Classical and Quantum Particles in Galilean and Poincaré Spacetime

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Abstract

Classical elementary particles are identified with modified coadjoint orbits of the pertinent symmetry group G. Quantum elementary particles are identified with irreducible projective unitary representations of G. We discuss the mathematical context of classifying both classical and quantum elementary particles. The common theme is that of *universal covers* and *central extensions*. These allow us to classify the modified coadjoint orbits and projective unitary representations of G in terms of ordinary coadjoint orbits and (non-projective) unitary representations, respectively, of a certain extension of G. These extensions are, generally speaking, both topological and algebraic in nature.

We apply the formalism to spacetime symmetry groups. It turns out that these symmetry groups account for the *mass* and *spin* of elementary particles. (Properties like electric charge arise from other symmetry groups.) In particular, we take G to be the connected component of the Galilei group or the Poincaré group. The classification of irreducible projective unitary representations of the Poincaré group is well known due to Wigner [43]. We state the results of this classification, and do not concern ourselves with the derivation. That of the Galilei group was later given by Bargmann [4] and Lévy-Leblond [24], the results of which we similarly state without calculation.

The coadjoint orbits of the Poincaré group were first calculated by Souriau [36]. Those of the Galilei group are given in Guillemin & Sternberg [13]. Next to the results of this classification, we also give explicit calculations due to the relative novelty and obscurity of this formalism.

In the last section we give a summary of the results and a physical interpretation thereof. We conclude that elementary particles are labelled by two numbers: a real number m called the *mass*, and for $m > 0$ a non-negative number s called *spin*, and for $m = 0$ a (possibly negative) number h called *helicity*. In quantum mechanics s and h are half-integer valued, while in classical mechanics they are real-valued.

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Contents

1 Introduction

Over the course of history, the idea has developed that all matter is composed of indivisible, *elementary particles*. Larger objects, like the ones we see in day-to-day life, are tremendous hodgepodges of different types of elementary particles. The idea is at least old enough to date back to the ancient Greeks, to whom we owe the term 'atom' (probably deriving from the word *atomos*, meaning "indivisible").

However, these so-called indivisible particles do not always live up to their name. In the early 1800's, chemist, meteorologist and physicist John Dalton (1766–1844) came up with the idea that each chemical element is comprised of *atoms*. At the time, atoms were thought of as being truly indivisible. At the turn of the century, however, Nobel laureate in physics, Sir Joseph J. Thomson (1856–1940) experimentally showed the existence of electrons, and that they were a part of atoms. This meant that atoms were in fact not truly *atomos*. In the early 1900's, Ernest Rutherford (1871–1937) (*et al.*) experimentally showed that the atom was further comprised of a dense core, called the atomic nucleus, about which the electrons 'orbit'. Around 1918, he also discovered that the atomic nucleus of the hydrogen atom was a positively charged particle. He called this the proton. Around 1940 it was shown that the atomic nucleus was compromised of particles we now call nucleons, namely neutrons and protons. In the 1960's the idea of quarks was developed by Murray Gell-Mann (1929–) and other physicists. These quarks compose the so called hadrons, which include the nucleons. These are particles composed of either two or three quarks, where, for instance, the proton is comprised of two so-called up-quarks, and one down-quark. Today, these quarks are what we call elementary particles.

But history has shown that it is often unwise to call a particle truly indivisible. And indeed, nowadays we view these terms through a more pragmatic lens. That being said, the Standard Model, our current theory describing all known elementary particles, is often considered to be most accurate scientific theory known to mankind.

In this thesis we discuss the problem of the mathematical classification of elementary particles. Our mathematical bread-and-butter is the concept of a *group action*.

Definition 1.1. Let G be a group with identity element $e \in G$, and let X be a set with arbitrary element $x \in X$. A ***group action*** of G on X is a map $\varphi : G \times X \to X$ that satisfies $\varphi(e, x) = x$, and $\varphi(gh, x) = \varphi(g, \varphi(h, x))$ for all $g, h \in G$. Writing $g \cdot x := \varphi(g, x)$, these laws have the form

$$e \cdot x = x, \qquad (gh) \cdot x = g \cdot (h \cdot x).$$

We will come to see that elementary particles are really different incarnations of group actions. The mathematical context of the physical framework will determine further properties that the group action must have. This means that, in particular, the group actions will take different forms depending on whether we are discussing classical mechanics or quantum mechanics. One example of such a property, which is shared by both the classical- and quantum framework, is that the group action should be continuous in some form. (Continuity arises in most physical theories from very elemental considerations on the nature of measurement [11, Sec.II].) It is moreover clear that the group G will play a big part in the form of these group actions, and hence in the types of elementary particles we will discover.

In Part I we describe the mathematical prerequisites for the classification. Starting with introductory remarks on symmetry in physics, we outline the structure of the spacetime symmetry groups: the (identity components of the) Galilei and Poincaré groups. Closing Section 2, we provide a motivation for the definition of quantum elementary particles as irreducible projective unitary representations. In Section 3 we discuss in detail the structure of the spacetime symmetry groups. Part II is dedicated to the study of classifying the quantum elementary particles. The first two of its sections, Sections 4 and 5, discuss the concept of central extensions and the application thereof to the spacetime symmetry groups, respectively. In Section 6 we discuss how central extensions and universal covering groups are used to classify quantum elementary particles. We close by stating the (partly well-known) result of this classification. In Part III we concern ourselves with the classification of classical elementary particles. A short overview of the classical formalism of mechanics in terms of symplectic geometry is discussed in Section 7. Due to the technical level of the material in this part we will state a lot of results without proof. In Section 8 we define (twisted) coadjoint orbits, and provide more details as to why classical elementary particles are identified with these entities. Borrowing from results in Part II, we calculate the (twisted) coadjoint orbits of the spacetime symmetry groups in Section 9. Closing the thesis, in Section 10 we summarise the relevant results and provide a physical interpretation.

Major references for this work have been [6, 13–15, 21, 22, 24, 25, 33, 36, 38]. References to specific parts of these texts (and others) are noted throughout the thesis.

Acknowledgements. I would like to express my most sincere gratitude towards Klaas Landsman for suggesting this topic to me, thereby introducing me to this fascinating subject, and, of course, for his guidance throughout this project and giving extremely useful feedback on the manuscript you are now reading!

Part I

Symmetry in physics

2 Symmetries and elementary particles

Symmetries can be observed everywhere in daily life. One obvious manifestation is *rotation*. Rotate a perfect sphere about its origin by any angle in any direction, and it will look exactly the same. Rotate a cube about its primary axes by 90°, and it will look the same. Closely related examples are reflection symmetries. For instance, a triangle with (at least) two equal sides has (at least) one reflective symmetry axis, that is, an axis showing the mirror image of the other on each side. But symmetries also occur in nature; many flora and fauna show symmetry. (I shall leave it to the reader to imagine them.) The left and right portions of a human body look the same, classically demonstrated by Leonardo da Vinci's (1452–1519) *'Vitruvian Man'* (see Figure 1). This drawing is named after the Roman architect Marcus Vitruvius Pollio, who lived in the first century BCE. And indeed, symmetry occurs markedly in architecture as well. From ancient temples to modern skyscrapers, its occurrence is unmistakable. There, and most everywhere else, symmetry is seen as a sign of *beauty*.

But what exactly is symmetry? The notion of symmetry in some of the examples above is somewhat vague, and we feel necessity for a more rigorous definition. For lack of such a rigorous definition, we quote the famous physicist and mathematician Hermann K. H. Weyl (1885–1955) [41]:

> *"[Symmetry is] invariance of a configuration of elements under a group of automorphic transformations."*

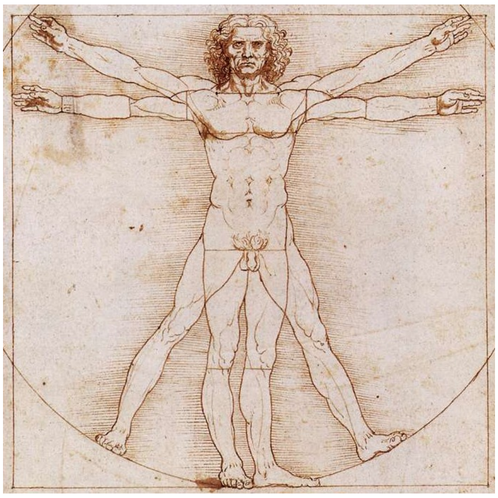


Figure 1: *The proportions of the human body according to Vitruvius*, as drawn by Leonardo da Vinci around 1490. It is supposed to illustrate the 'ideal' proportions of the human body, according to Marcus Vitruvius Pollio. (Original photograph taken from the public domain [44].)

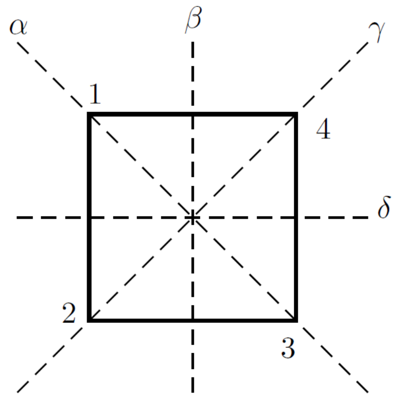


Figure 2: A square with vertices 1, 2, 3 and 4, and symmetry lines α, β, γ and δ.

As a purely mathematical notion we may state: a *symmetry* of a mathematical entity X is an invertible transformation $X \to X$ that preserves some property or structure of said entity. For example, then, rotations of 90°, 180° or 270° about the origin of a square are symmetries, since when performing these rotations the square returns exactly to the position it was before (especially when seen as a subset of the two-dimensional plane). Similar symmetries occur often in geometry, and are called *rotational symmetries*. But the square has additional symmetries. It can be reflected about four different axes; namely the two axes connecting the opposing corners (axes α and γ in Figure 2), and the two axes connecting the middles of the opposing ribs (axes β and δ). These are called *reflection symmetries*. In total this makes for eight transformations of the square under which it is invariant.

More abstractly, a symmetry may be a transformation that preserves, e.g., the *number* of elements (bijections of sets), *distances* (isometries on metric spaces), *linear structure* (linear isomorphisms), *angles* (orthogonal transformations), *multiplicative structure* (isomorphisms of groups), *topological structure* (homeomorphisms), *smooth structure* (diffeomorphisms), etc. In more concise terms: a symmetry is an *automorphism*, i.e., an isomorphism from a mathematical entity to itself.

Mathematically speaking, the concept of symmetry is captured by *group theory*. This can be seen by considering the natural properties of symmetries:

(G1) Given two symmetries, applying one after the other should again define a symmetry transformation (*closure*);

(G2) Given three symmetry transformations, the composition of the three should result in one well-defined transformation (*associativity*);

(G3) Doing nothing counts as a symmetry transformation (*identity*);

(G4) Lastly, any symmetry transformation can be undone by its inverse transformation (*inverses*).

Indeed, these properties define a *group*. Of a given mathematical entity X, we denote its group of all symmetries (i.e., automorphisms) by $\mathrm{Aut}(X)$, where the operation is that of composition. What the exact form of the elements of $\mathrm{Aut}(X)$ are will depend on the nature of X.

Symmetries, usually implemented via group theory, play a very important rôle in the natural sciences. For example, certain molecules may have rotational symmetries.

These may then be used to calculate certain chemical properties of the molecule. In solid state physics, similar techniques are used to calculate properties of crystals. But more importantly for the present thesis is the notion of symmetry in particle physics. For instance, one of the most fundamental symmetries of the universe is *charge-parity-time reversal* symmetry (abbreviated: CPT) in relativistic quantum field theory. The CPT transformation is comprised of three distinct discrete transformations. The first is charge conjugation. This transforms the charge of some given elementary particle to its negative. For instance, a particle with charge q becomes a particle with charge $-q$. Hence it transforms an electron into a positron, which is its antiparticle. The second is parity transformation, which transforms a system into its mirror image. This is achieved by sending every coordinate to its negation. The third and last is time reversal, which reverses the direction of time. This causes the momentum of each particle to be reversed. Performing these three transformations simultaneously comprises the full CPT transformation, which is believed to be a fundamental symmetry of the universe. (In any case, it is a symmetry of relativistic quantum field theory.) Note that, surprisingly, the charge, parity and time transformations in and of themselves do not define symmetries!

CPT is an example of a *discrete symmetry*. The transformations that represent the symmetry are finite in number. In fact, it is not hard to see that the CPT transformation forms its own inverse, so the corresponding group has only two elements. In this thesis, on the other hand, we shall mostly be interested in *infinite* symmetry groups. We have already seen an example of such a symmetry before; the rotational symmetries of a sphere. In particular, we are interested in so-called *Lie groups*, named after Norwegian mathematician Marius Sophus Lie (1842–1899), which are groups that simultaneously have the structure of a smooth manifold. (See Section 3 for the formal definition.) Symmetries based on Lie groups are prevalent, for example, in the Standard Model, which is the model in quantum field theory that describes all of the currently known elementary particles. These symmetries are somehow different from the ones we have encountered so far, in that they describe invariant properties of certain mathematical structures, rather than physical objects themselves.

We have seen everyday symmetries; of cubes and spheres and triangles, and more abstract symmetries; of molecules and crystals and charge-parity-and-time. One may wonder whether or not symmetry is a fundamental property of nature. May it be the case that we find a plethora of symmetries, simply because we are looking for them? Undeterred by this philosophical question, symmetry is undoubtedly a very useful tool in describing the universe, and we therefore consider it very much worth studying.

2.1 Galilean spacetime and principles of relativity

In both classical and quantum physics the general framework describes certain objects, specifically *particles*, in some background environment. This background is called *spacetime*. Points in spacetime are called *events*. For our purposes here, we take the mathematical structure of spacetime to be something like a smooth manifold endowed with some notion of distance, but possibly with additional structure.

One example is $\mathbb{R}^n$ as a vector space, endowed with the Euclidean metric (or standard inner product), which turns it into the Euclidean vector space. However, a vector space has a special element: the origin. This somehow seems undesirable in a physical model (where is the centre of the universe?), and therefore vector spaces may be replaced by *affine spaces*. These generalise Euclidean spaces in that the origin is 'removed' (by letting a vector space act on itself freely and transitively, to be thought of as translation), giving *Euclidean space* $\mathbb{E}^n$.

In Newtonian physics space and time are modelled by these affine spaces. Isaac Newton (1642–1726) believed that space and time are both *absolute*. Anyone occupying the spacetime would, despite their spatial position, always experience the same time as any other observer. Movement of objects is to be understood as movement with respect to some absolute frame[1].

An obvious first attempt at defining the appropriate spacetime for Newtonian physics is to take $\mathbb{E}^1 \times \mathbb{E}^3$, where the first component defines time, and the second component defines space[2]. However, this definition has some conceptual shortcomings, which we may understand by using the ideas of Galileo Galilei (1564–1642), an important Italian figure in physics and astronomy (but really in all of science). Consider, for instance, the positions of the letters on this page. Saying that they are the same at *this* very moment you are reading this as they were ten seconds ago, would imply that somehow the position of the paper has not changed at all. But really, you have probably moved this document around over the course of the past few seconds. Even if not (perhaps you are reading this digitally), the rotation of the Earth can certainly not be discounted, nor any other cosmic movement for that matter. Even though their positions with respect to the paper remains the same, it would be an amazing coincidence if the space $\mathbb{E}^3$, representing space, just so happened to co-move with the letters on this paper! The conclusion is that it is not useful to say that any point in space $\mathbb{E}^3$ is the *same* point (in the same $\mathbb{E}^3$) at any two distinct moments in time.

The solution to this is to consider spacetime as a so-called *fibre bundle* (see Figure 4 on page 69) over $\mathbb{E}^1$ *(time)*, with fibres $\mathbb{E}^3$ *(space)*. This is called *Galilean spacetime* [30, Sec.17.2], and we shall denote it by $\mathscr{G}$. In the bundle, time and space are disentangled, so that it is impossible to compare two spatial points when they occupy different fibres (i.e., occur at different times). Despite these conceptual intricacies, we will keep working with the idea of $\mathbb{E}^1 \times \mathbb{E}^3$ in mind.

For simplicity one often thinks of the Euclidean spaces $\mathbb{E}^n$ as *vector* spaces, as opposed to *affine* spaces, and we will do so from now on. Even though, as remarked earlier, vector spaces are conceptually inadequate as a model for space (or time), this can nevertheless be physically justified with the use of *coordinate systems*. A coordinate system is a nicely behaved smooth map that assigns to each point in an open subset of spacetime a point in $\mathbb{R}^4$. Such a point, say, $(t, x, y, z) \in \mathbb{R}^4$, is then called a *coordinate* for a certain event in spacetime. Since spacetime is modelled by a smooth manifold, at every point in spacetime there exists such a coordinate system, and the transitions between them are diffeomorphisms. In the case of Euclidean space $\mathbb{E}^n$ it is possible to pick one coordinate system that covers the entire space, and doing so amounts to specifying an 'origin', usually coinciding with the location of an observer. Newton's laws (or any other physical laws for that matter) are usually expressed in the resulting coordinates. This means that the form of these laws depend on the coordinate system. There are, then, certain coordinate systems in which Newton force law[3], $\mathbf{F} = m\mathbf{a}$, may be written down *"in its simplest form"* (all other

[1] This view is opposite to that of his nemesis Gottfried Wilhelm Leibniz (1646–1716), who believed that space and time were to be seen as the *relative* distances between physical objects. Movement of objects is then to be understood as movement with respect to other objects. We know that a car is moving because we see its motion with respect to the road, trees, and buildings. If there was a car in an otherwise empty space, we would not, Leibniz argues, be able to determine whether the car is moving at all.

[2] This space has been called *Aristotelian spacetime*, after ancient Greek philosopher Aristotle (384–322 BCE) [30, Sec.17.2].

[3] Newton's force law, also called Newton's second law (in the usual enumeration), is a relation

things being equal). These systems are called *Newtonian inertial reference frames* (or *inertial (reference) frames* for short), and in them, any particle on which no forces act will move with constant velocity, meaning that their path is a straight line that they traverse with constant speed. In other words, they are frames in which Newton's first law holds. Certainly not every coordinate system is an inertial reference frame. The classic example is that of a rotating coordinate system, in which there may be all kinds of 'fictitious' forces; namely the Coriolis-, centrifugal- and Euler forces. These cause particles to move in curved arcs, as opposed to straight lines, despite the fact that there is no actual force working on them.

The important notion is that it is always possible to switch between coordinate systems (in fact in a smooth way), as long as they both describe the same portion of spacetime. A transformation between coordinate systems can be realised by so-called transition functions. The following question arises:

Under which coordinate transformations are the laws of physics invariant?

And, specifically for our current discussion, under which coordinate transformations are Newton's laws of an inertial frame invariant? The answer is: under ***Galilean transformations***. These are characterised as follows:

1. First, we have time and space *translations*. In a given inertial frame we may represent an event in spacetime by a point $(t, \boldsymbol{x}) \in \mathbb{R}^4$, the first component representing time, and the latter components representing a position in space. A translation is represented by the map

$$(t, \boldsymbol{x}) \mapsto (t + s, \boldsymbol{x} + \boldsymbol{a}),$$

where $s \in \mathbb{R}$ and $\boldsymbol{a} \in \mathbb{R}^3$ are fixed.

2. Next, we have spatial *rotations* and *reflections*. The group of rotations and reflections in three-dimensional Euclidean space is denoted by $\mathrm{O}(3)$, the orthogonal group (see Section 3.2). This group acts on $\mathbb{R}^3$ canonically by rotating any given vector as prescribed by the group element. Then, a rotation is represented by the map

$$(t, \boldsymbol{x}) \mapsto (t, R\boldsymbol{x}),$$

where $R \in \mathrm{O}(3)$ is fixed, and $R\boldsymbol{x} \in \mathbb{R}^3$ denotes the vector $\boldsymbol{x}$ rotated according to R.

3. Lastly, we have the *uniform motions*, also called *(Galilean) boosts*. These are represented by the map

$$(t, \boldsymbol{x}) \mapsto (t, \boldsymbol{x} + \boldsymbol{v}t),$$

where $\boldsymbol{v} \in \mathbb{R}^3$ is a fixed vector, representing the *velocity* at which the new frame moves with respect to the original one.

It is of course possible to apply any combination of the above three transformations simultaneously. The fact that Newton's laws have the same form, even after applying these transformations, is called the *principle of Galilean relativity*. And every two inertial reference frames are related through a Galilean transformation. The principle thus states that we have a *symmetry of physical laws!* The symmetry group

between the force $\mathbf{F}$ acting on an object with mass m, and the acceleration $\mathbf{a}$ of the object. The acceleration is defined as the second time derivative of the position vector. Together with initial conditions, this law determines uniquely the motion of the object (but see [2, pp.3–4]). For this reason, it is known as an *equation of motion*.

here is called the ***Galilei group***, denoted Gal(3), which embodies the Galilean transformations. The elements of Gal(3) should therefore be characterised by a number $s \in \mathbb{R}$ describing temporal translation, a vector $\boldsymbol{a} \in \mathbb{R}^3$ describing spatial translation, another vector $\boldsymbol{v} \in \mathbb{R}^3$ describing the velocity of a uniform motion, and finally, an element $R \in \mathrm{O}(3)$ describing rotation or reflection. As a set we therefore define (following [24])

$$\mathrm{Gal}(3) := \{(s, \boldsymbol{a}, \boldsymbol{v}, R) : s \in \mathbb{R}, \boldsymbol{a}, \boldsymbol{v} \in \mathbb{R}^3, R \in \mathrm{O}(3)\}.$$

The group operation may not be obvious at a first glance, but it can be uncovered by considering the natural action of Gal(3) on the Galilean spacetime $\mathscr{G}$. Given an element $G = (s, \boldsymbol{a}, \boldsymbol{v}, R)$ in the group, and coordinates $(t, \boldsymbol{x})$ of an event in spacetime, the action is defined according to the following formula:

$$G(t, \boldsymbol{x}) := (t + s, R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}).$$

If $G' = (s', \boldsymbol{a}', \boldsymbol{v}', R')$ is another element in the group, we find that letting G' act after G gives the following coordinate:

$$\begin{aligned}(G'G)(t, \boldsymbol{x}) = G'(t + s, R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}) &= ((t + s) + s', R'(R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}) + \boldsymbol{v}'(t + s) + \boldsymbol{a}') \\ &= (t + (s + s'), R'R\boldsymbol{x} + (R'\boldsymbol{v} + \boldsymbol{v}')t + (R'\boldsymbol{a} + \boldsymbol{v}'s + \boldsymbol{a}')).\end{aligned}$$

Despite looking quite complicated, this motivates the definition of the following group operation on Gal(3):

$$(s', \boldsymbol{a}', \boldsymbol{v}', R') \cdot (s, \boldsymbol{a}, \boldsymbol{v}, R) := (s + s', R'\boldsymbol{a} + \boldsymbol{v}'s + \boldsymbol{a}', R'\boldsymbol{v} + \boldsymbol{v}', R'R). \tag{2.1}$$

It is easy to see that this makes Gal(3) into an actual group, with identity element $(0, 0, 0, I)$, where $I \in \mathrm{O}(3)$ is the identity rotation. The inverse of a given element can be calculated from (2.1), and is given by the following formula:

$$(s, \boldsymbol{a}, \boldsymbol{v}, R)^{-1} = (-s, R^{-1}(\boldsymbol{v}s - \boldsymbol{a}), -R^{-1}\boldsymbol{v}, R^{-1}).$$

2.2 Minkowski space and principles of special relativity

In special relativity the situation is slightly different: space and time are intertwined. This conclusion is reached after imposing the fundamental postulate that *the speed of light is the same in every inertial frame*. In particular, this leads us to abandon the notion of absolute time, and hence the fibre bundle structure of $\mathscr{G}$.

The concept of an isometry group is something we can define for a mathematical structure that has some notion of distance. One of the most elementary mathematical structures that has this notion is that of a *metric space*. Let X be some set, with arbitrary elements $x, y, z \in X$. A ***metric*** on X is a function $d : X \times X \to \mathbb{R}$ with the following properties:

1. The distance between the points x and y is the same as the distance between the points y and x; that is: $d(x, y) = d(y, x)$.

2. The distance between two points x and y is zero if and only if they are the same points; that is: $d(x, y) = 0$ if and only if $x = y$.

3. The triangle inequality holds, meaning that

$$d(x, y) \leqslant d(x, z) + d(z, y).$$

The pair (X, d) is called a ***metric space***.

In particular, we can now associate to (X, d) its *isometry group*, which contains all functions that 'preserve the metric'. Specifically, an *isometry* of (X, d) is a function $f : X \to X$ such that for all points $x, y \in X$ we have $d(f(x), f(y)) = d(x, y)$, that is, such that the distance between $f(x)$ and $f(y)$ is the same as the distance between x and y. The ***isometry group*** of (X, d), denoted $\mathrm{Isom}(X, d)$, is defined as the set of all its bijective isometries, endowed with the operation of composition. Of particular general interest to us is the isometry group of the Euclidean vector space $\mathbb{E}^n$. The metric of this space is defined via the usual inner product:

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}; \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{i=1}^{n} x_i y_i.$$

This inner product defines the Euclidean norm (also known as the ℓ^2 norm) by the formula $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$, and in turn, the Euclidean metric by the formula $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$. An isometry of Euclidean space is therefore a function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ we have $\|f(\boldsymbol{x}) - f(\boldsymbol{y})\| = \|\boldsymbol{x} - \boldsymbol{y}\|$. The isometry group $\mathrm{Isom}(\mathbb{E}^n)$ of the n-dimensional Euclidean vector space is called the ***Euclidean group***, and we shall denote it by $\mathrm{E}(n)$. This group is well understood. For one, we know its structure is that of a *semi-direct product*: $\mathrm{E}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$. (We will prove this in Section 3.3.) The Galilean group $\mathrm{Gal}(3)$ is not quite an isometry group (at least not of spacetime), but it is closely related to the Euclidean group $\mathrm{E}(3)$, as we will see in Section 3.3.

The spacetime symmetry group of special relativity, on the other hand, *is* exactly the isometry group of the special relativistic spacetime. This spacetime is the well-known four-dimensional *Minkowski space* $\mathbb{M}^4$, named after German mathematician Hermann Minkowski (1864–1909). The difference between this space and Euclidean space is the manner in which they measure distance. As a set, $\mathbb{M}^4$ is simply $\mathbb{R}^4$, but now it is endowed with the following form:

$$\langle \cdot, \cdot \rangle : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}; \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \langle \boldsymbol{x}, \boldsymbol{y} \rangle := x_1 y_1 - \sum_{i=2}^{4} x_i y_i.$$

This form defines something like a metric on Minkowski space, just as done above for the Euclidean spaces:

$$d(\boldsymbol{x}, \boldsymbol{y})^2 = (x_1 - y_1)^2 - \sum_{i=2}^{4} (x_i - y_i)^2.$$

Strictly speaking, however, this is not actually a metric. Namely, there may exist *distinct* non-zero vectors in spacetime whose distance in Minkowski space is nevertheless zero, violating the second axiom in the definition of a metric. In this physical context, these points are then said to be *lightlike* separated. It is even possible for the squared distance between two events to be negative, something that is not possible for a true metric (the distance between two points in a metric space is always non-negative); these points are then said to be *spacelike* separated. It is impossible to go from one of these point to the other, without breaking the speed of light speed limit. On the other hand, two events whose distance squared is positive *can* be reached without breaching the limit, and are called *timelike* separated.

This makes $\mathbb{M}^4$ into a *pseudo*metric space, and its distance function d is called a *pseudometric*. It is nevertheless possible to define an isometry group. And indeed, this group, denoted $\mathrm{Poin}(1,3) := \mathrm{Isom}(\mathbb{M}^4)$, is called the ***Poincaré group***, named after French mathematician Jules Henri Poincaré (1854–1912), and it is the fundamental spacetime symmetry group of special relativity. Its structure is similar to that of the Euclidean group, as we will see in Section 3.3.3. Just as for the Galilei group, the Poincaré group consists of spacetime translations:

$$(t, \boldsymbol{x}) \mapsto (t + s, \boldsymbol{x} + \boldsymbol{a}).$$

But now, instead of spatial rotations and Galilei boosts, we have *Lorentz transformations*. These can be thought of as the generalisation of orthogonal transformations on the Euclidean space to Minkowski space. They will be defined and discussed in more detail in Section 3.2.3.

2.3 Representation theory in quantum mechanics

We now have an idea what the symmetry groups are that describe the invariance of the laws of physics, both in a Newtonian and a relativistic setting. But how are these symmetries incorporated into the physical theory? To understand this, we need to have a general grasp of the underlying mathematical formalism for both classical- and quantum mechanics.

In quantum physics *pure states* may be identified with unit vectors of a complex Hilbert space $\mathscr{H}$. In the case that the "maximal amount of information is available" [18], the entire state space of a physical system can be defined in terms of these pure states. (In a statistical setting we need to introduce so-called *density operators*.) From such a pure state one usually deduces expectation values of physical quantities and probabilities for certain physical events to occur. Conventionally, these calculations are based on unit vectors in $\mathscr{H}$ (i.e., pure states). Still, any non-zero vector $\psi \in \mathscr{H}$ defines a pure state by multiplying it by the reciprocal of its norm, allowing us to think of even non-unit vectors as (non-normalised) physical states. In this formalism, these calculations turn out the same when multiplying the unit vectors by *complex phases*. Therefore, the physical state that an element $\psi \in \mathscr{H}$ represents is *invariant* under scalar transformations, meaning that for any non-zero complex number $\lambda \in \mathbb{C}$ the vector $\lambda\psi$ gives the same physical description of a system as does the state ψ.

This motivates the introduction of a relation $\sim$ on $\mathscr{H}$ that identifies vectors which only differ by scalar multiplication. That is, if ψ and ϕ are elements in $\mathscr{H}$, we say that they are equivalent, and in that case we write $\psi \sim \phi$ iff there exists a non-zero complex number $\lambda \in \mathbb{C}$ such that $\psi = \lambda\phi$. It is straightforward to verify that $\sim$ is an equivalence relation. The equivalence classes of $\mathscr{H} \setminus \{0\}$ under this equivalence relation will form the true state space of a quantum system[4], called the ***projective Hilbert space***:

$$\mathbf{P}(\mathscr{H}) := (\mathscr{H} \setminus \{0\})/{\sim}.$$

The equivalence class $[\psi]$ in $\mathbf{P}(\mathscr{H})$ of some non-zero vector $\psi \in \mathscr{H}$ is sometimes called the *ray* of ψ.

2.3.1 Symmetries of the quantum state space

Let us denote by $P : \mathscr{H} \setminus \{0\} \to \mathbf{P}(\mathscr{H})$ the canonical projection that sends a non-zero element in the Hilbert space to its equivalence class: $\psi \mapsto P(\psi) = [\psi]$. This map is trivially surjective. Given two elements $P(\psi)$ and $P(\phi)$ in the state space $\mathbf{P}(\mathscr{H})$, we

[4] The set-theoretic notation $A \setminus B$ is meant to denote the set A with all elements it shares with B taken out. So $\mathscr{H} \setminus \{0\}$, as a set, is the Hilbert space $\mathscr{H}$ without the origin.

define their ***transition probability*** as

$$\delta(P(\psi), P(\phi)) := \frac{|\langle\psi, \phi\rangle|^2}{\|\psi\|^2 \|\phi\|^2}, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ is the inner product of the Hilbert space. In the case that we restrict our attention to normalised vectors, as is usual, (2.2) can be simplified by leaving out the norms in the denominator. To make the notation somewhat less tedious, one often identifies $\psi \in \mathscr{H}$ with $P(\psi) \in \mathbf{P}(\mathscr{H})$, and in turn one may then write $\delta(\psi, \phi)$ to denote the value of $\delta(P(\psi), P(\phi))$. The usual physical interpretation of the transition probability $\delta(\psi, \phi)$ is that it represents the probability that (appropriate) measurements upon the state ψ will yield a result corresponding to the state ϕ.

We are interested in bijections $T : \mathbf{P}(\mathscr{H}) \to \mathbf{P}(\mathscr{H})$ that leave the transition probability invariant, in the sense that for all $\psi, \phi \in \mathbf{P}(\mathscr{H})$:

$$\delta(T\psi, T\phi) = \delta(\psi, \phi).$$

Such a map is called a ***projective automorphism***, or a *projective transformation* of $\mathscr{H}$, and these are exactly the symmetries of the quantum system. The inverse and composition of projective automorphisms again form projective automorphisms; again we find a group structure. The set of all projective automorphism together with composition is a group denoted by $\mathrm{Aut}(\mathbf{P}(\mathscr{H}))$, which is called the ***symmetry group*** of the quantum state space [33].

Symmetries of the quantum state space (i.e., projective automorphisms) arise already from certain physical considerations we have discussed before; in particular from the principles of relativity [43]. The form of the vectors in a Hilbert space may depend on the coordinate system of the observer. This is the case, for example, for the wave functions in the Hilbert space $L^2(\mathbb{R}^3)$ of square-integrable functions on $\mathbb{R}^3$ (which is the Hilbert space corresponding to, for example, a massive spinless particle in three-dimensional Euclidean space). By the principle of relativity, then, any inertial coordinate transformation (i.e., one described by an element of the Poincaré group) should not change the outcome of our experiments, and any wave function in one coordinate frame corresponds to some (non-unique) wave function in the other coordinate system. Say we have two wave functions ψ and ϕ in one coordinate frame, and two corresponding wave functions ψ' and ϕ' in the other coordinate system, respectively. The principle of relativity then states that

$$\delta(\psi, \phi) = |\langle\psi, \phi\rangle|^2 = |\langle\psi', \phi'\rangle|^2 = \delta(\psi', \phi'),$$

meaning that a Poincaré transformation defines a projective automorphism on the state space.

What exactly is meant by 'defines' is clarified by the following theorem from 1931 [42], due to Eugene P. Wigner (1902–1995).

Theorem 2.1 (Wigner's Theorem)**.** *Every projective automorphism T on $\mathbf{P}(\mathscr{H})$ arises from either a unitary or an anti-unitary operator U on $\mathscr{H}$, and U is determined uniquely by T up to a complex phase. (Cf. [33, Thm.3.3].)*

(The exact way in which the projective automorphisms arise from the unitary operators is explained in Section 6.2.) In other words, if the particular group element $L \in \mathrm{Poin}(1,3)$ is responsible for the Poincaré transformation from the unprimed system to the primed system, there is a unitary or an anti-unitary operator $D(L) : \mathscr{H} \to \mathscr{H}$, uniquely determined up to phase by L, which realises the transformation between the two coordinate systems: $\psi' = D(L)\psi$. Given two Poincaré transformations $L_1, L_2 \in \mathrm{Poin}(1,3)$, the wave function of the coordinate system obtained by applying either these two transformations simultaneously, or one after the other, should represent the same physical system. In other words, the two vectors $D(L_2L_1)\psi$ and $D(L_2)D(L_1)\psi$, assuming they are normalised, can only differ by a complex phase. This implies that

$$D(L_2)D(L_1) = \omega(L_2, L_1)D(L_2L_1),$$

where $\omega(L_2, L_1)$ is some complex number of unit modulus, depending on L_1 and L_2. The map D, which to every element of the symmetry group $\mathrm{Poin}(1,3)$ assigns a unitary or an anti-unitary operator on the Hilbert space, is known as a ***projective unitary representation*** of the group $\mathrm{Poin}(1,3)$. We will discuss these in more detail in Section 6, but first, more abstractly, in Section 4.2. It may so happen that $\omega(L_2, L_1) = 1$ for every two Poincaré transformations, in which case D becomes a *true* (or *ordinary*) unitary representation of the Poincaré group by the Hilbert space $\mathscr{H}$. Ordinary unitary representations are easier to calculate than projective ones, and have been extensively studied in the literature. For instance, the so-called 'irreducible' unitary representations (see below) of the special unitary group $\mathrm{SU}(2)$ are well-known, and sometimes even categorised in physics textbooks on quantum mechanics. It turns out that these give the desired representations of the rotation group $\mathrm{SO}(3)$. (See Section 3.2 for the definition of $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$.) The formal definition of a representation is as follows:

Definition 2.2. Let G be a group and let V be a vector space over a field k. A ***(linear) representation*** of G over V is a group homomorphism $\sigma : G \to \mathrm{GL}(V, k)$. Here $\mathrm{GL}(V, k) := \mathrm{Aut}(V)$ is the automorphism group of V (endowed with the operation of composition), containing all k-linear isomorphisms on V.

A representation is really just a different incarnation of a group *action* (recall Definition 1.1). Suppose that we have a representation $\sigma : G \to \mathrm{GL}(V, k)$. It may then be naturally associated to the group action

$$G \times V \to V; \quad (g, v) \mapsto \sigma(g)(v).$$

Note here that $\sigma(g) \in \mathrm{GL}(V, k)$, so it is in fact a linear map $V \to V$. It is easy to verify that this does indeed define a group action. This is the intuitive way to think about group representations.

In the context of the present thesis, we will be especially interested in so-called *irreducible* projective unitary representations of the spacetime symmetry groups. The physical motivation for this is the following [32]. Suppose that we have some Hilbert space $\mathscr{H}$ that represents some real physical system. The objects in this system are comprised of elementary particles, and so we expect certain subspaces of $\mathscr{H}$ to correspond to the Hilbert space of any of those single elementary particles. The pertinent symmetry group, say, the Poincaré group, can *act* on the Hilbert space $\mathscr{H}$ via a projective unitary representation D (in the sense of Definition 1.1). In particular, if $L \in \mathrm{Poin}(1,3)$, we can apply the operator $D(L)$ to every element in the Hilbert space to obtain the transformed Hilbert space $\mathscr{H}' = D(L)\mathscr{H}$, whose physical interpretation is the same as that of $\mathscr{H}$. There may then be subspaces of the Hilbert space $\mathscr{H}$ that, when transformed by $D(L)$, also remain physically invariant. Such systems are called *invariant subsystems*. We expect elementary particles to be such invariant subsystems. Therefore, if $\mathscr{H}_1 \subseteq \mathscr{H}$ is the Hilbert space corresponding to the states of a single elementary particle occupying the bigger system, we require that the transformed system $D(L)\mathscr{H}_1$ still corresponds to that particular single elementary particle. In particular, this means that if we restrict D to $\mathscr{H}_1$, we again obtain a projective unitary representation of the Poincaré group. The crucial point is now that an *elementary* particle has no further subsystem that is again an invariant subsystem. In the group theoretical terminology, this means that the restricted representation D to $\mathscr{H}_1$ is *irreducible*:

Definition 2.3. Consider a representation $\sigma : G \to \mathrm{GL}(V, k)$. A linear subspace $W \subseteq V$ is called an *invariant subspace* if $\sigma(g)(w) \in W$ for all $g \in G$ and $w \in W$. The representation is called ***irreducible*** if V is at least one-dimensional, and the only invariant subspaces are the trivial vector space and V itself.

It is therefore that elementary particles are mathematically identified with irreducible projective unitary representations of the pertinent symmetry group.

2.4 Coadjoint orbits in classical mechanics

The mathematical context of classical mechanics is somewhat different to the Hilbert space formalism of quantum mechanics. Instead of a projective Hilbert space, the rôle of the state space is now played by a *symplectic manifold* (or more generally, by a *Poisson manifold*). A classical symmetry is then a *symplectomorphism* of the state space (replacing the notion of projective automorphisms). The pertinent symmetry group can once again *act* on the state space, this time via so called *Hamiltonian group actions* (replacing the notion of a unitary representation). We forgo the formal definition of these terms for now (and postpone them to Section 7), since they rest on the concepts we will define (mostly) in Section 3. They are presented in Sections 7 and 8.

The notion of an elementary particle now becomes that of a symplectic manifold, together with a certain type of *transitive* symplectic action. In the literature this is sometimes called a *symplectic homogeneous space*. Transitivity simply means that every state in the system can be reached from another by virtue of a symmetry transformation acting on the manifold. See Section 8.4 in particular for the formal definition of classical elementary particles. We will see in Theorem 8.10 that these particular types of symplectic homogeneous spaces are classified by the *orbits* of a very particular type of action: the ***twisted coadjoint action***. These so-called ***(twisted) coadjoint orbits*** replace the notion of irreducible (projective) unitary representations.

3 Structure of the spacetime symmetry groups

The mathematical framework necessary for our discussions rests largely on the concept of a *Lie group*. We state the definition right away:

Definition 3.1. A ***Lie group*** is a group G, endowed with the structure of a smooth manifold such that the product map $G \times G \to G : (g, h) \mapsto gh$ and inversion map $G \to G : g \mapsto g^{-1}$ are smooth.

The general theory of Lie groups is quite abstract, but very powerful, allowing us (among other things) to define the notion of differentiation of functions on a group. We shall first give a short exposition of the general theory, and then in the next section move on to *matrix Lie groups*. For details on the theory of smooth manifolds we refer to the lecture notes [28].

Suppose that G is a Lie group with identity element $e \in G$. Since G is a manifold, we may consider the tangent space T_eG at the identity element, which will play an important rôle. The tangent space is a vector space with the same dimension as G (as a manifold), to be thought of as the 'velocities' of smooth curves running through some fixed point. Formally, the *tangent space* T_pG at any point $p \in G$ may be defined as the space of all derivations of smooth functions at that point: $\mathrm{Der}_p(C^\infty(G), \mathbb{R})$. Here we denote by $C^\infty(G)$ the vector space of all smooth functions on G, where addition and scalar multiplication is defined in a pointwise fashion. If we choose a tangent vector at every point on the manifold in a smooth way, the result is a vector field. If X is a vector field, we denote its value at the point p by X_p. More formally, a *vector field* is a smooth section of the tangent bundle TG. The space of all vector fields is linearly isomorphic to the space of all derivations $\mathrm{Der}(C^\infty(G))$, seen as a $C^\infty(G)$-module. A derivation $D \in \mathrm{Der}(C^\infty(G))$ is a linear map $D : C^\infty(G) \to C^\infty(G)$ such that $D(fg) = fD(g) + D(f)g$ for any two smooth functions $f, g \in C^\infty(G)$. Given the identification of vector fields with derivations, we may define a real bilinear operation with the use of their compositions:

$$[\cdot,\cdot] : \mathrm{Der}(C^\infty(G)) \times \mathrm{Der}(C^\infty(G)) \to \mathrm{Der}(C^\infty(G)); \quad (X, Y) \mapsto [X, Y] := X \circ Y - Y \circ X.$$

This operation, called the *commutator* of vector fields, is well defined (in the sense that $[X, Y]$ is again a vector field), skew-symmetric, and satisfies the so-called *Jacobi identity*:

$$\forall X, Y, Z \in \mathrm{Der}(C^\infty(G)) : \quad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Therefore, the space of all vector fields form a concrete example of what is called a *Lie algebra*:

Definition 3.2. A real bilinear operation, defined on some vector space, that is skew-symmetric and satisfies the Jacobi identity, i.e., that satisfies the following properties

1. $[X, Y] = -[Y, X],$ (skew-symmetry)
2. $[aX + bY, Z] = a[X, Y] + b[Y, Z],$ (bilinearity)
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0,$ (Jacobi identity)

for all $X, Y, Z \in \mathrm{Der}(C^\infty(G))$ and $a, b \in \mathbb{R}$, is called a ***(real) Lie bracket***. A ***Lie algebra*** is a real vector space endowed with a Lie bracket.

We want to associate a canonical Lie algebra $\mathrm{Lie}(G)$ to our Lie group G. This is done as follows. In any group we can define the *left translation map*:

$$\lambda_g : G \to G; \quad h \mapsto gh,$$

for any fixed element $g \in G$. By definition of a Lie group, this map is smooth, and in fact, it is a diffeomorphism. Its differential $\mathrm{d}_e\lambda_g : T_eG \to T_gG$ maps tangent vectors at the identity to tangent vectors at the point g. This map defines a vector field X by the formula $X_g := \mathrm{d}_e\lambda_g(v)$, which is uniquely determined by the tangent vector $v \in T_eG$. The vector field X is called the *left invariant extension* of v, and is denoted v^L. The space of all left invariant extensions is linearly isomorphic to the tangent space T_eG. This allows us to extend the commutator on vector fields to a 'commutator' of tangent vectors, leading us to the following definition.

Definition 3.3. Let G be a Lie group. The ***Lie algebra*** of G is the vector space $\mathrm{Lie}(G) := T_eG$, usually denoted by the lower case Fraktur letters, in this case $\mathfrak{g}$, together with the Lie bracket

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}; \quad (u, v) \mapsto [u, v] := [u^L, v^L]_e.$$

On the right hand side the expression $[u^L, v^L]_e$ denotes the value of the vector field $[u^L, v^L]$ at the identity.

The Lie algebra of a Lie group is an important instance of the general notion of a Lie algebra. Its importance arises because a great deal of the structure of G is encoded in its Lie algebra. This is in part due to the *exponential map*. Formally, this is the smooth map $\exp : \mathfrak{g} \to G$ that moves vectors in T_eG along the flow lines of their left invariant extension for one unit of time.

Definition 3.4. Let G and H be Lie groups. We say the map $F : G \to H$ is a ***homomorphism of Lie groups*** if it is a smooth group homomorphism. That is, if F is smooth and $F(g_1 g_2) = F(g_1)F(g_2)$ for all $g_1, g_2 \in G$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. We say the map $f : \mathfrak{g} \to \mathfrak{h}$ is a ***homomorphism of Lie algebras*** if it is linear and respects the Lie bracket structure. The latter means that $f([X_1, X_2]) = [f(X_1), f(X_2)]$ for all $X_1, X_2 \in \mathfrak{g}$. Note that on the left hand side the bracket is that of $\mathfrak{g}$, while on the right hand side it is that of $\mathfrak{h}$.

The exponential map now provides a relation between Lie group and Lie algebra homomorphisms.

Proposition 3.5. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Any Lie group homomorphism $F : G \to H$ induces a Lie algebra homomorphism by*

$$\mathrm{d}_e F : \mathfrak{g} = T_eG \to \mathfrak{h} = T_eH,$$

and the following diagram commutes[5]*:*

$$\begin{tikzcd} G \arrow[r, "F"] & H \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\mathrm{d}_e F"'] & \mathfrak{h} \arrow[u, "\exp"'] \end{tikzcd}.$$

For the sake of simplicity, in this thesis we shall restrict ourselves to *connected* Lie groups. A topological space is said to be ***connected*** when it cannot be written as the disjoint union of two non-empty open sets. Any Lie group contains a connected Lie group, via the result of the next elementary proposition.

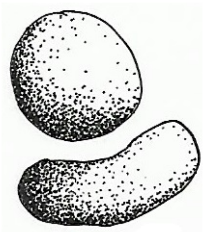
Proposition 3.6. *The identity component of any Lie group is a closed normal subgroup.*

Here, the *identity component* is the largest connected subset containing the identity element. The proof of the topological part can be found in, for example, [29, Lem.9.1.9], and the algebraic part is quite easy. Next to the notion of connectedness, we have the notion of *simple connectedness*. We say a topological space is ***simply connected*** when every continuous loop can be continuously contracted into a point. In this sense, a simply connected space has no 'holes' (cf. Figures 3b and 3c). Simply connected Lie groups behave nicely in that they allow for the converse of Proposition 3.5 to hold:

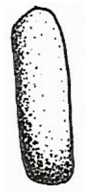
Proposition 3.7. *Let G and H be Lie groups, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Furthermore, suppose that G is simply connected. Then for any Lie* algebra *homomorphism $f : \mathfrak{g} \to \mathfrak{h}$ there exists a* unique *Lie* group *homomorphism $F : G \to H$, such that $\mathrm{d}_e F = f$.*

This proposition is proven in, for example, [45, Prop.1.20].

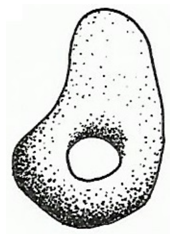
[5] A diagram is said to *commute* if the composition of any sequence of arrows with the same start- and end points gives the same map. In this particular instance commutativity means $F \circ \exp = \exp \circ \, \mathrm{d}_e F$.



(a) Disconnected space.



(b) Connected, simply connected space.



(c) Connected, non-simply connected space.

Figure 3: Illustrations of (from left to right) a disconnected space, a connected space that is also simply connected, and a connected space that is nevertheless not simply connected. (Note: in some conventions Figure 3a may be considered simply connected, while in others it may not. There will be no confusion for us.)

3.1 Matrix Lie groups and their Lie algebras

The generality of the above theory is vast, and abstract. There is, however, one very concrete family of Lie groups that will (more than) suffice for our purposes here. In fact, all of the Lie groups we will need are of this kind. These are the *matrix Lie groups*.

The *general linear group* $\mathrm{GL}(V, k)$ of any vector space V over the field k is the group of its automorphisms, i.e., the group of all k-linear isomorphisms mapping the vector space to itself. Its operation is that of composition. When it is clear what the underlying field is, we may write $\mathrm{GL}(V)$ instead. It is a well-known fact from linear algebra that

$$\mathrm{GL}(n, \mathbb{C}) := \mathrm{GL}(\mathbb{C}^n) = \{M \in \mathrm{M}_n(\mathbb{C}) : \det(M) \neq 0\},$$

where $\mathrm{M}_n(\mathbb{C})$ is the space of all $n \times n$ matrices with complex entries. In this case the operation of composition is realised via the usual matrix multiplication. $\mathrm{GL}(n, \mathbb{C})$ will serve as our '*proto*-Lie group'. In order for this group to be considered a Lie group, we need to at least define a topology on it. To do this, we identify the space of all matrices $\mathrm{M}_n(\mathbb{C})$ with the complex space $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$, together with the standard Euclidean topology (induced by the Euclidean metric). Convergence of matrices becomes a question of component-wise convergence. Since the determinant map $\det : \mathrm{M}_n(\mathbb{C}) \to \mathbb{C}$ is continuous (its expression is polynomial in terms of the matrix components) and $\mathbb{C} \setminus \{0\}$ is open, we find that $\mathrm{GL}(n, \mathbb{C}) = \det^{-1}(\mathbb{C} \setminus \{0\})$ is an open subset of $\mathrm{M}_n(\mathbb{C})$. It can therefore be endowed with the structure of a (real) smooth manifold of dimension $2n^2$. Moreover, since the multiplication and inversion operations in $\mathrm{GL}(n, \mathbb{C})$ are of polynomial nature, they are smooth, and hence we see that the general linear matrix groups define Lie groups in the sense of Definition 3.1.

Definition 3.8. A ***matrix Lie group*** G is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$, for some $n \in \mathbb{N}$. This means that, given a convergent sequence $(A_n)_{n \in \mathbb{N}}$ in G, either its limit lies in G or in $\mathrm{M}_n(\mathbb{C}) \setminus \mathrm{GL}(n, \mathbb{C})$. (The reason that the limit may not be an invertible matrix is that $\mathrm{GL}(n, \mathbb{C})$ is not closed in $\mathrm{M}_n(\mathbb{C})$.)

It is not obvious that matrix Lie groups (besides $\mathrm{GL}(n, \mathbb{C})$) are Lie groups in the sense of Definition 3.1. For a proof that they are, we refer to [15, Ch.3].

In the case of matrix groups, the relation between Lie groups and their Lie algebras, formed by the exponential map, is very pronounced. For matrix Lie groups the exponential map coincides with the *matrix exponential*. Given $X \in \mathrm{M}_n(\mathbb{C})$, its exponential is defined as the convergent power series

$$\exp(X) = e^X := \sum_{m=0}^{\infty} \frac{1}{m!} X^m.$$

We adopt the convention that $X^0 = I$, where $I \in \mathrm{M}_n(\mathbb{C})$ is the identity matrix. We state some properties of the matrix exponential, the proof of whose can be found in [14,15].

Proposition 3.9. *Let X and Y be two matrices in $\mathrm{M}_n(\mathbb{C})$, and $M \in \mathrm{GL}(n\mathbb{C})$ an invertible matrix. We denote the matrix transpose of X by $X^\top$, and the Hermitian transpose by $X^\dagger$. The matrix exponential has the following properties:*

1. $e^{X^\top} = (e^X)^\top$ *and* $e^{X^\dagger} = (e^X)^\dagger$;
2. $e^{MXM^{-1}} = Me^X M^{-1}$;
3. $\det(e^X) = e^{\mathrm{Tr}(X)}$;
4. $e^{X+Y} = \lim_{m\to\infty}(e^{X/m}e^{Y/m})^m$;
5. *If $XY = YX$ then the above simplifies to* $e^{X+Y} = e^X e^Y$;
6. $(e^X)^{-1} = e^{-X}$;
7. *And lastly, we have*
$$\left.\frac{\mathrm{d}}{\mathrm{d}t} e^{tX}\right|_{t=0} = X. \tag{3.1}$$

The Lie algebra can now be directly defined in terms of the exponential:

Definition 3.10. Let $G \subseteq \mathrm{GL}(n, \mathbb{C})$ be a matrix Lie group. The ***matrix Lie algebra*** $\mathrm{Lie}(G)$ of G, again denoted by $\mathfrak{g}$, is defined as the set

$$\mathfrak{g} := \{X \in \mathrm{M}_n(\mathbb{C}) : \forall t \in \mathbb{R} : e^{tX} \in G\},$$

together with the usual vector space structure of $\mathrm{M}_n(\mathbb{C})$ and the matrix commutator bracket:

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}; \quad (X, Y) \mapsto [X, Y] := XY - YX.$$

The following proposition proves that the matrix Lie algebra is a linear subspace of $\mathrm{M}_n(\mathbb{C})$, and that it is closed under the matrix commutator (i.e., that the bracket above is well-defined).

Proposition 3.11. *Let $\mathfrak{g}$ be the matrix Lie algebra belonging to some matrix Lie group G. Then the following holds:*

1. *The zero matrix* 0 *is an element of* $\mathfrak{g}$;
2. $\mathfrak{g}$ *is closed under real scalar multiplication of matrices;*
3. $\mathfrak{g}$ *is closed under component-wise matrix addition;*
4. *For every $g \in G$ and $X \in \mathfrak{g}$ we have $gXg^{-1} \in \mathfrak{g}$;*
5. $\mathfrak{g}$ *is closed under the matrix commutator, i.e., for all $X, Y \in \mathfrak{g}$ we have $[X, Y] \in \mathfrak{g}$.*

The proof makes use of Proposition 3.9 (see [14, Sec.16.5] for more details).

3.2 The classical Lie groups

There are several specific Lie groups that have an important place in the contemporary literature, and have come to be dubbed the 'classical Lie groups'. It is these classical groups (for lack of a better term) that play fundamental rôles in physical symmetries.

First we must note that the *real* invertible matrices $\mathrm{GL}(n, \mathbb{R})$ form a matrix Lie group. What are the Lie algebras $\mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{gl}(n, \mathbb{R})$ of $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{GL}(n, \mathbb{R})$, respectively? A matrix $X \in \mathrm{M}_n(\mathbb{C})$ is in $\mathfrak{gl}(n, \mathbb{C})$ if and only if for each $t \in \mathbb{R}$ the matrix e^{tX} is invertible, i.e., if and only if for each t we have $\det(e^{tX}) \neq 0$. Using Proposition 3.9 we find that this inequality holds if and only if $e^{\mathrm{Tr}(tX)} = e^{t\,\mathrm{Tr}(X)} \neq 0$, which we know to always be the case. Therefore $\mathfrak{gl}(n, \mathbb{C}) = \mathrm{M}_n(\mathbb{C})$. Similarly we find $\mathfrak{gl}(n, \mathbb{R}) = \mathrm{M}_n(\mathbb{R})$, the space of all $n \times n$ matrices with real entries. For both of the general linear groups we can define the ***special linear group***; the group of all matrices with unit determinant. For the complex case the notation is

$$\mathrm{SL}(n, \mathbb{C}) := \{M \in \mathrm{M}_n(\mathbb{C}) : \det(M) = 1\}.$$

Again using Proposition 3.9, we find that any elements $X \in \mathfrak{sl}(n, \mathbb{C})$ of its Lie algebra should satisfy $\det(e^{tX}) = e^{t\,\mathrm{Tr}(X)} = 1$, for every $t \in \mathbb{R}$. It follows that

$$\mathfrak{sl}(n, \mathbb{C}) = \{X \in \mathrm{M}_n(\mathbb{C}) : \mathrm{Tr}(X) = 0\}.$$

Being such basic examples of Lie groups, it would be useful to be able to view the additive groups $\mathbb{C}^n$ and $\mathbb{R}^n$ as matrix Lie groups. This can be done using the following homomorphism:

$$\Phi : \mathbb{C}^n \to \mathrm{GL}(n+1, \mathbb{C}); \quad \boldsymbol{z} \mapsto \begin{bmatrix} 1 & \boldsymbol{z} \\ 0 & I \end{bmatrix}.$$

Here I denotes the $n \times n$ identity matrix, and the element $\boldsymbol{z} \in \mathbb{C}^n$ is 'embedded' as a row vector into the matrix on the right hand side. As a group, we may therefore identify $\mathbb{C}^n \cong \mathrm{im}(\Phi)$ via the first isomorphism theorem for groups (the map is clearly injective). In this way, $\mathbb{C}^n$ is a matrix Lie group, because the limit of any of its convergent sequences has unit determinant. We similarly identify $\mathbb{R}^n$ with $\mathrm{im}(\Phi|_{\mathbb{R}^n})$. The Lie algebra of $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n$ as a vector space endowed with trivial bracket.

3.2.1 The orthogonal and unitary groups

The ***orthogonal group*** $\mathrm{O}(n)$ is defined as the set of all matrices whose inverse is the transpose:

$$\mathrm{O}(n) := \{R \in \mathrm{M}_n(\mathbb{R}) : RR^\mathsf{T} = R^\mathsf{T} R = I\}.$$

Its elements are called *orthogonal matrices*. Now, if $R \in \mathrm{O}(n)$ we find that $\det(R)^2 = \det(RR^\mathsf{T}) = \det(I) = 1$, and hence $\det(R) = \pm 1$. (The converse is not true; not any matrix with determinant ± 1 is orthogonal.) This naturally leads to the definition of the ***special orthogonal group***:

$$\mathrm{SO}(n) := \{R \in \mathrm{O}(n) : \det(R) = 1\}.$$

These two groups have a geometric interpretation. Namely, their elements, when acting on the Euclidean space in the usual way, preserve the Euclidean inner product. That is to say, if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $R \in \mathrm{O}(n)$ then $\langle R\boldsymbol{x}, R\boldsymbol{y}\rangle = \langle \boldsymbol{x}, \boldsymbol{y}\rangle$. In fact, every (not necessarily linear) function $\mathbb{R}^n \to \mathbb{R}^n$ that preserves the Euclidean inner product and has the origin as a fixed point corresponds uniquely to an element in $\mathrm{O}(n)$. (This claim is proven for $n = 2$ in [23, Ex.2.18].) The interpretation of $\mathrm{O}(n)$ is that it contains all the reflections and rotations about the origin of Euclidean space, whereas $\mathrm{SO}(n)$ contains just the rotations. To put it differently; the orthogonal group represents all isometries of Euclidean space that preserve the origin. The group $\mathrm{O}(n)$ has two connected components, corresponding to matrices with determinants ± 1, respectively. The identity component is therefore the special orthogonal group $\mathrm{SO}(n)$.

The complex analogue of the orthogonal group is the ***unitary group***:

$$\mathrm{U}(n) := \{U \in \mathrm{M}_n(\mathbb{C}) : UU^\dagger = U^\dagger U = I\}.$$

Here we use the dagger symbol $\dagger$ to indicate the *Hermitian transpose* of a matrix, which is its transpose where all of its components are complex conjugated. The elements of the unitary group are called *unitary matrices*. Again, we have a geometric interpretation; in this case the invariance of the standard complex inner product:

$$\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}; \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \langle \boldsymbol{x}, \boldsymbol{y} \rangle := \sum_{i=1}^{n} \overline{x_i} y_i.$$

Any unitary matrix U has a determinant with unit modulus: $|\det(U)| = 1$. Analogous to the special orthogonal group we define

$$\mathrm{SU}(n) := \{U \in \mathrm{U}(n) : \det(U) = 1\},$$

called the ***special unitary group***. The unitary groups $\mathrm{U}(n)$ are connected, and the special unitary group $\mathrm{SU}(n)$ is even simply connected (see the end of [14, Sec.16.2]).

3.2.2 . . . and their Lie algebras

What are the Lie algebras of the orthogonal and unitary groups? We start by calculating the algebra of $\mathrm{O}(n)$, which we denote by $\mathfrak{o}(n)$. A matrix $X \in \mathrm{M}_n(\mathbb{R})$ is an element of the Lie algebra if and only if for each $t \in \mathbb{R}$ we have $e^{tX} \in \mathrm{O}(n)$. This means that, in light of Proposition 3.9, in that case we should have $e^{-tX} = e^{tX^\mathsf{T}}$. We clearly see that any anti-symmetric matrix is an element of $\mathfrak{o}(n)$. On the other hand, if $X \in \mathfrak{o}(n)$ then the previous equation does hold for every $t \in \mathbb{R}$, and (3.1) gives $X^\mathsf{T} = -X$. Hence we have proved that the Lie group of $\mathrm{O}(n)$ contains exactly all anti-symmetric matrices:

$$\mathfrak{o}(n) = \{X \in \mathrm{M}_n(\mathbb{R}) : X^\mathsf{T} = -X\}.$$

The Lie algebra $\mathfrak{so}(n)$ of $\mathrm{SO}(n)$ should clearly be a subset of $\mathfrak{o}(n)$. Any matrix $X \in \mathfrak{so}(n)$ has to satisfy the equation $\det(e^{tX}) = e^{t\,\mathrm{Tr}(X)} = 1$, for all $t \in \mathbb{R}$. It follows that $\mathrm{Tr}(X) = 0$. However, this property already holds for any anti-symmetric matrix, and so the two Lie algebras are in fact the same: $\mathfrak{so}(n) = \mathfrak{o}(n)$. Of particular interest to us is the group $\mathrm{SO}(3)$ and its Lie algebra $\mathfrak{so}(3)$, as they correspond to the rotation group of three-dimensional Euclidean space. The standard basis for $\mathfrak{so}(3)$ is given by the following three matrices:

$$J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.2}$$

Together, these matrices satisfy the property that the commutator of either two gives the third: $[J_i, J_j] = \varepsilon_{ijk} J_k$, where ε_{ijk} is the *Levi-Civita symbol*. We use the Einstein summation convention to sum over repeating indices.

The Lie algebras $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ of $\mathrm{U}(n)$ and $\mathrm{SU}(n)$, respectively, are, as opposed to $\mathfrak{o}(n)$ and $\mathfrak{so}(n)$, distinct because of a subtle difference between the real transpose and the Hermitian transpose (especially in the sense that an anti-Hermitian matrix does not necessarily have zero trace.) Nevertheless, calculations analogous to the above can be done to find

$$\begin{aligned}\mathfrak{u}(n) &= \{X \in \mathrm{M}_n(\mathbb{C}) : X^\dagger = -X\},\\ \mathfrak{su}(n) &= \{X \in \mathfrak{u}(n) : \mathrm{Tr}(X) = 0\}.\end{aligned}$$

Of particular interest, so far perhaps for unclear reasons, is the group $\mathrm{SU}(2)$ and its Lie algebra $\mathfrak{su}(2)$. The Lie algebra contains all 2×2 anti-Hermitian matrices with zero trace:

$$\mathfrak{su}(2) = \left\{ \begin{bmatrix} ia & z \\ -\bar{z} & -ia \end{bmatrix} : a \in \mathbb{R}, z \in \mathbb{C} \right\},$$

where the overline denotes complex conjugation. A standard basis is given by the imaginary rescaled *Pauli matrices*:

$$S_1 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad S_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad S_3 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Just like the basis matrices J_1, J_2, J_3 for $\mathfrak{so}(3)$, these matrices satisfy the relations $[S_j, S_k] = \varepsilon_{jkl} S_l$. Indeed, we have a Lie algebra isomorphism $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ defined by the linear extension of $S_j \mapsto J_j$, for $j = 1, 2, 3$. The question is now whether we also have a Lie group isomorphism between $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$. This turns out to not be the case, but there is nevertheless an important relation between the two, as we will see in Section 3.4.1. One simple way to tell that $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$ cannot be isomorphic is by comparing their centres; they are not equal [14] (i.e., $\mathrm{Z}(\mathrm{SO}(3))$ is trivial, while $\mathrm{Z}(\mathrm{SU}(2)) \cong \mathbb{Z}/2\mathbb{Z}$).

3.2.3 The generalised orthogonal groups

We have interpreted the orthogonal group $\mathrm{O}(n)$ as the group of all rotations and reflections of the n-dimensional Euclidean space. A completely analogous construction can be made when $\mathbb{R}^n$ is equipped with a different inner product. Specifically, we consider the space $\mathbb{R}^{n+k}$ endowed with the inner product:

$$\langle \cdot, \cdot \rangle_{n,k} : \mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \to \mathbb{R}; \quad (\boldsymbol{x}, \boldsymbol{y}) \mapsto \langle \boldsymbol{x}, \boldsymbol{y} \rangle_{n,k} = \sum_{i=1}^{n} x_i y_i - \sum_{j=n+1}^{n+k} x_j y_j.$$

The ***generalised orthogonal group*** $\mathrm{O}(n, k)$ is defined as [15, Sec.1.2.3]

$$\mathrm{O}(n, k) := \{R \in \mathrm{M}_{n+k}(\mathbb{R}) : gRg = R^{-1}\},$$

where g is the matrix with the first n diagonal components equal to 1, and the last k diagonal components equal to -1. (Note that $g = g^{-1}$.) These generalised orthogonal matrices $R \in \mathrm{O}(n, k)$ have the property that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n+k}$ we have $\langle R\boldsymbol{x}, R\boldsymbol{y} \rangle_{n,k} = \langle \boldsymbol{x}, \boldsymbol{y} \rangle_{n,k}$. From the defining condition $gRg = R^{-1}$ we immediately find that any generalised orthogonal matrix has $\det(R) = \pm 1$. As one expects, we then define

$$\mathrm{SO}(n, k) := \{R \in \mathrm{O}(n, k) : \det(R) = 1\}.$$

Clearly Minkowski space $\mathbb{M}^4$ corresponds to the special case $n = 1$, $k = 3$. Its orthogonal group $\mathrm{O}(1, 3)$ is called the ***Lorentz group***, after Dutch physicist Hendrik

Antoon Lorentz (1853–1928). The elements of this group are called *Lorentz transformations.* The Lie algebra of the Lorentz group $\mathrm{O}(1,3)$ and the special Lorentz group $\mathrm{SO}(1,3)$ are calculated similarly to the above, and one finds:

$$\mathfrak{o}(1,3) = \mathfrak{so}(1,3) = \{X \in \mathrm{M}_4(\mathbb{R}) : gX^\mathsf{T} g = -X\}.$$

3.3 Spacetime symmetry groups as matrix Lie groups

3.3.1 The structure of the Euclidean group, semi-direct products

The geometric interpretation of the orthogonal groups makes it sound like they should have some relation with the Euclidean group. Translations in the Euclidean group do not preserve the Euclidean inner product, but they do preserve the Euclidean metric. On the other hand, any orthogonal transformation also preserves the Euclidean metric. It seems like the orthogonal group should somehow be embedded in the Euclidean group. But the orthogonal group certainly does not capture the entire Euclidean group, since spatial translations form Euclidean symmetries, but are not part of the orthogonal group as they move the origin (at least the non-trivial ones). However, these two types of transformations are, as it turns out, the only ones, and it can be shown that together they form the *entire* Euclidean group. In a more precise language, if we denote the group of all translations of n-dimensional Euclidean space by $\mathrm{T}(n)$, seen as a subgroup of the Euclidean group $\mathrm{E}(n)$, then we have $\mathrm{E}(n) = \mathrm{T}(n) \circ \mathrm{O}(n)$, by which we mean

$$\mathrm{E}(n) = \{t \circ R : t \in \mathrm{T}(n), R \in \mathrm{O}(n)\}.$$

Here we identify the matrices in $\mathrm{O}(n)$ with the corresponding linear maps $\mathbb{R}^n \to \mathbb{R}^n$ they represent. Geometrically, the above equation implies that every isometry of Euclidean space can be obtained by first applying an orthogonal transformation, and thereafter a translation. It is quite easy to see that the translation and orthogonal groups are both subgroups of the Euclidean group. Moreover, the translation group forms a *normal* subgroup of the Euclidean group. To denote this we write $\mathrm{T}(n) \lhd \mathrm{E}(n)$. This follows from the following fact; supposing we write $t_{\boldsymbol{a}} \in \mathrm{T}(n)$ to denote the translation along a vector $\boldsymbol{a} \in \mathbb{R}^n$, and taking $R \in \mathrm{O}(n)$ to be arbitrary, the following identity holds:

$$R \circ t_{\boldsymbol{a}} \circ R^{-1} = t_{R\boldsymbol{a}}. \tag{3.3}$$

This result can be verified easily by evaluating either side at some point in $\mathbb{R}^n$, and using the fact that orthogonal transformations are *linear*. Now, indeed, conjugating $t_{\boldsymbol{a}}$ with some general element $t \circ R$ of the Euclidean group, we find that $(t \circ R) \circ t_{\boldsymbol{a}} \circ (R^{-1} \circ t^{-1}) = t \circ t_{R\boldsymbol{a}} \circ t^{-1}$, which is again an element in the translation group. Together, the translation and orthogonal groups only share the identity transformation, as any orthogonal map necessarily preserves the origin. Therefore, the translational and orthogonal groups satisfy the conditions to form a *semi-direct product*:

Definition 3.12. Let G be a group with two subgroups: N and H, the former of which is a normal subgroup. If $G = NH = \{nh : n \in N, h \in H\}$ and $N \cap H = \{1_G\}$, we say that G is the ***(inner) semi-direct product*** of N and H, and in that case we write $G = N \rtimes H$.

The additive group $\mathbb{R}^n$ can be identified with the translation group $\mathrm{T}(n)$ in an obvious way, and so we identify $\mathbb{R}^n \lhd \mathrm{E}(n)$ as a normal subgroup. Given the discussion above, the Euclidean group is therefore a semi-direct product:

$$\mathrm{Isom}(\mathbb{E}^n) = \mathrm{E}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n).$$

What exactly is the structure of the Euclidean group, presented in this fashion? In light of this question, and because we will need the concept of semi-direct products to describe both the Galilean and Poincaré groups (done below), we shall go into some more detail. Next to the notion of *inner* semi-direct products, defined in Definition 3.12, we have the following different notion:

Definition 3.13. Let N and H be groups, and let $\tau : H \to \mathrm{Aut}(N)$ be a group homomorphism. Then the ***(outer) semi-direct product*** of N and H with respect to τ is defined as the group $N \rtimes_\tau H$, which is the set $N \times H$ endowed with the following operation:

$$(n, h)(m, k) = (n\tau(h)(m), hk).$$

We leave it to the reader to verify that $N \rtimes_\tau H$ defines a group.

Besides their notation, there may not seem to be an obvious connection between inner- and outer semi-direct product. However, the following two points hold, allowing us to think of inner semi-direct products as outer semi-direct products, and vice versa.

Proposition 3.14. *We have:*

1. *An inner semi-direct product $G = N \rtimes H$ is isomorphic to the outer semi-direct product $N \rtimes_\varphi H$, where $\varphi : H \to \mathrm{Aut}(N)$ sends an element $h \in H$ to the inner automorphism $\varphi(h) : N \to N; n \mapsto hnh^{-1}$.*
2. *An outer semi-direct product $G = N \rtimes_\tau H$ is an inner semi-direct product of the groups $\{(n, 1_H) : n \in N\} \cong N$ and $\{(1_N, h) : h \in H\} \cong H$.*

We shall give a proof for the first point, since we were initially given an *inner* semi-direct product structure of the Euclidean group. The proof of the second point is substantially easier, and is left to the reader. We will need the following lemma.

Lemma 3.15. *Let G be a group with subgroups N and H, so that $N \lhd G$. Then the following are equivalent:*

1. *$G = N \rtimes H$, i.e., $G = NH$ and $N \cap H = \{1_G\}$;*
2. *For each $g \in G$ there are unique elements $n \in N$ and $h \in H$ so that $g = nh$.*

Proof. Suppose first that G is the inner semi-direct product of N and H. Then any element $g \in G$ can be written as a product $g = nh$, for some $n \in N$ and $h \in H$. We are to show that n and h are unique. For that, suppose that there are two other elements $m \in N$ and $k \in H$ such that $g = nh = mk$. Rewriting this equation shows that $m^{-1}n = kh^{-1}$. Hence both sides of the equation are elements of the intersection $N \cap H = \{1_G\}$, from which uniqueness follows.

For the converse, we take it that every element $g \in G$ can be written uniquely as $g = nh$. The fact that $G = NH$ trivially follows. Lastly, take an element $g \in N \cap H$. Now the identity element 1_G can be written as gg^{-1}. It can also be written as the square of itself: $1_N 1_H = 1_G$, and since this decomposition is unique we must have $g = 1_G$. □

The proof of the proposition is now easy:

Proof of Proposition 3.14.1. We are to show that $G = N \rtimes H \cong N \rtimes_\varphi H$, where the function $\varphi : H \to \mathrm{Aut}(N)$ is the homomorphism described in the proposition statement. Using the above lemma, we introduce the map

$$\Phi : G \to N \rtimes_\varphi H; \quad g = nh \mapsto (n, h)$$

to define this isomorphism. The fact that $G = NH$, together with the lemma, makes Φ a well-defined bijection. In G we have the identity $nhmk = n\varphi(h)(m)hk$, for all $n, m \in N$ and $h, k \in H$, which ensures that Φ is a homomorphism:

$$\Phi((nh)(mk)) = \Phi(\overbrace{n\varphi(h)(m)}^{\in N}\overbrace{hk}^{\in H}) = (n\varphi(h)(m), hk) = \Phi(nh)\Phi(mk). \qquad \square$$

We return back to our study of the Euclidean group. Armed with Proposition 3.14.1, we conclude that the Euclidean group is isomorphic to, and hence may be identified with, the outer semi-direct product $\mathrm{E}(n) \cong \mathbb{R}^n \rtimes_\varphi \mathrm{O}(n)$. (We will, however, mostly use the notation of the inner semi-direct product.) It means that, when $(\boldsymbol{a}, R)$ and $(\boldsymbol{b}, S)$ are elements in the group $\mathbb{R}^n \rtimes_\varphi \mathrm{O}(n)$, their product reads

$$(\boldsymbol{a}, R)(\boldsymbol{b}, S) = (\boldsymbol{a} + \varphi(R)(\boldsymbol{b}), RS) = (\boldsymbol{a} + R\boldsymbol{b}, RS),$$

where we use $\varphi(R)(\boldsymbol{b}) = R\boldsymbol{b}$, which follows from (3.3) and the identification $\mathrm{T}(n) \cong \mathbb{R}^n$. Given that the direct product of two smooth manifolds can be endowed with a natural smooth structure, the additional fact that inner automorphisms are smooth then shows that the Euclidean group $\mathrm{E}(n)$ is a Lie group. Given the multiplication law above, we can also view it as a *matrix* Lie group. Namely, we have the homomorphism

$$\Phi : \mathrm{E}(n) \to \mathrm{GL}(n+1, \mathbb{C}); \quad (\boldsymbol{a}, R) \mapsto \begin{bmatrix} R & \boldsymbol{a} \\ 0 & 1 \end{bmatrix}.$$

On the right-hand side we view $\boldsymbol{a}$ as a column vector embedded into the matrix, and 0 as a row vector of length n. The map is clearly injective, and hence we see that $\mathrm{E}(n)$ can be viewed as the matrix group $\mathrm{im}(\Phi)$. To show that $\mathrm{E}(n)$ is a matrix Lie group, we need to show that it is *closed* in $\mathrm{GL}(n+1, \mathbb{C})$. For that, it is useful to note that $\det(\Phi(\boldsymbol{a}, R)) = \det(R)$. Now, if $(\Phi(\boldsymbol{a}_m, R_m))_{m\in\mathbb{N}}$ is a convergent sequence in the matrix group, it must be that $(R_m)_{m\in\mathbb{N}}$ is a convergent sequence as well, say, with limit $R \in \mathrm{M}_n(\mathbb{C})$. Since $\mathrm{O}(n)$ is a matrix Lie group, we know that R is either non-invertible, or is again an orthogonal matrix. In the first case, the limit of our original sequence $(\Phi(\boldsymbol{a}_m, R_m))_{m\in\mathbb{N}}$ will also be non-invertible, since $0 = \det(R) = \det(\Phi(\boldsymbol{a}, R))$, where $\boldsymbol{a}$ is the limit of $(\boldsymbol{a}_m)_{m\in\mathbb{N}}$. If, on the other hand, R is an orthogonal matrix, it is clear that $\Phi(\boldsymbol{a}, R)$ is again an element of the Euclidean group. We conclude that, with the identification $\mathrm{E}(n) \cong \mathrm{im}(\Phi)$, the Euclidean group is a matrix Lie group.

Now that we know the structure of the Euclidean group as a matrix Lie group, we turn to its matrix Lie algebra. The notation is $\mathfrak{e}(n)$, accordingly, and it is given by (cf. [13, p.124]):

$$\mathfrak{e}(n) = \left\{ \begin{bmatrix} X & \boldsymbol{a} \\ 0 & 0 \end{bmatrix} \in \mathrm{M}_{n+1}(\mathbb{R}) : X \in \mathfrak{o}(n), \boldsymbol{a} \in \mathbb{R}^n \right\}.$$

3.3.2 The structure of the Galilei group as a matrix Lie group

We now want to find the structure of the Galilei group $\mathrm{Gal}(3)$ in a similar fashion. The group can immediately be realised as a matrix group:

$$\Phi : \mathrm{Gal}(3) \to \mathrm{GL}(5, \mathbb{C}); \quad (s, \boldsymbol{a}, \boldsymbol{v}, R) \mapsto \begin{bmatrix} R & \boldsymbol{v} & \boldsymbol{a} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}.$$

Again, the map is clearly injective, so we identify $\mathrm{Gal}(3)$ with its image. The argument above used to show that the Euclidean group is a matrix Lie group can be repeated to show that $\mathrm{Gal}(3)$ is a matrix Lie group.

Is the Galilei group also a semi-direct product? The answer is *"yes"*, but the situation is a little more subtle. To find out, we first consider the subgroup

$$\mathrm{UGal}(3) := \{(0, 0, \boldsymbol{v}, R) \in \mathrm{Gal}(3)\} \leqslant \mathrm{Gal}(3),$$

called the group of *uniform Galilean motions*. Moreover, we identify $\mathbb{R}^3 \cong \{(0, 0, \boldsymbol{v}, I) \in \mathrm{Gal}(3)\}$ and $\mathrm{O}(3) \cong \{(0, 0, 0, R) \in \mathrm{Gal}(3)\}$. It is an easy exercise to show that under these identifications, $\mathbb{R}^3$ and $\mathrm{O}(3)$ form subgroups of the uniform Galilean motions. Now take an arbitrary element $(0, 0, \boldsymbol{v}, R) \in \mathrm{UGal}(3)$, and also $(0, 0, \boldsymbol{w}, I) \in \mathbb{R}^3$. Then

$$(0, 0, \boldsymbol{v}, R)(0, 0, \boldsymbol{w}, I)(0, 0, \boldsymbol{v}, R)^{-1} = (0, 0, R\boldsymbol{w}+\boldsymbol{v}, R)(0, 0, -R^{-1}\boldsymbol{v}, R^{-1}) = (0, 0, R\boldsymbol{w}, I) \in \mathbb{R}^3,$$

so in fact we have $\mathbb{R}^3 \lhd \mathrm{UGal}(3)$, i.e., $\mathbb{R}^3$ is a normal subgroup of $\mathrm{UGal}(3)$. It is also easy to see that $\mathbb{R}^3 \cap \mathrm{O}(3)$ is trivial, and $\mathbb{R}^3\,\mathrm{O}(3) = \mathrm{UGal}(3)$. Hence the group of uniform Galilean motions is an inner semi-direct product of $\mathbb{R}^3$ and $\mathrm{O}(3)$, and is therefore by Proposition 3.14.1 isomorphic to the Euclidean group:

$$\mathrm{UGal}(3) = \mathbb{R}^3 \rtimes \mathrm{O}(3) \cong \mathbb{R}^3 \rtimes_\varphi \mathrm{O}(3) \cong \mathrm{E}(3).$$

Note, however, that on the right hand side we interpret $\mathbb{R}^3$ as a group of *translations*, while on the left hand side it represents the *velocities* of uniform motion. Nevertheless, the isomorphism holds, giving us a useful description of the structure of the uniform Galilean motions.

The only thing that is missing from the group of uniform Galilean motions is the spacetime translations, which we identify with $\mathbb{R}^4 \cong \{(s, \boldsymbol{a}, 0, I) \in \mathrm{Gal}(3)\}$. Calculations similar to the ones above, show that $\mathbb{R}^4 \lhd \mathrm{Gal}(3)$, and furthermore, provide the structure of the Galilei group in terms of a double semi-direct product:

$$\mathrm{Gal}(3) = \mathbb{R}^4 \rtimes \mathrm{UGal}(3) \cong \mathbb{R}^4 \rtimes \mathrm{E}(3) \cong \mathbb{R}^4 \rtimes (\mathbb{R}^3 \rtimes \mathrm{O}(3)).$$

The Lie algebra of Gal(3) is [4, p.39]

$$\mathfrak{gal}(3) = \left\{ \begin{bmatrix} X & \boldsymbol{v} & \boldsymbol{a} \\ 0 & 0 & s \\ 0 & 0 & 0 \end{bmatrix} \in \mathrm{M}_5(\mathbb{R}) : s \in \mathbb{R}, \boldsymbol{a}, \boldsymbol{v} \in \mathbb{R}^3, X \in \mathfrak{o}(3) \right\}.$$

As mentioned above, we shall only consider the identity component of the Galilei group, which is found by simply replacing O(3) with its respective identity component, i.e., SO(3). To simplify terminology, from now on, when referring to the *Galilei group* we will mean its connected component: $\mathrm{Gal}(3) := \mathbb{R}^4 \rtimes (\mathbb{R}^3 \rtimes \mathrm{SO}(3))$. The latter term is the connected component of the Euclidean group, called the *special Euclidean group*, and denoted by $\mathrm{SE}(n) := \mathbb{R}^n \rtimes \mathrm{SO}(n)$. In this case, the Lie algebras of the connected components are the same as for their respective encompassing groups, since $\mathfrak{o}(n) = \mathfrak{so}(n)$.

A basis for a Lie algebra is often called a set of *generators*, since they can be seen to 'generate' the Lie group via the exponential map. Such a set of generators, when also given their commutation relations, is an easy way to describe the entire Lie algebra. In this case, the Lie algebra $\mathfrak{gal}(3)$ of the Galilei group is generated by the set $\{H, P_i, K_i, J_i : i = 1, 2, 3\}$ (following the notation of [25]). Here H generates the time translations (to be thought of as some sort of 'Hamiltonian'); the generators P_1, P_2, P_3 account for spatial translations; the generators K_1, K_2, K_3 account for Galilean boosts; and finally, we have the rotation generators J_1, J_2, J_3. These generators are subject to the following commutator relations:

$$[J_i, J_j] = \varepsilon_{ijk} J_k, \quad [J_i, K_j] = \varepsilon_{ijk} K_k, \quad [J_i, P_j] = \varepsilon_{ijk} P_k, \quad [K_i, H] = P_i,$$

for all $i, j \in \{1, 2, 3\}$ (k being summed over as a repeated index), and all other commutator relations being zero. (For a derivation of this result, cf., for example, [3,19,35].) These generators show a shimmer of the special orthogonal and translation groups. For instance, J_1, J_2, J_3 forms a set of generators for the Lie algebra $\mathfrak{so}(3)$, as we know.

3.3.3 The structure of the Poincaré group as a matrix Lie group

The Lorentz group $\mathrm{O}(1, 3)$, defined in Section 3.2.3, classifies all the isometries of Minkowski space that leave the origin fixed. As mentioned, we shall only concern ourselves with the identity component of the Poincaré group, which means we must replace the Lorentz group by $\mathrm{SO}(1, 3)$. However, this special generalized orthogonal group is not itself connected. The identity component of $\mathrm{SO}(1, 3)$ is given by $\mathrm{SO}^+(1, 3)$, which is the group of Lorentz transformations with unit determinant and that preserve the direction of time. What remains to be added to obtain the (connected component of the) Poincaré group is the spacetime translations, again represented by $\mathbb{R}^4$:

$$\mathrm{Poin}(1, 3) := \mathrm{Isom}(\mathbb{M}^4) = \mathbb{R}^4 \rtimes \mathrm{SO}^+(1, 3).$$

In the structure of the semi-direct product, the action of $\mathrm{SO}^+(1, 3)$ on $\mathbb{R}^4$ is simply the defining one. As such, as a matrix Lie group, it may be described as follows:

$$\mathrm{Poin}(1, 3) = \left\{ \begin{bmatrix} \Lambda & \boldsymbol{a} \\ 0 & 1 \end{bmatrix} \in \mathrm{M}_5(\mathbb{R}) : \boldsymbol{a} \in \mathbb{R}^4, \Lambda \in \mathrm{SO}^+(1, 3) \right\}.$$

Its matrix Lie algebra is given by [13, p.114]:

$$\mathfrak{poin}(1, 3) = \left\{ \begin{bmatrix} X & \boldsymbol{a} \\ 0 & 0 \end{bmatrix} \in \mathrm{M}_5(\mathbb{R}) : \boldsymbol{a} \in \mathbb{R}^4, X \in \mathfrak{so}^+(1, 3) \right\}.$$

The Poincaré algebra is in fact generated similarly to the Lie algebra of the Galilei group. We use the same letters for the generators, since they have a similar interpretation: $\{H, P_i, K_i, J_i : i = 1, 2, 3\}$. For the Poincaré algebra, the brackets are [40, p.61]

$$\begin{gathered} [J_i, J_j] = \varepsilon_{ijk} J_k, \quad [J_i, K_j] = \varepsilon_{ijk} K_k, \quad [J_i, P_j] = \varepsilon_{ijk} P_k, \\ [K_i, H] = P_i, \quad [K_i, P_j] = \delta_{ij} H, \quad [K_i, K_j] = -\varepsilon_{ijk} J_k, \end{gathered} \tag{3.4}$$

all others vanishing. Note that the first four relations are simply the non-vanishing ones from the Galilei algebra. In the literature, the generator H is sometimes denoted P_0, standing for the energy component of the four-momentum.

3.4 Universal covering groups

Another important concept we will need is that of a *universal covering*. Let X be a topological space. A ***covering space*** of X is a topological space C together with a continuous map $p : C \to X$, called the *covering map*, with the following property: for every $x \in X$ there is an open neighbourhood $U \subseteq X$ of x so that the preimage $p^{-1}(U)$ can be written as the union of disjoint open sets in C. That is, there exists a family $(V_i)_{i \in I}$ of disjoint open sets in C such that $p^{-1}(U) = \bigcup_{i \in I} V_i$. Moreover, for every $i \in I$ we have $V_i \cong U$, where the homeomorphism is given by p. If we now endow X and C with continuous group structures (so that they are topological groups) and we impose the additional condition that p should be a homeomorphism, we say that C (together with p) is a ***covering group*** of X. A *universal covering space* is a covering space C that is simply connected. A ***universal covering group*** is then a simply connected covering group. In the even more specific case that X is a Lie group, we require C to be a Lie group also, and p should be smooth. The universal covering group of a connected Lie group is the unique simply connected Lie group that has the same Lie algebra:

Theorem 3.16 (Lie's Third Theorem, [8,10])**.** *Every real finite-dimensional Lie algebra $\mathfrak{g}$ corresponds to a simply connected Lie group G such that $\mathrm{Lie}(G) = \mathfrak{g}$. In particular, for every connected Lie group there is a unique simply connected Lie group so that their Lie algebras are isomorphic.*

This means that, in a sense, there are *more* Lie groups than Lie algebras. Namely, there may be several distinct Lie groups that nonetheless share the same Lie algebra (up to isomorphism). The orthogonal groups form a prime example of this fact. The theorem tells us that, given a connected Lie group, up to isomorphism, there is only *one* simply connected Lie group that shares its Lie algebra.

To demonstrate the concept of universal covering groups, we prove that the additive Lie group $\mathbb{R}$ is the universal covering of $\mathrm{U}(1)$. Not only is this useful for illustrative purposes, but it will help simplify some constructions for us later on.

Lemma 3.17. *The universal covering group of $\mathrm{U}(1)$ is $\mathbb{R}$ with covering map $p : x \mapsto e^{2\pi i x}$.*

Proof. The map p is clearly a surjective Lie group homomorphism (i.e., a smooth homomorphism), and has discrete kernel $\ker(p) = \mathbb{Z}$. Let $z = e^{2\pi i y} \in \mathrm{U}(1)$ be arbitrary. The preimage $p^{-1}(\{z\})$ now consists of all real numbers that differ an integer amount from y. More generally, for a sufficiently small open neighbourhood $U \ni z$ we can write the preimage $p^{-1}(U)$ as a union of disjoint open sets over $\mathbb{Z}$. It is clear that each of these disjoint sets is isomorphic to U via p. $\square$

3.4.1 Universal covers of $\mathrm{SO}(3)$ and $\mathrm{SO}^+(1,3)$

In particular, we need to calculate the universal covers of the Galilei and Poincaré groups. Since $\mathbb{R}^n$ is connected and simply connected for every $n \in \mathbb{N}$, it is its own universal cover. We therefore need only calculate the universal covers of the rotation group $\mathrm{SO}(3)$ and the Lorentz group $\mathrm{SO}^+(1,3)$.

We have already seen that the Lie algebras of $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$ are isomorphic. We have also seen that the groups themselves are *not* isomorphic. However, since $\mathrm{SO}(3)$ is connected, and $\mathrm{SU}(2)$ is both connected and simply connected, a suspicion arises that $\mathrm{SU}(2)$ must be the universal cover of $\mathrm{SO}(3)$. And indeed, this turns out to be the case:

Theorem 3.18. *$\mathrm{SU}(2)$ is the universal covering group of $\mathrm{SO}(3)$. The covering map $\widetilde{p} : \mathrm{SU}(2) \to \mathrm{SO}(3)$ has kernel $\ker(\widetilde{p}) \cong \mathbb{Z}/2\mathbb{Z}$.*

The proof is well known, and for an explicit construction of the covering map we refer to [22, Prop.5.5]. The above theorem helps us in our study of the Galilei group, but for the Poincaré group we need the following:

Theorem 3.19. *$\mathrm{SL}(2,\mathbb{C})$ is the universal covering group of $\mathrm{SO}^+(1,3)$. The covering map $\widetilde{p} : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3)$ has kernel $\ker(\widetilde{p}) \cong \mathbb{Z}/2\mathbb{Z}$.*

For a proof (which is similar to the one of Theorem 3.18), see the entry [34] in the nLab.

Part II

Quantum particles

4 Central extensions and projective representations

Our goal in this section is to develop the necessary mathematical framework to classify projective representations in terms of *linear* representations. This is motivated by physical considerations, because, as we saw in Section 2.3, quantum particles are identified with certain types of projective representations of the spacetime symmetry group. We want to classify these representations. However, it is much easier to calculate and classify ordinary representations, so it is preferable to find a relation between projective- and ordinary representations, instead of simply calculating the projective ones. Here we discuss how this is done; for which we need to start with the seemingly unrelated concept of a *central extension*:

4.1 Central extensions

Consider a sequence of groups and group homomorphisms:

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} \dots \xrightarrow{f_{n-1}} G_{n-1} \xrightarrow{f_n} G_n.$$

We say such a sequence is ***exact*** if the image of each homomorphism is exactly the kernel of the next homomorphism. That is to say, for every $k \in \{1, \dots, n-1\}$ we have $f_k(G_{k-1}) = \operatorname{im}(f_k) = \ker(f_{k+1})$. A ***short exact sequence*** is an exact sequence of the form

$$1 \longrightarrow G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \longrightarrow 1.$$

Here 1 is the trivial group. The left- and rightmost arrows are fixed, being homomorphisms. Namely, the leftmost arrow sends the only element of 1 to the identity element $1_{G_1} \in G_1$, and the rightmost arrow sends every element of G_3 to the only element in 1. The fact that this sequence is exact means that $\{1_{G_1}\} = \ker(f)$, $\operatorname{im}(f) = \ker(g)$ and $\operatorname{im}(g) = G_3$. Hence f is injective and g is surjective. In this sense a short exact sequence is equivalent to an exact sequence

$$G_1 \hookrightarrow G_2 \twoheadrightarrow G_3.$$

Definition 4.1. A ***central extension*** of G is a short exact sequence

$$1 \longrightarrow A \xhookrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1 \tag{4.1}$$

such that the image of ι is contained in the *centre* of E: $\operatorname{im}(\iota) \subseteq \mathrm{Z}(E)$. More informally we may say that the group E is the central extension of G by A when it is clear what the maps ι and π are.

Note that if ι is injective then $\operatorname{im}(\iota) \cong A$ by the first isomorphism theorem for groups. Hence A must necessarily be abelian. Similarly, we obtain $E/\ker(\pi) \cong \operatorname{im}(\pi)$, which, in this case, reads $E/\operatorname{im}(\iota) \cong G$, informally written as $E/A \cong G$.

As with most mathematical structures, there is a trivial example. In this case, the ***trivial extension*** of G by any abelian group A is defined as the short exact sequence

$$1 \longrightarrow A \xhookrightarrow{\mathrm{i}_A} A \times G \xrightarrow{\mathrm{pr}_G} G \longrightarrow 1,$$

where $A \times G$ is the direct product group, $\mathrm{i}_A : a \mapsto (a, 1_G)$ is the inclusion map of A, and $\mathrm{pr}_G : (a, g) \mapsto g$ is the projection map onto G. This already shows that there are plenty central extensions of any given group G (at least as many as there are abelian groups).

It is useful to introduce some notion by which we can compare different central extensions. Then we can investigate whether two given extensions of G are really different. Suppose we have two central extensions E_1 and E_2 of G by A, as given by the following (commutative) diagram:

$$\begin{tikzcd}
 & & E_1 \arrow[dr, two heads, "\pi_1"] & & \\
1 \arrow[r] & A \arrow[ur, hook, "\iota_1"] \arrow[dr, hook, "\iota_2"'] & & G \arrow[r] & 1. \\
 & & E_2 \arrow[ur, two heads, "\pi_2"'] & &
\end{tikzcd}$$

We say that these extensions are *equivalent* if there exists an isomorphism $\Phi : E_1 \to E_2$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & E_1 \arrow[dr, two heads, "\pi_1"] \arrow[dd, "\Phi"] & & \\
1 \arrow[r] & A \arrow[ur, hook, "\iota_1"] \arrow[dr, hook, "\iota_2"'] & & G \arrow[r] & 1. \\
 & & E_2 \arrow[ur, two heads, "\pi_2"'] & &
\end{tikzcd}$$

This defines an equivalence relation on central extensions of G by A. To see this, note that reflexivity simply follows from the fact that the identity map on any group defines an isomorphism, and symmetry follows from the fact that isomorphisms are (by definition) invertible. For transitivity, consider a third central extension E_3 with corresponding maps $\iota_3 : A \to E_3$ and $\pi_3 : E_3 \to G$, and suppose that E_1 is equivalent to E_2, and E_2 is equivalent to E_3. Then we can find isomorphisms $\Phi : E_1 \to E_2$ and $\Psi : E_2 \to E_3$ with corresponding commutative diagrams. The composition $\Psi \circ \Phi : E_1 \to E_3$ now defines an isomorphism as well as a commutative diagram

$$\begin{tikzcd}
 & & E_1 \arrow[d, "\Phi"] \arrow[dr, two heads, "\pi_1"] & & \\
1 \arrow[r] & A \arrow[ur, hook, "\iota_1"] \arrow[r, hook, "\iota_2"] \arrow[dr, hook, "\iota_3"'] & E_2 \arrow[r, two heads, "\pi_2"] \arrow[d, "\Psi"] & G \arrow[r] & 1, \\
 & & E_3 \arrow[ur, two heads, "\pi_3"'] & &
\end{tikzcd}$$

which shows that E_1 and E_3 are also equivalent, proving that the relation is transitive.

A special case arises when a central extension is equivalent to the trivial extension. In that case, we say the extension is ***trivial*** (or *trivializable*). A seemingly different notion is that of a *split* extension. We say that the central extension (4.1) ***splits*** if there exists a group homomorphism $\sigma : G \to E$ so that $\pi \circ \sigma = \mathrm{id}_G$. This condition can be summarised by the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r, hook, "\iota"] & E \arrow[r, two heads, bend left, "\pi"] & G \arrow[l, dashed, bend left, "\sigma"] \arrow[r] & 1.
\end{tikzcd}$$

In fact:

Lemma 4.2. *A central extension is trivializable if and only if it splits.*

Proof. Suppose first that the extension E as given above in (4.1) splits. Then we have a homomorphism $\sigma : G \to E$ that is the right inverse of π. This defines a map $\Phi : A \times G \to E$ sending a pair $(a, g) \in A \times G$ to the product $\iota(a)\sigma(g) \in E$. Since the extension is central, Φ is a homomorphism. Moreover, we have $\Phi \circ \mathrm{i}_A = \iota$ and $\pi \circ \Phi = \mathrm{pr}_G$, so by the *short five lemma* (cf. [17, Lem.1.1]) it follows that Φ is an isomorphism, and hence that E is equivalent to the trivial extension $A \times G$.

Now suppose that E is trivializable. Then there exists an isomorphism $\Phi : A \times G \to E$ such that $\pi \circ \Phi = \mathrm{pr}_G$. Define $\sigma : G \to E$ by $\sigma(g) = \Phi(1_A, g)$. Now σ is clearly a homomorphism, and furthermore $\pi \circ \sigma(g) = \pi \circ \Phi(1_A, g) = \mathrm{pr}_G(1_A, g) = g$, showing that σ is a right inverse to π. Hence the extension splits, and we are done. $\square$

Given an abelian group A, a natural question is to ask how many inequivalent central extensions (4.1) of G there are. We shall try to answer this question by defining an object which in essence 'measures' the trivializability of a central extension. Motivated by Lemma 4.2, we may do this by considering sections (i.e., right inverses) $s : G \to E$ of π with the property that $s(1_G) = 1_E$, but which are not necessarily homomorphisms. (If it were, the extension would be trivial.) s induces a map [33]

$$\omega : G \times G \to \mathrm{im}(\iota) \cong A; \quad (g, h) \mapsto s(g)s(h)s(gh)^{-1}, \tag{4.2}$$

where, for typographical reasons, we shall temporarily write $s_g := s(g)$ for each $g \in G$. Note that, indeed, this map is well defined, since $\pi(s_g s_h s_{gh}^{-1}) = gh(gh)^{-1} = 1_E$, and hence $\mathrm{im}(\omega) \subseteq \ker(\pi) = \mathrm{im}(\iota)$. There are two properties that characterise ω; firstly, we have

$$\omega(1_G, 1_G) = s(1_G)s(1_G)s(1_G 1_G)^{-1} = 1_E 1_E 1_E^{-1} = 1_E,$$

and secondly:

$$\begin{aligned}\omega(g, h)\omega(gh, k) &= (s_g s_h s_{gh}^{-1})(s_{gh} s_k s_{ghk}^{-1}) = s_g s_h s_k s_{ghk}^{-1} \\ &= s_g s_h s_k s_{hk}^{-1} s_{hk} s_{ghk}^{-1} = s_g \omega(h, k) s_{hk} s_{ghk}^{-1} = \omega(g, hk)\omega(h, k),\end{aligned}$$

for all $g, h, k \in G$. (In the last step use that $\mathrm{im}(\iota) \subseteq \mathrm{Z}(E)$.) The map ω in a sense measures the degree to which s is (not) a homomorphism. Indeed, when s is a homomorphism (in which case the extension is trivial) ω would simply be the trivial map $(g, g) \mapsto 1_E$. This turns out to be the right concept that will help us classify central extensions (culminating in Theorem 4.6), and leads us to the following definition.

Definition 4.3. A map $\omega : G \times G \to A$ defined on a group G and an abelian group A that satisfies

$$\omega(1_G, 1_G) = 1_A, \qquad \text{and} \qquad \omega(g, h)\omega(gh, k) = \omega(g, hk)\omega(h, k)$$

for any $g, h, k \in G$ is called a ***(unital) 2-cocycle*** (or ***cocycle*** for short) on G with values in A.

A cocycle ω on G with values in A defines a group, denoted $A \times_\omega G$, defined as the set $A \times G$ together with the operation $(a, g) \cdot (b, h) := (\omega(g, h)ab, gh)$. The identity element of $A \times_\omega G$ is simply $(1_A, 1_G)$. Given an element $(a, g) \in A \times_\omega G$, its inverse is given by $(a, g)^{-1} = (\omega(g, g^{-1})^{-1}a^{-1}, g^{-1})$, as one easily verifies. The main task lies in proving associativity. For that, let $(a, g), (b, h), (c, k) \in A \times_\omega G$. Using the defining property of ω as a cocycle, we find

$$\begin{aligned}[(a, g) \cdot (b, h)] \cdot (c, k) &= (\omega(g, h)ab, gh) \cdot (c, k) = (\omega(g, h)\omega(gh, k)abc, ghk) \\ &= (\omega(g, hk)\omega(h, k)abc, ghk) = (a, g) \cdot (\omega(h, k)bc, hk) \\ &= (a, g) \cdot [(b, h) \cdot (c, k)],\end{aligned}$$

so we may conclude that $A \times_\omega G$ is indeed a group. It defines a central extension of G by A in the following way;

Construction 4.4. Let G be a group and let A be an abelian group. Let $\omega : G \times G \to A$ be a cocycle on G with values in A. ω induces a central extension of G by A:

$$1 \longrightarrow A \xhookrightarrow{\mathrm{i}_A} A \times_\omega G \xrightarrow{\mathrm{pr}_G} G \longrightarrow 1.$$

Proof. Clearly i_A and pr_G are injective and surjective, respectively, and form an exact sequence: $\mathrm{im}(\mathrm{i}_A) = A \times \{1_G\} = \ker(\mathrm{pr}_G)$. Now, i_A is a homomorphism, since

$$\mathrm{i}_A(ab) = (ab, 1_G) = (\omega(1_G, 1_G)ab, 1_G) = (a, 1_G) \cdot (b, 1_G) = \mathrm{i}_A(a) \cdot \mathrm{i}_A(b),$$

for any $a, b \in A$. Similarly, pr_G is a homomorphism, because

$$\mathrm{pr}_G((a, b) \cdot (b, h)) = \mathrm{pr}_G(\omega(g, h)ab, gh) = gh = \mathrm{pr}_G(a, g)\,\mathrm{pr}_G(b, h),$$

for any two elements $(a, b), (b, h) \in A \times_\omega G$.

That leaves us to show that the extension is central. Note that $\mathrm{i}_A(a) \cdot (b, h) = (\omega(1_G, h)ab, h)$ and $(b, h) \cdot \mathrm{i}_A(a) = (\omega(h, 1_G)ba, h)$. Hence it suffices to show that $\omega(1_G, h) = \omega(h, 1_G)$ for all $h \in G$. This follows from the cocycle property, which gives us $\omega(h, 1_G)\omega(h, h) = \omega(h, h)\omega(1_G, h)$. Since A is abelian, this reduces to the desired equality. (In fact $\omega(h, 1_G) = \omega(1_G, h) = 1_A$.) □

Since any cocycle defines a central extension, we want to figure out to what extent this construction captures all central extensions. The answer is, indeed, *all* central extensions, at least, up to equivalence.

To give a precise answer to this question, we need to investigate the cocycles more closely. Let us denote the set of all cocycles on G with values in some abelian group A by

$$\mathrm{Z}^2_{\mathrm{gr}}(G, A) := \{\text{cocycles } G \times G \to A\},$$

endowment of which with pointwise multiplication making it into an abelian group. As a first step, we will try to figure out when an extension $A \times_\omega G$ as in Construction 4.4 is trivial. By Lemma 4.2 we know this is equivalent to the extension splitting, i.e., the existence of a homomorphism $\sigma : G \to A \times_\omega G$ that is the right inverse of the projection pr_G. Most generally, such a map σ is of the form $\sigma(g) = (\alpha(g), \beta(g))$, for all $g \in G$, and where $\alpha : G \to A$ and $\beta : G \to G$ are some maps. Now since $\mathrm{pr}_G \circ \sigma = \beta$, we find that β must simply be the identity map on G. The other property of σ, namely it being a homomorphism, places restrictions on α. This restriction comes from the equation

$$(\alpha(g)\alpha(h)\omega(g, h), gh) = \sigma(g)\sigma(h) = \sigma(gh) = (\alpha(gh), gh),$$

which holds for all elements $g, h \in G$, and from the equation $\sigma(1_G) = (1_A, 1_G)$, which gives $\alpha(1_G) = 1_A$. Hence we have an explicit expression for the cocycle ω in terms of α:

$$\omega(g, h) = \alpha(gh)\alpha(g)^{-1}\alpha(h)^{-1}.$$

Clearly, any time a cocycle can be written as such, and the map α has the property that $\alpha(1_G) = 1_A$, the corresponding extension will split. A cocycle of this form is called a ***2-coboundary*** on G with values in A (or again just ***coboundary*** for short) [22]. In this terminology, an extension $A \times_\omega G$ is trivial if and only if ω is a coboundary.

We denote the set of all coboundaries by

$$\mathrm{B}^2_{\mathrm{gr}}(G,A) := \{\text{coboundaries } G \times G \to A\}.$$

More explicitly, if we denote the set of all *functions* $G \to A$ that map $1_G \mapsto 1_A$ by $\mathrm{B}^1_{\mathrm{gr}}(G,A)$, and for any such function $\alpha \in \mathrm{B}^1_{\mathrm{gr}}(G,A)$ we define the expression

$$\partial\alpha : G \times G \to A; \quad (g,h) \mapsto \alpha(gh)\alpha(g)^{-1}\alpha(h)^{-1},$$

then the set of coboundaries is exactly

$$\mathrm{B}^2_{\mathrm{gr}}(G,A) = \partial\mathrm{B}^1_{\mathrm{gr}}(G,A) = \{\partial\alpha : \alpha \in \mathrm{B}^1_{\mathrm{gr}}(G,A)\}.$$

Again, endowing $\mathrm{B}^2_{\mathrm{gr}}(G,A)$ with pointwise addition gives an abelian group. It is even a normal subgroup of the cocycles $\mathrm{Z}^2_{\mathrm{gr}}(G,A)$. To see this, it helps to note that $\partial(\alpha\beta) = \partial\alpha\partial\beta$, and normality follows automatically, because $\mathrm{Z}^2_{\mathrm{gr}}(G,A)$ is abelian.

Definition 4.5. The ***second cohomology group*** of G with values in A is defined as the quotient

$$\mathrm{H}^2_{\mathrm{gr}}(G,A) := \frac{\mathrm{Z}^2_{\mathrm{gr}}(G,A)}{\mathrm{B}^2_{\mathrm{gr}}(G,A)}.$$

For brevity, we shall usually call this group just the *cohomology group* of G with values in A. Its elements are called *cohomology classes*.

It is useful to understand when two cocycles ω and δ represent the same cohomology class. We know that this is the case if and only if the product $\omega\delta^{-1}$ is a coboundary, i.e., if and only if there exists a map $\alpha \in \mathrm{B}^1_{\mathrm{gr}}(G,A)$ such that $\omega\delta^{-1} = \partial\alpha$. (Note; here δ^{-1} is the *pointwise* inverse of δ, not the inverse map.) Naturally, this defines an equivalence relation between cocycles, where $\omega \sim \delta$ if and only if there exists a map α such that

$$\alpha(gh) = \omega(g,h)\delta(g,h)^{-1}\alpha(g)\alpha(h),$$

for all $g,h \in G$. (In that case ω and δ are sometimes called *cohomologous*.) Of course, the second cohomology group $\mathrm{H}^2_{\mathrm{gr}}(G,A)$ is isomorphic to the group of all equivalence classes of cocycles under this relation.

We have seen that the extension $A \times_\omega G$ is trivial if and only if ω is equivalent to a coboundary, i.e., if and only if ω is equivalent to the trivial cocycle $(g,g) \mapsto 1_A$. We will now see that the equivalence of cocycles and the equivalence of their respective central extensions fully matches up. When the extensions $A \times_\omega G$ and $A \times_\delta G$ are equivalent, it is easy to see that the cocycles are equivalent (by similar arguments to those used to show that coboundaries induce trivial extensions). Therefore, we are left to show the converse: equivalence of cocycles implies equivalence of extensions. Suppose that $\omega \sim \delta$. Then we are given a map $\alpha \in \mathrm{B}^1_{\mathrm{gr}}(G,A)$ so that $\omega\delta^{-1} = \partial\alpha$, which we use to define

$$\Phi : A \times_\delta G \to A \times_\omega G; \quad (a,g) \mapsto (a\alpha(g), g).$$

For all $a,b \in A$ and $g,h \in G$ we find

$$\begin{aligned}\Phi(a,g)\Phi(b,h) &= (a\alpha(g),g)(b\alpha(h),h) = (ab\alpha(g)\alpha(h)\omega(g,h), gh)\\ &= (ab\alpha(gh)\delta(g,h), gh) = \Phi(ab\delta(g,h), gh) = \Phi((a,g)(b,h)),\end{aligned}$$

and $\Phi(1_A, 1_G) = (1_A\alpha(1_G), 1_G) = (1_A, 1_G)$, so Φ defines a homomorphism. It is also straightforward to see that the diagram

$$\begin{tikzcd}[row sep=small] & & A \times_\delta G \arrow[dd, "\Phi"] \arrow[dr, two heads] & & \\ 1 \arrow[r] & A \arrow[ur, hook] \arrow[dr, hook] & & G \arrow[r] & 1 \\ & & A \times_\omega G \arrow[ur, two heads] & & \end{tikzcd}$$

commutes, and hence the extensions $A \times_\omega G$ and $A \times_\delta G$ are equivalent by the short five lemma. The conclusion is that the extensions induced by two cocycles are equivalent if and only if the cocycles themselves are equivalent.

But the main question still remains unanswered; does *every* central extension arise from a cocycle? More precisely, is every central extension equivalent to some extension $A \times_\omega G$, for some cocycle $\omega \in \mathrm{Z}^2_{\mathrm{gr}}(G, A)$? To see why the answer to this question is positive, we note that every extension (4.1) admits a section $s : G \to E$ of the projection map π, which we may choose to have the property that $s(1_G) = 1_E$. Any such section composes a cocycle $\omega : G \times G \to A$, as we have seen already in (4.2)[6]. Furthermore, this cocycle is uniquely determined by the extension E up to equivalence, meaning that a different choice of section s can only result in a cocycle that is cohomologous to ω. This follows from the fact that, for another section $s' : G \to E$ of π, there would exist a map $\alpha : G \to A$ so that $s' = s\alpha$. This map α can be explicitly constructed via the formula $\alpha(g) := s'(g)s(g)^{-1}$, which is well-defined because $\pi(\alpha(g)) = \pi(s'(g))\pi(s(g))^{-1} = gg^{-1} = 1_G$, so $\mathrm{im}(\alpha) \subseteq \ker(\pi) \cong A$. Now, if we denote the cocycle induced by the section s' by ω', we find that $\omega'\omega^{-1} = \partial(\alpha^{-1})$, so ω and ω' are indeed in the same cohomology class. As a last step we shall prove that the extension E is equivalent to extension $A \times_\omega G$, thus induced. For this, we define

$$A \times_\omega G \to E; \quad (a, g) \mapsto as(g).$$

The fact that this is a homomorphism follows from the definition of ω in terms of s, and the commutativity of the appropriate diagram from the fact that s is a section for the projection π. Using the short five lemma we may finally conclude that, indeed, *any* central extension of G by A is equivalent to some $A \times_\omega G$.

Our discussion in this section so far can be succinctly summarised by the following theorem.

Theorem 4.6. *The central extensions of G by A are classified up to equivalence by the second cohomology group $\mathrm{H}^2_{\mathrm{gr}}(G, A)$.*

4.2 Projective representations

We will now see the connection between projective representations (our main interest) and central qextensions.

The field k has an associated abelian *multiplication group*, denoted $k^\times$, which is simply the set $k \setminus \{0_k\}$ endowed with the multiplication of k. The map

$$\mathrm{diag} : k^\times \to \mathrm{GL}(V, k); \quad \lambda \mapsto \lambda\, \mathrm{id}_V$$

identifies the multiplication group $k^\times$ with scalar multiplication in the vector space V. This map clearly forms an injective group homomorphism, since $\ker(\mathrm{diag}) = \{1_k\}$. Hence $k^\times \cong \mathrm{diag}(k^\times)$. It is also easy to verify that $\mathrm{diag}(k^\times) \lhd \mathrm{GL}(V, k)$, i.e., that

[6] Even though this expression *looks* like a coboundary, it need not necessarily actually be one, because the section may not map into $\mathrm{im}(\iota) \cong A$.

$\mathrm{diag}(k^\times)$ is a normal subgroup of $\mathrm{GL}(V,k)$. The ***projective linear group*** of V is now defined as the quotient

$$\mathrm{PGL}(V,k) := \mathrm{GL}(V,k)/\mathrm{diag}(k^\times).$$

A ***projective representation*** of G over V is a group homomorphism $G \to \mathrm{PGL}(V,k)$. The canonical projection map

$$P : \mathrm{GL}(V,k) \to \mathrm{PGL}(V,k); \quad f \mapsto f \cdot \mathrm{diag}(k^\times)$$

of this quotient is by construction surjective, and has kernel $\ker(P) = \mathrm{diag}(k^\times)$. To put things together, we now have a short exact sequence

$$1 \longrightarrow k^\times \xhookrightarrow{\mathrm{diag}} \mathrm{GL}(V,k) \xrightarrow{P} \mathrm{PGL}(V,k) \longrightarrow 1.$$

Moreover, it is easy to check that $\mathrm{diag}(k^\times) \subseteq \mathrm{Z}(\mathrm{GL}(V,k))$, and so we have a central extension of the projective linear group $\mathrm{PGL}(V,k)$ in our hands!

What is the relation between ordinary (linear) representations and projective representations? Consider a group G with a projective representation $\rho : G \to \mathrm{PGL}(V,k)$. This representation cannot be directly converted into a linear representation of G. However, it *can* be used to create a linear representation of a central extension of G, which may be done as follows:

Theorem 4.7. *Let G be a group and let $\rho : G \to \mathrm{PGL}(V,k)$ be a projective representation. Then there exists a central extension E of G by $k^\times$ and a representation $\sigma : E \to \mathrm{GL}(V,k)$ so that the following diagram commutes:*

$$\begin{tikzcd}
1 \arrow[r] & k^\times \arrow[r, hook, "\iota"] \arrow[d, "\mathrm{id}"] & E \arrow[r, two heads, "\pi"] \arrow[d, "\sigma"] & G \arrow[r] \arrow[d, "\rho"] & 1 \\
1 \arrow[r] & k^\times \arrow[r, hook, "\mathrm{diag}"] & \mathrm{GL}(V,k) \arrow[r, two heads, "P"] & \mathrm{PGL}(V,k) \arrow[r] & 1.
\end{tikzcd}$$

Proof. (Note that the specification of ι and π belongs to the proof of the existence of the central extension.) We will first define the central extension of G, followed by the representation σ. Firstly, the set E is defined as follows:

$$E := \{(M,g) \in \mathrm{GL}(V,k) \times G : P(M) = \rho(g)\}.$$

We verify that E is a subgroup of the direct product $\mathrm{GL}(V,k) \times G$. It is clear that $(\mathrm{id}_V, 1_G) \in E$ since P and ρ are homomorphism, so they must map the identity elements of $\mathrm{GL}(V,k)$ and G, respectively, to the identity element of $\mathrm{PGL}(V,k)$. Now let (M,g) and (N,h) be arbitrary elements in E. Then $(M,g) \cdot (N,h)^{-1} = (MN^{-1}, gh^{-1}) \in E$, which can easily be deduced from the fact that P and ρ are homomorphisms. Indeed: E is a subgroup, and hence in particular a group itself.

We now define the maps ι and π. First:

$$\iota = \mathrm{i}_{k^\times} : k^\times \to E; \quad \lambda \mapsto (\mathrm{diag}(\lambda), 1_G) = (\lambda\, \mathrm{id}_V, 1_G).$$

It is clear that this is an injective homomorphism. The image of ι is calculated to be

$$\mathrm{im}(\iota) = P \circ \mathrm{diag}(k^\times) \times \{1_G\} \subseteq E,$$

which is obviously in the centre of E. Next we define

$$\pi = \mathrm{pr}_G : E \to G; \quad (M,g) \mapsto g.$$

Again, this is clearly a homomorphism. Take $g \in G$ arbitrary. We need to find a linear isomorphism $M : V \to V$ such that $P(M) = \rho(g)$. But since $\rho(g) \in \mathrm{PGL}(V, k) := \mathrm{im}(P)$, this can be done, which means that π is surjective. Furthermore, through an easy calculation we find $\ker(\pi) = \mathrm{im}(\iota)$, showing that the top row of the above diagram is a short exact sequence, and that E is a central extension of G. As a last step, we define the map

$$\sigma = \mathrm{pr}_{\mathrm{GL}(V,k)} : E \to \mathrm{GL}(V, k); \quad (M, g) \mapsto M.$$

This is a homomorphism, and for any $(M, g) \in E$ we now have

$$P \circ \sigma(M, g) = P(M) = \rho(g) = \rho \circ \pi(M, g),$$

showing that the diagram commutes. This concludes the proof. □

The theorem shows that every projective representation ρ of G by V lifts to *some* linear representation σ of the extension E constructed in the proof. We also know, from the preceding section, that this extension must be equivalent to some central extension $k^\times \times_\omega G$. The equivalence is realised by the cocycle ω induced (up to equivalence) by any section $s : \mathrm{PGL}(V, k) \to \mathrm{GL}(V, k)$, with $s(\mathrm{id}_V) = \mathrm{id}_V$, of the projection map P:

$$\omega : G \times G \to k^\times; \quad (g, h) \mapsto \partial(s \circ \rho)^{-1}(g, h) = s(\rho(g))s(\rho(h))s(\rho(gh))^{-1}$$

(of which well-definedness follows from $P \circ s = \mathrm{id}_{\mathrm{GL}(V,k)}$), and the homomorphism

$$k^\times \times_\omega G \to E; \quad (a, g) \mapsto (as(\rho(g)), g).$$

We therefore find that any projective representation $\rho : G \to \mathrm{PGL}(V, k)$ defines a unique equivalence class of central extensions of G by $k^\times$.

It is illustrative to consider the case that the second group cohomology is trivial:

Proposition 4.8. *Let G be a group with cohomology $\mathrm{H}^2_{\mathrm{gr}}(G, k^\times) = 1$. Then every projective representation $G \to \mathrm{PGL}(V, k)$ arises from an ordinary representation $G \to \mathrm{GL}(V, k)$.*

Proof. A projective representation $\rho : G \to \mathrm{PGL}(V, k)$ defines a cocycle $\omega = \partial(s \circ \rho)^{-1}$, as above. Since $\mathrm{H}^2_{\mathrm{gr}}(G, k^\times)$ is trivial, we know that ω is cohomologous to the trivial cocycle, i.e., ω is a coboundary. Hence we may find a map $\alpha \in \mathrm{B}^2_{\mathrm{gr}}(G, k^\times)$ so that $\omega = \partial\alpha$. Now the pointwise product $\sigma := \alpha \cdot (s \circ \rho)$ defines a representation of G by the vector space V:

$$\begin{aligned}
\sigma(g)\sigma(h) &= \alpha(g)\alpha(h)s(\rho(g))s(\rho(h)) \\
&= \alpha(g)\alpha(h)\omega(g, h)s(\rho(gh)) \\
&= \alpha(g)\alpha(h) \left[\alpha(gh)\alpha(g)^{-1}\alpha(h)^{-1}\right] s(\rho(gh)) \\
&= \alpha(gh)\alpha(gh)^{-1}\sigma(gh) = \sigma(gh).
\end{aligned}$$

The projective representation is now easily reconstructed via the formula $\rho = P \circ \sigma$. □

Lemma 4.9. *The projective representations $G \to \mathrm{PGL}(V, k)$ inducing (up to equivalence) a cocycle $\omega : G \times G \to k^\times$, are in bijective correspondence to linear representations $\sigma : k^\times \times_\omega G \to \mathrm{GL}(V, k)$ with the property that for all $a \in k^\times : \sigma(a, 1_G) =$* $\mathrm{diag}(a)$. *(Cf. [21, Prop.III.1.5.1].)*

Proof. Throughout the proof, let $s : \mathrm{PGL}(V, k) \to \mathrm{GL}(V, k)$ be a section of the projection map P. Given a projective representation $\rho : G \to \mathrm{PGL}(V, k)$, define $u : G \to \mathrm{GL}(V, k)$ by $u(g) = s(\rho(g))$. This map now satisfies

$$u(g)u(h) = \omega(g, h)u(gh),$$

where ω is the cocycle $\partial(s \circ \rho)^{-1}$ as defined above. One easily verifies that the map

$$\sigma : k^\times \times_\omega G \to \mathrm{GL}(V, k); \quad (a, g) \mapsto au(g)$$

defines a representation of the central extension $k^\times \times_\omega G$ that satisfies the desired property.

On the other hand, if given a representation $\sigma : k^\times \times_\omega G \to \mathrm{GL}(V, k)$ with the desired property, define $u : G \to \mathrm{GL}(V, k)$ by $u(g) = \sigma(1_k, g)$. Applying the projection map P after u now gives a projective representation $\rho = P \circ u : G \to \mathrm{PGL}(V, k)$. This construction is easily seen to be the left inverse of the construction in the previous paragraph.

We are therefore left to show that the first construction is the left inverse of the second construction. For that, let $\sigma : k^\times \times_\omega G \to \mathrm{GL}(V, k)$ be a representation with the desired property, and let $\rho = P \circ u$ be the induced projective representation. We now apply the first construction to ρ, which first gives a map $v : G \to \mathrm{GL}(V, k)$ given by $v(g) = s(\rho(g)) = (s \circ P)(\sigma(1_k, g))$, and in turn, a map $\kappa : k^\times \times G \to \mathrm{GL}(V, k)$ defined by $\kappa(a, g) = av(g) = a(s \circ P)(\sigma(1_k, g))$. Defining the section s in such a way that $(s \circ P)(\sigma(1_k, g)) = \sigma(1_k, g)$ (which we may do without loss of generality), it follows that $\kappa = \sigma$, and hence the result. $\square$

4.3 Central extensions of Lie groups and Lie algebras

Section 4.1 gave us an overview of the theory of central extensions of general groups, culminating in Theorem 4.6, showing that all central extensions of a group G by some abelian group A are classified (up to equivalence) by the second cohomology group $\mathrm{H}^2_{\mathrm{gr}}(G, A)$. The story changes when we want to describe the central extensions of *Lie* groups; in particular, the group $A \times_\omega G$ may not be a Lie group, although G and A are. This problem arises from the possibility that the cocycle $\omega : G \times G \to A$ need not be smooth, and the fact that it is inherited by the multiplication law of $A \times_\omega G$. In Section 4.3.3 we will see how to deal with this problem, eventually giving an analogue of Theorem 4.6 for (certain types of) Lie groups. But first, we shall investigate the problem of Lie *algebra* extensions.

4.3.1 Central extensions of Lie algebras

The discussion of Section 4.1 can be repeated for Lie algebras, with obvious modifications we shall present here, albeit somewhat more briefly. First, an *abelian* Lie algebra is one whose Lie bracket vanishes identically. This follows from the usual idea of commutativity, since $[X, Y] = [Y, X]$ implies $[X, Y] = 0$ by skew-symmetry. Consequently, the *centre* $\mathrm{Z}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is defined as all elements $X \in \mathfrak{g}$ so that $[X, Y] = 0$ for all other elements $Y \in \mathfrak{g}$. If $\mathfrak{g}$ and $\mathfrak{h}$ are two Lie algebras, we define their *commutator* as $[\mathfrak{g}, \mathfrak{h}] = \mathrm{span}\{[X, Y] : X \in \mathfrak{g}, Y \in \mathfrak{h}\}$. To say that $\mathfrak{g} \subseteq \mathfrak{h}$ is contained in the centre of $\mathfrak{h}$ can therefore simply be stated as $[\mathfrak{g}, \mathfrak{h}] = 0$, where 0 denotes the trivial Lie algebra (i.e., the trivial vector space). In analogue to the concept of a normal subgroup, we need the concept of an *ideal* of an algebra. In particular, if $\mathfrak{g}$ is a linear subspace of $\mathfrak{h}$, we say it is is an *ideal* in $\mathfrak{h}$ if $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{g}$, meaning that if $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$, then $[X, Y] \in \mathfrak{g}$.

Completely analogous to Definition 4.1, we now have [33]:

Definition 4.10. Let $\mathfrak{a}$ be an abelian Lie algebra, and $\mathfrak{g}$ another Lie algebra. A ***central extension*** of $\mathfrak{g}$ by $\mathfrak{a}$ is a short exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow \mathfrak{a} \xhookrightarrow{\iota} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0 \tag{4.3}$$

so that $[\mathfrak{a}, \mathfrak{e}] = 0$. Here we have identified $\mathfrak{a} \cong \mathrm{im}(\iota)$ as an ideal of $\mathfrak{e}$ via the first isomorphism theorem for Lie algebras. Invoking exactness in turn gives the isomorphism $\mathfrak{g} \cong \mathfrak{e}/\mathfrak{a}$.

The *trivial extension* of $\mathfrak{g}$ by $\mathfrak{a}$ is defined as the sequence

$$0 \longrightarrow \mathfrak{a} \hookrightarrow \mathfrak{a} \oplus \mathfrak{g} \twoheadrightarrow \mathfrak{g} \longrightarrow 0.$$

Here $\mathfrak{a} \oplus \mathfrak{g}$ is the *(outer) direct sum*, defined as the set $\mathfrak{a} \times \mathfrak{g}$ together with the component-wise vector space structure (i.e., just the direct sum of vector spaces) and the Lie bracket

$$[a \oplus X, b \oplus Y] = [a, b] \oplus [X, Y],$$

where we write $a \oplus X$ instead of $(a, X) \in \mathfrak{a} \oplus \mathfrak{g}$. In the case that $\mathfrak{a}$ is abelian, as assumed here, we find that

$$[a \oplus X, b \oplus Y] = 0 \oplus [X, Y],$$

which we may identify with $[X, Y] \in \mathfrak{g} \cong 0 \oplus \mathfrak{g}$.

Equivalence and splitting of central extensions of $\mathfrak{g}$ over $\mathfrak{a}$ are defined completely analogously to the abstract group case; we call two extensions $\mathfrak{e}_1$ and $\mathfrak{e}_2$ with inclusion maps ι_1, ι_2 and projection maps π_1, π_2, respectively, *equivalent* if there exists a Lie algebra isomorphism $\Phi : \mathfrak{e}_1 \to \mathfrak{e}_2$ so that the following diagram commutes:

$$\begin{tikzcd} & & \mathfrak{e}_1 \arrow[dd, "\Phi"] \arrow[dr, two heads, "\pi_1"] & & \\ 0 \arrow[r] & \mathfrak{a} \arrow[ur, hook, "\iota_1"] \arrow[dr, hook, "\iota_2"'] & & \mathfrak{g} \arrow[r] & 0. \\ & & \mathfrak{e}_2 \arrow[ur, two heads, "\pi_2"'] & & \end{tikzcd}$$

We call the extension (4.3) *split* if there exists a Lie algebra homomorphism $\sigma : \mathfrak{g} \to \mathfrak{e}$ that is a section of the projection map π. An analogue of Lemma 4.2 now holds (whose proof is obvious); any central extension is equivalent to the trivial extension if and only if it splits.

Definition 4.11. Let $\mathfrak{g}$ and $\mathfrak{a}$ be Lie algebras, where $\mathfrak{a}$ is abelian. A bilinear map $\Omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ with the properties

$$\Omega(X, Y) = -\Omega(Y, X), \quad \text{and} \quad \Omega(X, [Y, Z]) + \Omega(Y, [Z, X]) + \Omega(Z, [X, Y]) = 0,$$

holding for all $X, Y, Z \in \mathfrak{g}$, is called a ***2-cocycle*** (just ***cocycle*** for short) of $\mathfrak{g}$ with values in $\mathfrak{a}$. The latter property is to be thought of as a sort of Jacobi identity.

We want to prove a similar result to Theorem 4.6 for Lie algebra extensions. First we introduce some terminology, also not too dissimilar to that in Section 4.1. Let us write $\mathrm{Z}^2_{\mathrm{al}}(\mathfrak{g}, \mathfrak{a})$ for the set of all cocycles on $\mathfrak{g}$ with values in $\mathfrak{a}$. We endow this set with pointwise scalar multiplication and addition in $\mathfrak{a}$. In particular, the addition turns $\mathrm{Z}^2_{\mathrm{al}}(\mathfrak{g}, \mathfrak{a})$ into an abelian group.

Construction 4.12. Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{a}$ an abelian Lie algebra. Any cocycle $\Omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ defines a Lie algebra $\mathfrak{a} \oplus_\Omega \mathfrak{g}$, which forms a central extension

$$0 \longrightarrow \mathfrak{a} \hookrightarrow \mathfrak{a} \oplus_\Omega \mathfrak{g} \twoheadrightarrow \mathfrak{g} \longrightarrow 0,$$

with the inclusion map $a \mapsto a \oplus 0$ and projection map $a \oplus X \mapsto X$.

Proof. We define the underlying vector space of $\mathfrak{a} \oplus_\Omega \mathfrak{g}$ simply as the direct sum $\mathfrak{a} \oplus \mathfrak{g}$, and endow it with the following Lie bracket:

$$[a \oplus X, b \oplus Y]_\Omega := \Omega(X, Y) \oplus [X, Y],$$

for any $a, b \in \mathfrak{a}$ and all $X, Y \in \mathfrak{g}$. To verify that this makes $\mathfrak{a} \oplus_\Omega \mathfrak{g}$ a Lie algebra, for which it suffices to show that the above actually is a Lie bracket, we use the cocycle properties of Ω. In fact, each property will account for the corresponding property of the Lie bracket (showing that Ω need necessarily be a cocycle). We just verify skew-symmetry, and leave the other properties to the reader. For $a, b \in \mathfrak{a}$ and $X, Y \in \mathfrak{g}$ we verify

$$\begin{aligned}[a \oplus X, b \oplus Y]_\Omega &= \Omega(X, Y) \oplus [X, Y] = (-\Omega(Y, X)) \oplus (-[Y, X]) \\ &= -(\Omega(Y, X) \oplus [Y, X]) = -[b \oplus Y, a \oplus X]_\Omega.\end{aligned}$$

Knowing that $\mathfrak{a} \oplus_\Omega \mathfrak{g}$ is a Lie algebra, we prove that the above sequence is exact. To do so, we need to show the inclusion and projection maps are Lie algebra homomorphisms. They are clearly linear, so it suffices to show they respect the Lie brackets. The proof is obvious; the bracket $[a, b] = 0$ maps to $0 \oplus 0$, and the bracket of the images $a \oplus 0$ and $b \oplus 0$ gives the same result: $[a \oplus 0, b \oplus 0]_\Omega = \Omega(0, 0) \oplus [0, 0] = 0 \oplus 0$. Similarly for the projection map. Exactness follows trivially, as does centrality, since $\mathfrak{a}$ is an abelian ideal in $\mathfrak{a} \oplus_\Omega \mathfrak{g}$. □

To figure out what the algebra 'coboundaries' are, we start, just as in the group case, with figuring out what cocycle-induced extensions are trivial. Suppose that $\mathfrak{a} \oplus_\Omega \mathfrak{g}$, as constructed above, is trivial. Then we can find a splitting map $\sigma : \mathfrak{g} \to \mathfrak{a} \oplus_\Omega \mathfrak{g}$, which is a section of the projection map $\mathrm{pr}_\mathfrak{g}$. In most general form, σ maps $X \in \mathfrak{g}$ to $\alpha(X) \oplus \beta(X)$, where $\alpha : \mathfrak{g} \to \mathfrak{a}$ and $\beta : \mathfrak{g} \to \mathfrak{g}$ are some maps. Since σ is linear, both α and β are also linear. Moreover, since $\mathrm{pr}_\mathfrak{g} \circ \sigma = \mathrm{id}_\mathfrak{g}$ we must have that β is the identity on $\mathfrak{g}$, also. This fact alone already gives

$$[\sigma(X), \sigma(Y)]_\Omega = [\alpha(X) \oplus X, \alpha(Y) \oplus Y]_\Omega = \Omega(X, Y) \oplus [X, Y],$$

for arbitrary $X, Y \in \mathfrak{g}$. But since σ is a Lie algebra homomorphism, we also have

$$[\sigma(X), \sigma(Y)]_\Omega = \sigma([X, Y]) = \alpha([X, Y]) \oplus [X, Y].$$

Therefore, by component-wise equality, we have an expression for the cocycle:

$$\Omega(X, Y) = \alpha([X, Y]).$$

Conversely, when Ω takes this form for some arbitrary linear map $\alpha : \mathfrak{g} \to \mathfrak{a}$, it is straightforward to verify that Ω defines a cocycle, and that the map $\sigma : X \mapsto \alpha(X) \oplus X$ defines a splitting map for the extension $\mathfrak{a} \oplus_\Omega \mathfrak{g}$.

The preceding paragraph motivates the following definition; the ***2-coboundaries*** (abbreviated ***coboundaries***) of $\mathfrak{g}$ with values in $\mathfrak{a}$ are exactly the cocycles of the form $(X, Y) \mapsto \alpha([X, Y])$, where $\alpha : \mathfrak{g} \to \mathfrak{a}$ is any linear map. The set of all coboundaries is denoted $\mathrm{B}^2_{\mathrm{al}}(\mathfrak{g}, \mathfrak{a})$, which, when endowed with pointwise addition, forms a normal subgroup of $\mathrm{Z}^2_{\mathrm{al}}(\mathfrak{g}, \mathfrak{a})$.

Definition 4.13. The ***second cohomology group*** of $\mathfrak{g}$ with values in $\mathfrak{a}$ is defined as

$$\mathrm{H}^2_{\mathrm{al}}(\mathfrak{g}, \mathfrak{a}) := \frac{\mathrm{Z}^2_{\mathrm{al}}(\mathfrak{g}, \mathfrak{a})}{\mathrm{B}^2_{\mathrm{al}}(\mathfrak{g}, \mathfrak{a})}.$$

The natural equivalence between cocycles that is induced by the second cohomology group is that two cocycles $\Omega, \Delta \in \mathrm{Z}^2_{\mathrm{al}}(\mathfrak{g}, \mathfrak{a})$ are equivalent if and only if there exists a linear map $\alpha : \mathfrak{g} \to \mathfrak{a}$ such that

$$\Omega(X, Y) - \Delta(X, Y) = \alpha([X, Y])$$

for all $X, Y \in \mathfrak{g}$. The fact that equivalence of cocycles matches up with the equivalence of their respective cocycles is proved similarly to the group case; as a result we have the following theorem:

Theorem 4.14. *The second cohomology group $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{g}, \mathfrak{a})$ classifies the equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{a}$.*

An important special case for us is when $\mathfrak{a}$ is (isomorphic to) the abelian Lie algebra $\mathbb{R}$. Furthermore, we say a Lie group G is *semi-simple* if it has no connected normal abelian subgroups; we say a Lie algebra $\mathfrak{g}$ is *semi-simple* if it has no abelian ideals. In particular, any semi-simple Lie group induces a semi-simple Lie algebra. For any such Lie algebra we now have the following important result:

Lemma 4.15 (Whitehead's Lemma)**.** *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{g}, \mathbb{R})$ is trivial.*

For a proof we refer to [13, §52]. An important example where Whitehead's Lemma applies is the rotation group SO(3). Its Lie algebra $\mathfrak{so}(3)$ is semi-simple, and hence it has no non-trivial extensions: $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{so}(3), \mathbb{R}) = 0$. We will prove this also in Section 5.1.1, independently of Whitehead's Lemma.

4.3.2 Lie algebra extensions using structure constants

Lie algebras have a vector space structure. As such, we may find a (Hamel) basis; a linearly independent set of elements of the Lie algebra that span the entire space. Elements of such a basis for a Lie algebra are the *generators*. By bilinearity of the bracket, a full description of the Lie brackets of the generators gives a full description of the entire Lie algebra. Most generally, when $\{X_1, \ldots, X_n\}$ is a basis for an n-dimensional Lie algebra $\mathfrak{g}$, the Lie brackets are encoded by the *structure constants*: $\{C^k_{ij} : i, j, k = 1, \ldots, n\} \subseteq \mathbb{R}$, via

$$[X_i, X_j] = C^k_{ij} X_k,$$

where we again employ the Einstein summation convention, now also summing over an index that occurs both in sub- and superscript, meaning that the dummy index k is summed over between $k = 1$ and $k = n$ in this particular expression. These structure constants must satisfy certain obvious relations, forced by skew-symmetry and the Jacobi identity.

The specification of structure constants yields a very concrete method of constructing Lie algebras. We therefore want to find a way to construct the extension $\mathfrak{a} \oplus_\Omega \mathfrak{g}$ in terms of something similar to structure constants, hopefully giving for a very 'hands-on' approach of classifying central extensions. Let $\mathfrak{g}$ be the Lie algebra of the previous paragraph, and let $\mathfrak{a}$ be an abelian Lie algebra with basis $\{a_1, \ldots, a_m\}$, whose structure constants necessarily vanish. The set

$$\{a_1 \oplus 0, \ldots, a_m \oplus 0, 0 \oplus X_1, \ldots, 0 \oplus X_n\}$$

now forms a basis for the $(n + m)$-dimensional vector space $\mathfrak{a} \oplus \mathfrak{g}$. By definition, we have $[a \oplus X_i, b \oplus X_j]_\Omega = \Omega(X_i, X_j) \oplus [X_i, X_j]$ for any two basis elements X_i and X_j

of $\mathfrak{g}$, and any two arbitrary elements $a, b \in \mathfrak{a}$. This means that specifying the values $\Omega(X_i, X_j)$ for all pairs of generators now uniquely determines the extension $\mathfrak{a} \oplus_\Omega \mathfrak{g}$, up to equivalence.

We expand $\Omega(X_i, X_j) = B^l_{ij} a_l$ in terms of basis vectors in $\mathfrak{a}$. Hence we have, in total:

$$[a \oplus X_i, b \oplus X_j] = B^l_{ij} a_l \oplus C^k_{ij} X_k.$$

Of course, the coefficients (B^l_{ij}) will depend on the cocycle Ω, and once the basis is fixed, they are completely determined by it. Conversely, specifying a set of coefficients (B^l_{ij}) with the properties

$$C^p_{jk} B^l_{ip} + C^p_{ki} B^l_{jp} + C^p_{ij} B^l_{kp} = 0,$$
$$B^l_{ij} = -B^l_{ji},$$

corresponding to the Jacobi identity and skew-symmetry, respectively, define a cocycle $\Omega \in \mathrm{Z}^2_{\mathrm{al}}(\mathfrak{g}, \mathfrak{a})$ by linear expansion of $\Omega(X_i, X_j) = B^l_{ij} a_l$ (cf. [25, Sec.III.A.2]). Choosing such a set of coefficients therefore provides us with a way to define central extensions of $\mathfrak{g}$ by $\mathfrak{a}$.

Suppose now we have another cocycle $\Delta \in \mathrm{Z}^2_{\mathrm{al}}(\mathfrak{g}, \mathfrak{a})$ that is cohomologous to Ω. Clearly the extension $\mathfrak{a} \oplus_\Delta \mathfrak{g}$ can be taken to have the same basis as $\mathfrak{a} \oplus_\Omega \mathfrak{g}$, and the only thing that may differ are the coefficients for the expansion: $\Delta(X_i, X_j) = D^l_{ij} a_l$. The fact that Ω and Δ are cohomologous would mean that we could find a linear map $\alpha : \mathfrak{g} \to \mathfrak{a}$ so that

$$D^l_{ij} a_l = \Delta(X_i, X_j) = \Omega(X_i, X_j) - \alpha([X_i, X_j]) = B^l_{ij} a_l - C^k_{ij} \alpha(X_k).$$

If we in turn expand $\alpha(X_k) = \lambda^l_k a_l$ in the basis for $\mathfrak{a}$, we see that equivalence between Ω and Δ rests on the existence of coefficients (λ^l_k) that satisfy the equation

$$B^l_{ij} - D^l_{ij} = C^k_{ij} \lambda^l_k.$$

As a particular case of this equation, we have:

Corollary 4.16. *Consider a cocycle $\Omega \in \mathrm{Z}^2_{\mathrm{al}}(\mathfrak{g}, \mathfrak{a})$ with expansion $\Omega(X_i, X_j) = B^l_{ij} a_l$ as above. The extension $\mathfrak{a} \oplus_\Omega \mathfrak{g}$ is trivial if and only if we can find coefficients (λ^l_k) so that $B^l_{ij} = C^k_{ij} \lambda^l_k$, where (C^k_{ij}) are the structure constants of $\mathfrak{g}$.*

4.3.3 Central extensions of Lie groups

We want to build a theory of central extensions for Lie groups that is analogous to the theory presented in Section 4.1. As we noted before, the extensions $A \times_\omega G$ of Construction 4.4 may not be Lie groups, and we would therefore not call them extensions of Lie groups. An obvious remedy would be to consider only smooth cocycles, but this turns out to be too restrictive [38]. Instead, the fitting setting is that of ***e-smooth*** cocycles, i.e., cocycles that are smooth on a neighbourhood of the identity $(1_G, 1_G) \in G \times G$. The resulting cohomology group of G by A is denoted $\mathrm{H}^2_{\mathrm{es}}(G, A)$, containing the equivalence classes of e-smooth cocycles that differ by e-smooth coboundaries. For a proof that this cohomology group classifies Lie group extensions, we refer to [38, Prop.3.11].

We are particularly interested in calculating this cohomology group $\mathrm{H}^2_{\mathrm{es}}(G, A)$, preferably in terms of the Lie algebra cohomology group $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{g}, \mathfrak{a})$. At this point it is best to restrict ourselves to the case that the abelian group A is given by the circle group $\mathrm{U}(1)$, so that $\mathfrak{a} = \mathfrak{u}(1) \cong \mathbb{R}$. This is justified in a physical setting because, as we will see, all extensions that we will actually be interested in are performed over $\mathrm{U}(1)$. In this special (but plentiful) case we have a relation between Lie group and Lie algebra cohomologies:

Theorem 4.17. *If G is a connected, simply connected Lie group, then the Lie group cohomology $\mathrm{H}^2_{\mathrm{es}}(G, \mathrm{U}(1))$ and the corresponding Lie algebra cohomology $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{g}, \mathbb{R})$ are isomorphic.*

For a proof, see [22, Thm.5.55]. In conjunction with Whitehead's Lemma 4.15 this now gives:

Corollary 4.18. *The second cohomology group $\mathrm{H}^2_{\mathrm{es}}(G, \mathrm{U}(1))$ of a connected, simply connected, semi-simple Lie group G is trivial.*

Sadly, these results do not apply to the important case that $G = \mathrm{SO}(3)$, which is not simply connected. This is where Lie's Third Theorem 3.16 comes into play; or rather, the following refinement (cf. [22, Thm.5.41]):

Theorem 4.19. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Up to isomorphism, there exists a unique connected and simply connected Lie group $\widetilde{G}$ so that*

1. $\mathrm{Lie}(\widetilde{G}) = \mathfrak{g}$;
2. $G \cong \widetilde{G}/D$, *where* $D \lhd \mathrm{Z}(\widetilde{G})$;
3. $D \cong \pi_1(G)$.

Here, $\pi_1(G)$ is the (first) fundamental group of G, which measures the degree to which G is simply connected. (Specifically, G is simply connected if and only if $\pi_1(G)$ is trivial.) Note that, in particular, when $\widetilde{p}: \widetilde{G} \to G$ is the covering map, then $D = \ker(\widetilde{p})$, so that $\widetilde{G}$ is a central extension of G by D, i.e., we have a short exact sequence:

$$1 \longrightarrow D \hookrightarrow \widetilde{G} \xrightarrow{\widetilde{p}} G \longrightarrow 1.$$

The discrete group D will turn out to play a fundamental rôle in the classification of central extensions. The importance is realised through the concept of the *Pontryagin dual.* The ***Pontryagin dual*** $\widehat{A}$ of an abelian Lie group A is defined as the space of all continuous homomorphisms $A \to \mathrm{U}(1)$, endowed with pointwise operations. Its elements are often called *characters*, which are the irreducible unitary representations of A.

Theorem 4.20. *For a connected Lie group G whose Lie algebra $\mathfrak{g}$ has a trivial cohomology $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{g}, \mathbb{R})$, the map*

$$\widehat{D} \to \mathrm{H}^2_{\mathrm{es}}(G, \mathrm{U}(1)); \quad \chi \mapsto [c_\chi] \tag{4.4}$$

defines a surjective homomorphism, where $c_\chi := \chi \circ \partial \widetilde{s}^{-1}$, and D is the discrete group of the previous theorem. (Cf. [22, Thm.5.57] and [6, Lem.11].)

Proof. The covering map $\widetilde{p} : \widetilde{G} \to G \cong \widetilde{G}/D$ admits a section $\widetilde{s} : G \to \widetilde{G}$ so that $\widetilde{s}(1_G) = 1_{\widetilde{G}}$. The covering map is a homomorphism, and so combined with the equation $\widetilde{p} \circ \widetilde{s} = \mathrm{id}_G$ we find

$$\widetilde{p}(\widetilde{s}(g)\widetilde{s}(h)\widetilde{s}(gh)^{-1}) = \widetilde{p}(\widetilde{s}(g))\widetilde{p}(\widetilde{s}(h))\widetilde{p}(\widetilde{s}(gh))^{-1} = gh(gh)^{-1} = 1_G,$$

and hence $\widetilde{s}(g)\widetilde{s}(h)\widetilde{s}(gh)^{-1} \in \ker(\widetilde{p})$. A character $\chi \in \widehat{D}$ is a homomorphism from $D = \ker(\widetilde{p})$ to the torus $\mathrm{U}(1)$. Using previous notation to write $\partial\widetilde{s}^{-1}(g, h) =$

$\widetilde{s}(g)\widetilde{s}(h)\widetilde{s}(gh)^{-1}$, we define the map (4.4) as stated in the theorem. We verify that c_χ is actually a cocycle. Note that

$$c_\chi(g,h)c_\chi(gh,k) = \chi(\partial\widetilde{s}^{-1}(g,h))\chi(\partial\widetilde{s}^{-1}(gh,k)) = \chi(\partial\widetilde{s}^{-1}(g,h)\partial\widetilde{s}^{-1}(gh,k)),$$

so it suffices to show that

$$\partial\widetilde{s}^{-1}(g,h)\partial\widetilde{s}^{-1}(gh,k) = \partial\widetilde{s}^{-1}(g,hk)\partial\widetilde{s}^{-1}(h,k).$$

This is done by direct verification:

$$\begin{aligned}
\partial\widetilde{s}^{-1}(g,h)\partial\widetilde{s}^{-1}(gh,k) &= \widetilde{s}(g)\widetilde{s}(h)\widetilde{s}(gh)^{-1}\widetilde{s}(gh)\widetilde{s}(k)\widetilde{s}(ghk)^{-1} \\
&= \widetilde{s}(g)\widetilde{s}(h)\widetilde{s}(k)\widetilde{s}(ghk)^{-1} \\
&= \widetilde{s}(g)\widetilde{s}(h)\widetilde{s}(k)\widetilde{s}(hk)^{-1}\widetilde{s}(hk)\widetilde{s}(ghk)^{-1} \\
&= \widetilde{s}(g)\partial\widetilde{s}^{-1}(h,k)\widetilde{s}(hk)\widetilde{s}(ghk)^{-1} \\
&= \partial\widetilde{s}^{-1}(g,hk)\partial\widetilde{s}^{-1}(h,k),
\end{aligned}$$

where in the last step we use that $\partial\widetilde{s}^{-1}(h,k) \in \ker(\widetilde{p}) = D$, and the fact that D is abelian (it is contained in the centre of $\widetilde{G}$). We are left to show that c_χ is unital, for which it is sufficient to show that $c_\chi(1_G, 1_G) = 1$. This follows directly from $\widetilde{s}(1_G) = 1_{\widetilde{G}}$. Therefore $c_\chi \in \mathrm{Z}^2_{\mathrm{gr}}(G, \mathrm{U}(1))$. When we take the section $\widetilde{s}$ to also be e-smooth, it follows that $c_\chi \in \mathrm{Z}^2_{\mathrm{es}}(G, \mathrm{U}(1))$.

To make sure that this class of cocycles is independent on the chosen section, suppose we have another e-smooth section s of $\widetilde{p}$ with $s(1_G) = 1_{\widetilde{G}}$. We can now define the e-smooth map $\alpha : G \to \widetilde{G}$ by $\alpha(g) = \widetilde{s}(g)s(g)^{-1}$, so that $\widetilde{s} = \alpha s$. In fact α maps into D, because $\widetilde{p}(\alpha(g)) = \widetilde{p}(\widetilde{s}(g))\widetilde{p}(s(g))^{-1} = gg^{-1} = 1_G$. This means that the image of α is contained in the centre of $\widetilde{G}$. Now denote by ω_χ the cocycle induced by s. Then

$$\begin{aligned}
\omega_\chi(g,h)c_\chi(g,h)^{-1} &= \chi(s(g)s(h)s(gh)^{-1})\chi(\widetilde{s}(gh)\widetilde{s}(h)^{-1}\widetilde{s}(g)^{-1}) \\
&= \chi(s(g)s(h)s(gh)^{-1}s(gh)s(h)^{-1}s(g)^{-1}\alpha(g)^{-1}\alpha(h)^{-1}\alpha(gh)) \\
&= \chi(\partial\alpha^{-1}(g,h)) = \partial(\chi\circ\alpha^{-1})(g,h).
\end{aligned}$$

Therefore ω_χ and c_χ differ by coboundary, and are therefore cohomologous. This means that the two sections $\widetilde{s}$ and s define equivalent cocycles.

We are left to show that (4.4) is a surjection. Let $\omega \in \mathrm{Z}^2_{\mathrm{es}}(G, \mathrm{U}(1))$ be some arbitrary cocycle. Using the projection map $\widetilde{p}$, we define

$$c : \widetilde{G} \times \widetilde{G} \to \mathrm{U}(1); \quad c(x,y) = \omega(\widetilde{p}(x), \widetilde{p}(y)).$$

It turns out that this is a cocycle on $\widetilde{G}$ with values in $\mathrm{U}(1)$, and hence Corollary 4.18 tells us it must take the form of a coboundary: $c = \partial\alpha$, for some $\alpha : \widetilde{G} \to \mathrm{U}(1)$ with $\alpha(1_{\widetilde{G}}) = 1$. Whenever $x \in D = \ker(\widetilde{p})$ we find that, for each $y \in \widetilde{G}$, $c(x,y) = \omega(1_G, y) = 1$. But since $c = \partial\alpha$ this implies that $\alpha(x)\alpha(y) = \alpha(xy)$, so the restriction $\chi := \alpha|_D$ defines an element of the dual $\widehat{D}$. It now follows that the cocycles ω and c_χ differ by coboundary $\partial(\alpha\circ\widetilde{s})$, and therefore $[\omega] = [c_\chi]$, showing that (4.4) is surjective. □

In the case that $G = \mathrm{SO}(3)$ we know by Theorem 3.18 that $\widetilde{G} = \mathrm{SU}(2)$ and $D = \{\pm I\} \cong \mathbb{Z}/2\mathbb{Z}$. The dual of $\mathbb{Z}/2\mathbb{Z}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ itself. We therefore have the important result:

$$\mathrm{H}^2_{\mathrm{es}}(\mathrm{SO}(3), \mathrm{U}(1)) \cong \mathbb{Z}/2\mathbb{Z}.$$

This means that there is merely one non-trivial central extension of SO(3) by U(1). (Compare this with the fact that $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{so}(3),\mathbb{R}) = 0$.) Notwithstanding, this one non-triviality is sufficient to ensure the existence of *fermions* (at least mathematically speaking), cf. Summaries 10.2 and 10.3.

Lemma 4.21. *For a connected, simply connected Lie group G we have an isomorphism between $\mathrm{H}^2_{\mathrm{es}}(G,\mathrm{U}(1))$ and $\mathrm{H}^2_{\mathrm{es}}(G,\mathbb{R})$. In particular, a real-valued cocycle $\xi : G\times G \to \mathbb{R}$ defines a unique cohomology class of $\mathrm{U}(1)$-valued cocycles, represented by $\omega(g,h) = e^{i\xi(g,h)}$. (Cf. [6, Lem.7].)*

This lemma is actually a special case of [39, Cor.7.32], making use of Lemma 3.17 specifically. In particular, the U(1)-valued cocycles ω can be seen as the composition $p\circ\xi$ of the covering map $p : \mathbb{R} \to \mathrm{U}(1)$ and a real-valued cocycle ξ. (The factor 2π in Lemma 3.17 may obviously be removed without problem.) At the moment Lemma 4.21 may seem to have too narrow a use (it is only applicable to simply connected Lie groups), but in Section 6 we will come to see that it simplifies the situations we are interested in.

5 Extensions of the spacetime symmetry groups

Having discussed the necessary central extension theory for Lie groups, we are now ready to apply it to the Galilei and Poincaré groups.

5.1 Extensions of the Galilei group

5.1.1 Central extensions of the Galilei algebra

We have seen that the Lie algebra $\mathfrak{gal}(3)$ of the Galilei group has ten generators subject to the commutator relations

$$[J_i,J_j] = \varepsilon_{ijk}J_k,\quad [J_i,K_j] = \varepsilon_{ijk}K_k,\quad [J_i,P_j] = \varepsilon_{ijk}P_k,\quad [K_i,H] = P_i, \tag{5.1}$$

and all others vanishing. The Lie algebra $\mathfrak{gal}(3)$ contains several interesting subalgebras whose extensions—or rather, their lack thereof—we now calculate (following [25, Sec.III.A.3], but also see [16, Sec.12.5]). The formalism of Section 4.3.2 seems particularly appropriate here.

First, consider the special orthogonal algebra $\mathfrak{so}(3)$, generated by $\{J_1,J_2,J_3\}$. We are particularly interested in the case that $\mathfrak{a}$ is a *one*-dimensional Lie algebra (such as $\mathfrak{u}(1) = i\mathbb{R}$). In that case, $\mathfrak{a}$ is spanned by just one element, and we will treat it like $1\in\mathbb{R}$ to simplify notation. Furthermore, the coefficients (B^l_{ij}) of some cocycle $\Omega\in \mathrm{Z}^2_{\mathrm{al}}(\mathfrak{g},\mathbb{R})$ are written as $B_{ij} := B^1_{ij}$, so that the specification of a cocycle becomes a question of specifying an anti-symmetric $n\times n$ real matrix $B = (B_{ij})^n_{i,j=1}$, subject to some Jacobi-type identity. In the particular case that $\mathfrak{g} = \mathfrak{so}(3)$, this Jacobi identity takes the form

$$0 = \varepsilon_{pjk}B_{ip} + \varepsilon_{pki}B_{jp} + \varepsilon_{pij}B_{kp} = \pm\,\mathrm{Tr}(B)$$

which gives us nothing new since B is already anti-symmetric. Hence the matrix B has three free variables we can specify. This means that every central extension $\mathbb{R}\oplus_\Omega \mathfrak{so}(3)$ of $\mathfrak{so}(3)$ is of the form

$$[J_i,J_j]_\Omega = \varepsilon_{ijk}(B_k\oplus J_k),$$

where $B_1,B_2,B_3\in\mathbb{R}$ are some real numbers. However, the requirements of Corollary 4.16 are now clearly met, since $B_{ij} = \varepsilon_{ijk}B_k$, and therefore there are *no* non-trivial extensions of $\mathfrak{so}(3)$: $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{so}(3),\mathbb{R}) = 0$. This confirms Whitehead's Lemma 4.15, since $\mathfrak{so}(3)$ is semi-simple.

Next, we consider the Euclidean Lie algebra $\mathfrak{e}(3)$, whose generators are $\{J_i, P_i : i = 1, 2, 3\}$. Since the $\mathfrak{so}(3)$ portion of this algebra cannot be non-trivially extended, we leave the Lie brackets of J_1, J_2, J_3 unmodified. The only possible non-trivial modifications we can make are

$$[J_i, P_j]_\Omega = \varepsilon_{ijk}(B_k \oplus P_k), \quad \text{and} \quad [P_i, P_j]_\Omega = \varepsilon_{ijk} b_k \oplus 0,$$

where $b_1, b_2, b_3, B_1, B_2, B_3$ are arbitrary real numbers. In this case the Jacobi identity of Ω *does* impose restrictions; we have

$$0 = \Omega(J_i, [P_j, P_k]) + \Omega(P_j, [P_k, J_i]) + \Omega(P_k, [J_i, P_j]) = \Omega(P_k, \varepsilon_{ijp} P_p) - \Omega(P_j, \varepsilon_{ikq} P_q).$$

Taking, for instance, $i = k = 1$ and $j = 2$ gives $0 = \Omega(P_1, P_3) - \Omega(P_2, 0) = \Omega(P_1, P_3) = b_2$, and similar calculations show that the bracket of the generators P_1, P_2, P_3 cannot be non-trivially extended. Moreover, similar arguments as for $\mathfrak{so}(3)$ show that the remaining extension in the brackets $[J_i, P_j]$ is likewise trivial. We conclude that $\mathfrak{e}(3)$ has no non-trivial extensions, either: $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{e}(3), \mathbb{R}) = 0$.

We are finally ready to determine the central extensions of $\mathfrak{gal}(3)$ itself. The preceding discussion tells us that the brackets on the rotation generators, boost generators and spatial translation generators cannot be non-trivially extended. We use this knowledge by letting the cocycle Ω vanish on these generators, without any loss of generality. The Jacobi identity gives further restrictions, in particular on the time translation generator; we calculate

$$0 = \Omega(J_i, [J_j, H]) + \Omega(J_j, [H, J_i]) + \Omega(H, [J_i, J_j]) = \Omega(H, \varepsilon_{ijk} J_k),$$

so the brackets between the rotation and time translation generators cannot be extended. Replacing one of the rotation generators J_j in this equation by either the translation generator P_j or the boost generator K_j, further shows that $\Omega(H, P_k) = \Omega(H, K_k) = 0$.

Only the brackets between the boost and spatial translation generators leave room for a non-trivial extension. Invoking the Jacobi identity once more gives

$$0 = \Omega(J_i, [K_j, P_k]) + \Omega(K_j, [P_k, J_i]) + \Omega(P_k, [J_i, K_j]) = \varepsilon_{ikl}\Omega(P_l, K_j) + \varepsilon_{ijl}\Omega(P_k, K_l).$$

In the case that $i = j \neq k$, we find that the second term vanishes, so that $\varepsilon_{ikl}\Omega(P_l, K_i) = 0$. For $i \neq l$ this gives $\Omega(P_l, K_i) = 0$. More importantly, when i, j, k are distinct, we have

$$\varepsilon_{ikj}\Omega(P_j, K_j) + \varepsilon_{ijk}\Omega(P_k, K_k) = \pm(\Omega(P_j, K_j) - \Omega(P_k, K_k)) = 0,$$

showing that $\Omega(P_i, K_i)$ has the same value irrespective of i. But we are not able to place any restrictions on the actual *value* of $\Omega(P_i, K_i)$. Its value can therefore be freely prescribed. We set $\Omega(P_i, K_i) = M \in \mathbb{R}$. We will come to see that this number can be interpreted as the *mass* of an elementary particle, cf. Section 6.5. The extensions of $\mathfrak{gal}(3)$ therefore take the form

$$[K_i, P_j]_\Omega = M\delta_{ij} \oplus 0$$

(δ_{ij} is the Kronecker delta), all other generators commuting as usual. We denote the extension corresponding to the value M by $\mathfrak{gal}_M(3)$. This extension is not trivial, unless $M = 0$. Indeed, triviality amounts to the existence of a linear map $\alpha : \mathfrak{gal}(3) \to \mathfrak{gal}_M(3)$ so that $M = \Omega(P_i, K_i) = \alpha([P_i, K_i]) = \alpha(0) = 0$. Similar equations show that two extensions are equivalent if and only if their extension parameter M is the same. In terms of the second cohomology group we therefore have:

$$\mathrm{H}^2_{\mathrm{al}}(\mathfrak{gal}(3), \mathbb{R}) \cong \mathbb{R}.$$

In a more lax notation, the Lie brackets of the extended Lie algebra $\mathfrak{gal}_M(3)$ are as follows:

$$[J_i, J_j] = \varepsilon_{ijk} J_k, \quad [J_i, K_j] = \varepsilon_{ijk} K_k, \quad [J_i, P_j] = \varepsilon_{ijk} P_k,$$
$$[K_i, H] = P_i, \quad [K_i, P_j] = M\delta_{ij},$$

all others being zero.

5.1.2 Central extensions of the Galilei group

We have just seen that the Galilei algebra $\mathfrak{gal}(3)$ has an infinitude of inequivalent central extensions, namely the ones $\mathfrak{gal}_M(3)$, where M is some arbitrary real number, and the case $M = 0$ corresponding to the trivial extension. Surprisingly, this does not carry over to the Galilei *group*. It turns out, as we will now discover, that the Galilei group has only *one* non-trivial central extension. Throughout this section, we follow the arguments in [25, Sec.III.B].

To calculate the central extensions of the Galilei group, we use the following two elementary facts from Lie group theory:

Lemma 5.1. *Any connected Lie group is generated by any neighbourhood of the identity element. (Cf. [45, Lem.1.18].)*

Corollary 5.2. *Let G be a connected Lie group, and let $\mathfrak{g}$ be its Lie algebra. Then G is generated by the image $\exp(\mathfrak{g})$. More explicitly, if $g \in G$ is an arbitrary element, then there exists a finite number of elements $X_1, \ldots, X_n$ in the Lie algebra so that we may write*

$$g = \exp(X_1) \cdots \exp(X_n).$$

(Cf. [14, Cor.16.28].)

Let us denote by $\mathrm{Gal}_M(3)$ the centrally extended Galilei group corresponding to the extended Lie algebra $\mathfrak{gal}_M(3)$. Elements of $\mathrm{Gal}_M(3)$ are of the form (θ, g), where $\theta \in \mathbb{R}$ and $g \in \mathrm{Gal}(3)$. For the purposes of the current computations, we interpret the first component as a *phase*, rather than an element of the circle $e^{i\theta} \in \mathrm{U}(1)$. This does not make much difference, but only that we can treat the first component as an additive structure, and it is permitted via the result of Lemma 4.21. Given $g = (s, \boldsymbol{a}, \boldsymbol{v}, R)$ (in the notation of Section 2.1), we shall write the element of the extended group simply as $(\theta, g) = (\theta, s, \boldsymbol{a}, \boldsymbol{v}, R)$. Now we have [25, Eq.(3.31)]

$$(\theta, g) = e^{\theta I} e^{sH} e^{\boldsymbol{a}\cdot\boldsymbol{P}} e^{\boldsymbol{v}\cdot\boldsymbol{K}} R,$$

where $\boldsymbol{P} = (P_1, P_2, P_3)$ and $\boldsymbol{K} = (K_1, K_2, K_3)$. (Note that the rotation matrix $R \in \mathrm{SO}(3)$ is generated by the generators J_1, J_2, J_3 from the exponential map, so that we directly obtain the term R instead of an exponential involving $\boldsymbol{J} = (J_1, J_2, J_3)$.) The product between vectors $\boldsymbol{a} \in \mathbb{R}^3$ and triplets of matrices $\boldsymbol{P} \in \mathfrak{gal}(3)^3$ is defined in the obvious way: $\boldsymbol{a} \cdot \boldsymbol{P} = a_i P_i$.

Using the expression of arbitrary group elements in terms of these exponentials, and the so-called *Baker-Campbell-Hausdorff formula* (cf. [15, Ch.5]), which here is useful in the forms

$$e^{-Y} X e^{Y} = X + [X, Y] + \frac{1}{2}[[X, Y], Y] + \cdots, \tag{5.2}$$

$$e^{X} e^{Y} = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]+\cdots}, \tag{5.3}$$

we will find a description of the group law of $\mathrm{Gal}_M(3)$. Further terms in these formulas all contain nested expressions of $[X, Y]$. We calculate

$$[\boldsymbol{v}\cdot\boldsymbol{K}, \boldsymbol{a}\cdot\boldsymbol{P}] = v_i a_j [K_i, P_j] = v_i a_j M\delta_{ij} = \langle\boldsymbol{v}, \boldsymbol{a}\rangle M,$$

where we use the defining commutator relation $[K_i, P_j] = M\delta_{ij}$ of the extension $\mathfrak{gal}_M(3)$. Note: here $\langle\cdot,\cdot\rangle$ is simply the Euclidean inner product. The mass term M is contained in the centre of $\mathfrak{gal}_M(3)$, and hence commutes with everything. Equation (5.3), in conjunction with Proposition 3.9.5, therefore give

$$e^{\boldsymbol{v}\cdot\boldsymbol{K}}e^{\boldsymbol{a}\cdot\boldsymbol{P}} = e^{\boldsymbol{v}\cdot\boldsymbol{K}+\boldsymbol{a}\cdot\boldsymbol{P}+\frac{1}{2}\langle\boldsymbol{v},\boldsymbol{a}\rangle M} = e^{\boldsymbol{v}\cdot\boldsymbol{K}+\boldsymbol{a}\cdot\boldsymbol{P}}e^{\frac{1}{2}\langle\boldsymbol{v},\boldsymbol{a}\rangle M}. \tag{5.4}$$

In exactly the same way we calculate

$$e^{\boldsymbol{a}\cdot\boldsymbol{P}}e^{\boldsymbol{v}\cdot\boldsymbol{K}} = e^{\boldsymbol{v}\cdot\boldsymbol{K}+\boldsymbol{a}\cdot\boldsymbol{P}}e^{-\frac{1}{2}\langle\boldsymbol{v},\boldsymbol{a}\rangle M},$$

which we rewrite to

$$e^{\boldsymbol{v}\cdot\boldsymbol{K}+\boldsymbol{a}\cdot\boldsymbol{P}} = e^{\boldsymbol{a}\cdot\boldsymbol{P}}e^{\boldsymbol{v}\cdot\boldsymbol{K}}e^{\frac{1}{2}\langle\boldsymbol{v},\boldsymbol{a}\rangle M}. \tag{5.5}$$

Equations (5.4) and (5.5) now tell us how to commute the elements $e^{\boldsymbol{v}\cdot\boldsymbol{K}}$ and $e^{\boldsymbol{a}\cdot\boldsymbol{P}}$:

$$e^{\boldsymbol{v}\cdot\boldsymbol{K}}e^{\boldsymbol{a}\cdot\boldsymbol{P}} = e^{\boldsymbol{a}\cdot\boldsymbol{P}}e^{\boldsymbol{v}\cdot\boldsymbol{K}}e^{\langle\boldsymbol{v},\boldsymbol{a}\rangle M}. \tag{5.6}$$

Similarly, we calculate $[\boldsymbol{v}\cdot\boldsymbol{K}, sH] = sv_i[K_i, H] = sv_iP_i$, and therefore

$$[[\boldsymbol{v}\cdot\boldsymbol{K}, sH], sH] = s^2 v_i [P_i, H] = 0.$$

Only the first two terms in (5.2) survive (with $X = \boldsymbol{v}\cdot\boldsymbol{K}$ and $Y = sH$), and thus:

$$e^{-sH}\boldsymbol{v}\cdot\boldsymbol{K}e^{sH} = v_iK_i + sv_iP_i = \boldsymbol{v}\cdot(\boldsymbol{K}+s\boldsymbol{P}).$$

Proposition 3.9.2 now shows us how to commute e^{sH} and $e^{\boldsymbol{v}\cdot\boldsymbol{K}}$:

$$e^{\boldsymbol{v}\cdot\boldsymbol{K}}e^{sH} = e^{sH}e^{\boldsymbol{v}\cdot(\boldsymbol{K}+s\boldsymbol{P})}. \tag{5.7}$$

Suppose now that we have another group element $(\theta', g') = (\theta', s', \boldsymbol{a}', \boldsymbol{v}', R') \in \mathrm{Gal}_M(3)$, with the corresponding exponentiation $(\theta', g') = e^{\theta' I}e^{s'H}e^{\boldsymbol{a}'\cdot\boldsymbol{P}}e^{\boldsymbol{v}'\cdot\boldsymbol{K}}R'$. We want to calculate the product

$$(\theta', g')(\theta, g) = e^{\theta' I}e^{s'H}e^{\boldsymbol{a}'\cdot\boldsymbol{P}}e^{\boldsymbol{v}'\cdot\boldsymbol{K}}R'e^{\theta I}e^{sH}e^{\boldsymbol{a}\cdot\boldsymbol{P}}e^{\boldsymbol{v}\cdot\boldsymbol{K}}R. \tag{5.8}$$

At once it is clear that we may group the central terms $e^{\theta' I}$ and $e^{\theta I}$ together into the term $e^{(\theta+\theta')I}$. The commutator relations (5.1) of the Galilei algebra moreover give

$$R'e^{sH}e^{\boldsymbol{a}\cdot\boldsymbol{P}}e^{\boldsymbol{v}\cdot\boldsymbol{K}}R = e^{sH}e^{R'\boldsymbol{a}\cdot\boldsymbol{P}}e^{R'\boldsymbol{v}\cdot\boldsymbol{K}}R'R,$$

so that the product (5.8) can be simplified to

$$(\theta', g')(\theta, g) = e^{(\theta+\theta')I}e^{s'H}e^{\boldsymbol{a}'\cdot\boldsymbol{P}}e^{\boldsymbol{v}'\cdot\boldsymbol{K}}e^{sH}e^{R'\boldsymbol{a}\cdot\boldsymbol{P}}e^{R'\boldsymbol{v}\cdot\boldsymbol{K}}R'R.$$

Applying (5.7) to the term $e^{\boldsymbol{v}'\cdot\boldsymbol{K}}e^{sH}$, and thereafter the fact that the time and translation generators commute, we further simplify:

$$(\theta', g')(\theta, g) = e^{(\theta+\theta')I}e^{(s+s')H}e^{\boldsymbol{a}'\cdot\boldsymbol{P}}e^{\boldsymbol{v}'\cdot(\boldsymbol{K}+s\boldsymbol{P})}e^{R'\boldsymbol{a}\cdot\boldsymbol{P}}e^{R'\boldsymbol{v}\cdot\boldsymbol{K}}R'R.$$

Equation (5.5) now gives

$$e^{\boldsymbol{v}'\cdot(\boldsymbol{K}+s\boldsymbol{P})} = e^{s\boldsymbol{v}'\cdot\boldsymbol{P}}e^{\boldsymbol{v}'\cdot\boldsymbol{K}}e^{\frac{1}{2}s\boldsymbol{v}'^2 M}.$$

Substituting this into the previous equation gives

$$(\theta', g')(\theta, g) = e^{(\theta+\theta'+\frac{1}{2}s\boldsymbol{v}'^2 M)I} e^{(s+s')H} e^{(\boldsymbol{a}'+s\boldsymbol{v}')\cdot\boldsymbol{P}} e^{\boldsymbol{v}'\cdot\boldsymbol{K}} e^{R'\boldsymbol{a}\cdot\boldsymbol{P}} e^{R'\boldsymbol{v}\cdot\boldsymbol{K}} R'R$$

after some obvious rewriting. As a last step we apply (5.6) to find

$$e^{\boldsymbol{v}'\cdot\boldsymbol{K}} e^{R'\boldsymbol{a}\cdot\boldsymbol{P}} = e^{R'\boldsymbol{a}\cdot\boldsymbol{P}} e^{\boldsymbol{v}'\cdot\boldsymbol{K}} e^{\langle \boldsymbol{v}', R'\boldsymbol{a}\rangle M},$$

which allows us to simplify the product in the group to:

$$(\theta', g')(\theta, g) = e^{\theta+\theta'+\frac{1}{2}s\boldsymbol{v}'^2 M+\langle \boldsymbol{v}', R'\boldsymbol{a}\rangle M} e^{(s+s')H} e^{(\boldsymbol{a}'+s\boldsymbol{v}'+R'\boldsymbol{a})\cdot\boldsymbol{P}} e^{(R'\boldsymbol{v}+\boldsymbol{v}')\cdot\boldsymbol{K}} R'R,$$

which is the desired form. From this expression we can directly observe the expression for the group law of the central extension $\mathrm{Gal}_M(3)$:

$$(\theta', s', \boldsymbol{a}', \boldsymbol{v}', R')(\theta, s, \boldsymbol{a}, \boldsymbol{v}, R) = (\theta+\theta'+\frac{1}{2}s\boldsymbol{v}'^2 M+\langle \boldsymbol{v}', R'\boldsymbol{a}\rangle M, s+s', R'\boldsymbol{a}+\boldsymbol{v}'s+\boldsymbol{a}', R'\boldsymbol{v}+\boldsymbol{v}', R'R).$$

We immediately recognise the group operation of the base Galilei group (cf. (2.1)) in the last four components. It is also immediately clear from the first component that the ordinary Galilei group Gal(3) is not a subgroup of the central extension $\mathrm{Gal}_M(3)$. Defining the cocycle

$$\omega_M : \mathrm{Gal}(3) \times \mathrm{Gal}(3) \to \mathrm{U}(1); \quad (g, g') \mapsto e^{i\xi_M(g,g')} = e^{i(\frac{1}{2}s\boldsymbol{v}'^2 M+\langle \boldsymbol{v}', R'\boldsymbol{a}\rangle M)},$$

we see that $\mathrm{Gal}_M(3) \cong \mathrm{U}(1) \times_{\omega_M} \mathrm{Gal}(3)$. It appears that we have again obtained an infinitude of central extensions, this time of the Galilei group Gal(3). But, in actual fact, when $M, M' \in \mathbb{R} \setminus \{0\}$, the following map defines an isomorphism:

$$\mathrm{Gal}_M(3) \to \mathrm{Gal}_{M'}(3); \quad (\theta, g) \mapsto \left(\frac{M'}{M}\theta, g\right).$$

Clearly $M = 0$ corresponds to the trivial extension, for which the above map is not well-defined. We are therefore justified to speak of *the* non-trivial central extension $\mathrm{Gal}_M(3)$ of the Galilei group Gal(3). Notwithstanding, the value of M will turn out to play an important rôle in the classification of Galilean elementary particles.

5.1.3 Universal cover of the Galilei group

We have already seen in Theorem 3.18 that SU(2) is the universal cover of SO(3), with covering map $\widetilde{p} : \mathrm{SU}(2) \to \mathrm{SO}(3)$, which has kernel $\ker(\widetilde{p}) = \{\pm I\}$. Since for all $n \in \mathbb{N}$, the space $\mathbb{R}^n$ is connected and simply connected, the only non-simply connected part of the Galilei group is the rotation group. The universal covering group of the Galilei group Gal(3) is therefore given by

$$\widetilde{\mathrm{Gal}(3)} = \mathbb{R}^4 \rtimes \widetilde{\mathrm{SE}(3)} = \mathbb{R}^4 \rtimes (\mathbb{R}^3 \rtimes \widetilde{\mathrm{SO}(3)}) = \mathbb{R}^4 \rtimes (\mathbb{R}^3 \rtimes \mathrm{SU}(2)).$$

The covering map is what one would expect:

$$\widetilde{p} : \widetilde{\mathrm{Gal}(3)} \to \mathrm{Gal}(3); \quad (s, \boldsymbol{a}, \boldsymbol{v}, U) \mapsto (s, \boldsymbol{a}, \boldsymbol{v}, \widetilde{p}(U)).$$

5.2 Extensions of the Poincaré group

5.2.1 Central extensions of the Poincaré algebra

Recall the defining commutator relations (3.4) of the Poincaré algebra $\mathfrak{poin}(1,3)$, which we here reproduce for convenience:

$$\begin{aligned} [J_i, J_j] = \varepsilon_{ijk} J_k, \quad [J_i, K_j] = \varepsilon_{ijk} K_k, \quad [J_i, P_j] = \varepsilon_{ijk} P_k, \\ [K_i, H] = P_i, \quad [K_i, P_j] = \delta_{ij} H, \quad [K_i, K_j] = -\varepsilon_{ijk} J_k. \end{aligned} \tag{5.9}$$

A large amount of work on calculating the central extensions of $\mathfrak{poin}(1,3)$ has already been done in Section 5.1.1, where we calculated the central extensions of the Galilei algebra. We proceed, as before, by calculating restrictions of a cocycle $\Omega \in \mathrm{Z}^2_{\mathrm{al}}(\mathfrak{poin}(1,3), \mathbb{R})$. We already know that the first three brackets in (5.9) cannot be extended, as neither can any commutator with H. Just as with the Galilei algebra, we may therefore only have non-trivial extensions via

$$[K_i, P_j]_\Omega = \Omega(K_i, P_j) \oplus [K_i, P_j].$$

Only now, we know the explicit expression for the commutator between the boost and translation generators: $[K_i, P_j] = \delta_{ij} H$. In any case, the Jacobi identity for Ω gives

$$\begin{aligned} 0 &= \Omega(K_i, [J_j, P_k]) + \Omega(J_j, [P_k, K_i]) + \Omega(P_k, [K_i, J_j]) \\ &= \Omega(K_i, \varepsilon_{jkl} P_l) + \Omega(J_j, -\delta_{ki} H) + \Omega(P_k, -\varepsilon_{ijp} K_p). \end{aligned}$$

The middle term vanishes, so we have $\Omega(K_i, \varepsilon_{jkl} P_l) = \Omega(P_k, \varepsilon_{ijp} K_p)$. Clearly whenever $j = i$ the right hand side vanishes, while the left hand side may not. Picking i, k and l distinct, we have $\Omega(K_i, \pm P_l) = 0$. This remains for terms of the form $\Omega(K_i, P_i)$ to be non-trivial, however: picking j, k and i mutually distinct, we find $\Omega(K_i, P_i) = \Omega(P_k, \varepsilon_{ijp} K_p) = \pm\Omega(P_k, K_k)$. This means that we have one apparent degree of freedom, and we set $\Omega(K_1, P_1) = C$. Explicitly setting $i = 2$, $j = 3$ and $k = 2$ we find $\Omega(K_2, P_2) = -C$. Similarly, setting $i = 3$, $j = 2$ and $k = 1$ we find $\Omega(K_3, P_3) = C$. However, when we set $i = 2$, $j = 1$ and $k = 3$ we find $C = -C$, showing that $C = 0$, and proving that the Poincaré algebra $\mathfrak{poin}(1,3)$ cannot be non-trivially extended:

$$\mathrm{H}^2_{\mathrm{al}}(\mathfrak{poin}(1,3), \mathbb{R}) = 0.$$

5.2.2 Universal cover of the Poincaré group

Since the result $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{poin}(1,3), \mathbb{R}) = 0$ will be sufficient for our needs, we do not bother with calculating the explicit forms of central extensions of the Poincaré group itself.

Let $\widetilde{p} : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3)$ be the universal covering map as in Theorem 3.19. The universal cover of the Poincaré group reads

$$\widetilde{\mathrm{Poin}}(1,3) \cong \mathbb{R}^4 \rtimes \widetilde{\mathrm{SO}^+}(1,3) \cong \mathbb{R}^4 \rtimes \mathrm{SL}(2,\mathbb{C}),$$

with covering map

$$\widetilde{p} : \widetilde{\mathrm{Poin}}(1,3) \to \mathrm{Poin}(1,3); \quad (a, \Delta) \mapsto (a, \widetilde{p}(\Delta)).$$

6 Classifying the quantum elementary particles

We are now ready to apply the formalisms of universal covers (Section 3.4), central extensions (Section 4), and the pertinent results thereof (Section 5), to classify the elementary particles in both relativistic and non-relativistic quantum mechanics. First we elaborate on the mathematical definition of a quantum elementary particle.

6.1 Unitary operators on a Hilbert space

As always, mathematically speaking we regard a symmetry as a bijective map that preserves the structure of some object. In the case of Hilbert spaces, we have a linear structure and an inner product structure (the latter of which also induces a topological structure). The notion of symmetry is then realised by *unitary operators*. A ***unitary operator*** on a Hilbert space $\mathscr{H}$ is a bijective linear operator $U : \mathscr{H} \to \mathscr{H}$ so that for any two elements $\psi, \phi \in \mathscr{H}$ we have

$$\langle U\psi, U\phi\rangle = \langle \psi, \phi\rangle,$$

where $\langle \cdot, \cdot\rangle$ denotes the inner product of $\mathscr{H}$. This is equivalent to the condition that $UU^\dagger = U^\dagger U = \mathrm{id}_{\mathscr{H}}$, where $U^\dagger$ denotes the adjoint of U and $\mathrm{id}_{\mathscr{H}} : \psi \mapsto \psi$ is the identity operator on $\mathscr{H}$. If U is a unitary operator, then

$$\langle \psi, \phi\rangle = \langle UU^{-1}\psi, UU^{-1}\phi\rangle = \langle U(U^{-1}\psi), U(U^{-1}\phi)\rangle = \langle U^{-1}\psi, U^{-1}\phi\rangle,$$

so that U^{-1} is also a unitary operator. Similarly, if V is another unitary operator on $\mathscr{H}$, then the composition UV is again a unitary operator on $\mathscr{H}$. This shows that the unitary operators on a Hilbert space adhere to a group structure. In light of this fact, we define the ***unitary group***:

$$\mathrm{U}(\mathscr{H}) := \{\text{unitary operators on } \mathscr{H}\},$$

which, when endowed with the operation of composition, does indeed form a group. Recall from Section 3.2 that the *unitary matrix group* $\mathrm{U}(m)$ is defined as the group of all unitary complex $m \times m$ matrices:

$$\mathrm{U}(m) := \{U \in \mathrm{GL}(m, \mathbb{C}) : U^{-1} = U^\dagger\}.$$

In fact, when $\mathscr{H}$ is finite dimensional, $\mathrm{U}(\mathscr{H}) \cong \mathrm{U}(\dim(\mathscr{H}))$.

6.2 Symmetries of the state space

We have already seen in Section 2.3.1 that the corresponding notion of symmetry on the true quantum state space (i.e., the projective Hilbert spaces) are *projective automorphisms*.

Recall that the canonical projection $P : \mathscr{H} \setminus \{0\} \to \mathbf{P}(\mathscr{H})$ sends a non-zero element in the Hilbert space to its ray: $\psi \mapsto P(\psi) = [\psi]$. The transition probability $\delta : \mathbf{P}(\mathscr{H}) \times \mathbf{P}(\mathscr{H}) \to \mathbb{R}$ (see (2.2)) defines a topology on $\mathbf{P}(\mathscr{H})$, which is generated by the open balls $B_r(\psi) = \{\phi \in \mathbf{P}(\mathscr{H}) : \delta(\psi, \phi) < r\}$. This topology is the *final topology* induced by the projection map P, which is the topology $\{V \subseteq \mathbf{P}(\mathscr{H}) : P^{-1}(V) \subseteq \mathscr{H} \setminus \{0\} \text{ open}\}$ (i.e., the largest topology on $\mathbf{P}(\mathscr{H})$ so that P is continuous).

From Wigner's Theorem 2.1 we know that any projective automorphism $T \in \mathrm{Aut}(\mathbf{P}(\mathscr{H}))$ arises from either a unitary or an anti-unitary operator on $\mathscr{H}$, determined uniquely by T up to a complex phase. The precise way in which this happens is as follows. Given a unitary operator $U \in \mathrm{U}(\mathscr{H})$, we define the projective operator $\widehat{P}(U) : \mathbf{P}(\mathscr{H}) \to \mathbf{P}(\mathscr{H})$ by

$$\widehat{P}(U)(P(\psi)) := P(U\psi),$$

where $\psi \in \mathscr{H}$. We need to verify that this operator is well defined by checking that the right hand side does not depend on the representative ψ of the ray $P(\psi)$. For that, let $\psi, \phi \in \mathscr{H}$ be two elements in the same ray, meaning that $P(\psi) = P(\phi)$.

We can now find a non-zero complex number $\lambda \in \mathbb{C}$ so that $\phi = \lambda\psi$. Since U is linear, we find that $P(U(\phi)) = P(U(\lambda\psi)) = P(\lambda U(\psi)) = P(U(\psi))$. (More generally, we find that any injective linear operator on $\mathscr{H}$ respects the equivalence relation $\sim$.) This shows that $\widehat{P}(U)$ is well-defined. It turns out that $\widehat{P}(U)$ is in fact a projective automorphism:

$$\delta(\widehat{P}(U)(P(\phi)), \widehat{P}(U)(P(\psi))) = \delta(P(U\psi), P(U\phi)) = \frac{|\langle U\psi, U\phi\rangle|^2}{\|U\psi\|^2\,\|U\phi\|^2} = \frac{|\langle \psi, \phi\rangle|^2}{\|\psi\|^2\,\|\phi\|^2} = \delta(\phi, \psi).$$

Thus we see that $\widehat{P}(U) \in \mathrm{Aut}(\mathbf{P}(\mathscr{H}))$ for any $U \in \mathrm{U}(\mathscr{H})$. This motivates the definition of the *unitary projective transformations*:

$$\mathrm{PU}(\mathscr{H}) := \widehat{P}(\mathrm{U}(\mathscr{H})) \subseteq \mathrm{Aut}(\mathbf{P}(\mathscr{H})).$$

$\mathrm{PU}(\mathscr{H})$ is called the ***projective unitary group*** of $\mathscr{H}$. The unitary projective transformations do not necessarily give all of the projective automorphisms (some may arise from *anti*-unitary U), but since we are concerned with connected symmetry groups, it will suffice to only consider projective automorphisms that arise from unitary operators.

The unitary projective transformations $\mathrm{PU}(\mathscr{H})$ form a subgroup of the projective transformations $\mathrm{Aut}(\mathbf{P}(\mathscr{H}))$. To see this, let $\widehat{P}(U)$ and $\widehat{P}(V)$ projective unitary transformations. Then

$$\begin{aligned}\widehat{P}(U)\widehat{P}(V)(P(\phi)) &= \widehat{P}(U)\left(\widehat{P}(V)(P(\phi))\right) = \widehat{P}(U)(P(V\phi)) \\ &= P(U(V\phi)) = P(UV\phi) = \widehat{P}(UV)(P(\phi)).\end{aligned}$$

Hence $\widehat{P}(U)\widehat{P}(V) = \widehat{P}(UV)$, and since UV is again a unitary operator we see $\widehat{P}(U)\widehat{P}(V) \in \mathrm{PU}(\mathscr{H})$. Furthermore, we easily verify that $\widehat{P}(\mathrm{id}_{\mathscr{H}}) = \mathrm{id}_{\mathbf{P}(\mathscr{H})} \in \mathrm{PU}(\mathscr{H})$, which together with the above gives $\widehat{P}(U)^{-1} = \widehat{P}(U^{-1})$. Since U^{-1} is unitary when U is unitary, it follows that $\widehat{P}(U)^{-1} \in \mathrm{PU}(\mathscr{H})$. We conclude that $\mathrm{PU}(\mathscr{H})$ is indeed a subgroup of $\mathrm{Aut}(\mathbf{P}(\mathscr{H}))$.

Perhaps unsurprisingly, we can think of the projective unitary group $\mathrm{PU}(\mathscr{H})$ like a projective general linear group (cf. Section 4.2):

Lemma 6.1. *The sequence*

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{\mathrm{diag}} \mathrm{U}(\mathscr{H}) \xrightarrow{\widehat{P}} \mathrm{PU}(\mathscr{H}) \longrightarrow 1$$

is exact. Moreover, $\mathrm{U}(\mathscr{H})$ *defines a central extension of* $\mathrm{PU}(\mathscr{H})$ *by* $\mathrm{U}(1)$.

Proof. Above we have seen already that $\widehat{P}$ defines a homomorphism. It is in fact surjective, because by very definition $\mathrm{PU}(\mathscr{H}) = \mathrm{im}(\widehat{P})$. It is also clear that $\mathrm{diag} : \lambda \mapsto \lambda\,\mathrm{id}_{\mathscr{H}}$ is an injective homomorphism. The fact that diag is well-defined here follows from the fact that the inner product is linear in one component, and anti-linear in the other.

Let $\mathrm{diag}(\lambda) \in \mathrm{U}(\mathscr{H})$. Now:

$$\widehat{P}(\mathrm{diag}(\lambda))(P(\phi)) = P(\lambda\,\mathrm{id}_{\mathscr{H}}\,\phi) = P(\mathrm{id}_{\mathscr{H}}\,\phi) = \widehat{P}(\mathrm{id}_{\mathscr{H}})(P(\phi))$$

for each $\phi \in \mathscr{H}$, so that $\mathrm{diag}(\mathrm{U}(1)) \subseteq \ker(\widehat{P})$. The converse inclusion follows similarly. We conclude that the above sequence is in fact exact.

Lastly, note that $\mathrm{U}(1)$ is an abelian group. It is easy to see that $\mathrm{diag}(\mathrm{U}(1)) \subseteq \mathrm{Z}(\mathrm{U}(\mathscr{H}))$, so that $\mathrm{U}(\mathscr{H})$ is indeed a central extension of $\mathrm{PU}(\mathscr{H})$. □

6.3 Projective unitary representations on Lie groups

We can now formally define the notion of a projective unitary representation on a Lie group. To do this, we need to equip $\mathrm{U}(\mathscr{H})$ with a topology. The most fitting one turns out to be the *strong (operator) topology*, which is the topology in which a net $(u_\iota)_{\iota\in I}$ converges to some $u \in \mathrm{U}(\mathscr{H})$[7], if and only if the net $(u_\iota\psi)_{\iota\in I}$ converges to $u\psi$ in the Hilbert space $\mathscr{H}$, for each $\psi \in \mathscr{H}$. The reason that this topology is fitting is due to the following result:

Proposition 6.2. *If $u : G \to \mathrm{U}(\mathscr{H})$ is a homomorphism of some locally compact topological group G (such as a Lie group) to the unitary operators on a Hilbert space $\mathscr{H}$, then the following conditions are equivalent:*

1. *The map $G \times \mathscr{H} \to \mathscr{H}; (g, \psi) \mapsto u(g)\psi$ is continuous;*
2. *The map $G \to \mathrm{U}(\mathscr{H}); g \mapsto u(g)$ is continuous in the strong topology on $\mathrm{U}(\mathscr{H})$.*

Proof. Suppose that the first point is true. Then for every convergent net $(g_\iota, \psi_\iota)_{\iota\in I}$ in $G\times\mathscr{H}$, say, with limit (g, ψ), the net $(u(g_\iota)\psi_\iota)_{\iota\in I}$ in the Hilbert space converges to $u(g)\psi$. This means that in particular every net of the form $(g_\iota, \psi)_{\iota\in I}$, where $\psi \in \mathscr{H}$ is *fixed*, the net $(u(g_\iota)\psi)_{\iota\in I}$ converges to $u(g)\psi$. Since ψ is arbitrary, this is exactly what it means for $g \mapsto u(g)$ to be strongly continuous.

The converse direction is somewhat less trivial. Suppose the second point is true; for every convergent net $(g_\iota)_{\iota\in I}$ in the group (again with limit g) the net $(u(g_\iota))_{\iota\in I}$ converges strongly to $u(g)$. Furthermore, let $(\psi_\iota)_{\iota\in I}$ be an arbitrary net in the Hilbert space that converges to ψ. To show that $(g, \psi) \mapsto u(g)\psi$ is continuous, we need to show that $(u(g_\iota)\psi_\iota)_{\iota\in I}$ converges to $u(g)\psi$ in the Hilbert space. Using the triangle inequality we find

$$\begin{aligned}\|u(g_\iota)\psi_\iota - u(g)\psi\| &\leqslant \|u(g_\iota)\psi_\iota - u(g_\iota)\psi\| + \|u(g_\iota)\psi - u(g)\psi\| \\ &\leqslant \|u(g_\iota)\|\,\|\psi_\iota - \psi\| + \|u(g_\iota)\psi - u(g)\psi\|\,.\end{aligned}$$

The second term will clearly vanish when the limit is taken, by strong continuity of $g \mapsto u(g)$. The term $\|\psi_\iota \to \psi\|$ on its own will also vanish, since we assume ψ is the limit of $(\psi_\iota)_{\iota\in I}$. Therefore it suffices to show that the factor $\|u(g_\iota)\|$ is bounded. To do this, we appeal to the *Principle of Uniform Boundedness* [26, Thm.3.11]. We assume that G is locally compact, so that around the limit g we may find a compact neighbourhood K. Clearly the net $(g_\iota)_{\iota\in I}$ is eventually in K, so we can find $\nu \in I$ so that for all $\iota \geqslant \nu$ we have $g_\iota \in K$. Per hypothesis, the image $u(K) \subseteq \mathrm{U}(\mathscr{H})$ is compact in $\mathrm{U}(\mathscr{H})$.

For the Principle of Uniform Boundedness to apply, we need to show

$$\forall\varphi \in \mathscr{H} : \quad S_\varphi = \sup\{\|u(g_\iota)\varphi\| : \iota \geqslant \nu\} \leqslant \sup\{\|u(k)\varphi\| : k \in K\} < \infty.$$

Since $u(K)$ is compact, the image $u(K)\varphi = \{u(k)\varphi : k \in K\} \subseteq \mathscr{H}$ is bounded, so the above inequality holds. Therefore the Principle of Uniform Boundedness gives

$$S = \sup\{\|u(g_\iota)\| : \iota \geqslant \nu\} < \infty,$$

and the result follows; for $\iota \geqslant \nu$

$$\|u(g_\iota)\psi_\iota - u(g)\psi\| \leqslant S\,\|\psi_\iota - \psi\| + \|u(g_\iota)\psi - u(g)\psi\| \to 0. \qquad \square$$

[7] A *net* is a generalisation of a *sequence*; see for example [29, Sec.5].

We therefore endow $\mathrm{U}(\mathscr{H})$ with the strong operator topology. This directly endows the projective unitary operators $\mathrm{PU}(\mathscr{H})$ with a similar topology, to which we shall also refer as the strong operator topology. (This makes it into a topological group [33, Prop.3.11].)

Definition 6.3. Let G be a connected Lie group, as always. A ***unitary representation*** of G by a Hilbert space $\mathscr{H}$ is a continuous homomorphism $u : G \to \mathrm{U}(\mathscr{H})$. A ***projective unitary representation*** of G by $\mathscr{H}$ is a continuous homomorphism $\rho : G \to \mathrm{PU}(\mathscr{H})$.

The notion of irreducibility for unitary representations is just that of Definition 2.3. For projective unitary representations we say that $\rho : G \to \mathrm{PU}(\mathscr{H})$ is *irreducible* whenever one (and hence all) of the unitary representations $u : \mathrm{U}(1) \times_\omega G \to \mathrm{U}(\mathscr{H})$; $u(z, g) = z(s\circ\rho)(g)$ of the central extension $\mathrm{U}(1)\times_\omega G$ corresponding to ρ is irreducible.

6.4 Lifting projective unitary representations

Since $\mathrm{U}(\mathscr{H})$ is a central extension of $\mathrm{PU}(\mathscr{H})$ (see Lemma 6.1), we may ask if it is possible to 'lift' a projective unitary representation $\rho : G \to \mathrm{PU}(\mathscr{H})$ to a unitary representation $u : G \to \mathrm{U}(\mathscr{H})$, just like in Theorem 4.7. In light of Lemma 6.1 we have the following theorem for non-continuous unitary representations:

Theorem 6.4. *Let G be a group and let $\rho : G \to \mathrm{PU}(\mathscr{H})$ be a homomorphism. Then there exists a central extension E of G by $\mathrm{U}(1)$ and a homomorphism $u : E \to \mathrm{U}(\mathscr{H})$ so that the following diagram commutes:*

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{U}(1) \arrow[r, hook, "\iota"] \arrow[d, "\mathrm{id}"] & E \arrow[r, two heads, "\pi"] \arrow[d, "u"] & G \arrow[r] \arrow[d, "\rho"] & 1 \\
1 \arrow[r] & \mathrm{U}(1) \arrow[r, hook, "\mathrm{diag}"] & \mathrm{U}(\mathscr{H}) \arrow[r, two heads, "\widehat{P}"] & \mathrm{PU}(\mathscr{H}) \arrow[r] & 1.
\end{tikzcd}$$

The proof is almost exactly the same as that of Theorem 4.7. The difficulty is now in finding a similar theorem for *continuous* projective unitary representations. Central extensions provide the solution for *algebraically* lifting a projective (unitary) representation, but now we also have a *topological* obstruction.

The solution for this obstruction turns out to be the concept of a *universal central extension*. In the first place, we have the following lemma connecting the unitary representations of a connected Lie group G and its universal cover. Let $\widetilde{G}$ be this universal cover of G, with covering map $\widetilde{p} : \widetilde{G} \to G$ of Lie's Third Theorem 4.19, so that $D = \ker(\widetilde{p})$.

Lemma 6.5. *Every unitary representations $G \to \mathrm{U}(\mathscr{H})$ arises from another unitary representation $\widetilde{u} : \widetilde{G} \to \mathrm{U}(\mathscr{H})$ with the property that $D \subseteq \ker(\widetilde{u})$.*

Proof. Starting with a unitary representation $u : G \to \mathrm{U}(\mathscr{H})$, we straightforwardly define the unitary representation $\widetilde{u} := u \circ \widetilde{p}$. It is trivial to see that $\widetilde{u}$ satisfies the desired property, since $\ker(\widetilde{p}) = D$.

Conversely, suppose that we have a unitary representation $\widetilde{u} : \widetilde{G} \to \mathrm{U}(\mathscr{H})$ of the universal covering group, with the desired property. A section $\widetilde{s} : G \to \widetilde{G}$ of the covering map $\widetilde{p}$ with $\widetilde{s}(1_G) = 1_{\widetilde{G}}$ defines a map

$$\delta : G \times G \to D; \quad \delta := \partial\widetilde{s}^{-1}.$$

Now the fact that $D \subseteq \ker(\widetilde{u})$ makes the map $u := \widetilde{u} \circ \widetilde{s}$ into a homomorphism:

$$u(g)u(h) = \widetilde{u}(\widetilde{s}(g)\widetilde{s}(h)) = \widetilde{u}(\delta(g,h)\widetilde{s}(gh)) = \widetilde{u}(\delta(g,h))u(gh) = u(gh).$$

The second paragraph clearly describes a construction that is the left inverse of the construction in the first paragraph. Precisely speaking, every unitary representation u arises from a unitary representation $\widetilde{u}$ (with the desired property) via $u = \widetilde{u} \circ \widetilde{s}$. □

We now move on to a simple case where the problem of central extensions can be partially avoided. For this, we consider the situation of Theorem 4.20.

Proposition 6.6. *Let G be a connected Lie group with $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{g}, \mathbb{R}) = 0$, and $\mathscr{H}$ a separable Hilbert space (i.e., a Hilbert space with countable orthonormal basis). Every projective unitary representation $\rho : G \to \mathrm{PU}(\mathscr{H})$ is of the form $\rho = \widehat{P} \circ u \circ \widetilde{s}$, where $\widetilde{s}$ is a section of the covering map $\widetilde{p}$ preserving units, and $u : \widetilde{G} \to \mathrm{U}(\mathscr{H})$ is a unitary representation of the universal cover with the property that $u(D) \subseteq \mathrm{diag}(\mathrm{U}(1))$. More simply (but less accurately), the following diagram commutes:*

$$\begin{tikzcd}
1 \arrow[r] & D \arrow[r, hook] & \widetilde{G} \arrow[r, two heads, "\widetilde{p}"] \arrow[d, dashed, "u"] & G \arrow[r] \arrow[d, "\rho"] & 1 \\
1 \arrow[r] & \mathrm{U}(1) \arrow[r, hook, "\mathrm{diag}"] & \mathrm{U}(\mathscr{H}) \arrow[r, two heads, "\widehat{P}"] & \mathrm{PU}(\mathscr{H}) \arrow[r] & 1.
\end{tikzcd}$$

Proof. (We follow the proof of [22, Thm.5.59.1].) We start given a projective unitary representation $\rho : G \to \mathrm{PU}(\mathscr{H})$. Using a section $\widehat{s} : \mathrm{PU}(\mathscr{H}) \to \mathrm{U}(\mathscr{H})$ of the projection map $\widehat{P}$ we can define a cocycle $\omega = \partial(\widehat{s} \circ \rho)^{-1}$ on G with values in $\mathrm{U}(1)$ (as we know). Using Theorem 4.20 we may find a continuous homomorphism $\chi : D \to \mathrm{U}(1)$ so that $[\omega] = [c_\chi]$. By redefining the section $\widehat{s}$, we may therefore simply take $\omega = c_\chi$.

Using these ingredients, we define the map

$$u : \widetilde{G} \to \mathrm{U}(\mathscr{H}); \quad x \mapsto \chi\left(x(\widetilde{s} \circ \widetilde{p}(x))^{-1}\right)(\widehat{s} \circ \rho \circ \widetilde{p})(x). \tag{6.1}$$

Let $x \in \widetilde{G}$. Evaluating the covering map $\widetilde{p}$ at the point $x(\widetilde{s} \circ \widetilde{p}(x))^{-1}$ shows that this element is in the kernel D, so that the above expression for u is well-defined. An easy calculation

$$\begin{aligned}
u \circ \widetilde{s}(g) &= \chi\left(\widetilde{s}(g)\left(\widetilde{s} \circ \widetilde{p}\left(\widetilde{s}(g)\right)\right)^{-1}\right)\left(\widehat{s} \circ \rho \circ \widetilde{p}\right)\left(\widetilde{s}(g)\right) \\
&= \chi\left(\widetilde{s}(g)\widetilde{s}(g)^{-1}\right)(\widehat{s} \circ \rho)(g) = \widehat{s} \circ \rho(g)
\end{aligned}$$

shows that $\rho = \widehat{P} \circ u \circ \widetilde{s}$. It is furthermore clear that $u(D) \subseteq \mathrm{diag}(\mathrm{U}(1))$, since if $x \in D$ then $\widetilde{p}(x) = 1_G$, and hence $u(x) = \chi(x) \in \mathrm{diag}(\mathrm{U}(1))$. For the verification of u being a homomorphism we need the fact that $\omega = \partial(\widehat{s} \circ \rho)^{-1} = c_\chi = \chi \circ \partial \widetilde{s}^{-1}$. This gives the identity

$$(\widehat{s} \circ \rho)(\widetilde{p}(x)) = \chi\left(\widetilde{s}(\widetilde{p}(x))\widetilde{s}(\widetilde{p}(y))\widetilde{s}(\widetilde{p}(xy))^{-1}\right)(\widehat{s} \circ \rho)(\widetilde{p}(xy))(\widehat{s} \circ \rho)(\widetilde{p}(y))^{-1}.$$

We now calculate, using the above equation, that u respects the group multiplication in $\widetilde{G}$:

$$\begin{aligned}
u(x)u(y) &= \chi\left(x\widetilde{s}(\widetilde{p}(x))^{-1}\right)(\widehat{s}\circ\rho)(\widetilde{p}(x))\chi\left(y\widetilde{s}(\widetilde{p}(y))^{-1}\right)(\widehat{s}\circ\rho)(\widetilde{p}(y))\\
&= \chi\left(x\widetilde{s}(\widetilde{p}(x))^{-1}\right)\chi\left(\widetilde{s}(\widetilde{p}(x))\widetilde{s}(\widetilde{p}(y))\widetilde{s}(\widetilde{p}(xy))^{-1}\right)(\widehat{s}\circ\rho)(\widetilde{p}(xy))(\widehat{s}\circ\rho)(\widetilde{p}(y))^{-1}\\
&\quad\cdot\chi\left(y\widetilde{s}(\widetilde{p}(y))^{-1}\right)(\widehat{s}\circ\rho)(\widetilde{p}(y))\\
&= \chi\left(x\widetilde{s}(\widetilde{p}(y))\widetilde{s}(\widetilde{p}(xy))^{-1}\right)(\widehat{s}\circ\rho)(\widetilde{p}(xy))(\widehat{s}\circ\rho)(\widetilde{p}(y))^{-1}\chi\left(y\widetilde{s}(\widetilde{p}(y))^{-1}\right)(\widehat{s}\circ\rho)(\widetilde{p}(y))\\
&= \chi\left(x\left[y\widetilde{s}(\widetilde{p}(y))^{-1}\right]\widetilde{s}(\widetilde{p}(y))\widetilde{s}(\widetilde{p}(xy))^{-1}\right)(\widehat{s}\circ\rho)(\widetilde{p}(xy))\\
&= \chi\left(xy\widetilde{s}(\widetilde{p}(xy))^{-1}\right)(\widehat{s}\circ\rho)(\widetilde{p}(xy)) = u(xy).
\end{aligned}$$

In the last step we use that $y\widetilde{s}(\widetilde{p}(y))^{-1}\in D$ is an element in the centre of $\widetilde{G}$. It is further trivial to check that $u(1_{\widetilde{G}}) = \mathrm{id}_{\mathscr{H}}$. □

Sadly, this result does not apply to the Galilei group, since $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{gal}(3),\mathbb{R})\cong\mathbb{R}$ (see Section 5.1.1). To classify the projective unitary representations of the Galilei group, we need to do some more work. By Lemma 4.21 we know that the cohomology group $\mathrm{H}^2_{\mathrm{es}}(\widetilde{G},\mathrm{U}(1))$ of the universal cover $\widetilde{G}$ of some connected Lie group G is isomorphic to $\mathrm{H}^2_{\mathrm{es}}(\widetilde{G},\mathbb{R})$. Theorem 4.17 is an easy way to see that $\mathrm{H}^2_{\mathrm{es}}(\widetilde{G},\mathbb{R})$ is a finite dimensional vector space. For our purposes we need to consider a certain subset of this cohomology group. In particular, we need cocycles $\xi\in\mathrm{Z}^2_{\mathrm{es}}(\widetilde{G},\mathbb{R})$ so that for every $x\in\widetilde{G}$ we have $\xi(x,\delta)=\xi(\delta,x)$, whenever $\delta\in D=\ker(\widetilde{p})$. This property is respected by the cohomology relation: suppose that $\xi'\in\mathrm{Z}^2_{\mathrm{es}}(\widetilde{G},\mathbb{R})$ is cohomologous to ξ. Then we may find a coboundary $\partial\alpha\in\mathrm{B}^2_{\mathrm{es}}(\widetilde{G},\mathbb{R})$ so that $\xi'-\xi=\partial\alpha$. (Note: the notation $\partial\alpha$ in this case means $\partial\alpha(x,y)=\alpha(xy)-\alpha(x)-\alpha(y)$.) Now, since $D\subseteq\mathrm{Z}(\widetilde{G})$ we find every coboundary satisfies the aforementioned property: $\partial\alpha(x,\delta)=\partial\alpha(\delta,x)$. Hence:

$$\xi'(x,\delta)=\xi(x,\delta)+\partial\alpha(x,\delta)=\xi(\delta,x)+\partial\alpha(\delta,x)=\xi'(\delta,x),$$

as claimed. The property therefore gives a well defined subspace

$$\mathrm{H}^2_{\widetilde{p}}(\widetilde{G},\mathbb{R}) := \{[\xi]\in\mathrm{H}^2_{\mathrm{es}}(\widetilde{G},\mathbb{R}) : \forall x\in\widetilde{G},\delta\in\ker(\widetilde{p}) : \xi(x,\delta)=\xi(\delta,x)\}.$$

Suppose that $\xi_1,\dots,\xi_N\in\mathrm{Z}^2_{\mathrm{es}}(\widetilde{G},\mathbb{R})$ are some real-valued cocycles whose cohomology classes induce a basis for $\mathrm{H}^2_{\widetilde{p}}(\widetilde{G},\mathbb{R})$, for some $N\in\mathbb{N}$. The map

$$\xi^\star : \widetilde{G}\times\widetilde{G}\to\mathbb{R}^N;\quad (x,y)\mapsto(\xi_1(x,y),\dots,\xi_N(x,y))$$

defines a $\mathbb{R}^N$-valued cocycle on the covering group $\widetilde{G}$. This map is clearly smooth near the identity whenever $\xi_1,\dots,\xi_N$ are, and we therefore define:

Definition 6.7. The group $G^\star=\mathbb{R}^N\times_{\xi^\star}\widetilde{G}$ defined as a central extension of $\widetilde{G}$ by $\mathbb{R}^N$, i.e., as the product manifold $\mathbb{R}^N\times\widetilde{G}$ endowed with the operation (see Section 4.1)

$$(\boldsymbol{a},x)(\boldsymbol{b},x)=(\boldsymbol{a}+\boldsymbol{b}+\xi^\star(x,y),xy),$$

for $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^N$ and $x,y\in\widetilde{G}$, is called the ***universal central extension*** of G. Given the covering map $\widetilde{p}:\widetilde{G}\to G$, we define the *universal central covering map* in an obvious way:

$$p^\star : G^\star\to G;\quad (\boldsymbol{a},x)\mapsto\widetilde{p}(x).$$

We write the kernel of the universal central covering map $p^\star$ as $K:=\ker(p^\star)$. It is clear that $K=\mathbb{R}^N\times\ker(\widetilde{p})=\mathbb{R}^N\times D$. We therefore have an exact sequence

$$1\longrightarrow K\hookrightarrow G^\star\xrightarrow{p^\star}\!\!\!\!\rightarrow G\longrightarrow 1.$$

Moreover, by the very construction of the universal central extension $G^\star$, we have that $\xi^\star(\delta, x) = \xi^\star(x, \delta)$ for all $x \in \widetilde{G}$ and $\delta \in D$, so that $K \subseteq \mathrm{Z}(G^\star)$. The universal central extension is therefore, indeed, a central extension of G in the ordinary sense of Definition 4.1.

Using universal central extensions we may improve on Theorem 4.20:

Theorem 6.8. *Let G be a connected Lie group and let $G^\star = \mathbb{R}^N \times_{\xi^\star} \widetilde{G}$ be its universal central extension, with central subgroup $K = \mathbb{R}^N \times D$, as above. Take a section $s^\star : G \to G^\star$ of the universal central covering map with $s^\star(1_G) = 1_{G^\star}$. We have the following surjective homomorphism (cf. (4.4)):*

$$\widehat{K} \to \mathrm{H}^2_{\mathrm{es}}(G, \mathrm{U}(1)); \quad \chi \mapsto [\mu_\chi],$$

where $\mu_\chi : G \times G \to \mathrm{U}(1)$ is the cocycle defined by

$$\mu_\chi := \chi \circ \partial(s^\star)^{-1}.$$

Proof. (We follow the proof of [6, Lem.11], which is similar to that of Theorem 4.20.) We leave it to the reader to show that the map is a well defined homomorphism, and only show that the map in question is surjective. Take $\omega \in \mathrm{H}^2_{\mathrm{es}}(G, \mathrm{U}(1))$ to be an arbitrary cocycle. Since the cohomology class of μ_χ does not depend on the section $s^\star$, we take $s^\star(g) = (0, \widetilde{s}(g))$, where $\widetilde{s} : G \to \widetilde{G}$ is a section of the covering map $\widetilde{p}$ with the property that $\widetilde{s}(1_G) = 1_{\widetilde{G}}$. With this section we have

$$\partial(s^\star)^{-1}(g,h) = \left(\xi^\star(\widetilde{s}(g), \widetilde{s}(h)) - \xi^\star(\partial\widetilde{s}^{-1}(g,h), \widetilde{s}(gh)), \partial\widetilde{s}^{-1}(g,h)\right). \tag{6.2}$$

We extend the cocycle ω to a cocycle $\widetilde{\omega} := \omega \circ (\widetilde{p} \times \widetilde{p})$ of $\widetilde{G}$. Since $\widetilde{G}$ is simply connected, we know by Lemma 4.21 that $\widetilde{\omega}$ must be cohomologous to an exponentiated real-valued cocycle:

$$\widetilde{\omega}(x, y) = \partial\alpha(x, y) e^{i\tau(x,y)},$$

where $\alpha : \widetilde{G} \to \mathrm{U}(1)$ is some e-smooth map, and τ is a real-valued cocycle on $\widetilde{G}$. In the proof of [6, Lem.11] it is shown that

$$\tau(\kappa, x) = \tau(\kappa, x)$$

for all $x \in \widetilde{G}$ and $\kappa \in K$, and hence $\tau \in \mathrm{H}^2_{\widetilde{p}}(\widetilde{G}, \mathbb{R})$. Now, since the components of $\xi^\star$ are real-valued cocycles on $\widetilde{G}$ that span $\mathrm{H}^2_{\widetilde{p}}(\widetilde{G}, \mathbb{R})$, we find

$$\tau = \boldsymbol{w} \cdot \xi^\star := w_1 \xi_1 + \cdots + w_N \xi_N,$$

for some $\boldsymbol{w} \in \mathbb{R}^N$. Having fixed this vector $\boldsymbol{w}$, we define the character $\chi \in \widehat{K}$ via the following formula:

$$\chi(\boldsymbol{v}, x) := e^{i\langle \boldsymbol{w}, \boldsymbol{v} \rangle} \alpha(x).$$

χ defines a cocycle μ_χ as described in the theorem. Using (6.2) we find the explicit expression

$$\mu_\chi(g,h) = \chi \circ \partial(s^\star)^{-1}(g,h) = e^{i\left\langle \boldsymbol{w}, \xi^\star(\widetilde{s}(g), \widetilde{s}(h)) - \xi^\star(\partial\widetilde{s}^{-1}(g,h), \widetilde{s}(gh))\right\rangle} \alpha(\partial\widetilde{s}^{-1}(g,h)).$$

With the expression $\tau = \boldsymbol{w} \cdot \xi^\star$ the cohomology relation between $\widetilde{\omega}$ and $e^{i\tau}$ can be updated to

$$\widetilde{\omega}(x, y) = \partial\alpha(x, y) e^{i\boldsymbol{w} \cdot \xi^\star(x,y)}.$$

Applying this relation twice to the previous equation shows that μ and μ_χ are cohomologous: $\mu_\chi = \partial(\alpha \circ \widetilde{s})^{-1}\mu$, which completes the proof. $\square$

The advantage of this version of the theorem over Theorem 4.20 is that it also applies to Lie groups that have $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{g},\mathbb{R}) \neq 0$. It is an advantage that we need, since the Galilei group has non-trivial Lie algebra extensions: $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{gal}(3),\mathbb{R}) \cong \mathbb{R}$.

We now come to the main theorem of this section:

Theorem 6.9. *Let G be a connected Lie group. Any projective unitary representation $G \to \mathrm{PU}(\mathscr{H})$ on some seperable Hilbert space arises from a unitary representation $u : G^\star \to \mathrm{U}(\mathscr{H})$ of the universal central extension, with the property that $u(\mathbb{R}^N \times D) \subseteq \mathrm{diag}(\mathrm{U}(1))$. (Cf. [22, Thm.5.62] and [6, Thm.3].)*

Proof. The idea of the proof rests on the following construction: given a unitary representation $u : G^\star \to \mathrm{U}(\mathscr{H})$ with the desired property, we define

$$\rho : G \to \mathrm{PU}(\mathscr{H}); \quad g \mapsto \widehat{P}(u(\boldsymbol{a}, x)),$$

where $(\boldsymbol{a}, x) \in G^\star$ is some element such that $p(x) = g$. The condition $u(K) \subseteq \mathrm{diag}(\mathrm{U}(1))$ ensures that ρ is well defined. Ambiguity allows us to set $\rho(g) = \widehat{P} \circ u \circ s^\star$, where $s^\star : G \to G^\star$ is a section of the universal central covering map that satisfies $s^\star(1_G) = 1_{G^\star}$. We easily verify that ρ is a homomorphism:

$$\begin{aligned}
\rho(gh) &= \widehat{P}(u(s^\star(gh))) \\
&= \widehat{P}(u(s^\star(gh)))\widehat{P}(u(\partial(s^\star)^{-1}(g,h))) \\
&= \widehat{P}(u(s^\star(g))u(s^\star(h))) \\
&= \widehat{P}(u(s^\star(g)))\widehat{P}(u(s^\star(h))) \\
&= \rho(g)\rho(h).
\end{aligned}$$

Here we have used that $\partial(s^\star)^{-1}(g,h) \in K$, and hence $u(\partial(s^\star)^{-1}(g,h)) \in \mathrm{diag}(\mathrm{U}(1)) \subseteq \ker(\widehat{P})$.

We are left to show that the map $u \mapsto \widehat{P} \circ u \circ s^\star$ is surjective. For this, Theorem 6.8 comes in handy. Let $\rho : G \to \mathrm{PU}(\mathscr{H})$ be a projective unitary representation. We define the cocycle

$$\mu : G \times G \to \mathrm{diag}(\mathrm{U}(1)); \quad \mu = \partial(\widehat{s} \circ \rho)^{-1},$$

where $\widehat{s} : \mathrm{U}(\mathscr{H}) \to \mathrm{PU}(\mathscr{H})$ is a section of the projection map $\widehat{P}$ with $\widehat{s}(\mathrm{id}_{\mathscr{H}}) = \mathrm{id}_{\mathbf{P}(\mathscr{H})}$. By Theorem 6.8 there exists a character $\chi \in \widehat{K}$ which defines a cocycle μ_χ that is cohomologous to μ, meaning there is a map $\alpha : G \to \mathrm{U}(1)$ that makes for the cohomology relation

$$\mu(g,h) = \partial\alpha(g,h)\mu_\chi(g,h)$$

between μ and μ_χ. We put these ingredients together to define

$$u : G^\star \to \mathrm{U}(\mathscr{H}); \quad x \mapsto \chi\left(x(s^\star \circ p^\star(x)^{-1}\right)(\alpha \circ p^\star)(x)(\widehat{s} \circ \rho \circ p^\star)(x)$$

for $x \in G^\star$, cf. equation (6.1). Similar arguments as in the proof of Proposition 6.6 show that u is a unitary representation with the desired property. That leaves us to show that u, as defined by ρ, again gives rise to ρ via the construction $\widehat{P} \circ u \circ s^\star$. Using $p^\star \circ s^\star = \mathrm{id}_G$, an easy calculation transpires:

$$\begin{aligned}
u \circ s^\star(x) &= \chi\left(s^\star(x)(s^\star \circ p^\star(s^\star(x)))^{-1}\right)(\alpha \circ p^\star)(s^\star(x))(\widehat{s} \circ \rho \circ p^\star)(s^\star(x)) \\
&= \chi\left(s^\star(x)s^\star(x)^{-1}\right)\alpha(x)(\widehat{s} \circ \rho)(x) \\
&= \alpha(x)(\widehat{s} \circ \rho)(x).
\end{aligned}$$

Lastly, the fact that $\widehat{P} \circ \widehat{s} = \mathrm{id}_{\mathrm{PU}(\mathscr{H})}$ now clearly gives

$$\widehat{P} \circ u \circ s^{\star}(x) = \widehat{P}(\alpha(x))(\widehat{P} \circ \widehat{s} \circ \rho)(x) = \rho(x),$$

finishing the proof. $\square$

Since *irreducible* unitary representations of $G^{\star}$ automatically satisfy $u(\mathbb{R}^N \times D) \subseteq \mathrm{diag}(\mathrm{U}(1))$ [6, p.35], this condition poses no further restrictions on our classification.

6.5 Projective unitary representations of the Galilei group

We are now ready to apply the result of Theorem 6.9 to the Galilei group Gal(3). It should be noted that the construction of lifting a projective unitary representation preserves the irreducibility property. It therefore suffices to determine all irreducible unitary representations of the universal central extension.

The underlying strategy is the one that has first been outlined by Bargmann in his article [4, p.16], but also by Lévy-Leblond in [25]. In summary, it reads as follows. Take a connected Lie group G.

1. Determine all the inequivalent central extensions of G. In other words, determine all the inequivalent real- or U(1)-valued cocycles on G. Let us take real-valued cocycles, for simplicity's sake.
2. Extend these cocycles to the universal covering group $\widetilde{G}$ via a section $\widetilde{s} : G \to \widetilde{G}$ of the covering map $\widetilde{p} : \widetilde{G} \to G$. This preserves the equivalency of the cocycles (see the proof of [4, Thm.3.4]).
3. Determine all *irreducible* unitary representations of the central extensions of $\widetilde{G}$ so obtained, say, with cocycle ξ. The projections of the restrictions of these representations to $\widetilde{G}$ are projective unitary representations that correspond to the cocycle ξ.
4. Those irreducible unitary representations $u : \mathbb{R} \times_{\xi} \widetilde{G} \to \mathrm{U}(\mathscr{H})$ that satisfy $u(D) \subseteq \mathrm{U}(1)$ then give rise to all of the irreducible projective unitary representations of G.

For the Galilei group the following result is obtained. In it's totality, there are five different types of irreducible projective unitary representations. For a derivation, we refer to [25, Sec.IV] or [6, Sec.5.2]. We adopt the former reference's notation to a large extent. The first four of these unitary representations are actually (equivalent to) *ordinary* unitary representations of the Galilei groups:

1. The first type of representations are defined by two real positive numbers $p, \nu \in \mathbb{R}_{>0}$. The Hilbert spaces read $\mathscr{H}_{p,\nu} = L^2(S^1_\nu \times (\mathbb{R} \times S^2_p))$, where S^1_ν is the circle in the real plane with radius ν, and (recalling previous notation) S^2_p is the sphere with radius p so that $\mathbb{R} \times S^2_p = \{(E, \boldsymbol{p}) \in \mathbb{R}^4 : \|\boldsymbol{p}\| = p\}$. Here, for some subset $X \subseteq \mathbb{R}^n$, the Hilbert space $L^2(X)$ contains the square integrable functions with respect to the Lebesgue measure. The explicit form of the irreducible unitary representation reads [25, Eq.(4.21)]:

   $$u_{p,\nu}(s, \boldsymbol{a}, \boldsymbol{v}, U)\psi(\boldsymbol{\nu}, E, \boldsymbol{p}) = e^{i(sE + \langle \boldsymbol{p}, \boldsymbol{a}\rangle + \langle R_{\boldsymbol{p}}\boldsymbol{\nu}, \boldsymbol{v}\rangle)}\psi\left(\boldsymbol{\nu}_{-\alpha(U,\boldsymbol{p})}, E - \langle \boldsymbol{p}, \boldsymbol{v}\rangle, \widetilde{p}(U)^{-1}\boldsymbol{p}\right).$$

   We identify S^1_ν with the circle in the plane perpendicular to some reference point $(0, \boldsymbol{p}_0) \in \mathbb{R} \times S^2_p$. Furthermore, we define the rotation matrix $R_{\boldsymbol{p}}$ as one (of many) that satisfies $\boldsymbol{p} = R_{\boldsymbol{p}}\boldsymbol{p}_0$. The real number $\alpha(U, \boldsymbol{p})$ is the angle of rotation around the $\boldsymbol{p}_0$ axis, and $\boldsymbol{\nu}_{-\alpha(U,\boldsymbol{p})}$ is the vector obtained by rotating $\boldsymbol{\nu}$ in the plane by an angle $-\alpha(U, \boldsymbol{p})$.

2. The second class of representations is defined by two parameters; a positive real number $p \in \mathbb{R}_{>0}$ and a half integer $\sigma \in \mathbb{Z}/2$. The representation takes the form [25, Eq.(4.24)]

$$u_{p,\sigma}(s, \boldsymbol{a}, \boldsymbol{v}, U)\psi(E, \boldsymbol{p}) = e^{i(sE+\langle \boldsymbol{p},\boldsymbol{a}\rangle+\sigma\alpha(U,\boldsymbol{p}))}\psi(E - \langle \boldsymbol{p}, \boldsymbol{v}\rangle, \widetilde{p}(U)^{-1}\boldsymbol{p}),$$

defined on the Hilbert space $\mathscr{H}_{p,\sigma} = L^2(\mathbb{R} \times S^2_p)$.

3. The third (and actually also the fourth) class appears as the case $p = 0$ of the previous class. The labels are threefold; a real number $V \in \mathbb{R}$, a positive real number $k \in \mathbb{R}_{>0}$, and a half-integer $\xi \in \mathbb{Z}/2$. The Hilbert space and representation read, respectively, $\mathscr{H}_{V,k,\xi} = L^2(S^2_k)$ and [25, Eq.(4.26)]

$$u_{V,k,\xi}(s, \boldsymbol{a}, \boldsymbol{v}, U)\psi(\boldsymbol{k}) = e^{i(sV+\langle \boldsymbol{k},\boldsymbol{v}\rangle+\xi\alpha(U,\boldsymbol{k}))}\psi(\widetilde{p}(U)^{-1}\boldsymbol{k}).$$

4. The fourth case arises similarly via the further restriction $k = 0$. The labels are borrowed from the previous class, but we replace ξ by $l \in \mathbb{N}/2$. We denote the irreducible unitary representation of SU(2) with label j by $D_j : \mathrm{SU}(2) \to \mathrm{U}(\mathbb{C}^{2j+1}) \cong \mathrm{U}(2j + 1)$. For reference, see [22, Sec.5.8] or [14, Sec.17.8]. The Hilbert space of the fourth class is the same one as for the representation of SU(2) corresponding to $j = l$. The representation reads [25, Eq.(4.27)]

$$u_{V,l}(s, \boldsymbol{a}, \boldsymbol{v}, U) = e^{isV} D_l(\widetilde{p}(U)) \in \mathrm{U}(2l + 1).$$

The fifth and last class is the only one that describes irreducible representations that are projectively non-trivial, and is therefore the only one that is physically relevant. Irreducible unitary representations of central extensions of the universal covering group $\widetilde{\mathrm{Gal}(3)}$ give rise to all of the irreducible projective unitary representations of the Galilei group by restricting the ordinary representation to the covering group. We consider the extension corresponding to the parameter $M \in \mathbb{R}$. This may actually be viewed as a label for the representations. The Hilbert spaces are

$$\mathscr{H}_{m,j} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2j+1},$$

where we recall that $L^2(\mathbb{R}^3)$ are the square integrable functions on $\mathbb{R}^3$. Physicists will be familiar with the term *wave function* for these functions. The elements of $L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2j+1}$ are of the form $(\psi_1, \ldots, \psi_{2j+1})$, where each component is an element of $L^2(\mathbb{R}^3)$. Let $\psi \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2j+1}$ be a vector in the Hilbert space. The irreducible unitary representations of the extended Galilei group then read [25, Eq.(4.39)]:

$$u_{m,j}(\theta, s, \boldsymbol{a}, \boldsymbol{v}, U)\psi(\boldsymbol{p}) = e^{i\left(\theta+s\left(V+\frac{\boldsymbol{p}^2}{2m}\right)+\langle \boldsymbol{a},\boldsymbol{p}\rangle\right)} D_j(U)\psi\left(\widetilde{p}(U)^{-1}(\boldsymbol{p} + m\boldsymbol{v})\right),$$

where $(s, \boldsymbol{a}, \boldsymbol{v}, U) \in \widetilde{\mathrm{Gal}(3)}$, $\psi \in L^2(\mathbb{R}^3)$ and $\boldsymbol{p} \in \mathbb{R}^3$. (See also [24, Eq.(III.20)].) Here $D_j(U)$ is a $(2j + 1) \times (2j + 1)$ unitary matrix, and the product $D_j(U)\psi$ is defined in the obvious way. The representations $u_{m,j}$ are labelled by real number $V \in \mathbb{R}$, a non-zero real number $m := M \in \mathbb{R} \setminus \{0\}$, which is the extension parameter of the Galilei group, and a non-negative integer or half integer $j \in \mathbb{N}/2 \cup \{0\}$. Actually, one finds that the real parameter V does not affect the corresponding projective representation [24, p.780], so we set $V = 0$. Restricting to the covering group we may write [22, Eq.7.181]:

$$u_{m,j}(s, \boldsymbol{a}, \boldsymbol{v}, U)\psi(\boldsymbol{p}) = e^{i\left(s\frac{\boldsymbol{p}^2}{2m}+\langle \boldsymbol{a},\boldsymbol{p}\rangle\right)} D_j(U)\psi\left(\widetilde{p}(U)^{-1}(\boldsymbol{p} + m\boldsymbol{v})\right).$$

Note that the condition '$u(D) \subseteq \mathrm{U}(1)$' poses no further restrictions to these representations. In Section 10, Summary 10.2, we will summarise the results and provide a physical interpretation. For a more in-depth physical discussion of the results for the Galilei group we refer to [24, Sec.III].

6.6 Projective unitary representations of the Poincaré group

The (physically relevant) irreducible projective unitary representations of the Poincaré group were first classified by Wigner in 1939 [43]. Since the Lie algebra cohomology $\mathrm{H}^2_{\mathrm{al}}(\mathfrak{poin}(1,3),\mathbb{R})$ is trivial, instead of Theorem 6.9, we may even use Proposition 6.6. This proposition tells us that we need to find all irreducible unitary representations of the universal cover $\mathrm{Poin}^\star(1,3) = \mathbb{R}^4 \rtimes \mathrm{SL}(2,\mathbb{C})$ that are represented by complex phase factors on the kernel of the covering map. We now have four different families of irreducible unitary representations [21, Prop.IV.3.3.1]. (Also see [40].)

First, two similar classes labelled by a real number $m > 0$ and $j \in \mathbb{N}/2 \cup \{0\}$, and a sign $\pm$. In the $+$ case, the Hilbert space is $\mathscr{H}_{m,+,j}$, which is similar to the one for the Galilei group, only that the square integrable functions are now defined on the so-called *mass-shell*: $S_m^+ := \{(E,\boldsymbol{p}) \in \mathbb{R}^4 : E^2 - \boldsymbol{p}^2 = m^2, E > 0\}$. Upon further inspection, however, it is clear that $S_m^+ \cong \mathbb{R}^3$, so that the Hilbert space is the same as for the Galilei group:

$$\mathscr{H}_{m,+,j} = L^2(S_m^+) \otimes \mathbb{C}^{2j+1} \cong L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2j+1}.$$

For $a = (a_0, \boldsymbol{a}) \in \mathbb{R}^4$ and $\Delta \in \mathrm{SL}(2,\mathbb{C})$ one finds

$$u_{m,+,j}(a,\Delta)\psi(\boldsymbol{p}) = e^{i\left(a_0\sqrt{\boldsymbol{p}^2+m^2} - \langle \boldsymbol{a},\boldsymbol{p}\rangle\right)} D_j(b_{\boldsymbol{p}}^{-1}\Delta b_{\widetilde{p}(\Delta)^{-1}\boldsymbol{p}})\psi(\widetilde{p}(\Delta)^{-1}\boldsymbol{p}).$$

Here we denote by $b_{\boldsymbol{p}} \in \mathrm{SL}(2,\mathbb{C})$ the matrix so that $\widetilde{p}(b_{\boldsymbol{p}}) \in \mathrm{SO}^+(1,3)$ is the Lorentz transformation that maps $(m,0)$ to $(\sqrt{\boldsymbol{p}^2+m^2},\boldsymbol{p})$. Moreover, $D_j : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{U}(2j+1)$ is the irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ (cf. [15, Sec.4.6,Thm.4.32]).

The negative sign representation is the same in the sense that $u_{m,-,j}$ is defined by the same expression as $u_{m,+,j}$, but differing in the sense that the space S_m^+ is replaced by $S_m^- := \{(E,\boldsymbol{p}) \in \mathbb{R}^4 : E^2 - \boldsymbol{p}^2 = m^2, E < 0\}$, so that $\mathscr{H}_{m,-,j} = L^2(S_m^-) \otimes \mathbb{C}^{2j+1}$.

For $m = 0$ we get additional values for the discrete label: $j \in \mathbb{Z}/2$, and the representation changes. The Hilbert space is now the square integrable functions on the light cones $S_0^\pm = \{(E,\boldsymbol{p}) \in \mathbb{R}^4 : E^2 - \boldsymbol{p}^2 = 0, \pm E > 0\}$:

$$\mathscr{H}_{\pm,j} = L^2(S_0^\pm).$$

The explicit formula for the representation is a modified version of $u_{m,+,j}$, after substituting $m = 0$, and replacing the $\mathrm{SL}(2,\mathbb{C})$ representation D_j by the map d_j (defined below) [37]:

$$u_{\pm,j}(a,\Delta)\psi(\boldsymbol{p}) = e^{i\left(a_0\sqrt{\boldsymbol{p}^2} - \langle \boldsymbol{a},\boldsymbol{p}\rangle\right)} d_j(b_{\boldsymbol{p}}^{-1}\Delta b_{\widetilde{p}(\Delta)^{-1}\boldsymbol{p}})\psi(\widetilde{p}(\Delta)^{-1}\boldsymbol{p}).$$

We define $d_j(b_{\boldsymbol{p}}^{-1}\Delta b_{\widetilde{p}(\Delta)^{-1}\boldsymbol{p}})$ as the phase $e^{ij\alpha(\Delta,\boldsymbol{p})}$, where the real number $\alpha(\Delta,\boldsymbol{p})$ is defined analogously to the angle $\alpha(U,\boldsymbol{p})$ in Section 6.5 (see also [9, Eq.(B.67)] and surrounding text). (Note: there is also an elusive class of representations called the 'continuous spin' or 'infinite spin' representations, where j is a real number instead of being discrete. These representations are harder to construct, and are not believed to have any physical relevance, so we omit them here.)

The result of this section (or at least, part of it) is known as *Wigner's classification*. Cf. [21, Chap.IV,Sec.3.3]. See [5] for a generalisation of Wigner's classification to any spacetime dimension. It should be noted that the extension of $\mathrm{Poin}(1,3)$ to its universal covering $\mathrm{Poin}^{\star}(1,3)$ merely gives us the half-integer values for the label j. For the regular Poincaré group the representations are the same, only that we have $j \in \mathbb{N} \cup \{0\}$ (for $m > 0$) or $j \in \mathbb{Z}$ (for $m = 0$). See Summary 10.3.

Part III
Classical particles

7 The classical formalism

At this point we must elaborate on the comments made in Section 2.4, wanting to justify the definition of (connected) symplectic homogeneous spaces as classical elementary particles. And moreover, we want to confirm that these are classified by so-called *twisted coadjoint orbits*.

7.1 Classical state spaces

The model for state space in classical mechanics is a smooth manifold, together with additional structure.

Definition 7.1. Let M be a smooth manifold. A ***Poisson bracket*** on M is a Lie bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ (cf. Definition 3.2) so that for all $h \in C^\infty(M)$ the map

$$C^\infty(M) \to C^\infty(M); \quad f \mapsto \{h, f\}$$

defines a smooth vector field on M. The pair $(M, \{\cdot,\cdot\})$ is called a ***Poisson manifold***.

In any particular set of local coordinates $(x^1, \dots, x^m)$ on a Poisson manifold M, its Poisson bracket may be expressed in terms of the *Poisson tensor* B as follows:

$$\{f, h\}(x) = \sum_{i,j=1}^{m} B_{ij}(x) \frac{\partial f}{\partial x^i}(x) \frac{\partial h}{\partial x^j}(x),$$

for any $x \in M$, and each $f, h \in C^\infty(M)$. A Poisson manifold is called ***symplectic*** whenever its Poisson tensor is invertible at every point. In that case, the manifold is necessarily even-dimensional. The first half of the local coordinates are often interpreted as momentum coordinates, while the second half are interpreted as position coordinates. For simplicity, we shall restrict ourselves to Poisson manifolds that are symplectic, and in turn, assume that the state space is a connected symplectic manifold. The common definition of a symplectic manifold is as follows, and is of course equivalent to our previous definition.

Definition 7.2. A ***symplectic manifold*** is a smooth manifold M, together with a differential 2-form ω so that $\mathrm{d}\omega = 0$, and so that at each point $x \in M$ the bilinear form $\omega_x : T_xM \times T_xM \to \mathbb{R}$ is non-degenerate, meaning that if for fixed $u \in T_xM$ we have $\omega_x(u, v) = 0$ for every other $v \in T_xM$, then $u = 0$.

The notion of a smooth map between smooth manifolds can be sharpened to fit symplectic manifolds. Namely, a smooth map $f : M \to N$ between symplectic manifolds (M, ω) and (N, σ) is called ***symplectic*** when $f^*\sigma = \omega$, where f^* is the pullback map of f, so that the 2-form $f^*\sigma$ on M is defined via

$$(f^*\sigma)_x(v_1, v_2) = \sigma_{f(x)}(\mathrm{d}_x f(v_1), \mathrm{d}_x f(v_2)),$$

where $x \in M$ is any point on the manifold, and $v_1, v_2 \in T_xM$ are two points in the tangent space at x. We say f is a ***symplectomorphism*** if it is a symplectic diffeomorphism.

7.2 Smooth and symplectic group actions

Recall the definition of a group action at the very start of this thesis (Definition 1.1). The context of classical mechanics calls for a smoothness property of the group action. A ***smooth action*** (also called a *Lie group action*) of a Lie group G on a smooth manifold M is a group action that is smooth as a map on the product manifold $G \times M \to M$. A smooth manifold M endowed with such a group action is sometimes called a G-space. Of particular interest to us is the case that M is the Lie algebra $\mathfrak{g}$ of G, or its dual $\mathfrak{g}^*$. In any case, given a fixed point $x \in M$ in the manifold, the so-called *orbit map* $G \to M : g \mapsto g \cdot x$ is a smooth map. The *G-orbit* (or just *orbit*, when there is no confusion) of x is defined as the image of this map:

$$\mathrm{Orb}_G(x) := \{g \cdot x : g \in G\} \subseteq M.$$

Similarly, the *stabiliser* of x is defined as

$$\mathrm{Stab}_G(x) := \{g \in G : g \cdot x = x\} \subseteq G.$$

Stabilisers always form subgroups of G. Smoothness of the orbit map furthermore ensures that $\mathrm{Stab}_G(x)$ is a closed subgroup of G, so it forms a matrix Lie subgroup of G. In the literature one often encounters the notation G_x to denote the stabiliser of x in G. Borrowing from this notation, we denote the Lie algebra of G_x by $\mathfrak{g}_x$, called the *stabiliser algebra* of x. Directly from the definition of a matrix Lie algebra we find that $\mathfrak{g}_x$ is a Lie subalgebra of $\mathfrak{g}$.

Suppose now that we endow M with a symplectic structure. The smooth action of G on M is called ***symplectic*** if for each $g \in G$ the map $M \to M; x \mapsto g \cdot x$ is a symplectomorphism.

8 Coadjoint orbits

8.1 Some more representation theory

Given a representation ρ of G over a vector space V, we may naturally define its *dual representation*. This is a representation of G, denoted ρ^*, over the dual vector space V^*. To define it, we need the concept of a dual operator: given a linear operator $A : V \to V$ on V, we define its ***dual*** as the linear map $A^* : V^* \to V^*$ such that

$$A^*(T)(v) = T(Av)$$

for all $v \in V$ and $T \in V^*$. In the literature the value of a linear functional $T \in V^*$ at $v \in V$ is sometimes denoted by $\langle T, v\rangle := T(v)$. In this notation, the notion of a dual operator becomes similar to that of the adjoint operator. Namely, the dual operator A^* satisfies the formula:

$$\langle A^*T, v\rangle = \langle T, Av\rangle.$$

More generally, if $A : V \to W$ is a linear operator, its dual is defined as above, but now it is a map $A^* : W^* \to V^*$.

We can now apply this concept to group representations:

Definition 8.1. Let ρ be a representation of G over a vector space V. The ***dual representation*** of ρ is defined as the map

$$\rho^* : G \to \mathrm{GL}(V^*); \quad g \mapsto \rho(g^{-1})^*.$$

Here the expression $\rho(g^{-1})^*$ denotes the dual operator of $\rho(g^{-1}) \in \mathrm{GL}(V)$. It is not hard to see that ρ^* still defines a homomorphism. Let $g, h \in G$ be two arbitrary elements; then using the definition of the dual operator and the fact that ρ is a homomorphism gives

$$\langle \rho^*(gh)T, v\rangle = \langle T, \rho(h^{-1})\rho(g^{-1})v\rangle = \langle \rho(g^{-1})^*\rho(h^{-1})^*T, v\rangle = \langle \rho^*(g)\rho^*(h)T, v\rangle$$

for all $v \in V$ and $T \in V^*$. Hence $\rho^*(gh) = \rho^*(g)\rho^*(h)$, as desired.

8.2 The adjoint- and coadjoint representations

Consider again the general situation of a (not necessarily matrix) Lie group G with Lie algebra $\mathfrak{g}$ (to be identified with the tangent space T_eG together with the bracket defined via left-invariant extensions, see Section 3). Define $\varphi : G \to \mathrm{Aut}(G); g \mapsto \varphi_g$, where $\varphi(g) = \varphi_g$ is the inner automorphism $h \mapsto ghg^{-1}$. The differential of φ_g at the identity element $e \in G$ is denoted by

$$\mathrm{Ad}(g) = \mathrm{Ad}_g := \mathrm{d}_e(\varphi_g) : T_eG \to T_{\varphi_g(e)}G = T_eG.$$

With the identification $\mathfrak{g} \cong T_eG$, $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism. The map

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}); \quad g \mapsto \mathrm{Ad}_g$$

is called the ***adjoint representation*** of G. One property of the adjoint representation that we need later is the following:

Lemma 8.2. *Let $F : G \to H$ be a Lie group homomorphism. Then the following diagram commutes for each $g \in G$:*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{d}_eF"] \arrow[d, "\mathrm{Ad}(g)"'] & \mathfrak{h} \arrow[d, "\mathrm{Ad}(F(g))"] \\ \mathfrak{g} \arrow[r, "\mathrm{d}_eF"'] & \mathfrak{h}. \end{tikzcd}$$

Proof. We have the following elementary commutative diagram, stemming from the fact that F is a homomorphism:

$$\begin{tikzcd} G \arrow[r, "F"] \arrow[d, "\varphi_g"'] & H \arrow[d, "\varphi_{F(g)}"] \\ G \arrow[r, "F"'] & H. \end{tikzcd}$$

The result follows by taking the differential at the identity $e \in G$ for each of the arrows, which preserves the commutativity of the diagram. □

Given the adjoint representation, Definition 8.1 gives us a representation of G by the dual $\mathfrak{g}^*$, namely $\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*); g \mapsto \mathrm{Ad}(g^{-1})^*$. We give this representation a special name: the ***coadjoint representation***, and denote it by

$$\mathrm{Coad} : G \to \mathrm{Aut}(\mathfrak{g}^*); \quad g \mapsto \mathrm{Ad}(g^{-1})^*.$$

The fact that it is dual to the adjoint representation means that for all $X \in \mathfrak{g}$ and $F \in \mathfrak{g}^*$:

$$\langle \mathrm{Coad}(g)F, X\rangle = \langle F, \mathrm{Ad}(g^{-1})X\rangle.$$

Despite the level of abstractness in the preceding paragraphs, as always, the situation becomes more straightforward when we restrict ourselves to matrix Lie groups.

In that case the adjoint and coadjoint representations in fact have a very concrete form. Therefore, from now, let G be a matrix Lie group, whose identity element is the identity matrix I, and which has Lie algebra $\mathfrak{g}$. To figure out how $\mathrm{Ad}(g)$ acts on $\mathfrak{g}$, we consider the expression $\varphi_g(e^{tX}) = ge^{tX}g^{-1}$, for arbitrary $g \in G$, $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. Using (3.1) in Proposition 3.9, we determine

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(ge^{tX}g^{-1}\right)\bigg|_{t=0} = g\,\frac{\mathrm{d}}{\mathrm{d}t}e^{tX}\bigg|_{t=0} g^{-1} = gXg^{-1} \in \mathfrak{g}.$$

On the other hand, differentiating the right hand side of the equation using the chain rule gives[8]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_g(e^{tX})\right) = \mathrm{d}_{\exp(tX)}\varphi_g \circ \left(\frac{\mathrm{d}}{\mathrm{d}t}e^{tX}\right),$$

and with the special property of the exponential map that $\mathrm{d}_0 \exp = \mathrm{id}_{\mathfrak{g}}$, we find that evaluation at $t = 0$ reduces the equation to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_g(e^{tX})\right)\bigg|_{t=0} = \mathrm{d}_{\exp(0)}\varphi_g \circ \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX}\right)\bigg|_{t=0} = \mathrm{d}_I\varphi_g(X) =: \mathrm{Ad}(g)(X).$$

We therefore find that the adjoint map $\mathrm{Ad}(g)$, for any group element $g \in G$, acts as matrix conjugation on the the Lie algebra:

$$\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}; \quad X \mapsto gXg^{-1},$$

which is well-defined by Proposition 3.11.4. Note that, despite φ_g and $\mathrm{Ad}(g)$ both describing matrix conjugation, the former is defined on the *group* G, while the latter is defined on the *algebra* $\mathfrak{g}$. They are therefore not formally the same.

Determining the coadjoint representation is now easy; given a group element $g \in G$, the map $\mathrm{Coad}(g)$ sends a functional $F \in \mathfrak{g}^*$ to

$$\mathrm{Coad}(g)F : \mathfrak{g} \to \mathbb{C}; \quad X \mapsto F(\mathrm{Ad}(g^{-1})(X)) = F(g^{-1}Xg).$$

The ***coadjoint action*** is the group action associated to the coadjoint representation, which reads

$$G \times \mathfrak{g}^* \to \mathfrak{g}^*; \quad (g, F) \mapsto \mathrm{Coad}(g)F.$$

The orbits of this action in $\mathfrak{g}^*$ are what are called the ***coadjoint orbits***. Given a functional $F \in \mathfrak{g}^*$, we denote its orbit by $\mathscr{O}_G(F) := \{\mathrm{Coad}(g)F : g \in G\} \subseteq \mathfrak{g}^*\}$.

8.3 Coadjoint orbits of semi-direct products

In light of the structure of the Galilei and Poincaré groups, discussed in Section 3.3, we want to be able to find the coadjoint orbits of a semi-direct product. For our purposes here, let V be a finite-dimensional vector space, to be thought of as a translation group; and let L be another matrix Lie group, to be thought of as the Lorentz group, or an orthogonal group. Together with a homomorphism $\rho : L \to \mathrm{GL}(V)$ we consider the semi-direct product $G = V \rtimes_\rho L$. The Lie algebra of G is $\mathfrak{g} = \mathfrak{v} \rtimes_{\mathrm{d}\rho} \mathfrak{l}$, which is the vector space $\mathfrak{v} \oplus \mathfrak{l}$ together with the Lie bracket

$$[v \oplus X, w \oplus Y] = (\mathrm{d}\rho(X)w - \mathrm{d}\rho(Y)v) \oplus [X, Y],$$

[8] Given a smooth curve $\gamma : t \mapsto \gamma(t)$ on a smooth manifold M, its velocity is defined as the smooth curve $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(s) := \mathrm{d}_s\gamma(\partial/\partial t)$ in the tangent bundle TM, where $\partial/\partial t$ is the unit vector field on $\mathbb{R}$. Therefore if $f : M \to M$ is a smooth map, we have $\frac{\mathrm{d}(f\circ\gamma)}{\mathrm{d}t}(s) = \mathrm{d}_s(f\circ\gamma)(\partial/\partial t) = \mathrm{d}_{\gamma(s)}f \circ \mathrm{d}_s\gamma(\partial/\partial t) = \mathrm{d}_{\gamma(s)}f \circ \frac{\mathrm{d}\gamma}{\mathrm{d}t}(s)$, by the chain rule.

for $v, w \in \mathfrak{v}$ and $X, Y \in \mathfrak{l}$. Recall that the pushforward $\mathrm{d}\rho$ is defined as the differential of ρ at the identity of L. Assuming that V is connected, its abelian Lie algebra $\mathfrak{v}$ may be identified with V itself [38]. As always, we assume that L is a matrix Lie group, so we may as well restrict ourselves to the case that ρ is the action of matrix multiplication on V. Now the group G may be identified with the matrix group

$$G \cong \left\{ \begin{bmatrix} \Lambda & v \\ 0 & 1 \end{bmatrix} : \Lambda \in L, v \in V \right\},$$

and in turn, we identify its Lie algebra with [13, p.124]

$$\mathfrak{g} \cong \left\{ \begin{bmatrix} X & w \\ 0 & 0 \end{bmatrix} : X \in \mathfrak{l}, w \in V \right\}.$$

The representation ρ is in fact linear, so $\mathrm{d}\rho = \rho$ is the usual matrix action of $\mathfrak{l}$ on V, and the commutator of matrices here conforms with the previous expression for the Lie bracket on $\mathfrak{g}$. As a last preparatory note; we identify $\mathfrak{g}^*$ with $V^* \oplus \mathfrak{l}^*$, where the functional $p \oplus F \in V^* \oplus \mathfrak{l}^*$ maps an element $w \oplus X \in \mathfrak{g}$ to $p(w) + F(X) = \langle p, w \rangle + \langle F, X \rangle$.

We are now ready to calculate the coadjoint action of G on $\mathfrak{g}^*$. In matrix form the adjoint action is determined easily via the matrix conjugate; with abuse of notation:

$$\mathrm{Ad}(v, \Lambda)(w \oplus X) = \begin{bmatrix} \Lambda & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X & w \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Lambda & v \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \Lambda X \Lambda^{-1} & \Lambda w - \Lambda X \Lambda^{-1} v \\ 0 & 0 \end{bmatrix},$$

in other words: $\mathrm{Ad}(v, \Lambda)(w \oplus X) = (\Lambda w - \mathrm{Ad}(\Lambda) X v) \oplus \mathrm{Ad}(\Lambda) X$. We calculate the coadjoint action. For that, we determine

$$\mathrm{Ad}((v, \Lambda)^{-1})(w \oplus X) = \begin{bmatrix} \Lambda^{-1} X \Lambda & \Lambda^{-1} X v + \Lambda^{-1} w \\ 0 & 0 \end{bmatrix} = (\Lambda^{-1} X v + \Lambda^{-1} w) \oplus (\Lambda^{-1} X \Lambda).$$

Now the coadjoint action follows as

$$\begin{aligned} \langle \mathrm{Coad}(v, \Lambda)(p \oplus F), w \oplus X \rangle &= \left\langle p \oplus F, \mathrm{Ad}((v, \Lambda)^{-1})(w \oplus X) \right\rangle \\ &= \left\langle p \oplus F, (\Lambda^{-1} X v + \Lambda^{-1} w) \oplus (\Lambda^{-1} X \Lambda) \right\rangle \\ &= \left\langle p, \Lambda^{-1} X v \right\rangle + \left\langle p, \Lambda^{-1} w \right\rangle + \left\langle F, \Lambda^{-1} X \Lambda \right\rangle. \end{aligned}$$

In the last term we recognise the coadjoint action of L on $\mathfrak{l}^*$. The first term is independent of w, so we would like to interpret this expression as a functional on $\mathfrak{l}$. For this, we define the linear functional $p \wedge v : \mathfrak{l} \to \mathbb{C}; X \mapsto \langle p, \mathrm{d}\rho(X) v \rangle$, for $p \in V^*$ and $v \in V$. We therefore have $\langle p, \Lambda^{-1} X v \rangle = \langle \rho^*(\Lambda) p \wedge v, X \rangle$. Similarly $\langle p, \Lambda^{-1} w \rangle = \langle \rho^*(\Lambda) p, w \rangle$. Putting things together, we obtain

$$\langle \mathrm{Coad}(v, \Lambda)(p \oplus F), w \oplus X \rangle = \langle \rho^*(\Lambda) p, w \rangle + \langle \mathrm{Coad}(\Lambda) F + \rho^*(\Lambda) p \wedge v, X \rangle,$$

for all $w \in V$ and $X \in \mathfrak{l}$. In other words:

$$\mathrm{Coad}(v, \Lambda)(p \oplus F) = \rho^*(\Lambda) p \oplus (\mathrm{Coad}(\Lambda) F + \rho^*(\Lambda) p \wedge v), \tag{8.1}$$

for all $(v, \Lambda) \in G$ and $p \oplus F \in \mathfrak{g}^*$. From this expression it seems that the coadjoint orbits in $\mathfrak{g}^*$ are not simply the direct sums of the coadjoint orbits in V^* and $\mathfrak{l}^*$. Rather, we recognise the coadjoint action of L on $\mathfrak{l}^*$ (plus some extra term), and the dual action ρ^* of L on V^*. The following theorem gives a classification of the coadjoint orbits of $G = V \rtimes_\rho L$:

Theorem 8.3. *There is a bijection between coadjoint orbits $\mathscr{O}_G(p \oplus F)$ in $\mathfrak{g}^*$, and pairs of orbits $(\mathrm{Orb}_L(p), \mathscr{O}_{L_p}(F))$, where $\mathrm{Orb}_L(p)$ is the orbit of $p \in V^*$ under the dual action ρ^*, and $\mathscr{O}_{L_p}(F)$ is the coadjoint orbit of $F|_{\mathfrak{l}_p}$ in the stabiliser algebra dual $\mathfrak{l}_p^*$. (Cf. [21, Prop. IV.1.10.1].)*

Proof sketch. The method of classification (outlined in [13, §19]) starts by fixing an orbit $\mathrm{Orb}_L(p) = \{\rho^*(\Lambda)p : \Lambda \in L\}$ for some functional $p \in V^*$, and then investigating what orbits the second component of (8.1) constitutes. The orbit $\mathrm{Orb}_L(p)$ is fully determined by the first component of the coadjoint action because, in fact:

$$\mathscr{O}_G(p \oplus F) = \mathrm{Orb}_L(p) \times \{\mathrm{Coad}(\Lambda)F + \rho^*(\Lambda)p \wedge v : v \in V, \Lambda \in L\}. \tag{8.2}$$

However, the first component also influences the second via the term $\rho^*(\Lambda)p \wedge v$. To investigate this term, let $q \in \mathrm{Orb}_L(p)$ be some functional in the orbit of p, so that there exists a group element $\Delta \in L$ with $\rho^*(\Delta)p = q$. Now consider the stabiliser $L_q = \{\Lambda \in L : \rho^*(\Lambda)q = q\}$, with stabiliser Lie algebra $\mathfrak{l}_q$. Functionals on $\mathfrak{l}_q$ can be seen as restrictions of functionals on $\mathfrak{l}$, which is to say that the following map is surjective:

$$\pi_q : \mathfrak{l}^* \to \mathfrak{l}_q^*; \quad F \mapsto F|_{\mathfrak{l}_q}.$$

We claim that, for any two functionals $F_1, F_2 \in \mathfrak{l}^*$, we have $\exists v \in V : F_1 - F_2 = q \wedge v$ if and only if $\pi_q(F_1) = \pi_q(F_2)$. To see this, suppose first that $F_1 - F_2$ is of the form $q \wedge v$ for some vector $v \in V$. It suffices to show that for all $X \in \mathfrak{l}_q$ we have $\langle q \wedge v, X\rangle = 0$. Any element X of the stabiliser algebra $\mathfrak{l}_q$ satisfies, by the very definition of a matrix Lie algebra, the equation $\rho^*(e^{tX})q = q$ for all $t \in \mathbb{R}$. Differentiating this equation with respect to the variable t around $t = 0$ yields

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t} q\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho^*(e^{tX})q\right)\right|_{t=0} = \mathrm{d}_{\exp(0)}\rho^* \circ \left.\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{tX}\right)\right|_{t=0} q = \mathrm{d}\rho^*(X)q,$$

and hence

$$\langle q \wedge v, X\rangle = \langle q, \mathrm{d}\rho(X)v\rangle = \langle \mathrm{d}\rho(X)^* q, v\rangle = \langle -\mathrm{d}\rho^*(X)q, v\rangle = \langle 0, v\rangle = 0,$$

as desired. It is also useful to note that the stabiliser algebra $\mathfrak{l}_q$ in fact entirely consists of elements $X \in \mathfrak{l}$ that satisfy $\mathrm{d}\rho^*(X)q = 0$; by Proposition 3.5 we find that, if $X \in \mathfrak{l}$ does satisfy this equation, then for all $t \in \mathbb{R}$:

$$\rho^*(e^{tX})p = e^{t\,\mathrm{d}\rho^*(X)}p = \sum_{n=0}^{\infty} \frac{1}{n!} t^n\, \mathrm{d}\rho^*(X)^n p = p,$$

because for $n > 0$ the summand vanishes. Therefore, $\mathfrak{l}_q = \{X \in \mathfrak{l} : \mathrm{d}\rho^*(X)q = 0\}$. Further still, following [31], we define the linear map $\tau_q : \mathfrak{l} \to V^*$ by $X \mapsto -\mathrm{d}\rho^*(X)q$. Now clearly $\ker(\tau_q) = \mathfrak{l}_q$. The dual of this map $\tau_q^* : V^{**} \cong V \to \mathfrak{l}$, with the usual identification of V with its double-dual for finite-dimensional vector spaces, is given by $\tau_q^*(v) = q \wedge v$. To see this, note that the nature of the isomorphism $V \cong V^{**}$ means that any linear functional $\Phi : V^* \to \mathbb{C}$ is actually an evaluation map $q \mapsto \langle q, v\rangle$, at some vector $v \in V$. Therefore,

$$\langle \tau_q^* \Phi, X\rangle = \langle \Phi, -\mathrm{d}\rho^*(X)q\rangle = \langle -\mathrm{d}\rho^*(X)q, v\rangle = \langle q, \mathrm{d}\rho(X)v\rangle = \langle q \wedge v, X\rangle,$$

as claimed[9], i.e., $\mathrm{im}(\tau_q^*) = \{q \wedge v : v \in V\}$.

[9]Note that $\mathrm{d}\rho$ is a representation of the Lie algebra, so its dual is defined with respect to the addition of the underlying vector space. The inverse of X is then $-X$, and *not* the inverse matrix X^{-1}, so $\mathrm{d}\rho^*(X) = -\mathrm{d}\rho(X)^*$.

For the converse part of the claim, we need to show that every functional in the so-called *annihilator* $\mathfrak{l}_q^0 := \{F \in \mathfrak{l}^* : F|_{\mathfrak{l}_q} = 0\}$ is of the form $q \wedge v$ for some $v \in V$. That is, it is sufficient to show $\mathfrak{l}_q^0 = \operatorname{im}(\tau_q^*)$. We have already seen that $\mathfrak{l}_q^0 \supseteq \operatorname{im}(\tau_q^*)$ as linear subspaces of $\mathfrak{l}^*$, so in turn it suffices to show that the dimensions of these two spaces are equal. This is done in [31, Lem. 1]. Now, in particular, any difference of two functionals on $\mathfrak{l}$ that agree on $\mathfrak{l}_q$ can be written in the form $q \wedge v$; proving the claim.

Having done so, we finally establish the sought-after bijective correspondence. In this case it turns out to be easier to disassemble the coadjoint orbit $\mathscr{O}_G(p \oplus F)$ into the desired pairs. Clearly, the coadjoint orbit directly determines the orbit $\operatorname{Orb}_L(p)$ via the first component of the coadjoint action, as shown in (8.2). To distill the coadjoint orbit $\mathscr{O}_{L_p}(F)$ out of the second component of $\mathscr{O}_G(p \oplus F)$, we restrict ourselves to group elements $\Lambda \in L_p$, for which we have $\rho^*(\Lambda)p = p$. By the claim, we now know that $\operatorname{Coad}(\Lambda)F$ then coincides with $\operatorname{Coad}(\Lambda)F + \rho^*(\Lambda)p \wedge v$ on $\mathfrak{l}_p$. The latter expression is determined by $\mathscr{O}_G(p \oplus F)$, so in turn it fully determines the coadjoint orbit

$$\mathscr{O}_{L_p}(F) := \{\operatorname{Coad}(\Lambda)\pi_p(F) : \Lambda \in L_p\}$$

in $\mathfrak{l}_p^*$, where we use that $\pi_p(\operatorname{Coad}(\Lambda)F) = \operatorname{Coad}(\Lambda)\pi_p(F)$. More symbolically, the bijective correspondence is played by the map

$$\mathscr{O}_G(p \oplus F) \mapsto (\operatorname{Orb}_L(p), \mathscr{O}_{L_p}(F)). \qquad \square$$

The theorem does not help us calculate the actual form of the coadjoint orbits $\mathscr{O}_G(p \oplus F)$. Strictly speaking, for our goals of classifying the coadjoint orbits, this is not necessary. However, for the sake of completeness, we may use the following result:

Proposition 8.4. *The coadjoint orbit $\mathscr{O}_G(p \oplus F)$ is a fibre bundle over the cotangent bundle $T^* \operatorname{Orb}_L(p)$ with typical fibre $\mathscr{O}_{L_p}(F)$.*

The proper details and proof of this proposition go way beyond the level of this thesis. For a proof (and more context) we refer to [21, Thm.IV.1.10.4] and surrounding text. In the particular case that $F_0 \in \mathfrak{l}_p^0$, the fibres $\mathscr{O}_{L_p}(F_0)$ are trivial, and hence we have $\mathscr{O}_G(p \oplus F_0) \cong T^* \mathscr{O}_L(p)$. We only quickly recall the definition of a fibre bundle:

Definition 8.5. A ***fibre bundle*** is a quadruplet (E, B, π, F), where E, B and F are topological spaces, and a projection map $\pi : E \to B$, which is a continuous surjective map. The spaces B and F are called the *base* and *(typical) fibre*, respectively. The data satisfies the following condition: for every point $p \in E$ there exists an open neighbourhood $U \subseteq B$ of $\pi(p)$ and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times F$ so that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, hook, two heads, "\Phi"] \arrow[dr, two heads, "\pi"'] & & U \times F \arrow[dl, two heads] \\ & U. & \end{tikzcd}$$

The preimages $\pi^{-1}(\{x\})$, for $x \in B$, are called ***fibres*** (above x), and are all homeomorphic to F.

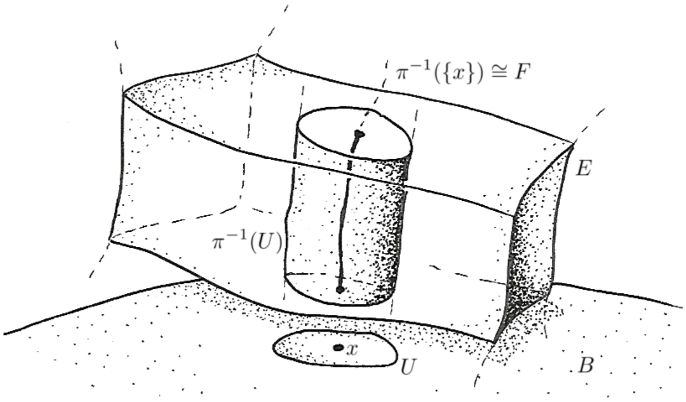


Figure 4: Illustration of a fibre bundle $\pi : E \to B$ with typical fibre F.

8.4 Hamiltonian actions and defining classical elementary particles

We need to further sharpen the notion of symplectic actions. Consider a symplectic action of G on M. The action induces a vector field on M for each element of the Lie algebra $\mathfrak{g}$. Namely, the *fundamental vector field* X_M on M induced by $X \in \mathfrak{g}$ is defined at the point $x \in M$ by

$$(X_M)_x := \frac{\mathrm{d}}{\mathrm{d}t} e^{tX} \cdot x \bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \varphi_{\exp(tX)}(x) \bigg|_{t=0},$$

where we recall the notation $\varphi_g : M \to M; x \mapsto g \cdot x$.

Definition 8.6. Let $\Phi : M \to \mathfrak{g}^*$ be a smooth map. For $X \in \mathfrak{g}$ define

$$\langle \Phi, X \rangle : M \to \mathbb{C}; \quad x \mapsto \langle \Phi(x), X \rangle.$$

We say Φ is a ***moment map*** for the G-space M if for all $X \in \mathfrak{g}$ the fundamental vector field X_M satisfies

$$\mathrm{d}_x \langle \Phi, X \rangle (v) = \omega((X_M)_x, v),$$

for all $x \in M$ and $v \in T_x M$ (see [12]), and moreover, such that for each $g \in G$ the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\varphi_g"] \arrow[d, "\Phi"'] & M \arrow[d, "\Phi"] \\ \mathfrak{g}^* \arrow[r, "\mathrm{Coad}(g)"'] & \mathfrak{g}^*. \end{tikzcd}$$

A symplectic action of G on M is called ***Hamiltonian*** if there exists a moment map for it.

The coadjoint representation is the prime example of a Hamiltonian action. Its moment map is simply the identity map on $\mathfrak{g}^*$.

The notion of a symplectic manifold M with a Hamiltonian G-action becomes the correct notion of a symmetry action on a classical state space. Hamiltonian actions on connected symplectic manifolds are therefore the classical analogue of projective unitary representations. The analogue of the irreducibility property here is that of *transitivity.* A group action is called ***transitive*** if it induces a single orbit. This means that, for every point $x \in M$, there exists another poin $y \in M$ and a group element $g \in G$ so that $g \cdot y = x$. In other words, every two elements in M can be reached from one another by letting the group G act on M. Classical elementary particles are therefore identified with *transitive Hamiltonian actions* on a connected symplectic manifold. In the next section, specifically Theorem 8.10, we will see how to classifies these actions in terms of a modified version of the coadjoint action.

8.5 Twisted coadjoint orbits and elementary particles

The structure of the coadjoint orbits alone does not suffice to classify the elementary particles we find in nature, and they can therefore not be the correct mathematical definition of an elementary particle. Instead, we need to modify them into *twisted* coadjoint orbits. (Throughout this section we follow [21, Chap.III.1]. Also see [20, Chap.1,Sec.4] and [7, Sec.3.6].)

At this point some of the theory in Section 4.1 makes its return in a different setting: Let G be a connected Lie group. A ***1-cocycle*** on G with values in $\mathfrak{g}^*$ is a smooth function $\gamma : G \to \mathfrak{g}^*$ that satisfies the following property:

$$\gamma(gh) = \gamma(g) + \mathrm{Coad}(g)\gamma(h),$$

for all $g, h \in G$. We denote the space of all such cocycles by $\mathrm{Z}^1_{\mathrm{Coad}}(G, \mathfrak{g}^*)$, which forms an abelian group under pointwise addition. A ***1-coboundary*** on G with values in $\mathfrak{g}^*$ is a cocycle $\gamma \in \mathrm{Z}^1_{\mathrm{Coad}}(G, \mathfrak{g}^*)$ of the form

$$\gamma(g) = \mathrm{Coad}(g)F_0 - F_0,$$

for some fixed functional $F_0 \in \mathfrak{g}^*$. Accordingly, the normal subgroup of 1-coboundaries is denoted $\mathrm{B}^1_{\mathrm{Coad}}(G, \mathfrak{g}^*)$.

A 1-cocycle $\gamma \in \mathrm{Z}^1_{\mathrm{Coad}}(G, \mathfrak{g}^*)$ defines a real-valued map $\Gamma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ via [21, Eq.III.1.24]:

$$\Gamma(X, Y) := -\left.\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \gamma(e^{tX}), Y \right\rangle\right|_{t=0}. \tag{8.3}$$

This map satisfies the Jacobi identity of the Lie algebra cocycles (see [7, Thm.3.6.2]). It is, however, not always anti-symmetric. Hence we have the following definition.

Definition 8.7. Any 1-cocycle $\gamma \in \mathrm{Z}^1_{\mathrm{Coad}}(G, \mathfrak{g}^*)$ that defines a real-valued Lie algebra cocycle $\Gamma \in \mathrm{Z}^2_{\mathrm{al}}(\mathfrak{g}, \mathbb{R})$ via (8.3) is called a ***symplectic cocycle***. The space of symplectic cocycles $\mathrm{Z}^1_{\mathrm{sym}}(G, \mathfrak{g}^*)$ induces the quotient

$$\mathrm{H}^1_{\mathrm{sym}}(G, \mathfrak{g}^*) := \frac{\mathrm{Z}^1_{\mathrm{sym}}(G, \mathfrak{g}^*)}{\mathrm{H}^1_{\mathrm{Coad}}(G, \mathfrak{g}^*)},$$

called the ***(first) symplectic cohomology group*** of G with respect to the coadjoint representation.

We now have an extension of Theorem 4.17:

Corollary 8.8. *For any connected, simply connected Lie group G with Lie algebra $\mathfrak{g}$, we have the following isomorphisms between different cohomology groups:*

$$\mathrm{H}^2_{\mathrm{es}}(G, \mathrm{U}(1)) \cong \mathrm{H}^2_{\mathrm{al}}(\mathfrak{g}, \mathbb{R}) \cong \mathrm{H}^1_{\mathrm{sym}}(G, \mathfrak{g}^*).$$

(See [21, Cor.III.1.3.7].)

Definition 8.9. Let G be a connected Lie group with coadjoint action $\mathrm{Coad}: G \to \mathrm{Aut}(\mathfrak{g}^*)$. Let $\gamma \in \mathrm{Z}^1_{\mathrm{sym}}(G, \mathfrak{g}^*)$ be a symplectic cocycle on G with values in $\mathfrak{g}^*$. The ***twisted coadjoint action*** on G with respect to γ is defined according to the following formula:

$$\mathrm{Coad}^\gamma : G \to \mathrm{Aut}(\mathfrak{g}^*); \quad \mathrm{Coad}^\gamma(g)F = \mathrm{Coad}(g)F + \gamma(g).$$

It is straightforward to verify that Coad^γ is a homomorphism. The ***twisted coadjoint orbits*** (also called Coad^γ*-orbits*) are the orbits of the twisted coadjoint action in $\mathfrak{g}^*$. An orbit trough a functional $F \in \mathfrak{g}^*$ is denoted by

$$\mathscr{O}^\gamma_G(F) := \{\mathrm{Coad}^\gamma(g)F : g \in G\}.$$

Finally, the reason that we consider twisted coadjoint orbits is justified by the following theorem.

Theorem 8.10. *Let M be a connected symplectic space with transitive Hamiltonian G-action. There exists a symplectic cocycle $\gamma \in \mathrm{Z}^1_{\mathrm{sym}}(G, \mathfrak{g}^*)$ so that M is isomorphic to a Coad^γ-orbit, i.e., $M \cong \mathscr{O}^\gamma_G(F)$ for some $F \in \mathfrak{g}^*$, or M is isomorphic to a twisted coadjoint orbit of the universal cover $\widetilde{G}$. (Cf. [21, Cor.III.1.4.8].)*

Recalling that symplectic manifolds with transitive Hamiltonian actions are exactly the phase spaces of elementary particles, we find that the correct mathematical notion for an elementary particle must indeed be that of the twisted coadjoint orbit.

We will now state the classical analogue of Theorem 6.9, which will help us classify the classical elementary particles. Given a central extension $\mathrm{U}(1) \times_\omega G$ of our connected Lie group G, we can identify its Lie algebra with $\mathbb{R} \oplus_\Omega \mathfrak{g}$. In turn, the functionals of its Lie algebra may be identified with $\mathbb{R} \oplus \mathfrak{g}^*$ as a vector space. The coadjoint action of the central extension $\mathrm{U}(1) \times_\omega G$ on $\mathbb{R} \oplus \mathfrak{g}^*$ is merely an action of G on $\mathbb{R} \oplus \mathfrak{g}^*$, since the group $\mathrm{U}(1)$ acts trivially [20, p.23]. (Any abelian group acts trivially through the (co)adjoint action, as easily seen from the definitions.)

Lemma 8.11. *The coadjoint action of G on $\mathbb{R} \oplus_\Omega \mathfrak{g}^*$ is of the form*

$$(g, a \oplus F) \mapsto a \oplus (\mathrm{Coad}(g)F + a\gamma(g)),$$

where $\gamma \in \mathrm{Z}^1_{\mathrm{sym}}(G, \mathfrak{g}^*)$.

Proof. (This is the proof of [20, Chap.1,Lem.7].) Let us denote by Coad_Ω the coadjoint action of G on $\mathbb{R} \oplus_\Omega \mathfrak{g}^*$, and by Coad the ordinary coadjoint action of G on $\mathfrak{g}^*$. We consider the projection map $p : \mathbb{R} \oplus_\Omega \mathfrak{g}^* \to \mathbb{R} \oplus_\Omega \mathfrak{g}^*/\mathfrak{g}^*$. Via the composition $p \circ \mathrm{Coad}_\Omega$, we let G act on $\mathbb{R} \oplus_\Omega \mathfrak{g}^*/\mathfrak{g}^*$. This must be trivial, since the extension is central. This tells us that the coadjoint action of G on $\mathbb{R} \oplus \mathfrak{g}^*$ mus leave the first component invariant. In particular, then, the coadjoint action on an element $0 \oplus F \in \mathbb{R} \oplus \mathfrak{g}^*$ must simply be the coadjoint action of G on $\mathfrak{g}^*$:

$$\mathrm{Coad}_\Omega(g)(0 \oplus F) = 0 \oplus \mathrm{Coad}(g)F.$$

More generally, the coadjoint action of G on $\mathbb{R} \oplus \mathfrak{g}^*$ must be of the form:

$$\mathrm{Coad}_\Omega(a \oplus F) = a \oplus (\mathrm{Coad}(g)F + a\gamma(g)),$$

where $\gamma : G \to \mathfrak{g}^*$ is some function. Multiplicativity of Coad_Ω gives that $\gamma \in \mathrm{Z}^1_{\mathrm{sym}}(G, \mathfrak{g}^*)$. ∎

The lemma and preceding theory motivates our main theorem:

Theorem 8.12. *Let G be a connected Lie group. There is a bijective correspondence between twisted coadjoint orbits of G and ordinary coadjoint orbits of the universal central extension $G^\star$.*

It is here that we find a remarkable resemblance to the quantum formalism. There, irreducible projective unitary representations of the symmetry group (i.e., elementary particles) are classified by ordinary irreducible unitary representations of a universal central extension. Here, too, the same reasoning works, once we replace the words "projective" by "twisted," and "irreducible unitary representation" by "coadjoint orbit".

9 Coadjoint orbits of the spacetime symmetry groups

We will now put the formalism of Section 8.3, and especially the result of Theorem 8.3, to use. In order to make our calculations simpler, we perform some standard identifications. For instance, we know that we can identify the Lie algebra of $V = \mathbb{R}^n$ with the abelian Lie algebra $\mathfrak{v} = \mathbb{R}^n$. In turn, the dual $(\mathbb{R}^n)^*$ is identified with $\mathbb{R}^n$ via the Riesz representation theorem with respect to the standard Euclidean inner product. That is, given a functional p on $\mathbb{R}^n$, we identify it with the unique vector $\boldsymbol{p} \in \mathbb{R}^n$ so that $p : \boldsymbol{a} \mapsto \langle \boldsymbol{p}, \boldsymbol{a} \rangle$, where on the right hand side we have the Euclidean inner product. (The Riesz representation theorem also applies to Minkowski space, and even to matrix Lie algebras with appropriate inner products defined on them.)

9.1 Coadjoint orbits of the Galilei group

For illustrative purposes, we calculate the coadjoint orbits of the un-extended Galilei group. As with the calculation of the extensions, we take a bottom-up approach by first calculating the coadjoint orbits of SO(3) and SE(3). To some extent we follow the general strategy and notation of [13, §19]. (Also see [1].)

9.1.1 Coadjoint orbits of SO(3)

We start with the coadjoint orbits of the special orthogonal group SO(3). Its Lie algebra $\mathfrak{so}(3)$ is spanned by the angular momentum matrices (3.2). As a three-dimensional vector space, we identify it with $\mathbb{R}^3$ via $\boldsymbol{a} \mapsto X_{\boldsymbol{a}} := a_i J_i$. This identification is often useful in the mechanics of rotations, because if $\boldsymbol{x} \in \mathbb{R}^3$ is another vector, then the matrix action $X_{\boldsymbol{a}}\boldsymbol{x}$ is the cross-product $\boldsymbol{a} \times \boldsymbol{x}$. (For a Lie algebra isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$, we may endow the latter with the Lie bracket $[\boldsymbol{a}, \boldsymbol{b}] = \boldsymbol{a} \times \boldsymbol{b}$.) The cross-product behaves pleasantly under multiplication by orthogonal matrices $R \in \mathrm{O}(3)$, because $R(\boldsymbol{a} \times \boldsymbol{x}) = (R\boldsymbol{a}) \times (R\boldsymbol{x})$. Using this identity, we find that the adjoint action of SO(3) on $\mathfrak{so}(3)$ is given by:

$$\mathrm{Ad}(R)(X_{\boldsymbol{a}})\boldsymbol{x} = (RX_{\boldsymbol{a}}R^{-1})\boldsymbol{x} = R(\boldsymbol{a} \times (R^{-1}\boldsymbol{x})) = (R\boldsymbol{a}) \times \boldsymbol{x}$$

for all $\boldsymbol{x} \in \mathbb{R}^3$, that is: $\mathrm{Ad}(R)(X_{\boldsymbol{a}}) = X_{R\boldsymbol{a}}$. Sticking to the identification $\mathfrak{so}(3) \cong \mathbb{R}^3$, we may even write $\mathrm{Ad}(R)(\boldsymbol{a}) = R\boldsymbol{a}$, so the adjoint action is in fact simply the matrix action of SO(3) on $\mathbb{R}^3$. To calculate the coadjoint action, we further identify $\mathfrak{so}(3)^* \cong \mathbb{R}^3$. Now, if $F \in \mathfrak{so}(3)^*$ is a functional corresponding to $\boldsymbol{F} \in \mathbb{R}^3$, and $\boldsymbol{a} \in \mathfrak{so}(3)$ is arbitrary, then

$$\langle \mathrm{Coad}(R)(F), \boldsymbol{a} \rangle = \langle F, \mathrm{Ad}(R^{-1})(\boldsymbol{a}) \rangle = \langle \boldsymbol{F}, R^{-1}\boldsymbol{a} \rangle.$$

Orthogonality of R gives $R^{-1} = R^\top$, and so $\langle \mathrm{Coad}(R)(F), \boldsymbol{a} \rangle = \langle R\boldsymbol{F}, \boldsymbol{a} \rangle$. Therefore, under the proper identifications (i.e., identification of F with $\boldsymbol{F}$), the coadjoint action of SO(3) on $\mathfrak{so}(3)^*$ coincides with the adjoint action:

$$\operatorname{Coad}(R)(\boldsymbol{F}) = R\boldsymbol{F}. \tag{9.1}$$

The classification of the coadjoint orbits in $\mathfrak{so}(3)^*$ therefore amounts to the classification of the orbits of the defining action of SO(3) on $\mathbb{R}^3$. These are, of course, well known. The origin constitutes an orbit by itself, and any non-zero vector $\boldsymbol{F} \in \mathbb{R}^3$ defines an orbit which is precisely the sphere whose radius is that of $\|\boldsymbol{F}\|$. The coadjoint orbits therefore are simply the 2-spheres in $\mathbb{R}^3$, parametrised as follows:

$$\mathscr{O}_{\mathrm{SO}(3)}(F) = \begin{cases} \{0\} & \text{if } \|\boldsymbol{F}\| = 0; \\ \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| = \|\boldsymbol{F}\|\} & \text{if } \|\boldsymbol{F}\| > 0. \end{cases}$$

Introducing the notation for the 2-sphere with radius r as $S^2_r := \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| = r\} \subseteq \mathbb{R}^3$, we may write the above as $\mathscr{O}_{\mathrm{SO}(3)}(F) = S^2_{\|\boldsymbol{F}\|}$.

9.1.2 Coadjoint orbits of SE(3)

Next up is the special Euclidean group. The Euclidean group is a semi-direct product $\mathrm{E}(3) = \mathbb{R}^3 \rtimes_\rho \mathrm{O}(3)$, where $\rho : \mathrm{O}(3) \to \mathrm{GL}(\mathbb{R}^3)$ is simply the defining action:

$$\rho(R)(\boldsymbol{x}) = R\boldsymbol{x}.$$

In light of Theorem 8.3, we first classify the dual orbits $\mathrm{Orb}_{\mathrm{SO}(3)}(p)$ of SO(3) in $(\mathbb{R}^3)^* \cong \mathbb{R}^3$. Through now familiar calculations we find that the dual action of SO(3) on $(\mathbb{R}^3)^*$ is simply given by matrix multiplication

$$\rho^*(R)\boldsymbol{p} = R\boldsymbol{p},$$

for a functional $p \in (\mathbb{R}^3)^*$ identified with $\boldsymbol{p} \in \mathbb{R}^3$. The dual orbits are therefore also 2-spheres, just like the coadjoint orbits: $\mathrm{Orb}_{\mathrm{SO}(3)}(\boldsymbol{p}) = \mathscr{O}_{\mathrm{SO}(3)}(\boldsymbol{p})$. The coadjoint action of SE(3) on $\mathfrak{se}(3)^*$ is now the following simplified version of (8.1):

$$\operatorname{Coad}(\boldsymbol{a}, R)(p \oplus F) = R\boldsymbol{p} \oplus (R\boldsymbol{F} + R\boldsymbol{p} \wedge \boldsymbol{a}). \tag{9.2}$$

Here we have once more identified $\mathfrak{so}(3)^* \cong \mathbb{R}^3$, which in this equation comes to light via the identification of $F \in \mathfrak{so}(3)^*$ with $\boldsymbol{F} \in \mathbb{R}^3$, and in turn the substitution of (9.1). To determine the coadjoint orbits, we distinguish between a few cases;

When $\boldsymbol{p} = 0$ we have $\mathrm{Orb}_{\mathrm{SO}(3)}(p) = \{0\}$, and the term $R\boldsymbol{p} \wedge \boldsymbol{a}$ vanishes. In turn, this case breaks down to the absolutely trivial case, when also $\boldsymbol{F} = 0$. Then the entire coadjoint orbit is $\mathscr{O}_{\mathrm{SE}(3)}(p \oplus F) = \{(0,0)\}$. In the less trivial case that $\boldsymbol{F}$ is non-zero, we find that the coadjoint orbit through $0 \oplus F$ is essentially the coadjoint orbit of F in $\mathfrak{so}(3)^*$, which we know to be the 2-sphere with radius $\|\boldsymbol{F}\|$. More precisely:

$$\mathscr{O}_{\mathrm{SE}(3)}(0 \oplus F) = \{0\} \times \mathscr{O}_{\mathrm{SO}(3)}(F) = \{0\} \times S^2_{\|\boldsymbol{F}\|} \cong S^2_{\|\boldsymbol{F}\|}.$$

The same transpires when we follow Theorem 8.3, since the stabiliser $\mathrm{SO}(3)_0$ of $\boldsymbol{p} = 0$ is clearly the entire group SO(3).

Lastly, we have the case that $\boldsymbol{p} \neq 0$. It is now appropriate to employ Proposition 8.4, and calculate the coadjoint orbits of the stabiliser $\mathrm{SO}(3)_{\boldsymbol{p}}$ (as opposed to directly calculating the coadjoint orbit from (9.2)). Elements in the stabiliser $\mathrm{SO}(3)_{\boldsymbol{p}}$ are clearly exactly the rotations $R \in \mathrm{SO}(3)$ about the $\boldsymbol{p}$-axis. Without loss of generality, we may take this axis to coincide with the z-axis. We now have an isomorphism

$\mathrm{SO}(3)_{\boldsymbol{p}} \cong \mathrm{SO}(2)$, where we identify the two-dimensional rotation group as a subgroup of $\mathrm{SO}(3)$, in the sense that it describes all rotations in the xy-plane $\mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$.

Therefore, we need to calculate the coadjoint orbits of $\mathrm{SO}(2)$. The elementary isomorphism $\mathrm{SO}(2) \cong \mathrm{U}(1)$, where we understand a rotation about the z-axis with angle α to correspond to the element $e^{i\alpha} \in \mathrm{U}(1)$, gives a Lie algebra isomorphism $\mathfrak{su}(2) \cong \mathfrak{u}(1) \cong \mathbb{R}$. (This also easily follows from the properties of matrices in $\mathfrak{su}(2)$.) By the very definition of the coadjoint action we find

$$\langle \mathrm{Coad}(e^{i\theta})F, X\rangle = \langle F, e^{-i\theta} X e^{i\theta}\rangle = \langle F, X\rangle,$$

so that $\mathrm{Coad}(z) = \mathrm{id}_{\mathbb{R}}$ for all $z \in \mathrm{U}(1)$. (The (co)adjoint actions are always trivial on abelian groups.) The coadjoint orbits of $\mathrm{SO}(2)$ are therefore simply points on the real line:

$$\mathscr{O}_{\mathrm{SO}(3)_{\boldsymbol{p}}}(x) \cong \mathscr{O}_{\mathrm{SO}(2)}(x) \cong \mathscr{O}_{\mathrm{U}(1)}(x) = \{x\},$$

where $x \in \mathbb{R}$.

Putting things together, we find through Proposition 8.4 that for $\boldsymbol{p} \neq 0$ the coadjoint orbit of $\mathrm{SE}(3)$ through $\boldsymbol{p} \oplus \boldsymbol{F}$ is a fibre bundle over the cotangent bundle $T^* \mathrm{Orb}_{\mathrm{SO}(3)}(\boldsymbol{p}) = T^* S^2_{\|\boldsymbol{p}\|}$, whose fibres consist of points $\mathscr{O}_{\mathrm{SO}(2)}(x) = \{x\}$ on the real line. This fibre bundle is trivial in the sense that it is diffeomorphic to the cotangent bundle $T^* S^2_{\|\boldsymbol{p}\|}$ itself, and hence we have

$$\mathscr{O}_{\mathrm{SE}(3)}(\boldsymbol{p} \oplus \boldsymbol{F}) \cong T^* S^2_{\|\boldsymbol{p}\|}.$$

Note that the cotangent bundle is not trivial (as a bundle) due to the hairy-ball theorem. Instead, as a set we can describe it as being isomorphic to the tangent bundle:

$$T^* S^2_r \cong \left\{(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^6 : \|\boldsymbol{p}\| = r, \langle \boldsymbol{q}, \boldsymbol{p}\rangle = 0\right\}.$$

In general, then, we see that the coadjoint orbits of $\mathrm{SE}(3)$ are parametrised by a positive real number $k \in \mathbb{R}_{>0}$ and a non-negative number $s \in \mathbb{R}_{\geqslant 0}$, and the orbits are of the form

$$\mathscr{O}^{k,s}_{\mathrm{SE}(3)} := \begin{cases} S^2_s & \text{if } k = 0; \\ T^* S^2_k & \text{if } k \neq 0. \end{cases} \tag{9.3}$$

9.1.3 Coadjoint orbits of the Galilei group

Finally, we are in a position to consider the Galilei group $\mathrm{Gal}(3) = \mathbb{R}^4 \rtimes_\rho \mathrm{SE}(3)$, where $\rho : \mathrm{SE}(3) \to \mathrm{GL}(\mathbb{R}^4)$ is simply the defining action:

$$\rho(\boldsymbol{v}, R)(t, \boldsymbol{x}) = (t, R\boldsymbol{x} + \boldsymbol{v}t).$$

With the proper identifications this becomes the matrix action of $\mathrm{GL}(4, \mathbb{R})$ on $\mathbb{R}^4$. We identify the dual algebra $\mathfrak{gal}(3)^*$ with the space $\mathbb{R}^4 \oplus \mathfrak{se}(3)^*$, where we in turn re-identify $\mathfrak{se}(3)^*$ with elements of the form $\boldsymbol{q} \oplus \boldsymbol{F}$, where $\boldsymbol{q} \in \mathbb{R}^3 \cong (\mathbb{R}^3)^*$ and $\boldsymbol{F} \in \mathbb{R}^3 \cong \mathfrak{so}(3)^*$. We denote elements of $\mathfrak{gal}(3)^*$ by $((E, \boldsymbol{p}), (\boldsymbol{q} \oplus \boldsymbol{F}))$.

Through easy matrix calculations, we find that on a functional $(E, \boldsymbol{p}) \in \mathbb{R}^4$ the dual action of $(R, \boldsymbol{v}) \in \mathrm{SE}(3)$ becomes:

$$\rho^*(\boldsymbol{v}, R)(E, \boldsymbol{p}) = (E - \langle R\boldsymbol{p}, \boldsymbol{v}\rangle, R\boldsymbol{p}). \tag{9.4}$$

For $\boldsymbol{p} = 0$ we have a singleton dual orbit:

$$\mathrm{Orb}_{\mathrm{SE}(3)}(E, 0) = \{(E, 0)\}. \tag{9.5}$$

For $\boldsymbol{p} \neq 0$ we can vary the first component through all the reals due to the term $\langle R\boldsymbol{p}, \boldsymbol{v}\rangle$. The second component is simply the orbit $\mathrm{Orb}_{\mathrm{SO}(3)}(\boldsymbol{p}) = S^2_{\|\boldsymbol{p}\|}$, and hence:

$$\mathrm{Orb}_{\mathrm{SE}(3)}(E, \boldsymbol{p}) = \mathbb{R} \times \mathrm{Orb}_{\mathrm{SO}(3)}(\boldsymbol{p}) = \mathbb{R} \times S^2_{\|\boldsymbol{p}\|}. \tag{9.6}$$

Following Theorem 8.3 as usual, we calculate the coadjoint orbits of the corresponding stabiliser subgroups to these dual orbits. We start with the easier case, where $\boldsymbol{p} = 0$. An element $(\boldsymbol{v}, R) \in \mathrm{SE}(3)$ is in the stabiliser of the point $(E, 0)$ whenever $\rho^*(\boldsymbol{v}, R)(E, 0) = (E, 0)$, which is always the case. The stabiliser is therefore the entire group, of which we know what the coadjoint orbits are (see the previous section). Hence by Proposition 8.4 we find a class of coadjoint orbits in the Galilei group that are fibre bundles over $T^* \mathrm{Orb}_{\mathrm{SE}(3)}(E, 0) = T^*\{(E, 0)\}$, with fibres $\mathscr{O}^{k,s}_{\mathrm{SE}(3)}$, for some $k \in \mathbb{R}_{>0}$ and $s \in \mathbb{R}_{\geqslant 0}$ (see (9.3)). As a zero-dimensional manifold, the singleton $\{(E, 0)\}$ has trivial (co)tangent bundle. Namely, since it contains but one point, the tangent bundle $T\{(E, 0)\}$ becomes simply the tangent space $T_{(E,0)}\{(E, 0)\}$, and since the manifold is zero-dimensional the tangent space is trivial as a vector space. We therefore find that the coadjoint orbit simply becomes the fibre:

$$\mathscr{O}^{0,r,s}_{\mathrm{Gal}(3)} := \mathscr{O}_{\mathrm{Gal}(3)}((E, 0), (\boldsymbol{q} \oplus \boldsymbol{F})) \cong \mathscr{O}^{r,s}_{\mathrm{SE}(3)},$$

where $k = \|\boldsymbol{q}\|$ and $s = \|\boldsymbol{F}\|$. This includes the trivial coadjoint orbit.

Now assume that $\boldsymbol{p} \neq 0$. Since the dual orbit is now of the form $\mathbb{R} \times S^2_{\|\boldsymbol{p}\|}$, we need only concern ourselves with points of the form $(0, \boldsymbol{p})$. Were an element $(\boldsymbol{v}, R)$ to be an element in the stabiliser of such a point, we would find that R is a rotation about the $\boldsymbol{p}$-axis, and that $\boldsymbol{v}$ is perpendicular to $R\boldsymbol{p} = \boldsymbol{p}$. The first restriction reduces the $\mathrm{SO}(3)$ part of the Euclidean group to a subgroup that is isomorphic to $\mathrm{SO}(2)$, as we have seen. The second restriction reduces $\boldsymbol{v}$ to, say, the xy-plane $\mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$. The stabiliser is therefore the *two*-dimensional Euclidean group: $\mathrm{SE}(3)_{(0,\boldsymbol{p})} \cong \mathrm{SE}(2)$.

We thus have to calculate the coadjoint orbits of $\mathrm{SE}(2) = \mathbb{R}^2 \rtimes_\sigma \mathrm{SO}(2)$, where σ is simply the action of matrix multiplication. Through familiar arguments we find that the dual action of σ is also matrix multiplication, so that the dual orbits are circles in the plane:

$$\mathrm{Orb}_{\mathrm{SO}(2)}(\boldsymbol{q}) = S^1_{\|\boldsymbol{q}\|},$$

where a functional $q \in (\mathbb{R}^2)^*$ is identified with the vector $\boldsymbol{q} \in \mathbb{R}^2$ just as in the three-dimensional case. Familiarly, we divide the situation up into two cases; first we take $\boldsymbol{q} = 0$. The corresponding stabiliser group is clearly the full group $\mathrm{SO}(2)$, of which we know the coadjoint orbits are just points on the real line. Thus $\mathscr{O}_{\mathrm{SE}(2)}(0 \oplus x) \cong \mathscr{O}_{\mathrm{SO}(2)}(x) = \{x\}$, for some $x \in \mathbb{R}$. In the case that $\boldsymbol{q} \neq 0$, the stabiliser is the trivial group $\{I\} \subseteq \mathrm{SO}(2)$, of which the coadjoint action is trivial. By Proposition 8.4 the coadjoint orbits are therefore simply the cotangent bundle of the circles: $\mathscr{O}_{\mathrm{SE}(2)}(\boldsymbol{q} \oplus x) \cong T^* S^1_{\|\boldsymbol{q}\|}$, which are well known to be isomorphic to the cylinders $\mathbb{R} \times S^1_{\|\boldsymbol{q}\|}$. Summarising, we have the following types of coadjoint orbits in $\mathrm{SE}(2)$:

$$\mathscr{O}^{r,h}_{\mathrm{SE}(2)} := \begin{cases} \{h\} & \text{if } r = 0; \\ \mathbb{R} \times S^1_r & \text{if } r \neq 0; \end{cases} \tag{9.7}$$

for $h \in \mathbb{R}$ and $r \in \mathbb{R}_{\geqslant 0}$, the latter number corresponding to the norm of $\boldsymbol{q}$.

Putting it together, we have the following new class of coadjoint orbits of the Galilei group. They are fibre bundles over $T^*(\mathbb{R} \times S^2_k)$ whose fibres are $\mathscr{O}^{r,h}_{\mathrm{SE}(2)}$. The coadjoint orbit through the point $((E, \boldsymbol{p}), (0 \oplus h))$ is of the form

$$\mathscr{O}^{k,0,0}_{\mathrm{Gal}(3)} := \mathscr{O}_{\mathrm{Gal}(3)}((E, \boldsymbol{p}), (0 \oplus h)) \cong T^*(\mathbb{R} \times S^2_k),$$

where $k = \|\boldsymbol{p}\|$. Since we have the following identity for tangent bundles: $T(\mathbb{R}\times S^2_k) \cong T\mathbb{R}\oplus TS^2_k$, we obtain a similar identification for the cotangent bundles: $T^*(\mathbb{R}\times S^2_k) \cong T^*\mathbb{R}\oplus T^*S^2_k$. Canonically identifying $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$, we therefore have

$$\mathscr{O}^{k,0,0}_{\mathrm{Gal}(3)} \cong \mathbb{R}^2 \times T^*S^2_k.$$

For $\boldsymbol{q} \neq 0$ we have a class of orbits that is isomorphic to the fibre bundle over $\mathbb{R}^2\times T^*S^2_k$ with fibres $\mathbb{R}\times S^1_r$, for some $r > 0$. We denote these coadjoint orbits by $\mathscr{O}^{k,r,0}_{\mathrm{Gal}(3)}$.

9.2 Coadjoint orbits of the extended Galilei group

The classical non-relativistic elementary particles are the coadjoint orbits of the universal central extension $\mathrm{Gal}^\star_M(3) = \mathbb{R}^5 \rtimes_\rho \widetilde{\mathrm{SE}(3)}$. We have already calculated the group structure of $\mathrm{Gal}_M(3) = \mathbb{R}^5 \rtimes \mathrm{SE}(3)$ in Section 5.1.2. The extension of this structure to the universal central extension is merely a matter of replacing rotation matrices by unitary matrices. In particular the defining action ρ of $\widetilde{\mathrm{SE}(3)}$ on $\mathbb{R}^5$ is given by the following equation:

$$\rho(\boldsymbol{v},U)(\theta,s,\boldsymbol{a}) = \left(\theta + \frac{1}{2}s\boldsymbol{v}^2M + \langle \boldsymbol{v}, \widetilde{p}(U)\boldsymbol{a}\rangle M, s, \widetilde{p}(U)\boldsymbol{a} + s\boldsymbol{v}\right). \tag{9.8}$$

Re-introducing the notation $\xi_M : \mathrm{Gal}(3)\times\mathrm{Gal}(3) \to \mathbb{R}$ for the real-valued cocycle of the extended Galilei group (with mass M), and modifying the notation to write

$$\xi_M(s,\boldsymbol{a},\boldsymbol{v},R) := \xi_M((s,\boldsymbol{a},\boldsymbol{v},R),(s,\boldsymbol{a},\boldsymbol{v},R)) = \frac{1}{2}s\boldsymbol{v}^2M + \langle \boldsymbol{v}, R\boldsymbol{a}\rangle M,$$

the action ρ may be written more concisely as

$$\rho(\boldsymbol{v},U)(\theta,s,\boldsymbol{a}) = (\theta + \xi_M(s,\boldsymbol{a},\boldsymbol{v},\widetilde{p}(U)), s, \widetilde{p}(U)\boldsymbol{a} + s\boldsymbol{v}).$$

We calculate the dual action ρ^* of $\widetilde{\mathrm{SE}(3)}$ on $(\mathbb{R}^5)^* \cong \mathbb{R}^5$. Take an element $(\Theta,E,\boldsymbol{p}) \in \mathbb{R}^5$ representing a functional in $(\mathbb{R}^5)^*$. By definition we have

$$\langle \rho^*(\boldsymbol{v},U)(\Theta,E,\boldsymbol{p}),(\theta,s,\boldsymbol{a})\rangle = \left\langle (\Theta,E,\boldsymbol{p}), \rho(-\widetilde{p}(U)^{-1}\boldsymbol{v},\widetilde{p}(U)^{-1})(\theta,s,\boldsymbol{a})\right\rangle.$$

Working out the action using (9.8) we find

$$\begin{aligned}\langle \rho^*(\boldsymbol{v},U)(\Theta,E,\boldsymbol{p}),(\theta,s,\boldsymbol{a})\rangle &= \left\langle \Theta, \theta + \frac{1}{2}s\langle \widetilde{p}(U)^{-1}\boldsymbol{v},\widetilde{p}(U)^{-1}\boldsymbol{v}\rangle M - \langle \widetilde{p}(U)^{-1}\boldsymbol{v},\widetilde{p}(U)^{-1}\boldsymbol{a}\rangle M\right\rangle \\ &\quad + \langle E,s\rangle + \langle \boldsymbol{p}, \widetilde{p}(U)^{-1}(\boldsymbol{a}-s\boldsymbol{v})\rangle \\ &= \left\langle \Theta, \theta + \frac{1}{2}s\boldsymbol{v}^2M - \langle \boldsymbol{v},\boldsymbol{a}\rangle M\right\rangle + \langle E,s\rangle + \langle \widetilde{p}(U)\boldsymbol{p}, \boldsymbol{a}-s\boldsymbol{v}\rangle.\end{aligned}$$

The next steps are a matter of simply rewriting this expression through the linearity of the situation. We find:

$$\rho^*(\boldsymbol{v},U)(\Theta,E,\boldsymbol{p}) = \left(\Theta, E + \frac{1}{2}\Theta\boldsymbol{v}^2M - \langle \widetilde{p}(U)\boldsymbol{p},\boldsymbol{v}\rangle, \widetilde{p}(U)\boldsymbol{p} - \Theta\boldsymbol{v}M\right). \tag{9.9}$$

Calculation of the dual orbits $\mathrm{Orb}_{\widetilde{\mathrm{SE}(3)}}(\Theta,E,\boldsymbol{p})$ is now straightforward. There are three types of orbits. First, assume that $\Theta = 0$ and $\boldsymbol{p} = 0$. The orbit is then the singleton

$$\mathrm{Orb}_{\widetilde{\mathrm{SE}(3)}}(0,E,0) = \{(0,E,0)\}.$$

These are clearly analogous to the dual SE(3)-orbits (9.5). We would do well to calculate the stabilisers of these points right away. In this case we clearly have $\widetilde{\mathrm{SE}(3)}_{(0,E,0)} = \widetilde{\mathrm{SE}(3)}$. If, on the other hand, $\boldsymbol{p} \neq 0$, then we find

$$\mathrm{Orb}_{\widetilde{\mathrm{SE}(3)}}(0, E, \boldsymbol{p}) = \mathrm{Orb}^{\|\boldsymbol{p}\|}_{\widetilde{\mathrm{SE}(3)}} := \{(0, V, \boldsymbol{q}) \in \mathbb{R}^5 : V \in \mathbb{R}, \|\boldsymbol{q}\| = \|\boldsymbol{p}\|\} \cong \mathbb{R} \times S^2_{\|\boldsymbol{p}\|},$$

which is exactly the dual orbit (9.6) of the ordinary Euclidean group through $(E, \boldsymbol{p})$. An element $(\boldsymbol{v}, U)$ is in the stabiliser of the point $(0, E, \boldsymbol{p})$ if and only if $\widetilde{p}(U)\boldsymbol{p} = \boldsymbol{p}$, which means that $\widetilde{p}(U)$ must be an element of the stabiliser $\mathrm{SO}(3)_{\boldsymbol{p}}$, i.e., $\widetilde{p}(U)$ must be a rotation about the axis spanned by $\boldsymbol{p}$. In turn we then also need $\boldsymbol{v}$ to be perpendicular to $\boldsymbol{p}$. Without loss of generality (recall that we are free to pick any functional in $\mathrm{Orb}_{\widetilde{\mathrm{SE}(3)}}(0, E, \boldsymbol{p})$), we may take $\boldsymbol{p}$ to be parallel to the z-axis. This restricts $\boldsymbol{v}$ to the xy-plane $\mathbb{R}^2 \times \{0\} \cong \mathbb{R}^2$. Considering the rotations around the z-axis as a subgroup isomorphic to $\mathrm{SO}(2) \subseteq \mathrm{SO}(3)$, the stabiliser is then isomorphic to the universal covering space of the *two*-dimensional Euclidean group:

$$\widetilde{\mathrm{SE}(3)}_{(0,E,\boldsymbol{p})} \cong \widetilde{\mathrm{SE}(2)} = \mathbb{R}^2 \rtimes \widetilde{\mathrm{SO}(2)}.$$

Here we view $\widetilde{\mathrm{SO}(2)}$ as the appropriate subgroup of $\widetilde{\mathrm{SO}(3)} = \mathrm{SU}(2)$.

The case that $\Theta \neq 0$ is the most complicated. It is useful to consider the case that $\boldsymbol{p} = 0$ first. In fact, this is without loss of generality, since the parameter $\boldsymbol{v}$ still allows us to vary the last component of the orbit throughout all of $\mathbb{R}^3$. We have

$$\rho^*(\boldsymbol{v}, U)(\Theta, E, 0) = \left(\Theta, E + \frac{1}{2}\Theta \boldsymbol{v}^2 M, -\Theta \boldsymbol{v} M\right).$$

Denoting the third component by$\boldsymbol{x} = -\Theta \boldsymbol{v} M$, we find that the expression

$$E - \frac{\boldsymbol{x}^2}{2\Theta M} = E$$

is a constant, whenever E is constant. Therefore orbits through $(\Theta, E, \boldsymbol{p})$ are of the form

$$\mathrm{Orb}_{\widetilde{\mathrm{SE}(3)}}(\Theta, E, \boldsymbol{p}) \cong \mathrm{Orb}^{M,V}_{\widetilde{\mathrm{SE}(3)}} := \left\{(\Theta, E, \boldsymbol{p}) \in \mathbb{R}^5 : E - \frac{\boldsymbol{p}^2}{2\Theta M} = V\right\}, \tag{9.10}$$

for some real number $V \in \mathbb{R}$, and $M \in \mathbb{R}\backslash\{0\}$ the extension parameter of the centrally extended Galilei group. Note that this is a completely different type of dual orbit to those that we found for the ordinary Galilei group, and moreover, the only one that depends on the parameter M. The stabiliser of these types of functionals must be the stabilisers of functionals of the form $(\Theta, E, 0)$, which is simply SU(2).

Following Theorem 8.3, we determine the coadjoint orbits of the stabiliser groups. This means calculating the coadjoint orbits of $\widetilde{\mathrm{SE}(3)}$, $\widetilde{\mathrm{SE}(2)}$ and $\widetilde{\mathrm{SO}(3)} = \mathrm{SU}(2)$. We have already calculated the coadjoint orbits of the groups SE(3), SE(2) and SO(3). The following lemma is therefore extremely useful.

Lemma 9.1. *Let G be a connected Lie group with universal covering $\widetilde{G}$. The coadjoint orbits of $\widetilde{G}$ are exactly the coadjoint orbits of G.*

Proof. We use Lemma 8.2. Since the covering map $\widetilde{p} : \widetilde{G} \to G$ is a homomorphism, for every $g \in G$ we have the following relation:

$$\mathrm{Ad}(\widetilde{p}(g)) \circ \mathrm{d}\widetilde{p} = \mathrm{d}\widetilde{p} \circ \mathrm{Ad}(g),$$

from which it follows that $\mathrm{Ad}(\widetilde{p}(g)) = \mathrm{Ad}(g)$, since the differential $\mathrm{d}\widetilde{p}$ serves as an isomorphism between $\mathfrak{g}$ and $\mathrm{Lie}(\widetilde{G})$. The adjoint action of $\widetilde{G}$ therefore coincides with the adjoint action of G, and it follows that the same holds for the coadjoint action. (See also [27, p.35].) $\square$

We therefore have the the following coadjoint orbits in the extended Galilei group; for the stabiliser SE(3) we have a coadjoint orbit that is a fibre bundle trivial fibre bundle over $T^* \operatorname{Orb}_{\widetilde{\mathrm{SE}(3)}}(0, E, 0) \cong \{(0, E, 0)\}$ with typical fibre $\mathscr{O}^{r,s}_{\mathrm{SE}(3)}$:

$$\mathscr{O}^{0,r,s}_{\mathrm{Gal}^\star(3)} := \mathscr{O}_{\mathrm{Gal}^\star(3)}((0, E, 0), (\boldsymbol{q} \oplus \boldsymbol{F})) \cong \mathscr{O}^{r,s}_{\mathrm{SE}(3)},$$

where $r = \|\boldsymbol{q}\|$ and $s = \|\boldsymbol{F}\|$.

For the stabiliser SE(2) the coadjoint orbit is a fibre bundle over $T^* \operatorname{Orb}_{\widetilde{\mathrm{SE}(3)}}(0, E, \boldsymbol{p}) = \mathbb{R}^2 \times T^* S^2_k$ with typical fibre $\mathscr{O}^{r,h}_{\mathrm{SE}(2)}$. For $r = 0$ this gives a class

$$\mathscr{O}^{k,0,0}_{\mathrm{Gal}^\star(3)} := \mathscr{O}_{\mathrm{Gal}^\star(3)}((0, E, \boldsymbol{p}), (0 \oplus h)) \cong \mathbb{R}^2 \times T^* S^2_k,$$

where $k = \|\boldsymbol{p}\|$. For $r > 0$ the typical fibre not trivial, but instead is isomorphic to the cylinder $\mathscr{O}^{r,h}_{\mathrm{SE}(2)} = \mathbb{R} \times S^1_r$, giving a class $\mathscr{O}^{k,r,0}_{\mathrm{Gal}^\star(3)}$.

And lastly, for the stabiliser SO(3) the coadjoint orbits are fibre bundles over $\operatorname{Orb}^{M,V}_{\widetilde{\mathrm{SE}(3)}}$ (see (9.10)), with typical fibres simply being spheres $\mathscr{O}^{s}_{\mathrm{SO}(3)} = S^2_s$. We denote this class of coadjoint orbits by $\mathscr{O}^{M,V,s}_{\mathrm{Gal}^\star(3)}$. This coadjoint orbit goes through the point $((\Theta, E, \boldsymbol{p}), (0 \oplus \boldsymbol{F}))$, where $s = \|\boldsymbol{F}\|$ and

$$V = E - \frac{\boldsymbol{p}^2}{2\Theta M}.$$

9.3 Coadjoint orbits of the Poincaré group

We calculate the coadjoint orbits of the ordinary Poincaré group. Following Theorem 8.3, we first determine the dual orbits of the defining action ρ of the Lorentz group $\mathrm{SO}^+(1,3)$ on $\mathbb{R}^4$. The dual action ρ^* is again matrix multiplication of $\mathrm{SO}^+(1,3)$ on $(\mathbb{R}^4)^* \cong \mathbb{R}^4$. There are four different types of dual orbits [21, p.411]. The first, and most obvious one, is the trivial orbit

$$\operatorname{Orb}^0_{\mathrm{SO}^+(1,3)} := \operatorname{Orb}_{\mathrm{SO}^+(1,3)}(0,0,0,0) = \{(0,0,0,0)\}.$$

The corresponding stabiliser is obviously the entire Lorentz group $\mathrm{SO}^+(1,3)$. Next we consider the orbits through points of the form $(E, 0) \in \mathbb{R}^4$:

$$\operatorname{Orb}^{m,\pm}_{\mathrm{SO}^+(1,3)} := \operatorname{Orb}_{\mathrm{SO}^+(1,3)}(\pm m, 0) = \{(E, \boldsymbol{p}) \in \mathbb{R}^4 : E^2 - \boldsymbol{p}^2 = m^2, \pm E > 0\}.$$

Since the property of the elements in this orbit clearly only depend on the length of $\boldsymbol{p}$, we see that the stabiliser is the rotation group SO(3). The next orbit type is

$$\operatorname{Orb}^{0,\pm}_{\mathrm{SO}^+(1,3)} := \operatorname{Orb}_{\mathrm{SO}^+(1,3)}(\pm 1, 0, 0, -1) = \{(E, \boldsymbol{p}) \in \mathbb{R}^4 : E^2 - \boldsymbol{p}^2 = 0, \pm E > 0\}.$$

In this case the stabiliser is the two-dimensional Euclidean group SE(2). Lastly, we have orbits through points with vanishing first component:

$$\operatorname{Orb}^{-m}_{\mathrm{SO}^+(1,3)} := \operatorname{Orb}_{\mathrm{SO}^+(1,3)}(0,0,0,m) = \{(E, \boldsymbol{p}) \in \mathbb{R}^4 : E^2 - \boldsymbol{p}^2 = -m^2\},$$

whose stabiliser is the two-dimensional Lorentz group $\mathrm{SO}^+(1,2)$.

For the trivial orbit $\mathrm{Orb}^0_{\mathrm{SO}^+(1,3)}$ the corresponding coadjoint orbits simply become the coadjoint orbits of the Lorentz group. But these are just the dual orbits we calculated moments ago.

Corresponding to the dual orbits $\mathrm{Orb}^{m,\pm}_{\mathrm{SO}^+(1,3)}$ are the coadjoint orbits of $\mathrm{SO}(3)$, which are the spheres S^2_s. The coadjoint orbits $\mathscr{O}_{\mathrm{Poin}(1,3)}((\pm m, 0), \boldsymbol{F})$ are fibre bundles over $T^* \mathrm{Orb}^{m,\pm}_{\mathrm{SO}^+(1,3)}$ with typical fibre S^2_s. The defining relation $E^2 - \boldsymbol{p}^2 = m^2$ of the dual orbit completely determines the first component of $(E, \boldsymbol{p})$ in terms of the second, and we therefore have

$$\mathrm{Orb}^{m,\pm}_{\mathrm{SO}^+(1,3)} \cong \mathbb{R}^3$$

via the map $\boldsymbol{p} \mapsto (\sqrt{\boldsymbol{p}^2 + m^2}, \boldsymbol{p})$. This gives the following class of coadjoint orbits:

$$\mathscr{O}^{m,\pm,s}_{\mathrm{Poin}(1,3)} \cong T^*\mathbb{R}^3 \times S^2_s \cong \mathbb{R}^6 \times S^2_s.$$

Next we determine the coadjoint orbits corresponding to the dual orbits $\mathrm{Orb}^{0,\pm}_{\mathrm{SO}^+(1,3)}$. We identify this orbit with $\mathbb{R}^3$, just as we did for $\mathrm{Orb}^{m,\pm}_{\mathrm{SO}^+(1,3)}$. The coadjoint orbits of the two-dimensional Euclidean group are $\mathscr{O}^{r,h}_{\mathrm{SE}(2)}$, labelled by $h \in \mathbb{R}$ and $r \in \mathbb{R}_{\geqslant 0}$, and given by (9.7). The corresponding coadjoint orbits in the Poincaré group are therefore fibre bundles over $T^*\mathbb{R}^3 \cong \mathbb{R}^6$ with fibres $\mathscr{O}^{r,h}_{\mathrm{SE}(2)}$. For $r = 0$ this gives

$$\mathscr{O}^{0,\pm,h}_{\mathrm{Poin}(1,3)} \cong \mathbb{R}^6,$$

while for $r > 0$ we have that $\mathscr{O}^{0,\pm,h,r}_{\mathrm{Poin}(1,3)}$ is a fibre bundle over $\mathbb{R}^6$ whose fibres are the cylinders $\mathbb{R} \times S^1_r$.

This leaves us to calculate the coadjoint orbits corresponding to the two-dimensional Lorentz group $\mathrm{SO}^+(1,2)$, which amounts to calculating the orbits of its canonical action on $\mathbb{R}^3$. This gives a class of orbits that is essentially the same as the dual orbits of $\mathrm{SO}^+(1,3)$. We omit the statement, since the corresponding coadjoint orbits of the Poincaré group are not believed to have physical significance (see Section 10.2.2).

The classical relativistic elementary particles are classified by the coadjoint orbits of the universal central extension $\mathrm{Poin}^\star(1,3) = \mathbb{R}^4 \rtimes_\rho \mathrm{SL}(2,\mathbb{C})$, which is simply the universal cover of the Poincaré group. (The symplectic cohomology of the Poincaré group is trivial [36, Prop.13.62].) By Lemma 9.1 we therefore already know the coadjoint orbits of $\mathrm{Poin}^\star(1,3)$.

10 Summary and physical interpretation

The point of this thesis was to give a thorough description of the mathematics underlying the physical concept of an elementary particle, with an emphasis on the classification of these particles. In particular, we have applied the resulting formalism to two different kinds of spacetime symmetry groups: the Galilei group and the Poincaré group (actually their identity components).

Summary 10.1. Let G be one of the spacetime symmetry groups (or any other connected Lie group, for that matter). A ***classical elementary particle*** is defined as a *transitive Hamiltonian G-space*, i.e., a *transitive Hamiltonian action* of G on some connected symplectic manifold. By Theorems 8.10 and 8.12 the classical elementary particles are classified by the *coadjoint orbits* of the universal central extension $G^\star$.

A ***quantum elementary particle*** is defined as an irreducible projective unitary representation of G. By Theorem 6.9 the quantum elementary particles are classified by *irreducible unitary representations* of $G^\star$. More practically, the quantum elementary particles are classified by irreducible unitary representations of central extensions of the universal cover $\widetilde{G}$ [4,25], where these central extensions are in turn classified by the second cohomology group $\mathrm{H}^2_{\mathrm{es}}(\widetilde{G}, \mathrm{U}(1)) \cong \mathrm{H}^2_{\mathrm{es}}(\widetilde{G}, \mathbb{R}) \cong \mathrm{H}^2_{\mathrm{al}}(\mathfrak{g}, \mathbb{R})$ (see Lemma 4.21 and Section 4.3.3).

In the classification of either classical- and quantum elementary particles we therefore need the universal central extension of the spacetime symmetry groups. For the calculation of the universal central extensions the second cohomology groups are an important part. We found in Section 5:

$$\mathrm{H}^2_{\mathrm{al}}(\mathfrak{gal}(3), \mathbb{R}) \cong \mathbb{R}, \qquad \mathrm{H}^2_{\mathrm{al}}(\mathfrak{poin}(1,3), \mathbb{R}) = 0.$$

Despite the Galilei algebra $\mathfrak{gal}(3)$ having an infinitude of non-trivial central extensions, we found in Section 5.1.2 that the Galilei *group* has only one non-trivial central extension (up to isomorphism). The universal central extensions read:

$$\mathrm{Gal}^\star_M(3) \cong \mathbb{R}^5 \rtimes_\rho \widetilde{\mathrm{SE}(3)} \cong \mathbb{R}^5 \rtimes_\rho \left(\mathbb{R}^3 \rtimes \widetilde{\mathrm{SO}(3)}\right)$$
$$\mathrm{Poin}^\star(1,3) \cong \mathbb{R}^4 \rtimes \widetilde{\mathrm{SO}^+}(1,3) \cong \mathbb{R}^4 \rtimes \mathrm{SL}(2,\mathbb{C}),$$

where ρ is as in (9.8), and $M \in \mathbb{R} \setminus \{0\}$ is the extension parameter of $\mathfrak{gal}_M(3)$.

We start with quantum elementary particles:

10.1 Quantum elementary particles

Summary 10.2. In Section 6.5 we state the classification of the irreducible unitary representations of the universal covering of the Galilei group. (For further references, please see [6,24,25] and [22, p.273].) The only physically relevant unitary representation is

$$u_{m,j}(t, \boldsymbol{a}, \boldsymbol{v}, U)\psi(\boldsymbol{p}) = e^{i\left(t\frac{\boldsymbol{p}^2}{2m} + \langle \boldsymbol{a}, \boldsymbol{p}\rangle\right)} D_j(U)\psi\left(\widetilde{p}(U)^{-1}(\boldsymbol{p} + m\boldsymbol{v})\right),$$

defined on the Hilbert space $L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2j+1}$. Here $D_j : \mathrm{SU}(2) \to \mathrm{U}(2j+1)$ are the irreducible representations of $\mathrm{SU}(2)$. The quantum elementary particles are therefore labelled by two numbers: m and j. The label m is called the ***mass*** of the particle, and can be any non-negative real number: $m \in \mathbb{R}_{\geqslant 0}$. Notably, the mass arises from the fact that the Galilei group has non-trivial central extensions, and indeed, a particle of mass m corresponds to unitary representations of the universal cover of $\mathrm{Gal}_m(3)$.

In the massive case ($m > 0$) the label j takes values in $\mathbb{N}/2 \cup \{0\}$, and is called the ***spin*** of the particle. In the massless case ($m = 0$) the label j instead takes values in $\mathbb{Z}/2$, and is called the ***helicity***. We consider the right representation for this case to be that of class 2. The representation is now of the form

$$u_{p,j}(s, \boldsymbol{a}, \boldsymbol{v}, U)\psi(E, \boldsymbol{p}) = e^{i(\langle \widetilde{p}(U)\boldsymbol{p}, \boldsymbol{a}\rangle - s(E + \langle \widetilde{p}(U)\boldsymbol{p}, \boldsymbol{v}\rangle) + j\alpha(U, \boldsymbol{p}))}\psi(E + \langle \widetilde{p}(U)\boldsymbol{p}, \boldsymbol{v}\rangle, \widetilde{p}(U)\boldsymbol{p}),$$

cf. [24, Eq.(IV.8)], where the Hilbert space is $L^2(\mathbb{R} \times S^2_p)$, for some $p \in \mathbb{R}_{>0}$.

Summary 10.3. In Section 6.6 we state the classification of the irreducible unitary representations of the universal cover of the Poincaré group. (For the original classification by Wigner, see [43]. We further refer to [21, Prop.IV.3.3.1].) The physically relevant representations are labelled by a number $m \in \mathbb{R}_{\geqslant 0}$, called the ***mass*** of the particle, and a discrete number j. The Hilbert space is $L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2j+1}$. For the massive case ($m > 0$) we have $j \in \mathbb{N}/2 \cup \{0\}$, and it is called the ***spin*** of the particle. The explicit form of the representation is then

$$u_{m,+,j}(a,\Delta)\psi(\boldsymbol{p}) = e^{i\left(a_0\sqrt{\boldsymbol{p}^2+m^2}-\langle \boldsymbol{a},\boldsymbol{p}\rangle\right)} D_j(b_{\boldsymbol{p}}^{-1}\Delta b_{\widetilde{p}(\Delta)^{-1}\boldsymbol{p}})\psi(\widetilde{p}(\Delta)^{-1}\boldsymbol{p}).$$

In the massless case ('$m = 0$') we have $j \in \mathbb{Z}/2$, where the Hilbert space is $L^2(S_0^{\pm})$. (For the explicit expression of the representation we refer to Section 6.6.)

10.2 Classical elementary particles

Next we have the classification of classical elementary particles, which is the same as the classification of the coadjoint orbits of the universal central extensions. Here we provide some more detail as to the origin of the interpretation, opening with Galilean particles:

10.2.1 Classical Galilean elementary particles

There are four types of coadjoint orbits through the universal central extension of the Galilei group:

$$\mathscr{O}^{0,r,s}_{\mathrm{Gal}^\star(3)}, \quad \mathscr{O}^{k,0,0}_{\mathrm{Gal}^\star(3)}, \quad \mathscr{O}^{k,r,0}_{\mathrm{Gal}^\star(3)}, \quad \mathscr{O}^{M,V,s}_{\mathrm{Gal}^\star(3)},$$

for $r, s \in \mathbb{R}_{\geqslant 0}$, $k \in \mathbb{R}_{>0}$, $V \in \mathbb{R}$ and $M \in \mathbb{R} \setminus \{0\}$. We pose that the physically most relevant orbit is $\mathscr{O}^{M,V,s}_{\mathrm{Gal}^\star(3)}$. Recall that the corresponding dual orbit reads (9.10):

$$\mathrm{Orb}^{M,V}_{\widetilde{\mathrm{SE}(3)}} := \left\{(\Theta, E, \boldsymbol{p}) \in \mathbb{R}^5 : E - \frac{\boldsymbol{p}^2}{2\Theta M} = V\right\}.$$

We interpret E as the total *energy* of the particle, $\boldsymbol{p}$ as the *momentum*, M as the *mass*, and V as the *internal energy*. Since the dual action of $\widetilde{\mathrm{SE}(3)}$ does not act on the first component of $(\Theta, E, \boldsymbol{p})$, see (9.9), and it acts merely as a scale factor for the momentum, we may set $\Theta = 1$ [25, p.250]. In that case, the defining relation of the dual orbit is interpreted as the classical energy-momentum relation between the total-, kinetic- and internal energy. Discarding the Θ component, the dual orbits are now four-dimensional parabolas whose offset from the origin is determined solely by V, so it is obvious that

$$\mathrm{Orb}^{M,V_1}_{\widetilde{\mathrm{SE}(3)}} \cong \mathrm{Orb}^{M,V_2}_{\widetilde{\mathrm{SE}(3)}}$$

for any two $V_1, V_2 \in \mathbb{R}$, and hence we set $V = 0$, without any loss of generality. Physically speaking, we interpret this as the non-absoluteness of internal energy as a quantity, meaning that the internal energy of an elementary particle is not a defining property. We relabel the coadjoint orbit $\mathscr{O}^{M,0,s}_{\mathrm{Gal}^\star(3)}$ to $\mathscr{O}^{m,s}_{\mathrm{Gal}^\star(3)}$, where $s \in \mathbb{R}_{\geqslant 0}$ is the ***spin*** and $m := M$ is the ***mass*** of the particle, recalling the extension label M of the centrally extended Galilei group. It is seen that the non-trivial central extensions of the Galilei group are responsible for the non-zero mass of these elementary particles. We similarly relabel the dual orbit:

$$\mathrm{Orb}^{m}_{\widetilde{\mathrm{SE}(3)}} := \left\{(E, \boldsymbol{p}) \in \mathbb{R}^4 : E - \frac{\boldsymbol{p}^2}{2m} = 0\right\} \cong \mathbb{R}^3.$$

We can now view the coadjoint orbit $\mathscr{O}^{m,s}_{\mathrm{Gal}^\star(3)}$ in a much simpler light:

$$\mathscr{O}^{m,s}_{\mathrm{Gal}^\star(3)} \cong T^*\,\mathrm{Orb}^{m}_{\widetilde{\mathrm{SE}(3)}} \times \mathscr{O}^{s}_{\mathrm{SO}(3)} \cong T^*\mathbb{R}^3 \times S^2_s \cong \mathbb{R}^6 \times S^2_s.$$

Since the coadjoint orbits are (up to isomorphism) the state spaces of elementary particles, we see that we have one component of the state space in S_s^2, representing the *intrinsic angular momentum* (whose absolute value is the spin), and a six-dimensional vector $(\boldsymbol{q}, \boldsymbol{p})$ in the cotangent bundle $T^* \operatorname{Orb}^m_{\widetilde{\mathrm{SE}(3)}} \cong \mathbb{R}^6$, where $\boldsymbol{q}$ is interpreted as the *position*, and $\boldsymbol{p}$ again as the momentum.

It remains for us to interpret the coadjoint orbits $\mathscr{O}^{0,r,s}_{\mathrm{Gal}^\star(3)}$, $\mathscr{O}^{k,0,0}_{\mathrm{Gal}^\star(3)}$ and $\mathscr{O}^{k,r,0}_{\mathrm{Gal}^\star(3)}$ that correspond to '$M = 0$', which, given the above interpretation of this label as the mass, we would like to interpret as *massless* particles. Such an interpretation is given in [13, §52,pp.440-441], which we quote. The orbits $\mathscr{O}^{0,r,s}_{\mathrm{Gal}^\star(3)}$ correspond to "particles of zero mass and finite velocity at infinity." The orbits $\mathscr{O}^{k,r,0}_{\mathrm{Gal}^\star(3)}$ are "particles at infinity with infinite velocity and mass zero." Lastly, we have the most physically relevant orbit: $\mathscr{O}^{k,0,0}_{\mathrm{Gal}^\star(3)}$, which correspond to "particles of zero mass and infinite velocity."

10.2.2 Classical Poincaré elementary particles

As a last part of the classification, we consider the classical relativistic elementary particles. We found four classes of coadjoint orbits of the (universal cover of the) Poincaré group; one corresponding to the coadjoint orbits of the Lorentz group, three others:

$$\mathscr{O}^{m,\pm,s}_{\mathrm{Poin}(1,3)}, \quad \mathscr{O}^{0,\pm,h}_{\mathrm{Poin}(1,3)}, \quad \mathscr{O}^{0,\pm,h,r}_{\mathrm{Poin}(1,3)}$$

and one more class corresponding to the coadjoint orbits of the two-dimensional Lorentz group. Only the second and third classes are believed to have physical significance [21, Prop.III.3.1.1]. (The coadjoint orbits arising from the two-dimensional Lorentz group are called *tachyonic*.)

First we consider the coadjoint orbits $\mathscr{O}^{m,\pm,s}_{\mathrm{Poin}(1,3)}$ corresponding to the dual orbits

$$\operatorname{Orb}^{m,\pm}_{\mathrm{SO}^+(1,3)} = \{(E, \boldsymbol{p}) \in \mathbb{R}^4 : E^2 - \boldsymbol{p}^2 = m^2, \pm E > 0\} =: S^\pm_m,$$

where we recall notation from Section 6.6. Just as for the Galilei group, E is interpreted as the total *energy*, $\boldsymbol{p}$ is interpreted as *momentum* so that $(E, \boldsymbol{p})$ is the *four-momentum*, and m is interpreted as the *mass*. The defining relation of the dual orbit is recognised as the relativistic mass-energy-momentum relation (in natural units). The orbit themselves are one sheet hyperboloids, which are isomorphic to $\mathbb{R}^3$. The coadjoint orbit becomes

$$\mathscr{O}^{m,\pm,s}_{\mathrm{Poin}(1,3)} \cong \mathbb{R}^6 \times S^2_s,$$

which corresponds to the state space of a particle with ***mass*** $m \in \mathbb{R}_{>0}$ and ***spin*** $s \in \mathbb{R}_{\geqslant 0}$. The other coadjoint orbits then correspond to *massless* particles. We have the coadjoint orbits

$$\mathscr{O}^{0,\pm,h}_{\mathrm{Poin}(1,3)} \cong \mathbb{R}^6$$

which describe massless particles with ***helicity*** $h \in \mathbb{R}$. The reason that the state spaces for the massless particles look like the ones for massive spinless particles is that the direction of the helicity is always along (or against) the momentum. Hence there is no need for an additional component to specify the direction of the helicity.

10.3 Discrete spin in classical mechanics

We have an incredible analogue between classical and quantum mechanics. In both cases the elementary particles are labelled by two numbers: the mass and spin (or helicity). The main difference is that in quantum mechanics the spin is *discrete*, meaning that in particular it takes half integer values, whereas in classical mechanics it is not. To get a 'true' analogy, we can restrict ourselves to *integral coadjoint orbits*:

Definition 10.4. Let G be a connected Lie group. We say a coadjoint orbit $\mathscr{O}_G(F_1) \subseteq \mathfrak{g}^*$ is ***integral*** if for some (and hence all) $F \in \mathscr{O}_G(F_1)$ its restriction to its own stabiliser algebra $F|_{\mathfrak{g}_F}$ is of the form

$$F(X) = i\,\frac{\mathrm{d}}{\mathrm{d}t}\chi(e^{tX})\bigg|_{t=0}$$

for each $X \in \mathfrak{g}_F$, and for some character $\chi : G_F \to \mathrm{U}(1)$. (Cf. [22, Def.5.47].)

In [22, p.163] it is shown that this integrability condition ensures that the classical spin s is also a non-negative half-integer, just as the quantum spin.

10.4 Elementary particles of the Standard Model

We have found that elementary particles are defined by two labels: the mass m, and for massive particles the spin s, while for massless particles we have the helicity h. For the quantum elementary particles, the spin and helicity are half-integer valued. In the Standard Model we know that every particle does indeed have half-integer valued spin. There are four types of massive particles ($m > 0$): for spin $s = \frac{1}{2}$ we have the *leptons* and *quarks*, which are the main constituents of 'ordinary matter'. The *Higgs boson* is a massive spinless particle: $s = 0$. Lastly, we have the *electroweak bosons*, with spin $s = 1$. Besides the massive particles, we have three types of massless particles ($m = 0$). Two of them have helicity $h = \pm 1$, which are the *photons* and *gluons*. Finally, we have the elusive *gravitons*, which are massless particles of helicity $h = \pm 2$.

We quote [22, p.272]: "On the one hand, this classification is a triumph of mathematical physics, but on the other hand, it fails to single out which cases actually occur in nature..." It should be noted, however, that there are various 'composit particles', i.e., particles that are composed of the elementary ones, which may have mass and spin other than those of the elementary particles (e.g., spin $s = \frac{5}{2}$).

	Quantum		**Classical**	
	$m > 0$, $j \in \mathbb{N}/2 \cup \{0\}$	$m = 0$, $h \in \mathbb{Z}/2$	$m > 0$, $s \in \mathbb{R}_{\geqslant 0}$	$m = 0$, $h \in \mathbb{R}$
Galilean	$L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2j+1}$	$L^2(\mathbb{R} \times S^2_p)$	$\mathbb{R}^6 \times S^2_s$	$\mathbb{R}^2 \times T^*S^1_{\lvert h \rvert}$
Poincaré	$L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2j+1}$	$L^2(S^{\pm}_0)$	$\mathbb{R}^6 \times S^2_s$	$\mathbb{R}^6$

Table 1: The classification of quantum- and classical elementary particles in Galilei- and Poincaré spacetime, with the corresponding state spaces. The label m is the *mass*, the labels j, s are the *spin*, and h is *helicity*.

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