Comment on "What is a Gauge Transformation in Quantum Mechanics?"

In a recent Letter [1], Rovelli addresses a technical issue in constrained quantization which is of great potential importance for quantum gravity, Yang-Mills theories, and other fundamental theories. Let $\mathcal{H}$ be the Hilbert space of unconstrained states of a quantum theory with quantized first-class constraints C. Rovelli defines two vectors ψ and φ in $\mathcal{H}$ to be related by a "complete gauge transformation" when $\psi - \phi \in L$, where $L \subset \mathcal{H}$ consists of all vectors of the form $\sum_i [\exp(it_i C) - 1]\rho_i$, $\rho_i \in \mathcal{H}$. He then proposes that the physical Hilbert space be $\mathcal{H}_{\mathrm{Ph}} = \mathcal{H}/L$, and he shows that when $\mathcal{H}$ is finite dimensional, $\mathcal{H}_{\mathrm{Ph}}$ is the same as the physical state space defined in Dirac's well-known theory of constrained quantization.

Rovelli's analysis of the situation in which $\mathcal{H}$ is infinite dimensional is based on his remark that "In infinite dimensions, the orthogonal complement $L_\perp$ of a subspace L may be trivial even if L is smaller than $\mathcal{H}$. But $\mathcal{H}/L$ exists nevertheless... ." This is true when L is not closed, but it is not clear what the inner product on $\mathcal{H}/L$ should be in that case, and how it is to be completed so as to become a Hilbert space.

I here wish to point out that the infinite-dimensional case may be handled [2] by modifying the inner product $\langle \, | \, \rangle$ on $\mathcal{H}$ (which is positive definite) into a positive semidefinite sesquilinear form $\langle \, | \, \rangle_0$, which in the infinite-dimensional case is defined only on a suitable dense subspace $\mathcal{D}$ of $\mathcal{H}$. This form has a nontrivial null space $\mathcal{N} = \{\psi \in \mathcal{H} \mid \langle \psi \mid \psi \rangle_0 = 0\}$, in terms of which the physical state space of the constrained system is the closure of $\mathcal{D}/\mathcal{N}$ in the inner product inherited from $\langle \, | \, \rangle_0$. When the dimension of $\mathcal{H}$ is finite, the space $\mathcal{N}$ coincides with Rovelli's L (this is immediate if one combines the theorem on p. 4614 of [1] with the analysis in section 1.3 of [3]).

In addition, one would like to specify the action of physical observables on $\mathcal{H}_{\mathrm{Ph}}$; recall that a Hilbert space as such conveys practically no physical information, since all Hilbert spaces of the same dimension are isomorphic. One may proceed by (i) defining a weak physical observable as an operator on $\mathcal{H}$ satisfying the modified Hermiticity condition $\langle \psi \mid A \mid \varphi \rangle_0 = \langle \varphi \mid A \mid \psi \rangle_0^*$; (ii) noting that this implies that A maps $\mathcal{N}$ into itself; (iii) concluding that A induces a well-defined operator A_{Ph} on $\mathcal{H}_{\mathrm{Ph}}$. In cases that are well understood, this procedure indeed turns out to yield the correct physical quantum observables [3].

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