Groupoids in geometric quantization

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Chapter 1

Introduction

Geometric quantization

The aim of this thesis is to introduce various new approaches to geometric quantization, with a particular rôle for groupoids. This introduction gives a general outline of the thesis and intends to explain its coherence. A more technical introduction explaining the results more precisely is found in separate introductions preceding each part.

Geometric quantization is a mathematical method to relate classical physics to quantum physics. In classical mechanics the information of the system under investigation is assembled in the concept of a phase space. Each point in this space corresponds to a possible state of the system. The observable quantities of the system correspond to functions on the phase space. Typical for classical mechanics is the existence of a Poisson bracket on the algebra of functions. An important source of commutative Poisson algebras are algebras of functions on symplectic manifolds. The dynamics of the system is determined by a special function, the Hamiltonian, and by the Poisson bracket.

The possible states of a quantum system correspond to unit vectors in a Hilbert space. The observable quantities correspond to certain operators on this Hilbert space. These operators form an operator algebra. Analogously to classical mechanics, there is a Lie bracket on this algebra, namely the commutator. The dynamics of a quantum system is determined by a special operator (the Hamiltonian) and the commutator.

The program of geometric quantization was initiated around 1965 by I. E. Segal, J. M. Souriau, B. Kostant, partly based on ideas of A. A. Kirillov going back to 1962 (cf. e.g. [38, 92] and references therein). Its aim is to construct a Hilbert space from a symplectic manifold, and to construct operators on that Hilbert space from functions on the symplectic manifold. The purpose is to relate the Poisson algebra to the operator algebra in such a way that the Poisson bracket relates to the commutator bracket. In this way one relates the two ways in which the dynamics of the classical and quantum systems are described.

Symmetry and geometric quantization

If a classical system has some symmetry, then one would hope that this symmetry is preserved under quantization. This idea does not work in general. Instead, it applies to a particular type of symmetries called Hamiltonian group actions. Geometric quantization turns a Hamiltonian group action on a symplectic manifold into a unitary representation on the quantizing Hilbert space. In this sense geometric quantization can be seen as a way to construct representations of groups. In the case of a Lie group G, the associated dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ can serve as a rich source of symplectic manifolds endowed with a Hamiltonian action. Indeed, the orbits of the coadjoint action of G on $\mathfrak{g}^*$ are symplectic manifolds and the coadjoint action is actually Hamiltonian. For simply connected solvable Lie groups all representations are obtained from geometric quantization of the coadjoint orbits. The idea that representations correspond to coadjoint orbits is called the "orbit philosophy" (cf. [38, 39]). Actually, the orbit philosophy goes much further in linking the representations of a Lie group to the coadjoint orbits, but we shall not discuss this. For compact Lie groups and non-compact semisimple Lie groups, geometric quantization is still a good method to obtain (some of) the representations, but the tight bond as described in the orbit philosophy has to be relaxed.

The advantage of the occurrence of a symmetry in a physical system is that the number of parameters used to describe the system can be reduced. For classical mechanics this mathematically boils down to a symplectic reduction of the symplectic manifold under the Hamiltonian action (cf. [50]). The obtained reduced symplectic manifold is called the Marsden-Weinstein quotient. On the other hand, if a quantum system has a symmetry one can perform a so-called quantum reduction. An important statement, the Guillemin-Sternberg conjecture, explains the relationship between geometric quantization of the Hamiltonian action on a symplectic manifold and the geometric quantization of the associated Marsden-Weinstein quotient (cf. [28]). The Guillemin-Sternberg conjecture states that quantization commutes with reduction or, more precisely, that the so-called quantum reduction of the geometric quantization should be isomorphic to the quantization of the Marsden-Weinstein quotient. It has been formulated and proved by Guillemin and Sternberg for compact Lie groups in the 80's of the previous century (cf. [28]). Other proofs were found in the 90's (cf. [52, 53, 62, 76]). But there are various new developments now ([47, 36, 35]) and this thesis is part of those developments.

Groupoids in geometric quantization

Suppose $(M, \mathcal{F})$ is a foliated manifold. This means that M is partioned by immersed submanifolds $F \in \mathcal{F}$. One could consider a longitudinal symplectic form ω on M; this means that the leafs of $(M, \mathcal{F})$ are symplectic manifolds. We find a suitable notion of geometric quantization of (M, ω) in Part III. Another track one can follow is to try to quantize the foliated manifold endowed with a transversal symplectic form. "Morally" this means one has a symplectic form on the leaf space $M/\mathcal{F}$ (which need not be smooth!). This is an important class of examples of Part IV. We shall explain these options some more later on in this introduction. First, note that groupoids play an important rôle in both approaches. In the first, groupoids occur as a generalized notion of symmetry. In the second, groupoids are used to describe the leaf space. Hence we shall use groupoids in both ways in this thesis, which explains the its title.

A short way to define a groupoid is as a category in which each arrow is invertible (supposing one knows what a category is). Groupoids were 'discovered' by H. Brandt in 1926 (according to [87]). Since then they have reappeared in several mathematical branches. Groupoids are like groups, but with a partial multiplication, i.e. only specified pairs of elements can be multiplied. In this sense they generalize the usual notion of symmetry related to groups and group actions. Groupoids are also in a sense like equivalence relations. In that sense they serve well as models describing the quotient of such a relation (for example a leaf space).

As already mentioned, in Part III groupoids occur as symmetries of foliated manifolds. We define Hamiltonian actions of groupoids on foliated manifolds with a longitudinal (pre)symplectic form. Next we define a suitable form of geometric quantization of these. This geometric quantization gives rise to a continuous field of Hilbert spaces, carrying a representation of the groupoid. In Part II we study such representations of groupoids on continuous fields of Hilbert spaces, developing basic parts of harmonic analysis for groupoids.

Representations of groupoids

The representation theory of groups is an established branch of mathematics. Unitary representations of e.g. nilpotent, compact and semi-simple Lie groups are all well understood (cf. e.g. [38, 40]). This contrasts with representations of groupoids. Although these occur at several places in mathematics, a separate structural treatment has been lacking so far. We provide such a treatment and hope it will be the starting point of more research in this direction.

We have chosen to study representations on continuous fields of Hilbert spaces. An important feature of such fields is that they are not necessarily locally trivial. There are some reasons for this choice. In the light of noncommutative geometry, the natural type of modules over a C^*-algebra A are the Hilbert A-modules (cf. [27]). For a space M, Hilbert $C_0(M)$-modules correspond to continuous fields of Hilbert spaces over M. Hence it is natural to represent a continuous groupoid $G \rightrightarrows M$ on a continuous field of Hilbert spaces over M. In this way, such representations also turn up in groupoid equivariant KK-theory (cf. [48]). Another reason is that the regular representation lives on a continuous field of Hilbert spaces that is not locally trivial, even for 'simple' étale groupoid (cf. the introduction of Part II).

We summarize the results of Part II. The first chapter begins by carefully comparing the different notions of continuity of groupoid representations. For unitary representations all these notions turn out to coincide (Lemma 5.1.6, 5.1.7 and 5.2.4). We prove that for proper groupoids any continuous representation is isomorphic to a unitary representation (Proposition 5.1.10). An important example of a representation of a groupoid is the regular representation. We show that this is continuous (Proposition 5.3.1). As another class of examples, in the next section we study families of groups. We describe the set of such representations on a given continuous field of Hilbert spaces as a space of continuous sections of a certain surjection (Proposition 5.4.4). The chapter finishes with a comparison the representations of a groupoid with the representations of the associated group of global bisections (Theorem 5.5.5). In the next chapter we prove an analogue of Schur's Lemma for groupoids (Lemma 6.2.7). Next, we show that for proper groupoids any unitary representation is square-integrable (Proposition 6.3.7). This is followed by two versions of the Peter-Weyl Theorem for groupoids (Theorem 6.4.6 and Theorem 6.5.5). We finish this chapter by comparing the K-theory of proper groupoids with their representation rings. The rôle of K-theory will increase towards the end of the thesis. The last chapter of Part II discusses the relation between representations of groupoids and representations of the Banach $*$-category of integrable functions on the groupoid (Theorem 7.2.3). Eventually, we also explain the relation between these representations and the representations of a certain C^*-category associated to the groupoid (Corollary 7.2.4).

Geometric quantization of Hamiltonian Lie algebroid actions

In Part III we extend the notion of geometric quantization to a new class of Hamiltonian actions; namely Hamiltonian Lie groupoid and Lie algebroid actions. As mentioned, the geometric quantization of such actions produces a representation of the groupoid on a continuous field of Hilbert spaces. As a result we extend our set of examples of representations of groupoids. Another motivation is the question whether there is a suitable orbit philosophy for representations of Lie groupoids. Finally, another important motivation is to formulate and prove a Guillemin-Sternberg conjecture for groupoids.

In part III we restrict ourselves to groupoids that have a smooth structure: Lie groupoids. The advantage of this is that one can associate a Lie algebroid to a Lie groupoid, just like a Lie group has an associated Lie algebra. Hamiltonian actions and momentum maps are defined in terms of this Lie algebroid.

Suppose $G \rightrightarrows M$ is a Lie groupoid. We quantize bundles $J : S \to M$ of symplectic manifolds over M. There are so-called internal symmetries $S_m \to S_m$ $(m \in M)$ of the fibers. These symmetries form a bundle of groups, which can be quantized if the symmetries are Hamiltonian (we call this internally Hamiltonian). But there might also be symmetries among the fibers $S_m \to S_n$ $(m \neq n \in M)$. The symmetry of the whole map $J : S \to M$ is described by an action of a Lie groupoid $G \rightrightarrows M$ on J. Thus, the manifold S is foliated by the orbits of this action $\mathcal{F} = S/G$. Such bundles of symplectic manifolds over M can be obtained as families of orbits the coadjoint action of $G \rightrightarrows M$ on the dual of the kernel of the anchor of the Lie algebroid associated to $G \rightrightarrows M$.

We assume an extension of the symplectic forms on the fibers to an $\mathcal{F}$-longitudinal presymplectic form on S to be given. In practice there is often more than one way to obtain such an extension. To quantize this to a representation of the groupoid we need some particular conditions on the momentum map (cf. Definition 8.4.3), which will be a section of the dual of the action Lie algebroid $\mathcal{A} \ltimes J$,

$$\mu : S \to (\mathcal{A} \ltimes J)^*.$$

If there exists such a map, then the action will be called Hamiltonian.

The main results of chapter 9 are Theorem 9.2.2 and Theorem 9.3.1. These show that there exists a suitable notion of prequantization for Hamiltonian actions of Lie groupoids and Lie algebroids. In the next chapter we consider bundles of Kähler manifolds $S \to M$, with a Hamiltonian action of a groupoid and an equivariant complex structure. In this situation, one cannot only construct a prequantization but also a geometric quantization (Theorem 10.1.3). Analogously to the case of Hamiltonian group actions, there is a notion of a Marsden-Weinstein quotient (Section 10.2) and a quantum reduction (Definition 10.3.1 and 10.3.7). Thus, there is a Guillemin-Sternberg conjecture (Theorem 10.3.2 and Corollary 10.3.8), which can be seen as the culmination of Part III. We prove this conjecture for regular proper Lie groupoids. We finish this part commenting on a possible orbit philosophy for Lie groupoids, using some examples.

Perhaps of the same interest as these theorems is our main example, which considers the canonical action of a gauge groupoid of a principal H-bundle $P \to M$ on a bundle of symplectic manifolds obtained by associating a symplectic manifold to $P \to M$ along a Hamiltonian H-action. This example is used as a concrete illustration at each step we take (cf. Examples 8.3.8, 8.3.11, 8.3.11, 8.4.13, 9.3.6, 9.4.6, 8.4.16, 10.1.8, and 10.3.5).

Noncommutative geometry and geometric quantization

As mentioned above, another direction in which we want to generalize geometric quantization is, for example, to foliated manifolds $(M, \mathcal{F})$ with a transversally symplectic form, which boils down to a symplectic form on the leaf space $M/\mathcal{F}$. But leaf spaces are in general neither Hausdorff nor smooth. A way to deal with such singular spaces is via noncommutative geometry (cf. [12]).

The idea of noncommutative geometry, founded by Alain Connes, is to replace the singular space by an algebra A that represents that space in a certain sense. The power of noncommutative geometry is that many constructions that usually rely on topological or smooth structures on the space can be done using the algebra A instead, without the necessity of an underlying space. For example, topological invariants like K-theory can be defined on C^*-algebras. In general, C^*-algebras are used to generalize topological constructions, whereas von Neumann algebras are seen as noncommutative measurable spaces. On the other hand, notions like a metric and a local index can be based on a so-called spectral triple, for which one needs a 'smooth subalgebra' $A_0 \subset A$. Also, for noncommutative differential geometry one uses such smooth subalgebras (see Chapter 11).

In part IV we investigate how the tools of noncommutative geometry can be used for a 'geometric' quantization of (noncommutative) algebras. Noncommutative algebras are usually only associated to quantum theory, but the point of view taken in this thesis is that they also arise in classical mechanics. For example, the Marsden-Weinstein quotient of a classical mechanical system with Hamiltonian symmetry can be singular. Instead of considering the quotient, one should study the groupoid algebra associated to the action or the algebra of invariant functions.

Clearly, if one wants to talk about symplectic algebras, one needs a notion of differential forms on an algebra. There is no unique way to do this. There are several so-called differential calculi on algebras. We focus on the derivation-based differential calculus introduced by Dubois-Violette (cf. [24]), but one should be aware that other choices are possible. In Chapter 11 we study Poisson algebras, in particular Poisson algebras associated to symplectic algebras. We introduce a notion of Hamiltonian actions on such algebras with a corresponding notion of a momentum map. The main results of this chapter discuss a generalized Marsden-Weinstein quotient for this setting (Theorem 11.1.15 and Theorem 11.4.2). The prequantization of such Hamiltonian actions can be constructed in terms of the differential calculus, using a noncommutative version of the theory of connections (Proposition 11.5.3). The rôle of prequantum line bundles is played by invertible bimodules. For the geometric quantization of such a prequantization we use again some noncommutative topology.

A noncommutative momentum map

Apart from the physical interpretation, geometric quantization is interesting as a means to construct representations of groups (and groupoids, cf. Part III). On the other hand, representations can also be constructed as the L^2-index of equivariant operators (cf. e.g. [13]). An important observation is that the index of an operator only depends on the homotopy class of its principal symbol. This gives rise to an interpretation of the index of an operator as a map in K-theory

$$K^0(T^*M) \to \mathbb{Z},$$

where the K-theory class of the principal symbol of the operator is mapped to the index of the operator. Via the Poincaré duality isomorphism this map relates to a map

$$K_0(M) \to \mathbb{Z},$$

where $K_0(M)$ is the so-called K-homology of M. This group can be interpreted as the group of homotopy classes of elliptic differential operators on M.

Before we discuss an equivariant version of this map, let's consider what should be the codomain of such a map. The index of an operator is the difference between the dimensions of the kernel and cokernel. For an equivariant operator we forget about the dimension and simply look at the kernel and the cokernel. In the L^2-index theorem these are endowed with representations of the group. Hence, if the group is compact the index can be interpreted as taking values in the representation ring of the group. A K-theoretic generalization of this representation ring is the K-theory of the C^*-algebra of the group. For compact groups these two rings coincide. Hence the map we need is a map

$$K_0^G(M) \to K_0(C^*(G)).$$

Such a map exists if the action of G on M is proper and cocompact, and is called the Baum-Connes analytical assembly map. It plays a crucial rôle in the Baum-Connes conjecture, which states that it is an isomorphism if $M = \underline{E}G$, the classifying space of proper actions of G (cf. [4]).

In [47] Landsman proposed to use the analytical assembly map to define geometric quantization for proper, cocompact Hamiltonian actions of non-compact groups. In this approach the geometric quantization is the image under the analytical assembly map of the K-homology class of an equivariant Spin^c-Dirac operator on M

coupled to the prequantization line bundle. In chapter 12 we push this approach a little further and use it to define geometric quantization for proper Hamiltonian actions on symplectic algebras. The main result is the construction of an analytical assembly map

$$KK_0^G(A,B) \to K_0(G \ltimes B),$$

for G-C^*-algebras A, B, where the action on A is proper and counital (Definition 12.5.1). For this purpose we review Rieffel's notion of proper action on C^*-algebras in Section 12.1. We discuss an interesting new class of examples of such actions based on proper actions on proper groupoids (Proposition 12.2.5). We introduce the notion of counital action. It generalizes the notion of cocompact actions on spaces. We show, for two large classes of examples, that they are proper and counital (Proposition 12.3.9 and Proposition 12.3.14).

Finally, we discuss our notion of noncommutative geometric quantization (Definition 12.6.1) using this notion of noncommutative analytical assembly map. We sketch an approach to geometric quantization in this setting. This gives rise to a noncommutative Guillemin-Sternberg-Landsman conjecture. Part IV is open-ended; an invitation for more research in this direction.

Part I

Preliminaries

Chapter 2

Groupoids

In this chapter we give an overview of the theory of continuous groupoids, Lie groupoids, and Lie algebroids. We focus on examples; certain important statements are not emphasized as such. In this chapter, as well as in the other preliminary chapters, our purpose is to get the reader acquainted with the mathematical objects that are important in this thesis and fix our notation for these objects, rather than providing proofs. These can be found in some of the following books, which can also serve as good introductions. The book [49] focuses on geometric aspects of Lie groupoids and Lie algebroids. The relationship between Lie groupoids and foliation theory is discussed in [55]. The book [87] explains the rôle of Lie groupoids in quantization and noncommutative algebra. The book [43] treats Lie groupoids and Lie algebroids in relation to strict deformation quantization. The recent overview [18] discusses integration of Lie algebroids. Finally, [67] should be consulted on Haar systems and C^*-algebras associated to groupoids.

We suppose the reader is familiar with the basics of topology, differentiable manifolds and measure theory.

2.1 Groupoids

Definition 2.1.1. A **groupoid** is a (small) category in which all arrows are invertible. In other (and considerably more) words, a groupoid is a septuple

$$(G_0, G_1, s, t, m, u, i)$$

consisting of

(i) a set of objects G_0;

(ii) a set of arrows G_1;

(iii) a source map $s : G_1 \to G_0$;

(iv) a target map $t : G_1 \to G_0$;

(v) an associative partial multiplication

$$m : G_2 := G_1 \,{}_s\!\times_t G_1 \to G_1, (g, h) \mapsto m(g, h) =: g\,h,$$

satisfying $s(h\,g) = s(g)$ and $t(h\,g) = t(h)$ for all $(h, g) \in G_2$, where

$$G_1\ {}_s\!\times_t G_1 = \{(g, h) \in G_1 \times G_1 \mid s(g) = t(h)\}$$

denotes a fibered product;

(vi) a unit map $u : G_0 \to G_1$ such that $u(x) =: 1_x$ is a left unit for $G^x := t^{-1}(x)$ and a right unit for $G_x := s^{-1}(x)$ for all $x \in G_0$, i.e. $u(x)\,g = g$ for all $g \in G^x$ and $g\,u(x) = g$ for all $g \in G_x$;

(vii) an inverse map $i : G_1 \to G_1$ such that $g^{-1} := i(g)$ is a 2-sided inverse of g for all $g \in G$, i.e. $g^{-1}\,g = u(s(g))$ and $g\,g^{-1} = u(t(g))$.

We shall denote a groupoid not by the septuple $(G_0, G_1, s, t, m, u, i)$, but simply by $G_1 \rightrightarrows G_0$.

Definition 2.1.2. A groupoid is **continuous** if G_1 and G_0 are topological spaces and the maps s, t, m, u and i are continuous.

Example 2.1.3 (Groupoids from a space)**.** Suppose X is a space. Then the **trivial groupoid** $X \rightrightarrows X$ is a continuous groupoid consisting of just one unit arrow $u(x)$ for every $x \in X$.

Also, one can consider the **pair groupoid** $X \times X \rightrightarrows X$. It has source map $s(y, x) := x$ and target map $t(y, x) := y$ for every pair $(y, x) \in X \times X$. Multiplication is given by $(z, y)(y, x) = (z, x)$, the unit map is $u(x) = (x, x)$ and inversion is defined by $(y, x)^{-1} = (x, y)$.

Every **equivalence relation** $R \subset X \times X$ is a continuous groupoid $R \rightrightarrows X$ in the subspace topology.

Suppose $G \rightrightarrows G_0$ is a groupoid. We use the notation $G_x^y := G_x \cap G^y$ for $x, y \in G_0$. The set G_x^x has the structure of a group and is called the **isotropy group** of G at $x \in G_0$. The set $t(G_x) = s(G^x) \subset G_0$ is the **orbit** through $x \in G_0$. A groupoid is **transitive** if G_0 consists of one orbit. To any groupoid $G \rightrightarrows G_0$ one associates the **orbit relation groupoid** $R_G := (t \times s)(G) \rightrightarrows G_0$. This is a particular example of an equivalence relation on G_0. It is a continuous groupoid if s and t are open. The **orbit set of a groupoid** $G \rightrightarrows G_0$ is the set of orbits of G, denoted by G_0/G. If $G \rightrightarrows G_0$ is continuous, then the orbits, isotropy groups and orbit set have an induced topology and the latter is called the **orbit space of a groupoid**.

Obviously, a pair groupoid $X \times X \rightrightarrows X$ is transitive. For an equivalence relation $R \subset X \times X$ as a groupoid $R \rightrightarrows X$, orbits correspond to equivalence classes.

Example 2.1.4. Suppose $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of a locally compact space X. We use the notation $U_{ij} := U_i \cap U_j$ for $i, j \in I$. Consider the groupoid

$$\coprod_{i,j \in I} U_{ij} \rightrightarrows \coprod_{i \in I} U_i,$$

where the source map is the inclusion $U_{ij} \to U_j$ and the target map is the inclusion $U_{ij} \to U_i$. Composition $U_{ij}\ {}_s\!\times_t U_{jk} \to U_{ik}$ is $(x, y) \mapsto x\,(= y)$. The unit is the identity $U_i \mapsto U_{ii} = U_i$. The inverse map is the identity $U_{ij} \to U_{ji}$. This groupoid is called the **cover groupoid** associated to the cover $\mathcal{U}$ of X. This cover groupoid is an **étale groupoid**, which means that $s : G \to X$ and $t : G \to X$ are local homeomorphisms.

Example 2.1.5 (Groups and actions)**.** Any topological group H can be seen as a continuous groupoid $H \rightrightarrows pt$ over a one-point set pt.

Suppose H acts from the left on a space X. Then one can construct the **action groupoid** $H \ltimes X \rightrightarrows X$, with $(H \ltimes X)_1 = H \times X$ and $(H \ltimes X)_0 = X$, $s(h,x) := x$, $t(h,x) := h \cdot x$, $(h', h \cdot x)(h,x) = (h'h, x)$, $u(x) := (e,x)$ and $(h,x)^{-1} = (h^{-1}, h \cdot x)$. This groupoid is continuous if the action is continuous.

A continuous groupoid $G \rightrightarrows G_0$ is **proper** if $t \times s : G \to G_0 \times G_0$ is a proper map. In particular, an action groupoid $G = H \ltimes X \rightrightarrows X$ is proper iff the action of H on X is proper. The action groupoid is transitive iff the action is transitive. The isotropy group G_x^x corresponds with the isotropy group of the action of H at $x \in X$.

Example 2.1.6 (The symmetry of a map)**.** The symmetries of an object X in a category $\mathcal{C}$ are given by the group of automorphisms $\mathrm{Aut}_{\mathcal{C}}(X)$ of the object. The group $\mathrm{Aut}_{\mathcal{C}}(X)$ is in general 'very large' and one instead studies morphisms $H \to \mathrm{Aut}_{\mathcal{C}}(X)$ for smaller groups H.

What is the symmetry of a map $f : X \to Y$ in $\mathcal{C}$? An automorphism of $f : X \to Y$ in the category of arrows in $\mathcal{C}$ consists of pair of automorphisms $\phi \in \mathrm{Aut}_{\mathcal{C}}(X)$ and $\psi \in \mathrm{Aut}_{\mathcal{C}}(Y)$ such that $\psi \circ f = f \circ \phi$. Suppose $\mathcal{C}$ is a category of sets, i.e. $\mathcal{C} \subset$ **Sets**. If f is surjective, then the automorphism ϕ of X fixes the automorphism ψ of Y, hence the automorphisms of f form a subgroup of the automorphisms of X. Since Y is a set, any automorphism (ϕ, ψ) of f decomposes as a family of isomorphisms $\{\phi_y : f^{-1}(y) \to f^{-1}(\psi(y))\}$. If $\psi(y) = y$, then ϕ_y is called an **internal symmetry of the map**, else it is called an **external symmetry** (cf. [86]). The union of all internal and external symmetries has the structure of a groupoid $\mathrm{Aut}(f) \rightrightarrows Y$. Indeed, one defines

$$\mathrm{Aut}_{\mathcal{C}}(f) := \bigcup_{y,y' \in Y} \mathrm{Iso}_{\mathcal{C}}(f^{-1}(y), f^{-1}(y')),$$

with obvious structure maps. We call $\mathrm{Aut}_{\mathcal{C}}(f) \rightrightarrows Y$ the **automorphism groupoid of a map** f. Again, the groupoid $\mathrm{Aut}_{\mathcal{C}}(f) \rightrightarrows Y$ is in general 'very large' and one instead studies morphisms $G \to \mathrm{Aut}_{\mathcal{C}}(f)$ for smaller groupoids $G \rightrightarrows Y$, which are called **groupoid actions** of $G \rightrightarrows Y$ on the map f.

Example 2.1.7. Suppose X is a topological space. The set of homotopy classes of paths $\gamma : [0,1] \to X$ form a groupoid $\pi_1(X) \rightrightarrows X$, called the **fundamental groupoid**. The source map is defined by $s([\gamma]) := \gamma(0)$ and the target map is $t([\gamma]) := \gamma(1)$. Composition is induced by concatenation of paths

$$\gamma' \cdot \gamma(t) := \begin{cases} \gamma(2t) & \text{if } t \leq 1/2 \\ \gamma'(2t-1) & \text{if } t > 1/2 \end{cases}$$

The unit $u(x)$ at $x \in X$ is defined by the constant path $[x]$ and the inverse is defined by $[\gamma]^{-1} := [\gamma^{-1}]$, where $\gamma^{-1}(t) := \gamma(1-t)$.

As a generalization of this, consider the fundamental groupoid $\pi_1(f) \rightrightarrows X$ of a surjective continuous map $f : X \to Y$, which consists of homotopy classes of paths restricted to the fibers $f^{-1}(y)$ for $y \in Y$.

We always assume G_0 to be Hausdorff in this text. A continuous groupoid $G \rightrightarrows G_0$ is **Hausdorff** if G is a Hausdorff space. This is not always the case. For example, consider the projection to the first coordinate axis $p_1 : \mathbb{R}^3\backslash\{0\} \to \mathbb{R}$. Then $\pi_1(p_1) \rightrightarrows \mathbb{R}$ is not Hausdorff. The orbit space G_0/G is also not Hausdorff in many examples (independently of G being Hausdorff or not). But, if $G \rightrightarrows G_0$ is proper, then G_0/G is Hausdorff.

Example 2.1.8 (Gauge groupoids)**.** Suppose H is a topological group and $\pi : P \to X$ a continuous principal H-bundle over a space X. Consider the groupoid $P \times_H P \rightrightarrows X$, where $P \times_H P$ is the orbit space of the diagonal action of H on $P \times P$. It obtains a groupoid structure from the pair groupoid $P \times P \rightrightarrows P$. Indeed, $s := \pi \circ pr_2$ and $t := \pi \circ pr_1$; composition $[p,q][q',r] = [p,q][q,h\cdot r] = [p,h\cdot r]$, where $p,q,q',r \in P$, $\pi(q) = \pi(q')$ and $h \in H$ is unique such that $h \cdot q' = q$. The unit is defined by $u(x) =: [p,p]$ for any $p \in \pi^{-1}(x)$ and the inverse is given by $[p,q]^{-1} := [q,p]$ for all $p,q \in P$. This groupoid is called the **gauge groupoid** of $P \to X$. There is a canonical action of $P \times_H P \rightrightarrows X$ on $\pi : P \to X$, given by $[p,q]\cdot q = p$ for $p,q \in P$.

In particular, the universal covering space $\tilde{X} \to X$ of a connected space X is a principal $\pi_1(X,x)$-bundle, where $\pi_1(X,x)$ is the fundamental group of X at a fixed point $x \in X$. The gauge groupoid $\tilde{X} \times_{\pi_1(X,x)} \tilde{X} \rightrightarrows X$ associated to $\tilde{X} \to X$ is isomorphic to the fundamental groupoid $\pi_1(X) \rightrightarrows X$.

Suppose $E \to M$ is a continuous complex rank n vector bundle. Let $F_E := \mathrm{GL}_M(\mathbb{C}^n, E) \to M$ denote the frame bundle of $E \to M$, i.e. the fiber $\mathrm{GL}_M(\mathbb{C}^n,E)_m$ at m equals the space $\mathrm{GL}(\mathbb{C}^n, E_m)$ of invertible linear maps $\mathbb{C}^n \to E_m$. This is a principal $\mathrm{GL}(\mathbb{C}^n)$-bundle. The associated gauge groupoid $F_E \times_{\mathrm{GL}(\mathbb{C}^n)} F_E \rightrightarrows M$ is obviously isomorphic to $\mathrm{GL}(E,E) \rightrightarrows M$, the **general linear groupoid** of E, where

$$GL(E,E)^n_m := Gl(E_m, E_n),$$

the space of invertible linear maps $E_m \to E_n$ for all $m,n \in M$. There are canonical actions of $\mathrm{GL}(E,E) \rightrightarrows M$ on $E \to M$ and on $F_E \to M$.

Let g be a hermitian metric on $E \to M$. Then we can analogously construct the unitary frame bundle $F_E^U := U(\mathbb{C}^n, E) \to M$ and the **unitary groupoid** $U(E) := U(E,E) \rightrightarrows M$ of E.

Gauge groupoids are transitive. Conversely, any transitive groupoid is isomorphic to the gauge groupoid of a principal bundle. Indeed, suppose $G \rightrightarrows X$ is a continuous transitive groupoid. Choose an $x \in X$. The source map $s : G^x \to X$ is a left principal bundle for the left action of G_x^x by multiplication. One can form the gauge groupoid $G_x \times_{G_x^x} G_x \rightrightarrows X$ of this bundle. It is easy to show that $G \cong G_x \times_{G_x^x} G_x$.

Example 2.1.9 (Families of groups, bundles of groups and group bundles)**.** Suppose $G \rightrightarrows X$ is continuous groupoid. The union of isotropy groups $\bigcup_{x\in X} G_x^x \subset G$ is a continuous groupoid over X in the subspace topology. It is denoted by $I_G \rightrightarrows X$ and is called the **isotropy groupoid** of G. Note that $s(g) = t(g)$ for all $g \in I_G$. Groupoids $G \rightrightarrows G_0$ for which $s(g) = t(g)$ for all $g \in G$ are called **families of groups**. If $G_1 \to G_0$ is a family of groups and a fiber bundle (as a space), then it is called a **bundle of groups**.

Given a space X and a topological group H, we can construct the basic example, $X \times H \to X$, a trivial family of groups over X. We call a continuous family of groups

$G \to X$ a **group bundle** if for every point $x \in X$ there exists a neighborhood $U \ni x$, a topological group H and an isomorphisms of families of groups $G|_U \to H \times U$ (with the obvious notion of morphism).

A group bundle with fiber H can be constructed from a principal H-bundle $P \to X$. Indeed, consider the associated bundle $P \times_H H \to X$ where H acts on itself by conjugation. The multiplication is defined by $[p, h] \cdot [p, h'] := [p, h\, h']$. One easily sees that this is well-defined. Idem dito for the unit map $u(x) = [p, e]$ for $x \in X$, $e \in H$ the unit and any $p \in P$. The inverse is also obtained from the inverse of H, $[p, h]^{-1} = [p, h^{-1}]$. The bundle of groups $P \times_H H \to X$ is isomorphic to the isotropy groupoid of $P \times_H P \rightrightarrows X$.

2.2 Lie groupoids

Definition 2.2.1. A groupoid $G \rightrightarrows M$ is **smooth** (or a **Lie groupoid**) if G and M are smooth manifolds, the maps s, t, m, u and i are smooth and s and t are submersions.

This last condition ensures that G_2 is a smooth manifold. Moreover, the orbits are smooth submanifolds of M, and G_m, G^m and G^m_n are smooth submanifolds of G for all $m, n \in M$. Note that the isotropy groupoid I_G need not be a smooth submanifold of G.

Example 2.2.2. Suppose M is a smooth manifold. Then the trivial groupoid $M \rightrightarrows M$ and the pair groupoid $M \times M \rightrightarrows M$ are smooth. Also, the cover groupoid of an open cover of M is smooth. Suppose that H is a Lie group that acts smoothly on M. Then the action groupoid $H \ltimes M \rightrightarrows M$ is smooth. If $P \to M$ is a smooth principal H-bundle then the gauge groupoid $P \times_H P \rightrightarrows M$ is smooth. Idem dito for the bundle of groups $P \times_H H \to M$. In particular, if $F(TM) \to M$ is the frame bundle of the tangent bundle $TM \to M$ of M, then $F(TM) \times_{\mathrm{GL}(n)} F(TM)$, the so-called the **jet groupoid** of M is smooth (where n is the dimension of M).

Example 2.2.3 (Foliations)**.** Suppose $\mathcal{F}$ is a regular foliation of a manifold M. Consider the projection on the leaf space $p : M \to M/\mathcal{F}$. The groupoid $\mathrm{Mon}(M, \mathcal{F}) := \pi_1(p) \rightrightarrows M$ is called the **monodromy groupoid** of $\mathcal{F}$. It can be endowed with a smooth structure. Indeed, one can use the fact that for any path $\gamma : [0, 1] \to M$ on a leaf of $\mathcal{F}$ there is a neighborhood $U \subset \mathrm{Mon}(M, \mathcal{F})$ of $[\gamma]$, such that $t \times s : U \to U_1 \times_{M/\mathcal{F}} U_0 \subset M \times M$ is a homeomorphism, for neighborhoods $U_i \subset M$ of the $\gamma(i)$, where $i = 0, 1$. Since the foliation is regular, one can find a U such that $U_1 \times_{M/\mathcal{F}} U_0$ is homeomorphic to an open set in $\mathbb{R}^{2m-d}$, where m is the dimension of M and d the dimension of $\mathcal{F}$.

Another smooth groupoid can be obtained by taking the quotient with respect to the relation induced by holonomy on the homotopy classes of paths; two homotopy classes of paths are equivalent if they have the same holonomy. This so-called **holonomy groupoid**, denoted by $\mathrm{Hol}(M, \mathcal{F}) \rightrightarrows M$, inherits a smooth structure from $\mathrm{Mon}(M, \mathcal{F}) \rightrightarrows M$.

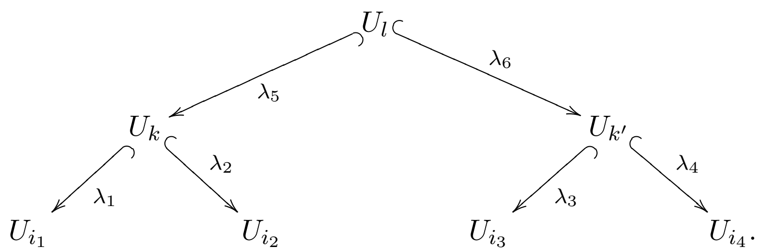
A typical property of the above groupoids is that they have discrete isotropy groups. Therefore groupoids with discrete isotropy groups are called **foliation groupoids**.

Example 2.2.4 (Orbifolds)**.** Suppose M is a Hausdorff space. An orbifold chart on M is a triple $(\tilde{U}, H, \phi)$ consisting of an open set $\tilde{U} \subset \mathbb{R}^n$, a finite group H that acts on $\tilde{U}$ and an H-invariant map $\phi : \tilde{U} \to M$ that induces a homeomorphism $\tilde{U}/H \cong \phi(\tilde{U})$. An embedding $(\tilde{U}, H, \phi) \hookrightarrow (\tilde{U}', H', \phi')$ of orbifolds charts is an embedding $\lambda : \tilde{U} \hookrightarrow \tilde{U}'$ such that $\phi'\lambda = \phi$. An orbifold atlas is a family $\mathcal{U} := \{(\tilde{U}_i, H_i, \phi_i)\}_{i \in I}$ of orbifold charts such that $\{\phi(\tilde{U}_i)\}_{i \in I}$ covers M and for every $i, j \in I$ and $m \in \phi_i(\tilde{U}_i) \cap \phi_j(\tilde{U}_j)$ there exist a $k \in I$ and embeddings $\lambda_i : (\tilde{U}_k, H_k, \phi_k) \hookrightarrow (\tilde{U}_i, H_i, \phi_i)$, $\lambda_j : (\tilde{U}_k, H_k, \phi_k) \to (\tilde{U}_j, H_j, \phi_j)$ such that $m \in \phi_k(\tilde{U}_k)$. Orbifold atlasses are equivalent if they have a common refinement. An **orbifold** is a Hausdorff space M endowed with an equivalence class of orbifold atlasses.

One can associate a Lie groupoid to an orbifold M with a fixed orbifold atlas $\mathcal{U}$ as follows. The space of objects is $G_0 := \coprod_{i \in I} \tilde{U}_i$. The space of arrows is

$$\{(\lambda_1, x, \lambda_2) \mid x \in \tilde{U}_k, \lambda_n : (\tilde{U}_k, H_k, \phi_k) \hookrightarrow (\tilde{U}_{i_n}, H_{i_n}, \phi_{i_n}) \text{ for } k, i_n \in I, n = 1, 2\}/\sim,$$

where two triples $(\lambda_1, x, \lambda_2)$ and $(\lambda_1', x', \lambda_2')$ are equivalent if there exists an embedding $\lambda : (\tilde{U}_{k'}, H_{k'}, \lambda_{k'}) \hookrightarrow (\tilde{U}_k, H_k, \lambda_k)$ such that $\lambda(x') = x$ and $\lambda_n \lambda = \lambda_n'$ for $n = 1, 2$. The source map is induced by $s(\lambda_1, x, \lambda_2) = \lambda_2(x) \in \tilde{U}_{i_2} \subset G_0$ and the target map is induced by $t(\lambda_1, x, \lambda_2) = \lambda_1(x) \in \tilde{U}_{i_1} \subset G_0$. The composition of two composable (classes of) triples $(\lambda_1, x, \lambda_2)$, $(\lambda_3, y, \lambda_4)$ is constructed as follows. Suppose $x \in \tilde{U}_k$ and $y \in \tilde{U}_{k'}$ for certain $k, k' \in I$. Since $\lambda_2(x) = \lambda_3(y)$, one has $\phi_k(x) = \phi_{k'}(y)$, hence there exists an orbifold chart with index $l \in I$ and embeddings $\lambda_5 : (\tilde{U}_l, H_l, \phi_l) \hookrightarrow (\tilde{U}_k, H_k, \phi_k)$ and $\lambda_6 : (\tilde{U}_l, H_l, \phi_l) \hookrightarrow (\tilde{U}_{k'}, H_{k'}, \phi_{k'})$,



Let $z \in \tilde{U}_l$ be the unique element such that $\lambda_5(z) = x$ (and $\lambda_6(z) = y$). Then composition is given by

$$[(\lambda_1, x, \lambda_2)] \cdot [(\lambda_3, y, \lambda_4)] := [(\lambda_1\lambda_5, z, \lambda_4\lambda_6)].$$

For example, suppose $M := U/H$ for an open set $U \subset \mathbb{R}^n$ and H a finite group acting freely on U. Denote the H-action by α. An orbifold atlas for M consisting of one single chart is given by (U, H, p), where $p : U \mapsto U/H$ is the canonical projection on the orbit space. Then $G_0 = U$. Embeddings $(U, H, p) \hookrightarrow (U, H, p)$ correspond to the maps $\alpha(h) : U \to U$ for $h \in H$. Hence $G_1 \rightrightarrows G_0$ is isomorphic to the action groupoid $H \ltimes U \rightrightarrows U$, using the isomorphism induced by $(\alpha(h), x, \alpha(h') \mapsto (h(h')^{-1}, x)$.

Equivalent atlasses give rise to Morita equivalent groupoids, as we shall see in Example 2.3.8. The groupoids constructed from orbifold atlasses are proper foliation groupoids. Moreover, they are **étale**, i.e. the source map $s : G \to M$ is a local diffeomorphism. A useful property of proper étale groupoids is that G_m and G^m are finite sets for all $m \in M$. One can show that any foliation groupoid is Morita equivalent to an étale groupoid (cf. Example 2.3.8).

2.3 Morphisms of groupoids

In this section we shall discuss a number of definitions and constructions for continuous groupoids. There are analogues for these for Lie groupoids, but we leave it to the reader to make the obvious necessary changes.

Definition 2.3.1. A **morphism of continuous groupoids**

$$(G \rightrightarrows G_0) \to (H \rightrightarrows H_0)$$

is a pair of continuous maps $\phi_1 : G \to H$ and $\phi_0 : G_0 \to H_0$ that commutes with the structure maps, i.e. $s \circ \phi_1 = \phi_0 \circ s$, $\phi_1(g \cdot g') = \phi_1(g) \cdot \phi_1(g')$, etcetera.

We denote the category of continuous groupoids with these morphisms by GPD.

Definition 2.3.2. The **pullback groupoid** $f^*G \rightrightarrows Y$ of a continuous groupoid $G \rightrightarrows X$ along a continuous map $f : Y \to X$ is defined as follows. The space of arrows is $f^*G := Y\,{}_f\!\times_t G\,{}_s\!\times_f Y$, the source map is projection on the third factor, the target map is projection on the first factor, composition is defined by $(y, g, y') \cdot (y', g', y'') := (y, g \cdot g', y'')$, the unit map is $u(y) := (y, u(f(y)), y)$ and inversion is defined by $(y, g, y')^{-1} := (y', g^{-1}, y)$.

Example 2.3.3. Suppose $G \rightrightarrows X$ is a continuous groupoid and $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of X. Consider the canonical continuous map $j : \coprod_{i\in I} U_i \to X$. The pullback groupoid along this map is denoted by $G[\mathcal{U}] := j^*G \rightrightarrows \coprod_{i\in I} U_i$.

The cover groupoid of an open cover $\mathcal{U}$ of a space X, discussed in Example 2.1.4, is an example of this construction. It equals $X[\mathcal{U}]$.

Suppose $G \rightrightarrows X$ is a continuous groupoid, Y a space and $J : Y \to X$ a continuous map. In Example 2.1.6 we defined an action of $G \rightrightarrows X$ on J as a map $G \to \mathrm{Aut}_{\mathcal{C}}(J)$. A different point of view is often more useful, in particular, in the topological category.

Definition 2.3.4. A **continuous left action of** $G \rightrightarrows X$ **on** $J : Y \to X$ is a continuous map

$$\alpha : G\,{}_s\!\times_J Y \to Y$$

satisfying

(i) $J(g \cdot y) = t(g)$ for all $(g, y) \in G\,{}_s\!\times_J Y$,

(ii) $1_{J(y)} \cdot y = y$ for all $y \in Y$,

(iii) $g \cdot (g' \cdot y) = (gg') \cdot y$ for all $(g, g') \in G^{(2)}$ and $y \in J^{-1}(s(g'))$,

using the notation $g \cdot y := \alpha(g, y)$.

We shall also use the notation $\alpha(g) := \alpha(g, .)$. There exists an analogous notion of a right action.

Example 2.3.5. Suppose $G \rightrightarrows X$ is a continuous groupoid. It acts from the left on $t : G \to X$ by left multiplication $l : G\,{}_t\!\times_s G \to G$, denoted by $l_g g' := g \cdot g'$. Analogously, $G \rightrightarrows X$ acts from the right on $s : G \to X$.

Suppose $G \rightrightarrows G_0$ is a continuous groupoid. Suppose G acts continuously from the left on a map $J : Y \to G_0$. The action is called **left principal** if the map

$$(g, y) \mapsto (g \cdot y, y)$$

is a homeomorphism

$$G\ {}_s\!\times_J Y \to Y\ {}_p\!\times_p Y,$$

where $G\backslash Y$ is endowed with the quotient topology, and $p : Y \to G\backslash Y$ is the projection on the orbit space. There is an analogous notion of right principal action. Suppose $H \rightrightarrows H_0$ is another continuous groupoid. A space Y is a (G, H)**-bibundle** if there is a left G-action on a map $J_G : Y \to G_0$ and a right H-action on a map $J_H : Y \to H_0$ that commute, i.e. $(g \cdot y) \cdot h = g \cdot (y \cdot h)$, $J_H(g \cdot y) = J_H(y)$ and $J_G(y \cdot h) = J_G(y)$. A **morphism of** (G, H)**-bibundles** Y, Y' is a (G, H)-equivariant continuous map $Y \to Y'$. An isomorphism class of a right principal (G, H)-bibundle can be interpreted as an arrow $G \to H$ in a category of groupoids. These arrows are called **Hilsum-Skandalis maps** or **generalized morphisms**. The category of continuous groupoids and generalized morphisms is denoted by . Composition of morphisms represented by a (G, H)-bibundle P and a (H, K)-bibundle Q is given by the fibered product $[P] \circ [Q] := [P \times_H Q]$. The unit morphism $U(G)$ at G is the class $[G]$ of G itself seen as a (G, G)-bibundle, with left and right multiplication as actions. One can show that a morphism given by a class of bibundles is an isomorphism if the representing bundles have principal left and right actions. In that case, one easily sees that $Y/H \cong G_0$ and $G\backslash Y \cong H_0$. Groupoids that are isomorphic in this category are called **Morita equivalent**. One can prove that a (G, H)-bibundle Y represents a Morita equivalence if it is left and right principal and $Y/H \cong G_0$ and $G\backslash Y \cong H_0$.

Example 2.3.6. Suppose H is a group and $P \to X$ is a continuous left principal H-bundle. The group H is Morita equivalent to the gauge groupoid $P \times_H P \rightrightarrows X$. Indeed, $[P]$ is an invertible generalized morphism $H \to P \times_H P$. Indeed, by definition, H acts from the left on P. The right action of $P \times_H P \rightrightarrows X$ on $P \to X$ is defined by $p \cdot [p, q] = q$.

Example 2.3.7. Suppose $(\phi_1, \phi_0) : (G \rightrightarrows G_0) \to (H \rightrightarrows H_0)$ is a continuous morphism of groupoids. This gives rise to a generalized morphism of groupoids $[G_0\ {}_{\phi_0}\!\times_t H] : (G \rightrightarrows G_0) \to (H \rightrightarrows H_0)$, where we view $G_0\ {}_{\phi_0}\!\times_t H$ as a (G, H)-bimodule as follows. The left action of $G \rightrightarrows G_0$ on the map $pr_1 : G_0\ {}_{\phi_0}\!\times_t H \to G_0$ is given by $g \cdot (s(g), h) := (t(g), \phi_1(g)h)$ for all $g \in G$, $h \in H^{\phi_0(s(g))}$. The right action of $H \rightrightarrows H_0$ on $s \circ pr_2 : G_0\ {}_{\phi_0}\!\times_t H \to H_0$ is given by right multiplication: $(x, h) \cdot h' := (x, h \cdot h')$. One easily sees that the action of $H \rightrightarrows H_0$ is right principal. This gives an inclusion functor

$$\mathsf{GPD} \to \mathsf{GPD_b}\,.$$

Example 2.3.8. Suppose $G \rightrightarrows X$ is a continuous groupoid and $j : Y \to X$ a continuous map. There is a canonical map $j^*G \to G$, which induces a generalized morphism $[Y\ {}_j\!\times_t G] : (j^*G \rightrightarrows Y) \to (G \rightrightarrows X)$. If $j(Y)$ intersect each G-orbit at least once, then it is actually a Morita equivalence with inverse $[G\ {}_s\!\times_j Y]$, endowed with the obvious actions.

In particular, we can conclude that for any continuous groupoid $G \rightrightarrows X$ and open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of X, the groupoids $G[\mathcal{U}] \rightrightarrows \coprod_{i \in I} U_i$ and $G \rightrightarrows X$ are Morita equivalent.

Suppose M is an orbifold with orbifold atlas $\{(\tilde{U}_i, H_i, \phi_i)\}_{i \in I}$. Suppose $\{(\tilde{U}'_i, H'_i, \phi'_i)\}_{i \in I'}$ is a refinement of this atlas, in the sense that there exist embeddings $(\tilde{U}'_j, H'_j, \phi'_j) \hookrightarrow (\tilde{U}_{i_j}, H_{i_j}, \phi_{i_j})$ for all $j \in I'$ and suitable $i_j \in I$. Denote the map induced by these embeddings by $\iota : \coprod_{j \in I'} \tilde{U}'_j \to \coprod_{i \in I} \tilde{U}_i$. Then one can prove that the groupoid associated to the second atlas is the pullback along ι of the groupoid associated to the first atlas. Hence these two groupoids are Morita equivalent. Thus, if two orbifold atlasses on M are equivalent, i.e. they have a common refinement, then the associated groupoids are Morita equivalent.

Example 2.3.9. Suppose M is a smooth manifold and $\mathcal{F}$ a regular foliation of M. Suppose $i_T : T \hookrightarrow M$ is a full transversal in the sense that it intersects each leaf at least once (transversally). By the previous example the holonomy groupoid $\mathrm{Hol}(M, \mathcal{F}) \rightrightarrows M$ is Morita equivalent to the pullback groupoid $i_T^* \mathrm{Hol}(M, \mathcal{F}) \rightrightarrows T$ obtained by restriction to the transversal T. Note that $i_T^* \mathrm{Hol}(M, \mathcal{F}) \rightrightarrows T$ is étale. In general, one can show that any foliation groupoid is Morita equivalent to an étale groupoid.

2.4 Lie algebroids

Definition 2.4.1. A **Lie algebroid** is a triple $(p, [.,.], \rho)$, where $p : \mathcal{A} \to M$ is a real smooth vector bundle, $[.,.] : \Gamma^\infty(\mathcal{A}) \times \Gamma^\infty(\mathcal{A}) \to \Gamma^\infty(\mathcal{A})$ is a Lie bracket, and $\rho : \mathcal{A} \to TM$ is a smooth map of vector bundles, called the **anchor**, that induces a Lie algebra homomorphism $\rho : \Gamma^\infty(\mathcal{A}) \to \mathfrak{X}^\infty(M)$, such that for all $X, Y \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$ the following Leibniz identity is satisfied

$$[X, f\,Y] = f\,[X, Y] + (\rho(X)\,f)Y.$$

Remark 2.4.2. Because of this last property, one actually has a Lie bracket on the sheaf of smooth local sections of $\mathcal{A}$.

Proposition 2.4.3. *For any Lie groupoid $G \rightrightarrows M$ there exists a canonical Lie algebroid structure on $u^*(T^sG) \to M$, where $T^sG := \ker Ts$.*

Proof. Denote the right multiplication by an element $g \in G$ by $r_g : G_{t(g)} \to G_{s(g)}$. A section $X \in \Gamma^\infty(u^*(T^sG))$ can be extended to a right-invariant vector field on G

$$(r^*X)(g) := T_{u(t(g))} r_g \; X(t(g))$$

on G. The bracket of two vector fields $X, Y \in \Gamma^\infty(u^*(T^sG))$ is defined by

$$[X, Y] := u^*[r^*X, r^*Y].$$

The fact that this a Lie bracket follows from the fact that the usual bracket of vector fields on G is a Lie bracket and the fact that the Lie bracket of two right-invariant vector fields is again right invariant.

The anchor $\rho : u^*(T^sG) \to TM$ is defined by $\rho := Tt \circ r^*$. The Leibniz identity follows from the Leibniz identity for vector fields on G. $\square$

The Lie algebroid described in the above proposition is called the **Lie algebroid associated to a Lie groupoid** $G \rightrightarrows M$ and is denoted by $(\mathcal{A}(G) \to M, [.,], \rho)$. In the above proof we could have used left-invariant vector fields, $T^t(G) := \ker(Tt)$ and $Ts \circ l^*$ as an anchor; this would give us an isomorphic Lie algebroid, with the following notion of morphism.

Definition 2.4.4. A **morphism of Lie algebroids** $(\mathcal{A}_1 \to M_1, [.,.]_1, \rho_1) \to (\mathcal{A}_2 \to M_2, [.,.]_2, \rho_2)$ is a vector bundle map (ϕ, f)

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\phi} & \mathcal{A}_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow[f]{} & M_2 \end{array},$$

that preserves the anchor,

$$Tf \circ \rho_1 = \phi \circ \rho_2$$

and that preserves the bracket, i.e. if

$$\phi \circ X_1 = \sum_{i \in I} c_i \, f^* X_2^i$$

and

$$\phi \circ Y_1 = \sum_{j \in J} d_j \, f^* Y_2$$

for index sets I, J, sections $X_1, Y_1 \in \Gamma(\mathcal{A}_1)$ and $\{X_2^i\}_{i \in I}, \{Y_2^j\}_{j \in J} \in \Gamma(\mathcal{A}_2)$ and $c_i, d_j \in C^\infty(M_1)$), then

$$\begin{aligned} \phi \circ [X_1, Y_1]_1 \quad = & \sum_{i \in I, j \in J} c_i \, d_j \, f^*([X_2^i, Y_2^j]_2) \\ & + \sum_{j \in J} \mathcal{L}_{\rho(X_1)} d_j \, f^*(Y_2^j) - \sum_{i \in I} \mathcal{L}_{\rho(Y_1)} c_i \, f^*(X_2^i). \end{aligned}$$

Example 2.4.5. Suppose M is a smooth manifold. The zero bundle $M \times \{0\} \to M$ is the Lie algebroid associated to the trivial groupoid $M \rightrightarrows M$. The tangent bundle $TM \to M$ is a Lie algebroid endowed with the Lie bracket of vector fields and the identity as an anchor. It is the Lie algebroid associated to both the pair groupoid $M \times M \rightrightarrows M$ and to the fundamental groupoid $\pi_1(M) \rightrightarrows M$.

Definition 2.4.6. A Lie groupoid $G \rightrightarrows M$ is said to **integrate the Lie algebroid** $(\mathcal{A}, [.,.], \rho)$ if this Lie algebroid is isomorphic to the Lie algebroid associated to $G \rightrightarrows M$.

Hence both the pair groupoid $M \times M \rightrightarrows M$ and the fundamental groupoid $\pi_1(M) \rightrightarrows M$ integrate the tangent bundle $TM \to M$ as a Lie algebroid. In many situations the fundamental groupoid is the preferred integration, since it is **s-simply connected**. This means that each s-fiber ($s^{-1}(m) = G_m$ for $m \in M$) is simply connected.

Example 2.4.7 (Lie algebras and action Lie algebroids)**.** Suppose $\mathfrak{h}$ is a Lie algebra. Then $\mathfrak{h}$ can be seen as a Lie algebroid over a point. It is well-known that there exists a Lie group H that integrates $\mathfrak{h}$. Moreover, there exists a simply connected integration H.

Suppose $\mathfrak{h}$ acts smoothly on a smooth manifold M. Denote the action by $\alpha : \mathfrak{h} \to \mathfrak{X}^\infty(M)$. The trivial bundle $M \times \mathfrak{h} \to M$ has a Lie algebroid structure. The Lie bracket of two sections $X, Y \in \Gamma(M \times \mathfrak{h})$ is defined by

$$[X,Y](m) := [X(m), Y(m)]_{\mathfrak{h}} - (\mathcal{L}_{\alpha(X)}Y)(m) + (\mathcal{L}_{\alpha(Y)}X)(m),$$

and the anchor is $-\alpha$. This Lie algebroid is called an **action Lie algebroid** and is denoted by $\mathfrak{h} \ltimes M \to M$. Suppose that the action of $\mathfrak{h}$ on M is obtained from an action of H on M (denoted by a) by $\alpha(X)(m) := \frac{d}{dt}\, a(\exp(t\,X))(m)|_{t=0}$. Then the action Lie algebroid $\mathfrak{h} \ltimes M \to M$ is isomorphic to the Lie algebroid associated to action groupoid $H \ltimes M \rightrightarrows M$.

Example 2.4.8 (Gauge Lie algebroids)**.** Suppose H is a Lie group with Lie algebra $\mathfrak{h}$ and $p : P \to M$ a principal H-bundle. The canonical action of H on TP makes the projection $TP \to P$ H-equivariant. Consider the quotient bundle $TP/H \to P/H \cong M$. It obtains a Lie algebroid structure from the one on $TP \to P$. Indeed, a section X of $TP/H \to P/H$ corresponds to an H-equivariant vector field $\tilde{X}$ on P. The Lie bracket of two H-equivariant vector fields on P is again H-equivariant. Hence we can define a Lie bracket on sections of $TP/H \to M$ by

$$\widetilde{[X,Y]} := [\tilde{X}, \tilde{Y}].$$

The anchor $TP/H \to TM$ is defined by $\rho := Tp : TP/H \to TM$, which is well-defined as p is H-invariant. The Lie algebroid $(TP/H, [.,.], Tp)$ is called the **gauge Lie algebroid** of $p : P \to M$ and is isomorphic to the Lie algebroid associated to the gauge groupoid $P \times_H P \rightrightarrows M$.

Example 2.4.9 (Bundles of Lie algebras)**.** A **bundle of Lie algebras** is a Lie algebroid with trivial (i.e. zero) anchor. If $\mathcal{A}$ is locally isomorphic to a trivial bundle of Lie algebras, then it is called a **Lie algebra bundle**. Suppose H is a Lie group with Lie algebra $\mathfrak{h}$ and $p : P \to M$ is a principal H-bundle. The associated bundle $P \times_H \mathfrak{h} \to M$ is a Lie algebra bundle, where H acts on $\mathfrak{h}$ by the adjoint action. The Lie bracket is defined by $[[p, X], [p, Y]] := [p, [X, Y]_{\mathfrak{h}}]$. This is isomorphic to the Lie algebroid associated to the Lie group bundle $P \times_H H \to M$. It is also isomorphic to the **isotropy Lie algebroid** of the gauge Lie algebroid TP/H, i.e. the kernel of the anchor of TP/H. The isotropy Lie algebroid only exists for **regular** Lie algebroids, that is, those Lie algebroids for which the anchor has locally constant rank. This condition is imposed to ensure that $\ker(\rho)$ is a smooth vector bundle (of locally constant rank). If this is the case, there exists a short exact sequence of Lie algebroids

$$0 \to \ker(\rho) \to \mathcal{A} \to \operatorname{im}(\rho) \to 0.$$

In particular, for the gauge Lie algebroid one has the short exact sequence

$$0 \to P \times_H \mathfrak{h} \to TP/H \to TM \to 0,$$

which is called the **Atiyah sequence** of $P \to M$.

Another general source of bundles of Lie algebras is provided by smooth deformations of Lie algebras. A smooth deformation of Lie algebras is a vector space V together with a family of Lie brackets $\{[.,.]_m : V \times V \to V\}_{m \in M}$ smoothly varying over a parameter manifold M. The bundle of Lie algebras that can be associated to such a deformation is the trivial bundle $M \times V \to M$, with Lie bracket $[X, Y](m) = [X(m), Y(m)]_m$.

Example 2.4.10. Suppose M is a smooth manifold and $B \subset TM$ is an regular integrable distribution on M, i.e. a subvector bundle of $TM \to M$ closed under the Lie bracket of vector fields. Then $B \to M$ is a Lie algebroid with the inclusion $i : B \hookrightarrow TM$ as the anchor and the Lie bracket of vector fields as bracket.

By Frobenius' Theorem there exists a regular smooth foliation $\mathcal{F}$ of M that integrates B. The monodromy groupoid $\mathrm{Mon}(M, \mathcal{F})$ of $\mathcal{F}$ integrates the Lie algebroid $(B \to M, [.,.], i)$. But the holonomy groupoid $\mathrm{Hol}(M, \mathcal{F}) \rightrightarrows M$ integrates $(B \to M, [.,.], i)$ as well. Actually, any s-simply connected Lie groupoid $G \rightrightarrows M$ integrating $(B \to M, [.,.], i)$ satisfies $\mathrm{Hol}(M, \mathcal{F}) < G < \mathrm{Mon}(M, \mathcal{F})$. Such groupoids $G \rightrightarrows M$ are all foliation groupoids (cf. Example 2.2.3).

Example 2.4.11. Suppose $E \to M$ is a smooth complex vector bundle. A derivation of $E \to M$ is a pair (D, v) consisting of a linear map $D : \Gamma^\infty(E) \to \Gamma^\infty(E)$ and a vector field $v \in \mathfrak{X}^\infty(M)$ satisfying a Leibniz identity

$$D(f\xi) = f\, D(\xi) + (v \cdot f)\xi,$$

for all functions $f \in C^\infty(M)$ and sections $\xi \in \Gamma^\infty(E)$. The set of derivations $\mathrm{Der}(E)$ can be interpreted as the space of sections of a Lie algebroid $\mathcal{D}(E) \to M$ with anchor $\rho(D, v) := v$ and Lie bracket

$$[(D_1, v_1), (D_2, v_2)] := ([D_1, D_2], [v_1, v_2]).$$

One can show that $\mathcal{D}(E)$ is isomorphic to the Lie algebroid $\mathcal{A}(\mathrm{GL}(E, E))$ associated to the general linear groupoid $\mathrm{GL}(E, E) \rightrightarrows M$.

2.5 Haar systems

Suppose X and Y are locally compact spaces and $p : Y \to X$ is a continuous surjection. A **continuous family of Radon measures on** $p : Y \to X$ is a family of Radon measures $\{\nu_x\}_{x \in X}$ on Y such that

- the support of ν_x is a subset of $p^{-1}(x) =: Y_x$ and
- for every function $f \in C_c(Y)$ the function

$$x \mapsto \int_{y \in Y_x} f(y)\, \nu_x(dy)$$

  is continuous $X \to \mathbb{C}$.

Suppose $G \rightrightarrows X$ is locally compact, second countable continuous groupoid.

Definition 2.5.1. A **left Haar system on** $G \rightrightarrows X$ is a continuous family of Radon measures $\{\lambda^x\}_{x\in X}$ on $t : G \to X$ that is left-invariant, i.e. for all $x, y \in X$, $h \in G_x^y$, and $f \in C_c(G)$,

$$\int_{g\in G^x} f(hg)\lambda^x(dg) = \int_{g\in G^y} f(g)\lambda^y(dg).$$

There is an analogous notion of right Haar system.

Example 2.5.2. Suppose X is a locally compact space. Trivial counting measures $\{\lambda^x\}_{x\in X}$ form a Haar system on the trivial groupoid $X \rightrightarrows X$. If ν is a Radon measure on X, then $\{\nu^x := \nu\}_{x\in X}$ is a Haar system on the pair groupoid $X \times X \rightrightarrows X$.

Example 2.5.3. If H is a locally compact group and κ a left Haar measure on H. Then κ is a Haar system on $H \rightrightarrows pt$. Suppose H acts on a locally compact space X. Then $\{\lambda^x := \kappa\}_{x\in X}$ forms a left Haar system on the action groupoid $H \ltimes X \rightrightarrows X$.

Suppose $p : P \to X$ is a left principal H-bundle. Suppose $x \in X$ and $\phi : P|_U \to U\times H$ is a local trivialization of $P \to X$ on a neighborhood U of x. The obvious Haar system on $U \times H \to U$ can be pushed forward to $P|_U$, that is $\kappa^x := (\phi^{-1})_*\kappa$. Since κ is left H-invariant this unambiguously defines a continuous family of H-invariant Radon measures on $p : P \to X$. Suppose ν is a Radon measure on X. We define a continuous family of Radon measures on $p \circ pr_2 : P \times P \to X$ by

$$\tilde{\lambda}^x := \int_{y\in X} \kappa^y \times \kappa^x \, \nu(dy),$$

which is H-invariant under the diagonal action of H and hence descends to a left Haar system $\{\lambda^x\}_{x\in X}$ on the gauge groupoid $P \times_H P \rightrightarrows X$.

Example 2.5.4. Suppose $p : G \to X$ is a locally compact continuous family of groups. By a classical result there exists a left Haar measure on each group $G_x := p^{-1}(x)$, unique up to multiplication by a positive constant. Renault proves that there is a specific choice of measures λ^x on G_x for $x \in X$ such that they form a Haar system if and only if p is open. Indeed, one should construct a continuous function $F : G \to \mathbb{R}$ that is compactly supported on the fibers and that satisfies $0 \leq F \leq 1$ and $F \circ u = 1$. Then the measures λ^x should be chosen such that $\int_{G_x} F\lambda^x = 1$ for every $x \in X$.

For example, consider a group bundle $p : G \to X$ on a space X with fibers isomorphic to a fixed compact group K. We can take $F = 1$. Then by the above procedure the measure λ^x has to come from the normalized Haar measure on K for each $x \in X$.

Example 2.5.5. Suppose $G \rightrightarrows M$ is a *Lie* groupoid. There exists a Haar system on $G \rightrightarrows M$. Indeed, one easily sees that there exists a strictly positive smooth density ρ on the manifold $\mathcal{A}(G) = u^*(T^tG)$. This can be extended to a G-invariant density $\tilde{\rho}$ on T^tG. Then we define a Haar system on $G \rightrightarrows M$ by

$$\lambda^x(f) := \int_{G^x} f\tilde{\rho},$$

for all $f \in C_c(G)$.

Suppose $G \rightrightarrows X$ is a locally compact groupoid endowed with a left Haar system $\{\lambda^x\}_{x\in X}$. Then s and t are open maps and the orbit relation groupoid $R_G \rightrightarrows X$ is a continuous locally compact groupoid. Suppose the isotropy groupoid $I_G \to X$ is endowed with a Haar system $\{\lambda_x^x\}_{x\in X}$ (not necessarily related to the Haar system on $G \rightrightarrows X$). These measures induce a left G-invariant continuous family of measures $\{\lambda_x^y\}_{(y,x)\in R_G}$ on $t \times s : G \to R_G$ by

$$\lambda_x^y := (l_g)_* \lambda_x^x,$$

for some $g \in G_x^y$ (independence of the choice g follows from left invariance of the Haar system on I_G).

Proposition 2.5.6. *(cf. [68]) If $\{\lambda^x\}_{x\in X}$ is a left Haar system on $G \rightrightarrows X$ and $\{\lambda_x^x\}_{x\in X}$ is a left Haar system on $I_G \rightrightarrows X$, then there exists a left Haar system $\{\nu^x\}_{x\in X}$ on $R_G \rightrightarrows X$ such that for all $x \in X$ there is a decomposition*

$$\lambda^x = \int_{(y,x)\in R_G} \lambda_x^y \, \nu^x(d(y,x)).$$

Obviously, one can go the other way around: given a left Haar system on $I_G \rightrightarrows X$ and on $R_G \rightrightarrows X$, one forms a left Haar system on $G \rightrightarrows X$, using the same formula.

Example 2.5.7. Suppose $p : P \to X$ is a left principal H-bundle for a locally compact group H. Recall the Haar system $\{\lambda^x\}_{x\in X}$ on $P \times_H P \rightrightarrows X$ that we defined in Example 2.5.3 given a Haar measure κ on H and a Radon measure ν on X. One can apply the above Proposition 2.5.6 with the continuous family of measures $\{\lambda_x^x := \psi_*^x \kappa\}_{x\in X}$ on $I_{P\times_H P} \to X$, where $\psi^x : H \hookrightarrow P \times_H H \xrightarrow{\cong} I_{P\times_H P}$ is the inclusion of H at the fiber of $I_{P\times_H P} \to X$ at x. One obtains the Haar system $\{\nu^x := \nu\}_{x\in X}$ on $R_G = X \times X \rightrightarrows X$ in the decomposition.

Example 2.5.8. Suppose a locally compact group H acts on locally compact space X. Given a Haar measure κ on H we constructed a Haar system on $G := H \ltimes X \rightrightarrows X$ in Example 2.5.3. Suppose we have constructed a continuous family of measures on the family $I_G \to X$ of isotropy groups of the action using Example 2.5.4. Applying Proposition 2.5.6 one obtains a measure ν_x on each orbit Gx such that the decomposition of Proposition 2.5.6 holds.

Definition 2.5.9. Suppose $G \rightrightarrows X$ is a groupoid endowed with a Haar system $\{\lambda_x\}_{x\in X}$. A **cutoff function for** $G \rightrightarrows X$ is a function $X \to \mathbb{R}_{\geq 0}$ such that

- the support of $(c \circ s)|_{t^{-1}K}$ is compact for all compact sets $K \subset M$;
- for all $x \in X$, $\int_{G^x} c(s(g))\lambda^x(dg) = 1$.

A cutoff for $G \rightrightarrows X$ exists iff $G \rightrightarrows X$ is proper (cf. [77]). Cutoff functions will be useful in averaging processes, cf. Section 4.3. If G^x is compact for all $x \in X$, then one can simply take $c(x) = 1/\lambda^x(G^x)$.

Chapter 3

Continuous fields of Banach and Hilbert spaces

This chapter contains an introduction to continuous fields of Banach spaces and continuous fields of Hilbert spaces. These play an important rôle in Part II. In contrast with the previous chapter, this chapter does contain some proofs. This is because ingredients of the proofs are needed in Part II. Most of this material of the first section can be found in [21]; the material of the second section is new.

3.1 Continuous fields of Banach and Hilbert spaces

Suppose X is a locally compact Hausdorff space.

Definition 3.1.1. A **continuous field of Banach spaces over** X is a family of Banach spaces $\{\mathcal{B}_x\}_{x\in X}$ and a space of sections $\Delta \subset \prod_{x\in X} \mathcal{B}_x$, such that

(i) the set $\{\xi(x) \mid \xi \in \Delta\}$ equals $\mathcal{B}_x$ for all $x \in X$.

(ii) For every $\xi \in \Delta$ the map $x \mapsto \|\xi(x)\|$ is in[1] $C_0(X)$.

(iii) Δ is locally uniformly closed, i.e. if $\xi \in \prod_{x\in X} \mathcal{B}_x$ and for each $\varepsilon > 0$ and each $x \in X$, there is an $\eta \in \Delta$ such that $\|\xi(y) - \eta(y)\| < \varepsilon$ on a neighborhood of x, then $\xi \in \Delta$.

Remark 3.1.2. By composing the map $x \mapsto \|\xi(x)\|_{\mathcal{B}_x}$ with the norm on $C_0(X)$ one obtains a norm

$$\|\xi\| = \sqrt{\sup_{x\in X} \|\xi(x)\|^2_{\mathcal{B}_x}}$$

on Δ. From (iii) it follows at once that Δ is complete in this norm.

There is a subclass of these continuous fields which has our special interest.

Definition 3.1.3. A **continuous field of Hilbert spaces over** X is a family of Hilbert spaces $\{\mathcal{H}_x\}_{x\in X}$ and a space of sections $\Delta \subset \prod_{x\in X} \mathcal{H}_x$ that form a continuous field of Banach spaces.

[1] $f \in C_0(X)$ iff f is continuous and for every $\varepsilon > 0$ there exists a compact set K such that $f(x) < \varepsilon$ if $x \notin K$

Example 3.1.4. Suppose $p : E \to X$ is continuous complex vector bundle endowed with a Hermitian metric $g : E \times E \to \mathbb{C}$. Then $(\{E_x\}_{x\in X}, \Gamma_0(E))$ is a continuous field of Hilbert spaces.

Example 3.1.5. Suppose $\mathcal{H}$ is a fixed Hilbert space and X a topological space. Then $(\{\mathcal{H}\}_{x\in X}, \Gamma_0(\mathcal{H})$ is a (trivial) continuous field of Hilbert spaces.

Remark 3.1.6. In the case of a continuous field of Hilbert spaces, the condition (ii) in Definition 3.1.1 can be replaced by the requirement that for any $\xi, \eta \in \Delta$ the map $x \mapsto \langle \xi(x), \eta(x)\rangle_{\mathcal{H}_x}$ is in $C_0(X)$. The field is called **upper (lower) semi-continuous** if $x \mapsto \|\xi(x)\|$ is just upper (lower) continuous for every $\xi \in \Delta$.

Lemma 3.1.7. *If $(\{\mathcal{B}_x\}_{x\in X}, \Delta)$ is a continuous field of Banach spaces, then Δ is a left $C_0(X)$-module.*

Proof. Suppose $f \in C_0(X)$ and $\xi \in \Delta$. Let $\varepsilon > 0$ and $x \in X$ be given. Define

$$V_x := \{y \in X \mid |f(x) - f(y)| < \frac{\varepsilon}{\|\xi(x)\| + 1} \text{ and } |\|\xi(x)\| - \|\xi(y)\|| < 1\}$$

Then, for $y \in V_x$

$$\|f(y)\xi(y) - f(x)\xi(y)\| < \frac{\varepsilon}{\|\xi(x)\| + 1}\|\xi(y)\| < \varepsilon.$$

Since $f(x)\xi \in \Delta$, we conclude by (iii) that $f\xi \in \Delta$. □

Actually Δ is a Banach C^*-module as we shall see in Section 4.2.

Lemma 3.1.8. *If $(\{\mathcal{B}_x\}_{x\in X}, \Delta)$ is a continuous field of Banach spaces, then there is a topology on the total space $\mathcal{B} := \coprod_{x\in X} \mathcal{B}_x$ such that Δ equals the set of continuous sections $\Gamma_0(\mathcal{B}) := \{\xi \in \Gamma(\mathcal{B}) \mid \|\xi\| \in C_0(X)\}$.*

Proof. For each $\varepsilon > 0$, $V \subset X$ open and $\xi \in \Delta$, we define

$$U(\varepsilon, \xi, V) := \{h \in \mathcal{B} \mid \|h - \xi(p(h))\| < \varepsilon \text{ and } p(h) \in V\},$$

where $p : \mathcal{B} \to X$ is the projection of the total space on the base. One easily sees that these sets form a basis for a topology on $\mathcal{B}$. Indeed, suppose that $U(\varepsilon_1, \xi_1, V_1)$ and $U(\varepsilon_2, \xi_2, V_2)$ are two of them and $h \in \mathcal{B}$ lies in the intersection. By (i) there is a $\xi \in \Delta$ such that $\xi(x) = h$, where $x = p(h)$. Let $\varepsilon_i' = \varepsilon_i - \|h - \xi_i(x)\|$ for $i = 1, 2$. Choose any $\varepsilon > 0$ such that $\varepsilon < \varepsilon_i'$ for $i = 1, 2$. Define

$$V := \{x \in V_1 \cap V_2 \mid \|\xi(x) - \xi_i(x)\| < \varepsilon_i - \varepsilon \text{ for } i = 1, 2\}.$$

Then $U(\varepsilon, \xi, V) \subset U(\varepsilon_1, \xi_1, V_1) \cap U(\varepsilon_2, \xi_2, V_2)$.

Suppose $\xi \in \prod_{x\in X} \mathcal{B}_x$ is a continuous section. Let $\varepsilon > 0$ and $x \in X$ be given. Define $h := \xi(x)$. There is a $\xi' \in \Delta$ such that $\xi'(x) = h$. Let V be any open neighborhood of x, then $W := \xi^{-1}U(\varepsilon, \xi', V)$ is open and on W we have $\|\xi' - \xi\| < \varepsilon$. By (iii) we conclude that $\xi \in \Delta$.

Conversely, suppose $\xi \in \Delta$. Let $U(\varepsilon, \eta, V)$ be an open set in $\mathcal{B}$, then

$$\begin{aligned} \xi^{-1}U(\varepsilon, V, \eta) &= p(U(\varepsilon, \eta, V) \cap \xi(V)) \\ &= \{x \in X \mid \|\xi(x) - \eta(x)\| < \varepsilon\} \end{aligned}$$

Note that $\xi - \eta \in \Delta$, hence $x \mapsto \|\xi(x) - \eta(x)\|$ is continuous. We conclude that the above set is open, so that $\xi \in \Gamma_0(\mathcal{B})$. □

Remark 3.1.9. As a short notation we sometimes denote a continuous field of Banach spaces $(\{\mathcal{B}_x\}_{x\in X}, \Delta_{\mathcal{B}})$ by $(\mathcal{B}, \Delta)$.

Lemma 3.1.10. *For any continuous field of Banach spaces* $(\mathcal{B}, \Delta)$ *the map* $\|.\| : \mathcal{B} \to \mathbb{R}_{\geq 0}$ *is continuous.*

Proof. Suppose $h \in \mathcal{B}_x$ for certain $x \in X$. Given $\varepsilon > 0$, take a $\xi \in \Delta$ such that $\xi(x) = h$ and

$$V := \|\xi\|^{-1}(\|h\| - \varepsilon/2, \|h\| + \varepsilon/2).$$

This is an open set, since $\|\xi\| : X \to \mathbb{R}_{\geq 0}$ is continuous. So, $h' \in U(\varepsilon/2, \xi, V)$, with $h' \in \mathcal{B}_{x'}$ implies

$$|\|h'\|_{x'} - \|h\|_x| \leq \|h' - \xi(x')\| + |\|\xi(x')\|_{x'} - \|h\|_x| \leq \varepsilon,$$

which finishes the proof. □

Definition 3.1.11. A **morphism** $\Psi : (\mathcal{B}^1, \Delta^1) \to (\mathcal{B}^2, \Delta^2)$ **of continuous fields of Banach spaces** is a family of bounded linear maps $\{\Psi_x : \mathcal{B}^1_x \to \mathcal{B}^2_x\}_{x\in X}$ such that the induced map $\Psi : \mathcal{B}_1 \to \mathcal{B}_2$ on the total spaces satisfies

$$\{\Psi \circ \xi \mid \xi \in \Delta^1\} \subset \Delta^2$$

and

$$x \mapsto \|\Psi_x\|$$

is a locally bounded map.

Here $\|\Psi_x\|$ is the operator norm of Ψ_x,

$$\|\Psi_x\| := \sup_{\|h\|_{\mathcal{B}^1_x}=1} \|\Psi_x(h)\|_{\mathcal{B}^2_x}.$$

The first condition has to be satisfied only on a dense subset of Δ^1 ([21], Proposition 5).

Lemma 3.1.12. *The map* $\Psi : \mathcal{B}^1 \to \mathcal{B}^2$ *is continuous iff* Ψ *is a morphism of continuous fields of Banach spaces.*

Proof. "$\Leftarrow$" Suppose $h \in U(\epsilon_2, \xi_2, V_2) \subset \mathcal{B}_2$ and $p(h) = x$. By (i), there is a $\xi_1 \in \Delta_1$ such that $\xi_1(x) = h$. Since $\Psi(\xi_1) \in \Delta_2$, the set defined by

$$V_1 := \{y \in X \mid \|\Psi(\xi_1) - \xi_2\|(y) < \varepsilon/2\} \cap V_2$$

is open. Let $f : X \to \mathbb{R}$ be a locally bounded function such that $\|\Psi(\xi)\| < f\|\xi\|$ for all $\xi \in \Delta$. Let $V_1' \subset V_1$ be a small enough neighborhood of x such that f has a supremum K on V_1', then

$$\Psi(U(\frac{\varepsilon_2}{2K}, \xi_1, V_1') \subset U(\varepsilon_2, \xi_2, V_2)$$

Indeed, for any $h' \in U(\frac{\varepsilon_2}{2K}, \xi_1, V_1')$ with $p(h') = y$ we have

$$\begin{aligned}\|\Psi(h') - \xi_2(y)\| &= \|\Psi(h') - \Psi(\xi_1(y)) + \Psi(\xi_1(y)) - \xi_2(y)\| \\ &\le \|\Psi(h' - \xi_1(y))\| + \|\Psi(\xi_1(y)) - \xi_2(y)\| \\ &= K\|h' - \xi_1(y)\| + \tfrac{\varepsilon_2}{2} \\ &= K\tfrac{\varepsilon_2}{2K} + \tfrac{\varepsilon_2}{2} = \varepsilon_2.\end{aligned}$$

"$\Rightarrow$" $\Psi(\Delta_1) \subset \Delta_2$ by Lemma 3.1.8. Let $x \in X$ be any element. By continuity $\Psi^{-1}(U(1, 0, X))$ is open, so it contains an open neighborhood $U(\varepsilon, 0, V)$, where V is an open neighborhood of x. Hence, $\|\Psi\|$ is bounded on V. □

The map $\Psi : (\mathcal{B}^1, \Delta^1) \to (\mathcal{B}^2, \Delta^2)$ is an (isometric) isomorphism of continuous fields of Banach spaces if all the Ψ_x are (isometric) isomorphisms and $\Psi(\Delta^1) = \Delta^2$. In fact, one can replace the second condition by $\Psi(\Lambda) \subset \Delta^2$ for a dense subset $\Lambda \subset \Delta^1$ ([21], Proposition 6).

Let $(\{\mathcal{B}_x\}_{x\in X}, \Delta)$ be a continuous field of Banach spaces over X and $J : Y \to X$ a continuous map. Define the **pullback continuous field** $J^*(\{\mathcal{B}_x\}_{x\in X}, \Delta)$ as follows. The fiber $(J^*\mathcal{B})_y$ at $y \in Y$ is the Banach space $\mathcal{B}_{J(y)}$. The space of sections $J^*\Delta$ is the closure of the linear space generated by elements of the form $f \cdot J^*\xi$ for all $\xi \in \Delta$ and $f \in C_0(Y)$ in the usual norm (cf. Remark 3.1.2), which takes the form

$$\|f \cdot J^*\xi\| := \sqrt{\sup_{y\in Y} |f(y)|^2 \, \|\xi(y)\|^2_{\mathcal{B}_{J(y)}}}$$

on generators. The continuous field thus obtained is denoted by $(J^*\{\mathcal{B}_x\}_{x\in X}, J^*\Delta)$.

3.2 Dimension and local pseudo-trivializations

The **dimension of a continuous field of Hilbert spaces** $(\mathcal{H}, \Delta)$ over X is the supremum of the function

$$\dim : X \to \mathbb{Z}_{\ge 0} \cup \{\infty\}, x \mapsto \dim(\mathcal{H}_x).$$

A continuous field of Hilbert spaces is **uniformly finite-dimensional** if it has finite dimension. One should distinguish between uniformly finite-dimensional and **finite-dimensional** continuous fields, which means that each fiber is finite dimensional.

Example 3.2.1. Consider the field over $\mathbb{R}$ with $\mathcal{H}_x := \mathbb{C}^n$ if $x \in [-n, -n+1) \cup (n-1, n]$ for all $n \in \mathbb{N}$ and $\mathcal{H}_0 = 0$. The topology on the field comes from the inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ on the first n coordinates. The inner product on each fiber is the standard Hermitian metric on $\mathbb{C}^n$. This field is finite-dimensional, but not uniformly finite-dimensional.

Definition 3.2.2. A continuous field $(\mathcal{H}, \Delta)$ is **locally trivial** if for every $x \in X$ there exist a neighborhood $U \ni x$, a Hilbert space $\mathcal{H}'$ and an isomorphism of continuous fields $\mathcal{H}|_U \to U \times \mathcal{H}'$.

Example 3.2.3. Locally trivial finite-dimensional continuous fields of Hilbert spaces are known as complex vector bundles with Hermitian metric (cf. Example 3.1.4 and 3.1.5).

Obviously a continuous field is not always locally trivial, see e.g. Example 3.2.1 or

Example 3.2.4 (Winding stair)**.** Consider the continuous field over $\mathbb{R}^2$ defined as follows. Fix any $d \in \mathbb{N}$ (the number of stairs). For $\vec{x} \in \mathbb{R}_{\geq 0} \times \{0\}$ let $\mathcal{H}_{\vec{x}} = 0$. For $\vec{x}$ in the sector between angles $i\,2\pi/d$ and $(i+1)\,2\pi/d$ (including the positive halfline at $(i+1)\,2\pi/d$, unless $i = d-1$) take $\mathcal{H}_{\vec{x}} = \mathbb{C}^{i+1}$, where $i = 0, \ldots d-1$. Again, the topology on the field comes from the inclusion $\mathbb{C}^i \hookrightarrow \mathbb{C}^{i+1}$ on the first i coordinates. The inner product on each fiber is the standard Hermitian inner product on $\mathbb{C}^i$. Obviously, the field is not locally trivial at the origin.

Therefore, we introduce the notion of local pseudo-trivializations.

Definition 3.2.5. A **local pseudo-trivialization** of a continuous field $(\mathcal{H}, \Delta)$ on an open set $U \subset X$ is a family of open sets $\{U_i\}_{0 \geq i \geq \dim(\mathcal{H}|_U)}$, such that

$$\bigcup_{0 \geq i \geq \dim(\mathcal{H}|_U)} U_i = U$$

and

$$\mathcal{H}|_U \simeq \sum_{0 \geq i \geq \dim(\mathcal{H}|_U)} U_i \times \mathbb{C}^i.$$

Such local pseudo-trivializations shall be useful in Section 5.4.

Lemma 3.2.6. *Suppose $(\mathcal{H}, \Delta)$ is a uniformly finite-dimensional continuous field of Hilbert spaces over X. Then for any $x \in X$ there exists a neighborhood U_x of x such that $(\mathcal{H}, \Delta)$ admits a local pseudo-trivialization on U_x.*

Proof. We prove by induction on the dimension d of $(\mathcal{H}, \Delta)$. Suppose $d = 1$. If $\mathcal{H}_x \neq 0$, then there exists a $\xi \in \Delta$ such that $\xi(x) \neq 0$. By continuity, $\xi(y) \neq 0$ for y in a neighborhood U of x. Obviously, the map $U \times \mathbb{C} \to \mathcal{H}|_U$ given by $(y, z) \mapsto z\,\xi(y)$ is a local (pseudo-)trivialization.

If $\mathcal{H}_x = 0$, then we proceed as follows. For every $y \in X$ for which with $\mathcal{H}_y \neq 0$ there exists a section ξ_y such that $\xi_y(y)$ spans $\mathcal{H}_y$. Since X is locally compact, there exists an open neighborhood U (with compact closure) and a subset $\{y_i\}_{i \in I} \subset X$ such that $\{\mathrm{supp}(\xi_{y_i})\}_{i \in I}$ forms a locally finite cover of $U \cup \mathrm{supp}(\mathcal{H})$. For any subsequence $\{y_j\}_{j \in J}$ converging to x, we add the $\lim_{j \to J} \xi_{y_j} \in \Delta$ to the set of sections indexed by I. By continuity the sum $\xi(y) := \sum_{i \in I} \xi_{y_i}(y)$ still spans $\mathcal{H}_y$ for y in a small enough open neighborhood U' of x. Again, the map $U' \times \mathbb{C} \to \mathcal{H}|_U$ given by $(y, z) \mapsto z\,\xi(y)$ is a local (pseudo-)trivialization.

Suppose $(\mathcal{H}, \Delta)$ has dimension d. If $\mathcal{H}_x \neq 0$, then there exists a $\xi \in \Delta$ such that $\xi(x) \neq 0$. Again, then $\xi(y) \neq 0$ for y in a neighborhood U of x. Hence $\mathcal{H}|_U \simeq \mathrm{span}\,\xi|_U \oplus \mathcal{H}'$ for some continuous field $(\mathcal{H}', \Delta')$ over U. The field $(\mathcal{H}', \Delta')$ has dimension $d-1$, so by the induction hypothesis there is an isomorphism

$$\phi : \sum_{i=0}^{d-1} U_i' \times \mathbb{C}^i \to \mathcal{H}'|_{U'}$$

on an open neighborhood U' of x. Hence, an isomorphism

$$\sum_{i=1}^{d} U'_{i-1} \times \mathbb{C}^i \to \mathcal{H}|_{U'}$$

is given by

$$(y, \vec{z}) \mapsto \phi(y, z_1, \dots, z_{i-1}) + z_i \, \xi(y).$$

If $\mathcal{H}_x = 0$, then we construct a local section ξ on a neighborhood of x as in the case $d = 1$. Proceed as above. □

Corollary 3.2.7. *A continuous field $(\mathcal{H}, \Delta)$ over a compact space X is uniformly finite-dimensional iff Δ is finitely generated over $C_0(X)$.*

Lemma 3.2.8. *For a uniformly finite-dimensional continuous field of Hilbert spaces over X the dimension is a lower semi-continuous function*

$$\dim : X \to \mathbb{Z}_{\geq 0} \subset \mathbb{R}.$$

That is, $\dim : X \to \mathbb{Z}_{\geq 0}$ *has a local minimum at every point.*

Proof. Suppose $(\mathcal{H}, \Delta)$ is such a continuous field of Hilbert spaces and $x \in X$. Choose sections $\xi_j^x \in \Delta$ for $j = 1, \dots, \dim(\mathcal{H}_x)$, such that $\{\xi_j^x(x)\}_{j=1}^{\dim(\mathcal{H}_x)}$ forms a basis of $\mathcal{H}_x$. Let V_x be the set on which their images stay linearly independent and non-zero. This set is open, since, for a local pseudo-trivializations ϕ,

$$x \mapsto \det(\phi^* \xi_1^x \mid \dots \mid \phi^* \xi_{\dim \mathcal{H}_x}^x) = \det((\langle \phi^* \xi_k^x, \phi^* \xi_l^x \rangle)_{kl})$$

is continuous. Indeed, this last expression is a polynomial in $\langle \phi^* \xi_k^x, \phi^* \xi_l^x \rangle$ for $1 \leq k, l \leq j$ which are continuous . □

Chapter 4

C^*-algebras, K-theory and KK-theory

In this chapter we introduce some theory of C^*-algebras, K-theory of C^*-algebras and KK-theory. There is a vast literature on C^*-algebras. As an introduction on C^*-algebras the author used [44]. Learning K-theory of C^*-algebras is possible using e.g. [27, 33, 5, 91]. Good references on KK-theory are [5] and [33]. Introductory texts are [32, 71].

4.1 C^*-algebras

Definition 4.1.1. A **complex Banach algebra** is an associative algebra A over the complex numbers $\mathbb{C}$, endowed with a norm $\|.\| : A \to \mathbb{R}$ such that A is a Banach space and $\|a\,b\| \leq \|a\|\,\|b\|$ for all $a, b \in A$.

A morphism of Banach algebras A and B is a bounded linear map $\phi : A \to B$ satisfying $\phi(a\,b) = \phi(a)\,\phi(b)$.

Definition 4.1.2. A **complex Banach $*$-algebra** is a complex Banach algebra $(A, \|.\|)$ together with an involutive conjugate-linear norm-preserving antihomomorphism $* : A \to A$. A **C^*-algebra** is a Banach $*$-algebra $(A, \|.\|, *)$ satisfying $\|a\,a^*\| = \|a\|\,\|a^*\| = \|a\|^2$.

A **morphism of complex Banach $*$-algebras** A and B is a morphism of Banach algebras $\phi : A \to B$ satisfying $\phi(a^*) = \phi(a)^*$. A **morphism of C^*-algebras** A and B is a morphism $A \to B$ of A and B as complex Banach $*$-algebras. We denote the ensuing category of C^*-algebras by $\mathsf{C^*\text{-}ALG}$.

Example 4.1.3. Suppose X is a locally compact space. Then the complex-valued functions $C_0(X)$ that vanish at infinity form a commutative C^*-algebra under pointwise multiplication of functions and involution given by $f^*(x) := \overline{f(x)}$. The norm is the supremum norm of functions. One can prove that any commutative C^*-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. This yields an equivalence of categories between the category of commutative C^*-algebras and the category of locally compact Hausdorff spaces with proper maps.

Example 4.1.4. Suppose $\mathcal{H}$ is a Hilbert space. The bounded linear operators on $\mathcal{H}$ form a C^*-algebra $\mathcal{B}(\mathcal{H})$. Indeed, the multiplication is given by composition. The norm is the operator norm $\|P\| := \sup_{h\in\mathcal{H},\|h\|=1} \|P(h)\|$ for all $P \in \mathcal{B}(\mathcal{H})$. The involution applied to an operator P is the adjoint operator P^* of P. In particular, if $\mathcal{H} = \mathbb{C}^n$, then $\mathcal{B}(\mathcal{H}) = M_n(\mathbb{C})$ is a C^*-algebra. Any norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. Conversely, one can prove that any C^*-algebra is a norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

A representation of a Banach $*$-algebra A is a morphism of Banach $*$-algebras $A \to \mathcal{B}(\mathcal{H})$. A representation of a C^*-algebra A is a representation of A as a Banach $*$-algebra.

Example 4.1.5. Suppose $\mathcal{H}$ is a Hilbert space. For elements $h, h' \in \mathcal{H}$ one can define the operator $h\langle h', .\rangle : \mathcal{H} \to \mathcal{H}$. The operators of this type generate a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. The norm-closure of this algebra is the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators. A given operator $T \in \mathcal{B}(\mathcal{H})$ is compact iff T maps the unit ball in $\mathcal{H}$ (which is not compact, if $\mathcal{H}$ is infinite-dimensional) to a set with compact closure.

Example 4.1.6. Suppose A is a Banach $*$-algebra. There exists a universal enveloping C^*-algebra B of A and a morphism $u : A \to B$ such that any morphism $A \to C$ factors through u. It is constructed by considering the sum

$$\pi := \bigoplus_{\pi\in\hat{A}} \pi : A \to \bigoplus_{\pi\in\hat{A}} \mathcal{B}(\mathcal{H}_\pi)$$

of all the representations of A. Then B is the closure of $\pi(A)$. From the universal property it follows that u gives a bijection between the representations of A and B.

Example 4.1.7. Consider a locally finite, directed graph $E = (E_0, E_1)$. For any path μ in E, let $s(\mu)$ denote the start vertex, $t(\mu)$ the end vertex of μ and $|\mu|$ the length of μ. There exists a **C^*-algebra $C^*(E)$ associated to the graph** E. It is the universal C^*-algebra generated by a family of partial isometries $\{S_e\}_{e\in E_1}$ and mutually orthogonal projections $\{p_v\}_{v\in E_0}$ on a Hilbert space $\mathcal{H}$ satisfying the relations

$$S_e^* S_e = p_{t(e)}, \quad \sum_{e\in E, s(e)=v} S_e S_e^* = p_v.$$

Example 4.1.8. Suppose H is a unimodular locally compact group and λ a Haar measure on H. The set of compactly supported complex-valued functions $C_c(H)$ forms a $*$-algebra over $\mathbb{C}$, with multiplication given by convolution

$$(f * f')(h) := \int_{k\in H} f(k) f'(k^{-1}h)\lambda(dh)$$

and the involution given by

$$f^*(h) := \overline{f(h^{-1})},$$

where $f, f' \in C_c(H)$ and $h \in H$.

Consider the Hilbert space $L^2(H, \lambda)$ of square-integrable functions on H. The map $\pi : C_c(H) \to \mathcal{B}(L^2(H, \lambda))$ given by

$$f \mapsto f * \cdot$$

is a morphism of $*$-algebras. One can check that $\|\pi(f)\| \leq \|f\|_{L^1(H)}$, hence $\pi(f) = f * \cdot$ is indeed a bounded operator on $L^2(H, \lambda)$. The **reduced C^*-algebra of the group** H $C_r^*(H)$ is the algebra obtained as the closure of $\pi(C_c(H)) \subset \mathcal{B}(L^2(H, \lambda))$ in the operator norm. On the other hand, one can consider the closure of $C_c(H)$ to a Banach $*$-algebra $L^1(H)$. This has a universal enveloping C^*-algebra (cf. Example 4.1.6): the (full) **C^*-algebra of the group** H, denoted by $C^*(H)$. There is a canonical bijection between the representations of $L^1(G)$ and the representations of $C^*(H)$.

If H is an Abelian group, then one can prove that $C_r^*(H) \cong C_0(\hat{H})$, where $\hat{H}$ denotes the unitary dual of H; the isomorphism is given by the Fourier transform.

Analogously, we shall define the reduced C^*-algebra of a groupoid (cf. Section 4.3).

Example 4.1.9. Other important examples of C^*-algebra are the algebras of adjointable and compact operators on a Hilbert C^*-module. These shall be discussed in Section 4.2.

Remark 4.1.10. (i) If A is a non-unital C^*-algebra, then one can construct a **unitization** C^*-algebra A^u. One defines $A^u := A \times \mathbb{C}$ with norm

$$\|(a, z)\| := \sup_{b \in A, \|b\| \leq 1} \|ab + zb\|$$

and involution

$$(a, z)^* := (a^*, \bar{z}).$$

(ii) The algebraic tensor product $A \otimes B$ of two C^*-algebra can be completed to a C^*-algebra in several norms. We shall not discuss this. We only remark that a C^*-algebra A is **nuclear** if for every C^*-algebra B the tensor product $A \otimes B$ has a unique norm, namely the so-called spatial norm. For later use, let's mention that $M_n(\mathbb{C})$ and $\mathcal{K}(\mathcal{H})$ are nuclear (but $\mathcal{B}(\mathcal{H})$ is not nuclear, for example). Hence, if A is a C^*-algebra, then $M_n(A) \cong M_n(\mathbb{C}) \otimes A$ and $A \otimes \mathcal{K}$ are well-defined unique C^*-algebras, where $\mathcal{K}$ denotes the C^*-algebra of compact operator on a standard separable Hilbert space (say $l^2(\mathbb{N})$).

(iii) Two $*$-homomorphisms $f, g : A \to B$ of C^*-algebras A and B are homotopic if there exists a family of $*$-homomorphisms $\{f_t : A \to B\}_{t \in [0,1]}$ such that $f_0 = f$, $f_1 = g$ and for all $a \in A$ the map $t \mapsto f_t(a)$ is norm-continuous. Another way to view such a homotopy is as a morphism of C^*-algebras $H : A \to C([0, 1], B)$ such that $(e_0)_* H = f_0$ and $(e_1)_* H = f_1$, where $e_0, e_1 : C([0, 1], B) \to B$ are the evaluation morphisms at respectively 0 and 1.

4.2 Banach/Hilbert C^*-modules

Let A be a C^*-algebra and A^+ the set of **positive elements** in A, i.e. elements of the form $a\, a^*$ for some $a \in A$.

Definition 4.2.1. A **left Banach A-module** is a Banach space Δ that has a left A-module structure $A \to \mathcal{B}(\Delta)$ and a linear map $\| \cdot \| : \Delta \to A^+$ such that for all $\xi, \eta, \chi \in \Delta$ and $a \in A$:

(i) the norm on Δ satisfies $\|\xi\|_\Delta = \sqrt{\|(\|\xi\|^2)\|_A}$,

(ii) $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$,

(iii) $\|a\xi\| = |a|\|\xi\|$, where $|a| := \sqrt{a^* a}$,

(iv) $\|\xi\| = 0$ iff $\xi = 0$.

Definition 4.2.2. A **left Hilbert A-module** is a Banach space Δ that has a left A-module structure $A \to \mathcal{B}(\Delta)$ and a sesquilinear pairing $\langle \cdot, \cdot \rangle : \Delta \times \Delta \to A$ such that for all $\xi, \eta, \chi \in \Delta$ and $a \in A$:

(i) the norm on Δ satisfies $\|\xi\|_\Delta = \sqrt{\| \langle \xi, \xi \rangle \|_A}$,

(ii) $\langle \xi, \eta + \chi \rangle = \langle \xi, \eta \rangle + \langle \xi, \chi \rangle$,

(iii) $\langle \xi, a\eta \rangle = a \langle \xi, \eta \rangle$,

(iv) $\langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*$,

(v) $\langle \xi, \xi \rangle > 0$ iff $\xi \neq 0$.

The pairing is also called the A-valued inner product. Obviously, every Hilbert A-module is a Banach A-module in the A-valued norm $\|\xi\|_\Delta = \sqrt{\langle \xi, \xi \rangle}$. There is an analogous notion of right Hilbert A-module. A Hilbert A-module is called **full** if the image of the A-valued inner product is dense in A.

Example 4.2.3. Suppose A is a C^*-algebra. Then A is a left A-module under left multiplication. The A-valued inner product is given by $\langle a, b \rangle := a^* b$ for all $a, b \in A$.

A **morphism of Banach A-modules** is a bounded linear operator $\Psi : \Delta_1 \to \Delta_2$ that intertwines the A-action. In the case that $A = C_0(X)$ for a locally compact space X, the boundedness of Ψ implies that $\|\Psi\|$ is a locally bounded map $X \to \mathbb{R}$.

Theorem 4.2.4. *There is an equivalence of categories of continuous fields of Banach (respectively Hilbert) spaces and left Banach (respectively Hilbert) $C_0(X)$-modules.*

Proof. (sketch, for a full proof see [21] §4), Suppose $(\mathcal{B}, \Delta)$ is a continuous field of Banach spaces. Then Δ is a $C_0(X)$-module, as proven in Lemma 3.1.7. Its completeness as a Banach space follows immediately from locally uniform completeness. This is one direction of the correspondence.

For the other direction, suppose Λ is a Banach $C_0(X)$-module. Define, for all $x \in X$

$$N_x := \{h \in \Lambda \mid \|h\|(x) = 0\}$$

and $\mathcal{B}_x := \Lambda / N_x$. Denote the projection by $\pi_x : \Lambda \to \Lambda/N_x$. Define the space of sections by

$$\Delta := \{\xi_\lambda := (x \mapsto \pi_x(\lambda)) \mid \lambda \in \Lambda\}.$$

We check that this is indeed a continuous field of Banach spaces.

(i) $\{\xi_\lambda(x) \mid \xi_\lambda \in \Delta\} = \Lambda/N_x$ trivially;

(ii) $x \mapsto \|\xi_\lambda(x)\| = \|\lambda\|(x)$ is by definition continuous;

(iii) suppose $\lambda \in \prod_{x\in X} \Lambda/N_x$ and suppose λ is locally uniformly close to sections in Δ. We want to show that this implies $\lambda \in \Delta$. Since Λ is complete as a Banach space it suffices to show globally uniformly close to a section in Δ. This one shows using a partition of unity argument. We omit the details.

If one begins with a Banach $C_0(X)$-module Λ, then produces a continuous field of Banach spaces, and from that again constructs a Banach $C_0(X)$-module, one trivially recovers Λ, up to isomorphism.

On the other hand, from a continuous field $(\{\mathcal{B}_x\}_{x\in X}, \Delta)$ one obtains the Banach $C_0(X)$-module Δ and once again this gives rise to a continuous field $(\{\Delta/N_x\}_{x\in X}, \Delta)$. An isomorphism $\Delta/N_x \to \mathcal{B}_x$ is given by $[\xi] \mapsto \xi(x)$. □

The well-known Serre-Swan theorem states that for (locally) compact Hausdorff spaces X there exists an equivalence of categories between finitely generated projective Hilbert $C(X)$-modules and locally trivial finite-dimensional continuous fields of Hilbert spaces (i.e. finite rank vector bundles) over X. Indeed, as mentioned, finitely generated Hilbert $C(X)$-modules Δ correspond to uniformly finite-dimensional continuous fields. Moreover, one can show that Δ being projective corresponds to the field being locally trivial.

Example 4.2.5. Suppose $\pi : Y \to X$ is a continuous surjection endowed with a continuous family of Radon measures $\{\nu_x\}_{x\in X}$ (cf. Section 2.5). For any $p \in \mathbb{R}_{\geq 1}$ consider the norm on $C_c(Y)$ given by

$$\|f\|_p := \sup_{x\in X} \|f|_{Y_x}\|_{L^p(Y_x,\nu_x)}.$$

Define $\Delta^p_\pi(Y)$ to be the closure of $C_c(Y)$ with respect to this norm. One easily sees that this is a Banach $C_0(X)$-module with $C_0(X)$-valued norm given by

$$\|f\|(x) := \|f|_{Y_x}\|_{L^p(Y_x,\nu_x)} = \left(\int_{Y_x} |f(y)|^p \nu_x(dy)\right)^{1/p}.$$

The continuous field associated to this Banach $C_0(X)$-module is denoted by

$$(\hat{L}^p_\pi(Y), \Delta^p_\pi(Y))$$

. The fiber at $x \in X$ equals $L^p(Y_x, \nu_x)$.

If $p = 2$, one obtains a Hilbert $C_0(X)$-module and hence a continuous field of Hilbert spaces. The $C_0(X)$-valued inner product is given on $C_c(Y)$ by

$$\langle f, f'\rangle (x) := \langle f|_{Y_x}, f'|_{Y_x}\rangle_{L^2(Y_x,\nu_x)} = \int_{Y_x} \overline{f(y)} f'(y)\nu_x(dy).$$

Suppose A is a C^*-algebra and Δ and Δ' are Hilbert A-modules. A morphism of Hilbert C^*-modules $T : \Delta \to \Delta'$ is **adjointable** if there exists a unique morphism $T^* : \Delta' \to \Delta$ satisfying

$$\langle \xi, T\eta\rangle = \langle T^*\xi, \eta\rangle\,,$$

for all $\xi \in \Delta'$ and $\eta \in \Delta$. Denote the **algebra of adjointable operators** $\Delta \to \Delta'$ by $\mathrm{Hom}_A(\Delta, \Delta')$. Note that adjointable maps are bounded. One can prove that

$\mathcal{B}_A(\Delta) := \mathrm{Hom}_A(\Delta, \Delta)$ is a C^*-algebra. Analogously to the case of Hilbert spaces, it has a $*$-subalgebra $\mathcal{K}_A(\Delta)$ of **A-compact operators on the Hilbert A-module**: the norm-closure of the algebra generated by operators of the form $\eta\langle\xi, .\rangle$ for $\eta, \xi \in \Delta$.

Suppose A and B are C^*-algebras.

Definition 4.2.6. A **left Hilbert (A, B)-bimodule** is a Banach space Δ with a left Hilbert A-module structure and a right representation $B \to \mathcal{B}(\Delta)$ of B.

There is an analogous notion of right Hilbert (A, B)-bimodule. Suppose C is another C^*-algebra, Δ_1 a Hilbert (A, B)-bimodule and Δ_2 a Hilbert (B, C)-bimodule. Consider the algebraic tensor product $\Delta_1 \otimes \Delta_2$. It has a canonical (A, C)-bimodule structure and an A-valued bilinear form given by

$$\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \rangle_A := \langle \xi_1, \eta_1 \langle \eta_2, \xi_2 \rangle_B \rangle_A .$$

Consider the equivalence relation on $\Delta_1 \otimes \Delta_2$ generated by $\xi\, b \otimes \eta \sim \xi \otimes b\, \eta$ for $\xi \in \Delta_1$, $\eta \in \Delta_2$ and $b \in B$. The quotient $\Delta_1 \otimes \Delta_2/\sim$ induces an A-valued inner product, that in turn gives rise to a norm. The completion of $\Delta_1 \otimes \Delta_2/\sim$ with respect to this norm is a Hilbert (A, C)-bimodule and is denote by $\Delta_1 \otimes_B \Delta_2$. There is a category of C^*-algebras, where the morphisms $A \to B$ are isomorphism classes of left Hilbert (B, A)-modules (with obvious notion of morphism) and the composition is the tensor product we just introduced. We denote this category by $\mathsf{C^*\text{-}ALG_b}$. If A and B are isomorphic in this category, then they are called **Morita equivalent C^*-algebras**.

A functor

$$\mathsf{C^*\text{-}ALG} \to \mathsf{C^*\text{-}ALG_b},$$

that is the identity on objects, is given by $(f : A \to B) \mapsto [B, f]$, where $[B, f]$ denotes the class of the canonical left Hilbert (B, A)-module B, with right A-action $A \to \mathcal{B}(B)$, given by $a \mapsto f(a)$.

Example 4.2.7. Suppose A is a C^*-algebra. Then A and $M_n(A)$ are Morita equivalent. A morphism is given by the class of A^n as a $(A, M_n(A))$-bimodule. The A-valued inner product is defined by

$$\langle (a_1, \cdots, a_n), (b_1, \ldots, b_n) \rangle = a_1^* b_1 + \ldots + a_n^* b_n,$$

for $a_1, \cdots, a_n, b_1, \ldots, b_n \in A$.

Remark 4.2.8. As a preparation for KK-theory (cf. Section 4.5) consider a possible notion of homotopy in the category $\mathsf{C^*\text{-}ALG_b}$ of C^*-algebras. A homotopy from $[\Delta_0] : A \to B$ to $[\Delta_1] : A \to B$ could be defined as a left Hilbert $(C([0, 1], B), A)$-bimodule such that for $i = 0, 1$

$$[B, e_i] \circ [\Delta] = [\Delta_i].$$

This is not an interesting notion though, since any $[\Delta] : A \to B$ is homotopic to the class of the trivial Hilbert (B, A)-module $[0]$. A homotopy is given by the Hilbert $(C([0, 1], B), A)$-module $\{\xi \in C([0, 1], \Delta) \mid \xi(1) = 0\}$. Non-trivial homotopy classes associated to Hilbert bimodules are obtained by considering homotopy classes of Hilbert bimodules endowed with a bounded operator with certain properties. This is the essence of KK-theory. Another idea might be to demand that the Hilbert modules are full.

4.3 The convolution algebra of a groupoid

Suppose $G \rightrightarrows X$ is a locally compact groupoid endowed with a Haar system $\{\lambda^x\}_{x\in X}$. An associative product on the space of compactly supported functions $C_c(G)$ is defined by convolution

$$(f * f')(g) := \int_{h\in G^{t(g)}} f(h)f'(h^{-1}g)\lambda^{t(g)}(dh),$$

for $f, f' \in C_c(G)$ and $g \in G$. An involution on $C_c(G)$ is given by

$$f^*(g) := \overline{f(g^{-1})},$$

for $f \in C_c(G)$ and $g \in G$.

We shall embed this algebra as a $*$-subalgebra of a C^*-algebra and then take the norm-closure to obtain a C^*-algebra associated to a groupoid, generalizing the construction for groups. Consider the continuous field of Hilbert spaces $(\hat{L}^2_s(G), \Delta^2_s(G))$ (cf. Example 4.2.5). The map $f \mapsto f*\cdot$ embeds $C_c(G)$ in the C^*-algebra $\mathcal{B}_{C_0(X)}(\Delta^2_s(G))$. The norm-closure of the image is the **reduced C^*-algebra of the groupoid** $G \rightrightarrows X$ (and the Haar system $\{\lambda^x\}_{x\in X}$). It is denoted by $C^*_r(G)$.

Remark 4.3.1. Different Haar systems give rise to Morita equivalent C^*-algebras. A generalized morphism of groupoids gives rise to a left Hilbert bimodule of the reduced C^*-algebras associated to the groupoid. That is, the correspondence $G \mapsto C^*_r(G)$ is functorial (cf. [59, 45])

$$\mathsf{GPD_b} \to \mathsf{C^*\text{-}ALG_b}$$

(fixing a Haar system for each groupoid). In particular, Morita equivalent groupoids induce Morita equivalent reduced C^*-algebras.

Example 4.3.2. Suppose X is a locally compact space. The C^*-algebra associated to $X \rightrightarrows X$ is equal to $C_0(X)$. Suppose ν is a Radon measure on X. This gives a Haar system on the pair groupoid. The C^*-algebra $C^*_r(X \times X \rightrightarrows X)$ is isomorphic to $\mathcal{K}(L^2(X,\nu))$. This C^*-algebra in turn is Morita equivalent to $\mathbb{C}$, which is no surprise since $X \times X \rightrightarrows X$ is Morita equivalent to $pt \rightrightarrows pt$ as a groupoid and $C^*_r(pt) = \mathbb{C}$.

Example 4.3.3. Suppose H is a locally compact unimodular group endowed with Haar measure κ. If $p : P \to X$ is a principal H-bundle, then $C^*_r(P \times_H P \rightrightarrows X) \cong \mathcal{K}(L^2(X,\nu)) \otimes C^*_r(H)$. This is Morita equivalent to $C^*_r(H)$. Again this is expected, since $P \times_H P \rightrightarrows X$ is Morita equivalent to $H \rightrightarrows pt$.

Example 4.3.4. Suppose $p : Y \to X$ is a continuous surjection. Consider the groupoid $G := Y\,{}_p\!\times_p Y \rightrightarrows X$. Suppose there exists a continuous family of Radon measures $\{\lambda^x\}_{x\in X}$ on p. Then $C^*_r(G)$ is isomorphic to the C^*-algebra $\mathcal{K}_{C_0(X)}(\Delta^2(Y))$ of compact operators on the Hilbert $C_0(X)$-module $\Delta^2(Y)$ of sections of the continuous field of Hilbert spaces $(\{L^2(p^{-1}(x))\}_{x\in X}, \Delta^2(Y))$.

4.4 K-theory of C^*-algebras

Suppose A is a unital C^*-algebra. The stabilization of A is the C^*-algebra $A \otimes \mathcal{K}$., for a 'standard' C^*-algebra of compact operators $\mathcal{K}$ on a countable Hilbert space. Let $P(A)$ denote the set of (compact) **projections** in $A \otimes \mathcal{K} \cong \mathcal{K}_A(A \otimes \mathcal{H})$, i.e. A-compact operators that satisfy $p^2 = p = p^*$. We say that $p \in P(A)$ and $q \in P(A)$ are equivalent $p \sim q$ if there exists a unitary $u \in A \otimes \mathcal{K}$ such that $u\,p\,u^* = q$. Note that there exists a canonical isomorphism $A \otimes \mathcal{K} \oplus A \otimes \mathcal{K} \to A \otimes (\mathcal{K} \oplus \mathcal{K}) \cong A \otimes \mathcal{K}$. We can use this isomorphism to define a semigroup structure on $P(A)/\sim$ by $[p] + [q] := [p+q]$.

Definition 4.4.1. The zeroth K-theory $K_0(A)$ of A is the Grothendieck group of the semi-group $P(A)/\sim$.

Remark 4.4.2. (i) A projection $p \in P(A)$ gives rise to a Hilbert A-module $p(A \otimes \mathcal{H})$. One can even show that $p(A \otimes \mathcal{H}) \cong p'(A^n)$ for a projection $p' \in M_n(A)$ for some $n \in \mathbb{N}$, and hence the module is finitely generated and projective, algebraically. Conversely, every finitely generated, projective A-module is isomorphic to a Hilbert A-module of the form $p(A \otimes \mathcal{H})$ for some $p \in P(A) \subset \mathcal{K}_A(A \otimes \mathcal{H})$. Compare this to the fact that, by the Kasparov absorption theorem, any *countably generated* (not necessarily projective) Hilbert A-module is the image $p(A \otimes \mathcal{H})$ of a *bounded* projection $p \in \mathcal{B}_A(A \otimes \mathcal{H})$. Consider, for example, the Hilbert $C_0(\mathbb{R})$-module $C_0^0(\mathbb{R}) = \{f \in C_0(\mathbb{R}) \mid f(0) = 0\}$. This is not projective, but it is the image of a non-trivial bounded projection in $\mathcal{B}_{C_0(\mathbb{R})}(C_0(\mathbb{R}) \otimes \mathcal{H})$.

(ii) Two projections $p, q \in P(A)$ are **homotopy equivalent** if there exists a norm-continuous path in $P(A)$ from p to q. One can show that homotopy equivalence equals the previously given equivalence relation on $P(A)$.

(iii) One can see that K_0 extends to a covariant functor from the category of unital C^*-algebras to the category of Abelian groups.

(iv) Suppose that A is non-unital. Consider the projection $p : A^u = A \times \mathbb{C} \to \mathbb{C}$. We define $K_0(A) := \ker(K_0(p))$. In this way K_0 extends to a functor on non-unital C^*-algebras.

Example 4.4.3. One has $K_0(\mathbb{C}) \cong \mathbb{Z}$ generated by the equivalence class of a rank one projector in $\mathcal{K}$.

Example 4.4.4. If $A = C_0(X)$ for some locally compact space X, then $K_0(C_0(X))$ equals the topological K-theory $K^0(X)$ of X. This is the Grothendieck group of the semi-group of isomorphism classes of complex vector bundles on X with direct sums as group operation, cf. Theorem 4.2.4 and the remarks on the Serre-Swan theorem after the proof.

Definition 4.4.5. The higher K-theory groups are defined by suspension, i.e.

$$K_n(A) := K_0(A \otimes C_0(\mathbb{R}^n)) \cong K_0(A \otimes C_0(\mathbb{R})^{\otimes n}),$$

for $n \in \mathbb{Z}_{\geq 0}$.

We list the following properties of K_n, $n \in \mathbb{Z}_{\geq 0}$:

(i) $K_n(A) \cong K_{n+2}(A)$ (Bott periodicity);

(ii) If $f, g : A \to B$ are homotopy equivalent, then $f_* = g_* : K_n(A) \to K_n(B)$ (homotopy invariance);

(iii) $K_n(A) \cong K_n(A \otimes \mathcal{K})$ (stability);

(iv) If $\{A_i\}_{i \in I}$ is a directed system of C^*-algebras, then $\varinjlim K_i(A_i) \cong K_i(\varinjlim(A_i))$ (continuity);

(v) A short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ canonically induces an exact sequence

$$K_n(A) \xrightarrow{i_*} K_n(B) \xrightarrow{j_*} K_n(C)$$

(half-exactness). One can construct so-called connecting homomorphisms $\delta_n : K_n(C) \to K_{n+1}(A)$ for each $n \in \mathbb{Z}_{\geq 0}$. Together with the Bott periodicity isomorphism β these give rise to an exact "hexagon"

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{i_*} & K_0(B) & \xrightarrow{j_*} & K_0(C) \\ {\scriptstyle \beta\circ\delta_1}\uparrow & & & & \downarrow{\scriptstyle \delta_0} \\ K_1(C) & \xleftarrow[j_*]{} & K_1(B) & \xleftarrow[i_*]{} & K_1(A). \end{array}$$

(vi) If A and B are Morita equivalent, then $K_n(A) \cong K_n(B)$.

Example 4.4.6. One easily sees that $K_1(\mathbb{C}) = K_0(C_0(\mathbb{R})) = K^0(\mathbb{R}) = \ker(i^* : K^0(S^1) \to K^0(pt)) = 0$, where $i : pt \to S^1$ is the inclusion of a point in the circle S^1. Using stability one can show $K_i(\mathcal{K}) \cong K_i(M_n(\mathbb{C})) \cong K_i(\mathbb{C})$ for $i = 0, 1$.

Example 4.4.7. Suppose H is a compact group, with unitary dual $\hat{H}$. One can show, using the Peter-Weyl theorem, that

$$C_r^*(H) \cong \hat{\bigoplus}_{\pi \in \hat{H}} M_{\dim(\pi)}(\mathbb{C})$$

where the hat on the plus, means that one has to use a suitable norm to complete the algebraic sum. From this it follows that

$$\begin{aligned} K_0(C_r^*(H)) &\cong K_0(\hat{\bigoplus}_{\pi \in \hat{H}} M_{\dim(\pi)}(\mathbb{C})) \\ &\cong \bigoplus_{\pi \in \hat{H}} K_0(M_{\dim(\pi)}(\mathbb{C})) \\ &\cong \bigoplus_{\pi \in \hat{H}} \mathbb{Z} = \mathcal{R}(H), \end{aligned}$$

where $\mathcal{R}(H)$ denotes the representation ring of H.

In Section 6.6 we shall consider the K-theory of groupoid C^*-algebras.

4.5 KK-theory

Recall the category $\mathsf{C}^*\text{-}\mathsf{ALG}_\mathsf{b}$ of C^*-algebras with as morphisms unitary isomorphism classes of Hilbert bimodules. We shall now decorate such bimodules with more structure, so-called generalized Fredholm operators and $\mathbb{Z}/2\mathbb{Z}$-gradings, to define a bifunctor KK.

Definition 4.5.1. A **Kasparov** (A, B)**-module** is a (countably generated) $\mathbb{Z}/2\mathbb{Z}$-graded right Hilbert (A, B)-bimodule $\mathcal{H}$ (the left action π of A is degree zero) endowed with an adjointable bounded operator $F \in \mathcal{B}_A(\mathcal{H})$ of degree 1 that satisfies $\pi(a)(F^2 - 1) \in \mathcal{K}_A(\mathcal{H})$ and $[\pi(a), F] \in \mathcal{K}_A(\mathcal{H})$ for all $a \in A$.

The set of Kasparov (A, B)-modules $(\mathcal{H}, \pi, F)$ is denoted by $E(A, B)$.

Example 4.5.2. Suppose $A = \mathbb{C}$ and $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ with $\mathcal{H}^0 = \mathcal{H}^1 = l^2_B := l^2(\mathbb{N}, B)$. Define

$$F := \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix},$$

with shift operators $S(b_1, b_2, \ldots) := (0, b_1, b_2, \ldots)$ and $T(b_1, b_2, \ldots) := (b_2, b_3, \ldots)$. Then

$$F^2 - 1 = \begin{pmatrix} ST - 1 & 0 \\ 0 & TS - 1 \end{pmatrix} = \begin{pmatrix} -p_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $p_1(b_1, b_2, \ldots) = (b_1, 0, 0, \ldots)$ is a rank one projection, hence compact operator in $\mathcal{K}_B(l^2_B)$. Thus, $(\mathcal{H}, \pi, F) \in E(\mathbb{C}, B)$ is a Kasparov module, with $\pi(z) = \mathrm{diag}(z)$.

Example 4.5.3. Suppose M is a smooth compact manifold, $E \to M$ and $F \to M$ smooth vector bundles over M and $P : \Gamma(E) \to \Gamma(F)$ an elliptic pseudo-differential operator of order zero (cf. e.g. [33]). It extends to a linear operator $L^2(M, E) \to L^2(M, F)$. Suppose $Q : L^2(M, E) \to L^2(M, F)$ is a parametrix for P. Let

$$\pi : f \mapsto M_f = \mathrm{diag}(f) : C(M) \to \mathcal{B}(L^2(M, E) \oplus L^2(M, F))$$

be the representation of $C(M)$ as multiplication operators. Then

$$\left(L^2(M, E) \oplus L^2(M, F), \pi, \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \right)$$

is a Kasparov module in $E(C(M), \mathbb{C})$, since

$$\begin{pmatrix} PQ - 1 & 0 \\ 0 & QP - 1 \end{pmatrix}, \left[M_f, \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \right]$$

are compact operators for all $f \in C(M)$.

A **morphism of Kasparov modules** is a morphism of (graded) Hilbert C^*-bimodules that intertwines the operators. We shall now extend the (trivial) notion of homotopy in $\mathsf{C^*\text{-}ALG_b}$ (cf. Remark 4.2.8) to homotopy of Kasparov modules. A **homotopy in** $E(A,B)$ **from** $(\mathcal{H}_0,\pi_0,F_0)$ **to** $(\mathcal{H}_1,\pi_1,F_1)$ is an element in $(\mathcal{H},\pi,F) \in E(A, C([0,1],B))$ such that for $i = 1,2$ the evaluation $e_i : C([0,1],B) \to B$ map induces isomorphisms $(e_i)_*(\mathcal{H},\pi,F) \cong (\mathcal{H}_i,\pi_i,F_i)$.

Example 4.5.4. A Kasparov module $(\mathcal{H},\pi,F)$ is **degenerate** if $\pi(a)(F^2-1) = 0$ and $[\pi(a),F] = 0$ for all $a \in A$. Degenerate Kasparov modules are homotopy equivalent to the zero Kasparov module $(0,0,0)$, cf. Remark 4.2.8.

Definition 4.5.5. The ***KK*-theory** $KK_0(A,B)$ **of the pair of** C^***-algebras** (A,B) is the quotient $E(A,B)/\sim$, where $\sim$ denotes homotopy equivalence.

It is an Abelian group with addition defined by

$$[\mathcal{H}_0,\pi_0,F_0] + [\mathcal{H}_1,\pi_1,F_1] := [\mathcal{H}_0 \oplus \mathcal{H}_1, \pi_0 \oplus \pi_1, F_0 \oplus F_1],$$

where $\oplus$ denotes the direct sum for graded modules and

$$-[\mathcal{H}_0,\pi_0,F_0] := [\mathcal{H}_0^{op}, \pi_0^{op}, -F_0].$$

Example 4.5.6. Recall the setting of Example 4.5.2. Note that the Kasparov module introduced in that example is the sum of two Kasparov modules

$$(p_1(l^2_B) \oplus 0, \pi, 0) + \left((1-p_1)l^2_B \oplus l^2_B, \pi, \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} \right).$$

The second module is degenerate: $ST = 1 : (1-p_1)l^2_B \to (1-p_1)l^2_B$ and $TS = 1 : l^2_B \to l^2_B$. Hence the whole Kasparov module is homotopy equivalent to the Kasparov module

$$(p_1(l^2_B) \oplus 0, \pi, 0).$$

Higher *KK*-theory groups are defined by suspension, analogously to *K*-theory; for $n \in \mathbb{Z}_{\geq 0}$

$$KK_n(A,B) := KK_0(A \otimes C_0(\mathbb{R}^n), B) \cong KK_0(A, B \otimes C_0(\mathbb{R}^n)).$$

Another way to view $KK_1(A,B)$ is a the set of homotopy classes of ungraded Kasparov (A,B)-mudules (i.e. without $\mathbb{Z}/2\mathbb{Z}$-grading).

KK-theory has the same type of properties as *K*-theory, but now in both variables. It is stable, homotopy invariant and continuous in each variable and it satisfies Bott periodicity $KK_n(A,B) \cong KK_{n+2}(A,B)$ for all $n \in \mathbb{Z}_{\geq 0}$. It is not half exact, but it is split exact (we shall not explain this).

Proposition 4.5.7. *For every* C^**-algebra* B*, one has* $KK_n(\mathbb{C},B) \cong K_n(B)$.

Suppose B is unital. Then a homomorphism $K_0(B) \to KK_0(\mathbb{C},B)$ is given by

$$[p]-[q] \mapsto \left[p(B \otimes \mathcal{H}) \oplus q(B \otimes \mathcal{H}), z \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, 0 \right].$$

Note that this Kasparov module is not degenerate, since $\pi(z)(F^2 - 1) = -\pi(z) \neq 0$. In favorable cases an inverse homomorphism $KK_0(\mathbb{C}, B) \to K_0(B)$ is given by

$$[\mathcal{H}, \pi, F] \mapsto [\ker(F)^0] - [\ker(F)^1],$$

where $[\ker(F)^0] - [\ker(F)^1]$ corresponds to the K-theory class determined by the Hilbert B-modules $\ker(F)^0$ and $\ker(F)^1$, which are finitely generated and projective (cf. Remark 4.4.2.(i), since F is a generalized Fredholm operator. We shall call this map 'index' in cf. Section 12.5 and Section 12.6, although it is not exactly an index in general (cf. [5]).

Example 4.5.8. Recall the setting of Examples 4.5.2 and 4.5.6. The KK-class of the Kasparov module $[\mathcal{H}, \pi, F] \in KK_0(\mathbb{C}, B)$ maps to

$$[p_1] - [0] \in K_0(B).$$

There exists, under some technical conditions, a map

$$\hat{\otimes}_B : KK_i(A, B) \times KK_j(B, C) \to KK_{i+j(\mathrm{mod}\,2)}(A, C),$$

$(i, j = 0, 1)$ called **Kasparov product** . This enables us to think of KK as a category with as objects C^*-algebras and as morphisms equivalence classes of Kasparov modules. The unit morphism at A is the class $[A \oplus 0, \pi, 0]$, where $\pi(a)b = a\,b$ for $a, b \in A$. Suppose $(\mathcal{H}_1, \pi_1, F_1)$ and $(\mathcal{H}_2, \pi_2, F_2)$ are Kasparov modules. On the underlying $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert C^*-bimodules the product is simply (the $\mathbb{Z}/2\mathbb{Z}$-graded version of) the composition $\mathcal{H}_1 \otimes_B \mathcal{H}_2$ introduced in Section 4.2. The problem arises with the definition of the operator on this tensor product.

If $F_2 = 0$, then the product is simply $F_1 \otimes 1$. If $F_1 = 0$ then the product is $1 \otimes F$, where F is a so-called F_2-connection. This is a generalization of the notion of connections on vector bundles in geometry. For $h_1 \in \mathcal{H}_1$ define

$$T_{h_1} : \mathcal{H}_2 \to \mathcal{H}_1 \otimes_B \mathcal{H}_2, h_2 \mapsto h_1 \otimes h_2$$
$$T_{h_1}^* : \mathcal{H}_1 \otimes_B \mathcal{H}_2 \to \mathcal{H}_2, h_1' \otimes h_2 \mapsto \langle h_1, h_1' \rangle\, h_2.$$

Definition 4.5.9. An F_2-**connection** is an operator $F : \mathcal{B}_C(\mathcal{H}_1 \otimes_B \mathcal{H}_2)$ satisfying

$$T_{h_1} F_2 - (-1)^{\delta(h_1)\delta(F_2)} F\, T_{h_1} \in \mathcal{K}_C(\mathcal{H}_2, \mathcal{H}_1 \otimes_B \mathcal{H}_2)$$
$$F_2\, T_{h_1}^* - (-1)^{\delta(h_1)\delta(F_2)} T_{h_1}^* F \in \mathcal{K}_C(\mathcal{H}_1 \otimes_B \mathcal{H}_2, \mathcal{H}_2)$$

for all $h_1 \in \mathcal{H}_1$, where the δ denotes the grading.

Example 4.5.10. Suppose B is unital. A very general and important example is the case that $\mathcal{H}_1 = p(B \otimes \mathcal{H})$. In that case one can use the canonical isomorphism

$$\mathcal{H}_1 \otimes_B \mathcal{H}_2 \cong (p \otimes 1)(\mathcal{H} \otimes_B \mathcal{H}_2)$$

to define the connection

$$F := (p \otimes 1)(1 \otimes F_2)(p \otimes 1).$$

This is called the **Grassmann connection**. It turns out to be very useful to compute Kasparov products of the form

$$K_0(B) \times KK_i(B, C) \to KK_i(\mathbb{C}, C)$$

for $i = 0, 1$ (that are also called index pairings in e.g. [33]) in Chapter 12.5.1.

Part II

Continuous representations of groupoids

Introduction

The purpose of this part is to study some of the basic theory of continuous representations in the context of groupoids. Most of this part has appeared previously in the form of a preprint [8]. Representations of groupoids occur naturally in geometry, since the parallel transport associated to a flat connection on a vector bundle is a representation of the fundamental groupoid of the base space (cf. Example 2.1.7). Also the monodromy associated to holomorphic extensions of functions is a representation of the fundamental groupoid. Another place where they occur is as vector bundles over an orbifold, since these correspond to representations of the groupoid representing the orbifold (cf. Example 2.2.4). Related to this is the fact that for a group acting on a space, equivariant vector bundles over that space correspond to representations of the associated action groupoid (cf. Example 2.1.5). Apart from these examples, we shall also be interested in representations of families of groups (cf. Example 2.1.9) and representations of gauge groupoids (cf. Example 2.1.8).

Some work on representations of groupoids on vector bundles was initiated by Westman in [89], [90]. We shall look at representations not only on continuous vector bundles, but on continuous fields of Hilbert spaces (cf. Chapter 3). Continuous fields of Hilbert spaces were introduced and studied by Dixmier and Douady [21]. They play an important rôle in noncommutative geometry, as they occur as Hilbert C^*-modules of commutative C^*-algebras (cf. Theorem 4.2.4). Moreover, they are a rich source of noncommutative C^*-algebras, which are obtained as the algebra of adjointable endomorphisms of such modules (cf. below Example 4.2.5). A reason why we not only consider representations on continuous vector bundles is the following. One should note that the regular representation of a groupoid $G \rightrightarrows M$ with Haar system is defined on a continuous field of L^2 functions on the target fibers. Even for very simple étale groupoids this is not a locally trivial field (consider e.g. the family of groups $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{R})\backslash\{(-1,0)\} \to \mathbb{R}$).

We shall develop an extension of harmonic analysis from continuous groups to continuous groupoids. It is investigated to which extent one can prove well-known statements from representation theory of groups, like Schur's Lemma and the Peter-Weyl theorem, in the context of representations of groupoids on continuous fields of Hilbert spaces. This turns out to be possible. Eventually, one can give an analogue of the decomposition of $L^2(G)$ for a compact group, for proper groups under suitable (and very strict) conditions.

In the spirit of Dixmier (cf. [22]) one would like to relate the representations of the groupoid to the representations of some operator algebra. Instead of an operator algebra we introduce a category of operators, a Banach $*$-category, which turns out to be most suitable in our situation. We relate the representations of a groupoid $G \rightrightarrows M$ to the representations of the continuous Banach $*$-category $\hat{L}^1(G)$. Let us mention that representations of groupoids were also studied by J. Renault [67]. But one should note that the representations discussed there are measurable representations on measurable fields of Hilbert spaces. These behave quite differently from continuous representations as studied in the present thesis. Renault discusses the relation of these measurable representations to representations of the C^*-algebra of $G \rightrightarrows M$.

As one will see in this part, proofs of theorems in representation theory of groupoids heavily rely on the representation theory of groups. The differences mostly arise in dealing with the global topology of the groupoid and its orbit foliation.

Chapter 5 introduces representations of groupoids on continuous fields of Hilbert spaces. We discuss several notions of continuity of representations and show how they relate. Then we treat two examples, namely the regular representation of a groupoid and representations of continuous families of groups. In the last part of this section we "embed" the theory of continuous groupoid representations in the theory of group representations. We discuss the topological group of global bisections of a groupoid and give a theorem that explains which representations of this group correspond to representations of the groupoid. Hence one could view the representation theory of groupoids as a way to understand some specific representations of certain 'infinite-dimensional' groups.

Chapter 6 treats harmonic analysis in the case of groupoids. We prove an analogue of Schur's Lemma and two versions of the Peter-Weyl Theorem. The last part of this section discusses the representation rings of a groupoid and the relation of those rings to the K-theory of the reduced C^*-algebra of the groupoid (cf. Section 4.4).

Chapter 7 discusses a continuous analogue of Renault's theorem that gives a bijection between measurable representations of $G \rightrightarrows M$ and non-degenerate representations of the Banach algebra $L^1(G)$. We construct a bijection between continuous representations of $G \rightrightarrows M$ and continuous non-degenerate representations of the Banach $*$-category $\hat{L}^1(G)$.

Chapter 5

Continuous representations of groupoids

5.1 Continuous representations of groupoids

In this section we introduce continuous representations of groupoids on continuous fields of Hilbert spaces. As far as we know this notion as we define it does not appear anywhere in the literature. We should mention the work of Westman [90, 89] though, who restricts himself to representations of locally trivial groupoids on vector bundles. Furthermore, there is a preprint by Amini [1], which treats continuous representations on Hilbert bundles, which is rather different from the notion of continuous field of Hilbert spaces as we use it. It seems as though his article does not give full attention to the 'continuity-issues' involved.

As for representations of groups there are several forms of continuity for such representations. We consider "normal", weak and strong continuity and in Section 5.2 also continuity in the operator norm. All these forms of continuity can be compared, cf. Lemma 5.1.6, Lemma 5.1.7 and Lemma 5.2.4, generalizing similar results for groups (cf. e.g. [26]). In Definition 5.1.8 we introduce the notion of a morphism of representations and we show in Proposition 5.1.10 that any representation of a proper groupoid is isomorphic to a unitary representation, generalizing a similar result for compact groups.

Let M be a locally compact space and $G \rightrightarrows M$ a continuous groupoid.

Definition 5.1.1. A **bounded representation** of $G \rightrightarrows M$ on a continuous field of Hilbert spaces $(\{\mathcal{H}_m\}_{m\in M}, \Delta)$ over M is a family of invertible bounded operators

$$\{\pi(g) : \mathcal{H}_{s(g)} \to \mathcal{H}_{t(g)}\}_{g\in G}$$

satisfying

(i) $\pi(1_m) = id_{\mathcal{H}_m}$ for all $m \in M$,

(ii) $\pi(gg') = \pi(g)\pi(g')$ for all $(g, g') \in G_2 = G\ {}_t\!\times_s G$,

(iii) $\pi(g^{-1}) = \pi(g)^{-1}$ for all $g \in G$ and

(iv) $g \mapsto \|\pi(g)\|$ is locally bounded.

We denote such a representation by a triple $(\mathcal{H}, \Delta, \pi)$. Recall from Lemma 3.1.8 that $\mathcal{H}$ can be endowed with a topology such that the sections Δ equals the set of continuous sections $\Gamma_0(\mathcal{H})$ of the projection $\mathcal{H} \to M$ onto the base space M.

Definition 5.1.2. A representation $(\mathcal{H}, \Delta, \pi)$ is **strongly continuous** if the map

$$g \mapsto \pi(g)\xi(s(g))$$

is continuous $G \to \mathcal{H}$ for all $\xi \in \Delta$. A representation is **weakly continuous** if the map

$$g \mapsto \langle \pi(g)\xi(s(g)), \eta(t(g))\rangle$$

is continuous $G \to \mathbb{C}$ for all $\xi, \eta \in \Delta$. A representation $(\pi, \mathcal{H}, \Delta)$ is **continuous** if

$$\Psi : (g, h) \mapsto \pi(g)h$$

is a continuous map $G\, {}_s\!\times_p \mathcal{H} \to \mathcal{H}$. The representation is **unitary** if the operators $\{\pi(g) : \mathcal{H}_{s(g)} \to \mathcal{H}_{t(g)}\}_{g \in G}$ are unitary.

For any $\xi, \eta \in \Delta^\pi$ we use the notation $\langle \xi, \pi\eta\rangle$ for the map $G \to \mathbb{C}$ given by

$$g \mapsto \langle \xi(t(g)), \pi(g)\eta(s(g))\rangle \, ,$$

which we call a **matrix coefficient**.

Condition (iv) of Definition 5.1.1 is perhaps somewhat strange at first sight. The following Example 5.1.3, Lemma 5.1.4 and Example 5.1.5 should clarify it. Moreover, recall that for morphism Ψ of continuous fields the map $m \mapsto \|\Psi_m\|$ has to be locally bounded too, cf. Definition 3.1.11.

Example 5.1.3. A simple example shows that $g \mapsto \|\pi(g)\|$ is not always continuous. Consider the groupoid $\mathbb{R} \rightrightarrows \mathbb{R}$, with a continuous representation on a field given by the trivial representation on $\mathbb{C}$ at each $x \in \mathbb{R}$ except in 0, where it is the zero representation. In this case, the norm of π drops from 1 to 0 at 0.

Lemma 5.1.4. *For any continuous representation $(\mathcal{H}, \pi, \Delta)$ the map $g \mapsto \|\pi(g)\|$ is lower semi-continuous $G \to \mathbb{R}$.*

Proof. Using the above definition and Lemma 3.1.10 we know that the map $(g, h) \mapsto \|\pi(g)h\|$ is continuous $G_s \times_p \mathcal{H} \to \mathbb{R}_{\geq 0}$. For any $g \in G$, let $\varepsilon > 0$ be given. Let $h' \in \mathcal{H}_{s(g)}$ be such that

$$|\|\pi(g)h'\| - \|\pi(g)\|| < \varepsilon/2.$$

by continuity there exists an open neighborhood $U \subset G_s \times_p \mathcal{H}$ of (g, h') such that $(g'', h'') \in U$ implies

$$|\|\pi(g'')h''\| - \|\pi(g)h'\|| < \varepsilon/2.$$

Take $V := pr_1(U) \subset G$. Then $g'' \in V$ implies, for an $h'' \in pr_2(U)$ such that $(g'', h'') \in U$, one has

$$\|\pi(g'')\| \geq \|\pi(g'')h''\| > \|\pi(g)h'\| - \varepsilon/2 > \|\pi(g)\| - \varepsilon,$$

and we are done. ☐

The function $g \mapsto \|\pi(g)\|$ is locally bounded if, for example, $(\mathcal{H}, \Delta)$ is uniformly finite-dimensional.

Example 5.1.5. A counterexample of a continuous representation of a proper groupoid where $g \mapsto \|\pi(g)\|$ is not locally bounded $G \to \mathbb{R}$, even though the restriction to G_m is bounded for each m, is as follows.

Consider the trivial bundle of groups $[0,1] \times \mathbb{Z}/2\mathbb{Z} \rightrightarrows [0,1]$. Define a continuous field of Hilbert spaces over $[0,1]$ by $\mathcal{H}_0 := \mathbb{C}^2 =: \mathcal{H}_1$ and $\mathcal{H}_x := \mathbb{C}^{2\,n}$ if $x \in [\frac{1}{n+1}, \frac{1}{n})$ for all $n \in \mathbb{N}$. The topology on the field is obtained from the inclusions $\mathbb{C}^{2n} \hookrightarrow \mathbb{C}^{2(n+1)}$ given by $\vec{v} \mapsto (0, \vec{v}, 0)$. Define, for every $n \in \mathbb{N}$ and $x \in [\frac{1}{n+1}, \frac{1}{n})$,

$$\pi(x, -1) := \text{diag}'(1/n, \ldots, 1/2, 1, 1, 2, \ldots n),$$

where diag′ denotes the matrix filled with zeros except the diagonal from the upper right corner to the lower left corner, where the above sequence is filled in. Furthermore, $\pi(0, -1) :=$ diag'$(1, 1)$. This representation is strongly continuous, but

$$\|\pi(x, -1)\| = n \text{ if } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right).$$

Hence $g \mapsto \|\pi(g)\|$ is not locally bounded at $(0, -1)$.

Lemma 5.1.6. *If a representation $(\pi, \mathcal{H}, \Delta)$ is strongly continuous, then it is weakly continuous. The converse implication holds if the representation is unitary.*

Proof. Suppose $(\pi, \mathcal{H}, \Delta)$ is strongly continuous. Suppose $\xi, \eta \in \Delta$ and $g \in G$. Write $n = t(g)$. Let $\varepsilon > 0$ be given. Let $\xi' \in \Delta$ be a section satisfying $\xi'(n) = \pi(g)\xi(s(g))$. Choose a neighborhood $U \subset M$ of n such that $n' \in U$ implies $|\langle \eta(n'), \xi'(n')\rangle_{\mathcal{H}_{n'}} - \langle \eta(n), \xi'(n)\rangle_{\mathcal{H}_n}| < \varepsilon/2$. This is possible since $\langle \eta, \xi'\rangle$ is continuous on M. Since π is strongly continuous there exists an open set $V \subset G$ containing g such that for all $g' \in V$ one has $t(g') \in U$ and

$$\|\pi(g')\xi(s(g')) - \xi'(t(g'))\|_{\mathcal{H}_{t(g')}} < \varepsilon/(2 \sup_{n' \in U} \|\eta(n')\|).$$

Hence, for all $g' \in V$

$$\begin{aligned} &|\langle \eta(t(g')), \pi(g')\xi(s(g'))\rangle_{\mathcal{H}_{t(g')}} - \langle \eta(n), \xi'(n)\rangle_{\mathcal{H}_n}| \\ &\leq |\langle \eta(t(g')), \pi(g')\xi(s(g'))\rangle_{\mathcal{H}_{t(g')}} - \langle \eta(t(g')), \xi'(t(g'))\rangle_{\mathcal{H}_{t(g')}}| \\ &\quad + |\langle \eta(t(g')), \xi'(t(g'))\rangle_{\mathcal{H}_{t(g')}} - \langle \eta(n), \xi'(n)\rangle_{\mathcal{H}_n}| \\ &< \|\eta(t(g'))\|\varepsilon/(2\sup_{n' \in U} \|\eta(n')\|) + \varepsilon/2 \leq \varepsilon. \end{aligned}$$

The converse implication is proven as follows. Suppose $(\pi, \mathcal{H}, \Delta)$ is weakly continuous and unitary. Let $U(\varepsilon, \eta, V)$ be a neighborhood of $\pi(g)\xi(s(g))$ in $\mathcal{H}$ for a given $g \in G$ and $\xi \in \Delta$, where $\eta \in \Delta$ satisfies $\eta(t(g)) = \pi(g)\xi(t(g))$. We compute for any $g' \in G$,

$$\begin{aligned}
&\|\eta(t(g')) - \pi(g')\xi(s(g'))\|_{\mathcal{H}_{t(g')}} \qquad (5.1.1)\\
&= |\langle\eta(t(g')), \eta(t(g'))\rangle - \langle\eta(t(g')), \pi(g')\xi(s(g'))\rangle \\
&\quad - \langle\pi(g')\xi(s(g')), \eta(t(g'))\rangle + \langle\pi(g')\xi(s(g')), \pi(g')\xi(s(g'))\rangle|^{1/2} \\
&\leq (|\langle\eta(t(g')), \eta(t(g'))\rangle - \langle\eta(t(g')), \pi(g')\xi(s(g'))\rangle| \\
&\quad + |\langle\xi(s(g')), \xi(s(g'))\rangle - \langle\pi(g')\xi(s(g')), \eta(t(g'))\rangle|)^{1/2} \qquad (5.1.2)
\end{aligned}$$

By weak continuity we can choose a neighborhood $W_g \subset G$ of g such that $g' \in W_g$ implies

$$|\langle\eta(t(g')), \pi(g')\xi(s(g'))\rangle - \langle\eta(t(g)), \pi(g)\xi(s(g))\rangle| < \varepsilon.$$

Since t is open and $\eta \in \Delta$, we can choose an open neighbor hood $W'_g \subset W_g$ of g such that

$$|\langle\eta(t(g')), \eta(t(g'))\rangle - \langle\eta(t(g)), \eta(t(g))\rangle| < \varepsilon$$

Hence the first two terms of Equation (5.1.2) are smaller than 2ε. Analogously, the last two terms of Equation (5.1.2) are also smaller than 2ε, which finishes the proof. □

Lemma 5.1.7. *If a representation $(\pi, \mathcal{H}, \Delta)$ is continuous, then it is strongly continuous. The converse holds if π is unitary.*

Proof. Suppose $(\pi, \mathcal{H}, \Delta)$ is continuous. Suppose $g \in G$ and $\xi \in \Delta$. There exists an open neighborhood $U(\varepsilon, \eta, V) \subset \mathcal{H}$ of $\pi(g)\xi(s(g))$ such that $\eta(t(g) = \pi(g)\xi(s(g))$. Then, by continuity of π there exists a neighborhood $W_g \subset G_s \times_p \mathcal{H}$ of g such that $g' \in W_g$ implies $\pi(W_g) \subset U(\varepsilon, \eta, V)$. Now, define a subset of G

$$W_G := \{g' \in G \mid (g', \xi(s(g'))) \in W_g\}.$$

This set is open since it equals $s^{-1}\xi^{-1}p_2(W_g) \cap p_1(W_g)$. If $g' \in W_G$, then

$$\|\eta(t(g')) - \pi(g')\xi(s(g'))\| < \varepsilon.$$

Conversely, suppose $(\pi, \mathcal{H}, \Delta)$ is strongly continuous and unitary. Suppose $(g, h) \in G_s \times_p \mathcal{H}$. Let $U(\varepsilon, \eta, V)$ be an open neighborhood of $\pi(g)h$ with $\eta(t(g)) = \pi(g)h$ as usual. Let ξ be any section in Δ such that $\xi(s(g)) = h$. Then by strong continuity there exists an open set $V_g \subset G$ such that $g' \in V_g$ implies $\|\eta(t(g')) - \pi(g')\xi(s(g'))\| < \varepsilon$. Define the set

$$W_{g,h} := \{(g', h') \in G_s \times_p \mathcal{H} \mid \|h' - \xi(s(g'))\| < \varepsilon, g' \in V_g\}.$$

It is easily seen to be open and $(g', h') \in W_{g,h}$ implies

$$\begin{aligned}
\|\eta(t(g')) - \pi(g')h'\| &\leq \|\eta(t(g')) - \pi(g')\xi(s(g'))\| + \|\pi(g')\xi(s(g')) - \pi(g')h'\| \\
&< \varepsilon + \|\pi(g')\|\|\xi(s(g')) - h'\| < 2\varepsilon,
\end{aligned}$$

which finishes the proof. □

Definition 5.1.8. A **morphism of continuous (unitary) representations**

$$(\mathcal{H}^1, \Delta^1, \pi_1) \to (\mathcal{H}^2, \Delta^2, \pi_2)$$

of a groupoid is a morphism $\Psi : (\mathcal{H}^1, \Delta^1) \to (\mathcal{H}^2, \Delta^2)$ of continuous fields of Hilbert spaces (cf. Definition 3.1.11) that intertwines the groupoid representations

$$\begin{array}{ccc} \mathcal{H}^1_{s(g)} & \xrightarrow{\pi_1(g)} & \mathcal{H}^1_{t(g)} \\ {\scriptstyle \Psi_{s(g)}}\downarrow & & \downarrow{\scriptstyle \Psi_{t(g)}} \\ \mathcal{H}^2_{s(g)} & \xrightarrow[\pi_2(g)]{} & \mathcal{H}^2_{t(g)}. \end{array}$$

Example 5.1.9. The trivial representation of a groupoid $G \rightrightarrows M$ is given by the continuous field $(\mathcal{H}, \Delta)$ that has fiber $\mathbb{C}$ over each $m \in M$ and a map $\pi : G \to U(M \times \mathbb{C}) \cong M \times U(\mathbb{C}) \times M$,

$$g \mapsto (t(g), 1, s(g)).$$

We give another example of a continuous unitary representation of a groupoid. For any continuous function $f : G \to \mathbb{R}$ we can construct the representation

$$\pi_f : g \mapsto (t(g), e^{2\pi i (f(t(g)) - f(s(g)))}, s(g)).$$

These representation are all isomorphic. Indeed, for $f, g : G \to \mathbb{R}$,

$$m \mapsto e^{2\pi i (f(m) - g(m))}$$

is an isomorphism $(\mathcal{H}, \Delta, \pi_g) \to (\mathcal{H}, \Delta, \pi_f)$. In particular all these representations are isomorphic to π_0, which is the trivial representation.

Proposition 5.1.10. *If $G \rightrightarrows M$ is a proper groupoid endowed with a Haar system $\{\lambda_m\}_{m \in M}$ (cf. Section 2.5), then any continuous representation $(\mathcal{H}, \Delta, \pi)$ is isomorphic to a unitary representation.*

Proof. Suppose $(\mathcal{H}, \Delta, \pi)$ is a non-zero continuous representation of G. Let $c : M \to \mathbb{R}_{>0}$ be a cutoff function (cf. Definition 2.5.9), with t and s interchanged). This exists, since $G \rightrightarrows M$ is proper. Define an inner product $\langle ., . \rangle^{new}$ on $\mathcal{H}$ by the following description: for all $m \in M$ and $h, h' \in \mathcal{H}_m$,

$$\langle h, h' \rangle^{new}(m) := \int_{G_m} \langle \pi(g)h, \pi(g)h' \rangle \, c(t(g)) \lambda_m(dg).$$

This inner product is G-invariant, since the Haar system and t are right invariant. It gives rise to a new topology on $\mathcal{H}$. The isomorphism is the identity on $\mathcal{H}$, which is easily seen to be continuous. Indeed, let $h \in \mathcal{H}$ and let $U(\varepsilon, \xi, V) \ni h$ be an open set in $\mathcal{H}$ with respect to the old norm. Then there exists a an open set V' such that $V' \subset V$, $p(h) \in V'$ and $g \mapsto \|\pi(g)\|$ is bounded on $t^{-1}V' \cap \operatorname{supp}(c \circ t)$. Since $c \circ t$ has compact support on each s-fiber, the function

$$m' \mapsto \int_{g \in G_{m'}} \|\pi(g)\| c(t(g)) \lambda_{m'}(dg)$$

is bounded on V'. Hence we can set

$$\delta := \frac{\varepsilon}{\sup_{m\in V'}\int_{g\in G_{m'}}\|\pi(g)\|c(t(g))\lambda_{m'}(dg)}.$$

Then $h' \in U(\delta, \xi, V')$ (in the old topology) implies

$$\begin{aligned}\|h'-\xi(m')\|_{m'}^{new} &= \int_{G_{m'}}\|\pi(g)(h'-\xi(m'))\|c(t(g))\lambda_{m'}(dg)\\ &\leq \int_{G_{m'}}\|\pi(g)\|c(t(g))\lambda_{m'}(dg)\|h'-\xi(m')\|\\ &\leq \varepsilon,\end{aligned}$$

which proves the continuity of the identity map.

The proof that the inverse (also the identity) is continuous proceeds similarly. One uses that

$$\begin{aligned}\|h'-\xi(m')\| &= \int_{G_{m'}}\|h'-\xi(m')\|c(t(g)))\lambda_{m'}(dg)\\ &= \int_{G_{m'}}\|\pi(g^{-1})\pi(g)(h'-\xi(m'))\|c(t(g))\lambda_{m'}(dg)\\ &= \sup_{g\in G_{m'}}\|\pi(g)\|\int_{G_{m'}}\|\pi(g)(h'-\xi(m'))\|c(t(g))\lambda_{m'}(dg)\end{aligned}$$

and local boundedness of $g \mapsto \|\pi(g)\|$. This finishes the proof. □

A representation $(\mathcal{H}, \Delta, \pi)$ is **locally trivial** if the continuous field $(\mathcal{H}, \Delta)$ is locally trivial. In [79] locally trivial representations of a groupoid $G \rightrightarrows M$ are called G-vector bundles. Representations of transitive groupoids are locally trivial.

5.2 Continuity of representations in the operator norm

In this section we go through quite some effort to define a suitable topology on the set of bounded linear operators $\{P : \mathcal{H}_m \to \mathcal{H}_n\}_{n,m\in M}$ for a continuous field of Hilbert spaces $(\{\mathcal{H}_m\}_{m\in M}, \Delta_{\mathcal{H}})$. This is done not only to be able to consider representations which are continuous in the operator topology, but the lower semi-continuous field of Banach spaces thus obtained also plays a crucial rôle in Section 7. At first reading one could consider skipping the proofs.

Let $(\{\mathcal{H}_m\}_{m\in M}, \Delta_{\mathcal{H}})$ be a continuous field of Hilbert spaces over M. Consider the continuous field of Banach spaces over $M \times M$ whose fiber at (n, m) is given by the bounded linear operators $\mathcal{H}_m \to \mathcal{H}_n$, i.e. $\mathcal{B}(\mathcal{H}, \mathcal{H})_{(n,m)} := \mathcal{B}(\mathcal{H}_m, \mathcal{H}_n)$. This is indeed a Banach space with the norm

$$\|P\| = \sup_{h\in\mathcal{H}_m, \|h\|_{\mathcal{H}_m}=1}\|P(h)\|_{\mathcal{H}_n}.$$

We define a space of sections $\Delta_{\mathcal{B}}$ of the field to consist of those maps $(n, m) \mapsto P(n, m)$ in $\prod_{(n,m)\in M\times M}\mathcal{B}(\mathcal{H}, \mathcal{H})$ such that

(i) for every $m \in M$ and $h \in \mathcal{H}_m$

$$n \mapsto P(n,m)h$$

is in $\Delta_{\mathcal{H}}$,

(ii) for every $n \in M$ and $\xi \in \Delta_{\mathcal{H}}$ the map

$$m \mapsto P(n,m)\xi(m)$$

is continuous $M \to \mathcal{H}_n$,

(iii) The map $(n,m) \mapsto \|P(n,m)\|$ locally bounded, and

(iv) P is adjointable, which means that there exists a $P^* : R \to \mathcal{B}(\mathcal{H},\mathcal{H})$, satisfying (i), (ii) and (iii), such that for all $\xi, \eta \in \Delta_{\mathcal{H}}$ one has $(\eta, P\xi) = (P^*\eta, \xi)$, more concretely: for all $(n,m) \in R$

$$\langle \eta(n), P(n,m)\xi(m)\rangle_{\mathcal{H}_n} = \langle P^*(m,n)\eta(n), \xi(m)\rangle_{\mathcal{H}_m} .$$

Lemma 5.2.1. *The pair* $(\{\mathcal{B}(\mathcal{H}_n,\mathcal{H}_m)\}_{(n,m)\in M\times M}, \Delta_{\mathcal{B}})$ *is a lower semi-continuous field of Banach spaces.*

Proof. First, we prove lower semi-continuity of the norm of a section $P \in \Delta_{\mathcal{B}}$. This follows from the fact that the map

$$(n,m,h) \mapsto \|P(n,m)h\|_{\mathcal{H}_n}$$

is a continuous map $M \times M \times_p \mathcal{H} \to \mathbb{R}$, analogously to the proof of Lemma 5.1.4. This last statement is proven as follows. Let $\varepsilon > 0$ be given. Suppose $(n,m,h) \in M \times M \times_p \mathcal{H}$. There exists a $\xi \in \Delta_{\mathcal{H}}$ such that $\xi(m) = h$. Then by condition (i), (ii), (iii) and continuity of $\|\xi\|$, there exists a neighborhood $W \in M \times M \times_p \mathcal{H}$ such that for any $(n',m',h') \in W$ the map $\|P\|$ is bounded on W and we have

$$\begin{aligned}
&|\|P(n',m')h'\| - \|P(n,m)h\||\\
&\quad\le |\,\|P(n',m')h'\| - \|P(n,m')h'\|\,| + |\,\|P(n,m')h'\| - \|P(n,m')\xi(m')\|\,|\\
&\qquad + |\,\|P(n,m')\xi(m')\| - \|P(n,m)\xi(m)\|\,|\\
&\quad\le \varepsilon + \|P(n,m')\|\varepsilon + \varepsilon.
\end{aligned}$$

Next, we prove that for every $P \in \mathcal{B}(\mathcal{H}_n,\mathcal{H}_m)$ and every $\varepsilon > 0$ there exist a $Q \in \Delta_{\mathcal{B}}$ such that $\|Q(n,m) - P\| < \varepsilon$. Suppose $P \in \mathcal{B}(\mathcal{H}_n,\mathcal{H}_m)$ and let $\varepsilon > 0$ be given. Let $\xi_1, \ldots, \xi_k \in \Delta_{\mathcal{H}}$ be such that for any $h \in \mathcal{H}_m$

$$\|h - \sum_{i=1}^{k} \langle \xi_i(m), h\rangle\, \xi_i(m)\| < \varepsilon.$$

Let $\eta_1, \ldots, \eta_l \in \Delta_{\mathcal{H}}$ be such that for any $h \in \mathcal{H}_n$

$$\|h - \sum_{i=1}^{l} \langle \eta_i(n), h\rangle\, \xi_i(n)\| < \varepsilon.$$

Define, for $(n', m') \in M \times M$,

$$Q(n', m')h := \sum_{i=1}^{k} \sum_{j=1}^{l} \langle \xi_i(m'), h \rangle \langle \eta_j(n), P\xi_i(m) \rangle \eta_j(n')$$

One easily checks that $Q \in \Delta_{\mathcal{B}}$. Furthermore,

$$\begin{aligned}
&\|Ph - Q(n,m)h\| \\
&\quad \leq \|Ph - \textstyle\sum_{i=1}^{k} \langle \xi_i(m), h \rangle P\xi_i(m)\| + \| \sum_{i=1}^{k} \langle \xi_i(m), h \rangle P\xi_i(m) \\
&\qquad - \textstyle\sum_{i=1}^{k} \sum_{j=1}^{l} \langle \xi_i(m'), h \rangle \langle \eta_j(n), P\xi_i(m) \rangle \eta_j(n')\| \\
&\quad < \|P\|\varepsilon + \varepsilon.
\end{aligned}$$

The last step is to show that $\Delta_{\mathcal{B}}$ is locally uniformly closed. Suppose

$$Q \in \coprod_{(n,m) in M\times M} \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m).$$

Suppose that for all $\varepsilon > 0$ and all $(n, m) \in M \times M$ there is a $Q' \in \Delta_{\mathcal{B}}$ such that

$$\|Q(n', m') - Q'(n', m')\| < \varepsilon$$

on a neighborhood V of (n, m). We shall now show that this implies $Q \in \Delta_{\mathcal{B}}$. Indeed, let $\varepsilon > 0$ be given and suppose $n \in M$. Then there exist Q' and V as above. Define $U := p_1(V)$. Then $n' \in U$ implies, for any $h \in \mathcal{H}_m$, that

$$\|Q(n', m)h - Q'(n', m)h\| \leq \|Q(n', m) - Q'(n', m)\|\|h\| < \varepsilon\|h\|.$$

Hence $n \mapsto \|Q(n, m)h\|$ is continuous. In a similar way one proves condition (ii) for Q which finishes the proof. $\square$

We shall see in Lemma 7.1.2 that $\mathcal{B}(\mathcal{H}, \mathcal{H})$ is a so-called lower semi-continuous Fell bundle over $M \times M \rightrightarrows M$ and therefore a (full) lower semi-continuous C^*-category over M. The collection of sets

$$\{U(\varepsilon, \xi, V) \mid \xi \in \Delta_{\mathcal{B}}, \varepsilon > 0, V \subset M \times M \text{ open}\},$$

as defined in Lemma 3.1.8 for a continuous field of Banach spaces, is in general a subbasis for the topology on $\coprod_{(n,m)\in M\times M} \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)$, instead of a basis. Since the field is not continuous in general, we do not have $\Delta = \Gamma_0(M \times M, \mathcal{B}(\mathcal{H}, \mathcal{H}))$. Consider the restriction of the total space $\mathcal{B}(\mathcal{H}, \mathcal{H})$ to the unitary operators, i.e.

$$U^{op}(\mathcal{H}) := \coprod_{(n,m)\in M\times M} U(\mathcal{H}_m, \mathcal{H}_n),$$

endowed with the subspace topology.

Lemma 5.2.2. *The total space $U^{op}(\mathcal{H})$ is a continuous groupoid over M.*

Proof. We show that the composition $\mathcal{B}(\mathcal{H},\mathcal{H})^{(2)} \to \mathcal{B}(\mathcal{H},\mathcal{H})$ is a continuous map. First note that for every $(P,Q) \in \mathcal{B}(\mathcal{H},\mathcal{H})^{(2)}$ the inequality $\|PQh\| \leq \|P\|\|Qh\|$ implies

$$\|PQ\| \leq \|P\|\|Q\|.$$

Suppose that $m,n,p \in M$, $P_2 \in \mathcal{B}(\mathcal{H}_p,\mathcal{H}_n)$, $P_1 \in \mathcal{B}(\mathcal{H}_n,\mathcal{H}_m)$ and $U(\varepsilon,Q,V)$ is an open neighborhood of P_2P_1 such that $Q(p,m) = P_2P_1$. There are $Q_1,Q_2 \in \Delta_{\mathcal{B}}$ such that $Q_1(n,m) = P_1$ and $Q_2(p,n) = P_2$. Choose $\varepsilon_i > 0$ and an open subset $V_i \subset M$ such that $P_i' \in U(\varepsilon_i,Q_i,V_i)$ implies $\|P_i'\|\varepsilon_i < \varepsilon/3$ for $i = 1,2$. Furthermore, note that by condition (i), for each $m' \in M$ and $h \in \mathcal{H}_{m'}$ the map $n' \mapsto Q_1(n',m')h$ is in $\Delta_{\mathcal{B}}$. Hence by condition (ii) the map for each $p',m' \in M$ the map $n' \mapsto Q_2(p',n')Q_1(n',m')$ is continuous. The map $(p',m') \mapsto Q_2(p',n')Q_1(n',m')$ is easily seen to be continuous too. Hence we can shrink V_1 and V_2 such that $(p',n',m') \in V_2 \times_M V_2$ implies

$$|\,\|Q_2(p',n)Q_1(n,m') - Q_2(p',n')Q_1(n',m')\|\,| < \varepsilon/3.$$

Define $Q \in \Delta_{\mathcal{B}}$ by $Q(p',m') := Q_1(p',n)Q_2(n,m')$ Suppose

$$(P_2',P_1') \in U(\varepsilon_2,Q_2,V_2)_s \times_t U(\varepsilon_1,Q_1,V_1),$$

then

$$\begin{aligned}
\|P_2'P_1' - Q(p',m')\| &= \|P_2'P_1' - Q_2(p',n)Q_1(n,m')\| \\
&\leq \|P_2'P_1' - Q_2(p',n')P_1'\| + \|Q_2(p',n')P_1' - Q_2(p',n')Q_1(n',m')\| \\
&\quad + \|Q_2(p',n')Q_1(n',m') - Q_2(p',n)Q_1(n,m')\| \\
&< \|P_2' - Q_2(p',n')\|\,\|P_1'\| + \|Q_2(p',n')\|\,\|P_1' - Q_1(n',m')\| + \varepsilon/3 \\
&< \varepsilon_2\|P_1'\| + \|Q_2(p',n')\|\varepsilon_2 + \varepsilon/3 < \varepsilon.
\end{aligned}$$

Proving that the other structure maps are continuous is similar, but easier. ☐

Definition 5.2.3. A representation $(\pi,\mathcal{H},\Delta)$ is **continuous in the operator norm** if the map

$$G \to \mathcal{B}(\mathcal{H},\mathcal{H}), g \mapsto \pi(g)$$

is continuous. If G is unitary, then the representation is continuous if

$$G \to U^{op}(\mathcal{H}), g \mapsto \pi(g)$$

is a continuous map of groupoids.

Lemma 5.2.4. *A representation is continuous if it is continuous in the operator norm. The converse implication is true if the representation Δ^π is finitely generated over $C_0(M)$ and unitary.*

Proof. Suppose $(g,h) \in G_s \times_p \mathcal{H}$ and let $n = t(g)$ and $m = s(g)$. Suppose $U(\varepsilon,V,\xi)$ is a neighborhood of $\pi(g)h$, with $\xi(n) = \pi(g)h$. Let $Q \in \Delta_{\mathcal{B}}$ be any section with $Q(n,m) = \pi(g)$, which exists since $(\mathcal{B}(\mathcal{H},\mathcal{H}),\Delta_{\mathcal{B}})$ is a lower semi-continuous field of Banach spaces. Let $\eta \in \Delta_{\mathcal{H}}$ be a section such that $\eta(m) = h$. By the conditions (i),

(ii) and (iii) above there exists a neighborhood $S \subset M \times M$ of (n,m) such that for all $(n',m') \in S$

$$\|\xi(n') - Q(n',m)h\| < \varepsilon/4,$$

the function $\|Q\|$ is bounded on S and

$$\|Q(n',m)\eta(m) - Q(n',m')\eta(m')\| < \varepsilon/4.$$

Define

$$\delta := \tfrac{\varepsilon}{4\sup_{(n',m)\in S}\|Q(n',m')\|},$$
$$W' := U(\delta,\eta,p_2(S)),$$
$$K := \sup_{h'\in W'}\|h'\|,$$

and

$$W := \pi^{-1}(U(Q, \frac{\varepsilon}{4K}, S)),$$

where $p_2 : M \times M \to M$ is the projection on the second entry. We claim that $(g',h') \in W_s \times_p W'$ implies $\pi(g')h' \in U(\varepsilon, V, \xi)$. Indeed, suppose $(g',h') \in W_s \times_p W'$ and $m' = s(g')$, $n' = t(g')$, then

$$\begin{aligned}\|\xi(n') - \pi(g')h'\| &\leq \|\xi(n') - Q(n',m)h\| + \|Q(n',m)\eta(m) - Q(n',m')\eta(m')\| \\ &+\|Q(n',m')\eta(m') - Q(n',m')h'\| + \|Q(n',m')h' - \pi(g')h'\| \\ &< \varepsilon/4 + \varepsilon/4 + \|Q(n',m')\|\delta + \|h'\|\tfrac{\varepsilon}{4K} < \varepsilon.\end{aligned}$$

We shall now prove the converse implication. Suppose $(\mathcal{H}^\pi, \Delta^\pi, \pi)$ is a strongly continuous unitary representation on a continuous field of Hilbert spaces with Δ^π finitely generated. There exist a finite set $\{\xi_i\}_{i\in I}$ of sections in Δ^π such that for each $m' \in M$ the set $\{\xi_i(m')\}_{i\in I}$ contains a (normalized) basis for $\mathcal{H}_{m'}$. Suppose $U(\varepsilon, Q, V)$ is a neighborhood of $\pi(g)$, $s(g) = m$, $t(g) = n$ and $Q(n,m) = \pi(g)$. Note that by condition (i) $n' \mapsto Q(n',m)\xi_i(m)$ is in Δ^π, so by strong continuity of π there exists an open set $U_i \subset G$ such that $g' \in U_i$ implies

$$\|\pi(g')\xi'(s(g')) - Q(t(g'),m)\xi_i(m)\| < \varepsilon/(2|I|).$$

Moreover, by condition (ii) we can shrink U_i such that $g' \in U_i$ implies that

$$\|Q(t(g'),m)\xi_i(m) - Q(t(g'),s(g'))\xi_i(s(g')\| < \varepsilon/(2|I|).$$

Hence

$$\|\pi(g')\xi'(s(g')) - Q(t(g'),s(g'))\xi_i(s(g') < \varepsilon/|I|$$

for $g' \in U_i$. Define $U := \bigcap_{i\in I} U_i$, then $g' \in U$ implies

$$\begin{aligned}&\|\pi(g') - Q(t(g'),s(g'))\| \\ &\quad= \sup_{h'\in\mathcal{H}_{s(g')},\|h'\|=1}\|\pi(g')h' - Q(t(g'),s(g'))h'\|_{\mathcal{H}_{t(g')}} \\ &\quad< \textstyle\sum_{i\in I}\|\pi(g')\xi_i(s(g')) - Q(t(g'),s(g'))\xi_i(s(g')\|_{\mathcal{H}_{t(g')}} \\ &\quad< \textstyle\sum_{i\in I}\varepsilon/|I| = \varepsilon,\end{aligned}$$

which finishes the proof. □

From these comparison lemmas (Lemma 5.1.6, Lemma 5.1.7 and Lemma 5.2.4) we can conclude that for unitary representations any of these topologies are equivalent. Hence, from now on, we shall not specify which notion we mean, but only say that a unitary representation is continuous (if it is).

5.3 Example: the regular representations of a groupoid

The following example considers the regular representation. In a different form it was studied by Renault (cf. [67]), but he considered $L^2(G)$ as a measurable field of Hilbert spaces. We are interested in representations on continuous fields of Hilbert spaces. Therefore, the statement of Proposition 5.3.1 is actually new. It generalizes the analogous statement for groups.

Suppose a continuous groupoid $G \rightrightarrows M$ is endowed with a left Haar system.

Proposition 5.3.1. *The **left regular representation** of a continuous groupoid $G \rightrightarrows M$ on $(\hat{L}^2_t(G), \Delta^2_t(G))$ (cf. Example 4.2.5) defined by (continuous extension of)*

$$(\pi_L(g)f)(g') = f(g^{-1}g'),$$

for $g \in G$, $f \in C_c(G^{s(g)})$ and $g' \in G^{t(g)}$, is a continuous unitary representation.

Proof. Unitarity is immediate from the G-invariance of the Haar system.

We have to check that for all $\xi \in \Delta^2_t(G)$ the map $g \mapsto \pi_L(g)\xi(s(g))$ is continuous $G \to \hat{L}^2_t(G)$. Let $g \in G$. Suppose a neighborhood $U(\varepsilon, \eta, V) \subset \hat{L}^2_t(G)$ of $\pi_L(g)\xi(s(g))$ is given, where $\varepsilon > 0$, V an open set in M and $\eta \in \Delta_t(G)$ is a section satisfying $\pi_L(g)\xi(s(g)) = \eta(s(g))$. There exist $\xi', \eta' \in C_c(G)$ such that $\|\eta - \eta'\|_{\hat{L}^2} < \varepsilon/3$, $\|\xi - \xi'\|_{\hat{L}^2} < \varepsilon/3$ and $\pi_L(g)\xi'(s(g)) = \eta'(s(g))$. To continue we first need the following lemma due to A. Connes [11].

Lemma 5.3.2. *If f is a compactly supported continuous function on $G^{(2)}$, then the map*

$$g \mapsto \int_{h \in G^{s(g)}} f(g,h)\lambda^{s(g)}(dh)$$

is continuous on G.

We restate the proof for completeness.

Proof. Since $G^{(2)}$ is closed in $G \times G$, there exists a continuous and bounded extension $\bar{f}$ of f to $G \times G$ (we suppose here that G is a normal space). The map $(g, m) \to \int_{h \in G^m} f(g,h)\lambda^m(dh)$ is continuous, as is proven as follows. Let (g, m) be any element in $G \times M$ and let $\varepsilon' > 0$ be given. Since the Haar system is continuous and $\bar{f}(g', \cdot)$ converges uniformly to $\bar{f}(g, \cdot)$ for $g' \to g$. we can choose a neighborhood $W \in G \times M$ such that $(g', m') \in W$ implies

$$\left| \int_{h \in G^m} \bar{f}(g', h)\lambda^m(dh) - \int_{h' \in G^{m'}} \bar{f}(g', h')\lambda^{m'}(dh') \right| < \varepsilon'/2$$

and

$$\left| \int_{h' \in G^{m'}} \bar{f}(g', h')\lambda^{m}(dh') - \int_{h' \in G^{m'}} \bar{f}(g, h')\lambda^{m'}(dh') \right| < \varepsilon'/2.$$

As a consequence,

$$\begin{aligned}
&\left|\int_{h\in G^m} \bar{f}(g',h)\lambda^m(dh) - \int_{h'\in G^{m'}} \bar{f}(g,h')\lambda^{m'}(dh')\right| \\
&\leq \left|\int_{h\in G^m} \bar{f}(g',h)\lambda^m(dh) - \int_{h'\in G^{m'}} \bar{f}(g',h')\lambda^{m'}(dh')\right| \\
&\quad + \left|\int_{h'\in G^{m'}} \bar{f}(g',h')\lambda^{m}(dh') - \int_{h'\in G^{m'}} \bar{f}(g,h')\lambda^{m'}(dh')\right| \\
&< \varepsilon'/2 + \varepsilon'/2 = \varepsilon'.
\end{aligned}$$

Restricting to $\{(g,m) \mid s(g) = m\} \subset G \times M$ gives the required result. □

Now, apply this lemma to the map

$$f(g',h') := |\xi'((g')^{-1}h') - \eta'(h')|^2.$$

As a result,

$$F(g') := \sqrt{\int_{h'\in G^{t(g')}} |\xi'((g')^{-1}h') - \eta'(h')|^2 \lambda^{t(g')}(dh')}$$

depends continuously on g'. Note that $F(g) = 0$, so that we can choose a neighborhood $U \subset G$ of g such that $F(g') < \varepsilon/3$ whenever $g' \in U$. Finally, intersect U with $t^{-1}(V)$ to obtain the required open set in G whose image is a subset of $U(\varepsilon, \eta, V)$. □

In the same way one proves that the **right regular representation** of G on $(\hat{L^2_s}(G), \Delta^2_s(G))$ given by

$$\pi_L(g)h(g') := h(g'g)$$

(where $h \in C_c(G_{s(g)})$ and $g' \in G_{t(g)}$) is strongly continuous and unitary.

Let's consider two very simple examples. Other examples will occur at other places of the text.

Example 5.3.3. Note that the left (and right) regular representation of the groupoid $M \rightrightarrows M$ is $m \mapsto 1_m : M \to M \times U(1)$. The left regular representation of the pair groupoid $M \times M \rightrightarrows M$

$$\pi_L : M \times M \to U(\hat{L}^2(M \times M)) \cong M \times U(L^2(M,\nu)) \times M$$

is given by

$$(m,n) \mapsto (m, 1_{L^2(M,\nu)}, n),$$

for a given Radon measure ν on M.

5.4 Example: continuous families of groups

The following example can give the reader a feeling for the issues on the global topology with continuous groupoid representations. We express the set of finite-dimensional continuous representations of a family of groups on a given continuous field of Hilbert spaces in terms of continuous sections of the family $\mathrm{Rep}^{\mathcal{H}}(G) \to M$ of the sets of finite-dimensional continuous representations of each of the groups.

Suppose H is a locally compact group. Let $\mathrm{Rep}(H)$ denote the set of non-zero continuous unitary representations of H. This set can be endowed with a topology. Indeed, one uses the Jacobson topology on the primitive spectrum of the C^*-algebra $C^*(H)$. We shall not go into the details, since there is an easier description of the case that has our interest. For $n \geq 1$, denote by $\mathrm{Rep}^n(H)$ the subspace of continuous non-zero unitary representations on $\mathbb{C}^n$ with standard inner product $\langle z, z' \rangle = \bar{z}z'$. Let $\mathrm{Rep}^0(H)$ be the set consisting of just the zero representation. Let $S(\mathbb{C}^n)$ denote the unit sphere in $\mathbb{C}^n$.

Lemma 5.4.1. *([22], 18.1.9) For every integer $n \geq 0$ a subbasis for the topology on $\mathrm{Rep}^n(H)$ is given by the sets*

$$U(\pi, \varepsilon, K) := \{\pi' \in \mathrm{Rep}^n(H) \mid \max_{g \in K} |\langle h', \pi(g)\, h\rangle - \langle h', \pi'(g)\, h\rangle| < \varepsilon, \forall h, h' \in S(\mathbb{C}^n)\},$$

for compact sets $K \subset H$, representations $\pi \in \mathrm{Rep}^n(H)$ and $\varepsilon > 0$.

We need the following technical notion. Suppose $p : N \to M$ is a continuous map.

Definition 5.4.2. We say a set $K \subset N$ is **p-open-compact** if the restriction $K \cap p^{-1}(m)$ is compact for all $m \in M$ and the image $p(K) \subset M$ is open. We say that $p : N \to M$ is **locally open-compact** if every $n \in N$ has a p-open-compact neighborhood.

Example 5.4.3. If $p : N \to M$ is a fiber bundle with locally compact fiber, then it is easy to show that p is locally open-compact.

Suppose $s : G \to M$ is a continuous family of groups (cf. Example 2.1.9). *Fix* a uniformly finite-dimensional continuous field of Hilbert spaces $(\mathcal{H}, \Delta)$. We proceed in a few steps to define the surjection $\mathrm{Rep}^{\mathcal{H}}(G) \to M$ and endow it with a suitable topology.

(1) Choose for each $m \in M$ a group $H_m \cong s^{-1}(m)$ and an isomorphism $\psi_m : s^{-1}(m) \to H_m$, fixing the group structure at each fiber. Endow $\coprod_{m \in M} H_m$ with the topology such that

$$\coprod_{m \in M} \psi_m : G \to \coprod_{m \in M} H_m$$

is a homeomorphism. Denote the canonical projection $\coprod_{m \in M} H_m \to M$ by s'.

(2) Suppose

$$\left\{ \phi_i : \mathcal{H}|_{U_i} \hookrightarrow U_i \times \sum_{n=0}^{\dim(\mathcal{H}_{U_i})} \mathbb{C}^n \right\}_{i \in I}$$

is a local pseudo-trivialization of $(\mathcal{H}, \Delta)$ (cf. Definition 3.2.5). Define for all $i, j \in I$ the homeomorphism

$$\gamma_{ij} := \phi_j(\phi_i)^{-1} : \mathrm{im}(\phi_i)|_{U_i \cap U_j} \to \mathrm{im}(\phi_j)|_{U_i \cap U_j}.$$

(3) For any $i \in I$, define

$$\mathrm{Rep}^{\mathcal{H}}(G|_{U_i}) := \coprod_{m \in U_i} \mathrm{Rep}^{\dim(\mathcal{H}_m)}(H_m)$$

and denote the canonical projection by

$$p_i : \mathrm{Rep}^{\mathcal{H}}(G|_{U_i}) \to U_i.$$

(4) For each $i \in I$ the following sets form a subbasis of a topology on $\mathrm{Rep}^{\mathcal{H}}(G)|_{U_i}$: For any $\xi, \eta \in \Delta$, $V \subset \mathbb{C}$ open and $K \subset \coprod_{m \in U_i} H_m$ s'-open-compact,

$$U(\xi, \eta, K, V) := \left\{ \pi \in \mathrm{Rep}^{\mathcal{H}}(G|_{U_i}) \mid \langle \xi, \pi\eta \rangle (K \cap H_{p_i(\pi)}) \subset V \right\}.$$

(5) Define

$$\mathrm{Rep}^{\mathcal{H}}(G) := (\coprod_{i \in I} \mathrm{Rep}^{\mathcal{H}}(G|_{U_i}))/\sim,$$

where $\mathrm{Rep}^{\mathcal{H}}(G|_{U_i})|_{U_i \cap U_j} \ni \pi_i \sim \pi_j \in \mathrm{Rep}^{\mathcal{H}}(G|_{U_j})|_{U_i \cap U_j}$ iff $\pi_j = \gamma_{ij}\pi_i\gamma_{ij}^{-1}$.

The space $\mathrm{Rep}^{\mathcal{H}}(G)$ is uniquely determined up to homeomorphism by the chosen local pseudo-trivialization of $(\mathcal{H}, \Delta)$ and isomorphisms $\{\psi_m : s^{-1}(m) \to H_m\}_{m \in M}$. One can see that $s : G \to M$ being locally open-compact implies that the topology of $\mathrm{Rep}^{\mathcal{H}}(G|_{U_i})$ restricted to each fiber is equivalent to the topology of Lemma 5.4.1.

Proposition 5.4.4. *Suppose that $s : G \to M$ is locally open-compact family of groups. Then there is a one-to-one correspondence between continuous representations of $s : G \to M$ on $(\mathcal{H}, \Delta)$ and continuous sections of $\mathrm{Rep}^{\mathcal{H}}(G) \to M$.*

Proof. A continuous unitary representation π of G on $(\mathcal{H}, \Delta)$ corresponds to a continuous section of $\mathrm{Rep}^{\mathcal{H}}(G)$, i.e. to a family of sections $\tilde{\pi}_i : U_i \to \mathrm{Rep}^{\mathcal{H}}(G)|_{U_i}$ given by

$$\tilde{\pi}_i(m) = \phi_i \circ \pi \circ (\psi_m^{-1} \times \phi_i^{-1}).$$

These are easily seen to be compatible, i.e. $\tilde{\pi}_j = \gamma_{ij}\tilde{\pi}_i\gamma_{ij}^{-1}$. It remains to show that each $\tilde{\pi}_i$ is continuous. Consider an open set $U(\xi, \eta, K, V)$ as above. Note that

$$\begin{aligned} \tilde{\pi}^{-1}(U(\xi, \eta, K, V)) &= \{m \in U_i \mid \langle \xi, \pi\eta \rangle |_{K \cap H_m} \subset V\} \\ &= s'(K \cap \{g \in \coprod_{m \in U_i} H_m \mid \langle \xi, \pi\eta \rangle (g) \subset V\}), \end{aligned}$$

which is open since K is s'-compact and π is continuous.

A continuous section $\tilde{\pi}$ of $\mathrm{Rep}^{\mathcal{H}}(G)$ determines a continuous unitary representation by

$$\pi(g) := \phi_i^{-1} \circ \tilde{\pi}_i \circ (\psi_{s(g)}(g) \times \phi_i) \in U(\mathcal{H}_{s(g)}),$$

where $i \in I$ such that $s(g) \in U_i$. We only need to show that $\pi|_{G_{U_i}}\psi^{-1}$ is continuous. Suppose $\xi, \eta \in \Gamma_0(\mathrm{im}(\phi_i))$. Given $g \in \coprod_{m \in U_i} H_m$ and $V \subset \mathbb{C}$, let K be an s'-compact neighborhood of g and $W \subset K$ an open neighborhood of g. Consider $U(\xi, \eta, K, V)$.

Define $W' := W \cap s^{-1}\tilde{\pi}^{-1}(U(\xi, \eta, K, U))$, which is open since s and $\tilde{\pi}$ are continuous. Then $g' \in W'$ implies

$$\langle \xi(s(g')), \pi(g')\eta(s(g')) \rangle = \langle \xi(s(g')), \tilde{\pi}(s(g'))(g')\eta(s(g')) \rangle \in U.$$

Moreover π is locally bounded, since $\mathcal{H}$ is uniformly finite-dimensional. This finishes the proof. □

Example 5.4.5. Consider a locally compact group H and a continuous principal H-bundle $\tau : P \to M$. From this we can construct a continuous bundle of groups $P \times_H H \to M$, where the action of H on H is given by conjugation. Consider a local trivialization $\{\chi_i : P|_{U_i} \to U_i \times H\}_{i \in I}$ of $P \to M$. Suppose $I = \mathbb{N}$. One can fix the group structure at each fiber of $P \times_H H \to M$ as follows: for every $m \in M$ choose the smallest $i \in I$ such that $m \in U_i$ and define

$$\psi_m : (P \times_H H)_m \to H, [p, h] \mapsto \chi_i(p)\, h\, \chi_i(p)^{-1}.$$

Given a representation $(\pi, \mathbb{C}^n) \in \mathrm{Rep}^n(H)$, one can construct a vector bundle $\mathcal{H} := P \times_\pi \mathbb{C}^n \to M$. Obviously, the trivialization of $P \to M$ gives rise to a trivialization $\{\phi_i : \mathcal{H}|_{U_i} \to U_i \times \mathbb{C}^n\}_{i \in I}$ of $\mathcal{H} \to M$, by $\phi_i([p, z]) = (\tau(p), \pi(\chi_i(p))z)$. Using these data one can form the bundle $\mathrm{Rep}^{\mathcal{H}}(P \times_H H) \to M$ and a topology on it. A continuous section of this bundle is given by

$$\tilde{\pi}_i(m) = (h \mapsto \pi(\gamma_{ij}^{-1} h \gamma_{ij}),$$

for all $i \in \mathbb{N}$, $m \in U_i$, $h \in H$ and the smallest $j \in \mathbb{N}$ such that $m \in U_j$. This section corresponds to the representation of $P \times_H H \to M$ on $(\mathcal{H}, \Delta)$ given by $\tilde{\pi}([p, h])[p, z] = [p, \pi(h)z]$.

Remark 5.4.6. One can "twist" $\mathcal{H} := P \times_\pi \mathbb{C}^n$ by another continuous field $(\mathcal{H}', \Delta')$, carrying the trivial representation of $P \times_H H \to M$, to obtain a representation on $\mathcal{H} \otimes \mathcal{H}'$. A similar construction is possible for any groupoid, cf. Lemma 6.6.7.

5.5 Representations of the global bisections group

For the reader who prefers representation theory of groups and wonders why one should be interested in representations of groupoids at all, the next section will be of particular interest. Namely, to any continuous groupoid is associated a topological group: the group of global bisections. For a large class of continuous groupoids (the ones we call locally bisectional) we establish a bijection between the continuous representations of the groupoid on continuous fields of Hilbert spaces and a specific type of continuous representations of the group of global bisections on Banach spaces. Hence the representation theory of such groupoids can be "embedded" in the representation theory of groups. From this point of view, the groupoid offers a way to study some representations of these groups of bisections.

Suppose $G \rightrightarrows M$ is a continuous groupoid. A continuous **global bisection** is a continuous map $\sigma : M \to G$ such that $t \circ \sigma = id_M$ and $\tilde{\sigma} := s \circ \sigma : M \to M$ is a homeomorphism. Denote the set of global bisections of $G \rightrightarrows M$ by $\mathrm{Bis}(G)$. This set has a group structure, cf. [87]. Moreover, it is even a topological group.

Lemma 5.5.1. Bis(G) *has the structure of a topological group in the compact-open topology.*

Proof. The multiplication is given by

$$(\sigma_1 \cdot \sigma_2)(m) := \sigma_1(m)\sigma_2(\tilde{\sigma}_1(m)).$$

The unit is given by the unit section $u : M \to G$ and the inverse is defined by

$$\sigma^{-1}(m) := (\sigma(\tilde{\sigma}^{-1}(m)))^{-1}.$$

The group laws are easily checked, for example

$$\begin{aligned}(\sigma \cdot \sigma^{-1})(m) &= \sigma(m)\sigma^{-1}(\tilde{\sigma}(m)) \\ &= \sigma(m)(\sigma(\tilde{\sigma}^{-1}\tilde{\sigma}(m)))^{-1} \\ &= 1_m.\end{aligned}$$

We prove that multiplication is continuous Bis$(G) \times$ Bis$(G) \to$ Bis(G). Suppose $\sigma_1 \cdot \sigma_2 \in U(C,V)$, where C is a compact set in M, V open in G and $U(C,V)$ the set of maps $\tau : M \to G$ that satisfy $\tau(C) \subset V$, i.e. $U(C,V)$ is in the standard subbasis of the topology on Bis(G). For each $m \in C$, let V_m be a neighborhood of $(\sigma_1 \cdot \sigma_2)(m) = \sigma_1(m)\sigma_2(\tilde{\sigma}_1(m))$. These V_m cover $\sigma_1 \cdot \sigma_2(C)$ which is compact by continuity of the multiplication in G and σ_1, σ_2. Let $\{V_i\}_{i \in I}$ be a finite subcover. The inverse image $m^{-1}(V_i)$ is open and contains a Cartesian product $W_i^1 \times W_i^2$ of open sets W_i^1, W_i^2 for each $i \in I$. Then $\sigma_1' \in U(C, \bigcup_{i\in I} W_i^1)$ and $\sigma_2' \in U(\tilde{\sigma}_1(C), \bigcup_{i\in I} W_i^2)$ implies $\sigma_1' \cdot \sigma_2' \in U(C,V)$. □

Example 5.5.2. The global bisection group of the pair groupoid $M \times M$ is the group of homeomorphisms of M.

Example 5.5.3. For the trivial group bundle $G \times M \to M$ (with fiber G) the group of global bisections is just the group of sections with the pointwise multiplication. In particular, if M is the circle $\mathbb{S}^1$ and G a Lie group then the group of global bisections is the loop group $C(\mathbb{S}^1, G)$ with its usual topology (cf. [65]).

Lemma 5.5.4. *A continuous unitary representation* $(\pi, \mathcal{H}, \Delta)$ *of a groupoid* $G \rightrightarrows M$ *canonically induces a continuous isometric representation of* Bis(G) *on* Δ.

Proof. Define the representation $\tilde{\pi}$ of Bis(G) by

$$(\tilde{\pi}(\sigma)\xi)(m) := \pi(\sigma(m))\xi(\tilde{\sigma}(m)),$$

where $\xi \in \Delta$, $m \in M$ and $\sigma \in$ Bis(G). This representation is isometric, since π is unitary:

$$\|\tilde{\pi}(\sigma)\xi\| = \sup_{m\in M} \|\pi(\sigma(m))\xi(\tilde{\sigma}(m))\|_{\mathcal{H}_m} = \|\xi\|.$$

Continuity is proven as follows. Suppose $\varepsilon > 0$ and $\xi \in \Delta$ are given. There exists a compactly supported section $\xi' \in \Delta_c := C_c(M)\Delta$ such that $\|\xi - \xi'\| < \varepsilon/6$. Denote the support of ξ' by K. Moreover, since π is continuous and unitary it is norm continuous and hence there exists an open set $V \subset G$ such that $g, g' \in V$

implies $\|\pi(g)\xi'(s(g)) - \pi(g')\xi'(s(g'))\| < \varepsilon/3$. Now, suppose that $\sigma, \sigma' \in U(K,V)$ and $\eta \in B(\xi, \varepsilon/6)$, then

$$\sup_{m \in M} \|\pi(\sigma(m))\eta(\tilde{\sigma}(m)) - \pi(\sigma'(m))\eta(\tilde{\sigma}'(m))\| < \varepsilon,$$

which finishes the proof. □

The obtained representation of $\mathrm{Bis}(G)$ is actually $C_0(M)$**-unitary** (or C^***-unitary** with respect to $C_0(M)$), in the sense that

$$\langle \tilde{\pi}(\sigma)\xi, \tilde{\pi}(\sigma)\eta \rangle = \langle \xi, \eta \rangle$$

for all $\sigma \in \mathrm{Bis}(G)$ and $\xi, \eta \in \Delta$.

For the following result we need a technical condition on groupoids. We call a continuous groupoid $G \rightrightarrows M$ **bisectional** if

(i) every $g \in G$ is in the image of a continuous global bisection;

(ii) for all compact sets $K \subset M$ and open sets $V \subset G$, the set $\bigcup_{\sigma \in U(K,V)} \mathrm{im}(\sigma) \subset G$ is open.

Theorem 5.5.5. *Suppose $G \rightrightarrows M$ is bisectional. Then there is a bijective correspondence between continuous unitary representations of G and continuous $C_0(M)$-unitary representations of* $\mathrm{Bis}(G)$ *on a Hilbert $C_0(M)$-module satisfying*

(i) $C_0(M)$-linearity, i.e.

$$\tilde{\pi}(\sigma)(f\,\xi) = \tilde{\sigma}^*(f)\,\tilde{\pi}(\sigma)(\xi)$$

for all $\sigma \in \mathrm{Bis}(G)$, $\xi \in \Delta$ and $f \in C_0(M)$ and

(ii) locality, i.e. if $\sigma(m) = 1_m$ for some $m \in M$, then $\|\tilde{\pi}(\sigma)\xi - \xi\|(m) = 0$

Proof. Given a representation $(\tilde{\pi}, \Delta)$ of $\mathrm{Bis}(G)$ as above, define a representation $\pi : G \to U(\mathcal{H})$ as follows. Form the continuous field of Hilbert spaces $(\{\mathcal{H}_m\}_{m \in M}, \Delta)$ associated to Δ (cf. Theorem 4.2.4). For any $g \in G$ and $h \in \mathcal{H}_{s(g)}$, define

$$\pi(g)h := (\tilde{\pi}(\sigma)\xi)(t(g)),$$

for any $\xi \in \Delta$ such that $\xi(s(g)) = h$ and $\sigma \in \mathrm{Bis}(G)$ such that $\sigma(t(g)) = g$, which exist by assumption. We now show that this definition does not depend on the choice of σ and ξ. Suppose ξ, ξ' satisfy $\xi(m) = h = \xi'(m)$. Let $\{U_i\}_{i \in \mathbb{N}}$ be a family of sets such that $\bigcap_{i \in \mathbb{N}} U_i = \{s(g)\}$ and $\{\chi_i : U_i \to [0,1]\}$ a family of functions such that $\chi_i(s(g)) = 0$ and $\chi_i(n) = 1$ for all $n \in M \backslash U_i$. Then

$$\begin{aligned}
(\tilde{\pi}(\sigma)\xi)(t(g)) - (\tilde{\pi}(\sigma)\xi')(t(g)) &= \lim_{i \to \infty} (\tilde{\pi}(\sigma)\chi_i(\xi - \xi'))(t(g)) \\
&= \lim_{i \to \infty} \chi_i(\tilde{\sigma}(t(g)))(\tilde{\pi}(\sigma)(\xi - \xi'))(t(g)) \\
&= 0,
\end{aligned}$$

since $\tilde{\pi}$ is $C_0(M)$-linear and $\tilde{\sigma}(t(g)) = s(g)$.

Suppose $\sigma(m) = \sigma'(m)$ for $\sigma, \sigma' \in \mathrm{Bis}(G)$ and $m \in M$. Then, by locality, for all $\xi \in \Delta$

$$\|\tilde{\pi}(\sigma^{-1}\sigma')\xi - \xi\|(m) = 0,$$

and hence $(\tilde{\pi}(\sigma)\xi)(m) = (\tilde{\pi}(\sigma')\xi)(m)$.

Unitarity of π follows at once from $C_0(M)$-unitarity of $\tilde{\pi}$.

Next, we prove continuity of π. Suppose $(g, h) \in G\,{}_s\!\times_p \mathcal{H}$ and $U(\varepsilon, \eta, V)$ open neighborhood of $\pi(g)h$, where $\eta(t(g)) = \pi(g)h$. We need to construct an open neighborhood of (g, h), which maps to $U(\varepsilon, \eta, V)$. Consider

$$B(\eta, \varepsilon) := \{\xi \in \Delta \mid \|\eta - \xi\| < \varepsilon\}.$$

Let $\sigma \in \mathrm{Bis}(G)$ be such that $\sigma(t(g)) = g$, which exists since G is bisectional. Define $\xi := \bar{\pi}(\sigma)^{-1}\eta$. By continuity of $\bar{\pi}$ there exists an open neighborhood $B(\xi, \delta)$ of ξ and an open neighborhood $U(K, W)$ of σ such that $\bar{\pi}(U(K, W) \times B(\xi, \delta)) \subset B(\eta, \varepsilon)$. Since $G \rightrightarrows M$ is bisectional, there exists an open neighborhood W' of g in $\bigcup_{\sigma \in U(K,W)} \mathrm{im}(\sigma)$.

Suppose that $(g', h') \in W'\,{}_s\!\times_p U(\xi, \delta, \tilde{\sigma}^{-1}(V))$, then

$$\pi(g')h' = (\bar{\pi}(\sigma')\xi')(t(g')) \in U(\varepsilon, \eta, V),$$

for some $\sigma' \in U(K, W)$ and $\xi' \in B(\xi, \delta)$.

One easily sees that the constructions given in this proof to obtain representations of G from representations of $\mathrm{Bis}(G)$ and vice versa in the proof of the above lemma are inverses of each other. □

Remark 5.5.6. As an intermediate step, one can also relate the representations of $G \rightrightarrows M$ to the representation of the inverse semi-group of continuous **local bisections** of $G \rightrightarrows M$. These are continuous maps $\sigma : U \to G$ for open $U \subset M$ such that $t \circ \sigma = id_U$ and $\tilde{\sigma} := s \circ \sigma : U \to U$ is a homeomorphism.

Chapter 6

Groupoid representation theory

Is there a Schur's Lemma for groupoids? Is there a Peter-Weyl theorem for groupoids? In this chapter we give answers to these questions. We discuss a way to generalize these statements, that are well-known for groups, to groupoids. It turns out that you need extra conditions on the groupoid for the statements to be true (unlike what is suggested in [1]). A crucial rôle is played by the functors that restrict representations of a groupoid to representations of its isotropy groups. This chapter shows that representation theory of groupoids is quite different from representation theory for groups, but many results can be carried over using some caution.

6.1 Decomposability and reducibility

Definition 6.1.1. (i) The **direct sum of a countable family of continuous fields of Hilbert spaces** $\{(\mathcal{H}^i, \Delta_i\}_{i\in I}$ is the continuous field of Hilbert spaces $(\mathcal{H}^{\oplus}, \Delta^{\oplus})$ whose fibers are given by $\mathcal{H}^{\oplus}_m := \bigoplus_{i\in I} \mathcal{H}^i_m$ and whose space of continuous sections $\Delta^{\oplus}$ is the closure of the pre-Hilbert $C_0(M)$-module of finite sums of sections $\sum_{j\in J} \xi_j$, where $J \subset I$ is a finite index set and $\xi_j \in \Delta_j$ for all $j \in J$.

(ii) The **direct sum of a countable family of continuous representations** $\{(\mathcal{H}^i, \Delta_i, \pi_i)\}_{i\in I}$ of a groupoid $G \rightrightarrows M$ is the representation of $G \rightrightarrows M$ on the direct sum of continuous fields of Hilbert spaces $(\mathcal{H}^{\oplus}, \Delta^{\oplus})$, given by the continuous extension of the map $\bigoplus_{j\in J} \pi_j : g \mapsto \sum_{j\in J} \pi_j(g)$ on finite sums by continuity.

(iii) We say that that a continuous unitary representation $(\mathcal{H}, \pi)$ of a groupoid G is **decomposable** if it is equivariantly isomorphic to a direct sum of representations of $(\mathcal{H}^1, \pi_1)$ and $(\mathcal{H}^2, \pi_2)$

$$(\mathcal{H}, \Delta) \cong \mathcal{H}^1 \oplus \mathcal{H}^2.$$

and **indecomposable** if this is not possible.

(iv) A **continuous subfield** of a continuous field of Hilbert spaces $(\mathcal{H}, \Delta)$ is a continuous field of Hilbert spaces $(\mathcal{H}', \Delta')$, such that $\mathcal{H}'_m \subset \mathcal{H}_m$ is a closed linear subspace with the induced inner product for all $m \in M$ and $\Delta' \subset \Delta$ a Hilbert $C_0(M)$-submodule.

(v) A **continuous subrepresentation** of a continuous unitary representation $(\mathcal{H}, \pi)$ of a groupoid G is a continuous subfield of $(\mathcal{H}, \Delta)$ stable under π.

(vi) A continuous unitary representation is **reducible** if it has a proper continuous subrepresentation. It is **irreducible** if it is not reducible.

Proposition 6.1.2. *If $(\mathcal{H}, \Delta, \pi)$ is a continuous locally trivial unitary representation and $(\mathcal{H}', \Delta', \pi')$ a locally trivial subrepresentation $(\mathcal{H}, \Delta, \pi)$, then $(\mathcal{H}, \Delta, \pi)$ decomposes as a direct sum of $(\mathcal{H}', \Delta', \pi')$ and another locally trivial subrepresentation.*

Proof. For each $m \in M$ let $\mathcal{H}''_m$ be the orthogonal complement with respect to the inner product. The family $\{\mathcal{H}''_m\}_{m\in M}$ forms a continuous field, with

$$\Delta'' := \{\xi \in \Delta \mid \xi(m) \in \mathcal{H}''_m \text{ for all } m \in M\},$$

since $\mathcal{H}$ is locally trivial. Moreover, $(\mathcal{H}'', \Delta'')$ is locally trivial too. Since π is unitary, this complement is G-invariant. ☐

Decomposability implies reducibility (irreducible implies indecomposable), but not vice versa. Indeed, a representation can contain a subrepresentation without being decomposable.

Example 6.1.3. Consider the trivial representation of $\mathbb{R} \rightrightarrows \mathbb{R}$ on $(\mathbb{R} \times \mathbb{C}, C_0(\mathbb{R}))$. It has a subrepresentation given by the continuous field of Hilbert spaces which is 0 at 0 and $\mathbb{C}$ elsewhere, with space of sections

$$C_0^0(M) := \{f \in C_0(M) \mid f(0) = 0\}.$$

This subrepresentation has no complement, since this would be a field that is $\mathbb{C}$ at 0 and zero elsewhere, whose only continuous section could be the zero section. Hence it would not satisfy condition (i) of Definition 3.1.1. Note that $\mathbb{R} \rightrightarrows \mathbb{R}$ is an example of a groupoid that has no continuous irreducible representations.

Definition 6.1.4. Define the **support of a continuous field of Hilbert spaces** $(\mathcal{H}, \Delta_{\mathcal{H}})$ by

$$\mathrm{supp}(\mathcal{H}, \Delta) := \{m \in M \mid \mathcal{H}_m \neq 0\}.$$

This last set equals

$$\{m \in M \mid \xi(m) \neq 0 \text{ for some } \xi \in \Delta_{\mathcal{H}}\}.$$

One easily sees that for all continuous fields of Hilbert spaces $(\mathcal{H}, \Delta_{\mathcal{H}})$ the support $\mathrm{supp}(\mathcal{H}, \Delta_{\mathcal{H}})$ is open in M.

Lemma 6.1.5. *(i) If the support of a continuous representation $(\mathcal{H}, \Delta_{\mathcal{H}}, \pi)$ of a groupoid $G \rightrightarrows M$ properly contains a closed union of G-orbits, then it is reducible.*

(ii) If the support of a continuous representation $(\mathcal{H}, \Delta_{\mathcal{H}}, \pi)$ of a groupoid $G \rightrightarrows M$ properly contains a clopen set of G-orbits, then it is decomposable.

Proof. Let $(\mathcal{H}, \Delta, \pi)$ be a continuous representation of $G \rightrightarrows M$. Suppose $U \subset M$ is a closed union of orbits. Define a new continuous field of Hilbert spaces by

$$\mathcal{H}'_m := \begin{cases} \mathcal{H}_m & \text{if } m \notin U \\ 0 & \text{if } m \in U \end{cases}$$

and

$$\Delta_{\mathcal{H}'} := \{\xi \in \Delta \mid \xi|_U = 0\},$$

The groupoid $G \rightrightarrows M$ represents on $(\mathcal{H}', \Delta')$ by

$$\pi'(g) := \begin{cases} \pi(g) & \text{if } s(g) \notin U \\ id_0 & \text{if } m \in U \end{cases}$$

One easily sees that $(\mathcal{H}', \Delta_{\mathcal{H}'}, \pi')$ is a continuous subrepresentation of $(\mathcal{H}, \Delta_{\mathcal{H}}, \pi)$. The second statement is proved analogously. □

The representation $(\mathcal{H}', \Delta_{\mathcal{H}'}, \pi')$ is called the **restriction of** $(\mathcal{H}, \Delta_{\mathcal{H}}, \pi)$ **to** U^c.

Example 6.1.6. If a groupoid $G \rightrightarrows M$ is proper and M, then the orbits are closed. Hence an irreducible representation must consist of one orbit that is clopen, since it is the support of a continuous field and the orbit of a proper groupoid. Therefore, a space $M \rightrightarrows M$ has an irreducible representation iff it has a discrete point $m \in M$.

6.2 Schur's lemma

In the previous section we have seen that in many cases of interest the irreducible representations do exist. Therefore, we introduce the weaker notion of internal irreducibility.

Definition 6.2.1. A continuous representation $(\pi, \mathcal{H}, \Delta)$ of a groupoid $G \rightrightarrows M$ is called **internally irreducible**, if the restriction of π to each of the isotropy groups is an irreducible representation.

Obviously, if a representation is irreducible, then it is internally irreducible. The converse does not hold as we have seen in Example 6.1.3.

Example 6.2.2. Suppose H is a topological group, $P \to M$ a continuous principal H-bundle and (π, V) an irreducible representation of H. Then, $P \times_H V \to M$ carries a canonical internally irreducible (but reducible, if $M \neq pt$ and Hausdorff) representation of the bundle of groups $P \times_H H \to M$ (cf. Section 5.4).

Example 6.2.3. If M is a topological space with a non-trivial rank 2 vector bundle $E \to M$. Then $E \to M$ is not internally irreducible as a representation of $M \rightrightarrows M$, even though it might be indecomposable.

Example 6.2.4. A morphism of internally irreducible continuous representations is not necessarily an isomorphism or the zero map. A counterexample is given by the following: let G be the constant bundle of groups $\mathbb{R} \times U(1) \rightrightarrows \mathbb{R}$. It represents internally irreducibly on the trivial rank one vector bundle $\mathcal{H} := \mathbb{R} \times \mathbb{C}$ over $\mathbb{R}$ by scalar multiplication. The map $\Psi : (x, z) \mapsto (x, x \cdot z)$ is an equivariant adjointable map $\mathcal{H} \to \mathcal{H}$, not equal to a scalar times the identity.

Remark 6.2.5. What one does see in this example is that Ψ is a function times the identity on $\mathcal{H}$, namely the function $\lambda : \mathbb{R} \to \mathbb{C}, x \mapsto x$, i.e. $\psi = \lambda 1_{\mathcal{H}}$. An alternative formulation of Schur's lemma for groupoids would be that an endomorphism of an internally irreducible representation $(\mathcal{H}, \pi)$ is a function $\lambda \in C(M)$ times the identity on $\mathcal{H}$. This we shall proof under some conditions in Lemma 6.2.7.

Notation 6.2.6. For a continuous groupoid $G \rightrightarrows M$ denote

(i) the set of isomorphism classes of continuous unitary representations by $\mathrm{Rep}(G)$;

(ii) the subset of isomorphism classes of indecomposable unitary representation by $\mathrm{IdRep}(G)$;

(iii) the subset of isomorphism classes of irreducible unitary representations by $\mathrm{IrRep}(G)$. For groups H this set is known as the unitary dual and denoted by $\hat{H}$;

(iv) the set of isomorphism classes of internally irreducible unitary representations by $\mathrm{IrRep}^i(G)$.

Lemma 6.2.7 (Schur's Lemma for groupoids)**.** *Suppose $(\pi_i, \mathcal{H}^i, \Delta^i)$ is an internally irreducible representation for $i = 1, 2$.*

(i) every equivariant endomorphism $\Psi : \mathcal{H}^1 \to \mathcal{H}^1$ is equal to a continuous function $\lambda \in C(M)$ times the identity on E, i.e. $\psi = \lambda\, 1_{\mathcal{H}^1}$.

(ii) If $\Phi : \mathcal{H}^1 \to \mathcal{H}^2$ is a morphism of representations then Φ_m is either an isomorphism or the zero map $\mathcal{H}^1_m \to \mathcal{H}^2_m$ for all $m \in M$.

(iii) If, furthermore, $\mathrm{Res}_m : \mathrm{IrRep}^i(G) \to \mathrm{IrRep}(G^m_m)$ is injective for every $m \in M$, then

$$\mathrm{Hom}_G(\mathcal{H}^1, \mathcal{H}^2) = \begin{cases} \textit{line bundle} & \textit{if } (\pi_1, \mathcal{H}^1, \Delta^1) \cong (\pi_2, \mathcal{H}^2, \Delta^2); \\ 0 & \textit{if } (\pi_1, \mathcal{H}^1, \Delta^1) \ncong (\pi_2, \mathcal{H}^2, \Delta^2). \end{cases}$$

The proof follows easily from the analogous statement for groups.

Example 6.2.8. Suppose $P \to M$ is a principal H-bundle for a group H. If $G \rightrightarrows M$ is the gauge groupoid $P \times_H P \rightrightarrows M$, then every irreducible representation is internally irreducible. Moreover, $\mathrm{Res}_m : \mathrm{IrRep}^i(G) \to \mathrm{IrRep}(G^m_m)$ is injective for all $m \in M$. Hence Schur's Lemma holds for all representations of these groupoids. Moreover, for two representations $E_i = P \times_H V_i \to M$ of $G \rightrightarrows M$ $(i = 1, 2)$, with V_1 and V_2 isomorphic representations of H.

$$\begin{aligned} \mathrm{Hom}_G(E_1, E_2) &= \mathrm{Hom}_G(P \times_H V_1, P \times_H V_2) \\ &\cong P \times_H \mathrm{Hom}_H(V_1, V_2) \\ &\cong P \times_H \mathbb{C} \end{aligned}$$

where we used Schur's Lemma for groups in the third equation. The group H acts on $\mathrm{Hom}_H(V_1, V_2)$ by $(h \cdot \phi)v_1 = h^{-1}\,\phi(h \cdot v)$.

Example 6.2.9. Consider the two-sphere as a groupoid $S^2 \rightrightarrows S^2$. It is proper and all indecomposable vector bundles over S^2 have rank one. These are internally irreducible representations, but obviously $\mathrm{Res}_m : \mathrm{IrRep}^i(S^2) \to \mathrm{IrRep}(\{m\})$ is not injective for any $m \in M$. Moreover, for non-isomorphic line bundles $L_1 \to S^2$ and $L_2 \to S^2$, one has

$$\mathrm{Hom}_M(L_1, L_2) \cong L_1^* \otimes L_2 \ncong 0.$$

6.3 Square-integrable representations

In this section we define the notion of square-integrability for continuous groupoid representations. In the end, we prove that for proper groupoids, with M/G compact, unitary representations are square-integrable, generalizing an analogous result for compact groups.

Suppose $G \rightrightarrows M$ is a locally compact groupoid endowed with a Haar system $\{\lambda_m\}_{m\in M}$, which desintegrates as $\lambda_m = \int_{n\in t(G_m)} \lambda_m^n \mu_m(dn)$, for a Haar system $\{\mu_m\}_{m\in M}$ on $R_G \rightrightarrows M$ and a continuous family of measures $\{\lambda_m^n\}_{(n,m)\in R_G}$ on $t \times s : G \to M \times M$ (cf. Proposition 2.5.6).

Using the family $\{\lambda_m^n\}_{(n,m)\in R_G}$ one can construct the continuous field of Hilbert spaces

$$(\hat{L}^2(G), \Delta^2(G)) := (\hat{L}^2_{t\times s}(G), \Delta^2_{t\times s}(G)),$$

over R_G, cf. Example 4.2.5.

Example 6.3.1. A simple example of this is the following (also see the example following Proposition 2.5.6). If M is a space and μ a Radon measure on M and H a Lie group with Haar measure λ. Then the trivial transitive groupoid $M \times H \times M \rightrightarrows M$ with isotropy groups H has a Haar system $\{\lambda_m = \mu \times \lambda\}_{m\in M}$. Obviously, this decomposes as $\lambda_m = \int_{n\in M} \lambda\mu(dn)$, hence

$$(\hat{L}^2(G), \Delta^2(G)) = \left(L^2(G,\lambda) \times (M \times M), C_0(M \times M, L^2(G,\lambda))\right).$$

Definition 6.3.2. A map $f : G \to \mathbb{C}$ is called **$\hat{L}^2(G)$-square integrable** if the induced map

$$(m, n) \mapsto (g \mapsto f(g), G_m^n \to \mathbb{C})$$

is in $\Delta^2(G)$.

Definition 6.3.3. (i) The **conjugate $(\bar{\mathcal{H}}, \bar{\Delta})$ of a continuous field of Hilbert spaces** $(\mathcal{H}, \Delta)$ is the family of Hilbert spaces is given by $\bar{\mathcal{H}}_m = \mathcal{H}_m$ as Abelian groups, but with conjugate complex scalar multiplication and the space of sections $\bar{\Delta} = \Delta$, but with conjugate $C_0(M)$-action.

(ii) The **conjugate representation** $(\bar{\mathcal{H}}, \bar{\Delta}, \bar{\pi})$ of a representation $(\mathcal{H}, \Delta, \pi)$ of $G \rightrightarrows M$ is the representation on the conjugate continuous field of Hilbert spaces $(\bar{\mathcal{H}}, \bar{\Delta})$ is given by $\bar{\pi}(g)h = \pi(g)h$, where $g \in G$ and $h \in \bar{\mathcal{H}}_{s(g)}$.

(iii) The **tensor product** $(\mathcal{H}^1 \otimes \mathcal{H}^2, \Delta^{\otimes}, \pi_1 \otimes \pi_2)$ **of two continuous continuous fields of Hilbert spaces** is the family of Hilbert spaces is given by $\mathcal{H}_m := \mathcal{H}_m^1 \otimes \mathcal{H}_m^2$. The space $\Delta^{\otimes}$ is the closure of the pre-Hilbert $C_0(M)$-module of all finite sums of sections $\sum_{j\in J} \xi_j \otimes \eta_j$ of $\xi_j \in \Delta^1$ and $\eta_j \in \Delta^2$.

(iv) The **tensor product of two representations** $(\mathcal{H}^1, \Delta^1, \pi_1)$ and $(\mathcal{H}^2, \Delta^2, \pi_2)$ of a groupoid $G \rightrightarrows M$ is the representation of $G \rightrightarrows M$ on $(\mathcal{H}^{\otimes}, \Delta^{\otimes})$ given by linearly extending the map $(\pi_1 \otimes \pi_2)(g)(h \otimes h') = \pi(g)h \otimes \pi(g)h'$ and then extending it continuously to the closure $(\mathcal{H}^{\otimes}, \Delta^{\otimes})$.

Definition 6.3.4. A continuous representation $(\pi, \mathcal{H}, \Delta)$ is **square-integrable** if the map

$$(\bar{\mathcal{H}} \otimes \mathcal{H}, \Delta^{\otimes}) \to (\hat{L^2}(G), \Delta^2(G))$$

given by

$$h_2 \otimes h_1 \mapsto (g \mapsto (h_2, \pi(g)h_1)_{\mathcal{H}_{t(g)}}$$

is a map of continuous fields of Hilbert spaces.

This means that the matrix coefficients $\langle \xi, \pi\eta \rangle$, defined by

$$(n, m) \mapsto (g \mapsto \langle \xi(n), \pi(g)\eta(m) \rangle)$$

for $\xi, \eta \in \Delta$ are $\hat{L}^2(G)$-square-integrable maps.

Example 6.3.5. For example, consider a topological space M. A (finite-dimensional) vector bundle $E \to M$ is a square-integrable representation of $M \rightrightarrows M$.

Example 6.3.6. Consider the family of continuous groups $G := (\mathbb{R} \times \mathbb{Z}/2\mathbb{Z}) \backslash (0, -1) \rightrightarrows \mathbb{R}$. One easily sees that the trivial representation $g \mapsto id_{\mathbb{C}}$ on $(\mathbb{R} \times \mathbb{C}, C_0(\mathbb{R}))$ is not square-integrable. But, note that G is not proper (although for every $m \in M$ the set $s^{-1}(m) = t^{-1}(m)$ is compact).

Proposition 6.3.7. *If $G \rightrightarrows M$ is proper and M/G compact, then every unitary representation is square-integrable.*

Proof. Suppose $(\mathcal{H}, \Delta, \pi)$ is a unitary representation and $\xi, \eta \in \Delta$. Given $\varepsilon > 0$, choose $\xi', \eta' \in C_c(M)\Delta$ such that $\|\xi - \xi'\| < \varepsilon'$ and $\|\eta - \eta'\| < \varepsilon'$, where

$$\varepsilon' = \frac{\min\{\varepsilon, 1\}}{3M \max\{\|\xi\|, \|\eta\|\}}$$

and

$$M = \max_{(n,m) \in R_G} \lambda^n_m(G^n_m),$$

which exists since M/G is compact. First note that $\langle \xi', \pi\eta' \rangle$ has compact support, since $G \rightrightarrows M$ is proper. Moreover,

$$\begin{aligned} \| \langle \xi, \pi\, \eta \rangle - \langle \xi', \pi\eta' \rangle \|_{\hat{L}^2} &\leq \| \langle (\xi - \xi'), \pi\, \eta \rangle \| + \| \langle \xi', \pi\, (\eta - \eta') \rangle \| \\ &\leq \max_{(n,m) \in R_G} \lambda^n_m(G^n_m)(\|\xi - \xi'\|\|\eta\| + \|\xi'\|\|\eta - \eta'\|) \\ &\leq \varepsilon'\|\eta\| + (\|\xi\| + \varepsilon')\varepsilon' \leq \varepsilon, \end{aligned}$$

which finishes the proof. ∎

Proposition 6.3.8. *If a continuous groupoid G has the property that for all $m \in M$ the restriction map*

$$\mathrm{Res}_m : \mathrm{IrRep}^i(G) \to \mathrm{IrRep}(G_m^m)$$

is injective, then for any two non-isomorphic internally irreducible unitary square-integrable representations $(\mathcal{H}, \Delta, \pi)$, $(\mathcal{H}', \Delta', \pi')$ and $\xi, \eta \in \Delta$, $\xi', \eta' \in \Delta'$,

$$\left\langle \langle \xi, \pi\eta \rangle , \langle \xi', \pi'\eta' \rangle \right\rangle_{\hat{L}^2(G)} = 0$$

Proof. This easily follows from the version of this statement for compact groups and the invariance of the Haar system. □

6.4 The Peter-Weyl theorem I

Suppose $G \rightrightarrows M$ is a continuous groupoid endowed with a Haar system $\{\lambda_m\}_{m \in M}$, which decomposes using a continuous family of measure $\{\lambda_m^n\}_{(n,m) \in R_G}$ as in Section 6.3. Let $\mathcal{E}(G) \subset \Delta^2(G)$ denote the $C_0(R_G)$-submodule spanned by the matrix coefficients (cf. Section 6.3) of all finite-dimensional representations of $G \rightrightarrows M$.

A generalization of the Peter-Weyl theorem as we are going to prove (cf. Theorem 6.4.6 and Theorem 6.5.5) appears not to be true for all continuous groupoids. Therefore, we introduce an extra condition:

Definition 6.4.1. For a continuous groupoid $G \rightrightarrows M$ the restriction map

$$\mathrm{Res}_m : \mathrm{Rep}(G) \to \mathrm{Rep}(G_m^m)$$

is **dominant** if for every $m \in M$ and every continuous unitary representation (π, V) of G_m^m there exists a continuous unitary representation $(\pi', \mathcal{H}, \Delta)$ of $G \rightrightarrows M$ such that (π, V) is isomorphic to a subrepresentation of $(\pi'|_{G_m^m}, \mathcal{H}_m)$.

Example 6.4.2. Suppose H is a group and $P \to M$ a principal H-bundle. Since $(P \times_H P)_m^m \cong H$ and $P \times_H P \rightrightarrows M$ are Morita equivalent, $\mathrm{Res}_m : \mathrm{Rep}(P \times_H P) \to \mathrm{Rep}((P \times_H P)_m^m)$ is dominant for all $m \in M$.

Example 6.4.3. Suppose H is a compact connected Lie group that acts on manifold M. Consider the action groupoid $G := H \ltimes M \rightrightarrows M$.

Proposition 6.4.4. *The restriction map* $\mathrm{Res}_m : \mathrm{Rep}(H \ltimes M) \to \mathrm{Rep}((H \ltimes M)_m^m)$ *is dominant for all $m \in M$.*

Proof. First we note that from every representation $(\pi, V) \in \mathrm{Rep}(H)$ we can construct a representation $\tilde{\pi} : H \ltimes M \to U(M \times V)$ of $H \ltimes M \rightrightarrows M$ on $M \times V \to M$ by $\tilde{\pi}(h, m) : (m, v) \mapsto (h \cdot m, \pi(h)v)$. Note that the isotropy groups of $H \ltimes M \rightrightarrows M$ coincide with the isotropy groups of the action. These are subgroups of H, hence the question is whether every representation of a subgroup of H occurs as the subrepresentation of the restriction of a representation of H.

Suppose K is a compact Lie subgroup of H. Fix a maximal tori $T_K \subset K$ and $T_H \subset H$ such that $T_K \subset T_H$, with Lie algebras $\mathfrak{t}_K$ and $\mathfrak{t}_H$. Note that $T_K \cong \mathfrak{t}_K/\Lambda_K$ and $T_H \cong \mathfrak{t}_H/\Lambda_H$ for lattices $\Lambda_K \subset \mathfrak{t}_K$ and $\Lambda_H \subset \mathfrak{t}_H$. There is an injective linear map $M : \mathfrak{t}_K \to \mathfrak{t}_H$ that induces the inclusion $\mathfrak{t}_K/\Lambda_K \hookrightarrow \mathfrak{t}_H/\Lambda_H$. Let P_K denote the integral weight lattice of T_K and P_H the integral weight lattice of T_H. Hence $q := M^T : \mathfrak{t}_H^* \to \mathfrak{t}_K^*$ is surjective map, mapping P_H onto P_K. Hence restriction of representations $\mathrm{Rep}(T_H) \to \mathrm{Rep}(T_K)$ is surjective too, since for tori irreducible representations correspond to integral weights.

The following argument is valid if one fixes positive root systems R_K^+, R_H^+ and hence fundamental Weyl chambers C_K^+, C_H^+ in a way specified in [30]. Suppose (π_λ, V) is an irreducible representation of K corresponding to the dominant weight $\lambda \in P_K \cap C_K^+$. One can choose any integral weight $\Lambda \in q^{-1}(\lambda) \cap P_H \cap C_H^+$; this set is non-empty, since q is surjective and the positive root systems have been fixed appropriately. Let π_Λ denote the irreducible representation of H associated to Λ. Then the multiplicity of π_λ in $\pi_\Lambda|_K$ is a positive integer (not necessarily 1), as follows from the Multiplicity Formula (3.5) in [30]. This finishes the proof. □

Example 6.4.5. A simple, but non-Hausdorff example of a proper groupoid which has a non-dominant restriction map is defined as follows. Consider $\mathbb{R} \times \mathbb{Z}/2\mathbb{Z} \rightrightarrows \mathbb{R}$ and identify $(x, 0)$ with $(x, 1)$ for all $x \neq 0$. Endow the obtained family of groups $(\mathbb{R} \times \mathbb{Z}/2\mathbb{Z})/\sim \rightrightarrows \mathbb{R}$, with the quotient topology. The non-trivial irreducible representation of $\mathbb{Z}/2\mathbb{Z}$ is not in the image of $\mathrm{Res}_0 : \mathrm{Rep}(G) :\to \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$.

We now prove a generalization of the Peter-Weyl theorem for groupoids. Consider the continuous field of Hilbert spaces $(\hat{L}^2(G), \Delta^2(G))$ associated to a groupoid $G \rightrightarrows M$. Let $\overline{\mathcal{E}(G)}$ denote the closure of $\mathcal{E}(G)$ to a Hilbert $C_0(R_G)$-module.

Theorem 6.4.6 (Peter-Weyl for groupoids I). *If $G \rightrightarrows M$ is a proper groupoid, M/G is compact and* Res_m *is dominant for all $m \in M$, then*

$$\overline{\mathcal{E}(G)} = \Delta^2(G).$$

Proof. Note that G_m^m is compact so Peter-Weyl for compact groups applies. Using the dominance property

$$\overline{\{\Theta(m,m) | \Theta \in \mathcal{E}(G)\}} = L^2(G_m^m, \lambda_m^m),$$

since $(\mathcal{H}, \Delta, \pi) < (\mathcal{H}', \Delta', \pi')$, implies $\langle \xi, \pi' \eta \rangle = \langle \xi, \pi\eta \rangle$ for $\xi, \eta \in \Delta$.

Note that $l_g^* : L^2(G_m^m, \lambda_m^m) \to L^2(G_m^n, \lambda_m^n)$ is an isometry for a chosen $g \in G_n^m$. Thus $\overline{\{l_g^*(\Theta(m,m)) | \Theta \in \mathcal{E}(G)\}} = L^2(G_m^n, \lambda_m^n)$. But, for all $h \in G_m^n$ and every continuous unitary finite-dimensional representation $(\mathcal{H}, \Delta, \pi)$

$$\begin{aligned} l_g^* \langle \xi, \pi\eta \rangle (h) &= \langle \xi(t(g)), \pi(gh)\eta(s(h)) \rangle_{\mathcal{H}^\pi_{t(g)}} \\ &= \textstyle\sum_{k=1}^{\dim(\mathcal{H}_n)} \langle \xi(m), \pi(g) e_k(n) \rangle_{E_n} \langle e_k(n), \pi(h)\eta(m) \rangle_{E_m}, \end{aligned}$$

where $e_1, \ldots, e_{\dim(\mathcal{H}_n)} \in \Delta$ are sections that form a basis of $\mathcal{H}$ at n. Thus $l_g^* \langle \xi, \pi\eta \rangle$ is a linear combination of matrix coefficients $\langle e_k, \pi\eta \rangle$ restricted to G_m^n, which implies $\overline{\{\Theta(n,m) | \Theta \in \mathcal{E}(G)\}} = L^2(G_m^n)$.

Let $f \in \Delta^2(G)$ and $\varepsilon > 0$ be given, then there exists a section $\tilde{f} \in \Delta^2(G)$ with compact support K such that $\|f - \tilde{f}\| < \varepsilon/2$, where the norm is the one associated to the $C_0(M)$-valued inner product. Moreover, for all $(m, n) \in R$ there are representations $(\mathcal{H}_{m,n}, \Delta_{m,n}, \pi_{m,n})$ and sections $u_{m,n}, v_{m,n} \in \Delta_{m,n}$, such that

$$\|\tilde{f} - (u_{m,n}, \pi_{m,n} v_{m,n})\|_{L^2(G_n^m)} < \varepsilon/2.$$

Since $\pi_{m,n}, u_{m,n}$ and $v_{m,n}$ are continuous we can find an open neighborhood $S_{m,n} \subset R$, such that still

$$\|\tilde{f} - (u_{m,n}, \pi_{m,n} v_{m,n})\|_{\hat{L}^2(G)|_{S_{m,n}}} < \varepsilon/2,$$

for all $(m,n) \in R$. These $S_{m,n}$ cover K, thus there is a finite subcover, which we denote by $\{S_i\}_{i \in I}$ to reduce the indices. Denote the corresponding representations by π_i and sections by u_i and v_i for $i \in I$. Let $\{\lambda_i\}$ be a partition of unity subordinate to $\{S_i\}$. Define $\tilde{u}_i = \sqrt{\lambda_i} u_i$ and $\tilde{v}_i = \sqrt{\lambda_i} v_i$, then

$$\phi = \sum_{i \in I} (\tilde{u}_i, \pi_i \tilde{v}_i)$$

is a finite sum of matrix coefficients and

$$\begin{aligned}
\|f - g\| &\leq \|f - \tilde{f}\| + \|\tilde{f} - \phi\| \\
&\leq \varepsilon/2 + \sup\nolimits_{(m,n)\in R} \|\tilde{f} - \textstyle\sum_{i \in I} (\tilde{u}_i, \pi_i \tilde{v}_i)\|_{L^2(G_n^m)} \\
&= \varepsilon/2 + \sup\nolimits_{(m,n)\in R} \|\textstyle\sum_{i \in I} \lambda_i \tilde{f} - \sum_{i \in I} (\sqrt{\lambda_i} u_i, \pi_i \sqrt{\lambda_i} v_i)\|_{L^2(G_n^m)} \\
&\leq \varepsilon/2 + \textstyle\sum_{i \in I} \lambda_i \sup_{(m,n)\in R} \|\tilde{f} - \sum_{i \in I} (u_i, \pi_i v_i)\|_{L^2(G_n^m)} \\
&\leq \varepsilon/2 + \textstyle\sum_{i \in I} \lambda_i \varepsilon/2 = \varepsilon,
\end{aligned}$$

which finishes the proof. □

Example 6.4.7. For a space M, $\overline{\mathcal{E}(M \rightrightarrows M)} = C_0(M)$ and $\overline{\mathcal{E}(M \times M \rightrightarrows M)} = C_0(M \times M)$ as Theorem 6.4.6 asserts.

Example 6.4.8. If H is a compact group and $P \to M$ an H-principal bundle. Then, for the bundle of groups $P \times_H H \to M$ one finds (cf. Example 6.2.2),

$$\begin{aligned}
\overline{\mathcal{E}(P \times_H H \rightrightarrows M)} &\cong \Gamma_0(P \times_H \overline{\mathcal{E}(H)}) \\
&\cong \Gamma_0(P \times_H L^2(H)) \\
&\cong \Delta^2(P \times_H H),
\end{aligned}$$

where in the second line we used the Peter-Weyl theorem for the group H.

6.5 The Peter-Weyl theorem II

In this section we shall try to find a decomposition of $(\hat{L}^2(G), \Delta^2(G))$ for proper groupoids $G \rightrightarrows M$, analogous to the case of compact groups H, where one has

$$L^2(H) \cong \bigoplus_{(\pi,V)\in\hat{H}} \bar{V} \otimes V$$

H-equivariantly.

Remark 6.5.1. There is a seemingly relevant proposition that asserts that

Proposition 6.5.2. *([79], Proposition 5.25) Any locally trivial countably generated representation $(\mathcal{H}, \Delta, \pi)$ of a proper groupoid $G \rightrightarrows M$ is a direct summand of the regular representation, after stabilizing, i.e. $\mathcal{H} \subset \hat{L}^2_s(G) \otimes \mathbb{H}$, G-equivariantly, where $\mathbb{H}$ denotes a standard separable Hilbert space, say $l^2(\mathbb{N})$.*

Example 6.5.3. The Serre-Swan theorem for vector bundles is a nice example of this. Consider the groupoid $M \rightrightarrows M$ for a compact space M. Locally trivial representations of this groupoid are vector bundles. The theorem states that any vector bundle is a direct summand of $\hat{L}^2(M)\otimes\mathbb{H} \cong M\times\mathbb{H}$. The Serre-Swan Theorem is actually somewhat stronger, since instead of $\mathbb{H}$ one could put a finite-dimensional vector space $\mathbb{C}^N$ for large enough $N \in \mathbb{N}$. This is because the projection onto the direct summand can be proven to proper in this case.

In general the direct summands will not add up to the whole of $\hat{L}^2_s(G) \otimes \mathbb{H}$. Moreover, stabilization is not something that occurs in the case of compact groups, where one simply has $L^2(H) \cong \bigoplus_{(\pi,V)\in\hat{H}} \bar{V}\otimes V$, not something involving $L^2(H)\otimes\mathbb{H}$.

The continuous field of Hilbert spaces $(\hat{L}^2(I(G)), \Delta^2(I(G)))$ is the pullback of $(\hat{L}^2(G), \Delta^2(G))$ to the diagonal $\{(m,m) \in R_G \mid m \in M\} \hookrightarrow R_G$. It carries a continuous unitary representation

$$\pi_{LR}(g)f(h) := f(g^{-1}hg),$$

where $g \in G_m^n$, $h \in G_n^n$ and $f \in L^2(G_m^m)$.

Lemma 6.5.4. *For any square-integrable continuous unitary representation* $(\mathcal{H}^\pi, \Delta^\pi, \pi)$ *of a groupoid* $G \rightrightarrows M$ *there is an equivariant map*

$$\Psi_\pi : (\bar{\mathcal{H}^\pi} \otimes \mathcal{H}^\pi, \Delta^\otimes) \to (\hat{L}^2(I(G)), \Delta^2(I(G))),$$

given by

$$h_2 \otimes h_1 \mapsto (g \mapsto (h_2, \pi(g)h_1)_{\mathcal{H}_{t(g)}}).$$

This map is a slight adaptation of the one introduced for the definition of square-integrability.

Proof. For equivariance we compute

$$\begin{aligned}\Psi(\pi(g)(h_1 \otimes h_2)) &= \Psi(\pi(g)h_1 \otimes \pi(g)h_2)\\ &= (g' \mapsto (\pi(g)h_1, \pi(g')\pi(g)h_2))\\ &= (g' \mapsto (h_1, \pi(g^{-1})\pi(g')\pi(g)h_2))\\ &= (g' \mapsto (h_1, \pi(g^{-1}g'g)h_2))\\ &= \pi_{LR}(g)(g' \mapsto (h_1, \pi(g')h_2))\end{aligned}$$

which finishes the proof. □

Theorem 6.5.5 (Peter-Weyl for groupoids II)**.** *Suppose* $G \rightrightarrows M$ *is a proper groupoid with* s *and* t *open maps and for every* $m \in M$

$$\mathrm{Res}_m : \mathrm{IrRep}^i(G) \to \mathrm{IrRep}(G_m^m)$$

is bijective. Then

$$\bigoplus_{\pi\in\mathrm{IrRep}^i(G)} \Psi_\pi : \bigoplus_{\pi\in\hat{G}} (\bar{\mathcal{H}^\pi}, \bar{\Delta^\pi}) \otimes (\mathcal{H}^\pi, \Delta^\pi) \to (\hat{L}^2(I(G)), \Delta^2(I(G))) \qquad (6.5.1)$$

is an isomorphism of representations.

Proof. Surjectivity of the map follows from Theorem 6.4.6. Injectivity follows from Proposition 6.3.8. □

Example 6.5.6. Consider the pair groupoid $M \times M \rightrightarrows M$ for a space M. It has just one irreducible and indecomposable continuous unitary representation, namely the trivial one $M \times \mathbb{C} \to M$. Suppose μ is a Radon measure on M. The isomorphism of continuous fields of Hilbert spaces $\hat{L}^2(I(M \times M)) \cong M \times \mathbb{C}$, is obviously $(M \times M)$-equivariant.

Example 6.5.7. Consider a principal H-bundle $P \to M$ for a compact group H and the associated gauge groupoid $G := P \times_H P \rightrightarrows M$. By Morita equivalence of H and $G \rightrightarrows M$, there is a bijection between unitary irreps (V, π) of H and unitary indecomposable, irreducible representations $P \times_H V \to M$ of G. Therefore, Res_m is bijective. Hence, by Theorem 6.5.5, one has the decomposition of formula 6.5.1. This is no surprise, since $I(P \times_H P) \cong P \times_H H$, where H acts on H by conjugation, hence

$$\begin{aligned} \hat{L}^2(I(P \times_H P)) &\cong P \times_H L^2(H) \\ &\cong P \times_H \bigoplus_{(\pi,V)\in\hat{H}} \overline{V^\pi} \otimes V^\pi \\ &\cong \bigoplus_{(\pi,V^\pi)\in\hat{H}} (P \times_H \overline{V^\pi}) \otimes (P \times_H V^\pi) \\ &\cong \bigoplus_{(\pi,\mathcal{H}^\pi)\in\mathrm{IrRep}^i(P\times_H P)} \overline{\mathcal{H}^\pi} \otimes \mathcal{H}^\pi. \end{aligned}$$

This is exactly the statement of Theorem 6.5.5.

Remark 6.5.8. Only for a few (types of) groupoids the map $\mathrm{Res}_m : \mathrm{IrRep}^i(G) \to \mathrm{IrRep}(G_m^m)$ is bijective for all $m \in M$. If the map is just surjective, then one could try to find a subset $\mathrm{PW}(G)$ of $\mathrm{IrRep}^i(G)$ that does map bijective to $\mathrm{IrRep}(G_m^m)$ for every $m \in M$. Then, if this set is well chosen, the decomposition of Theorem 6.5.5 holds with $\mathrm{IrRep}^i(G)$ replaced by $\mathrm{PW}(G)$. We call such a set a **PW-set** (or **Peter-Weyl set**) **for** $G \rightrightarrows M$.

Example 6.5.9. Suppose M is a space. The Peter-Weyl set for $M \rightrightarrows M$ is the trivial representation $M \times \mathbb{C} \to M$.

Example 6.5.10. If H is a compact group and $P \to M$ a principal H-bundle, then $G := P \times_H H \to M$ is a bundle of groups (cf. Example 2.1.9) and

$$\mathrm{PW}(G) := \{P \times_H V \mid (\pi, V) \in \mathrm{IrRep}(H)\}$$

is a Peter-Weyl set (cf. Example 6.2.2).

6.6 Representation rings and K-theory of a groupoid

Suppose $G \rightrightarrows M$ is a continuous groupoid and M/G is compact.

Definition 6.6.1. The set of isomorphism classes of finite-dimensional continuous unitary representations of $G \rightrightarrows M$, endowed with $\oplus$ and $\otimes$ form a unital semi-ring. Applying the Grothendieck construction one obtains the **representation ring of** $G \rightrightarrows M$, denoted by $\mathcal{R}_f(G)$. Denote the subring of locally trivial representations (projective Hilbert $C_0(M)$-modules) by $\mathcal{R}(G)$.

Example 6.6.2. Suppose M is a compact space. Consider the groupoid $M \rightrightarrows M$. By definition one has $K_0(M) = \mathcal{R}(M)$.

Example 6.6.3. Suppose H is a compact group. Consider the groupoid $G := H \rightrightarrows pt$. Then $\mathcal{R}_f(G) = \mathcal{R}(G)$ equals the usual representation ring $\mathcal{R}(H)$ of H.

Example 6.6.4. Suppose M is a compact space. Then for the pair groupoid $M \times M \rightrightarrows M$ one sees that $\mathcal{R}(M \times M) = \mathcal{R}_f(M \times M) \cong \mathbb{Z}$ generated by the trivial representation.

Example 6.6.5. suppose H is a compact group acting on a compact space M. Then the representation ring of the action groupoid $H \ltimes M \rightrightarrows M$ satisfies $\mathcal{R}(H \ltimes M) = K^0_G(M)$.

Example 6.6.6. One easily sees that Morita equivalent groupoids have isomorphic representation rings. Hence, for a group H and a principal H-bundle $P \to M$ one has

$$\mathcal{R}_f(P \times_H P) \cong \mathcal{R}(P \times_H P) \cong \mathcal{R}(H) \cong \mathcal{R}_f(H),$$

which generalizes the previous example.

Suppose $s, t : G \to M$ are open maps. Recall that the orbit relation of a groupoid $G \rightrightarrows M$ is denoted by $R_G \rightrightarrows M$.

Lemma 6.6.7. *The representation ring $\mathcal{R}_f(G)$ is a $\mathcal{R}_f(R_G)$-module via the inclusion $\mathcal{R}_f(R_G) \to \mathcal{R}_f(G)$ given by*

$$\pi_G(g) := \pi_{R_G}(t(g), s(g)).$$

Analogously, $\mathcal{R}(G)$ is a $\mathcal{R}(R_G)$-module.

Example 6.6.8. Suppose $s : G \to M$ is a continuous family of groups. Then, $\mathcal{R}_f(G)$ is a $\mathcal{R}_f(M)$-module and $\mathcal{R}(G)$ is a $K^0(M)$-modules.

For proper groupoids the representation ring is isomorphic to the K-theory of the reduced C^*-algebra of the groupoid (cf. Section 4.3), under some technical conditions. This was proved in [79] in a more general (twisted) setting. We give a summary of their proof. Suppose $G \rightrightarrows M$ is a proper groupoid and $c : G \to \mathbb{R}_{\geq 0}$ a cutoff function for $G \rightrightarrows M$ (cf. Definition 2.5.9). A bounded operator $P \in \mathcal{B}_{C_0(M)}(\Delta^2_t(G))$ on the Hilbert module $\Delta^2_t(G)$, corresponds to a family of operator $\{P_m\}_{m \in M}$. The average of P is defined by

$$P^G_m := \int_{g \in G^m} g \cdot P_{s(g)}\, c(s(g))\, \lambda^m(dg),$$

where $g \cdot P_m := \pi_L(g) P_m \pi_L(g^{-1})$.

Lemma 6.6.9. *([79]) The reduced C^*-algebra $C_r^*(G)$ is equal to the C^*-algebra of averaged compact operators $(\mathcal{K}_{C_0(M)}(\Delta_t^2(G)))^G$.*

One uses this result to prove:

Theorem 6.6.10. *([79]) If $G \rightrightarrows M$ is proper, M/G compact and $C_r^*(G) \otimes \mathbb{H}$ has an approximate unit consisting of projections, then $K_0(C_r^*(G)) \cong \mathcal{R}(G)$.*

Proof. (sketch) If $C_r^*(G) \otimes \mathcal{K}(\mathbb{H})$ has an approximate unit consisting of projections, then $K_0(C_r^*(G))$ is obtained from the semi-ring generated by projections in $C_r^*(G) \otimes \mathbb{H}$, i.e. averaged compact projections of $\hat{L}_t^2(G) \otimes \mathbb{H}$. But these correspond precisely to locally trivial unitary representations of G according to Proposition 6.5.2 and the Serre-Swan theorem. □

Example 6.6.11. Suppose M is a compact space. Then $C^*(M \rightrightarrows M) = C(M)$, and $K_0(C(M)) = K^0(M) = \mathcal{R}(M \rightrightarrows M)$. Also, for the pair groupoid one can show $C_r^*(M \times M \rightrightarrows M) \cong \mathcal{K}(L^2(M))$ (cf. [43]) and hence $K_0(C_r^*(M \times M \rightrightarrows M)) \cong K_0(\mathcal{K}(L^2(M))) \cong \mathbb{Z} \cong \mathcal{R}(M \times M \rightrightarrows M)$.

Example 6.6.12. For a compact group H we have seen in Example 4.4.7 that

$$K_0(C_r^*(H)) \cong \mathcal{R}(H)$$

. Theorem 6.6.10 generalizes this statement to proper groupoids (satisfying the mentioned condition).

Example 6.6.13. For a principal H-bundle $P \to M$ one can prove $C_r^*(P \times_H P \rightrightarrows M) \cong C_r^*(H) \otimes \mathcal{K}(L^2(M))$, hence

$$\begin{aligned} K_0(C_r^*(P \times_H P \rightrightarrows M)) &\cong K_0(C_r^*(H) \otimes \mathcal{K}(L^2(M))) \\ &\cong K_0(C_r^*(H)) \\ &\cong \mathcal{R}(H) \cong \mathcal{R}(P \times_H P \rightrightarrows M), \end{aligned}$$

by stability of K-theory.

Chapter 7

The groupoid convolution C^*-category

In [67] Renault established a bijective correspondence between representations of groupoids $G \rightrightarrows M$ on measurable fields of Hilbert spaces and the non-degenerate bounded representations of the Banach *-algebra $L^1(G)$ on Hilbert spaces, generalizing the analogous statement for groups. Since there is bijection between representations of $L^1(G)$ and representations of $C^*(G)$, which is the universal enveloping C^*-algebra of $L^1(G)$ (cf. Example 4.1.6), there is a bijection between measurable unitary representations of $G \rightrightarrows M$ and bounded non-degenerate representations of $C^*(G)$.

In this section we shall prove a different generalization suitable for continuous representations of groupoids. We give a bijective correspondence between continuous representations of groupoids on continuous fields of Hilbert spaces and continuous representations on continuous fields of Hilbert spaces of the continuous Banach *-category $\hat{L}^1(G)$. Moreover, we introduce the universal enveloping C^*-category of a Banach $*$-category and use this to define the C^*-category $C^*(G,G)$ of a groupoid. As a corollary we find a bijection between representations of $C^*(G,G)$ and the continuous representations of $G \rightrightarrows M$.

7.1 Fell bundles and continuous C^*-categories

First we need some terminology. We discuss the relation between continuous Fell bundles over groupoids (cf. [94, 60, 41]) and Banach $*$-categories and C^*-categories (cf. [23]).

A **(lower semi-)continuous Fell bundle over a groupoid** G is a (lower semi-)continuous field of Banach spaces $(\{\mathcal{B}_g\}_{g\in G}, \Delta)$ over G endowed with an associative bilinear product

$$\mathcal{B}_g \times \mathcal{B}_h \to \mathcal{B}_{gh}, (P,Q) \mapsto PQ$$

whenever $(g,h) \in G^{(2)}$ and an anti-linear involution

$$\mathcal{B}_g \to \mathcal{B}_{g^{-1}}, P \mapsto P^*$$

satisfying the following conditions for all $(g,h) \in G^{(2)}$ and $(P,Q) \in \mathcal{B}_g \times \mathcal{B}_h$

(i) $\|PQ\| \leq \|P\|\|Q\|$;

(ii) $\|P^*P\| = \|P\|^2$;

(iii) $(PQ)^* = Q^*P^*$;

(iv) P^*P is a positive element of $\mathcal{B}_{1_{s(g)}}$;

(v) the image of the multiplication $\mathcal{B}_g \times \mathcal{B}_h \to \mathcal{B}_{gh}, (P, Q) \mapsto PQ$ is dense;

(vi) multiplication $m^*\mathcal{B} \to \mathcal{B}$ and involution $\mathcal{B} \to \mathcal{B}$ are continuous maps of fields of Banach spaces.

where $\mathcal{B}$ denotes the total space of $(\mathcal{B}_{m\in M}, \Delta)$ endowed with the topology given by Δ and $m^*\mathcal{B}$ the pullback of the field $\mathcal{B}$ over G along $m : G^{(2)} \to G$.

Example 7.1.1. Our main example will be the following. Let $(\{\mathcal{H}_m\}_{m\in M}, \Delta_{\mathcal{H}})$ be a continuous field of Hilbert spaces over M. Consider the lower semi-continuous field of Banach spaces over $M \times M$ whose fiber at (n, m) is given by the bounded linear operators $\mathcal{H}_m \to \mathcal{H}_n$, i.e. $\mathcal{B}_{(n,m)} := \mathcal{B}(\mathcal{H}_n, \mathcal{H}_m)$. This field was introduced in Section 5.2.

Suppose $G \rightrightarrows M$ is a continuous groupoid with open $s, t : G \to M$ and $(\{\mathcal{H}_m\}_{m\in M}, \Delta, \pi)$ a representation of $G \rightrightarrows M$. Let $R_G \rightrightarrows M$ denote the orbit relation groupoid. Consider the pullback $(\{\mathcal{B}_{(n,m)}\}_{(n,m)\in R_G}, \Delta_{\mathcal{B}})$ of $\mathcal{B}(\mathcal{H}, \mathcal{H})$ along the inclusion $R_G \hookrightarrow M \times M$.

Lemma 7.1.2. *The lower semi-continuous field of Banach spaces*

$$(\{\mathcal{B}_{(n,m)}\}_{(n,m)\in R_G}, \Delta_{\mathcal{B}})$$

is a lower semi-continuous Fell bundle over R_G.

Proof. The continuity of the composition was proven in the proof of Lemma 5.2.2. Note that $\pi(g) : \mathcal{H}_{s(g)} \to \mathcal{H}_{t(g)}$ is an isomorphism of Hilbert spaces. Hence, the properties (i), (ii), (iii),(iv) and (v) follow from the fact that these are true for $\mathcal{B}(\mathcal{H})$, where $\mathcal{H} \cong \mathcal{H}_{t(g)} \cong \mathcal{H}_{s(g)}$. □

A (lower semi-)continuous Fell bundle A over a continuous equivalence relation $R \subset M \times M$ on M is a full **(lower semi-)continuous C^*-category over M**. Leaving out the C^*-norm equality (ii) we speak of a full **(lower semi-)continuous Banach $*$-category**. Because of the denseness condition (v), it is called a *full* (lower semi-)continuous Banach (or C^*-)category over M. The continuous field $\mathcal{B}(\mathcal{H}, \mathcal{H})$ over $M \times M$ is a continuous C^*-category. It is full iff $m \to \dim(\mathcal{H}_m)$ is constant.

Example 7.1.3. Let $G \rightrightarrows M$ be a locally compact groupoid endowed with a Haar system $\{\lambda^m\}_{m\in M}$. Suppose there exist a continuous families measures $\{\lambda^n_m\}_{(n,m)\in R_G}$ on G and $\{\mu_m\}_{m\in M}$ on M such that

$$\lambda^n = \int_{m\in s(G^n)} \lambda^n_m \mu_n(dm),$$

cf. Proposition 2.5.6.

Consider the continuous field of Banach spaces $(\hat{L}^1(G), \Delta^1(G)) := (\hat{L}^1_{t\times s}(G), \Delta^1_{t\times s}(G))$, cf. Example 4.2.5.

Lemma 7.1.4. *$(\hat{L}^1(G), \Delta^1(G))$ is a continuous Banach $*$-category over M, where the multiplication map $\hat{L}^1(G)^{(2)} \to \hat{L}^1(G)$ is the continuous extension of*

$$f * f'(g) := \int_{h \in G_k^m} f(gh^{-1}) f'(h) \lambda_k^m(dh),$$

for all $f \in C_c(G_m^n)$ and $f' \in C_c(G_k^m)$.

Proof. (sketch) One, indeed easily checks that $\|f * g\|(n,k) < \|f\|(n,m)\|g\|(m,k)$, so this extension is well-defined. □

Definition 7.1.5. A **strongly continuous representation** $(\mathcal{H}, \Delta, L)$ **of a continuous Banach $*$-category** A over a space M on a continuous field of Hilbert spaces $(\mathcal{H}, \Delta)$ over M is a continuous $*$-homomorphism

$$L : A \to \mathcal{B}(\mathcal{H}, \mathcal{H}),$$

such that $a \mapsto L(a)\xi(s(a))$ is continuous $A \to \mathcal{H}$ for every $\xi \in \Delta$ (with the obvious notion of homomorphism). A strongly continuous representation of a continuous C^*-category is a representation of this as a Banach $*$-category.

One has analogous definitions for weakly continuous representations and representations continuous in the operator norm.

Definition 7.1.6. A representation $(\mathcal{H}, \Delta, L)$ of a Banach $*$-category A is non-degenerate if $\overline{L(A)\mathcal{H}} = \mathcal{H}$.

Suppose $(A_{(n,m)\in R}, \Delta)$ is a continuous Banach $*$-category over M. Then a **universal enveloping C^*-category** of (A, Δ) is a continuous C^*-category (B, Δ) and a morphism $A \to B$ such that any morphism of continuous Banach $*$-categories $A \to C$ factors via B (compare to Example 4.1.6). It can be constructed as the closure of the A under the image of

$$\bigoplus_{\pi \in \hat{A}} \pi : A \mapsto \bigoplus_{\pi \in \hat{A}} \mathcal{H}_\pi,$$

where $\hat{A}$ denotes the set of isomorphism classes of continuous bounded non-degenerate representations of A.

Example 7.1.7. Suppose $G \rightrightarrows M$ is a continuous groupoid. The **continuous C^*-category** $C^*(G, G)$ **of** $G \rightrightarrows M$ is the universal enveloping C^*-category of $(\hat{L}^1(G), \Delta^1(G))$. One easily sees that this is a continuous C^*-category, since $(\hat{L}^1(G), \Delta^1(G))$ is a continuous Banach $*$-category. Analogously to the group case, one can also introduce the reduced C^*-category of a groupoid, but we shall not need this here.

7.2 Representations of $G \rightrightarrows M$ and $\hat{L}^1(G)$

We say $G \rightrightarrows M$ allows **Dirac sequences** $\{(\delta_k^g)_{k\in\mathbb{N}}\}_{g\in G}$ for the Haar system $\{\lambda_m^n\}_{(n,m)\in R_G}$, if

(i) $\delta_k^g \geq 0$ on $G_{s(g)}^{t(g)}$,

(ii) $\int_{g' \in G_m^n} \delta_k^g(g') \lambda_{s(g)}^{t(g)}(dg') = 1$ for all $k \in \mathbb{N}$,

(iii) for every open neighborhood $U \subset G_m^n$ of g and every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for $k > N$

$$\int_{g \in U^c} \delta_k^g(g') \lambda_m^n(dg') < \varepsilon.$$

Lemma 7.2.1. *If $(\mathcal{H}, \Delta, \pi)$ is a continuous unitary representation of $G \rightrightarrows M$, then $L_\pi : \hat{L}^1(G) \to \mathcal{B}(\mathcal{H}, \mathcal{H})$ given by*

$$f \mapsto \left((n,m) \mapsto \int_{G_m^n} f(g)\pi(g)\lambda_m^n(dg) \right)$$

is a non-degenerate strongly continuous representation of $(\hat{L}^1(G), \Delta^1(G))$ as a continuous Banach $*$-category on the continuous field of Hilbert spaces $(\mathcal{H}, \Delta)$.

Proof. By the properties of the Bochner integral one has

$$\begin{aligned} \|\pi(f)\|(n,m) &= \| \textstyle\int_{G_m^n} f(g)\pi(g)\lambda_m^n \| \\ &\leq \textstyle\int_{G_m^n} |f(g)| \|\pi(g)\| \lambda_m^n(dg) \\ &= \textstyle\int_{G_m^n} |f(g)| \lambda_m^n(dg) \end{aligned}$$

(Note that therefore, $\|\pi(f)\| = \sup_{(n,m)\in R} \|\pi(f)\|(n,m) \leq \|f\|_{\hat{L}^1(G)}$).

We now prove that L_π is a $*$-homomorphism. Suppose $f \in C_c(G_m^k)$ and $f' \in C_c(G_k^n)$, then

$$\begin{aligned} L_\pi(f * f') &= \int_{g \int G_m^n} (f * f')(g)\pi(g)\lambda_m^n(dg) \\ &= \int_{g \in G_m^n} \int_{h \in G_m^k} f(gh^{-1}) f'(h) \lambda_m^k(dh) \pi(g) \lambda_m^n(dg) \\ &= \int_{g \in G_k^n} f(g)\pi(g)\lambda_k^n(dg) \int_{h \in G_m^k} f'(h)\pi(h)\lambda_k^m(dh) \\ &= L_\pi(f) L_\pi(f'), \end{aligned}$$

by invariance of the Haar system. One easily checks that $L^\pi(f)^* = L^\pi(f^*)$.

Suppose $f \in C_c(G_m^n)$ is given. Suppose $F \in C_c(G)$ satisfies $F|_{G_m^n} = f$. Note that

$$\sup_{(n',m') \in R_G} \int_{G_m^n} \|F(g)\pi(g)\xi(s(g))\| \lambda_m^n(dg) \leq \|F\|_{\hat{L}^1(G)} \max_{m \in s(\mathrm{supp}(F))} \|\xi(m)\|.$$

Using this, one easily proves that L_π is strongly continuous.

The representation π_L is non-degenerate, since for any $m \in M$ and $h \in \mathcal{H}_m$

$$\lim_{k\to\infty} \|h - L(\delta_k^{1_m})h\| = \lim_{k\to\infty} \|h - \int_{g \in G_m^n} \delta_k^{1_m}(g)\pi(g)h \, \lambda_m^n(dg)\| = 0.$$

This finishes the proof. ∎

For $f \in C_c(G)$, $m \in M$ and $g, g' \in G^m$, we shall use the notation $f^g(g') := (\pi_L(g)f)(g') = f(g^{-1}g')$.

Lemma 7.2.2. *If $(\mathcal{H}, \Delta, L)$ is a strongly continuous non-degenerate representation of $(\hat{L}^1(G), \Delta^1(G))$, then*

$$\pi(g)(L(f)h) := L(f^g)h$$

defines a continuous unitary representation of $G \rightrightarrows M$ on the continuous field of Hilbert spaces $(\mathcal{H}, \Delta)$.

Proof. By non-degeneracy of L, the above formula defines π_L on a dense set. It extends to the whole of $\mathcal{H}$, since for all $g \in G$ and $h \in \mathcal{H}_{s(g)}$ one has

$$\begin{aligned} \|\pi(g)h\| &= \lim_{k\to\infty} \|L(\delta_g^k)h\| \\ &\leq \lim_{k\to\infty} B\|\delta_g^k\|\|h\| \\ &= B\|h\|, \end{aligned}$$

for a constant $B \in \mathbb{R} \geq 0$.

This is well-defined. Indeed, suppose $L(f)h = L(f')h'$ for $f \in L^1(G_m^n)$, $f' \in L^1(G_{m'}^n)$, $h \in \mathcal{H}_m$ and $h' \in \mathcal{H}_{m'}$. Let δ_k^g denote the translation of δ_k^m along $g \in G_m^{m'}$. One easily checks that

$$\|\delta_k^g * f - f^g\| \to 0$$

when $k \to \infty$. Then one has for all $k \in \mathbb{N}$:

$$\begin{aligned} \|L((f')g)h' - L(f^g)h\| &\leq \|L((f')^g)h' - L(\delta_k^g * f')h'\| \\ &\quad \|L(\delta_k^g * f')h' - L(\delta_k^g * f)h\| + \|L(\delta_k^g * f)h' - L(f^g)h\| \\ &\leq B\|(f')^g - \delta_k^g * f'\|\|h'\| + \|L(\delta_k^g)(L(f')h' - L(f)h)\| \\ &\quad +B\|(f)^g - \delta_k^g * f\|\|h\|. \end{aligned}$$

The second term is zero and the first and the last term go to zero as $k \to \infty$, hence $L((f')g)h' = L(f^g)h$.

π is a homomorphism. Indeed, for $(g, g') \in G^{(2)}$, $f \in L^1(G_m^{s(k)})$ and $h \in \mathcal{H}_m$ one has

$$\begin{aligned} \pi(gg')(L(f)h) &= L(f^{gg'})h \\ &= L((f^{g'})^g)h \\ &= \pi(g)L(f^{g'})h \\ &= \pi(g)\pi(g')(L(f)h). \end{aligned}$$

Furthermore, the following computation shows that $\pi(g)^* = \pi(g^{-1})$:

$$\begin{aligned}
\langle \pi(g)^* L(f)h, L(f')h' \rangle &= \langle h, L(f)^* \pi(g) L(f')h' \rangle \\
&= \langle h, L(f^*) L((f')^g) h') h' \rangle \\
&= \langle h, L(f^* * (f')^g) h' \rangle \\
&= \left\langle h, L((f^{g^{-1}})^* * f') h' \right\rangle \\
&= \left\langle h, L(f^{g^{-1}})^* L(f') h' \right\rangle \\
&= \left\langle L(f^{g^{-1}}) h, L(f') h' \right\rangle \\
&= \left\langle \pi(g^{-1}) L(f) h, L(f') h' \right\rangle,
\end{aligned}$$

where the fourth step follows from equivariance of the Haar system and the fact that

$$(f^g)^*(g') = f^*(g' g^{-1}).$$

The continuity of π follows from the fact that for any $F \in C_c(G)$, representing a section of $\hat{L}^1(G) \to R_G$, and any $\xi \in \Delta$, the section $m \mapsto L(F)(m,m)\xi(m)$ is again in Δ and that $g \mapsto F(s(g), s(g))^g$ is continuous, cf. Proposition 5.3.1. □

Theorem 7.2.3. *The correspondence $\pi \mapsto L_\pi$ is a bijection between the set of continuous unitary representations of $G \rightrightarrows M$ and the set of non-degenerate strongly continuous representations of $(\hat{L}^1(G), \Delta^1(G))$.*

Proof. The inverse correspondence is given by Lemma 7.2.2, which we denote by $L \mapsto \pi^L$ (not to be confused with the left regular representation π_L). Given a continuous unitary representation π of G, we compute

$$\begin{aligned}
\pi^{(L^\pi)}(g)(L^\pi(f)h) &= L^\pi(f^g)h \\
&= \textstyle\int_{g' \in G_m^n} f(g^{-1}g')\pi(g')h\lambda_m^n(dg') \\
&= \textstyle\int_{g' \in G_m^n} f(g')\pi(g)\pi(g')h\lambda_m^p(dg') \\
&= \pi(g)(L^\pi(f)h).
\end{aligned}$$

Conversely, suppose a non-degenerate strongly continuous representation L of $\hat{L}^1(G)$ is given. Then we have

$$\begin{aligned}
L^{(\pi^L)} L(f')h &= \textstyle\int_{g \in G_m^n} f(g)\pi^L(g)L(f')h\lambda_m^n(dg) \\
&= \textstyle\int_{g \in G_m^n} f(g)L((f')^g)h\lambda_m^n(dg) \\
&= L(\textstyle\int_{g \in G_m^n} f(g)(f')^g\lambda_m^n(dg))h \\
&= L(f * f')h = L(f)(L(f')h),
\end{aligned}$$

which finishes the proof. □

Corollary 7.2.4. *The bijective correspondence of Theorem 7.2.3 extends to a bijective correspondence between the set of continuous unitary representations of $G \rightrightarrows M$ and the set of non-degenerate strongly continuous representations of $C^*(G,G)$.*

Part III

Geometric quantization of Hamiltonian actions of Lie algebroids and Lie groupoids

Introduction

The aim of this part is to give a method to construct Hermitian representations of Lie algebroids and associated unitary representations of Lie groupoids. An important way of constructing representations of Lie algebras and Lie groups is by geometric quantization (cf. e.g. [92], [31], [38]). In this part this procedure will be generalized to Lie algebroids and Lie groupoids. The material of this part has appeared previously in [7] and also in [9].

A groupoid can be used as a model for a singular space as we shall see in the Part IV. This part takes a different perspective: groupoids model a generalized notion of symmetry. In this part we shall study the symmetry of maps $J : S \to M$ (cf. Example 2.1.6), i.e. morphisms of groupoids $G \to \mathrm{Aut}(J)$, called actions of $G \rightrightarrows M$. We shall construct representations of $G \rightrightarrows M$ from Hamiltonian actions of $G \rightrightarrows M$.

To a Lie groupoid $G \rightrightarrows M$ is associated a Lie algebroid $(\mathcal{A}, \rho, [.,.])$ (cf. Proposition 2.4.3). We shall often assume that Lie algebroids $\mathcal{A}$ are regular, i.e. $\mathrm{im}(\rho)) \subset TM$ has locally constant rank. This implies that the orbit foliation on M of the groupoid $G \rightrightarrows M$ is regular. A Lie groupoid with a regular orbit foliation is called regular. For example, transitive Lie groupoids, étale Lie groupoids and smooth bundles of Lie groups are regular. The regularity assumption is necessary to give proofs of some of the statements, but many constructions are possible to some extent in singular cases too.

We now give an outline of this part, including some more details on the content. In Chapter 8 we recall the notion of a Lie groupoid (and Lie algebroid) action on a map $J : S \to M$. The introduction of Hamiltonian actions of Lie algebroids (and Lie groupoids) proceeds in two steps. Suppose $J : S \to M$ is a surjective submersion endowed with a family of symplectic forms ω. First, we introduce the notion of an *internally Hamiltonian action* of a Lie algebroid on $(J : S \to M, \omega)$. The word "internal" refers to the fact that we only consider the symmetry of each of the fibers $J^{-1}(m)$ for $m \in M$, which is represented by the action of the isotropy Lie algebras $\mathcal{A}_m$ for $m \in M$. This action is internally Hamiltonian if there exists an *internal momentum map*, which is a map $\mu : S \to J^* \ker(\rho)^*$, satisfying certain natural conditions (cf. Definition 8.3.4 and Definition 8.2.4) generalizing the case of Hamiltonian Lie algebra actions.

The second step considers extensions of ω to a closed form $\tilde{\omega}$ on S (which we shall call a J-presymplectic form). There exist in general several inequivalent ways to construct such an extension $\tilde{\omega}$, given $(J : S \to M, \omega)$. One can proceed by defining the notion of a Hamiltonian action. An action will be called Hamiltonian if there exists a *momentum map*

$$\tilde{\mu} : S \to J^* \mathcal{A}^* = (\mathcal{A} \ltimes J)^*,$$

satisfying natural conditions (cf. Definition 8.4.3). We shall give many examples to motivate this definition. Some of the examples will return throughout this part of the thesis.

Chapter 9 is devoted to the construction of prequantization line bundles with a representation of the Lie algebroid, based on the data of a Hamiltonian Lie algebroid action. We introduce longitudinal Čech cohomology to study such line bundles endowed with a connection. The main result of this chapter is, summarizing Theorems

9.2.2 and 9.3.1,

Theorem. *If a Lie algebroid $\mathcal{A}$ acts in a Hamiltonian fashion on $(J : S \to M, \tilde{\omega})$ and $[\tilde{\omega}] \in H^{J,dR}(S)$ is integral, then there exists a prequantization line bundle carrying a Hermitian representation of $\mathcal{A}$.*

In the last section of this chapter we briefly discuss the possible integrability of such a representation to a representation of an integrating Lie groupoid for the Lie algebroid.

In the Chapter 10 we obtain a representation of the Lie algebroid through generalized Kähler quantization. To this effect we need $J : S \to M$ to be a bundle of compact Kähler manifolds. The main result is (cf. Theorem 10.1.3)

Theorem. *If a Lie algebroid $\mathcal{A}$ acts in a Hamiltonian fashion on $(J : S \to M, \tilde{\omega})$, $[\tilde{\omega}] \in H^{J,dR}(S)$ is integral and $J : S \to M$ is a bundle of Kähler manifolds, then there exists a geometric quantization Δ_Q, which corresponds to a continuous field of Hilbert spaces $(\Delta_Q, \mathcal{H}_Q)$ carrying a Hermitian representation of $\mathcal{A}$.*

Next, we study the symplectic reduction of Hamiltonian groupoid actions (a generalized Marsden-Weinstein quotient). We introduce an internal quotient $(I_G\backslash\mu^{-1}(0_M), \omega_0)$ and a 'full' quotient $(G\backslash\mu^{-1}(0_M), \tilde{\omega}_0)$. We also introduce internal quantum reduction $(\mathcal{H}_Q^{I_G}, \Delta_Q^{I_G})$ and full quantum reduction $(\mathcal{H}_Q^G, \Delta_Q^G)$.

Finally, we prove a "quantization commutes with reduction theorem" for regular proper Lie groupoids,

Theorem. *(cf. Theorem 10.3.2 and Corollary 10.3.8) If G is a proper groupoid acting in a proper, free and Hamiltonian fashion on a bundle of Kähler manifolds $(J : S \to M, \tilde{\omega})$, and $[\tilde{\omega}] \in H^{J,dR}(S)$ is integral, then there exist isomorphisms of continuous fields of Hilbert spaces*

$$(\mathcal{H}_Q^0, \Delta_Q^0) \stackrel{\cong}{\to} (\mathcal{H}_Q^{I_G}, \Delta_Q^{I_G})$$

and

$$(\mathcal{H}_Q^{00}, \Delta_Q^{00}) \stackrel{\cong}{\to} (\mathcal{H}_Q^G, \Delta_Q^G),$$

where $(\mathcal{H}_Q^0, \Delta_Q^0)$ denotes the geometric quantization of the internal Marsden-Weinstein quotient and $(\mathcal{H}_Q^{00}, \Delta_Q^{00})$ the geometric quantization of the full Marsden-Weinstein quotient.

The proof strongly relies on the 'quantization commutes with reduction theorem" for compact Lie groups.

The orbit method as developed by Kirillov (cf. [38]) is based on the idea that there should be a certain correspondence between the irreducible unitary representations of a Lie group and the coadjoint orbits in the dual of its Lie algebra. This method works very well for nilpotent Lie groups (cf. [14]) and compact Lie groups (the Borel-Weil theorem). There are also nice results for reductive Lie groups (cf. [83]) and even for quantum groups (cf. [39]). One might wonder if such a principle is also useful for Lie groupoids. In this part of the thesis we shall see that the answer is affirmative, although a smooth family of coadjoint orbits is not the only ingredient to construct a representation. One needs some more structure to take care of the global topology. Moreover, one should realize that the coadjoint orbits are submanifolds of the dual of the Lie algebroid of the *isotropy* groupoid (which equals the dual of the kernel of the anchor). Although the isotropy groupoid is in general not smooth, it plays an essential rôle in understanding the representation theory of G. Although in some examples it is clear that an orbit method is useful and true, we have not yet succeeded in formulating a general principle.

The theory presented here should be distinguished from the theory of symplectic groupoids and their prequantization (cf. [85]). Symplectic groupoids were introduced by Alan Weinstein and others in a program to geometrically quantize Poisson manifolds. This is not the purpose of this work. We do not assume any (quasi-)(pre-)symplectic structure on the Lie groupoid. Also, our notion of momentum map differs from the notion in e.g. [54].

Chapter 8

Hamiltonian Lie algebroid actions

8.1 Actions of groupoids and Lie algebroids

The material in this section is standard (see [49]), except for the introduction of internally symplectic and J-presymplectic actions of Lie groupoids and Lie algebroids.

Suppose $G \rightrightarrows M$ is a Lie groupoid. We shall assume throughout that $G \rightrightarrows M$ is source-connected. Suppose N is a smooth manifold and $J : N \to M$ a smooth surjective submersion. Then each fiber $J^{-1}(m)$ is a smooth manifold ($m \in M$). We also use the term **smooth family of manifolds** for $J : N \to M$.

Suppose α is a smooth left action of $G \rightrightarrows M$ on $J : N \to M$ (cf. Example 2.1.6 and Definition 2.3.4). One can show that the G-orbits in N are smooth submanifolds of N. Note that these orbits are equal to the fibers of the map

$$\tilde{J} := p \circ J : N \to M/G,$$

where $p : M \to M/G$ is the quotient map from M to the orbit space M/G. These orbits form a regular foliation of N, if G is a regular Lie groupoid.

Example 8.1.1. We mention three basic examples of groupoid actions here, all arising from a groupoid $G \rightrightarrows M$ itself. Firstly, one has the action of $G \rightrightarrows M$ on the identity map $M \to M$ by $g \cdot s(g) = t(g)$. Secondly, one has the action of $G \rightrightarrows M$ on $t : G \to M$ by multiplication $g \cdot g' := g\, g'$. Thirdly, one has a (in general non-smooth) action of G on the associated isotropy groupoid (which is in general not a smooth manifold, unless $G \rightrightarrows M$ is regular)

$$I_G := \{s^{-1}(m) \cap t^{-1}(m)\}_{m \in M} \to M$$

by conjugation $c(g)g' := g\, g'\, g^{-1}$.

Definition 8.1.2. Associated to a smooth groupoid action of $G \rightrightarrows M$ on $J : N \to M$ is an **action Lie groupoid** $G \ltimes J \rightrightarrows N$, analogously to the case of group actions (cf. Example 2.1.5). Its space of arrows is given by $G\, {}_s\!\times_J N$, the source map by $s(g, n) := n$, target map by $t(g, n) = g \cdot n$, multiplication by $(h, g \cdot n)(g, n) := (hg, n)$ and inversion by $i(g, n) := (g^{-1}, g \cdot n)$. If $G \rightrightarrows M$ is regular, then $G \rightrightarrows J \rightrightarrows S$ is regular.

Remark 8.1.3 (Notation)**.** Suppose $J : S \to M$ is a smooth surjective submersion (a smooth family of manifolds). The vector bundle $\ker(TJ) \subset TS$ is the integrable distribution underlying the foliation $\mathcal{F} := \{J^{-1}(m)\}_{m \in M}$ of S. We shall use the notation $T^J S := \ker(TJ)$, $T^{*,J}S =: \ker(TJ)^*$, $\mathfrak{X}^\infty_J(S) := \Gamma^\infty(\ker(TJ))$ and $\Omega^n_J(S) := \Gamma^\infty(\bigwedge^n \ker(TJ)^*))$. Moreover, there is an obvious differential d^J : $\Omega^n_J(S) \to \Omega^{n+1}_J(S)$, which gives rise to a generalized de Rham cohomology denoted by $H^n_{J,dR}(S)$.

Suppose α is an action of $G \rightrightarrows M$ on a smooth family of manifolds $J : S \to M$.

Definition 8.1.4. We define:

(i) $\omega \in \Omega^2_J(S)$ is a **smooth family of symplectic forms** (or a J-**longitudinal symplectic form**) if $d^J\omega = 0$ and $v \mapsto i_v\omega$ is a bijection $\mathfrak{X}^\infty_J(S) \to \Omega^1_J(S)$. In this case, $(J : S \to M, \omega)$ is a **smooth family of symplectic manifolds**.

(ii) The action is **internally symplectic** if it preserves the symplectic forms in the sense that
$$\alpha(g)^*\omega_{g\cdot\sigma} = \omega_\sigma,$$
for all $g \in I_G$ and $\sigma \in J^{-1}(s(g))$. This is just a "family version" of symplectic actions in the usual sense.

(iii) The action is said to be **symplectic** if
$$\alpha(g)^*\omega_{g\cdot\sigma} = \omega_\sigma,$$
for all $g \in G$ and $\sigma \in J^{-1}(s(g))$.

(iv) Let $\tilde{\omega} \in \Omega^2_{\tilde{J}}(S)$ be any closed 2-form extending $\omega \in \Omega^2_J(S)$. We call such a closed form $\tilde{\omega} \in \Omega^2_{\tilde{J}}(S)$ that restricts to a smooth family of symplectic forms $\omega \in \Omega^2_J(S)$ a J-**presymplectic form**.

Example 8.1.5. Let's consider a very simple example to get some acquaintance with these notions. Let $J := pr_3 : \mathbb{R}^3 \to \mathbb{R}$ be the projection on the third coordinate. Suppose $G = \mathbb{R}\times\mathbb{R}\times\mathbb{R} \rightrightarrows \mathbb{R}$ is the Lie groupoid with $t = pr_1$, $s = pr_3$, multiplication $(z, x, z')(z', x', z'') = (z, x + x', z'')$, $u(z) = (z, 0, z)$ and $(z, x, z')^{-1} = (z', -x, z)$. An action of $G \rightrightarrows \mathbb{R}$ on J is given by $(z, x, z') \cdot (x', y, z') := (x + x', y, z)$. Note that in this case $\tilde{J} : \mathbb{R}^3 \to \mathbb{R}/G = pt$. If $\omega = dx\wedge dy$, then the action is internally symplectic and symplectic. Any closed 2-form $\tilde{\omega}$ extending ω, e.g. $\tilde{\omega} = dx \wedge dy + z\, dx \wedge dz$, is J-presymplectic. If $\omega = (z^2 + 1)dx \wedge dy$, then the action is internally symplectic, but not symplectic.

The reader will find other (and more sophisticated) examples in the next sections.

Remark 8.1.6. A **smooth local bisection** of $G \rightrightarrows M$ is a map $\gamma : U \to G$ such that $s\circ\gamma = id|_U$ and $t\circ\gamma$ is a diffeomorphism onto its image, for an open set $U \subset M$. Note that
$$(\alpha(\gamma)^*\tilde{\omega}_{\gamma(m)\cdot\sigma})|_{T^J S} = \alpha(\gamma)^*\omega_{\gamma(m)\cdot\sigma},$$

for all open sets $U \subset M$, smooth local bisections $\gamma : U \to G$, $m \in U$ and $\sigma \in S_m := J^{-1}(m)$, since the local diffeomorphism $\alpha(\gamma)$ maps J-fibers to J-fibers. Hence the action is symplectic if

$$\alpha(\gamma)^* \omega_{\gamma(m)\cdot\sigma} = \omega_\sigma,$$

for all open sets $U \subset M$, local bisections $\gamma : U \to G$, $m \in U$ and $\sigma \in S_m := J^{-1}(m)$.

Now, let's view these definitions from a Lie algebroid perspective (cf. Section 2.4).

Definition 8.1.7. An **action of a Lie algebroid** $(\pi : \mathcal{A} \to M, \rho)$ **on a map** $J : N \to M$ is a map

$$\alpha : \Gamma^\infty(\mathcal{A}) \to \mathfrak{X}^\infty(N)$$

satisfying

(i) $\alpha(X + Y) = \alpha(X) + \alpha(Y)$;

(ii) $\alpha(fX) = (J^* f)\alpha(X)$;

(iii) $[\alpha(X), \alpha(Y)] = \alpha([X, Y])$;

(iv) $TJ(\alpha(X)) = \rho(X)$,

for all $X, Y \in \Gamma^\infty(\mathcal{A})$ and $f \in C^\infty(M)$.

Example 8.1.8. Any action of a Lie algebra $\mathfrak{g}$ on a manifold N is a Lie algebroid action of $\mathfrak{g} \to *$ on $N \to *$.

Example 8.1.9. Every Lie algebroid $(\mathcal{A} \to M, [\cdot,\cdot], \rho)$ acts on $J := id : M \to M$ via the anchor $\rho : \Gamma^\infty(\mathcal{A}) \to \mathfrak{X}^\infty(M)$.

Example 8.1.10. A Lie groupoid action α on a smooth map $J : N \to M$ gives rise to an action α' of the Lie algebroid $\mathcal{A}(G)$ on $J : N \to M$ by

$$\alpha'(X)(n) := \frac{d}{d\tau} \alpha(\exp(\tau X)_{J(n)}, n)|_{\tau=0}.$$

Suppose $(\pi : \mathcal{A} \to M, \rho)$ is a regular Lie algebroid. The image $\rho(\mathcal{A}) \subset TM$ of the anchor is an integrable distribution, which induces a foliation $\mathcal{F}_\rho$ on M. Suppose $\mathcal{A}$ acts on a smooth family of manifolds $J : S \to M$. Denote the projection of S on the leaf space $M/\mathcal{F}_\rho$ by $\tilde{J} : S \to M/\mathcal{F}_\rho$. Suppose $\tilde{\omega} \in \Omega^2_{\tilde{J}}(S)$ is a J-presymplectic form, with $\omega := \tilde{\omega}|_{T^J S}$. Note that $(\mathcal{L}_{\alpha(X)}\tilde{\omega})|_{T^J S} = \mathcal{L}_{\alpha(X)}\omega$ for all $X \in \Gamma^\infty(\mathcal{A})$.

Definition 8.1.11. We define:

(i) An action of $\mathcal{A}$ on a smooth family of symplectic manifolds $(J : S \to M, \omega)$ is **internally symplectic** if $\mathcal{L}_{\alpha(X)}\omega = 0$ for all $X \in \Gamma^\infty(\ker(\rho))$.

(ii) The action is said to be **symplectic** if $\mathcal{L}_{\alpha(X)}\omega = 0$ for all $X \in \Gamma^\infty(\mathcal{A})$.

Note that the action being symplectic implies it being internally symplectic. We shall see some examples of internally symplectic actions in Section 8.3 and many examples of symplectic actions in Section 8.4, since a Hamiltonian action as defined in this section is automatically symplectic.

Definition 8.1.12. Associated to a Lie algebroid action of $(\mathcal{A} \to M, \rho, [\cdot,\cdot])$ on $J : N \to M$ there is an **action Lie algebroid** $\mathcal{A} \ltimes J$. Denote the pullback of $\mathcal{A} \to M$ along $J : N \to M$ by $J^*\mathcal{A} \to N$. The space of sections $\Gamma^\infty(J^*\mathcal{A})$ is generated as a $C^\infty(N)$-module by sections of the form J^*X for $X \in \Gamma^\infty(\mathcal{A})$. A Lie bracket on the smooth sections is defined by

$$[f\, J^*X, g\, J^*Y] := f\, g J^*[X,Y] + f\,(\alpha(X)\cdot g)J^*Y - g\,(\alpha(Y)\cdot f)J^*X,$$

where $f, g \in C^\infty(N)$ and $X, Y \in \Gamma^\infty(\mathcal{A})$ and the anchor

$$\rho' : \Gamma^\infty(\mathcal{A} \ltimes J) \to \mathfrak{X}^\infty(N)$$

is given by

$$\rho'(f J^*X) := f\alpha(X).$$

Remark 8.1.13. Suppose a Lie groupoid $G \rightrightarrows M$ acts on a map $J : N \to M$. It induces an action of the Lie algebroid $\mathcal{A}(G)$ on $J : N \to M$ and the action Lie algebroid $\mathcal{A}(G) \ltimes J$ is isomorphic to the Lie algebroid $\mathcal{A}(G \ltimes J)$ associated to the action Lie groupoid.

8.2 Internally Hamiltonian actions

In this section we introduce the notion of internally weakly Hamiltonian Lie algebroid action. This notion and the notion of internally strongly Hamiltonian Lie algebroid action, introduced in the next section, should be seen as an intermediate stage towards defining Hamiltonian actions. They are separately treated for clarity and for their rôle in the orbit method. The reader could skip the coming two sections and proceed reading Section 8.4. Examples of internal Hamiltonian actions are postponed to the next section.

Remark 8.2.1. Note that the isotropy groupoid $I_G \to M$ (cf. Example 2.1.9) of a non-regular Lie groupoid $G \rightrightarrows M$ is not a smooth manifold. But for any G-orbit $Gm \subset M$ the restriction $I_G|_{Gm}$ is a smooth manifold. Hence $I_G \to M$ is a continuous family of smooth manifolds in the subspace topology, i.e. a surjective continuous map of topological spaces such that each fiber is a smooth manifold in the subspace topology. It is not a stratified space (in the sense of Whitney) in general. If $G \rightrightarrows M$ is a regular Lie groupoid, then $I_G \to M$ is a smooth family of Lie groups, i.e. $I_G \to M$ is a smooth family of manifolds and each fiber has a Lie group structure smoothly depending on $m \in M$. Let $\pi : \mathcal{A}(I_G) \to M$ be the smooth family of Lie algebras associated to $I_G \to M$. It is naturally isomorphic to the kernel $\ker(\rho) \to M$ of the anchor $\rho : \mathcal{A}(G) \to TM$ of the Lie algebroid of $G \rightrightarrows M$.

Example 8.2.2. Consider the gauge groupoid $P \times_H P \rightrightarrows M$ of a smooth principal H-bundle. Then $I_{P\times_H P} \cong P \times_H H \to M$ is a smooth family of Lie groups (cf. Example 2.1.9). The associated Lie algebroid $\mathcal{A}(I_{P\times_H P})$ is isomorphic to $P \times_H \mathfrak{h}$ (cf. Example 2.4.9).

Example 8.2.3. Consider the action of the circle $\mathbb{S}^1$ on the real plane $\mathbb{R}^2$ by rotation. Consider the action groupoid $G = \mathbb{S}^1 \ltimes \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$. The isotropy groupoid is a continuous family of Lie groups with fiber $\mathbb{S}^1$ at $(0,0)$ and zero fiber elsewhere.

Suppose that $(\pi : \mathcal{A} \to M, \rho)$ is a regular Lie algebroid that acts on a smooth family of symplectic manifolds $(J : S \to M, \omega)$. Denote the action by $\alpha : \Gamma^\infty(\mathcal{A}) \to \mathfrak{X}^\infty_{\tilde{J}}(S)$. Suppose that the action of $\mathcal{A}$ is internally symplectic (cf. Definition 8.1.11.i). Then $\alpha(X) \lrcorner \omega$ is closed, i.e.

$$d^J(\alpha(X) \lrcorner \omega) = 0$$

for all $X \in \Gamma^\infty(\ker(\rho))$. Indeed, this follows from the Cartan homotopy formula

$$\mathcal{L}_{\alpha(X)}\omega = d^J(\alpha(X) \lrcorner \omega) + \alpha(X) \lrcorner d^J\omega,$$

in which the last term is zero, since ω is longitudinally symplectic on S.

Definition 8.2.4. An internally symplectic action of a regular Lie algebroid $(\pi : \mathcal{A} \to M, \rho)$ on a smooth family of symplectic manifolds $(J : S \to M, \omega)$ is called **internally weakly Hamiltonian** if there exists a smooth map $\mu : S \to \ker(\rho)^*$, such that

$$\begin{array}{ccc} S & \xrightarrow{\mu} & \ker(\rho)^* \\ {\scriptstyle J}\downarrow & \swarrow{\scriptstyle p} & \\ M & & \end{array}$$

commutes and

$$d^J \langle \mu, J^* X \rangle = -\alpha(X) \lrcorner \omega,$$

for all $X \in \Gamma^\infty(\ker(\rho))$. The map μ is called an **internal momentum map** for the $\mathcal{A}$-action.

One can view μ as a section in $\Gamma^\infty(J^* \ker(\rho)^*)$.

Remark 8.2.5. The action of $G \rightrightarrows M$ on $(J : S \to M, \omega)$ is weakly Hamiltonian if the action of G_m^m on $S_m, \omega|_{S_m}$ is weakly Hamiltonian for each $m \in M$ *and* the momentum maps $\mu_m : S_m \to \mathfrak{g}_m$ can be pasted together to a smooth internal momentum map

$$\mu : S \to \mathcal{A}(I_G).$$

Remark 8.2.6. One should think of $[\alpha(X) \lrcorner \omega]$ as a cohomological obstruction to the existence of a momentum map. One has the following diagram

$$\begin{array}{ccccc} C^\infty_J(S) & \xrightarrow{d_0^J} & \Gamma^\infty(\ker(d_1^J)) & \longrightarrow & H^1_{J,dR}(S) \\ & \nwarrow{\scriptstyle \mu} & \uparrow{\scriptstyle -\alpha \lrcorner \omega} & \nearrow & \\ & & \Gamma^\infty(\ker(\rho)) & & \end{array}$$

where $\alpha \lrcorner \omega(X) := \alpha(X) \lrcorner \omega$ and the right diagonal arrow denotes the induced map on the quotient space. The vanishing of this map is a necessary condition for $\alpha \lrcorner \omega$ to lift to a map μ.

Definition 8.2.7. A symplectic Lie groupoid action is **internally weakly Hamiltonian** if the associated Lie algebroid action is internally weakly Hamiltonian.

Before we give examples of such actions we shall introduce the notion of internally strongly Hamiltonian actions in the next section.

8.3 The coadjoint action and internal momentum maps

In this section we introduce the notion of internally strongly Hamiltonian actions and treat several examples.

Let $G \rightrightarrows M$ be a regular Lie groupoid and $I_G \to M$ the associated isotropy Lie groupoid. Recall that $G \rightrightarrows M$ acts smoothly (from the left) on $I_G \to M$ by conjugation $G\,{}_s\!\times_p I_G \to I_G$,

$$c(g)g' := gg'g^{-1},$$

(cf. Example 8.1.1).

The action by conjugation induces an action of $G \rightrightarrows M$ on the smooth family of Lie algebras $\ker(\rho) \cong \mathcal{A}(I_G) \to M$ by

$$\mathrm{Ad}(g)X = \left.\frac{d}{d\tau}\right|_{\tau=0} c(g)\exp(\tau X),$$

where $X \in \mathcal{A}(I_G)_m$ and $g \in G_m$ for any $m \in M$. This action is called the **adjoint action of** $G \rightrightarrows M$ and is the generalization of the adjoint action for Lie groups.

In turn, this induces the **adjoint action** of the Lie algebroid $\mathcal{A}(G)$ on $\mathcal{A}(I_G) \to M$ (cf. Example 8.1.10)

$$\mathrm{ad}(X)Y = \left.\frac{d}{d\tau}\right|_{\tau=0} \mathrm{Ad}(\exp(\tau X))Y,$$

where $X \in \Gamma^\infty(\mathcal{A}(G))$ and $Y \in \Gamma^\infty(\mathcal{A}(I_G))$. Note that for $X \in \Gamma^\infty(\mathcal{A}(G))$ and $Y \in \Gamma^\infty(\mathcal{A}(I_G))$

$$\mathrm{ad}(X)Y = [X, Y].$$

Example 8.3.1. A trivial example is the pair groupoid $G = M \times M \rightrightarrows M$. Conjugation is given by $c((m,n),(n,n)) = (m,m)$. The kernel of the anchor is the zero bundle hence Ad is trivial on the fibers and $\mathrm{ad} : TM \to M \times \{0\}$ is the zero map.

Example 8.3.2. If G is a Lie group, then Ad and ad coincide with the usual notions.

Remark 8.3.3. The adjoint action as we discuss it here is an action of $\mathcal{A}$ on $\ker(\rho) \to M$ and not an action of $\mathcal{A}$ on $\mathcal{A} \to M$. But there is also the notion of an *action up to homotopy* (cf. [25]). It turns out that the map $\mathrm{ad}(X)Y := [X, Y]$ defines an action *up to homotopy* of $\mathcal{A}$ on $\mathcal{A} \to M$. We shall not use this structure in this thesis.

One defines the **coadjoint action of** $G \rightrightarrows M$ on the dual bundle $\mathcal{A}^*(I_G) \to M$ by

$$\langle \mathrm{Ad}^*(g)\xi, X\rangle := \left\langle \xi, \mathrm{Ad}(g^{-1})X\right\rangle,$$

where $\xi \in \mathcal{A}^*(I_G)_m$ and $g \in G_m$. Analogously, one defines the **coadjoint action of** $\mathcal{A}(G)$ on $\mathcal{A}^*(I_G) \to M$ by

$$\langle \mathrm{ad}^*(X)\xi, Y\rangle := \langle \xi, \mathrm{ad}(-X)Y\rangle,$$

which is obtained as the tangent map of Ad^* (cf. Example 8.1.10).

Definition 8.3.4. An internally weakly Hamiltonian action of a regular Lie algebroid $\mathcal{A}$ on a smooth family of symplectic manifolds $(J : S \to M, \omega)$ is **internally (strongly) Hamiltonian** if the momentum map $\mu : S \to \ker(\rho)^*$ is $\mathcal{A}$-equivariant with respect to the coadjoint action of $\mathcal{A}$ on $\ker(\rho)^*$, i.e.

$$\alpha(X) \cdot \langle \mu, Y \rangle = \langle \mathrm{ad}^*(X)\, \mu, Y \rangle .$$

From a Lie groupoid perspective this is

Definition 8.3.5. An internally weakly Hamiltonian Lie groupoid action of $G \rightrightarrows M$ on $(J : S \to M, \omega)$ is **internally (strongly) Hamiltonian** if the momentum map $\mu : S \to \ker(\rho)^*$ is G-equivariant with respect to the coadjoint action of G on $\ker(\rho)^*$, i.e.

$$\mu(g \cdot \sigma) = \mathrm{Ad}^*(g) \cdot \mu(\sigma).$$

Example 8.3.6. One can consider for example smooth families of Lie algebras $\pi : \mathfrak{g} \to M$; in particular, a bundle of Lie algebras $P \times_H \mathfrak{h}$, where H is a Lie group and $P \to M$ a principal H-bundle and the action of H on $\mathfrak{h}$ is the adjoint action (cf. Example 2.1.9). More about internally Hamiltonian actions of such bundles can be derived from Example 8.3.8.

In general one can remark the following. Suppose a smooth family of Lie algebras $\mathfrak{g} \to M$ acts on a smooth family of symplectic manifolds $S := \bigcup_{m \in M} S_m \to M$. Then a momentum map is a smooth map $S \to \mathfrak{g}^*$ that restricts to a momentum map in the classical sense on each fiber (cf. Remark 8.2.5). For example, for a smooth family of coadjoint orbits $\{\mathcal{O}_m \subset \mathfrak{g}_m^*\}_{m \in M}$,

$$S := \bigcup_{m \in M} \mathcal{O}_m \subset \mathfrak{g}_m^* \to M$$

carries a Hamiltonian action (namely the coadjoint action). The inclusion $S \hookrightarrow \mathfrak{g}^*$ is an internal momentum map.

Example 8.3.7 (The orbit method I)**.** Suppose $G \rightrightarrows M$ is a regular Lie groupoid with associated Lie algebroid $(\mathcal{A} \to M, \rho)$. Recall that there exists a coadjoint action of $G \rightrightarrows M$ on $p : \ker(\rho)^* \to M$. Note that, since $p : \ker(\rho)^* \to M$ is G-equivariant, it projects coadjoint orbits onto orbits in M. Hence a family of coadjoint orbits over M can be parametrized by the orbit space M/G. Consider such a family of coadjoint orbits

$$\{\mathcal{O}_{mG}\}_{mG \in M/G}$$

in $\ker(\rho)^*$. Suppose they form a smooth family

$$J : S := \bigcup_{mG \in M/G} \mathcal{O}_{mG} \to M.$$

For $m \in M$ the smooth manifold S_m has a symplectic structure given by the standard symplectic form on $S_m = \mathcal{O}_{mG} \cap \ker(\rho)^*_m$, which is a coadjoint orbit in the dual of the Lie algebra $\ker(\rho)_m$. Together these symplectic forms form a smooth family of symplectic forms $\omega \in \Omega^2_J(S)$. The inclusion

$$S := \bigcup_{mG \in M/G} \mathcal{O}_{mG} \hookrightarrow \ker(\rho)^*$$

is an internal momentum map for the coadjoint action on S which is therefore internally Hamiltonian. This is an important observation concerning the orbit method for Lie groupoids. We shall come back to this in Remark 8.4.19 and in Section 10.4.

Example 8.3.8. Suppose H is a Lie group and $\pi : P \to M$ a smooth principal H-bundle. Denote the action of H on P by α. Suppose H acts on a symplectic manifold (S, ω^S) in a Hamiltonian fashion with momentum map $\mu : S \to \mathfrak{h}^*$. Denote the action of H on S by β.

Let $G \rightrightarrows M$ be the gauge groupoid $P \times_H P \rightrightarrows M$. Define a smooth bundle of smooth manifolds by

$$S' := P \times_H S.$$

The map $\bar{\pi} : [p, \sigma] \mapsto \pi(p)$ is well defined $S' \to M$ and gives the bundle structure. The following observations and lemma will be necessary to endow $\bar{\pi} : S' \to M$ with the structure of a smooth bundle of symplectic manifolds.

Note that, since P is a principal H-bundle, the infinitesimal action

$$\alpha : P \times \mathfrak{h} \to T^\pi P$$

is an isomorphism of smooth vector bundles. Moreover, it is equivariant with respect to the adjoint action of H on $\mathfrak{h}$, hence it induces a diffeomorphism

$$\bar{\alpha} : P \times_H \mathfrak{h} \to T^\pi P/H.$$

Lemma 8.3.9. *Suppose a Lie group H acts properly and freely on a manifold N. Then*

$$T(N/H) \cong (TN)/\sim,$$

where the equivalence relation is generated by

$$hv \sim v$$

for all $v \in TN$ and $h \in H$ and

$$\gamma(X)_n \sim 0,$$

for all $X \in \mathfrak{h}$, $n \in N$, and where $\gamma : \mathfrak{h} \to \mathfrak{X}^\infty(N)$ denotes the infinitesimal action.

Proof. Consider the tangent map

$$TN \to T(N/H)$$

of the quotient map $N \to N/H$. It is surjective and the kernel is spanned by the elements mentioned above as one easily checks. □

Remark 8.3.10. One should compare this lemma to the fact that H acts on T^*N in Hamiltonian fashion with "classical" momentum map $\mu : T^*N \to \mathfrak{h}^*$ given by

$$\langle \mu(\theta), X \rangle = \langle \theta, \gamma(X) \rangle .$$

The Marsden-Weinstein quotient satisfies

$$T^*(N/H) \cong \mu^{-1}(0)/H$$

(cf. e.g., [31]). Taking duals at both sides proves the Lemma.

Another way the view the equivalence relation $\sim$ is as obtained from the induced action of TH (as a group) on TN.

Applying this lemma to $TS' = T(P \times_H S)$ one obtains

$$T(P \times_H S) \cong (TP \times TS)/\sim,$$

and restricting to the vertical tangent space one has

$$T^{\bar{\pi}}(P \times_H S) \cong (T^{\pi} P \times TS)/\sim.$$

The map α induces an isomorphism

$$(T^{\pi} P \times TS)/\sim \longrightarrow (P \times \mathfrak{h} \times TS)/\sim_1$$

with the equivalence relation generated by

$$[p, X, \beta'(X)] \sim_1 [p, 0, 0]$$
$$(h \cdot p, h \cdot X, h \cdot v) \sim_1 (p, X, v),$$

for all $p \in P$, $h \in H$, $X \in \mathfrak{h}$. The map

$$(P \times \mathfrak{h} \times TS)/\sim_1 \longrightarrow P \times_H TS$$

given by $[p, X, v] \to [p, v - \beta'(X)]$ is again an isomorphism of smooth vector bundles. So we conclude that

$$T^{\bar{\pi}}(P \times_H S) \cong P \times_H TS.$$

We can define a structure of a smooth bundle of symplectic manifolds on S' via this isomorphism by

$$\omega_{[p,\sigma]}([p, v_1], [p, v_2]) := \omega^S_{\sigma}(v_1, v_2),$$

for $p \in P$, $\sigma \in S$ and $v_1, v_2 \in T_{\sigma} S$. This well defined, since ω is H-invariant by assumption. One easily sees that this indeed gives a non-degenerate $\bar{\pi}$-2-form on $P \times_H S$ and that

$$d^{\bar{\pi}} \omega = 0.$$

Consider the left action of the gauge groupoid $G = P \times_H P \rightrightarrows M$ on $\bar{\pi} : P \times_H S \to M$ given by

$$[p, q] \cdot [q, \sigma] := [p, \sigma],$$

where we remark that if $t[p, q] = \pi([q', \sigma'])$, then one can always find a representative of the class $[q', \sigma']$ as above. Denote this action by γ.

Proposition 8.3.11. *The action of the gauge groupoid $G = P \times_H P \rightrightarrows M$ on $S' = P \times_H S \to M$ is internally Hamiltonian.*

Proof. Note that the Lie algebroid associated to $G \rightrightarrows M$ is isomorphic to $(TP)/H$ (cf. Example 2.4.8 and [43],[49]). Hence the dual of the kernel of the anchor is isomorphic to $T^{*,\pi} P/H$, which in turn is isomorphic to $P \times_H \mathfrak{h}^*$ using the map

$$(\bar{\alpha}')^* : T^{*,\pi} P/H \to P \times_H \mathfrak{h}^*$$

induced by the infinitesimal action α^* of $\mathfrak{h}$ on $T^* P$.

We give the momentum map via this isomorphism as a map

$$P \times_H S \to P \times_H \mathfrak{h}^*$$

defined by

$$\bar{\mu}[p, \sigma] := [p, \mu(\sigma)].$$

This is indeed well-defined, since μ is by assumption H-equivariant, hence

$$\begin{aligned} \bar{\mu}[h\,p, h\,\sigma] &:= [h\,p, \mu(h\,\sigma)] \\ &= [h\,p, \mathrm{Ad}^*(h)\mu(\sigma)] \\ &= [p, \mu(\sigma)]. \end{aligned}$$

Dually to $T^{\bar{\pi}}(P \times_H S) \cong P \times_H TS$ we have an isomorphism

$$k : T^{*,\bar{\pi}}(P \times_H S) \to P \times_H T^*S.$$

Finally, we check that for all $X \in \Gamma(P \times_H \mathfrak{h})$

$$\begin{aligned} d^J \langle \bar{\mu}, X \rangle &= d^J [p, \langle \mu, X \rangle] \\ &\mapsto [p, d^S \langle \mu, X \rangle] \\ &= [p, -\beta(X) \lrcorner\, \omega^S] \\ &\mapsto -\gamma(X) \lrcorner\, \omega, \end{aligned}$$

where the arrow on the second line refers to the isomorphism k and in the last line we again identify the action of $T^{\bar{\pi}}P/H$ on $P \times_H S$ with the action of $P \times_H \mathfrak{h}$ on $P \times_H S$ through the isomorphism $T^{\bar{\pi}}P/H \to P \times_H \mathfrak{h}$.

Finally we have to check equivariance of the momentum map $\bar{\mu}$. This is immediate if we again identify $T^{\bar{\pi}}P/H$ with $P \times_H \mathfrak{h}$ and $T^{*,\bar{\pi}}P/H$ with $P \times_H \mathfrak{h}^*$. □

Example 8.3.12. Suppose $\pi : E \to M$ is a smooth complex vector bundle endowed with a Hermitian metric h. Let $U(E)$ be the groupoid of unitary maps on the fibers $\{E_m \to E_n\}_{m,n \in M}$ (cf. Example 2.1.8). It has a smooth structure induced from the smooth structure on E and the smooth structure on $U(n)$ (cf. [49]). There exists a smooth family of symplectic structures $\omega \in \Omega^2_\pi(E)$, given by the imaginary part of h, after identifying $T^\pi E \to E$ with $\pi^* E \to E$.

Proposition 8.3.13. *The canonical action of $U(E)$ on (E, ω) is internally Hamiltonian.*

Proof. Let $F_U(E) \subset \mathrm{Hom}_M(M \times \mathbb{C}^n, E)$ be the unitary frame bundle of E, i.e. the principal bundle of unitary maps of the trivial bundle $M \times \mathbb{C}^n$ to E, where n is the rank of E. It is well-known that

$$F_U(E) \times_{U(n)} \mathbb{C}^n \cong E,$$

given by the map $(\Psi, z) \mapsto \Psi(z)$. Moreover, one easily checks that the map

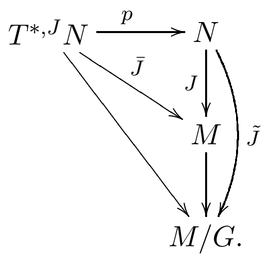
$$F_U(E) \times_{U(n)} F_U(E) \to U(E)$$

given by $[p,q] \mapsto ([q,z] \mapsto [p,z])$ is an isomorphism of the gauge groupoid of $F_U(E)$ with $U(E)$. Hence,

$$I(U(E)) \cong (F_U(E)\,{}_{\pi}\!\times_{\pi} F_U(E))/U(n).$$

Suppose ω' is the imaginary part of a Hermitian inner product on $\mathbb{C}^n$. The natural action of $U(n)$ on $(\mathbb{C}(n), \omega')$ is known to be Hamiltonian (cf. for example [31]). So the proposition follows from Proposition 8.3.11, where the silent assumption was that ω is induced from ω', as in the previous example. $\square$

Example 8.3.14. Suppose a regular Lie groupoid $G \rightrightarrows M$ acts on a surjective submersion $J : N \to M$. Denote the action by $\alpha : G\,{}_s\!\times_J N \to N$. Denote the composition of $J : N \to M$ and the quotient map $M \to M/G$ by $\tilde{J} : N \to M/G$. Let $p : T^{*,J}N \to N$ denote the projection and $\bar{J} := J \circ p$. Note that one has a commuting diagram



There exists an induced action of the groupoid on the map $\bar{J}$ given by

$$\tilde{\alpha}(g)\eta = T^{*,J}\alpha(g)^{-1}\eta,$$

where $\eta \in (T^{*,J}N)_{s(g)} := \bar{J}^{-1}(s(g))$. Moreover, there exists a canonical 1-form on $T^{*,J}N$ defined by

$$\tau := T^{*,J}p : T^{*,J}N \to T^{*,\bar{J}}(T^{*,J}N),$$

by abuse of notation (T^* is not a functor in general). This gives rise to a family of symplectic forms

$$\omega := d^{\bar{J}}\tau \in \Omega^{\bar{J}}(T^{*,J}N).$$

Proposition 8.3.15. *The action of $G \rightrightarrows M$ on $(\bar{J} : T^{*,J}N \to M, \omega)$ is internally Hamiltonian.*

Proof. We define an internal momentum map $\mu : T^{*,J}N \to \bar{J}^* \ker(\rho)$ by

$$\mu := -\tilde{\alpha}^*\tau,$$

where we use the same notation $\tilde{\alpha}$ for the induced action of the Lie algebroid of $G \rightrightarrows M$. The fact that the action is weakly internally Hamiltonian follows from

$$d^{\bar{J}}\left\langle \mu, \bar{J}^*X \right\rangle = -d^{\bar{J}}\left\langle \tau, \tilde{\alpha}(\bar{J}^*X)\right\rangle = -\tilde{\alpha}(X) \lrcorner\, d^{\bar{J}}\tau = -\tilde{\alpha}(X) \lrcorner\, \omega$$

for all $X \in \ker(\rho)$. Equivariance of the momentum map follows from

$$\begin{aligned} \left\langle \mu, \bar{J}^*X \right\rangle (\tilde{\alpha}(g)\eta) &= -\left\langle T^{*,J}\alpha(g)^{-1}\eta, \alpha(X)_{\alpha(g)^{-1}p(\eta)}\right\rangle \\ &= -\left\langle \eta, \alpha(\mathrm{Ad}(g)X)\right\rangle \\ &= \left\langle \mu, \bar{J}^*(\mathrm{Ad}(g)X)\right\rangle (\eta), \end{aligned}$$

for all $g \in G$, $\eta \in T^{*,J}N_{s(g)}$ and $X \in \ker(\rho)$. $\square$

Example 8.3.16. As a corollary of the previous example, every regular Lie groupoid $G \rightrightarrows M$ has three canonical internally Hamiltonian actions associated to it. Firstly, one has the action G on the base space M given by $g \cdot s(g) = t(g)$, with zero symplectic structure at each point $m \in M$. Secondly, consider the action on $T^{*,t}G \to M$, induced from the left action of G on $t : G \to M$ by left multiplication. Thirdly, one has the action on $T^{*,p}I_G$ induced from the conjugation action on $p : I_G \to M$ (if $G \rightrightarrows M$ is regular). This last one is, of course, related to the coadjoint action of G on $\ker(\rho)^* \subset T^{*,p}I_G$, which is internally Hamiltonian.

8.4 Hamiltonian actions and momentum maps

In this section we introduce Hamiltonian actions of Lie algebroids. A large part of the section will be devoted to examples justifying our terminology.

Let $\mathcal{A}$ be a Lie algebroid over M with anchor ρ. A **smooth n-cochain on $\mathcal{A}$** is a $C^\infty(M)$-multilinear antisymmetric map

$$\mu : \Gamma^\infty(\mathcal{A}) \times \ldots \times \Gamma^\infty(\mathcal{A}) \to C^\infty(M).$$

The space of smooth n-cochains is denoted $C^n(\mathcal{A})$. It is turned into a cochain complex by

$$\begin{aligned} d_{\mathcal{A}}\mu(X_1, \ldots, X_{n+1}) = & \sum_{i<j} (-1)^{i+j+1} \mu([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{n+1}) \\ & + \sum_{i=1}^{n+1} (-1)^i \rho(X_i) \cdot \mu(X_1, \ldots, \hat{X}_i, \ldots, X_{n+1}). \end{aligned}$$

The cohomology of the cochain complex is denoted $H^*(\mathcal{A})$ and is called the **Lie algebroid cohomology** of $\mathcal{A}$.

Remark 8.4.1. This not the only type of cohomology one could associate to Lie algebroids, see e.g., [19]. The cohomology groups discussed here are also called the (generalized) de Rham cohomology of $\mathcal{A}$. One could also define de Rham cohomology of $\mathcal{A}$ with coefficients in a representation of $\mathcal{A}$, but this is not needed in this thesis.

A morphism of Lie algebroids $\Phi : \mathcal{A} \to \mathcal{A}'$ (cf. Definition 2.4.4) induces a cochain map $\Phi^* : C^*(\mathcal{A}') \to C^*(\mathcal{A})$ and hence a map $\Phi^* : H^*(\mathcal{A}') \to H^*(\mathcal{A})$ on cohomology.

Example 8.4.2. Suppose $\mathcal{A} = TM$ for a smooth manifold M. Then $H^*(A)$ equals the de Rham cohomology $H^*_{dR}(M)$ of M, which explains the terminology.

If $\mathcal{A} = \mathfrak{g}$ is a Lie algebra, then $H^*(\mathcal{A})$ equals the Chevalley-Eilenberg cohomology $H^*(\mathfrak{g})$ of $\mathfrak{g}$.

Suppose a regular Lie algebroid $(\mathcal{A} \to M, [.,.], \rho)$ acts on a smooth surjective submersion $J : S \to M$. Let a J-presymplectic 2-form

$$\tilde{\omega} \in \Omega^2_{\tilde{J}}(S)$$

be given. Since the action α is a morphism of Lie algebroids $\mathcal{A} \ltimes J \to T^{\tilde{J}}S$, one has

$$d_{\mathcal{A} \ltimes J} \alpha^* \tilde{\omega} = \alpha^* d^{\tilde{J}} \tilde{\omega} = 0,$$

where $\mathcal{A} \ltimes J$ is the action Lie algebroid associated to the action of $\mathcal{A}$ on $J : S \to M$. Suppose the action is symplectic (cf. Definition 8.1.11.ii). Using Cartan's homotopy formula this implies that

$$d^J(\alpha(X) \lrcorner \tilde{\omega})|_{T^J S} = (\alpha(X) \lrcorner d^{\tilde{J}} \tilde{\omega})|_{T^J S} = 0.$$

Definition 8.4.3. An action α of a Lie algebroid $(\pi : \mathcal{A} \to M, \rho)$ on a map $J : S \to M$ is **Hamiltonian** if there exists a smooth section $\tilde{\mu} \in \Gamma^\infty((\mathcal{A} \ltimes J)^*)$, satisfying

$$d_{\mathcal{A} \ltimes J} \tilde{\mu} = -\alpha^* \tilde{\omega}, \tag{8.4.1}$$

$$d^J \langle \tilde{\mu}, J^* X \rangle = -\left(\alpha(X) \lrcorner \tilde{\omega}\right)|_{T^J S} \text{ for all } X \in \Gamma^\infty(\mathcal{A}). \tag{8.4.2}$$

$\tilde{\mu}$ is called a **momentum map** for the action.

Remark 8.4.4. Condition (8.4.1) is called the prequantization condition and has to be satisfied for an action to be prequantizable. Condition (8.4.2) is called the quantization condition and has to be satisfied for the prequantization to be quantizable. This terminology will be justified in the next sections.

Remark 8.4.5. The prequantization and quantization conditions state that $\tilde{\mu} \in \Gamma^\infty(\mathcal{A} \ltimes J)$ should be the simultaneous solution of an integration problem (8.4.1) for which $\alpha^* \tilde{\omega} \in H^2(\mathcal{A} \ltimes J)$ is the obstruction and a lifting problem (8.4.2) (which is an integration problem for each X) for which the map $X \mapsto (\alpha(X) \lrcorner \tilde{\omega})|_{T^J S} \in H^2_J(S)$ forms the obstruction. In particular, if these cohomology groups are zero, then all symplectic actions of $\mathcal{A}$ on $J : S \to M$ are Hamiltonian. There exist some vanishing results for Lie algebroid cohomology (cf. [15]).

Lemma 8.4.6. *If an action of $\mathcal{A}$ on $(J : S \to M, \tilde{\omega})$ is Hamiltonian, then it is internally strongly Hamiltonian.*

Proof. Note that condition (8.4.2) implies that that the action is internally weakly Hamiltonian, with internal momentum map $\mu := i^* \circ \tilde{\mu}$, where $i : \ker(\rho) \to \mathcal{A}$ is the inclusion.

We compute the left hand side of the prequantization condition (8.4.1):

$$d_{\mathcal{A}(G) \ltimes J} \tilde{\mu}(X, Y) = \langle \tilde{\mu}, [X, Y] \rangle - \alpha(X) \langle \tilde{\mu}, J^* Y \rangle + \alpha(Y) \langle \tilde{\mu}, J^* X \rangle . \tag{8.4.3}$$

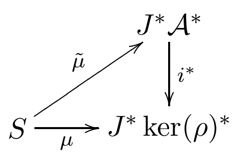
If $X \in \Gamma^\infty(\mathcal{A})$ and $Y \in \Gamma^\infty(\ker(\rho))$, then inserting (8.4.2) in (8.4.3) we obtain

$$\langle \tilde{\mu}, [X, Y] \rangle - \alpha(X) \langle \tilde{\mu}, J^* Y \rangle = 0. \tag{8.4.4}$$

But

$$\langle \tilde{\mu}, [X, Y] \rangle = \langle \tilde{\mu}, \mathrm{ad}(X) Y \rangle = \langle \mathrm{ad}^*(X) \tilde{\mu}, Y \rangle ,$$

hence (8.4.4) expresses $\mathcal{A}$-equivariance of $\tilde{\mu}$. We conclude that the momentum map is a lift of an internal momentum map $\mu : S \to J^* \ker(\rho)^*$, i.e. the diagram



commutes. □

Suppose $\mathcal{A}$ acts on $(J : S \to M, \tilde{\omega})$ in a Hamiltonian fashion. Then we can form a perturbation of $\tilde{\omega}$ by certain exact forms without changing the fact that the action is Hamiltonian.

Lemma 8.4.7. *For any $\beta \in \Omega^1_{\tilde{J}}(S)$, the action of $\mathcal{A}$ on $(J : S \to M, \tilde{\omega}')$, with*

$$\tilde{\omega}' := \tilde{\omega} + d^{\tilde{J}}\beta,$$

is Hamiltonian iff $\mathcal{L}_{\alpha(X)}\beta$ annihilates $T^J S$ for all $X \in \Gamma^\infty(\mathcal{A})$.

Proof. Define a momentum map by

$$\tilde{\mu}' := \tilde{\mu} + \alpha^*\beta.$$

One computes that Condition (8.4.1) is satisfied:

$$\begin{aligned} d_{\mathcal{A}\ltimes J}\tilde{\mu}' &= d_{\mathcal{A}\ltimes J}\tilde{\mu} + d_{\mathcal{A}\ltimes J}\alpha^*\beta \\ &= -\alpha^*\tilde{\omega} - \alpha^* d^{\tilde{J}}\beta \\ &= -\alpha^*\tilde{\omega}'. \end{aligned}$$

Idem dito for Condition (8.4.2)

$$\begin{aligned} d^J \left\langle \tilde{\mu}', X \right\rangle &= d^J \left\langle \tilde{\mu}, X \right\rangle + d^J \left\langle \alpha^*\beta, X \right\rangle \\ &= -(\alpha(X) \lrcorner\, \tilde{\omega})|_{T^J S} - (\alpha(X) \lrcorner\, d^{\tilde{J}}\beta)|_{T^J S} + \mathcal{L}_{\alpha(X)}\beta|_{T^J S} \\ &= -(\alpha(X) \lrcorner\, \tilde{\omega}')|_{T^J S}. \end{aligned}$$

□

Moreover, given a Hamiltonian action α of $\mathcal{A}$ on $(J : S \to M, \tilde{\omega})$ one can add certain closed forms to the momentum map and it is still a momentum map. Hence, the chosen momentum map is not unique.

Lemma 8.4.8. *Suppose $\tilde{\mu}$ is a momentum map and $\beta \in \Omega^1_{\tilde{J}}(S)$, then*

$$\tilde{\mu}' := \tilde{\mu} + \alpha^*\beta$$

is a momentum map for the action too iff $d^{\tilde{J}}\beta = 0$ and $\mathcal{L}_{\alpha(X)}\beta$ annihilates $T_J S$ for all $X \in \Gamma(\mathcal{A})$.

Proof. The proof is a calculation similar to the proof of the previous lemma. □

Definition 8.4.9. An action of a Lie groupoid $G \rightrightarrows M$ on a smooth family of symplectic manifolds $J : S \to M$ is **Hamiltonian** if the induced action of the associated Lie algebroid $\mathcal{A}(G)$ is Hamiltonian.

Example 8.4.10. For the case of an action of a smooth family of Lie algebras on a smooth family of symplectic manifolds, every internal momentum map is a momentum map. In particular, strongly Hamiltonian Lie algebra actions are Hamiltonian in our terminology too.

Example 8.4.11. Suppose M is a smooth manifold. Consider an integrable distribution $T\mathcal{F} \subset TM$ as a Lie algebroid over M. The differential in the Lie algebroid de Rham complex is the partial $\mathcal{F}$-de Rham differential $d^{\mathcal{F}}$. Consider the canonical action $\Gamma^\infty(T\mathcal{F}) \hookrightarrow \mathfrak{X}^\infty(M)$ of $T\mathcal{F}$ on $J := id : M \to M$. Suppose $\tilde{\omega}$ is a $\mathcal{F}$-partially closed 2-form. This is trivially a J-presymplectic form. A smooth section $\tilde{\mu} : M \to T^*\mathcal{F}$ is a momentum map iff $d^{\mathcal{F}}\tilde{\mu} = -\tilde{\omega}$ on M.

Example 8.4.12. Suppose a regular Lie algebroid $(\mathcal{A}_i \to M, \rho_i)$ acts on $(J_i : S_i \to M_i, \tilde{\omega_i})$ in a Hamiltonian fashion, with momentum map $\tilde{\mu}_i : S \to \mathcal{A}^*(G_i)$ for $i = 1, 2$. Then the Cartesian product of algebroids $\mathcal{A}_1 \times \mathcal{A}_2$ acts Hamiltonianly on

$$(J_1 \times J_2 : S_1 \times S_2 \to M_1 \times M_2, \tilde{\omega}_1 \times \tilde{\omega}_2)$$

with momentum map

$$\tilde{\mu}_1 \times \tilde{\mu}_2 : S_1 \times S_2 \to \mathcal{A}_1^* \times \mathcal{A}_2^*.$$

A particular example of this:

(i) Suppose M is a manifold endowed with a closed 2-form $\tilde{\omega}$. If $\beta \in \Omega^1(M)$ satisfies $d\beta = -\tilde{\omega}$, then $\beta : M \to T^*M$ is a momentum map for the action of TM on $(M \to M, \omega)$.

(ii) Suppose $\mathfrak{g}$ is a Lie algebra. Suppose $\mathcal{O} \subset \mathfrak{g}^*$ is a coadjoint orbit. The inclusion of the coadjoint orbit $i : \mathcal{O} \hookrightarrow \mathfrak{g}^*$ is a momentum map for the coadjoint action of $\mathfrak{g}$ on $\mathcal{O}$.

Consider the trivial Lie algebroid over M with fiber $\mathfrak{g}$. It is the Cartesian product $TM \times \mathfrak{g}$ of TM as a Lie algebroid over M and $\mathfrak{g}$ as a Lie algebroid over a point. Hence

$$\beta \times i : M \times \mathcal{O} \to T^*M \times \mathfrak{g}^*$$

is a momentum map for the action of $TM \times \mathfrak{g}$ on $pr_1 : M \times \mathcal{O} \to M$.

Example 8.4.13. Consider the situation of Example 8.3.8. That is, suppose a Lie group H acts in a Hamiltonian fashion on a symplectic manifold (S, ω^S), with momentum map $\mu : S \to \mathfrak{h}^*$. Suppose $\pi : P \to M$ is a principal H-bundle. Proposition 8.3.11 states that the action of the gauge groupoid $G = P \times_H P \rightrightarrows M$ on $J : S' := P \times_H S \to M$ is internally Hamiltonian. Given a connection on P we shall extend the symplectic form $\omega \in \Omega_J(S')$ on S' to a J-presymplectic form $\tilde{\omega} \in \Omega^2_{\tilde{J}}(S')$. Then we shall see that the action is Hamiltonian with respect to a well-chosen momentum map.

Suppose $\tau \in \Gamma^\infty(\bigwedge^1(P) \otimes \mathfrak{h})$ is a Lie algebra-valued connection 1-form on P. After identifying $T(P \times_H S) \cong (TP \times_H TS)/\!\sim$ as in Example 8.3.8, define

$$\tilde{\omega}_{[p,\sigma]}\big([w_1, v_1], [w_2, v_2]\big) := \omega^S_\sigma\big((v_1 - \beta(\tau(w_1)), v_2 - \beta(\tau(w_2))\big) - \langle \mu_\sigma, F_p(w_1, w_2)\rangle\,,$$

where F is the $\mathfrak{h}$-valued curvature 2-form on P.

Lemma 8.4.14. *$\tilde{\omega}$ is a well-defined 2-form in $\Omega^2_{\tilde{J}}(S')$.*

Proof. Note that for $X, Y \in \mathfrak{h}$

$$\tilde{\omega}\big([\alpha(X), \beta(X)], [\alpha(Y), \beta(Y)]\big) = 0.$$

Moreover, for all $h \in H$

$$\tilde{\omega}_{[h\,p, h\,\sigma]}\big([h\,w_1, h\,v_1], [h\,w_2, h\,v_2]\big) = \tilde{\omega}_{[p,\sigma]}\big([w_1, v_1], [w_2, v_2]\big),$$

since ω^S is H-invariant and

$$\begin{aligned} \langle \mu_{h\,\sigma}, F_{h\,p}(h\,w_1, h\,w_2)\rangle &= \langle \mathrm{Ad}^*(h)\,\mu_\sigma, \mathrm{Ad}(h)\,F_p(w_1, w_2)\rangle \\ &= \langle \mu_\sigma, F_p(w_1, w_2)\rangle\,, \end{aligned}$$

by H-equivariance of τ and μ. □

We shall omit here the proof that $\tilde{\omega}$ is closed since we shall later see that it is the curvature 2-form of a connection on a line bundle over $P \times_H S$. Obviously, $\tilde{\omega}$ restricts to ω on the vertical tangent space.

Remark 8.4.15. Given a smooth family of symplectic forms ω, there are in general many inequivalent ways of extending ω to a J-presymplectic form $\tilde{\omega}$. This is illustrated by the above example, where different choices of τ give rise to different $\tilde{\omega}$'s.

Proposition 8.4.16. *The action of the gauge groupoid $G = P \times_H P \rightrightarrows M$ on $(J : S' \to M, \tilde{\omega})$ is Hamiltonian.*

Proof. Define a momentum map $\tilde{\mu} : P \times_H S \to \mathcal{A}^* \cong T^*P/H$ by

$$\tilde{\mu} := \langle \mu, \tau\rangle\,,$$

where μ is the momentum map for the action of H on (S, ω^S). This is well-defined, since

$$\begin{aligned} \langle \mu, \tau\rangle\,(h\,p, h\,\sigma) &= \langle \mathrm{Ad}^*(h)\mu(\sigma), \mathrm{Ad}(h)\tau(p)\rangle \\ &= \langle \mu, \tau\rangle\,(p, \sigma). \end{aligned}$$

One easily sees that $\tilde{\mu}$ restricts to $\bar{\mu}$ (cf. Example 8.3.8) on the vertical tangent space. For H-equivariant vector fields w_1, w_2 on P, we compute

$$\begin{aligned} d_P\tilde{\mu}(w_1, w_2) &= \langle \mu, d_P\tau(w_1, w_2)\rangle \\ &= \langle \mu, F(w_1, w_2)\rangle - \langle \mu, [\tau(w_1), \tau(w_2)]_{\mathfrak{h}}\rangle \\ &= \langle \mu, F(w_1, w_2)\rangle + \beta(\tau(w_1)) \cdot \langle \mu, \tau(w_2)\rangle \\ &= \langle \mu, F(w_1, w_2)\rangle - \omega(\tau(w_1), \tau(w_2)), \end{aligned}$$

where the second equality follows from the curvature formula

$$F = d\tau + [\tau, \tau]_{\mathfrak{h}},$$

the third equality follows from H-equivariance of μ and the last equality follows from the fact that μ is a momentum map. Hence one has $d_{A \ltimes J}\tilde{\mu} = -\gamma^*\tilde{\omega}$, where γ denotes the action of $P \times_H P \rightrightarrows M$ on $P \times_H S \to M$.

We check the quantization condition (8.4.2) for $\tilde{\mu}$. Identify

$$T^J(P \times_H S) \cong P \times_H TS,$$

as in Example 8.3.8. For $w \in \mathfrak{X}(P)$ we compute

$$\begin{aligned} d^S \langle \tilde{\mu}, w \rangle &= d^S \langle \mu, \tau(w) \rangle \\ &= -\beta(\tau(w)) \lrcorner\, \omega^S, \end{aligned}$$

from which we conclude that

$$d^J \langle \tilde{\mu}, w \rangle = -\gamma(w) \lrcorner\, \tilde{\omega}.$$

□

Example 8.4.17. Suppose $\pi : E \to M$ is a complex vector bundle with Hermitian structure h. Consider the action of $U(E)$ on E as in Example 8.3.12. Let $F_U(E) \to M$ be the frame bundle of E. Suppose $F_U(E)$ is endowed with a connection τ. Then we can extend ω (cf. Example 8.3.12) to a closed form $\tilde{\omega}$ on E as in the above Example 8.4.13. As a consequence of Proposition 8.4.16 we have the following

Corollary 8.4.18. *The action of $U(E)$ on $(E, \tilde{\omega})$ is Hamiltonian.*

Remark 8.4.19 (The orbit method II)**.** Given a groupoid $G \rightrightarrows M$ one might want to construct its representation using geometric quantization. In Example 8.3.7 we showed that one can start with considering smooth bundles of coadjoint orbits

$$J : S = \{\mathcal{O}_{mG}\}_{mG \in M/G} \to M$$

of $G \rightrightarrows M$. These canonically possess a smooth family of symplectic forms for which the coadjoint action is internally strongly Hamiltonian. But for a geometric quantization one needs a Hamiltonian action as we shall see in Chapter 10.

The point is to find a suitable extensions of the smooth family of symplectic forms ω on the smooth family of coadjoint orbits to a J-presymplectic form $\tilde{\omega}$, such that the coadjoint action is Hamiltonian. As is illustrated in the previous examples, for the gauge groupoid this can be done using connections on the principal bundle.

We shall further discuss the implications of this for an orbit method for Lie groupoids in Section 10.4.

Chapter 9

Prequantization of Hamiltonian actions

9.1 Representations of Lie algebroids

In this section we introduce the notion of a representation of a Lie algebroid with a base manifold possibly different from the base manifold of $\mathcal{A}$. We also introduce a certain Picard group of Lie algebroid representations on line bundles and show that there exists an exact sequence involving this Picard group.

Suppose $(p : \mathcal{A} \to M, \rho)$ is a Lie algebroid and $\alpha : \Gamma^\infty(\mathcal{A}) \to \mathfrak{X}^\infty_{\tilde{J}}(S)$ is an action of $\mathcal{A}$ on $J : S \to M$. Suppose $E \to S$ is a smooth vector bundle over S. Let $\mathcal{D}(E)$ be the Lie algebroid associated to E as discussed in Example 2.4.11.

Definition 9.1.1. An **$\mathcal{A}$-connection on a complex vector bundle** $p : E \to S$, is a map of vector bundles $\pi : \mathcal{A} \ltimes J \to \mathcal{D}(E)$, such that $\alpha = \Theta \circ \pi$. If π preserves the Lie bracket, then it is called a **representation** or **flat $\mathcal{A}$-connection**.

Remark 9.1.2. This is a more general notion of a representation than usual, in the sense that we allow a base manifold which is not M. We shall see that prequantization defines a representation in this way. Quantization gives a representation on a continuous field of Hilbert spaces over M, which we shall come to later in this section.

Definition 9.1.3. A representation $\pi : \mathcal{A} \ltimes J \to \mathcal{D}(E)$ is **Hermitian** with respect to a Hermitian metric h on $E \to M$ if

$$h(\pi(X)\sigma, \tau) + h(\sigma, \pi(X)\tau) = \alpha(X)h(\sigma, \tau),$$

for all $\tau, \sigma \in \Gamma^\infty(E)$.

Proposition 9.1.4. *Any Hermitian representation of a Lie algebroid $\mathcal{A}$ on a line bundle $L \to S$ is of the form*

$$\pi(J^*X) := \nabla_{\alpha(X)} - 2\pi i \langle \tilde{\mu}, J^*X \rangle ,$$

where α is the action of $\mathcal{A}$ on $J : S \to M$, $\tilde{\mu} \in \Gamma^\infty((\mathcal{A} \ltimes J)^*)$ and ∇ a Hermitian $T^{\tilde{J}}S$-connection on S.

This proposition follows from the following lemma.

Lemma 9.1.5. *Let* $(p : \mathcal{A}' \to M, \rho)$ *be a Lie algebroid and let* ∇ *be a Hermitian* $\mathcal{A}$*-connection on a smooth complex vector bundle* $E \to M$*. Then there exist a Hermitian connection* ∇^E *on* $E \to M$ *and a section* μ *of* $\mathcal{A}^* \otimes \mathrm{End}(E)$ *such that* ∇ *is of the form*

$$\nabla_X = \nabla^E_X - 2\pi i \langle \mu, X \rangle,$$

for all $X \in \Gamma^\infty(\mathcal{A})$.

Proof. Let ∇^E be any Hermitian connection on E. It is well known that such a connection always exists. Consider the associated $\mathcal{A}$-connection defined by $\tilde{\nabla} : X \mapsto \nabla^E_{\rho(X)}$. Now,

$$\begin{aligned}(\nabla_X - \tilde{\nabla}_X)(fs) &= f\nabla_X s + \rho(X) fs - f\nabla^E_{\rho(X)} s - \rho(X) fs \\ &= f(\nabla_X - \tilde{\nabla}_X)s,\end{aligned}$$

hence $\nabla_X - \tilde{\nabla}_X$ is a zeroth order differential operator on E, i.e. $\nabla_X - \tilde{\nabla}_X \in \mathrm{End}(E)$. Moreover, $\nabla - \tilde{\nabla}$ is $C^\infty(M)$-linear, in the sense that

$$\nabla_{fX} - \tilde{\nabla}_{fX} = f(\nabla_X - \tilde{\nabla}_X),$$

Thus $\nabla - \tilde{\nabla} \in \Gamma^\infty(\mathcal{A}^* \otimes \mathrm{End}(E))$ by the Serre-Swan theorem. □

Isomorphism classes of smooth complex line bundles on a manifold S form a group $\mathrm{Pic}_{\bar{J}}(S)$ under the tensor product, with the trivial rank one line bundle as a unit, and inverse $[L]^{-1} = [L^*] := [\mathrm{Hom}_S(L, \mathbb{C})]$. If $\mathcal{A}$ acts on a map $J : S \to M$, then one can extend this structure to the set of isomorphism classes of Hermitian $\mathcal{A}$-representations on smooth complex line bundles over S. The product of $\pi : \mathcal{A} \ltimes J \to \mathcal{D}(L)$ and $\pi' : \mathcal{A} \ltimes J \to \mathcal{D}(L')$ is defined by

$$\pi \otimes 1 + 1 \otimes \pi' : A \ltimes J \to \mathcal{D}(L \otimes L'),$$

and the inverse is given by

$$\big\langle \pi^{-1}(X) s^*, s \big\rangle = -\langle s^*, \pi(X)(s) \rangle + d_{\mathcal{A}} \langle s^*, s \rangle.$$

The Hermitian structure on the tensor product $L_1 \otimes L_2$ of two line bundles (L_1, g_1), (L_2, g_2) is given by the formula

$$g(v_1 \otimes w_1, v_2 \otimes w_2) = g_1(v_1, v_2) g_2(w_1, w_2).$$

Definition 9.1.6. The **Hermitian Picard group** $\mathrm{Pic}_{\mathcal{A}}(J)$ **of the Lie algebroid action** of $\mathcal{A}$ on $J : S \to M$ is the group of isomorphism classes of Hermitian representations of $\mathcal{A}$ on line bundles over S, with product and inverse as described above.

Proposition 9.1.7. *There is an exact sequence of groups*

$$0 \to H^1(\mathcal{A} \ltimes J) \to \mathrm{Pic}_{\mathcal{A}}(J) \to \mathrm{Pic}(S) \xrightarrow{c_1^{\mathcal{A} \ltimes J}} H^2(\mathcal{A} \ltimes J). \tag{9.1.1}$$

Proof. The second arrow sends a closed section $\mu \in \Gamma^\infty((\mathcal{A} \ltimes J)^*)$ to the representation $X \mapsto \alpha(X) - 2\pi i\langle \mu, X\rangle$ on the trivial line bundle. This is well-defined and injective: suppose $\mu, \mu' \in \Gamma^\infty((\mathcal{A} \ltimes J)^*)$ give rise to isomorphic representations, i.e. there is a $f \in C^\infty(S)$ such that

$$(\alpha(X) - 2\pi i\langle \mu', J^*X\rangle)(f\sigma) = f(\alpha(X) - 2\pi i\langle \mu, J^*X\rangle)\sigma,$$

for all $X \in \Gamma^\infty(\mathcal{A})$ and all $\sigma \in C^\infty_{\tilde{J}}(S) = \Gamma^\infty(L)$. Using the Leibniz rule and $\langle d_{\mathcal{A}\ltimes J}(f), X\rangle = \alpha(X)f$ we obtain

$$\langle d_{\mathcal{A}} f, J^*X\rangle = \langle \mu - \mu', J^*X\rangle.$$

for all $X \in \Gamma^\infty(\mathcal{A})$. Thus the two representations are isomorphic iff there exists an $f \in C^\infty_{\tilde{J}}(S)$ such that $d_A f = \mu - \mu'$. The third arrow forgets the representation, so the sequence is exact at $\mathrm{Pic}_{\mathcal{A}}(S)$. The last arrow is the first $\mathcal{A}$-Chern class map $c_1^{\mathcal{A}\ltimes J}([L]) := [\alpha^* K]$, where K denotes the curvature 2-form of any connection ∇ on $L \to M$. If it is zero in $H^2(\mathcal{A} \ltimes J)$, then L carries a Hermitian $\mathcal{A}$-representation, cf. Theorem 9.3.1. □

Remark 9.1.8. One can generalize the notion of (higher) Chern classes of complex vector bundles to characteristic classes for complex representations of Lie algebroids (cf. [15]).

Definition 9.1.9. A **representation of a Lie algebroid $\mathcal{A}$ on a continuous field of Hilbert spaces** $(\mathcal{H}, \Delta)$ is a dense $C^\infty(M)$-submodule $\Delta^0 \subset \Delta$ and a bilinear map

$$\pi : \Gamma^\infty(\mathcal{A}) \times \Delta^0 \to \Delta^0$$

satisfying (we use the notation $\pi(X)\xi$ for $\pi(X, \xi)$)

(i) $\pi(X)(f\,\xi) = f\,\pi(X)\xi + (\rho(X)f)\,\xi$;

(ii) $\pi(f\,X)\xi = f\,\pi(X)\,\xi$;

(iii) $[\pi(X), \pi(Y)] = \pi([X, Y]_{\mathcal{A}})$,

for $\xi \in \Delta^0$, $f \in C^\infty(M)$ and $X, Y \in \Gamma^\infty(\mathcal{A})$.

Suppose $(\mathcal{H}, \Delta)$ is a finite-dimensional continuous field of Hilbert spaces over a smooth manifold M and $G \rightrightarrows M$ is a Lie groupoid. Suppose $\pi : G \to U(\mathcal{H})$ is continuous representation of $G \rightrightarrows M$. For any orbit $Gm \in G\backslash M$ the restriction $\mathcal{H}|_{Gm} \to Gm$ is a continuous Hermitian vector bundle. Any continuous vector bundle over a smooth manifold is continuously isomorphic to a smooth one, which is unique up to smooth isomorphism (cf. e.g. [3]). Hence we can define

Definition 9.1.10. A section $\xi \in \Delta$ is a **smooth section** for $\pi : G \to U(\mathcal{H})$, if

$$g \mapsto \pi(g)\xi(s(g))$$

is a smooth map $Gm \to \mathcal{H}|_{Gm}$ for all $m \in M$.

Denote the set of smooth sections by Δ^0. Then there is an induced representation of $\mathcal{A}(G)$ on $(\mathcal{H}, \Delta)$,

$$\pi : \Gamma^\infty(\mathcal{A}) \times \Delta^0 \to \Delta^0$$

given by

$$(\pi(X)\xi)(m) := \frac{d}{d\tau}\pi(\exp(\tau\, X)(m))\xi(m)|_{\tau=0}.$$

9.2 Longitudinal Čech cohomology

In this section we shall discuss how the class of a J-presymplectic form $[\tilde{\omega}] \in H^2_{\tilde{J},dR}(S)$ determines a class in the longitudinal Čech cohomology. This cohomology is defined analogously to the usual Čech cohomology (cf. [10]). Then we give a criterion for $\tilde{\omega}$ to be the curvature of a Hermitian connection on a complex line bundle.

Suppose $\mathcal{F}$ is a regular foliation of S. Consider the projection map on the orbit space $\tilde{J} : S \to S/\mathcal{F}$. Suppose $\mathcal{U}$ is a countable good foliation covering for S (i.e. for all $U \in \mathcal{U}$ the foliation restricted to U is diffeomorphic to a contractible open subset of $\mathbb{R}^q \times \mathbb{R}^{n-q}$, where $n = \dim(M)$ and q the dimension of the foliation. Let I be a countable ordered index set for $\mathcal{U}$. Denote the intersection of k sets $U_{i_1}, \ldots, U_{i_k}$ by $U_{i_1 \ldots i_k}$ for $i_1, \ldots, i_k \in I$. For $k \in \mathbb{Z}_{\geq 0}$ let $C^k_{\tilde{J}}(\mathcal{U}, \mathbb{R})$ be the vector space of smooth functions on $(k+1)$-fold intersections $U_{i_1, \ldots i_{k+1}}$ (where $i_1 < \cdots < i_{k+1}$) which are locally constant along the leaves of the foliation by $\tilde{J}$. Define a map

$$\delta_k : C^k_{\tilde{J}}(\mathcal{U}, \mathbb{R}) \to C^{k+1}_{\tilde{J}}(\mathcal{U}, \mathbb{R})$$

by the usual formula

$$\delta_k(f)|_{U_{i_1 \cdots i_{k+1}}} := \sum_{j=1}^{k+1} (-1)^j f|_{U_{i_1 \ldots \hat{i}_j \ldots i_{k+1}}}. \tag{9.2.2}$$

One checks that $\delta^2 = 0$. The cohomology of the complex is independent of the chosen good foliation cover and we call it the **longitudinal Čech cohomology** and denote it by $\check{H}^*_{\tilde{J}}(S, \mathbb{R})$

Consider the foliation Čech-de Rham double complex defined by

$$C^{k,l} := \prod_{i_1 < \ldots < i_{k+1}} \Omega^l_{\tilde{J}}(U_{i_1 \ldots i_{k+1}}),$$

with

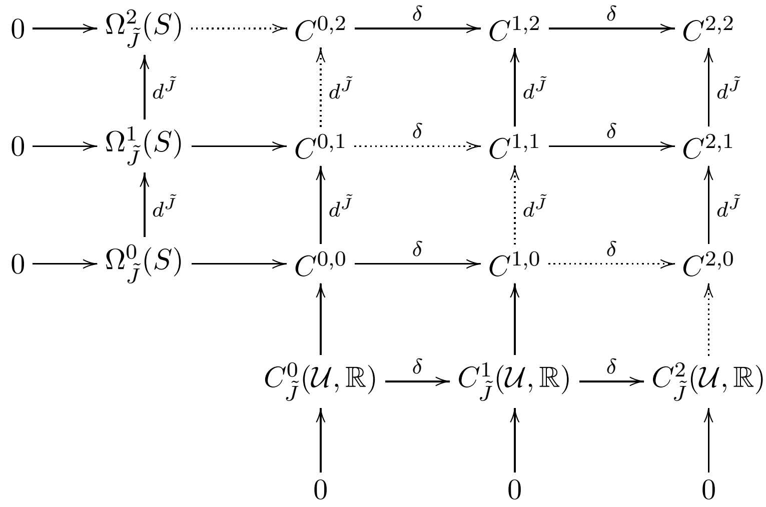
$$\delta_{k,l} : C^{k,l} \to C^{k+1,l}$$

the straightforward generalization of (9.2.2) and

$$d^{\tilde{J}}_{k,l} : C^{k,l} \to C^{k,l+1}$$

the restriction of $d^{\tilde{J}}$ to the $(k+1)$-fold intersections.

The augmented double complex partly shown here (ignore the fact that some arrows are dotted),



can be used to prove

Proposition 9.2.1. *There exists an isomorphism $H^*_{\tilde J,dR}(S) \cong \check H^*_{\tilde J}(S,\mathbb{R})$ between the foliation de Rham cohomology and the longitudinal Čech cohomology.*

The proof is analogous to the proof with the usual Čech-de Rham complex (cf. [10]).

Let $[\tilde\omega] \in H^2_{\tilde J,dR}(S)$ be the class of a J-presymplectic form. We shall concretely realize the above isomorphism to associate a degree 2 longitudinal Čech cohomology class to $[\tilde\omega]$. We shall follow the dotted arrows in the above diagram. Suppose $\mathcal{U}$ is a good foliation covering for S. Since $d^{\tilde J}\tilde\omega = 0$, for each $U_j \in \mathcal{U}$ there exists an $\eta_j \in \Omega^1_{\tilde J}(U_j)$ such that $d^{\tilde J}\eta_j = \tilde\omega|_{U_j}$. Since for all $U_j, U_k \in \mathcal{U}$ we have $d^{\tilde J}(\eta_j - \eta_k) = 0$ on the intersection U_{jk}, there exists an $f_{jk} \in C^\infty(U_{jk})$ such that $d^{\tilde J} f_{jk} = \eta_j - \eta_k$. One easily checks that $d^{\tilde J} f_{jk} + d^{\tilde J} f_{kl} - d^{\tilde J} f_{jl} = 0$ on U_{jkl}. Define

$$a := \{a_{jkl} := f_{jk} + f_{kl} - f_{jl}\}_{j,k,l\in I}.$$

Then $\delta(a_{jkl}) = 0$, hence a defines a class $[a]$ in $\check H^2_{\tilde J}(\mathcal{U},\mathbb{R})$.

There is an obvious definition of longitudinal Čech cohomology $\check H^*_{\tilde J}(S,\mathbb{Z})$ with values in $\mathbb{Z}$. But, one easily sees that $\check H^*_{\tilde J}(S,\mathbb{Z}) \cong \check H^*(S,\mathbb{Z})$, since a cocycle that is integral-valued and continuous is locally constant. We call a class $[\tilde\omega] \in H^2_{\tilde J,dR}(S)$ integer if the associated class in $\check H^2_{\tilde J}(S,\mathbb{R})$ is in the image of the canonical map

$$\check H^2_{\tilde J}(S,\mathbb{Z}) \longrightarrow \check H^2_{\tilde J}(S,\mathbb{R}).$$

Theorem 9.2.2. *A $\tilde J$-closed form $\tilde\omega \in \Omega^2_{\tilde J}(S)$ is the curvature 2-form of a $\tilde J$-partial Hermitian connection on a complex line bundle $L \to S$ iff $[\tilde\omega] \in H^2_{\tilde J,dR}(S)$ is integral.*

Proof. ($\Rightarrow$) Suppose a line bundle $L \to S$, Hermitian metric h and a Hermitian connection ∇ are given, such that $K^\nabla = \tilde\omega$. Suppose $\{(U_j, s_j)\}_{j\in I}$ form a normalized trivialization of $L \to S$, in the sense that $s_j : U_j \to L|_{U_j}$ is a section for all $j \in I$ such that $h(s_j, s_j) = 1$. This gives rise to a cocycle $\{c_{jk} : U_{jk} \to U(1)\}_{j,k \in I}$ defined by

$$s_k = c_{jk} s_j$$

for all $j, k \in I$.

To the curvature form $\tilde{\omega}$ of the connection ∇ is associated a Čech class as above. The local $\tilde{J}$-forms η_j ($j \in I$) give the partial connection with curvature $\tilde{\omega}$ by the formula

$$\nabla s_j = 2\pi i \eta_j \cdot s_j.$$

We use the notation of the proof of Proposition 9.2.1. From this formula one computes, using the Leibniz rule for connections, that

$$d^{\tilde{J}} f_{jk} = \eta_k - \eta_j = \frac{1}{2\pi i} \frac{d^{\tilde{J}} c_{jk}}{c_{jk}}.$$

One can easily show that the fact that ∇ is Hermitian implies that the function f_{jk} must be real-valued. Hence for all $j, k \in I$

$$c_{jk} = e^{2\pi i (f_{jk} + d_{jk})},$$

for a function $d_{jk} : U_{jk} \to \mathbb{R}$ locally constant along the leaves. The d_{jk} constitute a Čech 1-cocycle in $b \in C^1_{\tilde{J}}(\mathcal{U}, \mathbb{R})$. From the fact that $c_{jk} c_{kl} c_{jl}^{-1} = 1$ we deduce that

$$(f_{jk} + d_{jk}) + (f_{jk} + d_{jk}) - (f_{jk} + d_{jk}) \in \mathbb{Z},$$

hence $a - \delta(b) \in \mathbb{Z}$, which implies that $[a]$ is integer.
($\Leftarrow$) Suppose an integer class in $H^2_{J,dR}(S)$ is given. There exist an associated class in $\check{H}^2_{\tilde{J}}(\mathcal{U}, \mathbb{R})$. Choose a representative of this class such that the functions $a_{jkl} := f_{jk} + f_{kl} - f_{jl}$ (as above) have integer value. For all $j, k \in I$ define

$$c_{jk} := e^{2\pi i f_{jk}}$$

This defines is a cocycle, since a_{jkl} is integral, which gives a smooth complex line bundle

$$L = \Big(\bigcup_{j \in I} (U_j \times \mathbb{C})\Big)/\sim,$$

where the equivalence relation is given by $U_j \times \mathbb{C} \ni (m, z) \sim (m, c_{jk} z) \in U_k \times \mathbb{C}$ whenever $m \in U_{jk}$. The connection is given by

$$\nabla s_j = 2\pi i \eta_j \cdot s_j,$$

where s_j denote section $U_j \times \mathbb{C}$ which is constantly equal to 1. The Hermitian structure is given by

$$h((m, z_1), (m, z_2)) = \bar{z}_1 z_2.$$

A computation proves that ∇ is Hermitian with respect to h. □

Remark 9.2.3. One should relate this to the well-known fact that

$$c_1 : \mathrm{Pic}(S) \xrightarrow{\cong} \check{H}^2(S, \mathbb{Z})),$$

where $c_1([L]) := [\tilde{\omega}]$ is the first Chern class, which equals the class of the curvature 2-form of any connection on L. Moreover, as we remarked before, $\check{H}^2(S, \mathbb{Z}) \cong \check{H}^2_{\tilde{J}}(S, \mathbb{Z})$, hence the above proof is very similar to the proof of the fact that c_1 is an isomorphism. It is repeated here since $\tilde{\omega}$ gives rise to an element in $\check{H}^2_{\tilde{J}}(S, \mathbb{R})$ and not in $\check{H}^2(S, \mathbb{R})$ and for expositionary purposes. Summarizing one has the following commuting diagram

$$\begin{array}{ccccccc}
0 \longrightarrow \mathrm{Pic}_{\mathcal{A}}(J) \longrightarrow & \mathrm{Pic}(S) & & \xrightarrow{\quad c_1^{\mathcal{A}} \quad} & & H^2(\mathcal{A} \ltimes J) \\
& \cong \big\downarrow & & & & \big\uparrow \alpha^* \\
& \check{H}^2(S, \mathbb{Z}) & \xrightarrow{\cong} \check{H}^2_{\tilde{J}}(S, \mathbb{Z}) & \longrightarrow & \check{H}^2_{\tilde{J}}(S, \mathbb{R}) & \xrightarrow{\cong} H^2_{\tilde{J},dR}(S).
\end{array}$$

9.3 Prequantization representations

In this section we prove that under suitable assumptions there exists a prequantization representation associated to a Hamiltonian Lie algebroid action. Next, we discuss some examples and some properties of the prequantization representation.

Suppose a regular Lie algebroid $\mathcal{A}$ over M acts on a smooth map $J : S \to M$. Let $\alpha : \Gamma^\infty(\mathcal{A}) \to \mathfrak{X}^\infty_{\tilde{J}}(S)$ denote the action. Suppose that S is endowed with a J-presymplectic $\tilde{J}$-2-form $\tilde{\omega} \in \Omega^2_{\tilde{J}}(S)$. Suppose that the action is Hamiltonian with momentum map

$$\tilde{\mu} : S \to J^*\mathcal{A}^*.$$

Suppose, furthermore, that there exists a smooth complex line bundle $L \to S$ with a Hermitian metric h and a $\tilde{J}$-partial Hermitian connection ∇^L, such that the curvature J-2-form K^{∇_L} equals $\tilde{\omega}$. The triple $(L \to S, \nabla^L, h)$ is called a prequantization of the Hamiltonian action of $\mathcal{A}$ on $(J : S \to M, \tilde{\omega})$. We have seen in Theorem 9.2.2 that a prequantization exists if and only if the cohomology class of $\tilde{\omega}$ is integral.

Theorem 9.3.1. *There exists a Hermitian representation of the Lie algebroid $\mathcal{A}$ on $(L \to M, h)$ given by*

$$\pi(X) := \nabla_{\alpha(X)} - 2\pi i \left\langle \tilde{\mu}, J^*X \right\rangle.$$

Remark 9.3.2. Note that this formula is a generalization of the well-known Kostant formula in classical prequantization theory. The fact that it also applies to Lie algebroids was also used in [88]. Actually one only needs the prequantization condition (8.4.1), which is equivalent to $c_1^{\mathcal{A}}(L) = 0 \in H^2(\mathcal{A} \ltimes J)$. We fill in the last step of the exactness of the sequence 9.1.1.

Proof. For $X, Y \in \Gamma^\infty(\mathcal{A})$ one computes

$$\begin{aligned}
[\pi(X), \pi(Y)]_{\mathcal{D}(L)} - \pi([X, Y]_{\mathcal{A}}) &= [\nabla_{\alpha(X)}, \nabla_{\alpha(Y)}] \\
&\quad + 2\pi i\alpha(Y) \langle \tilde{\mu}, J^* X \rangle - 2\pi i\alpha(X) \langle \tilde{\mu}, J^* Y \rangle \\
&\quad - \nabla_{\alpha[X,Y]} + 2\pi i \langle \tilde{\mu}, J^*[X, Y] \rangle \\
&= 2\pi i K(\alpha(X), \alpha(Y)) + 2\pi i d_{A \ltimes J} \tilde{\mu}(X, Y) \\
&= 2\pi i \omega(\alpha(X), \alpha(Y)) + 2\pi i d_{A \ltimes J} \tilde{\mu}(X, Y) \\
&= 0.
\end{aligned}$$

So π is a homomorphism of Lie algebroids.

The representation being Hermitian is proven by computation. For $\sigma, \tau \in \Gamma^\infty(L)$ and $X \in \Gamma^\infty(\mathcal{A})$,

$$\begin{aligned}
h(\pi(X)\sigma, \tau) + h(\sigma, \pi(X)\tau) &= h(\nabla_{\alpha(X)}\sigma, \tau) - h(2\pi i \langle \mu, J^* X \rangle \sigma, \tau) \\
&\quad + h(\sigma, \nabla_{\alpha(X)}\tau) - h(\sigma, 2\pi i \langle \mu, J^* X \rangle \tau) \\
&= \alpha(X) h(\sigma, \tau)
\end{aligned}$$

since the connection is Hermitian and the metric h is sesquilinear. □

Definition 9.3.3. The above representation $(L \to M, h, \pi) \in \mathrm{Pic}_{\mathcal{A}}(J)$ is the **prequantization representation** of the Hamiltonian action of $\mathcal{A}$ on $J : S \to M$.

Example 9.3.4. The tangent bundle TM of a smooth manifold M is a Lie algebroid over M. It trivially acts on $J = id : M \to M$. The prequantization procedure boils down to a standard situation in differential geometry. Suppose M is endowed with an integral closed 2-form $\tilde{\omega} \in \Omega^2(M)$ (the J-presymplectic form). As we have seen, a momentum map for this action is a 1-form $\mu \in \Omega^1(M)$ satisfying $d\mu = -\tilde{\omega}$. A prequantum line bundle is a complex line bundle $L \to M$ endowed with a Hermitian connection ∇ whose curvature equals ω. The prequantization representation of $\mathcal{A} = TM$ is the flat connection $\nabla - 2\pi i \mu$.

For a regular integrable distribution as a Lie algebroid and its associated foliation $\mathcal{F}$ (cf. Example 8.4.11) a similar reasoning holds with the differential d replaced by a partial differential along the leaves of the foliation $\mathcal{F}$.

Example 9.3.5. Suppose $p : \mathfrak{g} \to M$ is a smooth family of Lie algebras $\mathfrak{g}_m$ ($m \in M$) as in Example 8.3.6. Suppose it acts in a Hamiltonian fashion on a smooth bundle $J : S \to M$ of coadjoint orbits $S_m := \mathcal{O}_m \subset \mathfrak{g}^*_m$. Then the inclusion $S \to \mathfrak{g}^*$ is a momentum map. We have a prequantization line bundle if we can paste prequantization line bundles $L_m \to S_m$ for each $\mathfrak{g}_m$ into a smooth bundle $L \to M$. A Hermitian representation on $L \to M$ is then given by $X \mapsto -2\pi i \langle \tilde{\mu}, J^* X \rangle$, where $\tilde{\mu} : S \to \mathfrak{g}^*$ is the inclusion.

Example 9.3.6. Let H be a Lie group H acting on a symplectic manifold (S, ω^S) in Hamiltonian fashion, with momentum map $\mu : S \to \mathfrak{h}^*$, and let $\pi : P \to M$ be a principal bundle endowed with an $\mathfrak{h}$-valued connection 1-form τ. The connection induces a decomposition of TP into a direct sum $\mathcal{H} \oplus \mathcal{V}$ of a horizontal bundle $\mathcal{H} := \ker(\tau)$ and a vertical bundle $\mathcal{V} := \ker(T\pi)$. In Example 8.4.13 we defined a

J-presymplectic 2-form $\tilde{\omega}$ on $J : S' := P \times_H S \to M$ and proved that the action of the gauge groupoid $P \times_H P \rightrightarrows M$ on $(J : S' \to M, \tilde{\omega})$ is Hamiltonian.

Suppose $(\pi_L : L \to S, \nabla^L, h)$ is a prequantization for the action of H on S. Consider the line bundle $P \times_H L \to P \times_H S$. We shall show that this forms a prequantization line bundle for the action of $P \times_H P \rightrightarrows M$ on $S' \to M$.

First, we shall explain the line bundle structure on $\pi : P \times_H L \to P \times_H S$. The map

$$\pi([p, z]) := [p, \pi_L(z)],$$

is well-defined, since π_L is H-equivariant. Addition is defined by finding representatives with equal first entry (this is always possible) and then adding the second entry, i.e.

$$[p, z_1] + [p, z_2] := [p, z_1 + z_2].$$

Scalar multiplication is defined by scalar multiplication on the second entry

$$\lambda[p, z] := [p, \lambda z].$$

A section $\theta \in \Gamma^\infty(P \times_H L)$ is represented by a pair (θ_1, θ_2) of H-equivariant maps $\theta_1 : P \times S \to P$ and $\theta_2 : P \times S \to L$, such that $\theta_1(p, \sigma) = h'(\sigma)p$ for some map $h' : S \to H$, and $\pi_L\theta_2$ equals the projection $P \times S \to S$. Indeed, this is the case iff $(\theta_1, \theta_2) : P \times S \to P \times L$ induces a section $P \times_H S \to P \times_H L$.

Since $(\pi_L : L \to S, \nabla^L, h)$ is a prequantization for the action of $\mathfrak{h}$ on (S, ω^S), the curvature of ∇^L equals ω^S and the representation of $\mathfrak{h}$ on $\Gamma^\infty(L)$ is given by Kostant's formula

$$X \mapsto \nabla^L_{\beta(X)} - 2\pi\, i \,\langle \mu, X \rangle\, .$$

We identify $T(P \times_H S) \cong (TP \times_H TS)/\!\sim$, cf. Example 8.3.8. For each H-equivariant vector field v on S and each H-equivariant vector field w on P, let $[v, w]$ denote the vector field induced on $P \times_H S$. For each $\theta = (\theta_1, \theta_2) \in \Gamma^\infty(P \times_H L)$ and $[w, v] \in \mathfrak{X}^\infty(P \times_H S)$, define

$$\nabla_{[w,v]}\theta := \big(\theta_1, \nabla^L_{v-\beta(\tau(w))}\theta_2 - \tau_h(w) \cdot \theta_2\big),$$

where $(\nabla^L\theta_2)(p, \sigma) := \nabla^L(\theta_2(p, \cdot))(\sigma)$ and $\tau_h(w) \in \mathcal{H}$ is the horizontal projection $w - \alpha(\tau(w))$ of w. Suppose that H is connected (we need this for equivariance of the connection ∇^L, cf. Corollary 9.3.11).

Lemma 9.3.7. *∇ is a connection on $P \times_H L$ with curvature $\tilde{\omega}$.*

Proof. First we check that ∇ is well-defined. Indeed, $\nabla_{[\alpha(X),\beta(X)]} = 0$, since $\tau(\alpha(X)) = X$ and $\tau_h(\alpha(X)) = 0$ and

$$\begin{aligned} h \cdot \nabla_{[w,v]}\theta &= \big(h \cdot \theta_1, h\nabla^L_{v-\beta(\tau(w))}\theta_2 - h \cdot (\tau_h(w)) \cdot \theta_2\big) \\ &= \big(\theta_1, \nabla^L_{h\cdot(v-\beta(\tau(w)))} h \cdot \theta_2 - \tau_h(w) \cdot \theta_2\big) \\ &= \big(\theta_1, \nabla^L_{v-\beta(\tau(w))} \cdot \theta_2 - \tau_h(w) \cdot \theta_2\big), \end{aligned}$$

by H-equivariance of $\theta, \nabla, w, v, \beta, \alpha$ and τ.

It is easy to check that ∇ is a connection. For example, for an H-invariant function $f \in C^\infty(P \times S)^H$ one computes

$$\begin{aligned}\nabla_{[w,v]} f\theta &= \big(\theta_1, \nabla^L_{v-\beta(\tau(w))} f\theta_2) + \tau_h(w) f\theta_2\big)\\ &= \Big(\theta_1, \big(f\nabla_{[w,v]} + \big(v - \beta(\tau(w))\big)\cdot f + \big(w - \alpha(\tau(w))\big)\cdot f\big)\theta_2\Big)\\ &= (f\nabla_{[w,v]}\theta + (w+v)f)\theta,\end{aligned}$$

since $(\alpha(\tau(w)) + \beta(\tau(w)))\cdot f = 0$ by H-invariance of f.

Now we shall compute the curvature of ∇. Note the two ways in which the brackets $[,]$ are used, namely as a commutator bracket and as a way to denote equivalence classes. Let $[w_1, v_1], [w_2, v_2]$ be vector fields on $P \times_H S$ and $\theta = (\theta_1, \theta_2) \in \Gamma^\infty(P \times_H L)$. We compute

$$\begin{aligned}\big[\nabla_{[w_1,v_1]}, \nabla_{[w_2,v_2]}\big]\theta = \Big(\theta_1, \big([\nabla^L_{v_1-\beta(\tau(w_1))}, \nabla^L_{v_2-\beta(\tau(w_2))}] + [\tau_h(w_1), \nabla^L_{v_2-\beta(\tau(w_2))}]\\ + [\nabla^L_{v_1-\beta(\tau(w_1))}, \tau_h(w_2)] + [\tau_h(w_1), \tau_h(w_2)]\big)\,\theta_2\Big)\end{aligned} \tag{9.3.3}$$

Note that

$$\begin{aligned}[\tau_h(w_1), \nabla^L_{v_2-\beta(\tau(w_2))}] &= \nabla^L_{\tau_h(w_1)\cdot\beta(\tau(w_2))}\\ &= \nabla^L_{\tau(w_1)\cdot\beta(\tau(w_2))},\end{aligned}$$

and in the same way we obtain

$$[\nabla^L_{v_1-\beta(\tau(w_1))}, \tau_h(w_2)] = -\nabla^L_{\tau(w_2)\cdot\beta(\tau(w_1))}.$$

We shall also need that

$$\begin{aligned}[\alpha(\tau(w_1)), w_2] &= -\alpha(w_2\cdot\tau(w_1));\\ [w_1, \alpha(\tau(w_2))] &= \alpha(w_1\cdot\tau(w_2));\\ [v_1, \beta(\tau(w_2))] &= 0;\\ [\beta(\tau(w_1)), v_2] &= 0,\end{aligned}$$

as follows from H-equivariance of w_1, w_2, v_1 and v_2.

On the other hand,

$$\begin{aligned}\nabla_{[[w_1,v_1],[w_2,v_2]]_{TP\times TS}}\theta &= \nabla_{[[w_1,w_2]_{TP},[v_1,v_2]_{TS}]}\theta\\ &= (\theta_1, \nabla^L_{[v_1,v_2]_{TS}-\beta(\tau([w_1,w_2]_{TP}))}\theta_2 - \tau_h([w_1,w_2]_{TP})\theta_2).\end{aligned} \tag{9.3.4}$$

Using the well-known formula

$$\begin{aligned}F(w_1, w_2) &= d\tau(w_1, w_2) + [\tau, \tau]_{\mathfrak{h}}(w_1, w_2)\\ &= \tau([w_1, w_2]) - w_1\cdot(\tau(w_2)) + w_2\cdot(\tau(w_1)) + [\tau(w_1, \tau(w_2)]_{\mathfrak{h}},\end{aligned}$$

we continue the calculation of (9.3.4)

$$\begin{aligned}\nabla_{[[w_1,v_1],[w_2,v_2]]_{TP\times TS}}\theta = \Big(\theta_1, (\nabla^L_{[v_1,v_2]_{TS}-\beta\left(F(w_1,w_2)+w_1\cdot\tau(w_2)-w_2\cdot\tau(w_1)-[\tau(w_1,\tau(w_2)]_{\mathfrak{h}}\right)} \\ + [w_1,w_2] - \alpha(F(w_1,w_2) + w_1\cdot\tau(w_2) - w_2\cdot\tau(w_1) \\ - [\tau(w_1,\tau(w_2)]_{\mathfrak{h}})\big)\,\theta_2\Big). \qquad (9.3.5)\end{aligned}$$

Note that θ_2 is equivariant, hence for any $X \in \mathfrak{h}$ one has

$$X\cdot\theta_2 = (\nabla^L_{\beta(X)} - 2\pi i\,\langle\mu, X\rangle + \alpha(X))\theta_2 = 0$$

In particular, this is true for $X = F(w_1, w_2)$.

Subtracting the identity (9.3.5) from (9.3.3) one obtains the curvature, using all the given equalities, namely

$$\begin{aligned}K([w_1,v_1],[w_2,v_2]) &= \frac{1}{2\pi i}\big([\nabla_{[w_1,v_1]},\nabla_{[w_2,v_2]}] - \nabla_{[[w_1,v_1],[w_2,v_2]]_{TP\times TS}}\big) \\ &= \frac{1}{2\pi i}\Big([\nabla^L_{v_1-\beta(\tau(w_1))},\nabla^L_{v_2-\beta(\tau(w_2))}] \\ &\quad - \nabla^L_{[v_1-\beta(\tau(w_1)),v_2-\beta(\tau(w_2))]_{TS}} - 2\pi i\,\langle\mu, F(w_1,w_2)\rangle\Big) \\ &= \omega\big(v_1-\beta(\tau(w_1)), v_2-\beta(\tau(w_2))\big) - \langle\mu, F(w_1,w_2)\rangle \\ &= \tilde{\omega}([w_1,v_1],[w_2,v_2]).\end{aligned}$$

This finishes the proof. □

A Hermitian metric h' on $P\times_H L$ is given by

$$h'([p,z_1],[p,z_2]) := h(z_1,z_2).$$

This is well-defined since the representation of H on L is unitary.

Lemma 9.3.8. *The connection ∇ on $P\times_H L \to S'$ is Hermitian with respect to Hermitian metric h'.*

Proof. This follows by computation: (in the notation introduced previously and $\theta = (\theta_1,\theta_2)$, $\theta' = (\theta'_1,\theta'_2)$)

$$\begin{aligned}h'(\nabla_{[w,v]}\theta,\theta') + h'(\theta,\nabla_{[w,v]}\theta') &= h(\nabla^L_{v-\beta(\tau(w))}\theta_2,\theta'_2) + h(\tau_h(w)\cdot\theta_2,\theta'_2) \\ &\quad + h(\theta_2,\nabla^L_{v-\beta(\tau(w))}\theta'_2) + h(\theta_2,\tau_h(w)\cdot\theta'_2) \\ &= (v-\beta(\tau(w)))\cdot h(\theta_2,\theta'_2) + \tau_h(w)\cdot h(\theta_2,\theta'_2) \\ &= [w,v]\cdot h'(\theta,\theta'),\end{aligned}$$

where in the third line we used the fact that ∇^L is Hermitian and in the last that θ_1, θ_2, θ'_1 and θ'_2 are H-equivariant. □

Corollary 9.3.9. *The triple $(P\times_H L \to P\times_H S, \nabla, h')$ is a prequantization for the action of $P\times_H P \rightrightarrows M$ on $(P\times_H S\to M,\tilde{\omega})$, with prequantization representation*

$$\mathcal{A}(P\times_H P)\cong TP/H \to \mathcal{D}(P\times_H L)$$

given by

$$\begin{aligned} w \mapsto \nabla_{\gamma(w)} - 2\pi i \langle \tilde{\mu}, w \rangle \\ = \nabla_{[w,0]} - 2\pi i \langle \mu, \tau(w) \rangle . \end{aligned}$$

Note that in Lemma 9.3.7 we have used the H-equivariance of the connection ∇^L. We shall now prove a more general result for source-connected Lie groupoids.

Lemma 9.3.10. *Consider the situation of Theorem 9.3.1. For any prequantization representation of $\mathcal{A}$ on a line bundle $L \to S$, the given connection ∇ on L is $\mathcal{A}$-equivariant.*

Proof. One computes for any $v \in \mathfrak{X}^\infty_{\tilde{J}}(S)$

$$\begin{aligned} [\pi(X), \nabla_v] &= [\nabla_{\alpha(X)}, \nabla_v] + 2\pi i \langle \tilde{\mu}, J^* X \rangle \\ &= 2\pi i \omega(\alpha(X), v) + \nabla_{[\alpha(X),v]} + 2\pi i v \cdot \langle \mu, J^* X \rangle \\ &= \nabla_{[\alpha(X),v]}, \end{aligned}$$

which means exactly that ∇ is $\mathcal{A}$-equivariant. □

The corollary that we tacitly used in the proof of Lemma 9.3.7 (in the particular case that $G \rightrightarrows M$ is a Lie group) is

Corollary 9.3.11. *If $G \rightrightarrows M$ is a source-connected Lie groupoid integrating $\mathcal{A}$, then ∇ is equivariant, in the sense that for $v \in T^J_\sigma S$ and each $g \in G_{J(\sigma)}$*

$$\nabla_{gv} = g \nabla_v g^{-1}.$$

Proof. Choose a connection ∇' on $\mathcal{A}$. Then there exists an exponential map $\exp_{\nabla'}$: $\mathcal{A} \to G$ (cf. [43]) Differentiating the expression

$$\nabla_{\exp_{\nabla'}(\tau X) v} = \exp_{\nabla'}(\tau X) \nabla_v \exp_{\nabla'}(-\tau X).$$

at $\tau = 0$ gives the equality in the proof above. □

9.4 Integrating prequantization representations

In this section we discuss the integrability of Lie algebroid representations. In particular, we consider the examples from the previous section.

Not every Lie algebroid integrates to a Lie groupoid. Precise conditions for the existence of an integrating Lie groupoid for a given Lie algebroid are given in [17]. Suppose $\mathcal{A}$ is a Lie algebroid and $\alpha' : \mathcal{A} \ltimes J \to \mathcal{D}(L)$ a Hermitian representation (e.g. obtained by prequantization). One would like to integrate such a representation to a representation of a Lie groupoid which has associated Lie algebroid $\mathcal{A}$.

Definition 9.4.1. A **representation of a Lie groupoid** $G \rightrightarrows M$ on a smooth complex vector bundle $E \to M$ is a smooth action of G on $\pi : E \to M$

$$\pi : G \,{}_s\!\times_\pi E \to E$$

that is linear, i.e.

$$\pi(g, \lambda \cdot e) = \lambda \cdot \pi(g, e)$$

and

$$\pi(g, e + f) := \pi(g, e) + \pi(g, f)$$

for all $g \in G$, $\lambda \in \mathbb{C}$ and $e, f \in E_{s(g)}$.

Remark 9.4.2. This notion generalizes the notion of H-equivariant vector bundle for a Lie group H. In particular, it is a continuous representation on a locally trivial field of Hilbert spaces $(\{E_m\}_{m\in M}, \Gamma_0(E))$ as discussed in Part II.

The representation π is **unitary** with respect to a Hermitian metric h on E if it preserves h, i.e.

$$h(\pi(g, e), \pi(g, f)) = h(e, f),$$

for all $g \in G$ and $e, f \in E_{s(g)}$. A unitary representation π can equivalently be given by a morphism of groupoids $G \to U(E)$, where $U(E)$ is the Lie groupoid of linear unitary maps $E_m \to E_n$ for all $m, n \in M$ (cf. Example 2.1.8).

Suppose G acts on a map $J : N \to M$. Suppose $E \to N$ a smooth complex vector bundle endowed with a Hermitian structure h.

Definition 9.4.3. A **(unitary)representation of** $G \rightrightarrows M$ on a smooth complex vector bundle $E \to N$ is a (unitary) representation of the action groupoid $G \ltimes J$ on $\pi : E \to N$.

Suppose $\mathcal{A}$ is integrable and $J : S \to M$ is proper, then by Proposition 3.5 and Proposition 5.3 in [56] the representation $\pi : \mathcal{A} \ltimes J \to \mathcal{D}(E)$ of the Lie algebroid $\mathcal{A}$ on the vector bundle $E \to S$ integrates to a unitary representation $G \ltimes J \to U(L)$ of the source-simply connected integrating Lie groupoid G of $\mathcal{A}$ on $E \to S$. The condition that J is proper will also arise in the next section about the quantization procedure. Note that one can prove that a proper smooth family of manifolds is a fiber bundle.

Example 9.4.4. A flat connection $\nabla - 2\pi i\mu$ on a line bundle $L \to M$ is a prequantization representation of TM as a Lie algebroid acting Hamiltonianly on (M, ω) as in Example 9.3.4. It integrates to a representation of a source-simply connected Lie groupoid integrating TM, for example the fundamental groupoid $\pi_1(M)$ of M. The representation is the parallel transport associated to the connection $\nabla - 2\pi i\mu$.

The only prequantization that lifts to a representation of the pair groupoid $M \times M$ (which also integrates TM) is the representation $d - 2\pi i\mu$ on the trivial line bundle $M \times \mathbb{C} \to M$.

Example 9.4.5. Recall the situation of Example 9.3.6. There exists a canonical unitary representation $\bar{\pi}$ of $P \times_H P \rightrightarrows M$ on $P \times_H L \to M$, defined by

$$\bar{\pi}([p, q])[q, z] := [p, z]$$

for suitable representatives.

Proposition 9.4.6. *The representation $\bar{\pi}$ integrates the prequantization representation given by*

$$w \mapsto \nabla_{\gamma(w)} - 2\pi\, i \, \langle \tilde{\mu}, w \rangle$$

(cf. Theorem 9.3.1), where ∇ is the connection we have defined in Lemma 9.3.7 and $\tilde{\mu} : P \times_H S \to T^*P/H$ is the momentum map for the Hamiltonian action of $P \times_H P \rightrightarrows M$ on $P \times_H S \to M$ (cf. Proposition 8.4.16).

Note therefore, that $\bar{\pi}$ does not depend on the chosen connection τ.

Proof. Suppose that $w \in \mathfrak{X}^\infty(P)$ is H-equivariant. It represents a smooth section in $\Gamma^\infty(\mathcal{A}(P \times_H P)) \cong \Gamma^\infty(TP/H)$. Note that the infinitesimal representation $\bar{\pi}' : \mathcal{A}(P \times_H P) \to \mathcal{D}(L)$ associated to the representation $\bar{\pi}$ is given by

$$\bar{\pi}'(w)\theta = (\theta_1, w\theta_2).$$

On the other hand,

$$\begin{aligned}
\pi(w)\theta &= \nabla_{\gamma(w)}\theta - 2\pi i \, \langle \tilde{\mu}, w \rangle \, \theta \\
&= \nabla_{[w,0]}\theta - 2\pi i \, \langle \tilde{\mu}, w \rangle \, \theta \\
&= (\theta_1, \nabla^L_{-\beta(\tau(w))}\theta_2 + (w - \alpha(\tau(w)))\theta_2 - 2\pi i \, \langle \mu, \tau(w) \rangle \, \theta_2) \\
&= (\theta_1, w \cdot \theta_2),
\end{aligned}$$

from which the lemma follows. ∎

Chapter 10

Quantization and symplectic reduction

10.1 Quantization through Kähler polarization

In this section we introduce Kähler quantization of Hamiltonian Lie algebroid actions. Next, we discuss the examples which we have been considering throughout this part of the thesis.

Suppose a regular Lie algebroid $(p : \mathcal{A} \to M, \rho)$ acts in a Hamiltonian fashion on a smooth surjective submersion $J : S \to M$, with a J-presymplectic 2-form $\tilde{\omega} \in \Omega^2_{\tilde{J}}(S)$, where $\tilde{J} : S \to M/\rho$. Denote the action by $\alpha : \Gamma^\infty(\mathcal{A}) \to \mathfrak{X}^\infty_{\tilde{J}}(S)$. Let $\tilde{\mu} : S \to J^*\mathcal{A}^*$ be a momentum map.

In this section we shall make the additional assumption that $J : S \to M$ is a smooth bundle of compact connected Kähler manifolds. Denote the almost complex structure by

$$j : T^J S \to T^J S.$$

The following conditions are satisfied: $\omega(j\cdot, \cdot) > 0$ and $\omega(j\cdot, j\cdot) = \omega$, where $\omega := \tilde{\omega}|_{T^J S}$.

Our final assumption is that the almost complex structure j is $\mathcal{A}$-equivariant, in the sense that

$$[\alpha(X), j(v)]_{T^{\tilde{J}}S} = j[\alpha(X), v]_{T^{\tilde{J}}S},$$

for all $X \in \Gamma^\infty(\mathcal{A})$ and $v \in \mathfrak{X}^\infty_J(S)$.

Let $T^{J,\mathbb{C}}S \to S$ denote the complexification $T^J S \otimes \mathbb{C} \to S$ of $T^J S \to S$. The complex extension of j is denoted by $j_{\mathbb{C}} : T^{J,\mathbb{C}}S \to T^{J,\mathbb{C}}S$.

Definition 10.1.1. The **Kähler polarization** $\mathcal{P}(S, j)$ of $(J : S \to M, \omega)$ is defined by

$$\mathcal{P}(S, j) := \{v \in T^{J,\mathbb{C}}S \mid j_{\mathbb{C}}(v) = -iv \in T^{J,\mathbb{C}}S\}.$$

Smooth sections of $\mathcal{P}(S, j)$ are called **polarized** sections of $T^{J,\mathbb{C}}(S)$.

Assume $(L \to S, \nabla^L, h)$ is a prequantization line bundle for the action of $\mathcal{A}$ on $(J : S \to M, \tilde{\omega})$. Denote the associated representation (see Section 9.3) by $\pi : \mathcal{A} \ltimes J \to \mathcal{D}(L)$.

Definition 10.1.2. The **geometric quantization** of a prequantization $(L \to S, \nabla, h)$ of the Hamiltonian action α of $\mathcal{A}$ on $(J : S \to M, \tilde{\omega})$ is given by

$$\Delta_Q^0 := \{\sigma \in \Gamma_c^\infty(L) \mid \nabla_v \sigma = 0 \text{ for all } v \in \Gamma^\infty(\mathcal{P}(S, j))\}.$$

We call the sections of $L \to S$ in Δ_Q^0 **holomorphic**. The space Δ_Q^0 is a $C_c^\infty(M)$-module, with $C_0(M)$-valued inner product

$$\langle \sigma, \sigma' \rangle (m) := \int_{S_m} h(\sigma, \sigma')\Omega_m,$$

where $\sigma, \sigma' \in \Gamma_c^\infty$ and $\{\Omega_m\}_{m \in M}$ is a smooth family of densities on $J : S \to M$ defined by

$$\Omega_m := \omega_m^{d_m}/(d_m!),$$

$(d_m := \dim(S_m)/2)$. The closure $\Delta_Q = \bar{\Delta}_Q^0$ is a Hilbert $C_0(M)$-module. Hence, it corresponds to a continuous field of Hilbert spaces $(\mathcal{H}_Q, \Delta_Q)$ over M. The continuous field $(\mathcal{H}_Q, \Delta_Q)$ is finite-dimensional, since S_m is compact and

$$\{\sigma \in \Gamma^\infty(S_m, L_m) \mid \nabla_v \sigma = 0 \text{ for all } v \in \Gamma^\infty(\mathcal{P}(S, j)|_{S_m})\}$$

is finite-dimensional for all $m \in M$.

Theorem 10.1.3. *The geometric quantization continuous field of Hilbert spaces $(\mathcal{H}_Q, \Delta_Q)$ carries a Hermitian representation of $\mathcal{A}$ (cf. Definition 9.1.9).*

Proof. We check that the representation π of $\mathcal{A}$ on $L \to M$ restricts to Δ_Q^0. From this the theorem follows. Suppose $\nabla_v \sigma = 0$ for all $v \in \Gamma^\infty(\mathcal{P}(S, j))$. Note that $\mathcal{A}$-equivariance of j implies that $[\alpha(X), v] \in \mathcal{P}(S, j)$ whenever $v \in \Gamma^\infty(\mathcal{P}(S, j))$. Indeed, suppose $v \in \Gamma^\infty(\mathcal{P}(S, j))$ and $X \in \Gamma^\infty(\mathcal{A})$ then

$$\begin{aligned} j_{\mathbb{C}}[\alpha(X), v] &= [\alpha(X), j_{\mathbb{C}} v] \\ &= [\alpha(X), -iv] \\ &= -i[\alpha(X), v]. \end{aligned}$$

Hence, the Δ_Q^0 is $\mathcal{A}$-invariant: for $\sigma \in \Delta_Q^0$

$$\begin{aligned} \nabla_v(\pi(X)\sigma) &= \nabla_v(\nabla_{\alpha(X)}\sigma - 2\pi i \langle \tilde{\mu}, J^* X \rangle \sigma) \\ &= \nabla_{\alpha(X)} \nabla_v \sigma - \nabla_{[\alpha(X), v]}\sigma - 2\pi i\, \tilde{\omega}(\alpha(X), v)\sigma \\ &\quad + 2\pi i \langle \tilde{\mu}, J^* X \rangle \nabla_v \sigma - 2\pi i (v \cdot \langle \tilde{\mu}, J^* X \rangle)\sigma \\ &= -2\pi i v \lrcorner (d^J \langle \tilde{\mu}, J^* X \rangle + \alpha(X) \lrcorner \tilde{\omega})\sigma \\ &= 0, \end{aligned}$$

by the quantization condition (8.4.2).

One easily sees that the representation is Hermitian since the prequantization representation is Hermitian and ω is invariant. Indeed, for all $m \in M$,$\sigma, \sigma' \in \Gamma^\infty(L)$

and $X \in \Gamma^\infty(\mathcal{A})$

$$\begin{aligned}
(\langle \pi(X)\sigma, \sigma' \rangle + \langle \sigma, \pi(X)\sigma' \rangle)(m) &= \int_{S_m} h(\pi(X)\sigma, \sigma') + h(\sigma, \pi(X)\sigma')\Omega_m \\
&= \int_{S_m} \alpha(X) \cdot h(\sigma, \sigma')\Omega_m \\
&= \big((TJ \circ \alpha(X)) \cdot (m' \mapsto \int_{S_{m'}} h(\sigma, \sigma')\Omega_{m'})\big)(m) \\
&= (TJ \circ \alpha(X)) \langle \sigma, \sigma' \rangle (m),
\end{aligned}$$

which finishes the proof. □

Remark 10.1.4. Of course, one might wonder whether one can integrate this representation to a representation of an integrating Lie groupoid. If $G \rightrightarrows M$ is a source-connected Lie groupoid integrating $\mathcal{A}$, then every quantization representation $(\mathcal{H}_Q, \Delta_Q)$ of $\mathcal{A}$ integrates to a continuous representation of $G \rightrightarrows M$ on $(\mathcal{H}_Q, \Delta_Q)$, for which Δ_Q^0 are the smooth sections.

Remark 10.1.5. We can endow the algebra $C^\infty(S)$ with a Poisson bracket using the smooth family of symplectic forms $\omega \in \Omega^2_J(S)$. For $f \in C^\infty(S)$ define the **Hamiltonian vector field** $X_f \in \mathfrak{X}^\infty_J(S)$ of f by

$$X_f \lrcorner\, \omega = d^J f$$

For $f, g \in C^\infty(S)$, $m \in M$ and $\sigma \in S_m$ define

$$\{f, g\} = \omega(X_f, X_g),$$

There is a quantization representation

$$Q : C^\infty(S) \to \operatorname{End}(\Delta_Q^0)$$

given by

$$f \mapsto \nabla_{X_f} - 2\pi\, i\, f.$$

This is known as family quantization. We have extended this to be able to consider actions of a Lie algebroid on $J : S \to M$ and to obtain a representations of $\mathcal{A}$ on the geometric quantization.

Example 10.1.6. In the case of the Hamiltonian action of an integrable distribution $T\mathcal{F} \subset TM \to M$ on $(J : M \to M, \omega)$, with momentum map $\mu : M \to \mathcal{F}^*$, a prequantization representation on a line bundle $L \to M$ with metric h and Hermitian connection ∇ is given by $\nabla - 2\pi i \mu$, where $d^{\mathcal{F}}\mu = -\omega$. Obviously, the quantization procedure is empty in this situation, since the fibers of J are points.

Example 10.1.7. If $\mathfrak{g} \to M$ is a smooth family of Lie algebras that acts in a Hamiltonian fashion on a bundle of symplectic manifolds $S \to M$ and there is a prequantization (L, ∇, h), then Kähler quantization is family Kähler quantization.

Example 10.1.8. Suppose H is a Lie group that acts in a Hamiltonian fashion on a symplectic manifold (S, ω_S) with momentum map μ. Suppose (L, ∇^L, h) is a prequantization of this action. Furthermore, suppose that P is a principal H-bundle, endowed with an $\mathfrak{h}$-valued connection 1-form τ. In Example 8.4.13 it was shown that there exists a closed form $\tilde{\omega}$ on $S' := P \times_H S$, such that the action of the gauge groupoid $P \times_H P$ on $(S', \tilde{\omega})$ is Hamiltonian. In Example 9.3.6 it was shown that there exists a prequantization $(P \times_H L \to S', \nabla, h')$ of this action.

Suppose that S is a compact Kähler manifold with H-equivariant almost complex structure $j : TS \to TS$.

Lemma 10.1.9. *The almost complex structure j induces a $(P \times_H P)$-equivariant family of almost complex structures*

$$j' : T^J S' \to T^J S'$$

on $J : S' \to M$.

Proof. We shall use the isomorphism $T^J(P \times_H S) \cong P \times_H TS$ from Proposition 8.3.11. Define the almost complex structure as a map $j' : P \times_H TS \to P \times_H TS$ by

$$j'([p, v]) = [p, j(v)].$$

This is obviously an almost complex structure:

$$j'(j'([p, v])) = [p, j(j(v))] = [p, -v] = -[p, v],$$

which is $P \times_H P$-equivariant by the computation

$$j'([p, q] \cdot [q, v]) = j'([p, v]) = [p, j(v)] = [p, q] \cdot [q, j(v)] = [p, q] \cdot j'([q, v]).$$

□

From the lemma we conclude that $J : S' \to M$ is a bundle of Kähler manifolds. So there exists a Kähler quantization $(\mathcal{H}_Q, \Delta_Q)$ of the prequantization

$$(P \times_H L \to S', \nabla, h')$$

of the action of $P \times_H P \rightrightarrows M$ on $(S' \to M, \tilde{\omega})$. This continuous fields of Hilbert spaces is in this case locally trivial, hence a vector bundle, that we denote by $Q' \to M$.

Let Q denote the representation space obtained by quantization of the action of H on S with prequantization (L, ∇^L, h). The associated vector bundle $P \times_H Q \to M$ carries a canonical representation of $P \times_H P \rightrightarrows M$.

Proposition 10.1.10. *The vector bundle $P \times_H Q \to M$ is $(P \times_H P)$-equivariantly isomorphic to the quantization bundle $Q' \to M$.*

Proof. Note that by definition there is a bijection between sections of $Q' \to M$ and holomorphic sections of $P \times_H L \to S'$.

Lemma 10.1.11. *There exists a canonical fiberwise linear bijection*

$$P \times_H \Gamma^\infty(L) \to \Gamma^\infty(P \times_H L),$$

where $\Gamma^\infty(P \times_H L)$ *is thought of as a vector bundle with fiber at* m *given by*

$$\Gamma^\infty(P \times_H L|_{P_m \times_H S}).$$

Proof. A bundle morphism $\Psi : P \times_H \Gamma^\infty(L) \to \Gamma^\infty(P \times_H L)$ is defined by

$$(m, [p, \eta]) \mapsto \Big(m, \big([p', \sigma] \mapsto \big[p, \eta\big((p(p')^{-1}) \cdot \sigma\big)\big]\big)\Big),$$

where $m \in M$, $p \in \pi^{-1}(m)$, $\eta \in \Gamma^\infty(L)$ and $p'p^{-1}$ is the unique element in H such that $(p'p^{-1}) \cdot p = p'$. This is well-defined. Indeed, for fixed $p \in P$ one has

$$\begin{aligned}[h\,p', h\,\sigma] &\mapsto \big[p, \eta\big((p(h\,p')^{-1}) \cdot h\sigma\big)\big] \\ &= \big[p, \eta\big((p(p')^{-1}) \cdot \sigma\big)\big]\end{aligned}$$

and

$$\begin{aligned}[h\,p, h \cdot \eta] &\mapsto \Big([p', \sigma] \mapsto \big[h\,p, h\,\eta(h^{-1}((h\,p)(p')^{-1}) \cdot \sigma)\big]\Big) \\ &= ([p', \sigma] \mapsto [p, \eta((p(p')^{-1}) \cdot \sigma)]).\end{aligned}$$

The map Ψ is obviously linear. A two-sided inverse is as follows. Suppose (θ_1, θ_2) is a section of $P_m \times_H L \to P_m \times_H S$ (cf. 9.3.6). Define a map $\Phi : \Gamma^\infty(P \times_H L) \to P \times_H \Gamma^\infty(L)$ by

$$[m, (\theta_1, \theta_2)] \mapsto \Big(\sigma \mapsto \big(m, \big[p, (p(\theta_1(p, \sigma))^{-1}) \cdot \theta_2(p, \sigma)\big]\big)\Big),$$

for a chosen $p \in P$. Straightforward calculations using the equivariance of θ_1 and θ_2 show that this is independent of the choice of $p \in P_m$ and that $\Psi \circ \Phi = 1$ and $\Phi \circ \Psi = 1$. □

On sections one obtains, for a smooth map $(m, \sigma) \mapsto \eta_m(\sigma)$ from $M \times S$ to Q and a smooth section $\xi \in \Gamma^\infty(P)$, a smooth section $\Psi(\xi, \eta) \in \Gamma^\infty(P \times_H L)$ given by

$$[p, \sigma] \mapsto [\xi(\pi(p)), \eta_{\pi(p)}((\xi(\pi(p))p^{-1}) \cdot \sigma)].$$

One easily checks that the sections $\eta_m \in \Gamma^\infty(L)$ are holomorphic for all $m \in M$ iff the image $\Psi([\xi, \eta])$ is holomorphic. Indeed, for $[\xi', v'] \in \mathcal{P}(S', j')$, with $\xi' : P \times S \to P$ and $v' : P \times S \to T^{\mathbb{C}}S$ such that $v'(p)$ is polarized for each $p \in P$, we have

$$\begin{aligned}\nabla_{[\xi', v']}\Psi(\xi, \eta)(p, \sigma) &= [\xi(p, \sigma), \nabla_{(\xi\xi'^{-1})v'(p, \cdot)}(\eta \circ \beta(\xi(\pi(p))p^{-1})(\sigma)] \\ &= [\xi(\pi(p)), 0] \\ &= 0,\end{aligned}$$

by equivariance of ∇^L. The reverse statement is proven by the same formula. □

Remark 10.1.12. We shall sketch a more general view of geometric quantization based on [47] and [29] and references in these papers. If the line bundle $L \to S$ is positive enough, then the quantization $(\mathcal{H}_Q, \Delta_Q)$ equals the index of a continuous G-equivariant family of Dolbeault-Dirac operators

$$\{\bar{\partial}_{L_m} + \bar{\partial}^*_{L_m} : \Omega^{0,\mathrm{even}}(S_m; L_m) \to \Omega^{0,\mathrm{odd}}(S_m; L_m)\}_{m \in M}$$

constructed from the connection ∇^L on the line bundle $L \to S$ and the family of almost complex structures on S.

For now, suppose that $G \rightrightarrows M$ is locally compact, σ-compact, endowed with a Haar system and that the action of $G \rightrightarrows M$ on $J : S \to M$ is proper. Then this family of Dolbeault-Dirac operators gives rise to a cycle in Kasparov's G-equivariant bivariant K-theory

$$[\{\bar{\partial}_{L_m} + \bar{\partial}^*_{L_m}\}_{m \in M}] \in KK_0^G(C_0(S), C_0(M)),$$

cf. Section 4.5 and Section 12.4 and e.g., [12], [78] or [64]. The Baum-Connes analytical assembly map for groupoids

$$\mu : KK_0^G(C_0(S), C_0(M)) \to K_0(C_r^*(G))$$

maps the class $[D] := [\{\bar{\partial}_{L_m} + \bar{\partial}^*_{L_m}\}_{m \in M}]$ to a class in the K-theory of the C^*-algebra $C_r^*(G)$ of the groupoid $G \rightrightarrows M$.

A different way to look at geometric quantization is to define $\mu([D])$ to be the geometric quantization of the Hamiltonian action of $G \rightrightarrows M$ on $(J : S \to M, \tilde{\omega})$. Under certain conditions, including G being a proper groupoid, $K_0(C_r^*(G))$ is isomorphic to the representation ring of G (cf. Theorem 6.6.10 and [79]). Hence, geometric quantization in this sense will then still yield (a formal difference of) representations of G. This approach gives new possibilities to generalize geometric quantization. Instead of requiring $J : S \to M$ to be a bundle of compact Kähler manifolds, one requires $J : S \to M$ to be endowed with a G-equivariant family of Spin^c-structures. One then proceeds by defining the geometric quantization as the image under the analytical assembly map of KK-cycle defined by the associated family of Spin^c-Dirac operators coupled to the prequantization line bundle. This generalizes the notion of family Spin^c-quantization (cf. [95]).

We shall generalize this approach in a different direction in Part IV.

10.2 Symplectic reduction

In this section we discuss a generalization of symplectic reduction to our setting. We reduce in stages, first internal symplectic reduction, then the 'entire' symplectic reduction.

Suppose $G \rightrightarrows M$ is a source-connected regular Lie groupoid. Suppose α is an *internally* strongly Hamiltonian left action of $G \rightrightarrows M$ on a smooth family of connected symplectic manifolds $(J : S \to M, \omega \in \Omega^2_J(S))$ with internal momentum map $\mu : S \to \mathcal{A}^*(I_G)$ (see Definition 8.3.4).

Denote the image of the zero section $0 : M \to \mathcal{A}^*(I_G)$ by 0_M. Suppose $0_M \subset \mathrm{im}(\mu)$ and μ and 0 are transversal, i.e. $T0(TM)$ and $T\mu(TS)$ are transversal in $T\mathcal{A}^*(I_G)$. Then $\mu^{-1}(0_M)$ is a manifold. Suppose, furthermore, that G_m^m acts freely and properly on $\mu^{-1}(0(m))$ for each $m \in M$. Then for each $m \in M$ the quotient manifold

$$S_m^{(0)} := G_m^m \backslash \mu^{-1}(0)$$

is a smooth manifold with a symplectic 2-form $\omega_m^0 \in \Omega^2(S_m)$ uniquely determined by the equation

$$p_m^* \omega_m^0 = i_m^* \omega|_{S_m},$$

(cf. [50]).

Lemma 10.2.1. *The map*

$$\bigcup_{m \in M} S_m^{(0)} = I_G \backslash \mu^{-1}(0_M) \to M$$

is a smooth family of symplectic manifolds.

Definition 10.2.2. The smooth family of symplectic manifolds

$$I_G \backslash \mu^{-1}(0_M) \to M$$

is called the **internal Marsden-Weinstein quotient** of the internally Hamiltonian groupoid action.

Example 10.2.3. In the case of a smooth family of Lie groups acting on a smooth family of symplectic manifolds the internal Marsden-Weinstein quotient is simply the family of Marsden-Weinstein quotients of the actions of the groups on the fibers (cf. [95]).

Example 10.2.4. Suppose H is a Lie group acting on a symplectic manifold (S, ω^S) in a Hamiltonian fashion, with momentum map μ. Suppose P is a principal H-bundle. Then $G := P \times_H P \rightrightarrows M$ acts on $(P \times_H S, \omega)$ in an internally Hamiltonian fashion (cf. Example 8.3.11), with momentum map $\bar{\mu}[p, s] := [p, \mu(s)] \in P \times_H \mathfrak{h}^*$. Recall that $I_{P \times_H P} \cong P \times_H H$. The internal Marsden-Weinstein quotient $I_G \backslash \bar{\mu}^{-1}(0_M)$ is symplectomorphic to $M \times (\mu^{-1}(0)/H)$ as a smooth bundle of symplectic manifolds (not to $P \times_H \mu^{-1}(0)$), using the map

$$I_G \backslash \bar{\mu}^{-1}(0_M) \longrightarrow M \times \mu^{-1}(0)/H, [p, \sigma] \mapsto (\pi(p), [\mu(\sigma)]).$$

We now turn our attention to the entire quotient.

Lemma 10.2.5. *The map*

$$G \backslash \mu^{-1}(0_M) \to G \backslash M$$

is a continuous family of symplectic manifolds in the quotient topology.

Remark 10.2.6. Note that the space $G \backslash M$ is in general neither Hausdorff nor a smooth manifold.

Proof. Since the momentum map $\mu : S \to \mathcal{A}^*(I_G)$ is equivariant and since for every $g \in G_m^n$ one has $\mathrm{Ad}^*(g)(0(m)) = 0(n)$, the smooth isomorphism $\alpha(g) : S_m \to S_n$ restricts to a smooth isomorphism

$$\alpha(g) : \mu^{-1}(0(m)) \to \mu^{-1}(0(n)).$$

This induces a well-defined action $\bar{\alpha}$ of G on the internal Marsden-Weinstein quotient $I_G \backslash \mu^{-1}(0_M) \to M$ given by

$$\bar{\alpha}(g)(G_m^m \sigma) := (G_n^n \alpha(g)\sigma).$$

Indeed, suppose $\sigma_1 = \alpha(g')\sigma_2$, for $\sigma_1, \sigma_1 \in \mu^{-1}(0(m))$ and $g' \in G_m^m$. Then $\alpha(g)\sigma_1 = \alpha(g\, g'\, g^{-1})\alpha(g)\sigma_2$. Actually, $\bar{\alpha}(g) = \bar{\alpha}(h)$ for all $g, h \in G_m^n$, as one checks by a similar computation.

Of course the action $\bar{\alpha}$ induces an equivalence relation on $I_G \backslash \mu^{-1}(0_M)$ and the quotient equals $G \backslash \mu^{-1}(0_M)$. For every $g \in G$, the map $\bar{\alpha}(g)$ is a symplectomorphism, since $\alpha(g)$ is a symplectomorphism. Hence there exists a canonical family of symplectic forms ω^{00} on $G \backslash \mu^{-1}(0_M) \to G \backslash M$. □

Definition 10.2.7. The continuous family of symplectic manifolds $(G \backslash \mu^{-1}(0_M) \to G \backslash M, \omega^{00})$ is called the **Marsden-Weinstein quotient** of the internally Hamiltonian action of $G \rightrightarrows M$ on $(J : S \to M, \omega)$.

Example 10.2.8. Consider a group H that acts on a manifold M. Denote the action by α. The action groupoid $H \ltimes M \rightrightarrows M$ acts in a Hamiltonian fashion on $(id : M \to M, 0)$, with momentum map given by any $\mu : M \to (\mathfrak{h} \ltimes M)^*$ such that $d^{\mathfrak{h} \ltimes M}\mu = d(\alpha^* \mu) = 0$. The Marsden-Weinstein quotient is defined iff $\mu = 0$ and then, obviously, is given by $(M/H \to M/H, 0)$. Note that M/H only is smooth if the action of H on M is proper and free.

Example 10.2.9. We continue Example 10.2.4. One easily sees that the Marsden-Weinstein quotient $(P \times_H P) \backslash \tilde{\mu}^{-1}(0_M) \to (P \times_H P) \backslash M, \tilde{\omega}^{00})$ is symplectomorphic to the Marsden-Weinstein quotient $(H \backslash \mu^{-1}(0), (\omega^S)^0)$.

10.3 Quantization commutes with reduction

Suppose $G \rightrightarrows M$ is a proper regular Lie groupoid. Suppose $\pi : G \to U(E)$ is a unitary representation of $G \rightrightarrows M$ on a continuous field of Hilbert spaces $(\mathcal{H}, \Delta)$. Define the continuous field of Hilbert spaces $(\mathcal{H}^{I_G}, \mathcal{H}^{I_G})$ of I_G-fixed vectors by

$$\mathcal{H}^{I_G} := \{h \in \mathcal{H} \mid \pi(g)h = h \text{ for all } g \in G_{p(h)}^{p(h)}\}$$

and

$$\Delta^{I_G} := \{\xi \in \Delta \mid \xi(m) \in \mathcal{H}_m^{I_G} \text{ for all } m \in M\}.$$

One easily checks that $(\mathcal{H}^{I_G}, \mathcal{H}^{I_G})$ carries a representation of $G \rightrightarrows M$. Indeed, for $g \in G_n^n$, $g' \in G_m^n$ and $h \in \mathcal{H}_m^{I_G}$ one has

$$\begin{aligned}\pi(g)\pi(g')h &= \pi(gg')h \\ &= \pi(g')\pi(((g')^{-1}gg'))h \\ &= \pi(g')h.\end{aligned}$$

Definition 10.3.1. The continuous field of Hilbert spaces $(\mathcal{H}^{I_G}, \Delta^{I_G})$ is called the **internal quantum reduction** of $\pi : G \to U(\mathcal{H})$.

By similar reasoning as above, the restriction of π to a map $G \to U(\mathcal{H}^{I_G})$ is I_G-invariant, i.e.

$$\pi(g)h = \pi(g')h$$

for all $g, g' \in G^n_m$, $n, m \in M$ and $h \in \mathcal{H}^{I_G}_m$.

Suppose $G \rightrightarrows M$ acts in Hamiltonian fashion on a smooth bundle of compact connected Kähler manifolds $J : S \to M$ endowed with a J-presymplectic form $\tilde{\omega}$, such that the complex structure j and Hermitian metric h are G-equivariant. Denote the momentum map by $\tilde{\mu} : S \to J^*\mathcal{A}^*$. Consider the internal Marsden-Weinstein quotient $(\mu^{-1}(0_M)/I_G, \omega^0)$. Suppose (L, ∇^L, h) is a prequantization of the G-action. Suppose $L^0 \to \mu^{-1}(0_M)/I_G$ is a line bundle such that

$$p^*L^0 = L|_{\mu^{-1}(0_M)}.$$

This is a strong condition, which is satisfied if the action of I_G on S is free. The line bundle L^0 has an induced prequantization connection ∇^0, since ∇^L is G-equivariant (cf. Corollary 9.3.11). Suppose $L^0 \to \mu^{-1}(0_M)/I_G$ has an induced Hermitian metric h^0. The triple (L^0, ∇^0, h^0) is a prequantization of the Hamiltonian action of $R_G \rightrightarrows M$ on the smooth bundle of symplectic manifolds $(I_G\backslash\mu^{-1}(0_M) \to M, \tilde{\omega}^0)$.

Moreover, the Kähler structure on $S \to M$ induces a Kähler structure on $I_G\backslash\mu^{-1}(0_M) \to M$. Denote the geometric Kähler quantization of the prequantization (L^0, ∇^0, h^0) of the internal Marsden-Weinstein quotient by $(\mathcal{H}^0_Q, \Delta^0_Q)$.

Theorem 10.3.2 (Quantization commutes with reduction I)**.** *If G is a proper regular Lie groupoid, then quantization commutes with internal reduction, i.e. there exists an isomorphism of continuous fields of Hilbert spaces*

$$(\mathcal{H}^0_Q, \Delta^0_Q) \overset{\cong}{\to} (\mathcal{H}^{I_G}_Q, \Delta^{I_G}_Q).$$

Remark 10.3.3. Note that for each $m \in M$ one can restrict the action of G on $J : S \to M$ to a Hamiltonian action of the isotropy Lie group G^m_m on $(S_m, \omega|_{S_m})$. Likewise, the momentum map, prequantization data, Kähler structure all restrict to S_m, and hence give rise to a quantization commutes with reduction statement as in the theorem for G^m_m, which is compact since G is proper. This theorem was first formulated and proven for compact Lie group actions by Guillemin and Sternberg (cf. [28]) and also goes under the name of "Guillemin-Sternberg conjecture". It is proven, in a more general form using Spin^c-Dirac operators, for compact Lie groups by Meinrenken (cf. [52]), Meinrenken and Sjamaar (cf. [53]), Tian and Zhang (cf. [76]) and Paradan (cf. [62]). For certain non-compact groups it is proved, in a somewhat different form using K-theory and K-homology (cf. Remark 10.1.12), by Hochs and Landsman (cf. [36]) and Hochs (cf. [35]). For families the theorem was proven in [95], also within the setting of Spin^c-Dirac operators and K-theory.

Remark 10.3.4. If the theorem holds, then the following diagram "commutes":

$$\begin{array}{ccc}
(\mathcal{H}_Q, \Delta_Q) & \overset{R}{\longmapsto} & (\mathcal{H}^0_Q, \Delta^0_Q) \cong (\mathcal{H}^{I_G}_Q, \Delta^{I_G}_Q) \\
{\scriptstyle Q}\uparrow & & \uparrow{\scriptstyle Q} \\
(S, \tilde{\omega}) & \underset{R}{\longmapsto} & ((I_G\backslash\mu^{-1}(0_M)), \omega^0),
\end{array}$$

where R denotes symplectic and quantum reduction and Q denotes quantization. One sometimes abbreviates the theorem by writing $[Q, R] = 0$.

Proof. We shall construct a morphism

$$\mathcal{H}^{I_G} \to \mathcal{H}^0.$$

The inclusion $i : \mu^{-1}(0_M) \hookrightarrow S$ induces a map

$$i^* : \Gamma^\infty(L)^{I_G} \to \Gamma^\infty(L|_{\mu^{-1}(0_M)})^{I_G},$$

where the superscript I^G means that we restrict to equivariant sections. These are the sections fixed under the action of I_G on $\Gamma^\infty(L)$ as a bundle over M. Moreover, the quotient map $p : \mu^{-1}(0_M) \to \mu^{-1}(0_M)/I_G$ induces an isomorphism

$$p^* : \Gamma^\infty(L^0) \to \Gamma^\infty(L|_{\mu^{-1}(0_M)})^{I_G}.$$

Because of the equivariance of the Kähler structure and the connection the composition $(p^*)^{-1} \circ i^*$ induces a map

$$\mathcal{H}_Q^{I_G} \to \mathcal{H}_Q^0,$$

which is the one we wanted to construct. This map is an isomorphism on each fiber, since the isotropy groups of a proper groupoid are compact groups for which the theorem is well established. Hence Ψ is a continuous isomorphism of continuous fields of Hilbert spaces. □

Example 10.3.5. Suppose H is a Lie group acting in Hamiltonian fashion on a Kähler manifold (S, h, j, ω^S). Suppose $\pi : P \to M$ is a principal H-bundle and $\tau \in \Gamma^\infty(\bigwedge^1(P) \otimes \mathfrak{h}$ a connection 1-form. As discussed in previous examples there exist a J-presymplectic form $\tilde{\omega}$ on $P \times_H S \to M$ and the action of the gauge groupoid $P \times_H P \rightrightarrows M$ on $(P \times_H S \to M, \tilde{\omega})$ is Hamiltonian. Suppose $(L \to S, \nabla^L, g)$ is prequantization of the action of H on S. Let Q_S denote the associated quantization vector space. In Example 10.1.8 we have seen that the prequantization and quantization of the H-action on S give rise to a prequantization $(P \times_H L, \nabla, g')$ and a quantization $Q' \to M$ of the action of the gauge groupoid $P \times_H P \rightrightarrows M$ on $(P \times_H S \to M, \tilde{\omega})$. In Example 10.2.4 we saw that the internal Marsden-Weinstein quotient is isomorphic to the trivial bundle $(M \times \mu^{-1}(0)/H \to M, \omega^0)$. The quantization of this bundle obviously equals the vector bundle $M \times Q(\mu^{-1}(0)/H, j^0) \to M$. The statement of Theorem 10.3.2 follows from $[Q, R] = 0$ for H plus the following observation

Lemma 10.3.6. *The internal quantum reduction of $Q' \to M$ is isomorphic to*

$$M \times Q_S^H.$$

Proof. In Example 10.1.8 we proved that $Q' \cong P \times_H Q(S, j)$. An element $[p, \xi] \in P \times_H Q(S, j)$ is fixed under all $[p', p] \in I_{P \times_H P}$ whenever ξ is fixed under all $h \in H$. Hence the statement follows. □

Suppose $\pi : G \to U(\mathcal{H})$ is a unitary representation of a proper regular Lie groupoid $G \rightrightarrows M$ on a continuous field of Hilbert spaces $(\mathcal{H}, \Delta)$.

Definition 10.3.7. The **quantum reduction** of $\pi : G \to U(\mathcal{H})$ is the quotient continuous field of Hilbert spaces $\mathcal{H}^G := G\backslash\mathcal{H}^{I_G}$ (this quotient is in general non-Hausdorff). The associated space of continuous sections is denoted by Δ^G.

Denote the geometric quantization of the continuous family of Kähler manifolds $(G\backslash\mu^{-1}(0_M) \to G\backslash M, \tilde{\omega}^0)$ by $(\mathcal{H}^{00}_Q, \Delta^{00}_Q)$.

Corollary 10.3.8 (Quantization commutes with reduction II)**.** *If all conditions for Theorem 10.3.2 are satisfied, then quantization commutes with symplectic reduction, i.e. there exists an isomorphism of continuous fields of Hilbert spaces*

$$(\mathcal{H}^{00}_Q, \Delta^{00}_Q) \overset{\cong}{\to} (\mathcal{H}^G_Q, \Delta^G_Q).$$

Example 10.3.9. As a (very) basic example we consider the pair groupoid $M \times M \rightrightarrows M$ for a manifold M, acting on $(id : M \to M, 0)$. A momentum map is any map $\mu \in \Omega^1(M)$ such that $d\mu = 0$. Since we assume $0_M \subset \mathrm{im}(\mu)$, μ has to be zero. Hence the Marsden-Weinstein quotient equals $(*, 0)$. Recall that quantization and prequantization line bundles coincide in this case. The only quantization representation of TM that integrates to a representation of $M \times M$ is the trivial representation on the trivial complex line bundle $M \times \mathbb{C} \to M$. Obviously, the quantum reduction of such a bundle is $\mathbb{C} \to *$. The Marsden-Weinstein quotient $(*, 0)$ is indeed quantized by $\mathbb{C}$.

Example 10.3.10. The previous example is a special case of gauge groupoid considered in the previous Examples 8.3.8, 8.3.11, 8.4.13, 9.3.6, 9.4.6, 10.1.8, 10.3.5, where $P \to M$ is a principal H-bundle and H a Lie group. One easily sees that the full quantum reduction of

$$Q' \cong P \times_H Q_S$$

is isomorphic to Q_S^H. Moreover, Marsden-Weinstein quotient equals $\mu^{-1}(0)/H$. Hence, in this example, it is particularly clear how $[Q, R] = 0$ for gauge Lie groupoids reduces to $[Q, R] = 0$ for Lie groups.

10.4 The orbit method

To investigate and illustrate a possible orbit method we treat some examples in this section. We have not yet arrived at a full formulation. The basic principle of Kirillov's orbit method for Lie groups G is that there should be a bijective correspondence or at least a correspondence between the coadjoint orbits in the dual of the Lie algebra $\mathfrak{g}^*$ and the irreducible unitary representations of G. A way to establish such a correspondence is through geometric quantization.

The coadjoint orbits of a Lie groupoid $G \rightrightarrows M$ lie within the dual of the kernel of the anchor of the Lie algebroid $\ker(\rho)^*$. As discussed in Remark 8.3.7 and Remark 8.4.19 a smooth family of such orbits

$$\{\mathcal{O}_{G\,m}\}_{G\,m \in G\backslash M}$$

carry an internally Hamiltonian action of $G \rightrightarrows M$ (namely the coadjoint action). Before being able to apply geometric quantization we need to extend the family of symplectic forms on

$$S := \bigcup_{G\,m \in G\backslash M} \mathcal{O}_{G\,m}$$

to a J-presymplectic form. These extensions are in general not unique.

Nevertheless, for example, in the case of the gauge groupoid the representation of the Lie groupoid does not depend on the choice of the extension as we have seen in Proposition 9.4.6. Therefore the basic Kirillov orbit principle is true for gauge groupoids of principle H-bundles, if it is true for the Lie group H, as we shall see in the following example.

Example 10.4.1. Suppose H is a Lie group and $P \to M$ an H-principal bundle. Recall that the isotropy groupoid of the gauge groupoid $P \times_H P \rightrightarrows M$ is isomorphic to $P \times_H H \to M$, where the action of H on H is by conjugation. Hence the bundle of Lie algebras $\mathcal{A}(I_{P\times_H P}) \to M$ is isomorphic to $P \times_H \mathfrak{h} \to M$, where the action of H on $\mathfrak{h}$ is the adjoint action. Moreover, the dual bundle $\mathcal{A}(I_{P\times_H P})^* \rightrightarrows M$ is isomorphic to $P \times_H \mathfrak{h}^* \rightrightarrows M$, where the action of H on $\mathfrak{h}^*$ is the coadjoint action. From a coadjoint orbit $\mathcal{O} \subset \mathfrak{h}^*$ one can construct a bundle $P \times_H \mathcal{O} \subset P \times_H \mathfrak{h}^*$, which is easily seen to correspond to a coadjoint orbit in $\mathcal{A}(I_{P\times_H P})^*$.

Lemma 10.4.2. *There is a bijective correspondence between coadjoint orbits of the Lie group H and coadjoint orbits of the gauge groupoid $P \times_H P$ given by*

$$\mathcal{O} \mapsto P \times_H \mathcal{O}$$

for $\mathcal{O} \subset \mathfrak{h}^*$.

If we choose a connection on P, then we can extend the symplectic structure on $\mathcal{O}$ to a J-presymplectic form, such that the action of the gauge groupoid is Hamiltonian. In Proposition 9.4.6 we proved that the geometric quantization of this action does not depend on the choice of connection. Actually, the representation of $P \times_H P \rightrightarrows M$ is obtained by geometric quantization of the action of H on $\mathcal{O}$ by Morita equivalence. In general, since H and $P \times_H P \rightrightarrows M$ are Morita equivalent, there exists a bijection between irreducible unitary representations of H and irreducible unitary representations of $P \times_H P \rightrightarrows M$. From the above lemma we see that there is also a bijection between the coadjoint orbits of H and $P \times_H P \rightrightarrows M$. From this we conclude that for as far as the orbit method (using geometric quantization) works for H, it works for $P \times_H P \rightrightarrows M$ too.

Example 10.4.3. The previous example gives rise to an examples where the orbit method for Lie groupoids fails. If the isotropy groups are discrete, we are in trouble.

Consider the fundamental groupoid $\pi_1(M) \rightrightarrows M$ of a smooth connected manifold M. The associated Lie algebroid is TM, hence there is only one coadjoint orbit in $\ker(\rho) = M$, namely the zero orbit. In general, our geometric quantization procedure gives rise to many non-isomorphic irreducible unitary representations of TM, namely the flat connections on vector bundles $E \to M$. For a fixed vector bundle $E \to M$, these representations integrate to isomorphic unitary representations of $\pi_1(M) \rightrightarrows M$, given by parallel transport. From Morita equivalence it follows that there exist a bijective correspondence between unitary irreducible representations of the fundamental *group* $\pi_1(M, m)$ $(m \in M)$ and unitary irreducible representations of $\pi_1(M) \rightrightarrows M$ given by

$$V \mapsto P \times_H V$$

Hence, in this case, there is no bijection between smooth families of coadjoint orbits of $\pi_1(M) \rightrightarrows M$ and irreducible unitary representations of $\pi_1(M,m)$, unless $\pi_1(M,m)$ is trivial (or has only one trivial unitary irreducible representation).

Remark 10.4.4. Another hope one might have is for a correspondence between symplectic leaves of the Poisson structure on $\mathcal{A}^*$ and isomorphism classes of irreducible unitary representations of the groupoid. The same Example 10.4.3 shows this will not work. In general, we already noted that this Poisson structure on $\mathcal{A}^*$ does not have a specific rôle in (our version of) geometric quantization. The main reason for this is that there is no canonical action of $G \rightrightarrows M$ on $\mathcal{A}^* \to M$.

The next two examples show some more use of our geometric quantization procedure to construct representations and discuss the implications of these examples for an orbit method for Lie groupoids.

Example 10.4.5. Now we shall consider a non-regular groupoid. A simple example is given by the action groupoid $G := S^1 \ltimes \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ of the action of the circle S^1 on the plane $\mathbb{R}^2$ by rotation around the origin. The dual of the bundle of Lie algebras associated to the isotropy groupoid is given by

$$(\mathcal{A}^*(I_G))_{(x,y)} \cong \begin{cases} \mathbb{R} & \text{if } (x,y) = (0,0) \\ 0 & \text{if } (x,y) \neq (0,0) \end{cases}$$

The only *smooth* bundle of coadjoint orbits is the trivial one $\{0\} \times \mathbb{R}^2 \cong \mathbb{R}^2$. If the orbit method would claim a bijective correspondence between smooth families of coadjoint orbits and irreducible unitary representations, then all such representations of $G \rightrightarrows \mathbb{R}^2$ would be trivial at the origin, but this is not true. Anyway, the irreducible representations of $G \rightrightarrows \mathbb{R}^2$ are obtained by geometric quantization. A G-invariant J-presymplectic on $\mathbb{R}^2$ is necessarily zero. A smooth momentum map $\mathbb{R}^2 \to \mathcal{A}^*(G)$ given by any constant map $f : \mathbb{R}^2 \to \mathbb{R}$. The prequantum line bundle L is necessarily the trivial one (since $\mathbb{R}^2$ is contractible) and the prequantization(=quantization) representation

$$\mathcal{A}(G) \cong \mathbb{R} \times \mathbb{R}^2 \to (\mathfrak{u}(1) \times \mathbb{R}^2) \oplus T\mathbb{R}^2 \cong \mathcal{D}(L)$$

is given by

$$(X, (r, \alpha) \mapsto (2\pi i f(r,\alpha)X, X\frac{d}{d\alpha})$$

(we use polar coordinates (α, r)) which integrates to a representation of

$$S^1 \ltimes \mathbb{R}^2 \to U(L) \cong \mathbb{R}^2 \times U(1) \times \mathbb{R}^2$$

given by

$$(\beta, (r,\alpha)) \mapsto ((r, \alpha+\beta), e^{2\pi i(f(r,\alpha+\beta)-f(r,\alpha))}, (r,\alpha)),$$

whenever f is integral valued.

Example 10.4.6. For continuous families of Lie groups our geometric quantization procedure works too, although one should proceed with caution. For example, consider the 2-sphere $S^2 \subset \mathbb{R}^3$. It can be seen as a continuous family of Lie groups under the projection $S^2 \to [-1,1]$ given by $(x,y,z) \mapsto x$. The dual of the associated bundle of Lie algebras is given by

$$(\mathcal{A}^*(S^2)_x \cong \begin{cases} \mathbb{R} & \text{if } x \in]-1,1[\\ 0 & \text{if } x = \pm 1 \end{cases}$$

The image of any continuous section $\theta : [-1,1] \to \mathcal{A}^*(S^2)$ is a continuous family of coadjoint orbits (which are points). A momentum map is given by inclusion $\mu : \theta([-1,1]) \hookrightarrow \mathcal{A}^*(S^2)$. A prequantum line bundle is again necessarily trivial $L = [-1,1] \times \mathbb{C}$. The prequantum representation is given by

$$(x, X) \mapsto 2\pi i \langle \mu, X \rangle .$$

The remarkable feature of this example is that one can allow a θ, and hence μ, which is *not* continuous at $x = \pm 1$, namely $\theta(x) = k$ for $x \in]0,1[$ and a fixed $k \in \mathbb{Z}$ and $\theta(\pm 1) = 0$, and still find a continuous representation after integration:

$$(x, \alpha) \mapsto e^{2\pi i \mu \alpha}.$$

This is a particular instance of the fact there sometimes exist non-continuous vector fields v on a manifold M that still induce homeomorphisms $\exp(v) : M \to M$. Realizing this fact, an orbit method should allow families of coadjoint orbits that are non-continuous at certain points.

Part IV

Noncommutative geometric quantization

Introduction

The purpose of this part is to provide a very general framework for geometric quantization and symplectic reduction. For singular spaces the usual constructions have to be dealt with carefully. The idea is to set up a general approach using noncommutative geometry. Instead of the manifold or space one uses a suitable algebra of functions on that space. There might even not be a space lurking in the background at all.

Our starting point will be a Poisson algebra. We shall in particular be interested in Poisson algebras coming from so-called symplectic algebras. For Poisson algebras it is quite easy to define a reduced Poisson algebra. In this we follow the work of Sniatycki (cf. [72]). This in contrast to the case of symplectic reduction, where things are much more difficult.

To define a notion of symplectic algebras, we need a notion of differential forms on algebras. There are several differential calculi for algebras. In Section 11.2 we focus on a derivation based differential calculus, introduced by Dubois Violette [24]. We recall and extend the work of Masson [51] on pullback and pushforward for such a differential calculus in Section 11.3. Then, in Section 11.4, we discuss symplectic reduction in this setting. In the final section of Chapter 11 we use the notion of central invertible bimodules (cf. [51]) to extend prequantization to symplectic algebras.

The approach to geometric quantization that we propose is based on a suggestion by Landsman (cf. [47, 36]). Before explaining this approach step by step let's summarize it in one sentence: he suggests to view the geometric quantization of a Hamiltonian action as the image of the K-homology class of the Spin^c-Dirac operator on the underlying manifold coupled to the prequantization line bundle under the Baum-Connes analytical assembly map. This approach embeds the Guillemin-Sternberg conjecture within a more general program of proving the functoriality of geometric quantization in a certain sense (cf. [47]). In this program geometric quantization is to be viewed as a functor from classical dual pairs (cf. e.g. [43]) to Kasparov KK-theory (cf. Section 4.5).

This transition from the traditional discussion of geometric quantization to the approach mentioned above proceeds in a few steps. The usual approach to geometric quantization, in particular Kähler quantization, is to consider the holomorphic sections of the prequantization line bundle with respect to some equivariant complex (Kähler) structure on the symplectic manifold M. This is the approach we followed in the previous part. A well-known fact is that in favorable cases this space of holomorphic sections coincides with the index of the Dolbeault operator associated to the complex structure (Kodaira's vanishing Theorem). So a first generalizing step is to define the geometric quantization simply as the index of the Dolbeault operator. However, a manifold M need not always possess an equivariant Kähler structure. A somewhat weaker condition is for M to have an equivariant Spin^c-structure. Therefore, one weakens the definition of geometric quantization to being the index of a Spin^c-Dirac operator on M. This approach makes sense for compact groups, and the index naturally carries a (virtual) representation of the group.

To enlarge the scope of geometric quantization to non-compact groups (and noncompact spaces) Landsman suggested to consider proper cocompact actions. This makes available the machinery of the Baum-Connes conjecture (cf. [4]). The purpose of the Baum-Connes conjecture is to describe the K-theory of the C^*-algebra of a locally compact group by topological means. Indeed, the topological space under consideration is the classifying space $\underline{E}H$ of *proper* actions of a group H. One considers the K-homology $K^i(\underline{E}H)$ of this space and a map

$$\mu_{BC} : K_i^H(\underline{E}H) \to K_i(C^*(H)),$$

$i = 0, 1$, called the Baum-Connes analytical assembly map. The group $K_i(C^*(H))$ should be seen as a generalized representation ring $\mathcal{R}(H)$ of H. Indeed, for compact H it equals $\mathcal{R}(H)$ (cf. Example 4.4.7). As we have seen in Theorem 6.6.10, under some conditions the same is true for proper groupoids. The Baum-Connes conjecture states that this map μ_{BC} is an isomorphism. Morally, this means that any generalized representation is the index of a generalized equivariant Fredholm operator on $\underline{E}H$.

Landsman's suggestion is to use the map

$$\mu_{BC} : K_0^H(M) \to K_0(C^*(H)),$$

which is defined for any locally compact Hausdorff space M on which H acts properly and cocompactly, to define the geometric quantization of a proper, cocompact, Hamiltonian action on M. This will be the image of the class in $K_0^H(M)$ of the Spin^c-Dirac operator coupled to the prequantization line bundle under the Baum-Connes assembly map μ_{BC}. This approach is indeed successful, cf. [36, 35].

Our plan is to extend the approach another step further. We would like to apply it to Hamiltonian actions on (noncommutative) symplectic algebras. We construct a Baum-Connes assembly map

$$\mu_{BC} : KK_i^H(A, B) \to K_i(H \ltimes B),$$

($i = 0, 1$) for C^*-algebras A endowed with a proper, counital action of H. For this purpose we review the notion of proper actions on C^*-algebras by Rieffel (cf. [69]). We introduce the notion of counital action, generalizing cocompact actions to the noncommutative setting. Our main examples of proper, counital actions are certain actions on $C_0(M)$-algebras and reduced groupoid C^*-algebras.

For a general C^*-algebra A there are no general ways known to construct a class in $KK_0^H(A, \mathbb{C})$, like a Spin^c-Dirac operator. Because of this we shall suggest a very general notion of geometric quantization. We even propose a version of geometric quantization *with coefficient in a C^*-algebra*. This part is open-ended; we outline a direction for continuation of the research suggested by our previous results.

Chapter 11

Algebraic momentum maps

11.1 Hamiltonian actions on Poisson algebras

In this section we introduce Poisson algebras and algebraic momentum maps for Hamiltonian actions on Poisson algebras.

Suppose A is an algebra. By an algebra we shall in this section always mean an associative algebra over $\mathbb{C}$. A **derivation** on A is a linear map $D : A \to A$ satisfying

$$D(a\,b) = D(a)b + aD(b),$$

for all $a, b \in A$. Denote the set of derivations on A by $\mathrm{Der}(A)$. The set $\mathrm{Der}(A)$ endowed with the commutator bracket $[D, D'] := D\,D' - D'\,D$ is a Lie algebra. Furthermore $\mathrm{Der}(A)$ is a module over the center $Z(A)$ of A.

Example 11.1.1. Suppose M is a smooth manifold. One can prove that derivations of $C^\infty(M)$ correspond to smooth vector fields on M.

Example 11.1.2. For any algebra A and $a \in A$, the map $\mathrm{ad}(a) : A \to A$, $b \mapsto a\,b - b\,a$ is a derivation, a so-called **inner derivation**. Denote the Lie algebra of inner derivations by $\mathrm{Der}^i(A)$. There is an exact sequence

$$0 \longrightarrow Z(A) \longrightarrow A \xrightarrow{\mathrm{ad}} \mathrm{Der}(A) \longrightarrow \mathrm{Der}(A)/\mathrm{Der}^i(A) \longrightarrow 0.$$

Remark 11.1.3. The classes in $\mathrm{Der}(A)/\mathrm{Der}^i(A)$ are called outer derivations. This set of outer derivations equals the first Hochschild cohomology $HH^1(A, A)$ of A. Hochschild cohomology $HH^n(A, A)$ is the cohomology of the cocomplex $C^n(A, A) := \mathrm{Hom}(A^{\otimes n}, A)$ for $n \in \mathbb{N}$ (and $C^0(A, A) = A$) with codifferential $b_n : C^n(A, A) \to C^{n+1}(A, A)$

$$\begin{aligned} b_n\, f(a_1, \ldots, a_{n+1}) =& a_1 f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\ &+ (-1)^{n+1} f(a_1, \ldots, a_n) a_{n+1} \end{aligned}$$

and $b_0(a) = ad(a)$. Obviously, $HH^0(A, A) = \ker(b_0) = Z(A)$.

Example 11.1.4. Suppose H is a finite group acting on a smooth manifold M. Consider the action Lie groupoid $H \ltimes M \rightrightarrows M$. The outer derivations of the convolution algebra $C_c^\infty(H \ltimes M)$ were computed in [61] as part of a computation of the whole Hochschild cohomology. Let M^h denote the fixed point set of $h \in H$; let $\pi_0(M^h)$ denote the set of the connected components of M^h, $\mathrm{conj}(H)$ the set of conjugacy classes in H and $Z(h)$ the centralizer of $h \in H$. Then, the outer derivations can be identified via an isomorphism (cf. e.g. [61]) of $HH^1(C^\infty(H \ltimes M), C^\infty(H \ltimes M))$ with

$$\bigoplus_{[h]\in \mathrm{conj}(H)} \bigoplus_{M^h_\alpha \in \pi_0(M^h)} \Gamma^\infty\left(M^h_\alpha, \bigwedge^{1-\dim(M)+\dim(M^h_\alpha)} T(M^h_\alpha)\right)^{Z(h)}.$$

Note that for $h = e$ is the unit of H one obtains the smooth invariant vector fields $\mathfrak{X}^\infty(M)^H$ as a summand. This identification is also extended to orbifolds in [61].

Example 11.1.5. Let $\theta \in [0, 1)$ be irrational. Let $\mathbb{T}^2_\theta$ denote the algebra

$$\mathbb{T}^2_\theta := \{\sum_{k,l\in\mathbb{Z}} c(k, l) u^k v^l \mid c \in \mathcal{S}(\mathbb{Z}^2)\},$$

where $\mathcal{S}(\mathbb{Z}^2)$ denotes the set of Schwartz functions on $\mathbb{Z}^2$ and u and v are unitaries, satisfying $u\,v = e^{2\pi i\theta} v\,u$. This algebra is called the . The derivations of $\mathbb{T}^2_\theta$ can be computed (cf. [27] and references therein). In particular, if $|1 - e^{2\pi\, i\, n\, \theta}|^{-1} = O(n^k)$ for some $k \in \mathbb{N}$ (this is called a Diophantine condition), then

$$\mathrm{Der}(\mathbb{T}^2_\theta) = \mathbb{C}\delta_1 \oplus \mathbb{C}\delta_2 \oplus \mathrm{Der}^i(\mathbb{T}^2_\theta),$$

where $\delta_1(u^k v^l) := k\, u^k v^l$ and $\delta_2(u^k v^l) := l\, u^k v^l$ for all $k, l \in \mathbb{Z}$. The lie algebra $\mathrm{Der}^i(\mathbb{T}^2_\theta)$ is generated over $\mathcal{S}(\mathbb{Z}^2)$ by internal derivations of the form

$$\mathrm{ad}(u^k v^l)(u^{k'} v^{l'}) = (1 - e^{2\pi\, i\, k\, l'}) u^{k+k'} v^{l+l'},$$

for $k, k', l, l' \in \mathbb{Z}$.

Suppose $\mathfrak{g}$ is a Lie algebra. An **action of the Lie algebra $\mathfrak{g}$ on the algebra** A is a Lie algebra homomorphism

$$\alpha : \mathfrak{g} \to \mathrm{Der}(A).$$

The algebra A is a (possibly noncommutative) **Poisson algebra** if there exists a Lie bracket

$$\{.,.\} : A \times A \to A,$$

satisfying the Leibniz identity

$$\{a, b\,c\} = b\{a, c\} + \{a, b\}c,$$

for all $a, b, c \in A$. This is equivalent to $\{a, .\}$ being a derivation on A for all $a \in A$. The derivation $D_a := \{a, .\}$ is called the **Hamiltonian derivation** associated to $a \in A$.

Example 11.1.6. If P is a Poisson manifold, then $A = C^\infty(P)$ is a commutative Poisson algebra.

Example 11.1.7. Any algebra A is a Poisson algebra with the commutator bracket $\{a, b\} := [a, b] = a\,b - b\,a$ as Poisson bracket.

Example 11.1.8. Suppose $\alpha : G \to \mathrm{Aut}(A)$ is an action of a group G on a Poisson algebra A. Suppose the action is a **Poisson action**, i.e. $g(\{a, b\}) = \{g \cdot a, g \cdot b\}$ for all $a.b \in A$. Then the linear subspace A^G of G-invariant elements of A is a Poisson algebra. In the same way, if X is an orbifold represented by a proper étale groupoid $G \rightrightarrows M$, then a G-invariant Poisson structure on M turns $C^\infty(M)^G$ into a Poisson algebra.

Remark 11.1.9. More examples are given in Section 11.4. In that section, we show how Poisson algebras can be constructed from symplectic algebras.

Definition 11.1.10. An action $\alpha : \mathfrak{g} \to \mathrm{Der}(A)$ is **Hamiltonian** if there exists an **algebraic momentum map**

$$\mu : \mathfrak{g} \to A$$

satisfying

(i) $\alpha(X) = \{\mu(X), .\}$ for all $X \in \mathfrak{g}$;

(ii) $\mu([X, Y]) = \{\mu(X), \mu(Y)\}$ for all $X, Y \in \mathfrak{g}$;

(iii) $\mu(\mathfrak{g}) \subset Z(A)$.

Remark 11.1.11. From the first condition we conclude that any derivation that comes from the $\mathfrak{g}$-action is Hamiltonian. The second condition states that μ is a Lie algebra homomorphism. The last condition is needed for the important Proposition 11.5.3 to hold.

Remark 11.1.12. A different notion of momentum map based on so-called double derivations can be found in [20]. An advantage of their notion is that the center $Z(A)$ of A does not play a particular rôle. This is good because the center might very well be trivial. On the other hand, the advantage of our notion is that it stays closer to the classical (commutative, geometrical) case. Another difference that will turn up in the next sections is that the authors of [20] work with De Rham differential calculus, whereas we prefer a derivation based differential calculus.

Example 11.1.13. Suppose $\mathfrak{g}$ is a Lie algebra that acts in Hamiltonian fashion on a Poisson manifold $(P, \{., .\})$, with momentum map $\bar{\mu} : P \to \mathfrak{g}^*$. Then the induced action of $\mathfrak{g}$ on $A := C^\infty(P)$ is Hamiltonian with momentum map $\mu : \mathfrak{g} \to C^\infty(P)$ given by $\mu(X) := (p \mapsto \bar{\mu}(X)(p))$.

Example 11.1.14. Suppose a Lie algebra $\mathfrak{g}$ acts on a Poisson algebra A in Hamiltonian fashion, with momentum map $\mu : \mathfrak{g} \to A$. Suppose that H is a finite discrete group acting on A preserving the Poisson bracket, and such that the actions of H and $\mathfrak{g}$ commute. Then the action of $\mathfrak{g}$ on A^H is Hamiltonian with momentum map

$$\mu'(X) := \sum_{h \in H} \frac{1}{|H|} \mu(X).$$

The quotient $A/(\mathrm{im}(\mu)A)$ induces a $\mathfrak{g}$ action defined by $\alpha(X)[a] := [\alpha(X)a]$. This is well-defined, since

$$\begin{aligned}\alpha(X)(\mu(Y)a) &= \alpha(X)(\mu(Y))a + \mu(Y)\alpha(X)(a)\\ &= -\mu([X,Y])a + \mu(X)\alpha(X)(a) \in \mathrm{im}(\mu)A\end{aligned}$$

for all $X, Y \in \mathfrak{g}$ and $a \in A$.

Theorem 11.1.15. *Suppose A is a Poisson algebra with a Hamiltonian action of a Lie algebra $\mathfrak{g}$. Let $\mu : \mathfrak{g} \to Z(A)$ be a momentum map. Then the algebra*

$$A^0 := (A/(\mathrm{im}(\mu)A))^{\mathfrak{g}}$$

of $\mathfrak{g}$-invariant elements in $A/(\mathrm{im}(\mu)A)$ (i.e. $\alpha(X)[a] = [0]$ for all $X \in \mathfrak{g}$) is a Poisson algebra.

Proof. A bracket on A^0 is defined by

$$\{[a],[b]\} := [\{a,b\}],$$

for all $a, b \in A$. This well defined, since for all $[a],[b] \in A^0$ and $X \in \mathfrak{g}$

$$\begin{aligned}\{a, \mu(X)b\} &= \mu(X)\{a,b\} - \{\mu(X), a\}b\\ &= \mu(X)\{a,b\} - (\alpha(X)\,a)b \in A\ \mathrm{im}(\mu).\end{aligned}$$

The bracket on A^0 is Poisson, since the bracket on A is Poisson. □

The algebra A^0 is called the **reduced Poisson algebra** with respect to the given $\mathfrak{g}$-action and momentum map μ.

Remark 11.1.16. Suppose $\mathfrak{g}$ integrates to a connected Lie group G. Then the action of $\mathfrak{g}$ on A gives rise to an action of G on A. The reduced Poisson algebra $(A/(\mathrm{im}(\mu)A))^{\mathfrak{g}}$ equals $(A/(\mathrm{im}(\mu)A))^G$. Hence, for actions of non-connected Lie groups it makes sense to define the reduced Poisson algebra by $(A/(\mathrm{im}(\mu)A))^G$.

Remark 11.1.17. In some cases it might even be better not to consider $(A/(\mathrm{im}(\mu)A))^G$, but the invariants $(M(A/(\mathrm{im}(\mu)A)))^G$ of the multiplier algebra $M(A/(\mathrm{im}(\mu)A))$ (if A is a C^*-algebra). Indeed, if $\mathbb{Z}$ acts on $C_0(\mathbb{R})$ by translation, then $C_0(\mathbb{R})^{\mathbb{Z}}$ is $\{0\}$, but $(M(C_0(\mathbb{R})))^{\mathbb{Z}} \cong C_b(\mathbb{R})^{\mathbb{Z}}$ contains all $\mathbb{Z}$-periodic functions. The problem, in general, is to lift the Poisson bracket to these algebras.

Remark 11.1.18. There is another way in which $(A/\,\mathrm{im}(\mu)A)^G$ might turn out "too small". Consider, for example, the action of $\mathbb{Z}$ on S^1 by irrational rotations. Then $C(S^1)^{\mathbb{Z}}$ contains just constant functions. In this case one should consider the crossed product $G \ltimes (A/\,\mathrm{im}(\mu)A)$ instead of this subalgebra of invariant elements. The points is that this algebra does not inherit a Poisson bracket, but a so-called noncommutative Poisson structure (cf. [6, 93, 73, 74, 75]). This is a class $\Pi \in HH^2(A,A)$ satisfying $[\Pi,\Pi] = 0$, where the bracket is the Gerstenhaber bracket on Hochschild cohomology. For example, in the case of the noncommutative torus (cf. Example 11.1.5)

$$\Pi = \delta_1 \wedge \delta_2$$

is a noncommutative Poisson structure (cf. [93]). It would be very interesting to generalize the notion of momentum map to this setting.

11.2 Noncommutative differential forms

As a preparation for symplectic algebras, we introduce derivation based differential calculi in this section. These calculi were introduced by Dubois-Violette [24]. We first need to introduce the universal differential calculus on an algebra.

Suppose A is an (associative) unital algebra over $\mathbb{R}$ or $\mathbb{C}$. Define the **noncommutative 1-forms** on A by $\Omega^1_{nc}(A) = \ker(m)$, where $m : A \otimes A \to A$ denotes the multiplication of the algebra. This is an (A, A)-bimodule. One can prove that $\mathrm{Der}(A) \cong \mathrm{Hom}_{(A,A)}(\Omega^1_{nc}(A), A)$. Define an (A, A)-bimodule morphism $d : A \to \Omega^1_{nc}(A)$ by

$$d(a) := a \otimes 1 - 1 \otimes a,$$

for all $a \in A$. We shall use the notation da for $d(a)$. Define

$$\Omega^{\bullet}_{nc}(A) := T_A(\Omega^1_{nc}(A)),$$

the universal tensor algebra over A of $\Omega^1_{nc}(A)$. The differential d extends to a differential $d : \Omega^{\bullet}_{nc}(A) \to \Omega^{\bullet+1}_{nc}(A)$ by

$$d(a \otimes b) = d\,a \otimes_A d\,b = (1 \otimes a - a \otimes 1)(1 \otimes b - b \otimes 1),$$

for all $a, b \in A$ and

$$d(\omega_1 \otimes \ldots \otimes \omega_n) := \sum_{i=1}^{n} (-1)^{i+1} \omega_1 \otimes \ldots \otimes d\omega_i \otimes \ldots \otimes \omega_n,$$

for all $\omega_i \in \Omega^1_{nc}(A)$, $i = 1, \ldots, n$, for all $n \in \mathbb{N}$. One can see that $\Omega^{\bullet}_{nc}(A)$ is generated by elements of the form

$$da_1\, da_2 \ldots da_n$$

and

$$a_0\, da_1\, da_2 \ldots da_n$$

for $a_0, \ldots, a_n \in A$ and $n \in \mathbb{Z}_{\geq 0}$.

Suppose $D \in \mathrm{Der}(A)$ is a derivation. A map $i_D : \Omega^{\bullet}_{nc}(A) \to \Omega^{\bullet-1}_{nc}(A)$ is given by

$$i_D(a_0\, da_1 \ldots da_n) := \sum_{i=1}^{n} (-1)^{i+1} a_0\, da_1 \ldots D(a_i) \ldots da_n$$

and

$$i_D(da_1 \ldots da_n) := \sum_{i=1}^{n} (-1)^{i+1} da_1 \ldots D(a_i) \ldots da_n$$

for $a_0, \ldots, a_n \in A$ and $n \in \mathbb{Z}_{\geq 0}$. Finally, one can form a complete Cartan calculus defining a Lie derivative by

$$\mathcal{L}_D(a_0\, da_1 \ldots da_n) = D(a_0)\, da_1 \ldots da_n + \sum_{i=1}^{n} a_0\, da_1 \ldots d(D(a_i)) \ldots da_n,$$

for $D \in \mathrm{Der}(A)$, $a_0, \ldots, a_n \in A$ and $n \in \mathbb{Z}_{\geq 0}$.

The differential calculus thus formed is the **universal differential calculus** on A. Indeed, it is universal in the following sense. Given any graded differential algebra $(\Omega^\bullet, d')$ with $\Omega^0 = A$, there exists a unique morphism of graded differential algebras

$$(\Omega^\bullet_{nc}(A), d) \to (\Omega^\bullet, d').$$

The cohomology of $(\Omega^\bullet_{nc}(A), d)$ is zero except at degree zero, where it is k. One obtains a more interesting cohomology theory with the following definition of so-called **noncommutative De Rham forms**

$$\mathrm{DR}^\bullet(A) := \Omega^\bullet_{nc}(A)/[\Omega^\bullet_{nc}(A), \Omega^\bullet_{nc}(A)],$$

where $[.,.]$ denotes the graded commutator. The maps d, i_D and $\mathcal{L}_D$ descend to $\mathrm{DR}^\bullet(A)$ for all $D \in \mathrm{Der}(A)$, forming a a Cartan calculus with the usual identities. This differential calculus is the **noncommutative De Rham differential calculus**. If $A = C^\infty(M)$ for a smooth manifold M, then $\mathrm{DR}^\bullet(A)$ can be identified with the de Rham complex $\Omega^\bullet(M)$ on M.

We shall be interested in a different differential calculus, which forms a suitable framework for defining symplectic algebras and symplectic reduction. Note that $\mathrm{Der}(A)$ is a $Z(A)$-module. For $n \in \mathbb{N}$, define

$$\underline{\Omega}^n_{\mathrm{Der}}(A) := \mathrm{Hom}_{Z(A)}(\textstyle\bigwedge^n_{Z(A)} \mathrm{Der}(A), A)$$

and

$$\Omega^0(A) := A.$$

For $n \in \mathbb{N}$ a differential $d : \underline{\Omega}^n_{\mathrm{Der}}(A) \to \underline{\Omega}^{n+1}_{\mathrm{Der}}(A)$ is given by

$$\begin{aligned} d\omega(D_0, \ldots, D_n) := &\sum_{i=0}^{n} (-1)^i D_i(\omega(D_0, \ldots, \hat{D}_i, \ldots, D_n)) \\ &+ \sum_{0 \le i < j \le n}^{n} (-1)^{i+j}([D_i, D_j]\omega(D_0, \ldots, \hat{D}_i, \ldots, \hat{D}_j, \ldots, D_n)) \end{aligned}$$

Note that this is well defined, since $Z(A)$ is stable under $\mathrm{Der}(A)$ and for all $D_1, D_2 \in \mathrm{Der}(A)$ and $z \in Z(A)$ one has $[D_1, zD_2] = D_1(z)D_2 + z[D_1, D_2]$. The graded differential algebra thus obtained is called the **derivations based differential calculus**, cf. [24]. It has a minimal differential subalgebra

$$(\Omega^\bullet_{\mathrm{Der}}(A), d) \subset (\underline{\Omega}^\bullet_{\mathrm{Der}}(A), d)$$

called the **minimal derivations based differential calculus**. It equals the image of the unique map of graded differential algebras $\Omega^\bullet_{nc}(A) \to \Omega^\bullet_{\mathrm{Der}}(A)$, defined by extension of

$$da \mapsto (D \mapsto D(a)).$$

There is an obvious Cartan calculus on $\underline{\Omega}^\bullet_{\mathrm{Der}}(A)$ and $\Omega^\bullet_{\mathrm{Der}}(A)$, based on

$$i_D\omega(D_1, \ldots, D_n) := \omega(D, D_1, \ldots, D_n)$$

for $D, D_1, \ldots, D_n \in \mathrm{Der}(A)$ and $\omega \in \Omega^{n+1}_{\mathrm{Der}}(A)$.

11.3 Pullback and pushforward of forms

In order to be able to define symplectic reduction we introduce in this section the notion of submanifold ideal and quotient manifold subalgebra and discuss the behavior of forms with respect to these structures. Except for Proposition 11.3.4, this material can be found in [51].

Suppose A is an algebra and $I \subset A$ a linear subspace. Define

$$\mathrm{Der}(A, I) := \{D \in \mathrm{Der}(A) \mid D(I) \subset I\}$$

and

$$\mathrm{Der}(A, I)_0 := \{D \in \mathrm{Der}(A) \mid D(A) \subset I\}.$$

If I is an ideal, one easily sees that there is a canonical exact sequence

$$0 \longrightarrow \mathrm{Der}(A, I)_0 \hookrightarrow \mathrm{Der}(A, I) \longrightarrow \mathrm{Der}(A/I). \tag{11.3.1}$$

Definition 11.3.1. An ideal $I \subset A$ is a **submanifold ideal** if $\mathrm{Der}(A, I) \to \mathrm{Der}(A/I)$ is surjective and $Z(A/I) = Z(A)/(I \cap Z(A))$.

Hence for submanifold ideals $I \subset A$ we can complete the exact sequence (11.3.1) to a short exact sequence

$$0 \to \mathrm{Der}(A, I)_0 \to \mathrm{Der}(A, I) \to \mathrm{Der}(A/I) \to 0.$$

Proposition 11.3.2. *If $I \subset A$ is a submanifold ideal, then there is a canonical surjective* ***pushforward map of graded differential algebras***

$$p_* : \Omega^\bullet_{\mathrm{Der}}(A) \to \Omega^\bullet_{\mathrm{Der}}(A/I)$$

given by

$$p_*(a_0\, da_1, \ldots, da_n) = p(a_0)\, d(p(a_1)), \ldots, d(p(a_n)),$$

for $a_0, \ldots, a_n \in A$ and $n \in \mathbb{N}$.

Proof. For $D \in \mathrm{Der}(A/I)$ and $a \in A$

$$d(p(a))(D) := p(da(\tilde{D})),$$

where $\tilde{D} \in \mathrm{Der}(A, I)$ lifts D. This does not depend on the choice of the lift, since $\tilde{D} \in \mathrm{Der}(A, I)_0$ implies $da(\tilde{D}) = \tilde{D}(a) \in I$. Moreover $d(p(a))$ is $Z(A/I)$ linear, since

$$d(p(a))(z\, D) = p(\widetilde{z\, D}(a)) = p(\tilde{z}\tilde{D}(a)) = z\, \tilde{D}(a) = z\, d(p(a))(D),$$

for $D \in \mathrm{Der}(A/I)$ and $a \in A$, where $\tilde{z} \in Z(A)$ is a lift of $z \in Z(A/I)$.

One easily sees that $[p_*, d] = 0$. ☐

Suppose $J \subset Z(A)$ is a central subset of A. Consider the ideal $I := J\, A$ generated by J. We call an ideal of this form a **central ideal**. Note that, in this case, $J\, \mathrm{Der}(A) \subset \mathrm{Der}(A, I)_0$.

Definition 11.3.3. Suppose $I = JA \subset A$ is a central ideal for a central subset $J \subset A$. Then I is a **nice submanifold ideal** if I is a submanifold ideal of A and

$$\mathrm{Der}(A, I)_0 = J\, \mathrm{Der}(A).$$

In this case the above exact sequence 11.3.1 becomes the short exact sequence

$$0 \to J\ \mathrm{Der}(A) \to \mathrm{Der}(A, I) \to \mathrm{Der}(A/I) \to 0.$$

It is important that J is central since $\mathrm{Der}(A)$ is a $Z(A)$-module, but not an A-module in general.

Proposition 11.3.4. *If $I = JA$ is a nice submanifold ideal, then there is a canonical surjective* ***pushforward map of graded differential algebras***

$$p_* : \underline{\Omega}^\bullet_{\mathrm{Der}}(A) \to \underline{\Omega}^\bullet_{\mathrm{Der}}(A/I)$$

given by

$$p_*\omega(D_1, \ldots, D_n) = p(\omega(\tilde{D}_1, \ldots, \tilde{D}_n)),$$

where $\tilde{D}_i \in Der(A, I)$ is a lift of $D_i \in \mathrm{Der}(A/I)$ along the map $\mathrm{Der}(A, I) \to \mathrm{Der}(A/I)$ for $i = 1, \ldots, n$.

Proof. Note that p_* does not depend on the choice of the lifts $\tilde{D}_i$. Indeed, if $\tilde{D} \in J\,\mathrm{Der}(A)$, i.e. $\tilde{D} = j\,D$ for some $j \in J$ and $D \in \mathrm{Der}(A)$, then

$$\begin{aligned} \omega(\tilde{D}, \tilde{D}_1, \ldots, \tilde{D}_{n-1}) &= \omega(jD, \tilde{D}_1, \ldots, \tilde{D}_{n-1}) \\ &= j\,\omega(D, \tilde{D}_1, \ldots, \tilde{D}_{n-1}) \in I \end{aligned}$$

for all $\tilde{D}_1, \ldots, \tilde{D}_{n-1} \in \mathrm{Der}(A, I)$ and $\omega \in \Omega^n_{\mathrm{Der}}(A)$.

Moreover, $p_*\omega$ is $Z(A/I)$-linear since

$$\begin{aligned} p_*\omega(p(z)\,D_1, \ldots, D_n) &= [\omega(\widetilde{p(z)\,D}_1, \ldots, \tilde{D}_n)] \\ &= [\omega(z\,\tilde{D}_1, \ldots, \tilde{D}_n)] \\ &= p(z)p_*\omega(D_1, \ldots, D_n) \end{aligned}$$

for all $\tilde{D}_1, \ldots, \tilde{D}_n \in \mathrm{Der}(A, I)$ and $z \in Z(A)$. □

Let's focus on pullbacks now. Consider an algebra A with a subset $B \subset A$. Define

$$\mathrm{Der}(A, B)^0 := \{D \in \mathrm{Der}(A) \mid D(B) = 0\}.$$

Note the difference with $\mathrm{Der}(A, B)_0$. Suppose $B \subset A$ is a subalgebra. The Lie subalgebra $\mathrm{Der}(A, B)^0$ is a Lie ideal in $\mathrm{Der}(A, B)$. There is an exact sequence of Lie algebras

$$0 \to \mathrm{Der}(A, B)^0 \to \mathrm{Der}(A, B) \to \mathrm{Der}(B)$$

where the arrows are, respectively, inclusion and restriction of derivations to B.

Definition 11.3.5. The subalgebra $B \subset A$ is a **quotient manifold subalgebra** if

(i) $Z(B) = Z(A) \cap B$;

(ii) $\mathrm{Der}(A, B) \to \mathrm{Der}(B)$ is surjective, which implies that is there is a short exact sequence of Lie algebras

$$0 \to \mathrm{Der}(A, B)^0 \to \mathrm{Der}(A, B) \to \mathrm{Der}(B) \to 0$$

and $\mathrm{Der}(B) \cong \mathrm{Der}(A, B)/\,\mathrm{Der}(A, B)^0$.

(iii) $B = \{a \in A | D(a) = 0 \text{ for all } D \in \mathrm{Der}(A, B)^0\}$.

The subalgebra of $\underline{\Omega}^\bullet_{\mathrm{Der}}(A)$ defined by

$$\underline{\Omega}^\bullet_{\mathrm{Der}}(A, B)_{\mathrm{bas}} := \{\omega \in \underline{\Omega}^\bullet_{\mathrm{Der}}(A) \mid \mathcal{L}_D\omega = 0, i_D\omega = 0 \text{ for all } D \in \mathrm{Der}(A, B)^0\}$$

is called the algebra of B-**basic forms** on A.

Proposition 11.3.6. *If $i : B \hookrightarrow A$ is a quotient manifold subalgebra, then there exists a canonical* ***pullback map of differential algebras***

$$i^* : \underline{\Omega}^\bullet_{\mathrm{Der}}(A, B)_{\mathrm{bas}} \to \underline{\Omega}^\bullet_{\mathrm{Der}}(B)$$

given by

$$i^*\omega(D_1, \dots, D_n) := \omega(\tilde{D}_1, \dots, \tilde{D}_n),$$

where $\tilde{D}_i$ is a lift of D_i along $\mathrm{Der}(A, B) \to \mathrm{Der}(B)$ *for all $i = 1 \dots, n$.*

Proof. By the definition of basic forms, the value of $i^*\omega$ does not depend on the choice of the lift. By condition (iii) and the Cartan formula, $\omega(\tilde{D}_1, \dots, \tilde{D}_n)$ actually lies in B. For example on 1-forms we have, for $D \in \mathrm{Der}(A, B)^0$ and $\tilde{D} \in \mathrm{Der}(A, B)$,

$$\begin{aligned} D(\omega(\tilde{D})) &= i_D d(i_{\tilde{D}}\omega) = -i_{\tilde{D}} i_D d\omega + i_D \mathcal{L}_{\tilde{D}}\omega \\ &= \mathcal{L}_{\tilde{D}} i_D\omega + i_{[\tilde{D}, D]}\omega = 0. \end{aligned}$$

The $Z(B)$-linearity follows from condition (i). □

11.4 Hamiltonian actions on symplectic algebras

In this section we introduce symplectic algebras. We relate this notion to Poisson algebras. The main result of this section and of this part is Theorem 11.4.2, which deals with the construction of the symplectic reduction of a Hamiltonian action on a symplectic algebra.

Suppose A is a **symplectic algebra**, i.e. an algebra A together with a noncommutative 2-form $\omega \in \underline{\Omega}^2_{\mathrm{Der}}(A)$ that is closed ($d\omega = 0$) and non-degenerate, i.e.

$$D \mapsto i_D\omega$$

is a bijection of $Z(A)$-modules $\mathrm{Der}(A) \to \underline{\Omega}^1_{\mathrm{Der}}(A)$.

Suppose that (A, ω) is a symplectic algebra. Given $a \in A$, the **Hamiltonian derivation** D_a of a is defined by

$$i_{D_a}\omega = d\,a$$

or equivalently

$$\omega(D_a, D) = D(a)$$

for all $D \in \mathrm{Der}(A)$.

Proposition 11.4.1. *Suppose (A, ω) is a symplectic algebra. Then A is a Poisson algebra, with Poisson bracket defined by*

$$\{a, a'\} := \omega(D_a, D_{a'}),$$

for all $a, a' \in A$.

Proof. The proof is completely analogous to the classical (geometric) case. For example, for $a, b, c \in A$ we obtain the Leibniz rule

$$\begin{aligned}\{a\,b, c\} &= \omega(D_{a\,b}, D_c) = d(a\,b)(D_c) \\ &= da\,b(D_c) + a\,db(D_c) = \{a, c\}b + a\{b, c\}.\end{aligned}$$

Similarly, the Jacobi identity follows from $d\omega = 0$. □

The conditions for an action α of a Lie algebra $\mathfrak{g}$ on an algebra A to be Hamiltonian with respect to a momentum map $\mu : \mathfrak{g} \to A$ translate to

(i) $i_{\alpha(X)}\omega = -d(\mu(X))$ or $\alpha(X) = -D_{\mu(X)}$;

(ii) $\mu([X, Y]) = \omega(D_{\mu(X)}, D_{\mu(Y)}) = -\alpha(X)\mu(Y)$;

(iii) $\mathrm{im}(\mu) \subset Z(A)$,

for all $X, Y \in \mathfrak{g}$. For any algebra A endowed with an action of a Lie algebra $\mathfrak{g}$ we define the subalgebra of $\mathfrak{g}$-invariants by

$$A^{\mathfrak{g}} := \{a \in A \mid \alpha(X)a = 0 \text{ for all } X \in \mathfrak{g}\}.$$

Theorem 11.4.2. *Suppose (A, ω) is a symplectic algebra and a Lie algebra $\mathfrak{g}$ acts in a Hamiltonian fashion on A, with momentum map $\mu : \mathfrak{g} \to A$.*

(a) If $\mathrm{im}(\mu)A$ is a nice submanifold ideal of A, then the algebra

$$A/(\mathrm{im}(\mu)A)$$

has a canonical closed 2-form $p_*\omega \in \underline{\Omega}^2_{\mathrm{Der}}(A/(\mathrm{im}(\mu)A))$.

(b) The algebra $A/(\mathrm{im}(\mu)A)$ induces an action α' of $\mathfrak{g}$.

(c) If

$$A^0 := (A/(\mathrm{im}(\mu)A))^{\mathfrak{g}}$$

is a quotient manifold algebra of $A/\mathrm{im}(\mu)A$, then A^0 can be endowed with a 2-form

$$i^*p_*\omega \in \underline{\Omega}^2_{\mathrm{Der}}\left((A/(\mathrm{im}(\mu)A))^{\mathfrak{g}}\right).$$

(d) If $i^*p_*\omega$ is symplectic, then the associated Poisson structure coincides with the Poisson structure from Theorem 11.1.15.

Proof. (a) follows from Proposition 11.3.4. For (b) note that the Leibniz rule and condition (ii) imply $\alpha(X)(\mu(Y)a) \in \operatorname{im}(\mu)A$. The existence of the closed form $i^*p_*\omega$ of (c) follows from Proposition 11.3.6 and the fact that $p_*\omega$ is a basic form. Indeed, for all $D \in \operatorname{Der}(A/\operatorname{im}(\mu)A, (A/\operatorname{im}(\mu)A)^{\mathfrak{g}})^0 = \alpha'(\mathfrak{g})$ one has

$$\mathcal{L}_D i^* p_* \omega = i^* p_* \mathcal{L}_{\tilde{D}} \omega = 0$$

and

$$i_D i^* p_* \omega = i^* p_* i_{\tilde{D}} \omega = -i^* p_* d(\mu(X)) = 0,$$

where $\tilde{D} = \alpha(X) \in \alpha(\mathfrak{g})$ is a lift of D. (d) follows immediately. □

We call $(A^0, i^*p_*\omega)$ the **algebraic Marsden-Weinstein quotient** with respect to the given $\mathfrak{g}$-action and momentum map μ. Obviously, results (iii) and (iv) are weaker than one would like. We leave this as an open problem.

11.5 Algebraic prequantization

In this section we construct the prequantization of a Hamiltonian Lie algebra action on a symplectic Lie algebra.

A **central bimodule** over an algebra A (cf. [51] and references therein) is an (A, A)-bimodule M such that

$$z\,m = m\,z$$

for all $m \in M$ and $z \in Z(A)$. Consider the category with as objects algebras and as arrows $A \to B$ isomorphism classes of central (B, A)-bimodules. Composition of a central (A, B)-module M_1 and a central (B, C)-module M_2 is given by the tensor product $M_1 \otimes_B M_2$ over B, cf. Section 4.5. We say a central bimodule L is invertible if the isomorphism class $[L]$ is invertible in this category. The set of invertible projective central (A, A)-bimodules is called the **Picard group** of A over $Z(A)$. It is denoted by $\operatorname{Pic}_{Z(A)}(A)$.

Example 11.5.1. If $A = C(X)$ for a compact topological space X. Then $\operatorname{Pic}_{Z(A)}(A)$ equals the group $\operatorname{Pic}(X)$ of line bundles over X. It can be shown to be isomorphic to $\check{H}^2(X, \mathbb{Z})$ (see also Section 9.2).

A **connection on a central bimodule** M (cf. [51] and references therein) is an element

$$\nabla \in \underline{\Omega}^1_{\operatorname{Der}}(A) \otimes_A \operatorname{End}(M),$$

satisfying

$$\nabla_D(a\,m) = D(a)\,m + a\nabla_D(m)$$

and

$$\nabla_D(m\,a) = m\,D(a) + \nabla_D(m)a$$

for all $D \in \operatorname{Der}(A)$, $a \in A$ and $m \in M$, using the notation $\nabla_D = \nabla(D)$. Analogously to the geometric case, there is a **curvature 2-form** $R_\nabla \in \underline{\Omega}^2_{\operatorname{Der}}(A)$ defined by

$$R_\nabla(X, Y)m := \frac{1}{2\pi i}(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(m)$$

for $X, Y \in \operatorname{Der}(A)$ and $m \in M$.

Example 11.5.2. If $M = pA^n$ for a projection $p \in M^n(A)$, then $p\,(d \otimes id_{A^n})\,p$ is a connection on M, called the **Grassmann connection** (cf. Example 4.5.10).

Obviously, there is an analogous notion of connection replacing $\underline{\Omega}^1_{\mathrm{Der}}(A)$ by $\Omega^1_{\mathrm{Der}}(A)$. We shall focus on the first notion.

Suppose (A, ω) is a symplectic algebra, with $\omega \in \underline{\Omega}^2_{\mathrm{Der}}(A)$. A **prequantization module** for (A, ω) is an invertible projective central right (A, A)-bimodule $L \in \mathrm{Pic}_{Z(A)}(A)$, together with a connection ∇ with curvature 2-form

$$R_\nabla = \omega \in \underline{\Omega}^2_{\mathrm{Der}}(A).$$

Suppose $\mathfrak{g}$ a Lie algebra that acts in a Hamiltonian fashion on A, with algebraic momentum map $\mu : \mathfrak{g} \to A$.

Proposition 11.5.3. *If (L, ∇) is a prequantization module for (A, ω), then L carries a representation of $\mathfrak{g}$ given by*

$$a \mapsto \nabla_{\alpha(X)} - 2\pi i\, \mu(X).$$

Proof. The proof is a simple computation similar to the one in the proof of Theorem 9.3.1. Crucial is the fact that $\mathrm{im}(\mu) \subset Z(A)$. □

How should one proceed to quantize such a prequantization? This shall be discussed in Section 12.6. As mentioned in the introduction a central rôle is played by the Baum-Connes analytical assembly map. The construction of this map we shall discuss in the first five sections of the next chapter.

Chapter 12

Noncommutative analytical assembly maps

12.1 Proper actions

In this section we review proper actions of groups on C^*-algebras and introduce new classes of examples thereof.

Suppose G is a locally compact unimodular group. Let $\alpha : G \to \mathrm{Aut}(A)$ be a continuous left action of G on a C^*-algebra A. For a subalgebra $A_0 \subset A$ let $M(A_0)$ denote the subalgebra of the multiplier algebra $M(A)$ of A consisting of $m \in M(A)$ that satisfy $m\, A_0 \subset A_0$

Definition 12.1.1. (cf. [69]) The action α is **proper** with respect to a dense G-invariant $*$-subalgebra A_0 of A if

(i) for all $a, b \in A_0$

$$x \mapsto \alpha_x(a)\, b$$

is in $L^1(G, A, \lambda)$ with respect to a Haar measure λ on G and

(ii) for all $a, b, c \in A_0$ there exists an element $d \in M(A_0)^G$ such that

$$\int_G \alpha_x(a\, b)\, c\, \lambda(dx) = d\, c.$$

Remark 12.1.2. In more recent work [70] Rieffel introduced a more intrinsic notion of proper action without reference to a subalgebra A_0. For our purposes it is convenient to stick to the old definition. A special rôle for a subalgebra of a C^*-algebra is certainly within the spirit of noncommutative geometry.

A large class of examples of proper actions on C^*-algebras is of the following kind. First we recall a definition by G. Kasparov (cf. [37]). Suppose M is a locally compact space endowed with a continuous action of a locally compact group G.

Definition 12.1.3. A C^*-algebra A is a $C_0(M)$**-algebra** if there exists a homomorphism

$$\pi : C_0(M) \to Z(M(A))$$

and an approximate unit $\{u_i\}$ in $C_c(M)$ such that

$$\lim_{i\to\infty} \|u_i a - a\| = 0.$$

A **G-$C_0(M)$-algebra** is a $C_0(M)$-algebra A on which G acts continuously such that

$$x \cdot (\pi(f)a) = \pi(x \cdot f)(x \cdot a). \tag{12.1.1}$$

One can prove that G-$C_0(M)$-algebras correspond to G-equivariant upper semi-continuous fields of C^*-algebras over M.

Proposition 12.1.4. *If G acts properly on M and A is a G-$C_0(M)$-algebra then G acts properly on A with respect to $A_0 := C_c(M)A$.*

Proof. Equation (12.1.1) implies that A_0 is G-invariant. The existence of the approximate unit implies that A_0 is dense in A. The fact that π maps $C_c(M)$ into the *center* of the multiplier algebra of A implies that A_0 is a $*$-subalgebra of A. Moreover, for $f, f' \in C_c(M)$ and $a, a' \in A$

$$x \cdot (\pi(f)a)\pi(f')a' = \pi((x \cdot f)f')a\, a',$$

hence $x \mapsto x \cdot (\pi(f)a)\pi(f')a'$ has compact support, say $C_{f\,f'}$. One has

$$\int_G x \cdot (\pi(f)a)\pi(f')a'\, dx = \int_{C_{f\,f'}} \pi(x \cdot f)a\, dx\, \pi(f')a',$$

where $\int_{C_{f\,f'}} \pi(x \cdot f)a\, dx$ is in A and hence a multiplier of A_0. □

Example 12.1.5. Consider a space M on which G acts properly and a C^*-algebra B. Let A be the C^* algebra $C_0(M, B)$ of continuous function with values in B that vanish at infinity. Then A is a G-$C_0(M)$-algebra, with

$$(\pi(f)\theta)(m) := f(m)\,\theta(m)$$

and

$$(x \cdot \theta)(m) := \theta(x^{-1} \cdot m)$$

for all $x \in G$, $f \in C_0(M)$ and $\theta \in A$. This is Example 2.6 in [69].

12.2 Example: proper actions on groupoid C^*-algebras

In this section we introduce another large class of examples. It comes from proper group actions on proper groupoids.

Suppose $H \rightrightarrows M$ is a proper groupoid and G a locally compact unimodular group. Suppose G **acts continuously on the groupoid** $H \rightrightarrows M$ (cf. e.g. [49]), i.e. G acts continuously on H and on M in the usual sense such that the action behaves well with respect to the groupoid structure maps: for all $x \in G$, $m \in M$ and $h, h' \in H$ one has

$$\begin{aligned} s(x \cdot h) &= x \cdot s(h), \\ t(x \cdot h) &= x \cdot t(h), \\ x \cdot (h\,h') &= (x \cdot h)(x \cdot h'), \\ u(x \cdot m) &= x \cdot u(m), \\ (x \cdot h)^{-1} &= x \cdot h^{-1}. \end{aligned}$$

Remark 12.2.1. The quotient $H/G \rightrightarrows M/G$ of such an action of a group G on a groupoid $H \rightrightarrows M$ inherits a groupoid structure from $H \rightrightarrows M$. An example of this is the gauge groupoid of a principal G-bundle $P \to M$. This is the quotient of the pair groupoid $P \times P \rightrightarrows P$ under the diagonal action of H.

Remark 12.2.2. Consider the group of continuous global bisections $\mathrm{Bis}(H)$ of $H \rightrightarrows M$ (cf. Section 5.5). This group gives rise to "inner" actions on $H \rightrightarrows M$. Indeed, one has the left action of $\mathrm{Bis}(H)$ on H given by $l_\sigma h := \sigma(t(h))h$, the right action given by $r_\sigma h := h(\sigma(s(h)))^{-1}$ and hence conjugation $r \circ l = l \circ r$.

Suppose $H \rightrightarrows M$ is endowed with a Haar system $\{\lambda^m\}_{m\in M}$ that is G-invariant, in the sense that

$$\int_{H^m} f(x \cdot h)\lambda^m(dh) = \int_{H^{x\cdot m}} f(h)\lambda^{x\cdot m}(dh)$$

for all $m \in M$ and $x \in G$. LConsider $C^*_r(H)$, the reduced C^*-algebra associated to $H \rightrightarrows M$ and $\{\lambda^m\}_{m\in M}$. Suppose that the action of G is proper and that the groupoid $H \rightrightarrows M$ itself is proper. We shall show that there exists an induced action of G on $C^*_r(H)$ which is proper.

Define $A := C^*_r(H)$ and $A_0 := C_c(H)$. The action of G on A_0 is defined in the obvious way by

$$(x \cdot f)(h) := f(x^{-1} \cdot h),$$

where $x \in G$, $f \in A_0$ and $h \in H$. We can extend this action to A, since $\|x \cdot f\| = \|f\|$ in the reduced C^*-algebra norm.

Lemma 12.2.3. *The reduced C^*-algebra norm is G-invariant on A_0, i.e. for all $x \in G$ and $f \in A_0$*

$$\|x \cdot f\| = \|f\|$$

Proof. Because of G-invariance of the Haar system, one has

$$\begin{aligned} \|x \cdot f\| &= \sup_{\|\xi\|_{\hat{L}^2_t(H)}=1} \|(x \cdot f) * \xi\|_{\hat{L}^2_t(H)} \\ &= \sup_{\|\xi\|_{\hat{L}^2_t(H)}=1} \|f * (x^{-1} \cdot \xi)\|_{\hat{L}^2_t(H)}. \end{aligned}$$

But, since $\|x^{-1} \cdot \xi\|_{\hat{L}^2_t(H)} = \|\xi\|_{\hat{L}^2_t(H)}$, the above term equals $\|f\|$. □

Lemma 12.2.4. *The action of G on A is strongly continuous.*

Proof. It suffices to show continuity on A_0 at the unit in G. Suppose $f \in A_0$ and $x \in G$. Then,

$$\begin{aligned} \|x \cdot f - f\| &= \sup_{\xi \in \hat{L}^2_t(H), \|\xi\|=1} \|(x \cdot f - f) * \xi\| \\ &\leq \|x \cdot f - f\|_{\hat{L}^2_t(H)} \\ &= \sup_{m\in M} \sqrt{\int_{H^m} |f(x^{-1} \cdot h) - f(h)|^2 \lambda^m(dh)}. \end{aligned}$$

By continuity of f one has for any $\varepsilon > 0$ that there is an open neighborhood U of e in G such that $x \in U$ implies $|f(x^{-1} \cdot h) - f(h)| < \varepsilon / \sup_{m\in M} \lambda^m(\mathrm{supp}(f))$. From this the statement easily follows. □

Proposition 12.2.5. *If G acts properly on a proper groupoid $H \rightrightarrows M$, then the induced action of G on the reduced C^*-algebra $C_r^*(H)$ is proper with respect to $C_c(H)$.*

Proof. One easily checks that $x \cdot (f * f') = (x \cdot f) * (x \cdot f')$ for all $x \in G$ and $f \in A_0 = C_c(H)$. For any $f, f' \in A_0$ the support of $x \mapsto (x \cdot f) * f'$ is compact, since the G-action is proper and $H \rightrightarrows M$ is a proper groupoid.

Next, we want to prove property 12.1.1.ii. Suppose $f, f' \in A_0$ and consider

$$\begin{aligned} \int_G (x \cdot f) * f'(h')\, dx &= \int_G \int_{H^{t(h')}} f(x^{-1} \cdot h)\, f'(h^{-1}h')\, \lambda^{t(h')}(dh)\, dx \\ &= \int_{H^{t(h')}} \int_G f(x^{-1} \cdot h)\, dx\, f'(h^{-1}h')\, \lambda^{t(h')}(dh) \\ &= \tilde{f} * f'(h'), \end{aligned}$$

where

$$\tilde{f}(h) := \int_G f(x^{-1} \cdot h)\, dx.$$

Note that $\tilde{f} \in C_b(H)$. In [79] Proposition 4.3 it is proved that $A = C_r^*(H)$ can be identified with the compact operators on $\hat{L}_s^2(H)$ made H-equivariant by averaging. From Lemma 4.4 in the same paper one can conclude that the multiplier algebra $M(A)$ equals the H-equivariant (adjointable) bounded operators on $\hat{L}_s^2(H)$. If $\pi_l(f) \in C_r^*(H)$ is an H-equivariant and bounded operator then

$$\begin{aligned} \int_G \pi_l(x \cdot f) dx &= \pi_l(\int_G x \cdot f\, dx) \\ &= \pi_l(\tilde{f}) \end{aligned}$$

is an H-equivariant bounded operator, hence an element of $M(A)$.

Left to show is that $\tilde{f}$ is a left multiplier of A_0, in the sense that $\tilde{f}A_0 = A_0$. We first prove this for the case that $H \rightrightarrows M$ is the pair groupoid $M \times M \rightrightarrows M$ with the diagonal action of G. Note that, in this case, convolution of $\tilde{f}, f'$ with $f, f' \in C_c(M \times M)$ is given by

$$\tilde{f} * f'(m,p) \int_M \tilde{f}(m,n) f'(n,p) dn,$$

w.r.t. a Radon measure on M. Consider the map $M \times M \times M \to \mathbb{C}$ given by $(m,n,p) \mapsto \tilde{f}(m,n)f'(n,p)$. The support of this map is

$$G\, \mathrm{supp}(f) \times M \cap M \times \mathrm{supp}(f').$$

Since the G-action on M is proper, this set is compact. Hence the support of $\tilde{f} * f'$ is compact.

The map $t \times s : G \to M \times M$ is a proper groupoid homomorphism. Hence for $f, f' \in C_c(G)$ the support of $\tilde{f} * f'$ is contained in

$$(t \times s)^{-1}\big((t \times s)(G\, \mathrm{supp}(f)) \times M \cap M \times (t \times s)(\mathrm{supp}(f'))\big),$$

which is compact. We conclude that the support of $\tilde{f} * f'$ is compact since it is a closed subset of a compact set. $\square$

Example 12.2.6. As an example of Proposition 12.2.5, consider a proper action of G on a space M with a Radon measure μ. Then the diagonal action of G on the pair groupoid $M \times M \rightrightarrows M$ is proper. Hence the induced action of G on $C^*(M \times M)$ is proper. Since $C^*(M \times M)$ is canonically isomorphic to the C^*-algebra of compact operator on $L^2(M, \mu)$ we have a proper action of G on $\mathcal{K}(L^2(M, \mu))$. This generalizes Rieffel's Example 2.1 of [69].

Example 12.2.7. If H is a compact group. Then H acts on itself from the left, for example by left multiplication or by conjugation. Both actions induce a proper action of H on $C_r^*(H)$ with $A_0 = C_c(H)$.

Example 12.2.8. Suppose M is a space with commuting proper actions of locally compact groups G and H. Consider the action groupoid $H \ltimes M \rightrightarrows M$. A proper action of G on $H \ltimes M \rightrightarrows M$ is defined by

$$g \cdot (h, m) := (h, g \cdot m),$$

where $g \in G$, $h \in H$ and $m \in M$. By Proposition 12.2.5 the induced action of G on

$$C^*(H \ltimes M) \cong H \ltimes C_0(M)$$

is proper. This is Example 2.5 in [69].

Example 12.2.9. Suppose X is an orbifold and $H \rightrightarrows M$ a proper étale groupoid representing X, i.e. there exists a homeomorphism $M/H \to X$. A group G acts properly on X iff it corresponds to a proper action of G on $H \rightrightarrows M$. Hence it induces a proper action of G on the C^*-algebra $C^*(H)$ by Proposition 12.2.5.

Definition 12.2.10. We call the action of G on A **amenable** if the natural projection $G \ltimes A \to G \ltimes_r A$ is an isomorphism of C^*-algebras. The action is K**-amenable** if $K_i(G \ltimes A) \to K_i(G \ltimes_r A)$ is an isomorphism of groups for $i = 0, 1$.

Obviously amenability implies K-amenability. A proper action of G on M induces an amenable (and proper) action of G on $C_0(M)$. Also, if G is amenable, then any action of G on a C^*-algebra is amenable. But not any proper action of G on a C^*-algebra is amenable.

Example 12.2.11. In [69] Rieffel gives the example of G acting on $A := \mathcal{K}(L^2(G))$ by conjugation with the regular representation, which is a special case of Proposition 12.2.5. In this case,

$$G \ltimes A \cong C^*(G) \otimes \mathcal{K}$$

and

$$G \ltimes_r A \cong C_r^*(G) \otimes \mathcal{K}.$$

Hence if $C^*(G) \ncong C_r^*(G)$, then the action is not amenable. An example of this is G being the free group on two generators.

Lemma 12.2.12. *If G acts properly on M and A is a G-$C_0(M)$-algebra, then the action of G on A is amenable.*

Proof. This is an immediate consequence of the fact that $G \ltimes M$ is proper groupoid, hence amenable and Theorem 3.4 in [2], which states (among other things) that, if the action of G on M is amenable, then the action of G on A is amenable. □

12.3 Hilbert C^*-modules associated to proper actions

The main ingredient in the definition of a noncommutative Baum-Connes analytical assembly map for a action of a group G on a C^*-algebra A is a certain Hilbert $G \ltimes_r A$-module associated to this action. For this to exist, the action has to be proper. In this section we shall construct this module, following Rieffel [69] very closely up to Corollary 12.3.5. For the actual construction of the analytical assembly map we shall need an extra condition. This condition corresponds in the commutative case to the action being cocompact. Since compactness of a space X corresponds to the commutative C^*-algebra $C_0(X)$ being unital, we shall name actions satisfying this condition counital.

Suppose G is a locally compact unimodular group acting on a C^*-algebra A. Consider the convolution $*$-algebra $C_c(G, A)$ with convolution defined by ($f, g \in C_c(G, A)$, $x \in G$)

$$f * g(x) := \int_G \alpha_y(f(y^{-1}\, x))\, g(y)\, \lambda(dy)$$

and involution defined by

$$f^*(x) = \alpha_x(f(x^{-1}))^*,$$

where λ is a Haar measure on G.

Consider the right representation $C_c(G, A) \to \mathcal{B}(L^2(G, A, \lambda))$ of $C_c(G, A)$ defined by

$$g \mapsto . * g$$

This is just the integration $\int \pi_L(x)\, g(x)\, \lambda(dx)$ of the left regular (covariant) representation π_L of G on $L^2(G, A, \lambda)$:

$$\pi_L(y) f(x) := \alpha_y(f(y^{-1}\, x)).$$

The closure of the image of $C_c(G, A)$ under this map is called the **reduced crossed product**, denoted by $G \ltimes_r A$. Obviously, if $A = \mathbb{C}$ we get the reduced group C^*-algebra (cf. Example 4.1.8).

Suppose a locally compact group G acts properly on a C^*-algebra A with respect to A_0. We shall now construct the needed Hilbert $G \ltimes_r A$-module using an appropriate closure of A_0. Define a sesquilinear form on A_0 with values in $G \ltimes_r A$ by ($a, b \in A_0, x \in G$)

$$\langle a, b \rangle\, (x) := \alpha_x(a^*)\, b.$$

Denote the space of finite linear combinations of elements $\langle a, b \rangle \in G \ltimes_r A$ by E_0.

Lemma 12.3.1. *The space E_0 is a *-subalgebra of $G \ltimes_r A$.*

Proof. A computation shows that for $a, b, c, d \in A_0$ and $x \in G$ one has

$$\begin{aligned} \langle a, b \rangle * \langle c, d \rangle\, (x) &= \int_G \alpha_y\, \langle a, b \rangle\, (y^{-1}\, x)\ \langle c, d \rangle\, (y)\, \lambda(dy) \\ &= \left\langle a, \int_G \alpha_y(b\, c^*)\, d\ \lambda(dy) \right\rangle (x), \end{aligned} \tag{12.3.2}$$

where the integral is in A_0 by Definition 12.1.1.ii. Hence E_0 is closed under multiplication.

Furthermore, for all $a, b \in A_0$

$$\langle a, b\rangle^* (x) = \alpha_x(\alpha_{x^{-1}}(a^*)\, b)^* = \langle b, a\rangle\, (x),$$

hence E_0 is closed under $*$. ☐

Lemma 12.3.2. *A right action of E_0 on A_0 is given by ($a \in A_0$, $f \in E_0$)*

$$a \cdot f := \int_G \alpha_x(a)\; f(x)\, \lambda(dx).$$

Proof. Suppose $a, b, c \in A_0$, $f = \langle b, c\rangle \in E_0$, then

$$a \cdot f = \int_G \alpha_x(a\, b^*)c\, \lambda(dx)$$

which is in A_0 by Definition 12.1.1.ii.

A small computation shows for $a \in A_0$ and $f, g \in E_0$ that

$$\begin{aligned}
a \cdot (f * g) &= \int_G \alpha_x(a)\, (f * g)(x)\, \lambda(dx) \\
&= \int_G \int_G \alpha_x(a)\, \alpha_y(f(y^{-1}\, x))\, (g(y))\, \lambda(dx)\, \lambda(dy) \\
&= \int_G \alpha_y(\int_G \alpha_x(a)\, f(x)\, \lambda(dx)\, (g(y))\, \lambda(dy) \\
&= (a \cdot f) \cdot g.
\end{aligned}$$

Similarly, one checks that $a \cdot f^* = (a^* \cdot f)^*$. Hence one has a right action of E_0 on A_0. ☐

Lemma 12.3.3. *The pair $(A_0, \langle ., .\rangle)$ forms a full pre-Hilbert E_0-module (or rigged E_0-space).*

Proof. From Equation 12.3.2 it follows at once that for $a, b, c, d \in A_0$, $f = \langle c, d\rangle \in E_0$

$$\langle a, b \cdot f\rangle = \langle a, b \cdot \langle c, d\rangle\rangle = \langle a, b\rangle * \langle c, d\rangle = \langle a, b\rangle * f.$$

Left to prove is the positivity of $\langle a, a\rangle \in G \ltimes_r A$ for all $a \in A_0$. For any $\xi \in L^2(G, A, \lambda)$ we compute

$$\begin{aligned}
\langle \langle a, a\rangle\, \xi, \xi\rangle_{L^2} &= \int_G (\langle a, a\rangle * \xi)^*(x)\xi(x)\lambda(dx) \\
&= \int_G \left(\int_G \alpha_y \langle a, a\rangle\, (y^{-1}\, x)\xi(y)\lambda(dy)\right)^* \xi(x)\lambda(dx) \\
&= \int_G \left(\int_G \alpha_x(a^*)\alpha_y(a)\xi(y)\lambda(dy)\right)^* \xi(x)\lambda(dx) \\
&= \int_G \xi(y)^*\alpha_y(a^*)\lambda(dy) \int_G \alpha_x(a)\xi(x)\lambda(dx) \geq 0,
\end{aligned}$$

which finishes the proof. ☐

Let E denote the closure of E_0 in $G \ltimes_r A$.

Lemma 12.3.4. *The algebra E is an ideal in $G \ltimes_r A$.*

Proof. We show that E is a right ideal (and hence a two-sided ideal). Suppose $a, b \in A_0$ and $f \in C_c(G, A)$, then

$$\begin{aligned} \langle a, b\rangle * f &= \int_G \alpha_y \langle a, b\rangle (y^{-1} \cdot) f(y) \, \lambda(dy) \\ &= \int_G \langle a, \alpha_y(b) \, f(y)\rangle \, \lambda(dy), \end{aligned}$$

and this last integral converges in E, since f is compactly supported. By continuity we conclude that E is a right ideal in $G \ltimes_r A$. ☐

Let $\bar{A}_0$ denote the closure of A_0 with respect to the norm $\|a\| := \sqrt{\| \langle a, a\rangle \|_{G \ltimes_r A}}$. The action of E_0 on A_0 extends to an action of E on $\bar{A}_0$ by continuity. The right action of E on $\bar{A}_0$ extends uniquely to a right action of $G \ltimes_r A$, since the action of E on $\bar{A}_0$ is non-degenerate (i.e. $E \, \bar{A}_0$ is dense in $\bar{A}_0$). Indeed, suppose $f \in G \ltimes_r A$ and $a \in \bar{A}_0$. There exists a sequence $\{e_i \, a_i \in E \, \bar{A}_0\}_{i \in \mathbb{N}}$ that converges to a. Define

$$a \cdot f := \lim_{i \to \infty} a_i \, (e_i \cdot f).$$

Corollary 12.3.5. *The pair $(\bar{A}_0, \langle ., .\rangle)$ forms a Hilbert $G \ltimes_r A$-module.*

Definition 12.3.6. We call an action of G on A which is proper with respect to A_0 **quantizable** if the associated Hilbert $G \ltimes_r A$-module $(\bar{A}_0, \langle ., .\rangle)$ is rank one and projective.

If this is the case, then a proper, quantizable action of G on A determines a class

$$[(\bar{A}_0, \langle ., .\rangle)] \in K_0(G \ltimes_r A).$$

If the action is K-amenable then $[(\bar{A}_0, \langle ., .\rangle)]$ induces a class in $K_0(G \ltimes A)$. The following Lemma and two Propositions treat the classes of examples that we have been considering before.

Definition 12.3.7. Suppose G is a locally compact group with Haar measure λ. We say an action of G on a C^*-algebra A is **counital** with respect to λ and a dense subalgebra A_0, if $M(A_0)^G$ is unital and if there exists a positive element $k \in A_0$ such that

$$\int_G \alpha_x(k) a \, \lambda(dx) = a$$

for all $a \in A_0$.

Lemma 12.3.8. *If an action of G on a C^*-algebra A is proper and counital with respect to λ and a dense subalgebra A_0, then the action of G on A is quantizable.*

Proof. Suppose $k = (k')^* k'$ for some $k' \in A_0$ The Hilbert $G \ltimes_r A$-module $(\bar{A}_0, \langle ., .\rangle)$ is rank one projective iff there exists an element $k' \in \bar{A}_0$ such that

$$1_{\bar{A}_0} = |k'\rangle\langle k'|.$$

So, in particular, the fact that for all $a \in A_0$ one has

$$|k'\rangle\langle k'|a = \int_G \alpha_x((k')^* k')\, a\, \lambda(dx) = a$$

implies that $(\bar{A}_0, \langle .,. \rangle)$ is rank one projective. ☐

Proposition 12.3.9. *Suppose the action of G on a proper groupoid $H \rightrightarrows M$ is proper and cocompact, then the action of G on $A := C_r^*(H)$ is proper and counital with respect to $A_0 := C_c(H)$, and hence quantizable.*

Proof. The first part is Proposition 12.2.5. As for the second part, let $l \in C_c(H)$ be a positive function with support on each G-orbit. Such a function exists since the G-action is cocompact. Then, the function $k \in C_c(H)$ defined by

$$k(h) := \frac{l(h)}{\int_G l(x^{-1} \cdot h) dx},$$

for all $h \in H$, has the desired property. ☐

The following definition is a variation on the notion of a quasi-local algebra (cf. e.g. [63]) in the case of a $C_0(M)$-algebra.

Definition 12.3.10. A $C_0(M)$-algebra A is said to have **local units** if for every $f \in C_c(M)$ there exists $u_f \in A$ such that for all $a \in A$

$$\pi(f)\, u_f\, a = \pi(f)\, a.$$

The element u_f is called a local unit for f.

Example 12.3.11. Suppose B is a C^*-algebra. The $C_0(M)$-algebra $A := C_0(M, B)$ has local units iff B is unital.

Remark 12.3.12. Interpreting A as the space of continuous sections

$$A \cong \Gamma_0(M, \coprod_{m \in M} A_m)$$

of an upper semi-continuous field of C^*-algebras $\{A_m\}_{m \in M}$, the above notion of having local units can be stated as follows. For any compact set $C \subset M$ there exists a continuous section $u_C \in A$ such that $u_C(m)$ equals the unit of the fiber A_m at m for all $m \in C$. So, in particular, A_m has to be unital for all $m \in M$.

Definition 12.3.13. A $C_0(M)$-algebra A is said to have G**-equivariant local units** if it has local units and if for every $f \in C_c(M)$ and $x \in G$ the element $x \cdot u_f$ is a local unit for $x \cdot f$.

In terms of continuous fields, as in the above remark, this means that x applied to the unit of A_m should equal the unit of $A_{x \cdot m}$.

Proposition 12.3.14. *Suppose G acts properly and cocompactly on M. If A is a G-$C_0(M)$-algebra that has G-equivariant local units, then the G action on A is proper and counital with respect to $A_0 := C_c(M)A$, and hence quantizable.*

Proof. The first part is proven in Proposition 12.1.4. As for the second part, choose any $f' \in C_c(M)$ with support on all G-orbits. Let $u_{f'}$ be a local unit for f'. Define $f \in C_c(M)$ by

$$f(m) := \frac{f'(m)}{\int_G f'(x^{-1} \cdot m) dx}$$

and

$$k := \pi(f)\, u_{f'} \in A_0.$$

Then, for any $\pi(g)a \in A_0$ ($g \in C_c(M)$ and $a \in A$), one has

$$\begin{aligned} \int_G (x \cdot k)\pi(g)a\, dx &= \int_G \pi(x \cdot f)(x \cdot u_{f'})\pi(g)a\, dx \\ &= \int_G \pi(x \cdot f)\, dx\, \pi(g)a \\ &= \pi(g)a, \end{aligned}$$

which finishes the proof. □

12.4 Equivariant KK-theory

In addition to Section 4.5, where we introduced KK-theory, we use this section to introduce equivariant KK-theory. It will play a central rôle in our definition of the noncommutative Baum-Connes analytical assembly map and in our definition of noncommutative geometric quantization.

Suppose G is a locally compact group that acts on C^*-algebras A and B. A continuous action of G on a right Hilbert B-module $\mathcal{H}$ is a representation $U : G \to \mathcal{B}(\mathcal{H})$, continuous in the strong operator topology and satisfying

$$g \cdot (h\, b) = (g \cdot h)(g \cdot b),$$

for all $b \in B$, $g \in G$ and $h \in \mathcal{H}$. For any two Hilbert B-modules $\mathcal{H}_1$ and $\mathcal{H}_2$ carrying G-actions U_1 respectively U_2, a morphism $F \in \mathcal{B}_B(\mathcal{H}_1, \mathcal{H}_2)$ is called G-continuous if $g \mapsto g \cdot F := U_2(g)\, F U_1\,(g^{-1})$ is norm-continuous $G \to \mathcal{B}_B(\mathcal{H}_1, \mathcal{H}_2)$.

Definition 12.4.1. A **G-Kasparov (A, B)-module** is a Kasparov (A, B)-module $(\mathcal{H}, \pi, F)$ with a continuous action U of G on $\mathcal{H}$, such that

(i) the grading is G-equivariant;

(ii) F is G-continuous;

(iii) π is G-equivariant;

(iv) $\pi(a)(g \cdot F - F) \in \mathcal{K}_B(\mathcal{H})$ for all $a \in A$ and $g \in G$.

Denote the set of G-Kasparov (A, B)-modules by $E_G(A, B)$. The notions of morphism and homotopy equivalence for G-Kasparov modules are analogous to the non-equivariant case (cf. Section 4.5). The **G-equivariant KK-theory** $KK_0^G(A, B)$ is the set of homotopy classes of G-Kasparov modules in $E_G(A, B)$. The group $KK_1^G(A, B)$ is defined as the set of homotopy classes of *ungraded* G-Kasparov modules.

Example 12.4.2. Suppose $A := C_0(\mathbb{R})$, $B = \mathbb{C}$, $\mathcal{H} := L^2(\mathbb{R})$ and π the canonical action of A on $\mathcal{H}$. Let $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the **Hilbert transform**

$$F(h)(x) := \frac{i}{\pi} \lim_{\varepsilon \downarrow 0} \int_{|t|>\varepsilon} \frac{h(t-x)}{t} dt.$$

Another way to understand this operator F is as follows: it is the Fourier transform of the operator $M_f : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ 'multiplication by the function f', where

$$f(x) = \begin{cases} -1 & \text{if } x \in \mathbb{R}_{<0} \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x \in \mathbb{R}_{>0}, \end{cases}$$

that is $f(x) = \mathrm{Sign}(x)$ and $F = P^{-1} M_f P$, where P denote the Fourier transform and P^{-1} the inverse Fourier transform.

The action of $\mathbb{Z}$ on $\mathbb{R}$ by translations induces an action of $\mathbb{Z}$ on $C_0(\mathbb{R})$ and a representation of $\mathbb{Z}$ on $L^2(\mathbb{R})$. The triple $(L^2(\mathbb{R}), \pi, F)$ forms an ungraded $\mathbb{Z}$-Kasparov $(C_0(\mathbb{R}), \mathbb{C})$-module and hence a class in $KK_1^{\mathbb{Z}}(C_0(\mathbb{R}), \mathbb{C})$ (cf. [81], p. 38).

Example 12.4.3. Any G-equivariant morphism $f : A \to B$ of G-C^*-algebras gives rise to a G-Kasparov (A, B)-module analogous to the construction just above Example 4.2.7.

Remark 12.4.4. Most properties of K-theory and KK-theory (cf. the list under Definition 4.4.5 and the remarks in Section 4.5) extend to equivariant KK-theory. In particular, there exists an equivariant version of the Kasparov product

$$KK_i^G(A, B) \times KK_i^G(B, C) \to KK_i^G(A, C),$$

for $i = 0, 1$.

We shall now briefly discuss **Kasparov's descent map** in KK-theory

$$j_r : KK_i^G(A, B) \to KK_i(G \ltimes_r A, G \ltimes_r B),$$

for $i = 0, 1$. Suppose $(\mathcal{H}, \pi, F)$ is a G-Kasparov (A, B)-module. Denote the covariant representation of G on $\mathcal{H}$ by U, the action of G on A by α and the action of G on B by β. Consider the space $C_c(G, \mathcal{H})$. We can endow it with a right action by $C_c(G, B)$ (for $\xi \in C_c(G, \mathcal{H})$, $f \in C_c(G, B)$ and $x \in G$)

$$(\xi \cdot f)(x) := \int_G U(y)(\xi(y^{-1} x))\, f(y)\, \lambda(dy)$$

and a left action of $C_c(G, A)$ by ($\xi \in C_c(G, \mathcal{H})$, $g \in C_c(G, A)$ and $x \in G$)

$$(\tilde{\pi}(g)\xi)(x) := \int_G \pi(\beta_y(g(y^{-1} x)))\, \xi(y)\, \lambda(dy).$$

One defines a $G \ltimes_r B$-valued inner product on $C_c(G, \mathcal{H})$ by the formula ($\xi, \eta \in C_c(G, \mathcal{H})$, $x \in G$)

$$\langle \xi, \eta \rangle (x) = \int_G \left\langle U(y)(\xi(y^{-1} x)), \eta(y) \right\rangle_B \lambda(dy).$$

Positivity of this inner product follows using Kasparov's stability theorem (cf. Lemma 3.9 of [37]). Denote the closure of $C_c(G,\mathcal{H})$ with respect to the norm associated to the $G\ltimes_r B$-inner product by $G\ltimes_r\mathcal{H}$. Then, by continuity, one obtains a right Hilbert $(G\ltimes_r A, G\ltimes_r B)$-bimodule structure on $G\ltimes_r\mathcal{H}$. An operator on $G\ltimes_r\mathcal{H}$ is defined by ($\xi\in C_c(G,\mathcal{H})$, $x\in G$)

$$\tilde{F}(\xi)(x) := F(\xi(x)).$$

Proposition 12.4.5. *The triple $(G\ltimes_r\mathcal{H},\tilde{\pi},\tilde{F})$ is a Kasparov $(G\ltimes_r A, G\ltimes_r B)$-module.*

For a proof see [37] Theorem 3.11. This cycle defines the descent map j_r by $j_r([\mathcal{H},\pi,F]) := [G\ltimes_r\mathcal{H},\tilde{\pi},\tilde{F}]$. Analogously, there exists a descent map for the unreduced C^*-algebra case:

$$j : KK_i^G(A,B)\to KK_i(G\ltimes A, G\ltimes B),$$

for $i=0,1$.

12.5 The noncommutative analytical assembly map

We are now ready to define the noncommutative analytical assembly map.

Suppose a locally compact group G acts on a C^*-algebra A in a proper and counital (or, more generally, quantizable) way with respect to A_0. Denote the isomorphism class of the associated projective $(G\ltimes_r A)$-module by

$$[\bar{A}_0]\in K_0(G\ltimes_r A).$$

Definition 12.5.1. Suppose G acts properly and counitally on A with respect to A_0 and G acts on B. The **noncommutative analytical assembly map** with respect to A_0

$$\mu_{BC} : KK_i^G(A,B)\to\longrightarrow K_i(G\ltimes_r B)$$

is defined as follows; for any $[\mathcal{H},\pi,F]\in KK_i^G(A,B)$, it is the 'index' isomorphism

$$KK_i(\mathbb{C},G\ltimes_r B)\xrightarrow{\cong} K_i(G\ltimes_r B)$$

(cf. Section 4.5) applied to the Kasparov product of $[\bar{A}_0]\in KK_0(\mathbb{C},G\ltimes_r A)$ and $j_r([\mathcal{H},\pi,F])\in KK_i(G\ltimes_r A, G\ltimes_r B)$, that is

$$\mu_{BC}([\mathcal{H},\pi,F]) := \mathrm{index}([\bar{A}_0]\hat{\otimes}_{G\ltimes_r A} j_r([\mathcal{H},\pi,F]))\in K_i(G\ltimes_r B). \qquad (12.5.3)$$

By the distributivity of the Kasparov product and the fact that j_r is a homomorphism of Abelian groups, this map μ_{BC} is a homomorphisms of Abelian groups. If the action is K-amenable, then we can use the class in $K_0(\mathbb{C},G\ltimes A)$ determined by $[\bar{A}_0]$ and the unreduced version of the descent map j to define an analytical assembly map

$$\mu_{BC} : KK_i^G(A,B)\to K_i(G\ltimes B)$$

analogously to Formula 12.5.3.

The operator F is called **properly supported** if for all $a\in A_0$ there exists a $a'\in A_0$ such that $\pi(a')\,F\,\pi(a) = F\,\pi(a)$.

Example 12.5.2. Suppose a locally compact group G acts properly and cocompactly on a space M, A is a G-$C_0(M)$-algebra that has G-equivariant local units and B is a C^*-algebra. Then every G-Kasparov (A, B)-module $(\mathcal{H}, \pi, F)$ is homotopy equivalent to a Kasparov module $(\mathcal{H}, \pi, F')$ with F properly supported and G-equivariant. The proof of this is analogous to the proof of Lemma 2 en Proposition 1 in [82].

Example 12.5.3. Suppose a locally compact group G acts properly and cocompactly on a proper *étale* groupoid $H \rightrightarrows M$. Then every G-Kasparov $(C_r^*(H), B)$-module $(\mathcal{H}, \pi, F)$ is homotopy equivalent to a Kasparov module $(\mathcal{H}, \pi, F')$ with F properly supported and G-equivariant. The proof of this is analogous to the proof of Lemma 2 en Proposition 1 in [82]. The extra condition of $H \rightrightarrows M$ being étale is necessary to be able to find a function analogous to h in [82].

Suppose G is a locally compact group that acts properly on a C^*-algebra A with respect A_0. If $(\mathcal{H}, \pi, F)$ is a G-Kasparov (A, B)-module and F is G-equivariant and properly supported. Then there exists an alternative definition of the noncommutative analytiical assembly map. Indeed, consider the dense subspace $\pi(A_0)\mathcal{H} \subset \mathcal{H}$. A right $C_c(G)$-module structure on $\pi(A_0)\mathcal{H}$ is given by

$$(a \cdot h) \cdot f := \int_{g \in G} f(g) U(g^{-1})(a \cdot h) \lambda(dh),$$

for all $a \in A$, $h \in \mathcal{H}$ and $f \in C_c(G)$. A $C_c(G)$ valued inner product is given by

$$\big\langle (a \cdot h), (a' \cdot h') \big\rangle (g) := \big\langle (a \cdot h), U(g)(a' \cdot h') \big\rangle ,$$

for all $a, a' \in A$, $h, h' \in \mathcal{H}$ and $g \in G$. Using the C^*-norm obtained from this inner product, the space $\pi(A_0)\mathcal{H}$ can be completed to a Hilbert $C_r^*(G)$-module $\mathcal{E}$. The operator F extends to an operator $\mathcal{F}$ on $\mathcal{E}$ satisfying $\mathcal{F}^2 - 1 = 0$. The proof of this is completely analogous to the proof in the commutative case as found for example in [82].

Proposition 12.5.4. *The Kasparov* $(\mathbb{C}, G \ltimes_r B)$*-modules* $(\mathcal{E}, 1, \mathcal{F})$ *and*

$$\bar{A}_0 \,\hat{\otimes}_{G \ltimes_r A}\, j_r(\mathcal{H}, \pi, F)$$

are homotopy equivalent.

Proof. The proof is analogous to the proof of Lemma 8 in [82]. □

Example 12.5.5. Suppose H is a locally compact group and K a compact subgroup. Consider the action groupoid $G := K \ltimes H \rightrightarrows H$ of the left action of K on H by left multiplication. There exists a proper, cocompact action of H on $G \rightrightarrows H$ given by $h' \cdot (k, h) := (k, h(h')^{-1})$ for $h' \in H$ and $(k, h) \in G = K \ltimes H$. By Proposition 12.3.9 this induces a quantizable action of H on $C_r^*(G)$. Hence one can construct an analytical assembly map

$$\mu_{BC} : KK_0^H(C_r^*(K \ltimes H), \mathbb{C}) \to K_0(C_r^*(H)). \tag{12.5.4}$$

We shall construct H-Kasparov $(C_r^*(K \ltimes H), \mathbb{C})$-modules that represent classes of $KK_0^H(C_r^*(K \ltimes H), \mathbb{C})$. Suppose (π, V) is a representation of K. This induces a representation $\bar{\pi}$ of $K \ltimes H \rightrightarrows M$ on the trivial vector bundle $V \times H \to H$ defined by

$$\bar{\pi}(k,h)(v,h) := (\pi(k)v, k \cdot h),$$

for $k \in K$, $v \in V$ and $h \in H$.

Suppose λ is a Haar measure on H. As explained in [67], the measurable representation $\bar{\pi}$ of $K \ltimes H$ on $V \times H$ gives rise to representation $\tilde{\pi}$ of $C_r^*(K \ltimes H)$ on $L^2(H, V, \lambda)$. Obviously, there exists a representation of H on $L^2(H, V \times H, \lambda)$. One easily checks that the triple $(L^2(H,V,\lambda), \tilde{\pi}, 0)$ is an H-Kasparov $(C_r^*(K \ltimes H), \mathbb{C})$-module and hence represents a class

$$[L^2(H,V,\lambda), \tilde{\pi}, 0] \in KK_0^H(C_r^*(K \ltimes H, \mathbb{C})$$

Strictly speaking we should have started with an element in the representation ring of K, and the KK-cycle would consist of the direct sum corresponding to two representations representing the formal difference.

The K-theory of the C^*-algebra of a compact group K is isomorphic to its representation ring (cf. Example 4.4.7),

$$K_0(C_r^*(K)) \cong \mathcal{R}(K).$$

In particular, an isomorphism is given as follows. A representation (π, V) of K corresponds to the projection in $C_r^*(K)$ induced by the function on K given by

$$p_\pi : k \mapsto \dim(V) \langle v, \pi(k)v \rangle \,,$$

where $v \in V$ with $\|v\| = 1$. The function $p_\pi \in C(K)$ is interpreted as projection operator in $C_r^*(H)$ by the formula

$$\bar{p}_\pi(f)(h) := \int_K p_\pi(k)\, f(k^{-1}h)\, dk,$$

where $f \in C_c(H)$.

Lemma 12.5.6. *The image of* $[L^2(H,V,\lambda), \tilde{\pi}, 0] \in KK_0^H(C_r^*(K \ltimes H), \mathbb{C})$ *under the assembly map* μ *(Equation 12.5.4) is* $[\bar{p}_\pi]$.

Hence the composition is the induction map

$$K_0(C_r^*(K)) \to KK_0^H(C_r^*(K \ltimes H), \mathbb{C}) \to K_0(C_r^*(H))$$

given by $[p_\pi] \mapsto [\bar{p}_\pi]$. This map is well-known, but is usually obtained via $KK_0^H(C_0(H/K), \mathbb{C})$ instead of $KK_0^H(C_r^*(K \ltimes H)$. But $C_0(H/K)$ is Morita equivalent to $C_r^*(K \ltimes H)$, since the groupoids $H/K \rightrightarrows H/K$ and $K \ltimes H \rightrightarrows H$ are Morita equivalent,

$$\begin{array}{ccccc} & & KK_0^H(C_r^*(K \ltimes H), \mathbb{C}) & & \\ & \nearrow & & \searrow & \\ K_0(C_r^*(K)) & & \downarrow \cong & & K_0(C_r^*(H)). \\ & \searrow & & \nearrow & \\ & & KK_0^H(C_0(H/K), \mathbb{C}) & & \end{array}$$

Using the commutativity of this diagram the proof of the Lemma follows from the commutative case (cf. [82]).

Example 12.5.7. Suppose X is an orbifold, represented by a proper étale Lie groupoid $H \rightrightarrows M$, i.e. $X \cong M/H$ (cf. Example 2.2.4). Suppose a Lie group G acts properly and cocompactly on X. Then G acts properly and cocompactly on $H \rightrightarrows M$. Hence, by Proposition 12.3.9, G acts properly on $A := C^*_r(H)$ with respect to $A_0 := C_c(G)$ and we have constructed an analytical assembly map

$$\mu_{BC} : KK^G_0(C^*_r(H), \mathbb{C}) \to K_0(C^*_r(G)).$$

12.6 Noncommutative geometric quantization

The initial data for noncommutative geometric quantization are as follows. Suppose

(i) (A_0, ω) is a symplectic algebra (cf. Section 11.4);

(ii) A_0 lies dense in a C^*-algebra A;

(iii) G a connected Lie group acting properly on A with respect to A_0 (cf. Section 12.1);

(iv) the action of G on A is counital with respect to A_0 (cf. Definition 12.3.7);

(v) $\mathfrak{g}$ is the Lie algebra of G;

(vi) the induced action of $\mathfrak{g}$ on A_0 is Hamiltonian, with momentum map $\mu : \mathfrak{g} \to A_0$ (cf. Section 11.4);

(vii) (L, ∇) is a prequantum module for (A_0, ω) (cf. Section 11.5);

(viii) L is a right pre-Hilbert (A, A)-module with respect to an A-valued inner product $\langle ., . \rangle$;

(ix) ∇ is Hermitian with respect to $\langle ., . \rangle$;

(x) the unitary representation of $\mathfrak{g}$ on L (cf. Proposition 11.5.3) is integrable to a representation of G on L.

Then the completion of L to a Hilbert (A, A)-module $(\bar{L}, \langle ., . \rangle)$ determines a class $[\bar{L}] \in KK^G_0(A, A)$. Instead of the action being counital, we can more generally require it to be quantizable. In the spirit of Bott's definition of geometric quantization and the extensions made by Landsman (cf. [47]) we define the following.

Suppose B is another C^*-algebra with a G-action.

Definition 12.6.1. For any class

$$[\mathcal{H}, \pi, F] \in KK^G_i(A, B)$$

($i = 0$ or 1) the **noncommutative geometric quantization with coefficients in** B **of** $(L, \langle ., . \rangle, \nabla)$ **with respect to** $[\mathcal{H}, \pi, F]$ is the analytical assembly map

$$KK^G_i(A, B) \xrightarrow{\mu_{BC}} K_i(G \ltimes_r B)$$

applied to the Kasparov product over A

$$([\bar{L}], [\mathcal{H}, \pi, F]) \mapsto [\bar{L}]\hat{\otimes}_A[\mathcal{H}, \pi, F],$$
$$KK^G_0(A, A) \otimes KK^G_i(A, B) \to KK^G_i(A, B).$$

More concretely, the noncommutative geometric quantization with coefficients in B is

$$\operatorname{index}\big([\bar{A}_0]\hat{\otimes}_{G\ltimes_r A} j_r([\bar{L}]\hat{\otimes}_A[\mathcal{H},\pi,F])\big) \in K_i(G\ltimes_r B).$$

In this definition one could replace the reduced C^*-algebra by the full one, if the action is K-amenable.

Example 12.6.2. The definition of geometric Kähler quantization restricts itself to the case $A = C_0(M)$, with M a Kähler manifold. In this approach the choice of the Kasparov triple $[\mathcal{H},\pi,F]$ is the one associated to the Dolbeault operator constructed from the equivariant complex structure on the Kähler manifold (cf. e.g. Part III).

Example 12.6.3. A generalization of the above example is geometric Spin^c-quantization. This approach also restricts itself to the case $A = C_0(M)$, but now for manifolds M with an equivariant Spin^c-structure. The Kasparov triple is the one associated to the Spin^c-Dirac operator.

Example 12.6.4. If $(M,\mathcal{F})$ is a foliated manifold, then one can consider the C^*-algebra $C^*(M,\mathcal{F})$ of the holonomy groupoid $\operatorname{Hol}(M,\mathcal{F})$ (cf. Example 2.2.3 and Section 4.3). Under the assumption that the projection $p : M/\mathcal{F} \to pt$ is K-oriented (cf. [34]), there exists a fundamental class

$$p! \in KK_0(C^*(M,\mathcal{F}),\mathbb{C}).$$

In the case of a group H acting on the foliation (groupoid), it would be very interesting to construct an H-equivariant version of this

$$p! \in KK_0^H(C^*(M,\mathcal{F}),\mathbb{C}),$$

and use it in our proposed geometric quantization scheme.

Example 12.6.5. We shall use the notation of [63]. Consider a locally finite, directed graph $E = (E_0, E_1)$ and an action of a locally compact group G on E. Let E^* denote the set of finite paths in E, $s(\mu)$ the start vertex of path μ, $t(\mu)$ the end vertex of μ and $|\mu|$ the length of μ. Let $A := C^*(E)$ denote the C^*-algebra associated to E (cf. Example 4.1.7).

The action of G on E induces an action on $C^*(E)$ by

$$x\cdot S_e = S_{x\cdot e},\, x\cdot p_v = p_{x\cdot v},$$

for all $x \in G$, $e \in E_1$ and $v \in E_0$.

There is another action on $C^*(E)$, namely the so-called gauge action of the circle (or 1-torus) $\mathbb{T}$ defined by

$$\alpha\cdot S_e = \exp(2\pi\, i\, \alpha)\, S_e,\, \alpha\cdot p_v = p_v$$

for all $e \in E_1$, $v \in E_0$ and $\alpha \in \mathbb{T}$.

In [63] a class $[V,F] \in KK_1(C^*(E), C^*(E)^{\mathbb{T}})$ is constructed. The Hilbert $C^*(E)^{\mathbb{T}}$-module V is the closure of A_0 under the norm obtained from the $C^*(E)^{\mathbb{T}}$-valued inner product given by

$$\langle a,b\rangle := \int_{\mathbb{T}} \alpha\cdot(a^* b)\, d\alpha.$$

One can show that any element in A_0 can be written as a finite sum of elements of the form $S_\mu S_\nu^*$, where $\mu = e_1 \dots e_{|\mu|}$ and

$$S_\mu := S_{e_1} \dots S_{e_{|\mu|}}$$

and the same for ν. The operator F is defined by

$$F := \frac{D}{\sqrt{1+D^2}},$$

where D is given by the simple formula

$$D(S_\mu S_\nu^*) := (|\mu| - |\nu|) S_\mu S_\nu^*,$$

for all $\mu, \nu \in E^*$. From this formula one sees at once that D is G-equivariant, hence it determines a class in $KK_1^G(C^*(E), C^*(E)^{\mathbb{T}})$. This can serve as a canonical class for graph C^*-algebras. Hence, we could of the geometric quantization of actions on graphs.

Suppose the action of G on $C^*(E)$ is proper and counital with respect to a subalgebra A_0. Then, we can consider its image under the noncommutative assembly map with respect to A_0

$$KK_1^G(C^*(E), C^*(E)^{\mathbb{T}}) \to K_1(G \ltimes_r C^*(E)^{\mathbb{T}}).$$

To what extent can we expect this map to be non-trivial?

Lemma 12.6.6. *If the action of G on E is free and every vertex receives an edge, then*

$$K_1(G \ltimes_r C^*(E)^{\mathbb{T}}) = 0.$$

Proof. Firstly, $C^*(E)^{\mathbb{T}}$ is Morita equivalent to $C^*(Z \times E)$, where Z is the graph with edges and vertices indexed by $\mathbb{Z}$ and

$$s(e_k) = v_k, t(e_k) = v_{k+1}.$$

The product graph is defined by $(Z \times E)_i := Z_i \times E_i$ for $i = 0, 1$ and

$$t(z, e) = (t(z), t(e)), s(z, e) = (s(z), s(e)).$$

There is an obvious action of G on $Z \times E$ and $G \ltimes_r C^*(E)^{\mathbb{T}}$ is Morita equivalent to $G \ltimes_r C^*(Z \times E)$. Since the action is free, $G \ltimes_r C^*(Z \times E)$ is isomorphic to

$$C^*((Z \times E)/G) \otimes \mathcal{K}(l^2(G)),$$

(cf. [42]) which equals $C^*(Z \times (E/G)) \otimes \mathcal{K}(l^2(G))$. Hence, since K_1 is invariant under Morita equivalence and stable,

$$K_1(G \ltimes C^*(E)^{\mathbb{T}}) \cong K_1(C^*(Z \times (E/G))).$$

But $K_1(C^*(Z \times (E/G))) = \mathbb{Z}^{(\text{number of loops in } Z \times (E/G))}$ (cf. [63]) and one easily sees that there are no loops in $Z \times (E/G)$. □

We are not yet able to formulate a noncommutative Guillemin-Sternberg conjecture in full generality, but let's comments on how we think one should proceed from here. As is clear from the previous examples, the main problem is to define canonical classes in $[D] \in KK_i^G(A, B)$ for a class of C^*-algebras. This class of C^*-algebras should be closed under symplectic reduction, in the sense that a suitable closure of $\bar{A}^0$ of the symplectic reduction A^0 should be a C^*-algebra in our class again.

Suppose we have found a class of C^*-algebras closed under reduction and we have the initial data for a geometric quantization of such a C^*-algebra A as described above. Furthermore, suppose L^0 is a prequantum Hilbert (A^0, A^0)-bimodule.

Conjecture 12.6.7 (**noncommutative Guillemin-Sternberg-Landsman Conjecture with coefficients** in B)**.** *The classes* $[L]\hat{\otimes}_A[D]$ *and* $[L^0]\hat{\otimes}_{\bar{A}^0}[D^0]$ *map to the same class in the lower-right corner of the diagram*

$$\begin{array}{ccc} [L]\hat{\otimes}_A[D] \in KK_i^G(A,B) & \xrightarrow{\mu^G_{BC}} & K_i(C^*(G \ltimes_r B)) \\ & & \downarrow (\int_G)_* \\ [L^0]\hat{\otimes}_{\bar{A}^0}[D^0] \in KK_i(\bar{A}^0, B) & \xrightarrow[\mu^{\{e\}}_{BC}]{} & K_i(B), \end{array} \tag{12.6.5}$$

where $(\int_G)_*$ *is the map induced by the map* $C_c(G, B) \to B$ *given by* $f \mapsto \int_G f\lambda$*, for a Haar measure* λ *of* G*, and* $\{e\}$ *is the trivial group.*

Remark 12.6.8 (Discrete groups)**.** Sometimes it is possible to relate these classes through some map

$$V : KK_i^G(A, B) \dashrightarrow KK_i(\bar{A}^0, B)\ ,$$

for example, if G is discrete. One would like to obtain a commutative diagram

$$\begin{array}{ccc} [L]\hat{\otimes}_A[D] \in KK_i^G(A,B) & \xrightarrow{\mu^G_{BC}} & K_i(C^*(G \ltimes_r B)) \\ \downarrow V & & \downarrow (\int_G)_* \\ [L^0]\hat{\otimes}_{\bar{A}^0}[D^0] \in KK_i(\bar{A}^0, B) & \xrightarrow[\mu^{\{e\}}_{BC}]{} & K_i(B). \end{array} \tag{12.6.6}$$

Indeed, for discrete groups G, the momentum map μ is always the zero map and hence $A^0 = A^G$. In the case that $B = \mathbb{C}$ and $A = C_0(M)$ for a manifold M, the proof of the conjecture splits in two steps (cf. [36]). In this case there exists a map V called the Valette map (in [36]), involving a form of averaging. First, one shows that the above diagram commutes for this V, i.e. the analytical assembly map is natural. Then one shows that the (equivariant) fundamental class $[D]$ is mapped by V to the fundamental class of the "quotient" A^G.

The first step can take a nice form, if one defines the Marsden-Weinstein quotient by $G \ltimes_r A$ instead of A^G as suggested in Remark 11.1.18 (G is still discrete, hence

$\mu = 0$). Then, Diagram 12.6.6 takes the form

$$
\begin{array}{ccccc}
[D] \in KK_i^G(A,B) & & \xrightarrow{\mu_{BC}} & & K_i(C^*(G \ltimes_r B)) \\
 & \searrow^{j_r} & & \nearrow^{\mathrm{index} \circ [\bar{A}_0] \hat{\otimes}_{G \ltimes_r A}} & \\
V \Big\downarrow & & KK_i(G \ltimes_r A, G \ltimes_r B) & & \Big\downarrow (\sum_G)_* \\
 & \swarrow_{(\sum_G)_*} & & & \\
[D^0] \in KK_i(G \ltimes_r A, B) & & \xrightarrow[\mathrm{index} \circ [\bar{A}_0] \hat{\otimes}_{G \ltimes_r A}]{} & & K_0(B),
\end{array}
$$

where V is defined as the composition $(\sum_G)_* \circ j_r$. The commutativity of this diagram follows from the functorial properties of the Kasparov product.

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Index

Samenvatting in het Nederlands

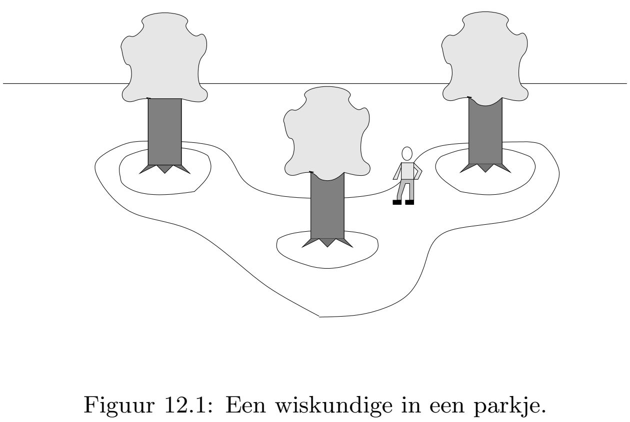
Dit proefschrift behandelt generalisaties van meetkundige kwantisatie gebruikmakend van groepoïden en niet-commutatieve meetkunde.

Dat was de Nederlandstalige samenvatting voor kenners. Voor meer informatie verwijs ik deze groep naar de introductie en de rest van het proefschrift. Voor alle andere geïnteresseerden zal ik deze volzin in wat volgt woord voor woord uitleggen. Aan de hand van een parkje en een badkamervloer zal ik de notie van groepoïde uitleggen en iets over symmetrie zeggen in de eerste 2 secties. Daarna volgt een schematische uitleg van wat meetkundige kwantisatie is in de derde sectie. Het zal blijken dat dit de klassieke en kwantummechanica met elkaar in verband brengt. Dat wordt geïllustreerd met behulp van een magneetje ronddraaiend op een stokje en met behulp van een ronddraaiend atoom. Ten slotte leg ik uit wat commutatief en niet-commutatief is en wat de rol van niet-commutatieve meetkunde in dit proefschrift is in de vierde sectie.

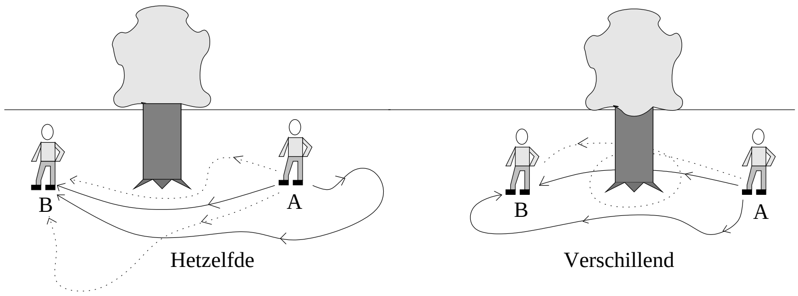
Groepoïden

Wat een groepoïde is laat zich het best uitleggen aan de hand van een voorbeeld: de *fundamentaalgroepoïde*. Stel, een wiskundige is vastgelopen in zijn werk en besluit een wandelingetje door een parkje te maken (zie Figuur 12.1). In het parkje staan wat bomen en hij wandelt graag een beetje om die bomen heen.

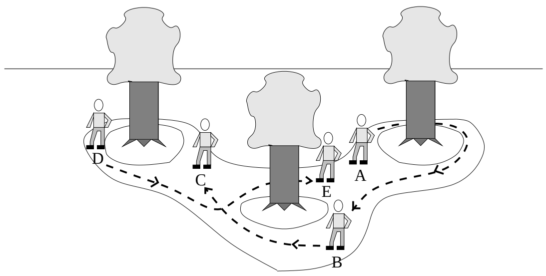
De fundamentaalgroepoïde van het park is de verzameling van wandelingetjes van een punt naar een ander punt. Kijk nu naar het linker plaatje in Figuur 12.2.



Figuur 12.1: Een wiskundige in een parkje.



Figuur 12.2: Links vier 'dezelfde' en rechts drie 'echt verschillende' wandelingen.



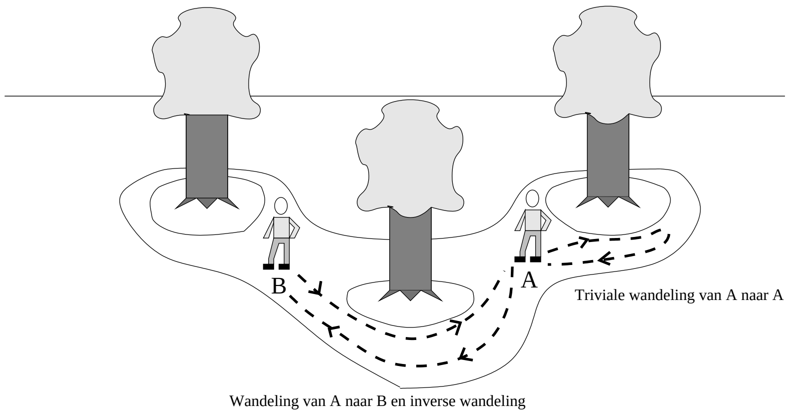
Figuur 12.3: De wandelingen van A naar B en van B naar C kunnen worden samengesteld tot een wandeling van A naar C. De wandeling van A naar B kan niet worden samengesteld met de wandeling van D naar E.

Daar ziet u 4 wandelingetjes van A naar B (de 2 gestippelde en 2 doorgetrokken lijnen). Als we dat als allemaal verschillende wandelingetjes zouden rekenen, dan werd de verzameling van wandelingetjes wel erg groot.

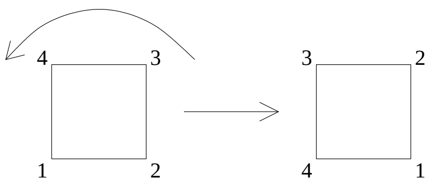
Gelukkig blijkt onze wiskundige een topoloog te zijn en typisch voor topologen is dat ze tamelijk onnauwkeurig naar zo'n parkje kijken. Ze zijn niet zo geïnteresseerd in afstanden, maar vooral in de obstakels, in dit geval de bomen. Dus ziet hij de wandelingetjes van A naar B in het linker plaatje allemaal als hetzelfde wandelingetje. Dat maakt de verzameling wandelingetjes al wat overzichtelijker. De wandelingetjes van A naar B in het rechter plaatje vindt hij daarentegen echt verschillend, aangezien ze op verschillende wijze langs de boom gaan; de (gestippelde) derde gaat zelfs één keer helemaal om de boom heen.

Wat maakt deze verzameling van alle wandelingetjes nu tot een groepoïde? De belangrijkste eigenschap is dat je twee wandelingetjes kunt samenstellen, *maar niet altijd!* De wandeling van punt A naar punt B en de wandeling van punt B naar C vormen samen een wandeling van punt A naar C (zie Figuur 12.3). Dit is de *samenstelling*. De wandeling van A naar B kan echter niet samengesteld worden met de wandeling van D naar E. Het eindpunt B van de eerste wandeling is namelijk niet hetzelfde als het beginpunt D van de tweede wandeling. Dit is typisch voor een groepoïde: je kunt wel samenstellen maar alleen als het eindpunt van de één gelijk is aan het beginpunt van de ander.

Een andere eigenschap van groepoïden die zich makkelijk laat illustreren aan de hand van dit voorbeeld is dat je op ieder punt een *triviale* wandeling kunt maken



Figuur 12.4: Een triviale wandeling van A naar A en een wandeling van A naar B met zijn inverse wandeling van B naar A.



Figuur 12.5: Een symmetrie van een vierkant: rotatie linksom van 90 graden.

(zie Figuur 12.4). Dat is dus een wandelingetje van een punt naar zichzelf, waarbij je niet om bomen heen loopt. Onze topoloog ziet in feite geen verschil tussen zo'n wandeling en een pas op de plaats!

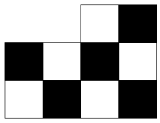
Tenslotte heeft iedere wandeling een *inverse* wandeling: als je die twee wandelingetjes na elkaar loopt (samenstelt), maak je een triviale wandeling. Concreet betekent dat dat de inverse wandeling van een wandeling van A naar B gewoon dezelfde wandeling is alleen dan omgekeerd, van B naar A. Zie maar in Figuur 12.4: als ik eerst van A naar B loop langs het aangegeven pad en dan weer terug van B naar A langs hetzelfde pad, dan loop ik dus in lus. Die lus gaat niet om bomen heen en is dus een triviale wandeling.

De drie genoemde aspecten, samenstellen, inverse en triviale wandeling, vormen de kern van een formele definitie van groepoïden. Er blijken talloze manieren te zijn waarop deze groepoïden voorkomen in de wiskunde.

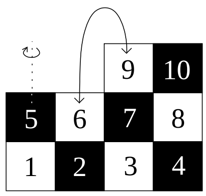
Acties

Eén van die manieren waarop groepoïden voorkomen is als middel om symmetrie te beschrijven. Een symmetrie van een object is een handeling met dat object zodat je voor en na geen verschil ziet. Dat is misschien anders dan u eerst over symmetrie nadacht, maar een voorbeeld maakt het duidelijk. Een symmetrie van een vierkant is bijvoorbeeld rotatie van 90 graden linksom[1] (zie Figuur 12.5). De verzameling

[1] Kunt u nog andere symmetrieën van het vierkant noemen (denk aan spiegelen)?



Figuur 12.6: Een badkamervloer.



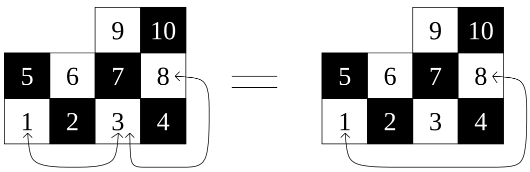
Figuur 12.7: Een interne symmetrie: rotatie van tegel 5; een externe symmetrie: verwissel tegel 6 en 9.

van al deze symmetrieën heeft de structuur van wat wiskundigen een *groep* noemen. Voor het vierkant draagt deze groep de prozaïsche naam "D_4". Wiskundigen zeggen dat er een *actie* is van de groep D_4 op het vierkant.

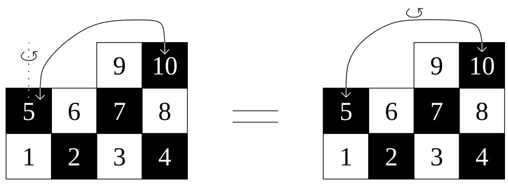
Bekijk nu het volgende plaatje van een stijlvolle betegeling van een badkamervloer met zwarte en witte tegels in Figuur 12.6. De vloer heeft geen enkele symmetrie[2]. Toch ziet het oog een hoop regelmaat en symmetrie als je de tegels afzonderlijk beschouwt. Als je er één los wrikt[3] en 90 graden draait, dan ziet de vloer er onveranderd uit. Zo'n soort symmetrie van een afzonderlijke tegel heet een *interne symmetrie* (zie Figuur 12.7). Als je twee witte of zwarte tegels zou loswrikken en verwisselen ziet de vloer er ook onveranderd uit. Zo'n soort symmetrie tussen twee verschillende tegels heet een *externe symmetrie*. Het belangrijke verschil met het eerder genoemde vierkant is dat we de vloer beschouwen als bestaand uit 10 losse objecten (de tegels). De interne en externe symmetrieën vormen samen een groepoïde. Er is inderdaad een "samenstelling". We kunnen bijvoorbeeld eerst de tegel op positie 1 verwisselen met de tegel op positie 3, en dan de tegel op positie 3 met de tegel op positie 8 (Zie Figuur 12.8). Dit is hetzelfde als de tegel op positie 1 verwisselen met de tegel op positie 8. Of we kunnen, bijvoorbeeld, eerst de tegel op positie 5 90 graden rechtsom draaien en vervolgens de tegel op positie 5 verwis-

[2]Probeer er maar eens één te vinden (dat lukt dus niet).

[3]Voorzichtig, de tegel mag niet breken!



Figuur 12.8: Een voorbeeld van samenstelling: verwisselen van de tegels op 1 en 3 en dan op 3 en 8 is hetzelfde als verwisselen van de tegels op 1 en 8.



Figuur 12.9: Een ander voorbeeld van samenstelling: roteren op 5 en dan verwisselen met 10 kan ook in één handeling.

selen met de tegel op positie 10 (Zie Figuur 12.9). Wederom kan dat ook in één handeling. Daarentegen het verwisselen van tegel 6 en 9 en daarna het roteren van tegel 5 kan niet worden samengesteld tot één handeling. Dit is opnieuw typisch voor groepoïden. Je kunt wel samenstellen, maar niet alles. Net als in het voorbeeld van het vierkant noemen wiskundigen dit een *actie*, alleen dan een actie van de groepoïde van interne en externe symmetrieën op de badkamervloer. Het is daarbij belangrijk dat we aangeven dat we de badkamervloer niet als een geheel zien (want dan waren er dus geen symmetrieën) maar als opgebouwd uit tegels.

Representaties en meetkundige kwantisatie

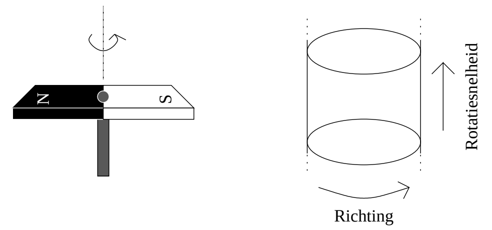
Natuurkundigen zijn beroepshalve wellicht minder geïnteresseerd in badkamertegels of boswandelingen. Symmetrie speelt echter sowieso een belangrijke rol in hun leven. Het gaat dan om de symmetrie van het systeem dat zij bestuderen. Deze systemen vallen in twee categorieën: klassiek en kwantum. Klassieke systemen zijn macroscopisch, zoals biljartballen en hemellichamen, en worden beschreven door de klassieke mechanica. Kwantum systemen zijn juist heel klein, bijvoorbeeld atomen en elementaire deeltjes, en worden beschreven door de kwantummechanica.

In de klassieke mechanica wordt de verzameling van alle mogelijke toestanden waarin een systeem zich zou kunnen bevinden de *faseruimte*[4] genoemd. Bijvoorbeeld, voor een magneetje op een stokje (zie Figuur 12.10, links) is deze ruimte een cirkel (om de richting van het magneetje aan te geven) en een lijn (om de snelheid waarmee het magneetje ronddraait aan te geven). Deze cirkel en lijn vormen samen een (oneindig lange) cilinder (zie Figuur 12.10, rechts). Iedere toestand van het systeem correspondeert dus met een punt op deze cilinder. Zo'n cilinder heeft veel symmetrie, hetgeen in berekeningen over het gedrag van het magneetje een belangrijke rol kan spelen.

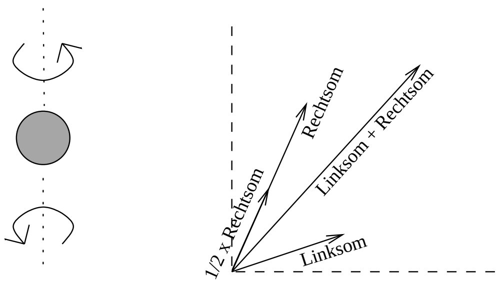
In de kwantummechanica gaat het er veel vreemder aan toe. Een zeer curieuze eigenschap van de kwantummechanica is dat de onderliggende ruimte van mogelijke toestanden van het systeem *lineair* is. Dat houdt in dat je twee toestanden kunt *optellen*. Een atoom kan (in tegenstelling tot de magneet van figuur 12.10) in een toestand zijn waarbij het "rechts om zijn as draait" + "links om zijn as draait" [5] (Zie Figuur 12.11). Minstens zo vreemd is het dat toestanden als 1/2 keer "rechtsom

[4]Dit is een symplectische variëteit. Dat is heel belangrijk, maar toch zullen we hier voor de duidelijkheid verder slechts over faseruimten spreken.

[5]Hierdoor ontstaan paradoxen zoals dat een kat in de toestand "dood"+"levend", dood *en* levend zou zijn. Deze paradox staat bekend als "Schrödingers kat").



Figuur 12.10: Een magneet draaiend op een stokje. De faseruimte is een cilinder.



Figuur 12.11: Een atoom draaiend om zijn as. De mogelijke toestanden vormen een lineaire ruimte.

draaiend" ook zijn toegestaan. Deze eigenschappen zijn typisch voor een zogeheten *lineaire ruimte*[6].

Symmetrieën van zo'n lineaire ruimte zijn een speciaal soort acties van groepen: wiskundigen noemen ze *representaties*. Groepen zijn uitgebreid bestudeerd; er bestaan classificaties van allerlei mogelijke groepen[7]. Ook de representaties van die groepen zijn bestudeerd en geclassificeerd in veel gevallen. De representaties geven niet zo veel informatie over de lineaire ruimte, maar vaak des te meer over de groep zelf.

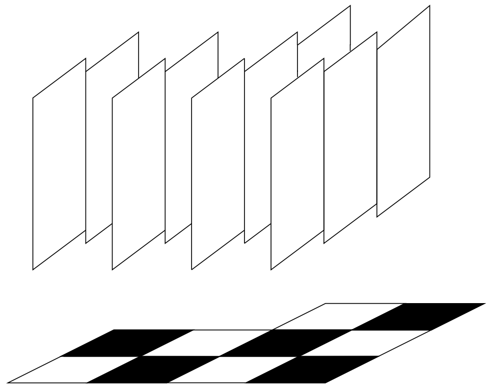
Dit proefschrift behandelt in deel II representaties van groepoïden. Een representatie van een groepoïde is niet op een lineaire ruimte, maar op een heel *veld* van lineaire ruimten. In het bovenstaand voorbeeld zou dat er voor iedere badkamertegel één zijn (Zie Figuur 12.12).

Deel III van het proefschrift bespreekt een methode om dit soort representaties te construeren, namelijk de zogeheten *meetkundige kwantisatie*. Meetkundige kwantisatie is een procedure op het kruispunt van de wiskunde en de natuurkunde[8]. Het is voor natuurkundigen interessant omdat het een verbintenis laat zien tussen klassieke mechanica en kwantummechanica. Het geeft een recept hoe je van een faseruimte van een systeem in de klassieke mechanica een lineaire ruimte maakt voor een kwantummechanisch systeem. Het mooie is dat een symmetrie van het klassieke

[6]In de kwantummechanica is in feite sprake van een Hilbert ruimte, d.w.z. een lineaire ruimte met nog een extra eigenschap.

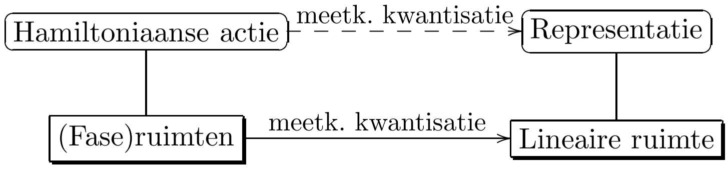
[7]Naast bovengenoemde D_4 bestaan er talloze, met namen als D_3, D_5, S_7, $\mathbb{R}$, $\mathbb{T}$, $SO(3)$, $U(2)$, enz.

[8]Lees hierover in de oratie *Op het kruispunt* van mijn promotor Klaas Landsman te vinden op http://www.math.ru.nl/~landsman/oratieRU.pdf



Figuur 12.12: Een veld van lineaire ruimten 'over' de badkamervloer. Dat wil zeggen voor iedere tegel één lineaire ruimte. De lineaire ruimte staat hier als een vierkant afgebeeld, alhoewel een lineaire ruimte eigenlijk oneindig uitgestrekt is.

systeem[9] door meetkundige kwantisatie een symmetrie van een kwantum systeem oplevert. Een symmetrie van een lineaire ruimte heet een *representatie*.



Dit is voor wiskundigen interessant, omdat zij geïnteresseerd zijn in het construeren van representaties *an sich*. In deel III wordt het recept voor meetkundige kwantisatie uitgebreid naar een bepaald soort faseruimten, waarvan de symmetrie door groepoïdeacties beschreven wordt. Dit culmineert in de formulering en het bewijs van het *kwantisatie-commuteert-met-reductie* vermoeden[10], uitgebreid naar groepoïden. Dat was één van de eerst geformuleerde doelen van mijn promotieonderzoek.

Niet-commutative meetkunde

Gaandeweg werd een ander doel van het onderzoek het uitbreiden van de procedure van meetkundige kwantisatie: van meetkunde naar *niet-commutatieve meetkunde*. Dit project is nog niet voltooid, maar deel IV behandelt een aantal belangrijke stappen. Wat meetkunde is, dat weet u waarschijnlijk nog van de middelbare school. Maar wat is nu niet-commutatieve meetkunde? Wat commuteert er niet? Wat is commuteren überhaupt? Het is tijd om dit te verhelderen.

Uit elke ruimte (zoals bijvoorbeeld de bovengenoemde faseruimten) kan op heel natuurlijke manier een *algebra* verkregen worden.[11] Een algebra is een lineaire ruimte

[9]mits deze *Hamiltoniaans* is; een bepaalde eigenschap die ik hier niet verder uit ga leggen, maar die wel heel belangrijk is.

[10]Ook wel het Guillemin-Sternberg vermoeden genoemd, naar de eerste vermoeders.

[11]Dit is onzorgvuldig geformuleerd; het gaat hier om topologische ruimten en zogeheten C^*-algebra's.

waarin je ook twee punten met elkaar kan vermenigvuldigen: $a \times b$. Het typische van de aldus verkregen algebra's is dat ze *commutatief* zijn, dat wil zeggen

$$a \times b = b \times a.$$

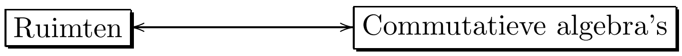
Vergelijk dit bijvoorbeeld met getallen, daarvan is de vermenigvuldiging ook commutatief, bijvoorbeeld

$$3 \times 5 = 5 \times 3$$

en

$$7\frac{1}{3} \times \sqrt{2} = \sqrt{2} \times 7\frac{1}{3}.$$

We kunnen uit een ruimte een commutatieve algebra maken, maar we kunnen ook omgekeerd uit een commutatieve algebra een ruimte maken:



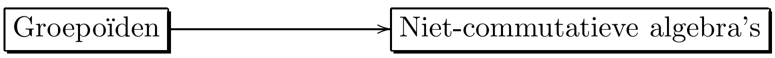
Dit diagram en de volgende diagrammen zult u helaas als abstract gegeven moeten aannemen.

Maar, er bestaan ook *niet-commutatieve* algebra's! Daarvoor geldt dus dat soms

$$a \times b \neq b \times a.$$

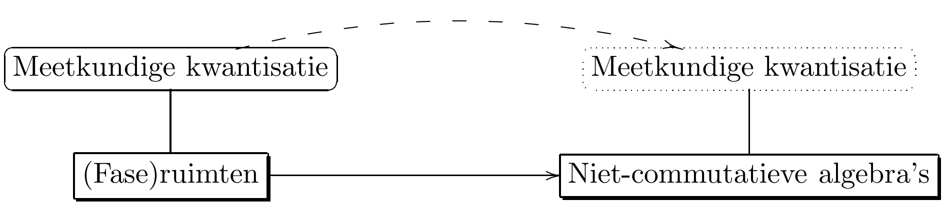
Niet-commutativiteit treedt vooral op als a en b, 'handelingen' ('operaties') zijn. Als ik bijvoorbeeld neem a is $+5$ en b is $\times 3$, dan commuteren ze niet: losgelaten op bijvoorbeeld 2 krijgen we $(2+5) \times 3 = 21$ en $(2 \times 3) + 5 = 11$, en $21 \neq 11$ bent u hopelijk met mij eens.[12]

Een belangrijk feit is dat sommige niet-commutatieve algebra's gemaakt kunnen worden met behulp van een groepoïde.



Deze uit groepoïden verkregen niet-commutatieve algebra's spelen een belangrijke rol in deel IV.

Het doel van niet-commutatieve meetkundigen is de wiskundige technieken voor gewone ruimten via commutatieve algebra's uit te breiden naar niet-commutatieve algebras. In deel IV wordt dit gedaan voor meetkundige kwantisatie. Tot nog toe werd meetkundige kwantisatie alleen toegepast op faseruimten[13] (corresponderend met commutatieve algebra's). Het doel in deel IV is het uitbreiden van meetkundige kwantisatie naar niet-commutatieve algebra's.



[12]Probeer ook maar eens voor andere getallen dan 2. Een minder wiskundig voorbeeld: Als ik mij uitkleed en daarna een douche neem, dan is het resultaat anders dan als ik eerst een douche neem en mij daarna pas uitkleed (als u niet bent overtuigd, probeer dit wiskundig experiment thuis zelf eens)! U kunt ook zelf voorbeelden verzinnen.

[13]Ik bedoel hier eigenlijk symplectische variëteiten (en meetkundige generalisaties daarvan).

Conclusie

We hebben nu dus twee manieren gezien waarop we meetkundige kwantisatie kunnen generaliseren: naar faseruimten met symmetrie gegeven door groepoïdeacties en naar niet-commutatieve faseruimten. Leest u nu de eerste regel van deze Nederlandse samenvatting nog eens en, als uw interesse gewekt is, de rest van dit proefschrift.

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Curriculum vitae

Rogier Bos was born on the 7th of august 1978 in Ter Aar, a small village in the Netherlands. He is the second child of three of Elly van Zuiden and Hans Bos. He lived and went to school in successively Boxmeer, Leiden and Zoetermeer. In Zoetermeer he obtained his 'Gymnasium' certificate at the Erasmus College in 1996. In the same year he moved to Utrecht, where he began his studies with a TWIN-program in mathematics and physics. He obtained a master degree in mathematics in 2001 with a 'minor' in physics and philosophy. His master thesis *Operads in deformation quantization* was written under guidance of prof. dr. I. Moerdijk.

In 2002 he joined the mathematical physics group of prof. dr. N. P. Landsman at the University of Amsterdam as a Ph. D. student. In 2004 this group moved to the Radboud University of Nijmegen. Rogier's work on groupoids, noncommutative geometry and geometric quantization resulted in the book *Groupoids in geometric quantization* that you are reading at the moment.

From september 2007 to september 2008 he will be a postdoctoral researcher at the Instituto Superior Técnico in Lisbon.