NEWTON AND HILBERT ON THE FOUNDATIONS OF GEOMETRY

A CASE STUDY IN THE PHILOSOPHY OF MATHEMATICS

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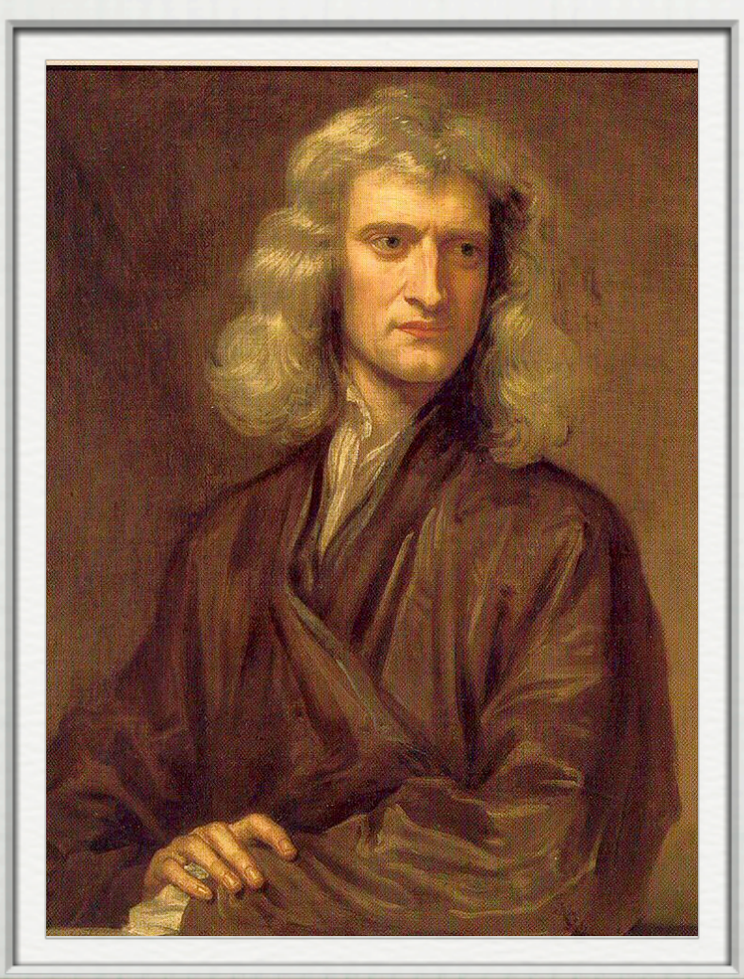


Nearer the Gods, no mortal may approach

Edmund Halley, Editor's Preface to *Principia* (1687)

The end result of my study of Newton has served to convince me that with him there is no measure. He has become for me wholly other, one of the tiny handful of supreme geniuses who have shaped the categories of the human intellect, a man not finally reducible to the criteria by which we comprehend our fellow beings.

Richard Westfall, *Never at Rest: A Biography of Isaac Newton* (CUP, 1980)

Hilbert's so-called Festschrift [Grundlagen der Geometrie, 1899] is perhaps one of the most celebrated works in the history of mathematics.

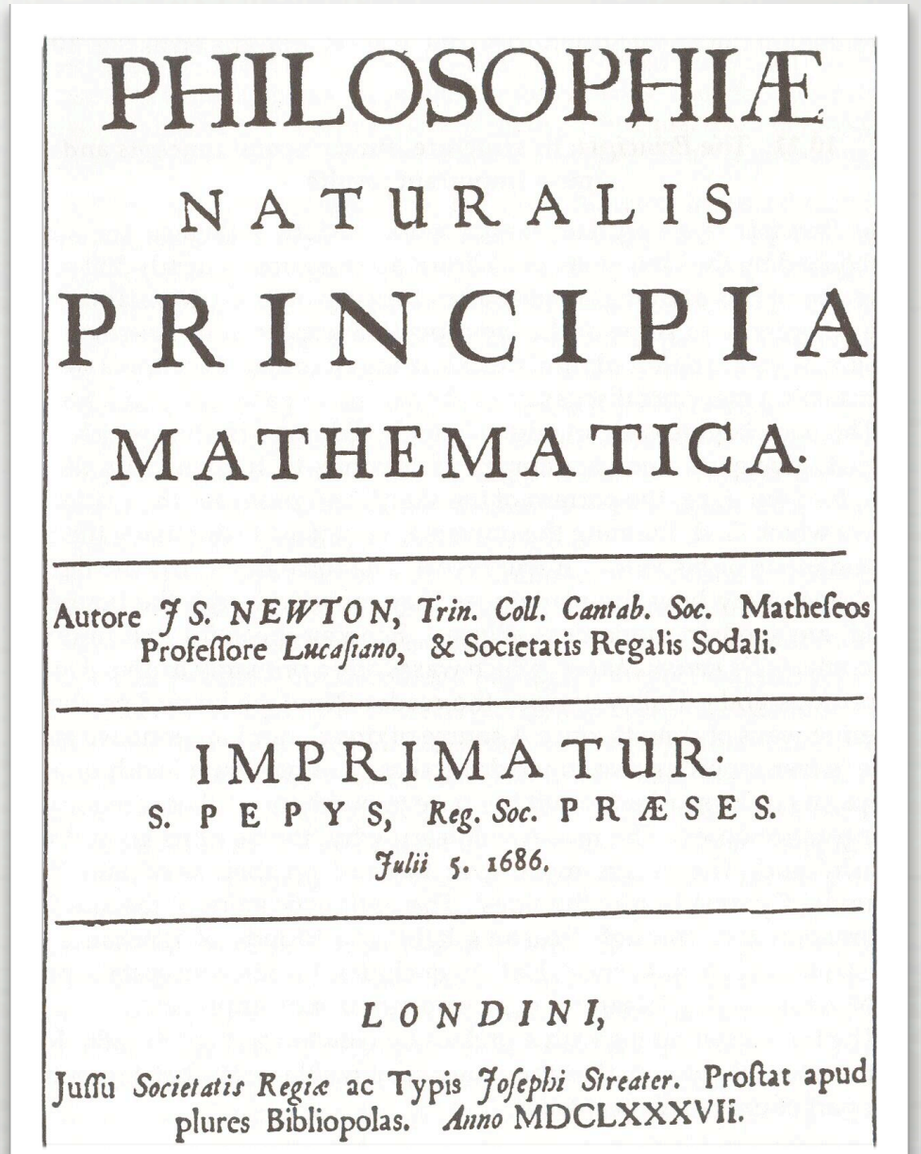
Michael Hallett & Ulrich Majer, *David Hilbert's Lectures on the Foundations of Geometry 1891-1902* (Springer, 2004)

Grundlagen der Geometrie stands as a watershed in the development of modern mathematics and logic. Though the subject-matter of the work is geometry, its lasting influence concerns more broadly the role of axioms in mathematical theories, and the systematic treatment of such metatheoretical questions as consistency and independence.

Patricia Blanchette, The Frege-Hilbert Controversy (SEP, 2007)

PRINCIPIA

- ☐ OPPOSED TO DESCARTES' PRINCIPIA PHILOSOPHIAE (1644)
- ☐ ROLE MODEL: HUYGENS (1673) "VIR SUMMUS HUGENIUS"
- ☐ GRAND AIM: "INJECTION OF CERTAINTY INTO NATURAL PHILOSOPHY THROUGH MATHEMATICS" (GUICCIARDINI)
- ☐ "UNCOMPROMISING CONVICTION CONCERNING THE PRIMACY OF MATHEMATICS IN THE DOMAIN OF NATURAL PHILOSOPHY" (FEINGOLD)



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore *J S. NEWTON, Trin. Coll. Cantab. Soc.* Matheſeos Profeſſore *Lucaſiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,

Juſſu *Societatis Regiæ* ac Typis *Joſephi Streater.* Proſtat apud plures Bibliopolas. *Anno* MDCLXXXVII.

The ancients considered mechanics in a twofold respect; as rational, which proceeds accurately by demonstration; and practical. To practical mechanics all the manual arts belong, from which mechanics took its name. But as artificers do not work with perfect accuracy, it comes to pass that mechanics is so distinguished from geometry, that what is perfectly accurate is called geometrical; what is less so, is called mechanical. But the errors are not in the art, but in the craftsmen. He that works with less accuracy is an imperfect mechanic; and if any could work with perfect accuracy, he would be the most perfect mechanic of all [mechanicus omnium perfectissimus i.e. God]*; for the description of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn; for it requires that the learner should first be taught to describe these accurately, before he enters upon geometry; then it shows how by these operations problems may be solved. To describe right lines and circles are problems, but not geometrical problems. The solution of these problems is required from mechanics; and by geometry the use of them, when so solved, is shown; and it is the glory of geometry that from those few principles, brought from without, it is able to produce so many things. Therefore geometry is founded in mechanical practice (...)*

NEWTON, PRINCIPIA (1687), AUTHOR'S PREFACE

NEWTON VERSUS TRADITION

- TRADITIONAL VIEW:
  MECHANICS IS APPLICATION OF GEOMETRY (AS ALREADY ESTABLISHED) TO MOTION

  AXIOMS ARE SELF-EVIDENT

- NEWTON: GEOMETRY IS PREDICATED ON MECHANICS

  AXIOMS ARE VALIDATED BY MOTIONS IN NATURE

  ALL SUCH MOTIONS PROVIDE GEOMETRY WITH ITS SUBJECT MATTER (AGAINST DESCARTES)

NEWTON'S RADICAL EMPIRICISM

- *In all philosophy we must begin from phenomena and admit no principles of things, no causes, no explanations, except those which are established through phenomena (1713)*
- *to treat God from phenomena is certainly a part of natural [experimental] philosophy (Principia,* 1687)
- *Newton does not merely reject the mechanical philosophy of Descartes, Leibniz and Huygens; he transforms what they take to be a purely* a priori *questions (...) into empirical questions (...)*

  The resulting picture, then, is of a Newton whose rejection of Cartesian metaphysical foundations for physics and empirical research is so thoroughgoing that he rejects the very idea of a priori *knowledge, even of the most basis facts about the divine being (...) [and mathematics!]. [Newton] adopts the radical view that our knowledge of such matters is purely empirical.*

  Andrew Janiak, *Newton as Philosopher* (2008)

GRUNDLAGEN DER GEOMETRIE

GOALS OF HILBERT'S GRUNDLAGEN DER GEOMETRIE:

- **COMPLETE SET OF AXIOMS FOR EUCLIDEAN GEOMETRY**
  EUCLID NOT STRICTLY AXIOMATIC-DEDUCTIVE: OFTEN GAPS AND APPEALS TO INTUITION
- **PROOF OF (RELATIVE) CONSISTENCY OF THESE AXIOMS**
  SET OF AXIOMS A IS CONSISTENT IF NO CONTRADICTION CAN BE DERIVED;
  CONSISTENCY OF A RELATIVE TO THEORY T (E.G. REALS) PROVED BY INTERPRETING A IN T
- **STUDY OF INDEPENDENCE OF AXIOMS**
  E.G. PARALLEL POSTULATE; INDEPENDENCE PROVED THROUGH CONSISTENCY

KEY LOGICAL INNOVATIONS BY HILBERT:

- NON-LOGICAL TERMS IN AXIOMS (POINT, LINE, LIES ON, ..) ARE UNINTERPRETED ("PLACE-HOLDERS", NO MEANING)
- AXIOMS HAVE NO SUBJECT MATTER AND NO TRUTH-VALUE
- NON-LOGICAL TERMS CAN BE INTERPRETED IN MANY WAYS; TRUTH OF AXIOM DEPENDS ON SPECIFIC INTERPRETATION

HILBERT TO FREGE, 29.12.1899

Sie sagen weiter: "Ganz anders sind wohl die Erklärungen in Par. 1, wo die Bedeutungen Punkt, Gerade, ... nicht angegeben, sondern als bekannt vorausgesetzt werden". Hier liegt wohl der Cardinalpunkt des Misverständnisses. Ich will nichts als bekannt voraussetzen; ich sehe in meiner Erklärung in Par. 1 die Definition der Begriffe Punkt, Gerade, Ebenen, wenn man wieder die sämtlichen Axiome (...) als die Merkmale hinzunimmt. (...) Sie schreiben: "Aus der Wahrheit der Axiome folgt, dass sie einander nicht widersprechen." Ich sage gerade umgekehrt: Wenn sich die willkürlich gesetzten Axiome nicht einander widersprechen mit sämtlichen Folgen, so sind sie wahr, so existieren die durch die Axiome definierten Dinge. Das ist für mich das Criterium der Wahrheit und der Existenz. (...) Wenn ich unter meinen Punkten irgendwelche Systeme von Dingen, z.B. das System: Liebe, Gesetz, Schornsteinfeger ..., denke und dann meine sämtlichen Axiome als Beziehungen zwischen diesen Dingen annehme, so gelten meine Sätze, z.B. der Pythagoras, auch von diesen Dingen.

HILBERT'S DILEMMA

- HILBERT WANTS IT BOTH WAYS:
  - CONCLUSIONS OF MATHEMATICS ARE CERTAIN
  - YET ITS SUBJECT MATTER IS EMPIRICAL IN ORIGIN

  "WHEN SOME ARBITRARILY POSED AXIOMS DO NOT CONTRADICT EACH OTHER AND THEIR CONSEQUENCES, THEY ARE TRUE, AND THE THINGS DEFINED BY THESE AXIOMS EXIST. THIS IS MY CRITERION OF TRUTH AND EXISTENCE" (LETTER TO FREGE, 1899)

  "EACH GROUP [OF HILBERT'S AXIOMS OF GEOMETRY] EXPRESSES CERTAIN INTERCONNECTED BASIC FACTS ABOUT OUR EXPERIENCE"(GDG, 1899)

- COMPATIBILITY SUGGESTED BY MOTTO TO GRUNDLAGEN DER GEOMETRIE:

  "SO FÄNGT DENN ALLE MENSCHLICHE ERKENNTNIS MIT ANSUCHAUUNGEN AN, GEHT VON DA ZU BEGRIFFEN UND ENDET MIT IDEEN (KANT, KRITIK DER REINEN VERNUNFT)

NEWTON'S DILEMMA

- NEWTON WANTS IT BOTH WAYS, TOO:
  - SUBJECT MATTER OF MATHEMATICS IS EMPIRICAL
  - YET ITS CONCLUSIONS ARE CERTAIN

  "GEOMETRY IS FOUNDED IN MECHANICAL PRACTICE" (1686)

  "FOR THE FORCE OF GEOMETRY AND ITS EVERY MERIT LAY IN THE UTTER CERTAINTY OF ITS MATTERS, AND THAT CERTAINTY [LIES] IN ITS SPLENDIDLY COMPOSED DEMONSTRATIONS" (LATE 1710'S)

  "I THEREFORE URGE GEOMETERS TO INVESTIGATE NATURE MORE RIGOROUSLY, AND THOSE DEVOTED TO NATURAL SCIENCE TO LEARN GEOMETRY FIRST. HENCE THE FORMER SHALL NOT ENTIRELY SPEND THEIR TIME IN SPECULATIONS OF NO VALUE TO HUMAN LIFE, NOR SHALL THE LATTER, WHILE WORKING ASSIDUOUSLY WITH AN ABSURD METHOD, PERPETUALLY FAIL TO REACH THEIR GOAL" (1670)

- COMPATIBILITY ARISES FROM "MECHANICUS OMNIUM PERFECTISSIMUS"

Does Newton emerge as a creative and innovative philosopher of mathematics? Clearly, the answer is no. (...) Newton was a mathematician, and as many great mathematicians he was an innovator, an opportunist (...) His mathematical practice is unsystematic, it appears as a patchwork of problem-solving techniques which is difficult to schematize. (...) Newton's position is ultimately extremely contradictory.

Niccolò Guicciardini, *Isaac Newton on Mathematical Certainty and Method* (2009)

Hilbert was primarily a mathematician and as a mathematician he was frequently impatient with philosophers. So although his concern for justifying mathematical practice led him to make extended philosophical remarks concerning the nature of mathematics, these remarks do not amount to a systematically articulated position.

Mary Tiles, *Mathematics and the Image of Reason* (1991)

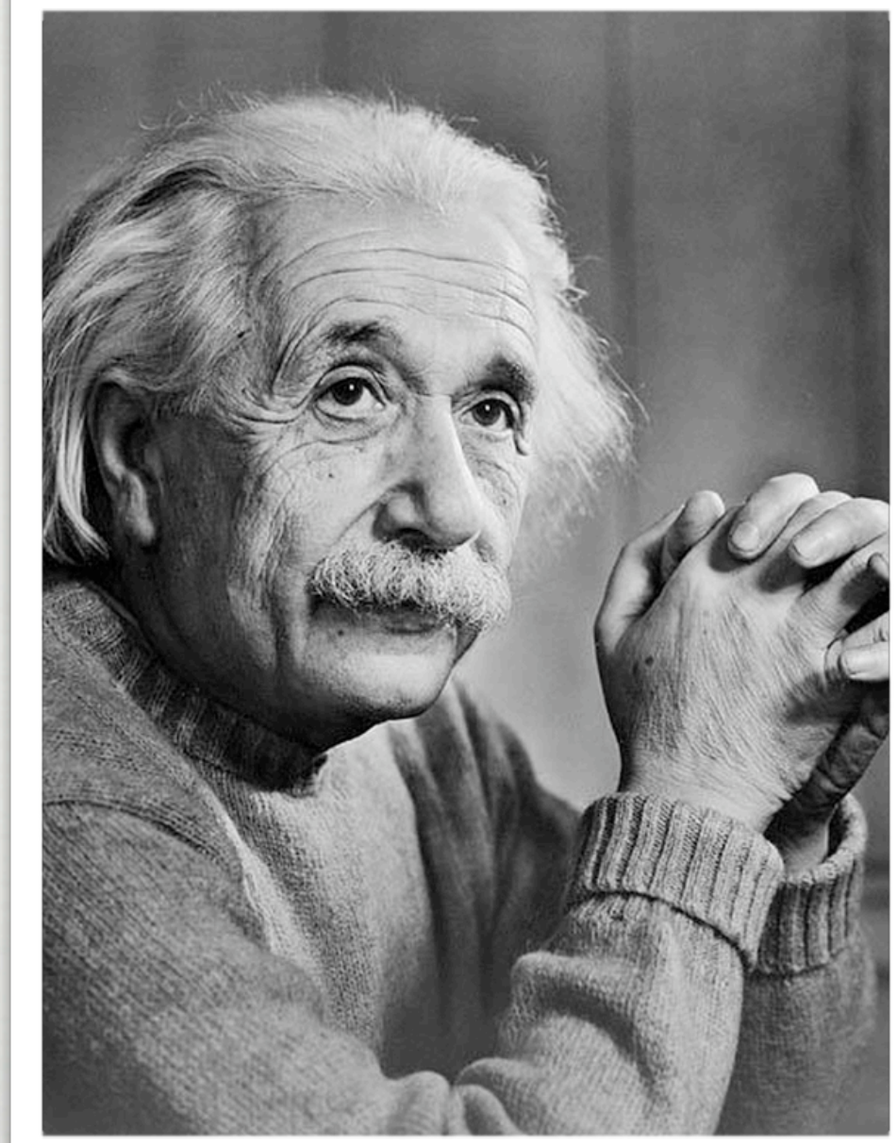
EINSTEIN'S SYNTHESIS

> As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality. (...)

The matter of which geometry treats is first defined by the axioms. These axioms are free creations of the human mind. (...)

To be able to make assertions about the relations of real objects, geometry must be stripped of its merely logical-formal character by the co-ordination of real objects of experience with the empty conceptual frame-work of axiomatic geometry. (...) Geometry thus completed is evidently a natural science.

(Geometry and Experience, 1921)



VON NEUMANN'S SYNTHESIS

POST-GÖDELIAN DIAGNOSIS: MATHEMATICAL RIGOR VARIABLE AND TIME-DEPENDENT NOTION

GOAL OF AXIOMATIZATION (AT LEAST OF PHYSICAL THEORIES) IS NEITHER TO ACHIEVE RIGOR NOR TO DERIVE TRUE STATEMENTS

IT IS A PRAGMATIC PROCEDURE INTENDED TO ISOLATE AND MATHEMATIZE THE BASIC ASSUMPTIONS OF A SCIENTIFIC MODEL OF (SOME PART OF) NATURE

HEALTHY MATHEMATICAL THEORIES ARISE IN THIS WAY

[I]t is *not* necessarily true that the mathematical method is something absolute, which was revealed from on high, or which somehow, after we got hold of it, was evidently right and has stayed evidently right ever since. To be more precise, maybe it *was* evidently right after it was revealed, but it certainly didn't stay evidently right ever since. There have been very serious fluctuations in the professional opinion of mathematicians on what mathematical rigor is. To mention one minor thing: In my own experience, which extends over only some thirty years, it has fluctuated so considerably, that my personal and sincere conviction as to what mathematical rigor is, has changed at least twice. And this is in a short time of the life of one individual! (von Neumann (1961b), p. 480)

> The variability of the concept of rigor shows that something else besides mathematical abstraction must enter into the makeup op mathematics [namely] its quite peculiar relationship to any science which interprets experience on a higher than purely descriptive level. (...) As a mathematical discipline travels far from its empirical source, it is beset with grave dangers. It becomes more and more purely aestheticizing, more and more purely *l'art pour l'art*.

EPILOGUE: HILBERT'S SIXTH PROBLEM

"THE INVESTIGATIONS ON THE FOUNDATIONS OF GEOMETRY SUGGEST THE PROBLEM: TO TREAT IN THE SAME MANNER, BY MEANS OF AXIOMS, THOSE PHYSICAL SCIENCES IN WHICH MATHEMATICS PLAYS AN IMPORTANT PART, IN THE FIRST PLACE PROBABILITY THEORY AND MECHANICS" (PARIS, 1900)

SUCCESSFUL EXAMPLES:

- ☐ SPECIAL THEORY OF RELATIVITY (MINKOWSKI, 1909)
- ☐ GENERAL THEORY OF RELATIVITY (HILBERT, 1915)
- ☐ QUANTUM MECHANICS (VON NEUMANN, 1932)
- ☐ PROBABILITY THEORY (KOLMOGOROV, 1933)

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