DIMENSIONAL REDUCTION AT HIGH TEMPERATURE REVISITED

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Received 20 February 1989

We give a perturbative analysis of dimensional reduction in the infinite-temperature limit of thermal field theory in the imaginary-time formalism. In analogy with the ordinary (Appelquist–Carazzone) decoupling theorem, we seek a class of renormalization schemes in which the heavy (non-static) modes decouple. This class turns out to consist of temperature-dependent schemes, so that the appropriate context to study dimensional reduction at high T is seen to be the finite-temperature renormalization group. As a corollary, it follows that dimensional reduction to all orders occurs only in exceptional cases; in scalar field theories and QCD such a reduction is valid only up to a given (low) order of renormalization-group improved perturbation theory.

1. Introduction

It has been stated by many authors [1–7] that in the infinite-temperature limit thermal field theories exhibit an effective reduction from four to three dimensions. This dimensional reduction (DR) is usually analyzed in perturbation theory, and is argued to take place on account of the decoupling theorem of Appelquist and Carazzone (AC) [8].

The basic reasoning is as follows: in the free propagators $[\boldsymbol{k}^2+m^2+(2\pi nT)^2]^{-1}$ of imaginary-time perturbation theory, the term $2\pi nT$ acts like a "mass" in a three-dimensional theory. According to the AC theorem all non-static modes ($n\neq 0$) decouple in the limit $T\to\infty$, leaving the static (=three-dimensional) sector as the effective theory.

However, one should realize that the AC theorem is only valid in a particular class of renormalization schemes. As a matter of fact, it turns out that in trying to demonstrate the decoupling of the non-static modes, one is forced to adopt a temperature-dependent renormalization scheme (RS). Consequently, one has T-dependent renormalized parameters, e.g. $m=m(T)$, etc. In such an RS a generalized AC theorem (i.e. extended to an infinite set of masses) can be shown to hold up to terms of the order m^2/T^2 [9,10]. Dimensional reduction, then, actually takes place only if $\lim_{T\to\infty} m(T)/T=0$. Insertion of the well-known [4] "generated mass", $m\propto\lambda^N T$, for static fields shows that this is not expected to be the case.

The main purpose of this paper is to show that DR should be investigated by fully taking into account the consequences of the renormalization group (RG) at finite temperature. In particular, the precise definition of temperature-dependent masses and their implications for DR will be discussed.

2. Dimensional reduction and renormalization

To explain the important role of thermal renormalization in the subject of DR, we investigate a ϕ^4 theory with a bare field ϕ_B. Following Jourjine [5], we rewrite the euclidean action, defined as a functional of periodic fields [11], as a three-dimensional vacuum theory with an infinite number of massive fields. If we turn directly to renormalized fields, this is achieved by substituting

$$\phi_B(\tau,\boldsymbol{x})=Z_3^{1/2}T^{1/2}\sum_{n=-\infty}^{+\infty}\phi_n(\boldsymbol{x})\exp(\mathrm{i}\omega_n\tau), \qquad (1)$$

where $\omega_n=2\pi nT$ are the well-known Matsubara frequencies [4,11]. The renormalization program is

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(North-Holland Physics Publishing Division)

completed by introducing the renormalized mass and coupling constant as $m_B^2 = Z_1 Z_3^{-1} m^2$ and $\lambda_B = Z_2 \times Z_3^{-2}\lambda$.

Perturbation theory is set up with an action cast into the form

$$S_E = \int d^3x \, (\mathscr{L}_f^s + \mathscr{L}_b^s + \mathscr{L}_{ct}^s + \mathscr{L}_f^n + \mathscr{L}_b^n + \mathscr{L}_{ct}^n). \quad (2)$$

The static (s) and nonstatic (n) free lagrangians, given by

$$\mathscr{L}_f^s = \int d^3x \, [\tfrac{1}{2}(\nabla\phi_0)^2 + \tfrac{1}{2}m^2\phi_0^2],$$

$$\mathscr{L}_f^n = \int d^3x \, \tfrac{1}{2} \sum_{n\neq 0} [\nabla\phi_n \nabla\phi_{-n} + (m^2+\omega_n^2)\phi_n\phi_{-n}], \quad (3)$$

respectively, are used to generate the free propagators of the theory. The interaction part ("basic lagrangian" [12]) $\mathscr{L}_b^s = (\lambda T/4!)\phi_0^4$ produces the purely static interaction vertex, and $\mathscr{L}_b^n = (\lambda T/4!) \times \sum'_{n_i} \delta(n_1+n_2+n_3+n_4)\phi_{n1}\phi_{n2}\phi_{n3}\phi_{n4}$ yields the non-static and mixed ones. Here $\sum'$ denotes that the static term is excluded.

The subtractions, which remove the ultra-violet (UV) divergences, and, if possible, also the leading corrections to dimensional reduction (see below), are implemented by counterterms. For the static part we have

$$\mathscr{L}_{ct}^s = \tfrac{1}{2}(Z_3-1)(\nabla_i\phi_0)^2 + \tfrac{1}{2}(Z_1-1)m^2\phi_0^2 + \frac{1}{4!}(Z_2-1)\lambda T\phi_0^4, \quad (4)$$

and it is straightforward to write down the expression for $\mathscr{L}_{ct}^n$.

It is to be remarked that this theory with an infinite number of fields is renormalizable, because it is essentially a four-dimensional finite-temperature field theory. At $T>0$ there will be no UV divergences additional to the $T=0$ ones. In particular, renormalizations at different temperatures T are connected by finite reparametrizations [13,14]. The renormalization constants Z_i are determined by suitable normalization conditions on the amputated one-particle-irreducible (1PI) two- and four-point vertex functions in momentum space with vanishing external energies. At finite T such conditions do not have a direct physical interpretation [15]. We stress that the renormalized parameters λ and m are defined by, and thus depend on, the particular choice of these conditions, which uniquely define a RS.

In the present case, the counterterms in principle consist of two independent terms, i.e. $Z_i - 1 = \delta Z_i^s + \delta Z_i^n$, where δZ_i^s receives only contributions from purely static graphs, while δZ_i^n corresponds to the additional non-static mode summations. For DR to take place it is necessary that non-negligible contributions from the latter to static Green functions are removed by a proper choice of the non-static renormalization constants δZ_i^n. As already stated, this requirement leads to a temperature-dependent prescription. Obviously, there is no specific constraint for the static renormalization constants δZ_i^n.

Before turning to explicit calculations, we will give an RS which has proved to be very useful in the treatment of DR [16,10,8]. This scheme consists of a minimal subtraction (MS) for the purely static graphs at scale ν, while 1PI (sub)graphs containing at least one non-static internal propagator are subtracted at momentum scale μ and at temperature τ. In terms of 1PI vertex functions $\Gamma_0^{(n)}$ with n static external lines, this scheme implies the normalization conditions

$$\Gamma_0^{(2)}(\boldsymbol{p}^2=\mu^2, \tau) = \Gamma_0^{(2)}(\boldsymbol{p}^2=\mu^2, \tau)_{\text{static,MS}}, \quad (5)$$

$$\frac{\partial}{\partial \boldsymbol{p}^2}\Gamma_0^{(2)}(\boldsymbol{p}^2=\mu^2, \tau) = \frac{\partial}{\partial \boldsymbol{p}^2}\Gamma_0^{(2)}(\boldsymbol{p}^2=\mu^2, \tau)_{\text{static}}, \quad (6)$$

$$\Gamma_0^{(4)}(s=t=u=\tfrac{4}{3}\mu^2, \tau) = \Gamma_0^{(4)}(s=t=u=\tfrac{4}{3}\mu^2, \tau)_{\text{static}}, \quad (7)$$

with $s=(\boldsymbol{p}_1+\boldsymbol{p}_2)^2$, $t=(\boldsymbol{p}_1-\boldsymbol{p}_3)^2$ and $u=(\boldsymbol{p}_1-\boldsymbol{p}_4)^2$.

The RG invariance of the bare theory leads to renormalization group equations (RGE) for the renormalized vertex functions. Extending the analysis in the real-time formalism [14,17] to the imaginary-time case, we simply have

$$\left(\frac{N}{2}\alpha_\tau + \beta_\tau\frac{\partial}{\partial\lambda} + \gamma_\tau m^2\frac{\partial}{\partial m^2} + \tau\frac{\partial}{\partial\tau}\right)\Gamma_{n_1\ldots n_N}^{(N)} = 0 \quad (8)$$

for a 1PI function carrying N external legs with frequencies $n_1, \ldots, n_N$. A similar equation holds with τ replaced by μ. The RG functions are

$$\alpha_\tau = -\tau\frac{\partial}{\partial\tau}\log Z_3, \quad \beta_\tau = \tau\frac{\partial\lambda}{\partial\tau}, \quad \gamma_\tau = m^{-2}\tau\frac{\partial m^2}{\partial\tau}, \quad (9)$$

and similar expressions for α_μ, etc. These RG functions can be derived in a standard manner from the renormalization constants, which in turn can be found by computing the two- and four-point vertex functions with static external legs, satisfying the conditions (5)–(7). Solving (9) gives the renormalized parameters $m=m(\tau)$, etc.

3. One-loop results

To show that DR indeed takes place one must show that Feynman graphs with internal non-static propagators can be neglected relative to static graphs, which are generated by the three-dimensional static action. We obviously need a calculational method for the divergent expressions, which admits a clean separation between regularized static and non-static contributions. The standard way of using a contour integral [11] is not suited here, because it leads to a regularization of the total sum. Consequently, direct comparison with calculations in the static theory, with its own different regularization scheme, is out of the question. Therefore, we introduce the following way of calculating graphs. We first perform the momentum integrations using dimensional regularization. Hereby we introduce a scale ν, and set $D=3-2\epsilon$ ($\epsilon>0$). Then the non-static mode summation, which is now automatically regularized, is performed, after which one can make an expansion in m/T, or alternatively in T/m, and take $\epsilon\to 0$ [9]. Instead, for simplicity, we will first make the high-temperature expansion and then perform the summation, cf. refs. [18,19]. This procedure rigorously leads to the same result [9].

For the non-static self-energy tadpole diagram with vanishing external frequency we obtain

$$\Gamma_0^{(2)}(T)_{\rm ns} =\lambda\left\{\frac{T^2}{24}+\frac{m^2}{32\pi^2}\left[\frac{2}{D-3}-\gamma+\ln\left(\frac{4\pi T^2}{\nu^2}\right)\right] +\mathrm{O}\left(\frac{m^2}{T^2}\right)\right\}. \qquad (10)$$

It is remarkable that the pole comes from the ζ-function rather than a Γ-function, as in $D=4$. The static mode gives $\Gamma_0^{(2)}(T)_{\rm s}=-\lambda mT/8\pi$.

An application of the RS outlined in the previous section shows that we must choose $\tau=T$ in order to remove the non-suppressed term $\propto T^2$ in (10).

In a similar way one may compute the four-point function as

$$\Gamma_0^{(4)}(s,t,u,T)_{\rm ns} =\frac{3\lambda^2T}{32\pi^2}\nu^{3-D}\left[\frac{2}{D-3}-\gamma+\ln\left(\frac{4\pi T^2}{\nu^2}\right) -\frac{\zeta(3)}{16\pi^2T^2}\left[m^2+\tfrac{1}{18}(s+t+u)\right] +\mathrm{O}\left(\frac{m^4}{T^4},\frac{s^2}{T^4},\frac{t^2}{T^4},\frac{u^2}{T^4}\right)\right], \qquad (11)$$

and for the static part one finds

$$\Gamma_0^{(4)}(s,t,u,T)_{\rm s} =\frac{-\lambda^2T^2}{8\pi s^{1/2}}\arcsin\left[\left(1+\frac{4m^2}{s}\right)^{-1/2}\right] +(s\to t)+(s\to u). \qquad (12)$$

The renormalized non-static contribution is minimized by choosing $\tau=T$, $\mu=0$. From these subtractions for the non-static parts of the graphs it is straightforward to find the RG functions (9). Using the RG invariance of the bare mass and coupling, as well as formula (9) for α_T one derives $\beta_T=3\lambda^2/16\pi^2$ and $\gamma_T=\lambda T^2/12m^2+\lambda/16\pi^2$, from which

$$g(T)=\frac{1}{\ln(\Lambda^2/T^2)}, \qquad (13)$$

$$m^2(T)=g^{1/3}\Lambda^2[C+\Gamma(\tfrac{1}{3})-\tfrac{4}{9}\pi^2\gamma(\tfrac{1}{3},1/g)], \qquad (14)$$

in terms of $g=3\lambda/32\pi^2$ and the RG invariants $\Lambda^2=T_0^2\exp[1/g(T_0)]$ and C. Using the asymptotic expansion for the incomplete γ-function in (14) gives

$$m^2(T)=Cg^{1/3}\Lambda^2+\tfrac{4}{9}\pi^2gT^2+\mathrm{O}(g^2)T^2, \qquad (15)$$

with $g=g(T)$. The appearance of the second term in (15) is obviously a direct consequence of the fact that we absorbed the first term in (10) into a mass counterterm. As already claimed before, the corrections of order m/T do not vanish in the limit $T\to\infty$, but rather approach $\frac{4}{9}\pi^2g+\mathrm{O}(g^2)$ (in view of triviality problems, this result is valid for $Cg^{1/3}\ll T\ll\Lambda$ [9]).

The fact that such corrections are always present, and cannot all be removed by an improved RS, can be inferred by a simple dimensional analysis for the one-loop UV-convergent six-point function $\Gamma_0^{(6)}$. If we take, for simplicity, all external momenta equal to zero, we have

$$\Gamma_0^{(6)} \propto \lambda^3 \frac{T^3}{m^3} \Big[\underbrace{1}_{\text{static}} + \underbrace{\mathrm{O}\Big(\frac{m^3}{T^3}\Big)}_{\text{non-static}} \Big] , \tag{16}$$

where $m = m(T)$ defined by the RS. From this result it is immediately clear that complete DR does not occur. Nevertheless, the reduced theory still yields a valid approximation up to a certain low order in the coupling constant, depending on the particular Green function under consideration [9].

4. Concluding remarks

Let us point out, in the context of the AC theorem, why non-static modes do not decouple, while a heavy field in vacuum does. For a vacuum field theory the light mass m is an input parameter found to be small from experiment. However, from an RG analysis similar to the one given above one would immediately find that $m \propto \lambda^n M$, where M is the large mass of the decoupling field. Clearly, this ruins the decoupling for the same reasons as in the finite-T case. This dilemma can be resolved by fine-tuning the *bare* mass, that is, giving it an M dependence so as to make the *renormalized* mass m independent of M. The fact that both m and M are fixed parameters, with the bare mass unknown from the beginning, legitimates this procedure. The need for such fine-tuning is just the hierarchy problem [12]. *On the other hand, the bare mass, being temperature independent by definition, cannot be fine-tuned with a temperature-dependent term.* Therefore one can never get rid of the generated T-dependent masses.

Of course, the fact that the static fields acquire a mass as such has fully been recognized in the literature. In the conventional treatment of DR, this mass is added by hand to the 3D static action. The main lesson to be learnt from our approach, based on the RG at finite T, is that such an inclusion induces corrections to DR of order m^2/T^2 as a feedback mechanism. This feature causes DR to fail in many cases, including ϕ^4 and QCD.

The main conclusion of this letter is that in general the appearance of temperature-dependent masses indicates the breakdown of the DR process in higher-order perturbation theory. Only in exceptional cases like QED, in which there is no mass generation for the static spatial gauge field [1], complete DR occurs [9].

The T-dependent masses acquire their proper meaning by means of the RG at finite T, and cannot be removed from the theory by a redefinition of the T-independent bare mass. This is in some sense closely related to the hierarchy problem.

In our opinion, the results of our investigation imply that results obtained non-perturbatively in a 3D theory, especially $(\mathrm{QCD})_3$, cannot be directly extrapolated to the high-temperature 4D theory. An extensive account of dimensional reduction at high temperature, with applications to QED and QCD, will appear elsewhere [9].

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