ on T^*G is defined by

$$i_{X_f}\omega = \mathrm{d}f = \frac{\partial f}{\partial p_a}\, \mathrm{d}p^a + (L_a f)\, \theta^a, \tag{2.3}$$

and the Poisson bracket of two such functions is given by

$$\begin{aligned} \{f, g\} &= -\omega(X_f, X_g) = X_f g \\ &= \left(L_a g + C_{ab}{}^c p_c \frac{\partial g}{\partial p_b} \right) \frac{\partial f}{\partial p_a} - \frac{\partial g}{\partial p_a} L_a f. \end{aligned} \tag{2.4}$$

Thus the Poisson bracket of two coordinate functions p_a and p_b is

$$\{p_a, p_b\} = C_{ab}{}^c p_c. \tag{2.5}$$

The contraints for the system, in these coordinates, are

$$p_i = 0, \quad i = 1, \dots, \dim H, \tag{2.6}$$

which satisfy

$$\{p_i, p_j\} = C_{ij}{}^k p_k, \tag{2.7}$$

with $C_{ij}{}^k$ the structure constants of H, and are thus first class. The "constraint surface", Γ_p, is $T^*G \restriction_{p_i = 0} \simeq G \times \mathfrak{m}^*$, where we have chosen a specific reductive decomposition, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, of the Lie algebra $\mathfrak{g}$ ($\mathfrak{h}$ is the Lie algebra of H) with

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}. \tag{2.8}$$

The hamiltonian vector field X_{p_i} associated to the constraint p_i is given by

$$X_{p_i} = L_i - C_{ai}{}^c p_c \frac{\partial}{\partial p_a}, \tag{2.9}$$

and hence on the constraint surface it is

$$X_{p_i} \upharpoonright \Gamma_p = L_i - C_{\alpha i}{}^\beta p_\beta \frac{\partial}{\partial p_\alpha}, \quad \alpha = 1, \dots, \dim \mathfrak{m}, \tag{2.10}$$

which is tangent to Γ_p, as it should be. The commutator of two such vector fields satisfies

$$\left[X_{p_i}, X_{p_j}\right] \upharpoonright \Gamma_p = C_{ij}{}^k X_{p_k} \upharpoonright \Gamma_p, \tag{2.11}$$

so that, by Frobenius' theorem (see ref. [17] for example) the system $\{X_{p_i} \upharpoonright \Gamma_p\}_{i=1}^{\dim \mathrm{H}}$ is integrable and foliates Γ_p (in fact $[X_{p_i}, X_{p_j}] = C_{ij}{}^k X_{p_k}$ even off the constraint surface so that the gauge orbits are defined on all of $\mathrm{T^*G}$).

Now $\mathfrak{m}^*$ carries a representation of the subgroup H – the so-called co-isotropic representation π_{co}:

$$\left(\pi_{\mathrm{co}}(T_i) p\right)_\alpha = -C_{i\alpha}{}^\beta p_\beta, \tag{2.12}$$

so that

$$X_{p_i} \upharpoonright \Gamma_p = L_i - \left(\pi_{\mathrm{co}}(T_i) p\right)_\alpha \frac{\partial}{\partial p_\alpha}, \tag{2.13}$$

and $X_{p_i} \upharpoonright \Gamma_p$ generate the global transformation of Γ_p,

$$\left(x(t), p(t)\right) = \left(x\, \mathrm{e}^{tT_i}, \mathrm{e}^{-t\pi_{\mathrm{co}}(T_i)} p\right), \tag{2.14}$$

so that the set $\{X_{p_i} \upharpoonright \Gamma_p\}_{i=1}^{\dim \mathrm{H}}$ generates the action of H,

$$(x, p) \mapsto \left(xh, \pi_{\mathrm{co}}(h^{-1}) p\right), \quad h \in \mathrm{H}. \tag{2.15}$$

By definition of a vector bundle associated to the principle fibre bundle (G, G/H, p, H) (see e.g. ref. [18]), the quotient space of Γ_p by this H action is precisely the associated vector bundle $\mathrm{G} \times_{\mathrm{H}} \mathfrak{m}^*$, where $\mathfrak{m}^*$ is an H-module by eq. (2.12). But this vector bundle is just $\mathrm{T^*(G/H)}$ [19], so that we have shown that $\Gamma_p/\mathrm{H} = \mathrm{T^*(G/H)}$. We have thus achieved the desired goal of showing how the constraints p_i reduce the phase space $\mathrm{T^*G}$ to $\mathrm{T^*(G/H)}$.

The reduction of observables proceeds straightforwardly in this case. The starting point is the space of observables on the unconstrained phase space,

$\mathscr{A} = C^\infty(T^*G)$. One firstly takes the quotient of this space by the ideal consisting of functions vanishing of Γ_p, $\mathscr{I} = \{f \in C^\infty(T^*G) \mid f \restriction_{\Gamma_p} = 0\}$ so that $\mathscr{A}/\mathscr{I} \equiv \mathscr{B} = C^\infty(\Gamma_p)$. The true observables are the gauge-invariant functions, i.e. $\mathscr{A}_{\text{red}} = \mathscr{B} \restriction_{\text{gauge invariant}}$ where gauge invariant means

$$X_{p_i} \restriction_{\Gamma_p} f = \{p_i, f\} \restriction_{\Gamma_p} = 0, \tag{2.16}$$

but since $\Gamma_p/H = T^*(G/H)$, as demonstrated above, we get finally

$$\mathscr{A}_{\text{red}} = C^\infty(T^*(G/H)). \tag{2.17}$$

One may also perform these steps in opposite order, i.e. by defining the gauge-invariant functions in $C^\infty(T^*G)$ as those satisfying eq. (2.16), and then $\mathscr{A}_{\text{red}}$ as $\mathscr{A}_{\text{gauge invariant}}/\mathscr{I}$.

3. Dirac quantisation

Having treated the system classically, we now turn to the quantisation of the model and show how to quantise motion on the classical configuration space G/H treated as constrained motion on G, following Dirac's procedure: we identify the extended Hilbert space (i.e. the relevant quantisation of the classical configuration space G) and identify the true Hilbert space as wave functions in the extended Hilbert space, subject to the quantised constraint $\pi(p_i) = 0$. We are particularly concerned to demonstrate how the inequivalent quantisations of G/H emerge in the treatment of G/H as a constrained system. As we shall show, the requirement that the classical constraints p_i be quantisable (i.e. that $\pi(p_i)$ are well-defined operators) forces us to quantise G in a non-trivial manner.

In order to see why this is the case, let us try to quantise motion on G in a naïve fashion and show why this is inadequate for our purposes. We shall take quantisation of G to mean (i) identification of a subalgebra of the Poisson bracket algebra of functions on phase space, $C^\infty(T^*G)$ (one takes only a subalgebra since one cannot quantise all classical observables [4,17]); (ii) replacement of such functions f with operators $\hat{f}$; (iii) finding a unitary representation of this operator algebra. The C^*-algebra/group theoretic approach to this scheme is disucssed in detail in ref. [7] and when applied to the configuration space G, leads to the quantisable functions being C^∞ functions on configuration space and symbols of right-invariant vector fields on G (if $G = \mathbb{R}^n$, this just means functions linear in p), with associated algebra of quantum observables $G \ltimes C^\infty(G)$ (this denotes the crossed product of $C^\infty(G)$ and G, with respect to the obvious automorphic action of the latter on the former [6]: heuristically this algebra is generated by $C^\infty(G)$ and the Lie algebra of

G, with commutation relations $[T_a, g] = R_a g$ for $g \in C^\infty(G)$; cf. eq. (3.2) below). The unique irreducible representation of this algebra is on $L^2(G)$ and is given by

$$
\begin{aligned}
(\pi(f)\psi)(x) &= f(x)\psi(x), \quad f \in C^\infty(G), \quad \psi \in L^2(G), \\
(\pi(y)\psi)(x) &= \psi(y^{-1}x), \quad y \in G,
\end{aligned}
\tag{3.1}
$$

so that the symbols f_{R_a} of right-invariant vector fields R_a on G are quantised by

$$
\begin{aligned}
\left(\pi(f_{R_a})\psi\right)(x) &= \frac{d}{dt}\psi(e^{-tT_a}x)\,|_{t=0} \\
&= \pi_L(T_a)\psi(x).
\end{aligned}
\tag{3.2}
$$

π_L is the left-regular representation of G on $L^2(G)$: $(\pi_L(y)\psi)(x) \equiv \psi(y^{-1}x)$, cf. eq. (3.1).

However, the constraints p_i are the symbols of *left*-invariant vector fields which means that they are not automatically quantised within the scheme. Of course, left-invariant vector fields can be expressed as products of right-invariant vector fields and functions in $C^\infty(G)$, however this will lead to ordering ambiguities on quantisation. There is however a more serious problem with the naïve approach: even if one could find a consistent ordering prescription, one is led to a unique quantisation of G/H and the infinite set of inequivalent quantisations labelled by representations of H [1,5] do not emerge *.

A resolution of these difficulties is to treat G as the homogeneous space $G \times H/\tilde{H}$ where $\tilde{H} \simeq H$ is the diagonal subgroup of $H \times H \subset G \times H$: an element $(x, h) \in G \times H$ acts on $y \in G$ sending y to xyh^{-1}; the little group of this action at the identity $\mathrm{id} \in G$ consists of elements of the form $(h, h) \in G \times H$, so that defining $\tilde{H} = \{(h, h) \in G \times H \mid h \in H\}$, we indeed have that $G \simeq G \times H/\tilde{H}$.

The inequivalent quantisations of a particle moving on $G \times H/\tilde{H}$ (which we shall write as $\tilde{G}/\tilde{H}$ from now on) are labelled by representations χ of $\tilde{H}$ and are realised on the Hilbert space ${}^\chi\mathscr{H} = L^2(G, \mathscr{H}_\chi)$ (i.e. functions on G with values in $\mathscr{H}_\chi$) where the Hilbert space $\mathscr{H}_\chi$ carries an irreducible representation π_χ of H (but now regarded as a representation of $\tilde{H}$). Applying eq. (1.6) of ref. [7] and choosing

* It is, of course, possible to quantise G in such a way that operators associated to symbols of left-invariant vector fields are well defined – instead of the left action of G on itself in the scheme used in ref. [7], one uses the right action. One finds a similar representation of the canonical group on $\mathscr{H} = L^2(G)$, so that in particular the quantisation of p_i is $\pi(p_i) = \pi_R(T_i)$ (π_R being the right-regular representation of G on $L^2(G)$ – $(\pi_R(y)\psi)(x) \equiv \psi(xy)$) and the Hilbert space of the constrained system is $\mathscr{H}^Q = \{\psi \in \mathscr{H} \mid \pi_R(T_i)\psi = 0\}$ or equivalently $\psi(xh) = \psi(x)\ \forall h \in H$. However $\mathscr{H}^Q$ is not invariant under $\pi_R(T_\alpha)$ (the remaining generators of G) so that one does not have a representation of "momentum observables" in this quantisation. Furthermore this quantisation also suffers from the problem that the inequivalent quantisations do not appear.

the cross section $s(x)=s(x, \mathrm{id})$ from G to $\tilde{\mathrm{G}}$, the Hilbert space carries a representation of $\tilde{\mathrm{G}} \ltimes \mathrm{C}^\infty(\mathrm{G})$ which we shall denote ${}^\chi\pi$, with $\tilde{\mathrm{G}}$ represented by

$$({}^\chi\pi((x, h))\psi)(y) = \pi_\chi(h)\psi(x^{-1}yh), \quad \psi \in {}^\chi\mathscr{H}, \tag{3.3}$$

and $\mathrm{C}^\infty(\mathrm{G})$ represented by

$$({}^\chi\pi(f)\psi)(y) = f(y)\psi(y). \tag{3.4}$$

The reason for using this quantisation of G is that the operators corresponding to the constraints p_i are well defined on the Hilbert space ${}^\chi\mathscr{H}$,

$${}^\chi\pi(p_i) = \frac{\mathrm{d}}{\mathrm{d}t}\, {}^\chi\pi\big((\mathrm{id}, \mathrm{e}^{tT_i})\big)\big|_{t=0}, \tag{3.5}$$

i.e.

$$({}^\chi\pi(p_i)\psi)(y) = \big(\pi_{\mathrm{R}}(T_i) + \pi_\chi(T_i)\big)\psi(y). \tag{3.6}$$

Note that π_{R} acts on the argument y of ψ whereas π_χ acts on its (suppressed) index in $\mathscr{H}_\chi$. We may thus define the constrained Hilbert space as

$$\mathscr{H}^\chi = \big\{\psi \in {}^\chi\mathscr{H} \mid {}^\chi\pi(p_i)\psi = 0\big\}, \tag{3.7}$$

or equivalently

$$\mathscr{H}^\chi = \big\{\psi \in {}^\chi\mathscr{H} \mid \psi(xh) = \pi_\chi(h^{-1})\psi(x)\big\}. \tag{3.8}$$

$\mathscr{H}^\chi$ has the inner product inherited from that on ${}^\chi\mathscr{H}$. Since H is compact (and normalised so that its volume is 1) it follows that this Hilbert space is precisely that given in refs. [6,7] as a representation π^χ of $\mathrm{G} \ltimes \mathrm{C}^\infty(\mathrm{G}/\mathrm{H})$ induced by the representation χ of H. As explained there, this is one of the inequivalent quantisations of the classical configuration space $\mathrm{Q} = \mathrm{G}/\mathrm{H}$, and we have therefore shown that there is a one-to-one correspondence between inequivalent quantisations of $\mathrm{G} \times \mathrm{H}/\tilde{\mathrm{H}}$ and those of G/H. In particular we have a representation of the group G (whose self-adjoint generators give the "momentum" observables appropriate to G/H),

$$(\pi^\chi(x)\psi)(y) \equiv ({}^\chi\pi((x, \mathrm{id}))\psi)(y) = \psi(x^{-1}y), \tag{3.9}$$

(since these commute with the constraint) and $\mathrm{C}^\infty(\mathrm{Q}) \subset \mathrm{C}^\infty(\mathrm{G})$ is represented by

$$\big(\pi^\chi(\tilde{f})\psi\big)(y) = \tilde{f}(y)\psi(y), \quad \tilde{f} \in \mathrm{C}^\infty(\mathrm{G}/\mathrm{H}). \tag{3.10}$$

Clearly ${}^{\chi}\pi((x, \mathrm{id})) = \pi^{\chi}(x)$ in eq. (3.9) is the usual representation of G induced by π_χ of H [6,7]. Notice that only those functions in $C^\infty(G)$ that satisfy

$$f(xh) = f(x) \quad \forall h \in H \text{ and } \forall x \in G \tag{3.11}$$

commute with the constraints and are therefore defined on $\mathscr{H}^{\chi}$. These functions are precisely the ones (now called $\tilde{f}$) which are defined on G/H.

In summary, we have shown that in order to Dirac quantise motion on the configuration space G/H treated as constrained motion on G, the requirement that the classical constraints p_i be represented by well-defined operators in the quantum theory leads to the quantisation of G as the coset space $G \times H/\tilde{H}$, this being the minimal one for which the constraints are amongst the algebra of observables. It is then found that the known inequivalent quantisations of G/H are in one-to-one correspondence with the inequivalent quantisations of $G \times H/\tilde{H}$. In particular, the observables of the extended system which commute with the constraints are precisely those which arise when quantising G/H directly.

4. Classical BRST

In sect. 3 it was shown that, by starting with an extended space of states (the Hilbert space ${}^{\chi}\mathscr{H}$) and an extended set of obsevables, the physical Hilbert space and observables are recovered by imposing the quantised constraints. A natural next step is to enlarge the state space and observables one stage further and analyse the system of a particle moving on G/H within the BRST framework (cf. ref. [20]).

As the classical BRST formalism has been treated before in great generality [11–13] we discuss it here very briefly for completeness: one extends the classical algebra of observables, $C^\infty(T^*G)$, by adding $2d_H$ anti-commuting variables $\{c^i\}$, $\{b_i\}$, $i = 1, \ldots, d_H$. These have the following Poisson brackets:

$$\{c^i, b_j\} = \delta^i_j. \tag{4.1}$$

The "algebra of observables" is now isomorphic to the supermanifold $\mathscr{A}_{BRST} = C^\infty(T^*G) \otimes \Lambda(\mathfrak{h} \oplus \mathfrak{h}^*)$, where $\Lambda(V)$ is the exterior algebra of a vector space V [12,21] *. Having extended the algebra of observables, one has to find a set of criteria for selecting the "physical" observables. If these criteria are satisfactory, the space of physical observables should be isomorphic to $C^\infty(T^*(G/H))$. In order

* Although only $C^\infty(T^*G)$ has physical meaning, we still refer to this supermanifold as the algebra of observables.

to identify the physical observables one needs to introduce two new observables: the classical BRST charge

$$\Omega = c^i p_i - \tfrac{1}{2} C_{ij}{}^k c^i c^j b_k \tag{4.2}$$

and the classical ghost charge

$$N_{\mathrm{gh}} = c^i b_i \tag{4.3}$$

with Poisson brackets

$$\{\Omega, N_{\mathrm{gh}}\} = \Omega, \qquad \{\Omega, \Omega\} = 0. \tag{4.4}$$

Ω generates BRST transformations $\delta_{\mathrm{BRST}} f = \{\Omega, f\}$, $f \in \mathscr{A}_{\mathrm{BRST}}$, which are nilpotent on account of eq. (4.4). A classical observable A is called BRST closed if $\{A, \Omega\} = 0$ and BRST exact if $A = \{B, \Omega\}$ for some observable B. A is said to have ghost number n if $\{N_{\mathrm{gh}}, A\} = nA$.

It is shown in refs. [11–13] that the space of BRST cohomology classes at ghost number zero (i.e. the set of observables of ghost number zero which are BRST closed, with two observables being identified if their difference is BRST exact) is isomorphic to the algebra of smooth functions on the reduced phase space, i.e. to $\mathrm{C}^\infty(\mathrm{T}^*(\mathrm{G}/\mathrm{H}))$ in our case.

In sect. 2 we discussed the unconstrained phase space $\mathrm{T}^*\mathrm{G}$ and its reduction to the physical phase space $\mathrm{T}^*(\mathrm{G}/\mathrm{H})$, but the idea of a BRST-extended classical phase space is rather illusive since $\mathrm{C}^\infty(\mathrm{T}^*\mathrm{G}) \otimes \Lambda(\mathfrak{h} \oplus \mathfrak{h}^*)$ is not the space of functions on some space in any very obvious way. Nonetheless, the classical BRST cohomology has a "classical" meaning: applying the results of ref. [22], we find that (at non-negative ghost number *) it is isomorphic to the ordinary Lie algebra cohomology of H with values in $\mathrm{C}^\infty(\Gamma_p)$ (cf. sect. 2), which is to be regarded here as an infinite-dimensional $\mathfrak{h}$-module under the action (2.13), i.e.

$$\mathrm{H}^*_{\mathrm{BRST\ classical\ observable}} = \mathrm{H}^*(\mathfrak{h}, \mathrm{C}^\infty(\mathrm{G} \times \mathfrak{m}^*)). \tag{4.5}$$

In other words, since $\Gamma_p = \mathrm{G} \times \mathfrak{m}^*$ carries a right action of H (eq. (2.15)), $\mathrm{C}^\infty(\Gamma_p)$ is a H-module under the trivially induced action, which gives rise to an $\mathfrak{h}$-module by differentiation. This characterisation of BRST cohomology is obvious for the zeroth cohomology space since the algebra of observables is the space of H-invariant functions on Γ_p.

* The classical BRST cohomology vanishes at negative ghost number under a regularity assumption which is satisfied in our case.

5. Quantum BRST

The aim of this section is to show how the physical states and observables occur within the quantum BRST framework. In particular we shall show how the non-trivially induced representations arise.

5.1. AN ILLUSTRATIVE EXAMPLE

In sect. 3 we showed that one can quantise the observables of the extended system (i.e. those corresponding to motion on G) in such a way that the classical constraints $p_i = 0$ are well-defined operators. The space of states and the inner product on such states which resulted was shown to be that which would have been found on direct quantisation of motion on G/H (i.e. the physical states were isometrically embedded in the extended Hilbert space). It might be hoped that by quantising an algebra of classical BRST observables (as in sect. 2 we do not expect to quantise all classical observables) and imposing the quantised version of the classical BRST conditions on the BRST extended state space one could identify the physical states with their correct inner product. Specifically: the physical states might be expected to be identified with the cohomology classes of the BRST operator of ghost number zero. Unfortunately it does not seem possible to implement this idea in general (cf. ref. [23]).

Rather than treat the general problem, it is instructive to treat a simple example (discussed previously in refs. [14,24]) as the problems which arise, and a resolution of them, may been seen in this case. The example is a particle moving on the configuration space $Q = \mathbb{R}^n$ thought of as constrained motion on $G = \mathbb{R}^n \times S^1$ (i.e. $H = S^1 \simeq SO(2)$). Let us call local coordinates on $\mathbb{R}^n \times S^1$, $(\underline{q}, \theta)$, then the classical constraint is $p_\theta = 0$. In order to BRST quantise this system we introduce one ghost coordinate c and its conjugate momentum b. An irreducible representation of all the operators of the BRST extended system is on the indefinite metric space of functions of $\underline{q}$, θ, c with inner product

$$\langle \Psi, \tilde{\Psi} \rangle = \int_{-\infty}^{\infty} \mathrm{d}\underline{q} \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi} \int \mathrm{d}c\; \Psi^*(\underline{q}, \theta, c) \tilde{\Psi}(\underline{q}, \theta, c). \tag{5.1}$$

$\int \mathrm{d}c$ is Berezin integration, $\underline{q}$ is represented by multiplication and the conjugate momentum to q^α is represented by $\hat{p}_\alpha = -i(\partial/\partial q^\alpha)$; c is represented by multiplication and b by $\hat{b} = \partial/\partial c$; there is a slight complication in the treatment of the S^1 where the Heisenberg group is not appropriate [3,25], but the result is as expected in that one can represent p_θ by $\hat{p}_\theta = -i(\partial/\partial\theta)$, defined and self-adjoint on absolutely continuous periodic functions on the interval $[0, 2\pi]$. The skew-adjoint ghost number operator is

$$\hat{N}_{\mathrm{gh}} = \frac{1}{2}\left(c\frac{\partial}{\partial c} - \frac{\partial}{\partial c}c \right), \tag{5.2}$$

(we are not free to add a constant to $\hat{N}_{gh}$ without destroying its skew-adjointness *) and the BRST charge is

$$\hat{\Omega} = -ic\frac{\partial}{\partial\theta}, \tag{5.3}$$

which is self-adjoint on the above-mentioned domain, tensored with functions of c.

We may expand any function $\Psi(\underline{q}, \theta, c)$ as

$$\Psi(\underline{q}, \theta, c) = \psi_0(\underline{q}, \theta) + c\psi_1(\underline{q}, \theta) \tag{5.4}$$

with

$$\hat{N}_{gh}\psi_0 = -\tfrac{1}{2}\psi_0, \qquad \hat{N}_{gh}(c\psi_1) = \tfrac{1}{2}(c\psi_1). \tag{5.5}$$

Thus we note that the eigenvalues of $\hat{N}_{gh}$ are $\pm\frac{1}{2}$ and there are no states of ghost number zero.

We now analyse the cohomology of the BRST operator $\hat{\Omega}$: at ghost number $-\frac{1}{2}$, $\hat{\Omega}\psi_0 = 0 \Rightarrow (\partial\psi_0/\partial\theta)(\underline{q}, \theta) = 0$, and the fact that there are no states at ghost number $-\frac{3}{2}$ (so that no state of ghost number $-\frac{1}{2}$ may be written as $\hat{\Omega}\chi$) means that the cohomology classes at ghost number $-\frac{1}{2}$ are in one-to-one correspondence with functions on $\mathbb{R}^n$. One might therefore be tempted to interpret these cohomology classes as the physical states. However, in order for this identification to be correct, the inner product on the space of such states should be the "physical" inner product – at the very least we require that the inner product is positive definite. However it may easily be seen that the norm of the cohomology classes at ghost number $-\frac{1}{2}$ with respect to the inner product $\langle , \rangle$ is zero.

A resolution of this difficulty, suggested in refs. [14–16], may be found by consideration of the cohomology classes at ghost number $+\frac{1}{2}$: clearly all states of ghost number $+\frac{1}{2}$ are BRST closed and (by use of Fourier series, for example) it may be seen that the only states which are not of the form $\hat{\Omega}\chi$ for some χ are $\Psi = c\psi_1(\underline{q}, \theta)$ where $(\partial\psi_1/\partial\theta)(\underline{q}, \theta) = 0$ (for the required χ would be $\chi = \theta\psi_1$, but this is not in the domain of self-adjointness of $\hat{\Omega}$ **). We thus see that the space of cohomology classes at ghost number $\pm\frac{1}{2}$ are the same and may both be identified with functions on $\mathbb{R}^n$.

As at ghost number $-\frac{1}{2}$, it may easily be checked that the norm (with respect to $\langle , \rangle$) of any state in a BRST cohomology classes at ghost number $+\frac{1}{2}$ is zero. Thus *one cannot take the physical states to be eigenstates of ghost number*.

* In refs. [12,23], $\hat{N}_{gh}$ is indeed modified by subtracting its smallest eigenvalue ($-\frac{1}{2}$ in this example) from it, so that zero is now in its spectrum, at the expense of destroying its skew-adjointness, but without resolving any of the issues raised here.

** That this reasoning is correct may be seen by taking the inner product with a BRST closed function: if $\tilde{\Psi}(\underline{q}, \theta, c) = \psi_0(\underline{q})$ (i.e. $\hat{\Omega}\tilde{\Psi} = 0$) then $\langle \tilde{\Psi}, \Psi \rangle = \int \mathrm{d}\underline{q}\ \psi_0^*(\underline{q})\psi_1(\underline{q}, \theta)$ which is non-zero in general. However if one could write $\Psi = \hat{\Psi}\chi$ for some χ then $\langle \tilde{\Psi}, \Psi \rangle = \langle \tilde{\Psi}, \hat{\Omega}\chi \rangle = \langle \hat{\Omega}\tilde{\Psi}, \chi \rangle = 0$.

We are thus left with the question of just how one should identify physical states. As far as the present authors are aware, there is no completely satisfactory way of doing this, with various possibilities being ad hoc in different ways (cf. ref. [26] for a discussion of the corresponding situation in string theory). We therefore take a pragmatic view and adopt a procedure which (as we shall show) leads to the correct identification of the true physical states.

Following refs. [14–16] we note that one can identify a positive norm subspace in the BRST cohomology in such a way that the inner product of two states in the subspace is the "physical" inner product. This subspace does not have definite ghost number. The idea is that given a state $\Psi = \psi_0(\underline{q}, \theta)$ in the BRST cohomology at ghost number $-\frac{1}{2}$ one associates to it the state

$$\Psi_{\text{phys}} = \psi_0 + \tfrac{1}{2}c\psi_0. \tag{5.6}$$

Thus if we take as the representative of a given cohomology class that function which is independent of θ (of course the inner product will be independent of which representative of the cohomology class we choose), we find that

$$\begin{aligned}\langle \Psi_{\text{phys}}, \tilde{\Psi}_{\text{phys}} \rangle &= \int \frac{\mathrm{d}\theta}{2\pi}\, \mathrm{d}\underline{q}\, \mathrm{d}c\, \left(\psi_0^* + \tfrac{1}{2}c\psi_0^*\right)\left(\tilde{\psi}_0 + \tfrac{1}{2}c\tilde{\psi}_0\right) \\ &= \int \mathrm{d}\underline{q}\, \psi_0^* \tilde{\psi}_0,\end{aligned} \tag{5.7}$$

and hence that this subspace may be isometrically identified with the physical state space.

This simple example has shown that one cannot choose the quantum states to be the cohomology classes of ghost number zero or even to be eigenstates of the "renormalised" (by one half) ghost number operator. A resolution of the difficulty was to choose a linear subspace of the cohomology spaces at ghost number $+\frac{1}{2}$ and $-\frac{1}{2}$. This subspace could be identified with the physical states with their correct positive definite inner product.

Before we discuss the physical observables it is useful to examine the remaining BRST cohomology classes. We may take states of the form

$$\Psi_{-} = \psi_0 - \tfrac{1}{2}c\psi_0 \tag{5.8}$$

where $\partial\psi_0/\partial\theta = 0$ as representatives of these classes. These states have negative norm and are orthogonal to the physical state space as defined above. Thus we see that the space of BRST chomology classes breaks up into two orthogonal subspaces, one, V_+, of positive definite inner product, and one, V_-, of negative definite inner product. (Clearly any state $\Psi = \psi_0 + c\psi_1$ may be written as a linear combination of states in V_+ and V_-.)

In order that we may truly identify V_+ as the physical states we must check that the "physical observables" are represented on them. This is the case since the fundamental operators $\hat{x}$ and $\hat{p} = -i(\partial/\partial x)$ clearly commute with the BRST charge and furthermore they do not mix positive and negative norm subspaces, for if $\Psi_{\text{phys}} \in V_+$ and $\Psi_- \in V_-$, then

$$\langle \Psi_{\text{phys}}, \hat{x}\Psi_- \rangle = \langle \Psi_{\text{phys}}, \hat{p}\Psi_- \rangle = 0. \tag{5.9}$$

It should be noted that $\hat{x}$ and $\hat{p}$ which are "physical" operators in a heuristic sense commute with $\hat{\Omega}$ and have ghost number zero. This once again emphasises the mis-match between physical states and physical observables in quantum BRST: physical operators are of ghost number zero, but physical states are not eigenstates of ghost number.

5.2. THE GENERAL CASE

The features observed in the simple example occur generally in the class of coset spaces addressed in this paper. This allows us to show how the physical states of such models may be identified within the space of states of the BRST system. As explained below, the physical states *are* represented by cohomology classes of the BRST operator, but in order to get the correct inner product, one cannot take the states at ghost number zero in general, but may take a linear subspace of cohomology classes at lowest and highest ghost number. The formalism we employ shows that the non-trivially induced representations occur naturally, and arise as a consequence of a choice of representation of the constraint algebra.

What we require, then, is a quantisation of the BRST-extended system, including ghosts, in which the BRST charge is self-adjoint, the ghost number operator is skew-adjoint and which allows us to then identify the physical states. Thus for the general case of motion on G/H (G unimodular and H compact), we introduce dim H ghosts, c^i, and their conjugate momenta b_i. As in the subsect. 5.1, the quantisation of the classical BRST charge, Ω will be denoted $\hat{\Omega}$, so that the classical BRST charge

$$\Omega = c^i p_i - \tfrac{1}{2} C_{ij}{}^k c^i c^j b_k \tag{5.10}$$

(where p_i are the classical constraints as in sect. 2) is quantised by

$$\hat{\Omega} = \hat{c}^i\, {}^{\chi}\pi(p_i) - \tfrac{1}{2} C_{ij}{}^k \hat{c}^i \hat{c}^j \hat{b}_k, \tag{5.11}$$

where this operator is represented on the space ${}^{(\chi)}\mathscr{H} = {}^{\chi}\mathscr{H} \otimes \Lambda(\mathfrak{h})$ (${}^{\chi}\mathscr{H}$ is defined in sect. 3), i.e. functions of the form

$$^{\chi}\Psi(x, c) = \sum_{r=0}^{\dim \mathrm{H}} {}^{\chi}\psi_{i_1 \dots i_r}(x) c^{i_1} c^{i_2} \dots c^{i_r}, \tag{5.12}$$

where each function ${}^{\chi}\psi_{i_1\ldots i_r}(x)$ is in $L^2(G, \mathscr{H}_\chi)$ (i.e. the Hilbert space ${}^{\chi}\mathscr{H}$ associated to the quantisation induced by the representation χ of $\tilde{H} \simeq H$). b_i is represented by $\hat{b}_i = \partial/\partial c^i$.

The (indefinite) inner product is

$$\langle \Psi, \tilde{\Psi} \rangle = \int \mathrm{d}c^N \, \mathrm{d}c^{N-1} \ldots \mathrm{d}c^1 (\Psi, \tilde{\Psi})_\chi, \tag{5.13}$$

where $(\Psi, \tilde{\Psi})_\chi$ is the (positive definite) inner product on ${}^{\chi}\mathscr{H}$.

It may be checked that $\hat{\Omega}^2 = 0$ and that $\hat{\Omega}$ is essentially self-adjoint on the domain of functions of the form (5.12) such that the ${}^{\chi}\psi_{i_1\ldots i_r}$ are in a common domain of essential self-adjointness of the constraints $\pi(p_i)$. The closure of $\hat{\Omega}$ thus defines the self-adjoint BRST operator. The ghost number operator is represented by

$$\hat{N}_{\mathrm{gh}} = \tfrac{1}{2}\left(\hat{c}^i \hat{b}_i - \hat{b}_i \hat{c}^i\right) \tag{5.14}$$

and is skew-adjoint.

In sect. 2 it was remarked that the classical BRST observable cohomology is related to the Lie algebra cohomology of $\mathfrak{h}$ with values in $C^\infty(\Gamma_p)$. The presentation above shows that quantum BRST state cohomology is also related to the Lie algebra cohomology of $\mathfrak{h}$, as is well known [24,27,28]. Specifically, the BRST state cohomology is isomorphic to the Lie algebra cohomology of $\mathfrak{h}$ with values in the representation space of the constraints: the operators ${}^{\chi}\pi(p_i)$ are defined on a dense domain in ${}^{\chi}\mathscr{H} = L^2(G, \mathscr{H}_\chi)$, and the representation is given in eq. (3.6) by

$$\left({}^{\chi}\pi(p_i)\psi\right)(y) = \left(\pi_{\mathrm{R}}(T_i) + \pi_\chi(T_i)\right)\psi(y), \tag{5.15}$$

i.e. this representation is the tensor product of π_{R} on $L^2(G)$ with π_χ on $\mathscr{H}_\chi$. Although the representative $\pi_{\mathrm{R}}(h)$ of the group element $h \in H$ is defined on the whole of $L^2(G)$, the representative of a Lie algebra element is unbounded and naturally defined on $C_c^\infty(G) \subset L^2(G)$ (C_c^∞ denotes C^∞ functions of compact support – a common domain of essential self-adjointness for the $\pi_{\mathrm{R}}(T_i)$). Hence the quantum BRST state cohomology is (cf. eq. (4.5))

$${}^{\chi}H^*_{\text{BRST quantum state}} \simeq H^*\left(\mathfrak{h}, C_c^\infty(G, \mathscr{H}_\chi)\right). \tag{5.16}$$

In order to identify convenient representatives of the cohomology classes of $\hat{\Omega}$, we use the fact that there is a Hodge theorem for the quantum BRST complex [24]: one defines the operators

$$\hat{\Omega}^* = \hat{b}^i \, {}^{\chi}\pi(p_i) - \tfrac{1}{2} C_{ij}{}^k \hat{b}^i \hat{b}^j \hat{c}_k, \tag{5.17}$$

and the "BRST laplacian"

$$\hat{\Delta} = \hat{\Omega}\hat{\Omega}^* + \hat{\Omega}^*\hat{\Omega}. \tag{5.18}$$

It is found that any BRST cohomology class may be represented by a unique "harmonic form" – i.e. a state satisfying

$$\hat{\Delta}\Psi \text{ or, equivalently } \hat{\Omega}\Psi = \hat{\Omega}^*\Psi = 0. \tag{5.19}$$

The latter two conditions mean [24] that any cohomology class may be represented by a state satisfying

$$^{\chi}\pi(p_i)\Psi = 0, \quad c^i C_{ij}{}^k b_k \Psi = 0. \tag{5.20}$$

Notice that the Hodge theorem in [24], which was derived for a finite-dimensional representation of the constraints, may be extended to the infinite-dimensional case – one merely required that $\hat{\Omega}$ be self-adjoint on a dense domain, which we have arranged to be the case. The fact that the representation of the constraints is not specified means that the theorem applies both to the trivally and non-trivially induced case.

We may now see that the cohomology classes at ghost number $N_g = -\frac{1}{2}d_H$ (i.e. with no powers of c in their expansion) may be (non-isometrically) identified with those states in $L^2(G, \mathscr{H}_\chi)$ satisfying

$$^{\chi}\pi(p_i)\Psi^{\chi} = 0, \tag{5.21}$$

as may the states at ghost number $N_g = +\frac{1}{2}d_N$.

As in the example discussed in subsect. 5.1, one cannot take a physical state to be an eigenstate of ghost number in general, but we may take the physical state to be

$$\Psi_+ = \Psi_{\text{phys}} = \psi_0 + \tfrac{1}{2}c^1 \dots c^N \psi_0, \tag{5.22}$$

where

$$^{\chi}\pi(p_i)\psi_0 = 0. \tag{5.23}$$

As in the earlier example of $\mathbb{R}^n \times S^1$, we see that the space V of BRST cohomology classes at ghost numbers $N_g = \pm\frac{1}{2}d_H$ breaks up into orthogonal positive and negative norm subspaces $V = V_+ \oplus V_-$ where

$$\Psi_- = \psi_0 - \tfrac{1}{2}c^1 \dots c^N \Psi_0 \in V_-. \tag{5.24}$$

We also see that the quantisation of the "physical" observables, $G \ltimes C^\infty(G/H)$, do not mix V_+ and V_- since, as we showed in sect. 3, the physical observables commute with the constraints and hence with $\hat{\Omega}$ and for such an observable, $\hat{O}$, $\langle \Psi_{\text{phys}}, \hat{O}\Psi_{-} \rangle = 0$.

6. Conclusion

In sect. 3 we showed that the requirement that the constraints p_i be unambiguously quantised observables in the Dirac approach to quantum theory necessitates the treatment of the unconstrained configuration space G as $\mathrm{G} \times \mathrm{H}/\tilde{\mathrm{H}}$. This, in turn, leads to the appearance of inequivalent quantisations when treating motion on G/H as a (first-class) constrained system.

The BRST treatment of this model was not so straightforward as a result of the unsatisfactory nature of the definition of physical states. A potential source of the difficulties lies in the fact that (following the proposal in ref. [12]) we have quantised the functions on phase space $C^{\infty}(\mathrm{T}^*\mathrm{G})$ on a conventional, positive inner product Hilbert space, ${}^{\chi}\mathscr{H}$, and subsequently quantised the BRST-extended phase space $\mathscr{A}_{\mathrm{BRST}}$ (cf. sect. 4) by simply taking the tensor product of ${}^{\chi}\mathscr{H}$ with $\Lambda(\mathfrak{h})$, the representation space of the ghost algebra $\Lambda(\mathfrak{h} \otimes \mathfrak{h}^*)$, which has an indefinite inner product. That this strategy is problematic was pointed out in ref. [23], and a solution was suggested in refs. [14,29]: the "matter observables" (i.e. the classical extended phase space) ought to be quantised on an indefinite-metric space, in which case examples suggest that the BRST cohomology works out satisfactorily. Indeed the success of the BRST method in gauge field theories is based on the fact that the gauge field is covariantly quantised on an indefinite-metric space. It would be very interesting to develop this idea in the case at hand, i.e. to study indefinite-metric representations of generic crossed product C^*-algebras $\tilde{\mathrm{G}} \ltimes C^{\infty}(\tilde{\mathrm{G}}/\tilde{\mathrm{H}})$ (where in our case $\tilde{\mathrm{G}} = \mathrm{G} \times \mathrm{H}$ and $\tilde{\mathrm{H}} = \mathrm{H}$) (cf. ref. [30] for the indefinite-metric representations of the canonical commutation relations).

Alternatively one may enlarge the BRST phase space by adding a Lagrange multiplier plus its conjugate momentum as well as a new ghost–antighost pair for each constraint. This is the idea of the so-called Batalin–Fradkin–Vilkovisky (BFV) formalism (see e.g. refs. [31–33]): the algebra $\mathscr{A}_{\mathrm{BRST}}$ is to be replaced by $\mathscr{A}_{\mathrm{BFV}} = C^{\infty}(\mathrm{T}^*\tilde{\mathrm{G}}) \otimes \Lambda(\mathfrak{h} \oplus \mathfrak{h}^* \oplus \mathfrak{h} \oplus \mathfrak{h}^*)$ (cf. the text below eq. (4.1)). Note that the range of the Lagrange multipliers has been taken to be compact if H is.

One of the virtues of doubling the number of ghosts is that the quantum ghost number operator now has integral spectrum, leaving some hope that one may retrieve the physical state space of the constrained system as the space of cohomology classes of the BFV-extended BRST operator at ghost number zero (it may be that one has to both double the number of ghosts and quantise the "matter" in an indefinite metric space, cf. ref. [14]).

Leaving this for the future, we would instead like to upgrade the quantum BRST formalism, while retaining positive definite metric quantisations of the matter observables, by putting forward a tentative suggestion for a reformulation of the physicality conditions which has the merit of emphasising the duality between physical states and operators in the quantum theory.

A form of duality has been discussed by Henneaux [14] who showed that, at least for finite-dimensional representation spaces, the set of physical operators (irrespective of ghost numbers) is isomorphic to the set of operators on physical states (of all ghost numbers). However, as we have already discussed, as soon as one tries to cut down the physical states by some restriction on the ghost number, this duality disappears as physical operators are naturally of ghost number zero whereas positive norm physical states are not eigenstates of ghost number, in general.

Inspired by the algebraic approach to quantum theory, we suggest an alternative way of defining physical states which uses the fact that any state $|\psi\rangle$ in an (indefinite) Hilbert space carrying a representation of the BRST and ghost operators defines a linear functional ω_ψ on the space, $\mathscr{A}$, of bounded operators on ${}^{\chi}\mathscr{H}\otimes\Lambda(\mathfrak{h})$:

$$\omega_\psi \equiv \langle\psi|A|\psi\rangle, \quad A\in\mathscr{A}. \tag{6.1}$$

Such functionals generate a certain vector space $\mathscr{A}'$, which is a subspace of the algebraic dual of $\mathscr{A}$.

As is well known, $\mathscr{A}$ is graded by the ghost number, i.e.

$$\mathscr{A} = \oplus_{n=-\infty}^{\infty}\mathscr{A}^{(n)}, \tag{6.2}$$

with $A\in\mathscr{A}^{(n)}$ if $[\hat{N}_{\mathrm{gh}}, A]=nA$. The restricted dual $\mathscr{A}'$ inherits this grading, with $\omega\in\mathscr{A}'_{(n)}$ if ω vanishes on all of $\mathscr{A}^{(m)}$ for $m\neq n$. Equivalently,

$$\omega\in\mathscr{A}'_{(n)} \Leftrightarrow \omega\Big(\big[\hat{N}_{\mathrm{gh}}, A\big]-nA\Big)=0\ \forall A\in\mathscr{A}. \tag{6.3}$$

We propose using the usual derivation δ on the space of operators:

$$\delta A=\delta_{\mathrm{BRST}}A=i[A,\hat{\Omega}]_{\pm}, \tag{6.4}$$

and defining the usual operator cohomology with respect to δ, with a physical operator being a cohomology class of ghost number zero, i.e.

$$\big[\hat{N}_{\mathrm{gh}}, A\big]=0. \tag{6.5}$$

The alteration we suggest is to define the BRST state homology by defining an operator ∂ on the space of linear functionals via

$$\partial\omega(A)\equiv\omega(\delta A). \tag{6.6}$$

Clearly $\partial^2=0$ since $\delta^2=0$. As δ maps $\mathscr{A}^{(n)}$ into $\mathscr{A}^{(n+1)}$, ∂ maps $\mathscr{A}'_{(n)}$ into $\mathscr{A}'_{(n-1)}$.

Thus we can define the state homology classes, H_n^{BRST}, at ghost number n, as those $\omega \in \mathscr{A}'_{(n)}$ satisfying

$$\partial\omega = 0 \tag{6.7}$$

with

$$\omega_1 \equiv \omega_2 \Leftrightarrow \omega_1 - \omega_2 = \partial\omega_3 \quad \text{for some } \omega_3. \tag{6.8}$$

An element of H_n^{BRST} is called pure if it can be represented in the form (6.1) for some $|\psi\rangle$. Physical states are then defined as the pure elements of H_0^{BRST}.

This definition is motivated by the way states are defined in C^*-algebra theory and has the advantage that the state and operator cohomology are automatically dual. Our hope is that the physical states, thus defined, are positive and correspond to the pure states on the constrained algebra $G \ltimes C^\infty(G/H)$ in the superselection sector χ, and that, by varying χ in the above construction, one gets all pure states. This conjecture is under investigation.

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