

Randomness? What randomness?

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Dedicated to Gerard 't Hooft, on the 20th anniversary of his Nobel Prize^b

Abstract

This is a review of the issue of randomness in quantum mechanics, with special emphasis on its ambiguity; for example, randomness has different antipodal relationships to determinism, computability, and compressibility. Following a (Wittgensteinian) philosophical discussion of randomness in general, I argue that deterministic interpretations of quantum mechanics (like Bohmian mechanics or 't Hooft's Cellular Automaton interpretation) are strictly speaking incompatible with the Born rule. I also stress the role of outliers, i.e. measurement outcomes that are *not* 1-random. Although these occur with low (or even zero) probability, their very existence implies that the no-signaling principle used in proofs of randomness of outcomes of quantum-mechanical measurements (and of the safety of quantum cryptography) should be reinterpreted statistically, like the second law of thermodynamics. In appendices I discuss the Born rule and its status in both single and repeated experiments, and review the notion of 1-randomness introduced by Kolmogorov, Chaitin, Martin-Löf, Schnorr, and others.

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1 Introduction

‘Quantum mechanics commands much respect. But an inner voice tells me that it is not Mr. Right. The theory delivers a lot, but it hardly brings us closer to the mystery of the Old Man. Anyway, I’m sure *he* does not play dice. (Einstein to Born, 1926).¹

‘In our scientific expectations we have grown antipodes. You believe in God playing dice and I in perfect laws in the world of things existing as real objects, which I try to grasp in a wildly speculative way.’ (Einstein to Born, 1944).²

Einstein’s idea of ‘perfect laws’ that should in particular be deterministic is also central to ‘t Hooft’s view of physics, as exemplified by his intriguing *Cellular Automaton Interpretation of Quantum Mechanics* (‘t Hooft, 2016). The aim of this paper is to provide arguments against this view,³ but even if these turn out to be unsuccessful I hope to contribute to the debate about the issue of determinism versus randomness by providing a broad view of the latter.⁴ My point of view is that, as argued in §2, randomness is a family resemblance in the sense of Wittgenstein (Sluga, 2006), but a special one that is always defined through its anti-pole, which may change according to the specific use of the term.

The anti-pole defining which particular notion of randomness is meant may vary even within quantum mechanics, and here two main candidates arise (§3): one is *determinism*, as emphatically meant by Born (1926) and most others who claim that randomness is somehow ‘fundamental’ in quantum theory, but the other is *compressibility* or any of the other equivalent notions defining what is called 1-randomness in mathematics as its anti-pole (see Appendix B for an explanation of this). The interplay between these different notions of randomness is the topic of §5, in which I argue that one cannot eat one’s cake and have it, in the sense of having a deterministic hidden variable theory underneath quantum mechanics that is strictly compatible with the Born rule. I also argue, more wildly, that Einstein’s prohibition of superluminal signaling should be demoted from an absolute to a statistical law, much as the second law of thermodynamics. This analysis relies on an in-depth (re)view of the Born rule, presented in Appendix A.

¹The German original is: ‘Die Quantenmechanik ist sehr achtung-gebietend. Aber eine innere Stimme sagt mir, daß das doch nicht der wahre Jakob ist. Die Theorie liefert viel, aber dem Geheimnis des Alten bring sie uns kaum näher. Jedenfalls bin ich überzeugt, daß *der* nicht würfelt.’ (Translation by the author.) The source is Einstein’s letter to Max Born from December 4, 1926, see Einstein & Born (2005), p. 154. Note the italics in *der*, or *he* in English: Einstein surely means that it is not God but the physicists who play dice. Since some authors claim the opposite, it may be worth emphasizing that this complaint against the indeterminism of quantum mechanics as expressed in Born (1926), to which Einstein replies here (see also §3 below for the specific passage that must have upset Einstein), forms the sole contents of this letter; Einstein’s objections to the non-locality of the theory only emerged in the 1930s. Even Pais (1982, p. 440), who should have known better, confuses the issue by misattributing the ‘God does not play dice’ quotation to a letter by Einstein to Lanczos from as late as March 21, 1942, in which at one stroke he complains that ‘It seems hard to look into God’s cards. But I cannot for a moment believe that He plays dice and makes use of “telepathic” means (as the current quantum theory alleges He does).’

²Letter dated November 7, 1944. Strangely, this letter is not contained in the *Einstein–Born Briefwechsel 1916–1955* cited in the previous footnote; the source is Born (1949, p. 176) and the translation is his.

³What I will not argue for here is the real reason I do not believe in perfect laws, namely the idea of *Emergence*, according to which there are no fundamental laws, let alone perfect ones: every (alleged) law originates in some lower substratum, which itself is subject to laws originating in yet another realm. As I like to say: ‘Nothing in science makes sense except in the light of emergence’ (free after Theodosius Dobzhansky, who famously wrote that ‘nothing in biology makes sense except in the light of evolution.’)

⁴See Bricmont *et al* (2001), Bub & Bub (2018), Cassirer (1936), Frigg (2016), Loewer (2001), Nath Bera *et al* (2017), Svozil (1993, 2018), and Vaidman (2014) for other perspectives on randomness in physics.

2 Randomness as a family resemblance

‘The idea that in order to get clear about the meaning of a general term one had to find the common element in all its applications has shackled philosophical investigation; for it has not only led to no result, but also made the philosopher dismiss as irrelevant the concrete cases, which alone could have helped him to understand the usage of the general term.’ (Wittgenstein, *Blue Book*, §§19–20).

‘I’m saying that these phenomena have no one thing in common in virtue of which we use the same word for all – but there are many kinds of *affinity* between them. (...) we see a complicated network of similarities overlapping and criss-crossing: similarities in the large and in the small. I can think of no better expression to characterize these similarities than “family resemblances”; for the various resemblances between members of a family – build, features, colour of eyes, gait, temperament, and so on and so forth – overlap and criss-cross in the same way.’ (Wittgenstein, *Philosophical Investigations*, §§65–67).

Though he did not mention it himself, randomness seems a prime example of a phenomenon Wittgenstein would call a ‘family resemblance’.⁵ Independently, as noted by historians Lüthy & Palmerino (2016) on the basis of examples from antiquity and medieval thought,⁶ the various different meanings of randomness (or chance) can all be identified by their antipole. Combining these ideas, I submit that randomness is not just any old Wittgensteinian family resemblance, but a special one that is *always* defined *negatively*.

- To begin with, in Aristotle’s famous example of a man who goes to the market and walks into his debtor, the randomness of the encounter derives from the fact that the man did *not* go to the market in order to meet his debtor (but instead went there to buy food). Similarly for the man who digs a hole in his garden to plant a tree and finds a treasure. Even the birth of female babies (and certain other ‘chance substances’ for which he literally uses the Greek word for ‘monsters’) was identified by The Philosopher as a *failure* of purpose in Nature. Thus what actually happened in all these examples was accidental because (as we would say it) it was *not* intended, or, in Aristotelian parlance, because there was *no* final cause. By the same token, Aristotle found the atomistic cosmos of Democritus “random” because it was purposeless, ridiculing him for making the cosmic order a product of chance.
- In contrast, half a century later Epicurus found the atomic world not random at all and introduced randomness through the ‘swerve’, immortalized by Lucretius:

‘When the atoms are traveling straight down through empty space by their own weight, at quite indeterminate times and places they swerve ever so little from their course, just so much that you can call it a change of direction. If it were not for this swerve, everything would fall downwards like raindrops through the abyss of space. No collision would take place and no impact of atom on atom would be created. Thus nature would never have created anything.’ (*De Rerum Natura*, Book II).⁷

⁵This may be worth emphasizing, since even first-rate philosophers like Eagle (2005) still try to nail it down, ironically citing other philosophers who also did precisely that, but allegedly in the ‘wrong’ way!

⁶See Vogt (2011) for a comprehensive survey of the historical usage of randomness and chance etc., including references to original sources. See also Lüthy & Palmerino (2016) for a brief summary.

⁷See Lucretius (1951), p. 66. See also Greenblatt (2011) for the thrilling rediscovery of *De Rerum Natura*.

This was, so to speak, the first complaint against determinism (the goal of Epicurus/Lucretius was to make room for free will in a world otherwise seen as effectively dead because of the everlasting sequence of cause and effect), and indeed, in the context of our analysis, the key point is that the swerve is random because it is ‘indeterminate’, or because the atoms *depart* from their natural straight course.

- Neither of these classical meanings is at all identical with the dominant usage from medieval times to the early 20th century, which was exemplified by Spinoza, who claimed that not only miracles, but also circumstances that have concurred by chance are reducible to ignorance of the true causes of phenomena, for which ultimately the will of God (‘the sanctuary of ignorance’) is invoked as a placeholder.⁸ Thus Spinozist randomness lies in the *absence* of full knowledge of the entire causal chain of events.
- In the Leibniz–Clarke correspondence,⁹ the latter, speaking for Newton, meant *involuntariness* by randomness, which constraint he and Newton, against Leibniz, denied at least God could have. Leibniz, on the other hand, in some sense ahead of his time (yet in another following Epicurus/Lucretius), used the word ‘random’ to designate the *absence* of a determining cause—a possibility which he (unlike Epicurus/Lucretius) denied on the basis of his principle of sufficient reason.¹⁰ This is clear from an interesting passage which is not widely known and predates Laplace:

‘One sees then that everything proceeds mathematically - that is, infallibly - in the whole wide world, so that if someone could have sufficient insight into the inner parts of things, and in addition has remembrance and intelligence enough to consider all the circumstances and to take them into account, he would be a prophet and would see the future in the present as in a mirror.’¹¹

- Arbuthnot, a younger contemporary and follower of Newton, may have been the first author to explicitly address the role of randomness in the deterministic setting of Newtonian physics. In the Preface to his translation of Huygens’s path-breaking book *De Ratiociniis in Ludo Aleae* on probability theory,¹² he wrote:

‘It is impossible for a Die, with such determin’d force and direction, not to fall on such determin’d side, only I don’t know the force and direction which makes it fall on such determin’d side, and therefore I call it Chance, which is nothing but the want of Art.’

And similarly, but highlighting the alleged negativity of the concept even more:

⁸See *Ethics*, Part I, Appendix.

⁹See also Lüthy & Palmerino (2016), §2.7 for part of the following analysis. The Leibniz–Clarke correspondence is available in many editions, such as Vailati (1997), and the online edition Bennett (2017).

¹⁰Hacking (1990, Ch. 2) calls this the *doctrine of necessity* and shows it pervaded early modern thought.

¹¹The German original is quoted and referenced by Cassirer (1936): ‘Hieraus sieht man nun, das alles mathematisch, d.i. ohnfehlbar zugehe in der ganzen weiten Welt, so gar, dass wenn einer eine genugsame Einsicht in die inneren Teile der Dinge haben könnte, und dabei Gedächtnis und Verstand genug hätte, um alle Umstände vor zu nehmen und in Rechnung zu bringen, würde er ein Prophet sein, und in dem Gegenwärtigen das Zukünftige sehen, gleichsam als in einem Spiegel.’ English translation by the author.

¹²The Latin original *De Ratiociniis in Ludo Aleae* is from 1657 and Arbuthnot’s English translation *On the Laws of Chance* appeared in 1692. The quotation is taken from Diaconis & Skyrms (2018), p. 9.

‘*Chance*, in atheistical writings or discourse, is a sound utterly insignificant: It imports no determination to any *mode of Existence*; nor in deed to *Existence* itself, more than to *non existence*; it can neither be defined nor understood; nor can any Proposition concerning it be either affirmed or denied, excepting this one, “That it is a mere word.”’ (De Moivre, 1718).¹³

So this is entirely in the medieval spirit, where ignorance—this time relative to Newton’s physics as the ticket to full knowledge—is seen as the origin of randomness.

- A century later, and like Arbuthnot and De Moivre in a book on probability theory (*Essai philosophique sur les probabilités*, from 1814), Laplace portrayed his demon to make the point that randomness arises in the *absence* of such an intellect.¹⁴

‘An intelligence which could comprehend all the forces that set nature in motion, and all positions of all items of which nature is composed—an intelligence sufficiently vast to submit these data to analysis—it would embrace in the same formula the movements of the greatest bodies in the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as well as the past, would be present to its eyes.’

Note that Leibniz’ prophet appeals to the logical structure of the universe that makes it deterministic, whereas Laplace’s intelligence knows (Newtonian) physics.¹⁵ In any case, it is important to note that Laplacian randomness is defined *within a deterministic world*,¹⁶ so that its anti-pole is not indeterminism but full knowledge (and computing power, etc.). Indeed, less well known than the above quotation is the following one, from the very same source:

‘All events, even those which on account of their insignificance do not seem to follow the great laws of nature, are a result of it just as necessarily as the revolutions of the sun.’¹⁷

- The ignorance interpretation of randomness and chance still lurks behind the probabilities introduced in the 19th century in statistical mechanics, which in my view were therefore wholeheartedly construed in the medieval and early modern spirit.¹⁸

¹³Quoted by Hacking (1990, p. 13) from De Moivre’s *Doctrine of Chance*, originally written in English.

¹⁴The translation is from Laplace (1902), p. 4. See van Strien (2014) for history and analysis.

¹⁵However, van Strien (2014) argues that Laplace also falls back on Leibniz (and gets the physics wrong by not mentioning the momenta that the intelligence should know, too, besides the forces and positions).

¹⁶Famously: ‘The world W is Laplacian deterministic just in case for any physically possible world W' , if W and W' agree at any time, then they agree at all time.’ (Earman, 1986, p. 13).

¹⁷Quoted by Hacking (1990), p. 11.

¹⁸It is often maintained that these probabilities are *objective*, which might cast doubt over the idea that they originate in ignorance. I personally find the distinction between “objective” and “subjective” chances quite unhelpful in the context of fundamental physics, a left-over from (by now) irrelevant and outdated attempts to define probabilities “objectively” as relative frequencies (see also Appendix B) as opposed to interpreting them “subjectively” as credences à la Ramsey. For example, Loewer (2001, 2004) claims that the probabilities in statistical mechanics are objective and are as objective as those in quantum mechanics. But both are predicated on a subjective choice of observables, based on ignoring microscopic and quantum-mechanical degrees of freedom, respectively. *Given* that choice, the probabilities are *subsequently* objective, but the *choice* itself is surely subjective (see also Heisenberg quoted in §3 below). Instead of a pointless table-tennis game between “objective” and “subjective”, a better term would be “perspectival”. For example, I agree with Rovelli (2017) that time’s arrow is perspectival (and hence is not “objective”). See also Jaynes (1985), pp. 118–120, for similar comments on terminology in philosophy of probability.

- Hacking (1990) explains how the doctrine of necessity began to erode in the 19th century, largely through the use of statistics in population studies and biology.¹⁹ In philosophy, the century culminated in the famous words of Peirce (1892, p. 337):

‘I believe I have thus subjected to fair examination all the important reasons for adhering to the theory of universal necessity, and shown their nullity. (...) If my argument remains unrefuted, it will be time, I think, to doubt the absolute truth of the principle of universal law.’

This partly paved the way for the claim of irreducible randomness in quantum mechanics, although the influence of population studies and biology on intrinsic developments in physics should perhaps not be overestimated. However, the insight that probability and statistics gave rise to their own laws (as opposed to the fear that randomness is pretty much the same as lawlessness), which partly dated back to the previous two centuries (Hacking, 2006), surely made quantum theory possible.

- The randomness of variations in heritable traits that were introduced almost simultaneously with the rise of statistical physics in Darwin’s theory of evolution by natural selection meant something completely different from Laplace etc., best expressed by the geneticist Theodosius Dobzhansky a century later (Merlin, 2010):

‘Mutations are random changes because they occur independently of whether they are beneficial or harmful.’

Indeed, both historically and actually, the anti-pole to Darwin’s randomness of variations is Lamarck’s goal-orientedness thereof,²⁰ intended to strengthen the species (like the proverbial sons of the blacksmith who according to Lamarck inherit his strong muscles). In particular, it does not matter if the variations are of known or unknown origin, or fall under a deterministic or an indeterministic kind of physics.

- Continuing our detour into biology, the well-known Hardy–Weinberg equilibrium law in population genetics, which gives the relative frequencies of alleles and genotypes in a given (infinite) population, is based on the assumption of *random mating*. This means that mating takes place between pairs of individuals who have *not* selected each other on the basis of their genetic traits (i.e. there is *no* sexual selection).
- Eagle (2005, p. 775–776) proposes that ‘randomness is maximal *unpredictability*’ (which agrees with criterion 3 below for binary strings), and argues that this is equivalent to a random event being ‘probabilistically *independent* of the current and past states of the system, given the probabilities supported by the theory.’
- Most people, especially those without scientific training,²¹ asked to mention a random event they have encountered, typically mention what is called a *coincidence*. This notion goes back at least to Aristotle, but a modern definition is the following:

¹⁹Though also inspired by the eventual rise of quantum mechanics, Kern (2004) gives a completely different history of causality and uncertainty in the 19th and 20th centuries, tracking the changing explanatory roles of these factors in the the study of murder as documented by more than a hundred novels.

²⁰As such, Spinoza’s philosophical analysis, modern physics, and (evolutionary) biology all contributed to the downfall of the Aristotelian (and subsequently Christian) final causes my analysis started from.

²¹Though Stephen Hawking was not adverse to the fact that he was born on January 8th, 1942, exactly 300 years after the death of Galileo Galilei, and, had he been able to note it, would undoubtedly have rejoiced in the equally remarkable fact that he died in 2018 on the birthday of Albert Einstein (March 14).

‘A coincidence is a surprising concurrence of events, perceived as meaningfully related, with no apparent causal connection.’
(Diaconis & Mosteller, 1989, p. 853).²²

The subjective nature of this definition (as in ‘surprising’, ‘perceived’, ‘meaningful’, and ‘apparent’) may repel the scientist, but ironically, the aim of this definition is to debunk coincidences. Because of the contrast between coincidences in daily life we are now talking about and coincidences in the physical setting of bipartite correlation experiments à la ERR–Bohm–Bell, it is interesting to briefly recall how most if not all everyday coincidences can be rejected on the basis of the above definition:²³

1. *The concurrence of events was not at all as surprising as initially thought.* This argument is either based on the inability of most people to estimate probabilities correctly (as in the well-known birthday problem), or, if the events were really unlikely, on what Diaconis and Mosteller call *the law of truly large numbers*.²⁴

‘Rare events occur with high frequency in the presence of large numbers of events’.

2. *Against first appearances there was a causal connection, either through a common cause or through direct causation.* On a par with the first explanation, in daily life this is the second way to debunk coincidences, but in physics it may fail even when the first explanation is unlikely; indeed, the point of Bell’s Theorem is of course that both kinds of causation are *excluded* in the ERR–Bohm–Bell setting! This case will be taken up in §3 and elsewhere in this paper.

- Finally, serving our aim to compare physical and mathematical notions of randomness, I preview the three equivalent definitions of 1-randomness (see Appendix B and references for details) and confirm that also they fit into our general picture. What is being defined here is randomness of binary strings, and our discussion here is so superficial that I will not even distinguish between finite and infinite ones.

1. A string x is 1-random if its shortest description is x itself, i.e., there exists *no* lossless compression of x (in the sense of a computer program that outputs x and whose length is shorter than the length of x): thus x is *incompressible*.
2. A string x is 1-random if it *fails* all tests for patterns (within a suitable class).
3. A string x is 1-random if there exists *no* successful (computable) gambling strategy on the digits of x ; roughly speaking, these digits are *unpredictable*.

Moving towards the main goal of the paper, I now continue our list of examples (i.e. of the principle that randomness is a family resemblance whose different meanings are always defined negatively through their anti-poles) in the context of quantum mechanics, which is rich enough by itself to provide its own family of different meanings of randomness (all duly defined negatively), although these may eventually be traceable to the above cases.²⁵

²²Perhaps confusingly, Ramsey (1926) attributes any coincidence to chance and hence would probably even take the Diaconis–Mosteller definition to define a *non*-coincidence.

²³For details see Landsman (2018), so far available in Dutch only. In practice it usually does not help to argue that the events in question were not ‘meaningfully related’, since at that point semi-intellectual opponents will invoke “synchronicities” (Jung, 1952) and the like, and leave the domain of science (as Jung himself squarely admitted). Even if one goes along with that, discussions tend to become circular.

²⁴In a slightly different phrasing this “law” is also called *The Improbability Principle* (Hand, 2015).

²⁵It will be clear from the way I discuss quantum mechanics that this paper will just be concerned with

3 Randomness in quantum mechanics

Already the very first (scholarly) exposition of the issue of randomness in quantum mechanics by Born made many of the major points that are still relevant today:

‘Thus Schrödinger’s quantum mechanics gives a very definite answer to the question of the outcome of a collision; however, this does not involve any causal relationship. One obtains *no* answer to the question “what is the state after the collision,” but only to the question “how probable is a specific outcome of the collision” (in which the quantum-mechanical law of [conservation of] energy must of course be satisfied). This raises the entire problem of determinism. From the standpoint of our quantum mechanics, there is no quantity that could causally establish the outcome of a collision in each individual case; however, so far we are not aware of any experimental clue to the effect that there are internal properties of atoms that enforce some particular outcome. Should we hope to discover such properties that determine individual outcomes later (perhaps phases of the internal atomic motions)? Or should we believe that the agreement between theory and experiment concerning our inability to give conditions for a causal course of events is some pre-established harmony that is based on the non-existence of such conditions? I myself tend to relinquish determinism in the atomic world. But this is [also] a philosophical question, for which physical arguments alone are not decisive.’ (Born, 1926, p. 866).²⁶

Given the fact that Born was the first to discuss such things in the open literature, it is remarkable how perceptive his words are: he marks the opposition of randomness to determinism, recognizes the possibility of hidden variables (with negative advice though), and understands that the issue is not just a technical one. *Bravo!* Having said this, in line with the previous section our aim is, of course, to confront the anti-pole of determinism with other possible anti-poles to randomness as it is featured by quantum mechanics.

The introduction of fundamental probabilities in quantum theory is delicate in many ways, among which is that fact that the Schrödinger equation is even more deterministic than Newton’s laws.²⁷ Hence what is meant is randomness of *measurements outcomes*; since it is not our aim (here) to solve the measurement problem—for which see Landsman (2017), Chapter 11—I simply assume that *i*) measurement is a well-defined laboratory practice, and *ii*) measurements have outcomes. In all that follows, I also accept the statistical predictions of quantum mechanics for these outcomes (which are based on the Born rule reviewed in Appendix A). Even so, the claim of *irreducibility* of randomness, which is typical for all versions of the Copenhagen Interpretation (and for any mainstream view held by physicists) is almost incomprehensible, since one of the pillars of this interpretation is Bohr’s doctrine of classical concepts, according to which the apparatus is described classically; randomness of measurement outcomes is then seen as a consequence of the very definition of a measurement. But this suggests that randomness should be *reducible* to ignorance about the quantum-mechanical degrees of freedom of the apparatus:

process randomness, as opposed to *product randomness* (Eagle, 2005). There is certainly a distinction between the two, but the former would take us into the quagmire of the measurement problem.

²⁶Translation by the author. The reference is to the German original.

²⁷In the sense that the solution is not only uniquely determined by the initial state, but, by Stone’s Theorem, even exist for all $t \in \mathbb{R}$ (in other words, incomplete motion is impossible), cf. Earman (2009).

‘these uncertainties (...) are simply a consequence of the fact that we describe the experiment in terms of classical physics.’ (Heisenberg, 1958, p. 53)

Ironically, Bell’s Theorem(s), which arose in opposition to the the Copenhagen Interpretation (Cushing, 1994), did not only prove Einstein (as the leading opponent of this interpretation) *wrong* on the issue that arguably mattered most to him (namely locality in the sense defined later in a precise way by Bell), but also proved Bohr and Heisenberg *right* on the *irreducibility* of randomness, at least if we grant them randomness *per se*. Indeed, suppose we equate *reducibility* of randomness with the existence of a “Laplacian” deterministic hidden variable theory (i.e. use the anti-pole of determinism), and assume, as the Copenhagenists would be pleased to, the conjunction of the following properties:

1. The *Born rule* and the ensuing statistical predictions of quantum mechanics;
2. *Hidden signal-locality* (i.e. the impossibility of active superluminal communication if one knows the state λ of the underlying deterministic theory);²⁸
3. *Free will* (or *free choice*), that is, the independence of the choice of measurement settings from the state of the system one measures using these settings, in a broad sense of ‘state’ that includes the prepared state as well as the ‘hidden’ state λ .²⁹

Then both Bell’s (1964) Theorem and the Free Will Theorem imply that such a deterministic hidden variable theory cannot exist, and hence that the Laplacian interpretation of randomness does not apply to quantum mechanics (granting assumptions 1–3). This warrants the Copenhagen claim of *irreducible* or *non-Laplacian* or *Leibnizian* randomness.

Viable deterministic hidden variable theories compatible with the Born rule therefore have to choose between giving up either *hidden signal-locality* or *free will*. Given this choice, we may therefore distinguish between theories that:

- give up *hidden signal-locality*, like Bohmian mechanics;
- give up *free will*, like the cellular automata interpretation of ’t Hooft (2016).

In both cases the statistical predictions of quantum mechanics are recovered by averaging the hidden variable or state with respect to a probability measure μ_ψ on the space Λ of hidden variables, given some (pure) quantum state ψ (which is assumed to be known). The difference is that in the first case the hidden state λ determines the outcomes given the settings, whereas in the second case λ determines the outcomes as well as the settings.

- In Bohmian mechanics the hidden variable is position x , and $d\mu_\psi = |\psi(x)|^2 dx$ is the Born probability for outcome x with respect to the expansion $|\psi\rangle = \int dx \psi(x)|x\rangle$.³⁰
- Similarly, in ’t Hooft’s theory the hidden state n is identified with a basis vector $|n\rangle$ in some Hilbert space H , and once again the measure $\mu_\psi(n) = |c_n|^2$ is given by the Born probability for outcome n with respect to the expansion $|\psi\rangle = \sum_n c_n |n\rangle$.

In Bohmian mechanics (’t Hooft does not need this!) such averaging also restores (surface) *signal-locality*, i.e., the impossibility of superluminal communication on the basis of *actual* measurement outcomes (which is a theorem of quantum theory, though a much more delicate one than is usually thought, as I will argue in §5), see also Valentini (2002).

²⁸This is often called *Parameter Independence*, as in e.g. the standard textbook by Bub (1997).

²⁹This assumption even makes sense in a super-deterministic theory, where the settings are not free. See Cator & Landsman (2014) or Landsman (2017), Chapter 6, for a discussion of this point as well as for a unified view of both Bell’s Theorem (from 1964) and the Free Will Theorem of Conway and Kochen.

³⁰Bohmians call this choice of μ_ψ the *quantum equilibrium condition*. It was first written down by Pauli.

4 Probabilistic preliminaries

‘O False and treacherous Probability,
Enemy of truth, and friend to wickednesse;
With whose bleare eyes Opinion learns to see,
Truth’s feeble party here, and barrenesse.
When thou hast thus misled Humanity,
And lost obedience in the pride of wit,
With reason dar’st thou judge the Deity,
And in thy flesh make bold to fashion it.
Vaine thoght, the word of Power a riddle is,
And till the vayles be rent, the flesh newborne,
Reveales no wonders of that inward blisse,
Which but where faith is, every where findes scorne;
Who therfore censures God with fleshly sp’rit,
As well in time may wrap up infinite

Philip Sidney (1554–1586), *Coelica*, Sonnet CIV.³¹

My aim is to give a critical assessment of the situation described in the previous section. My analysis is based on the interplay between the single-case probability measure μ on an outcome space X , which for the purpose of this paper will be the Born measure $\mu = \mu_a$ on the spectrum $X = \sigma(a)$ of some self-adjoint operator a , and hence are provided by *theory* (see also Appendix A), and the probabilities defined as long-run frequencies for outcome sequences $x = (x_1, x_2, \dots)$ of the Bernoulli process defined by (X, μ) , which are given by *experiment*. To obtain clean mathematical results, I assume experiments can be repeated infinitely often. This is clearly an idealization, which is regulated by *Earman’s Principle*:

‘While idealizations are useful and, perhaps, even essential to progress in physics, a sound principle of interpretation would seem to be that no effect can be counted as a genuine physical effect if it disappears when the idealizations are removed.’ (Earman, 2004, p. 191)

Indeed, finite-size effects would confirm the picture I sketch (see Appendix B), much as the law(s) of large numbers have finite-size approximants like the Chernoff-Hoeffding bound. In particular, zero probability of some infinite sequence comes down to very low probability for the corresponding finite strings, and probability 1 for infinite sequences amounts to very high probability for finite strings.³² Thus I use the canonical probability measure μ^∞ on the infinite product space $X^\mathbb{N}$ of all infinite sequences $x = (x_1, x_2, \dots)$, where $x_n \in X$, and $X^\mathbb{N}$ are canonically equipped with the cylindrical σ -algebra $\mathcal{S} \subset \mathcal{P}(X^\mathbb{N})$.³³

³¹The first four lines of this poem are printed on the last page of Keynes (1921), without any source.

³²One has to distinguish between *outcomes* with probability zero and *properties* that hold with probability zero. Indeed, *every* single outcome (in the sense of an infinite bitstream produced by a fair quantum coin flip) has probability zero, and hence this property alone cannot distinguish between random outcomes (in whatever sense, e.g. 1-random) and non-random outcomes (such as deterministic outcomes, or, more appropriately in our technical setting, computable ones), or indeed between any kind of different outcomes. On the other hand, not being random is a *property* that holds with probability zero, in that the *set* of all outcomes that are not random has probability zero (equivalently, the event consisting of all outcomes that are random happens almost surely, i.e. with probability 1). And yet outcomes with this probability zero property exist and may occur (in the finite case, with very small but positive probability). See also §5.

³³ Here $\mathcal{P}(Y)$ denotes the power set of Y . The σ -algebra \mathcal{S} is generated by all subsets of the form

To see how one may recover or verify μ from the long-term frequencies governed by the product measure μ^∞ , for any function $f : X \rightarrow \mathbb{R}$, define $f^{(N)} : X^N \rightarrow \mathbb{R}$ by

$$f^{(N)}(x_1, \dots, x_N) = \frac{1}{N}(f(x_1) + \dots + f(x_N)). \quad (4.1)$$

Then (by the ergodic theorem), for any $f \in C(X)$, almost surely with respect to μ^∞ ,

$$\lim_{N \rightarrow \infty} f^{(N)} = \langle f \rangle_\mu \equiv \int_X d\mu(x) f(x), \quad (4.2)$$

times the unit function 1_{X^N} on X^N . This is for continuous functions f , but a limit argument extends it to characteristic functions $1_A : X \rightarrow \underline{2} \equiv \{0, 1\}$, where $A \subset X$, so that

$$\lim_{N \rightarrow \infty} 1_A^{(N)} = \mu(A), \quad (4.3)$$

times 1_{X^N} , again μ^∞ -almost surely. If we define the ‘probability’ of A within some infinite sequence (x_1, x_2, \dots) as the relative frequency of A , i.e., as the limit as $N \rightarrow \infty$ of the number of x_n within (x_1, \dots, x_N) that lie in A divided by N , then (4.3) states that for almost all sequences in $X^{\mathbb{N}}$ with respect to the infinite product measure μ^∞ this ‘probability’ of A equals its Born probability. This is useful, since the latter is a purely mathematical quantity, whereas the former is experimentally accessible (at least for large N).

In what follows, I specialize this setting to a fair coin flip, that is, $X = \underline{2} = \{0, 1\}$ and $\mu(0) = \mu(1) = 1/2$. Hence $\underline{2}^{\mathbb{N}}$ is the space of infinite binary sequences, equipped with the probability measure μ^∞ induced by μ (as I shall argue in §5 below, this situation cannot in fact arise classically, at least not in a deterministic theory). We have:

Theorem 4.1 *Almost every binary sequence $x \in \underline{2}^{\mathbb{N}}$ is 1-random with respect to μ^∞ .*

See e.g. Calude (2010), Corollary 6.32. Thus the set E of all sequences that are *not* 1-random has probability zero, i.e. $\mu^\infty(E) = 0$, but this by no means implies that E is “small” in any other sense: set-theoretically, it is as large as its complement, i.e. the set of all 1-random sequences, which makes it bizarre that (barring a few exceptions related to Chaitin’s number Ω) not a single one of these 1-random sequences can actually be *proved* to be 1-random, cf. Appendix B. Theorem 4.1 has further amazing consequences:

Corollary 4.2 *With respect to μ^∞ , almost every infinite outcome sequence x of a fair coin flip is Borel normal,³⁴ incomputable,³⁵ and contains any finite string infinitely often.³⁶*

This follows because any 1-random sequence has these properties *with certainty*, see Calude (2010), §6.4. The relevance of Bernoulli processes for quantum theory comes from Theorem 5.1 in the next section, whose second option almost by definition yields these processes.

$B = \prod_{n=1}^N A_n \times \prod_{m=N+1}^\infty X$, where $A_n \subset X$ is Borel measurable and $N < \infty$. The probability measure μ on X then defines a probability measure μ^∞ on \mathcal{S} , whose value on B is defined by $\mu^\infty(B) = \prod_{n=0}^N \mu(A_n)$. See e.g. Dudley (1989), especially Theorem 8.2.2, for this and related constructions (due to Kolmogorov).

³⁴This means that any group $x_1 \cdots x_n$ of digits in x occurs with relative frequency equal to 2^{-n} .

³⁵This means that there is no computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ taking values in $\{0, 1\}$ that produces x , where computability is meant in the old sense of recursion theory, or, equivalently, in the sense of Turing machines or indeed modern computers. One could also say that there is no algorithm for f .

³⁶This is a strong version of the “infinite monkey (typewriter) theorem”, to the effect that if some monkey randomly hits the keys of a typewriter for a very long time, the probability that it produces a play by Shakespeare (or whatever other canonical text) is positive. See also the “Boltzmann Brain argument.”

5 Critical analysis and claims

The relevance of the material in the previous section comes from the following result.

Theorem 5.1 *The following procedures for repeated identical measurements are equivalent (in giving the same possible outcome sequences with the same probabilities):*

1. *Quantum mechanics is applied to the whole run (with classically recorded outcomes).*
2. *Quantum mechanics is just applied to single experiments (with classically recorded outcomes), upon which classical probability theory takes over to combine these.*

See Appendix A for the proof, which culminates in eq. (A.35), showing that the Born probability μ_a for single outcomes induces the Bernoulli process probability μ_a^∞ on the space $\sigma(a)^\mathbb{N}$ of infinite outcome sequences. As mentioned before, I specialize to fair quantum coin flips producing 50-50 Bernoulli processes, of which there are examples galore: think of measuring the third Pauli matrix $\sigma_z = \text{diag}(1, -1)$ in a state like $\psi = (1, 1)/\sqrt{2}$. In that case, Theorem 5.1 and Corollary 4.2 trivially have the following implication:

Corollary 5.2 *With respect to the product measure μ_a^∞ coming from the Born measure μ_a , almost every infinite outcome sequence x of a fair quantum coin flip is 1-random and therefore Borel normal, incomputable, and contains any finite string infinitely often.*

The Born rule therefore implies very strong randomness properties of outcome sequences, albeit merely with probability one (i.e. almost surely) with respect to μ_a^∞ . This makes it pointless to try and prove that outcome sequences of quantum measurements are 1-random: firstly, many sequences are not, and secondly, of those that are, the property is (generically) unprovable.³⁷ In any case, in the spirit of our general philosophy of randomness it is natural to compare randomness of infinite measurement outcome sequences as defined by:³⁸

1. *1-randomness* (with compressible sequences as its anti-pole), as suggested by the Born rule and the above analysis of its mathematical implications;³⁹
2. *indeterminism*, as suggested by Born himself, and in his wake also by Bell’s Theorem and the Free Will Theorem (seen as proofs of indeterminism under assumptions 1–3).

To make this comparison precise, we once again need the case distinction between hidden variable theories giving up hidden signal-locality like Bohmian mechanics and those giving up free will, like ’t Hooft’s theory. In the usual EPR–Bohm–Bell setting, let (x, y) be the outcome for given settings $(a, b) \in \underline{k} \times \underline{k}$ (assuming each setting can take k possible values, typically $k = 2$ or $k = 3$), where (x, a) are Alice’s and (y, b) are Bob’s (outcomes, settings).

³⁷See Abbott *et al* (2019) and Kovalsky *et al* (2018) for empirical tests of approximate 1-randomness.

³⁸In the literature on quantum cryptography and quantum random number generators (QRNG) one also find notions of (“free”) randomness for single qubits, see e.g. the reviews Brunner *et al* (2019), Herrero-Collantes & Garcia-Escartin (2017), Ma *et al* (2016), and Pironio (2018). The two (closely related) definitions used there fall into our general scheme of anti-polarity, namely randomness as *unpredictability* defined as minimizing the probability that an eavesdropper correctly guesses the qubit, and randomness as the absence of correlations between the qubit in question and anything outside its forward lightcone (which is heavily Leibnizian). These are related, since guessing is done through such correlations of the qubit. Certification of both cryptographic protocols and QRNGs is defined in these terms, backed by proofs of indeterminism à la Bell, so that in all versions non-locality plays a crucial role.

³⁹I am by no means the first to relate quantum theory to algorithmic randomness: see, for example, Bendersky *et al* (2014, 2016, 2017), Calude (2004), Svozil (1993, 2018), Senno (2017), Yurtsever (2001), and Zurek (1989). The way my work relates to some of these references will become clear in due course.

- In Bohmian mechanics, narrowed down to the above context, one may regard the hidden state $\lambda \in \Lambda$ as a function $\lambda(a, b) = (x, y)$ that determines the outcome from the settings, which are supposed to be “freely chosen”, and hence are given independently of λ . If we number the consecutive runs of the experiment by $n \in \mathbb{N} = \{1, 2, \dots\}$, then the outcome sequence is determined by some function

$$f : \mathbb{N} \rightarrow \Lambda \times \underline{k} \times \underline{k}; \quad (5.1)$$

$$n \mapsto (\lambda_n, a_n, b_n) \quad (5.2)$$

that gives the value $\lambda_n \in \Lambda$ of the hidden state as well as the settings (a_n, b_n) , from which the outcome (x_n, y_n) follows. It will be useful to split $f = (f_1, f_2)$ into two parts, where $\lambda_n = f_1(n)$ for some $f_1 : \mathbb{N} \rightarrow \Lambda$, and likewise $(a_n, b_n) = f_2(n)$.

- In ’t Hooft’s theory, λ determines all of $(a, b, x, y) = (a(\lambda), b(\lambda), x(\lambda), y(\lambda))$, but the settings (a, b) do not determine the outcome (x, y) ; one needs to know λ for this. In this case, the outcome sequence is therefore simply determined by a function

$$g : \mathbb{N} \rightarrow \Lambda; \quad (5.3)$$

$$n \mapsto \lambda_n. \quad (5.4)$$

A key point in the analysis of these functions f and g is the requirement that both theories reproduce the statistical predictions of quantum mechanics given through the Born rule relative to some pure state ψ . As already noted, this is achieved by requiring that λ is averaged with respect to some probability measure μ_ψ on Λ . If the experimental run is to respect this averaging, as it has to if its long-run frequencies are to reproduce the quantum-mechanical ones, then the maps f_1 and g must be ‘typical’ for the Born-like measure μ_ψ (cf. Dürr, Goldstein, & Zanghi, 1992; Callender, 2007; Norsen, 2018) which seems to mean that (the range of) f_1 and g should pass sufficiently many statistical tests based on μ_ψ (such as the law of large numbers). Technically, I would say that the samplings from Λ must be 1-random, assuming we use the tests used in the second definition of 1-randomness in §2, which include all the usual ones (Li & Vitányi, 2008, §2.4).⁴⁰ Anything deterministic, like computable samplings, will only contribute output sequences that are atypical for μ_ψ : even the *set* of all such sequences has measure zero.⁴¹

The requirement that the functions f_1 and g randomly sample μ_ψ necessarily introduces an element of randomness into the hidden variable theories in question, which seems somewhat at odds with their deterministic character. Indeed, there are two possibilities:

- This sampling is provided by the hidden variable theory. In that case, the above argument shows that the theory must contain an irreducibly random ingredient.
- The sampling is not provided by the theory. In that case, the theory fails to determine the outcome of any specific experiment and just provides averages of outcomes.

⁴⁰We only defined 1-randomness for outcome sequences of fair coin flips, but the definition can be extended to other measures, see Downey & Hirschfeldt (2010), §6.12 and references therein.

⁴¹In this light I draw attention to an important result of Senno (2017), Theorem 3.2.7, see also Bendersky *et al* (2017), which is entirely consistent with the above analysis: If the functions f_1 and g are computable (within a computable time bound), then Alice and Bob can signal superluminally. In other words, where mathematically speaking averaging the hidden state over the probability measure μ_ψ suffices to guarantee (surface) signal-locality even in a theory without hidden signal-locality (Valentini, 2002), if this averaging is done by sampling Λ in a long run of repeated measurements, then this sampling must at least be incomputable. If not, the mathematical averaging is not reproduced, to one’s detriment.

Either way, like standard (Copenhagen Interpretation) quantum mechanics the hidden variable theory cannot be deterministic, and the question is what has been gained. Surely, the source of indeterminism has been shifted: in Copenhagen quantum mechanics it is in the outcomes of experiments whereas in ‘deterministic’ hidden variable theories it is in the assignment of the hidden variable to each run of the experiment, i.e. in the sampling of the Born-like measure μ_ψ , but since in both Bohmian mechanics and ’t Hooft’s theory this measure literally *is* the Born measure while in Copenhagen quantum mechanics the outcomes of experiments equally well sample the Born measure, the difference between the two cases is almost metaphysical. Therefore, it would be fair to say that

truly deterministic hidden variable theories compatible with the Born rule do not exist.

It is instructive to compare this with classical coin tossing, in which the role of the hidden state is played by the initial conditions (cf. Diaconis & Skyrms, 2018, Chapter 1, Appendix 2). The 50-50 chances making the coin fair should be obtained by averaging over the initial conditions, i.e., by sampling with respect to a suitable probability distribution. By the above argument, this sampling cannot be done deterministically, for otherwise the outcome sequences appropriate to a fair coin do not obtain: it must be done in a genuinely random way. This is impossible classically, so that *fair classical coins* do not exist (as confirmed by the experiments of Diaconis c.s., see Diaconis & Skyrms (2018), Chapter 1).

In response to this argument, both the Bohmians and ’t Hooft go for the second option and blame the randomness in question on the initial conditions,⁴² whose specification is indeed usually seen as lying outside the range of a deterministic theory.⁴³ As explained by both parties (Dürr, Goldstein, & Zanghi, 1992; ’t Hooft, 2016), the randomness in the outcomes of measurement on quantum system, including the Born rule, is a direct consequence of the above randomness in initial conditions. But in a Laplacian deterministic theory one can either predict or retrodict and these procedures should be equivalent; so *within the context of a deterministic hidden variable theory of the kinds under discussion*, Copenhagen attributing the origin of randomness to the *outcomes* of measurement and our hidden variable theorists attributing it to the *initial conditions* for measurement, are equivalent. This makes it impossible to regard the hidden variable theories in question as deterministic underpinnings of (Copenhagen) quantum mechanics, as they are intended.⁴⁴

Bohmians (but not ’t Hooft) even have an additional problem, namely the function $f_2 : \mathbb{N} \rightarrow \underline{n} \times \underline{n}$ that provides the measurement settings. This is left outside the theory, but Bohmians still have to account for both the ‘freedom’ of choosing these settings and their randomness (which for example is used in all arguments around Bell’s Theorem, cf. Esfeld, 2015). Bell tried to kill two birds with the same stone by saying that the settings had to be ‘at least effectively free for the purpose at hand’, and clarifying this as follows:⁴⁵

⁴²Indeed, N runs of the same experiment need not be done subsequently but can be done simultaneously.

⁴³The Bohmians seem to be split on the origin of the quantum equilibrium distribution, compare e.g. Dürr, Goldstein, & Zanghi (1992) with Colin & Valentini (2014). See also Callender (2007) and Norsen (2018). Let me emphasize that the origin of μ_ψ or its dependence on μ_Ψ is not my concern here; the problem I address is the need to randomly sample it and the justification for doing so.

⁴⁴A different point was made by Pauli (1953, p. 38) against the pilot wave theory of De Broglie: ‘The hypothesis of a general probability distribution for the hidden variables that is determined by the single [wave] function ψ is not justified from the point of view of a deterministic scheme: it is borrowed from a theory which is based on the totally different hypothesis that the [wave] function provides a complete description of the system.’ I am indebted to Anthony Valentini and Guido Bacciagaluppi for this reference.

⁴⁵The source is Bell (1985), p. 105, which corresponds to p. 86 of the reprinted version in Bell, Gottfried, and Veltman (2001). The author learnt it from the talk by Tim Maudlin at the ’t Hooft 2019 conference.

‘Suppose that the instruments are set at the whim, not of experimental physicists, but of mechanical random generators. (...) Could the input of such mechanical devices be reasonably be regarded as sufficiently free for the purpose at hand? I think so. Consider the extreme case of a “random” generator which is in fact perfectly deterministic in nature and, for simplicity, perfectly isolated. In such a device the complete final state perfectly determines the complete initial state—nothing is forgotten. And yet for many purposes, such a device is precisely a “forgetting machine”. (...) To illustrate the point, suppose that the choice between two possible [settings], corresponding to a and a' , depended on the oddness of evenness of the digit in the millionth decimal place of some input variable. Then fixing a or a' indeed fixes something about the input—i.e., whether the millionth digit is odd or even. But this peculiar piece of information is unlikely to be the vital piece for any distinctly different purpose, i.e., it is otherwise rather useless. (...) In this sense the output of such a device is indeed a sufficiently free variable for the purpose at hand.’

In view of von Neumann’s warning about mechanical (i.e. pseudo) random generators:

‘Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin’ (von Neumann, 1951),

see also Markowsky (2014), Bell’s statement was questionable already at the time of writing, but today we know for sure that mechanical random generators leave a loophole in the EPR–Bohm–Bell experiment: as soon as just one of the functions defining the settings (i.e. either of Alice or of Bob) is computable,⁴⁶ there is a model that is local (in the sense of Bell). See Bendersky *et al* (2016) or Senno (2017), Theorem 2.2.1 This implies that Bohmian mechanics (as well as other deterministic hidden variable theories that leaves the settings free) requires even more randomness than the sampling of the (Born) probability measure μ_ψ , which further undermines the claim that it is a deterministic theory.

The analysis given so far focused on the necessity of correctly sampling a probability measure μ : if, so far in the context of hidden variable theories, where $\mu = \mu_\psi$, this is not done correctly, quantum-mechanical predictions such as signal-locality may be threatened. But in general there *are* measurement outcome sequences that fail to sample μ : events with very low or even zero probability can and do occur. This is even the whole point of the “law of truly large numbers” quoted in §2, which is true for infinite sequences, but may be clearer for (very) long finite sequences, where (in the spirit of Earman’s Principle) zero probability is replaced by very low probability. In general, the no-signaling (or signal-locality) property of quantum mechanics states that the probability

$$P_\psi(x | a, b) := \sum_y P_\psi(x, y | a, b) \quad (5.5)$$

is independent of b , where $P_\psi(x, y | a, b)$ is the *Born probability* that measurement of observables determined by the settings a and b give outcomes x and y . Indeed, we have

$$P_\psi(x | a, b) = P_{\psi|_A}(x | a), \quad (5.6)$$

where $\psi|_A$ is the restriction of the state ψ on $B(H_A \otimes H_B)$ to Alice’s part $B(H_A)$. Similarly, the Born probability $P_\psi(y | a, b)$ should be independent of a and in fact equals $P_{\psi|_B}(y | b)$.

⁴⁶The computable function is subject to a time bound, but this is true for all pseudo-random generators.

However, I have repeatedly noted that the *empirical probability* extracted from a long measurement outcome sequence coincides with the corresponding Born probability only almost surely with respect to the induced probability measure on the space of outcome sequences, and hence outliers violating the property (5.5) exist (for finite sequences even with positive probability). If one such run is found, the door to superluminal signaling is open, at least in principle. To see this, recall that the crudest form of determinism is what is called *predictability* by Cavalcanti & Wiseman (2012), i.e. the property that

$$P_\psi(x, y \mid a, b) \in \{0, 1\}. \quad (5.7)$$

It is easy to show that the conjunction of predictability and signal-locality implies factorization and hence, for random settings, the Bell (CHSH) inequalities, and therefore for suitable states ψ this conjunction contradicts the statistical predictions of quantum mechanics as expressed by the Born rule. Accepting the latter, signal-locality therefore implies unpredictability and hence some (very weak) form of randomness. There are many other results in this direction, ranging from e.g. Barrett, Kent, & Hardy (2005) to Wolf (2015),⁴⁷ involving varying definitions of randomness, all coming down to the implication

$$\text{no signaling} \Rightarrow \text{randomness}, \quad (5.8)$$

assuming free choice or random settings.⁴⁸ What I simply argue for is the contrapositive

$$\text{no randomness} \Rightarrow \text{signaling}. \quad (5.9)$$

This argument is just heuristic, since the terms are used differently in both cases: to prove (5.8) one typically uses Born probabilities and other theoretical entities, whereas for (5.9) I use probabilities obtained from outcome sequences as limiting relative frequencies: (5.9) then comes from low-probability sequences that *violate* (5.5), which is *satisfied* with probability one by sufficiently random sequences. Indeed, this difference is the whole point:

*signal-locality is a statistical property, like the second law of thermodynamics.*⁴⁹

⁴⁷Note that the aim of Barrett, Kent, & Hardy (2005) is to prove security of some quantum key distribution protocol on the basis of signal-locality *even if quantum mechanics turns out to be incorrect*, whereas the present paper investigates the role of statistical outliers *assuming quantum mechanics is correct*. These aims are closely related, of course, since too many (how many?) outliers may make one question the theory.

⁴⁸See Hermens (2019) for arguments for (5.8) and (5.9) even without assuming free choice.

⁴⁹Having said this, giving an actual signaling protocol for an outcome sequence (in the EPR–Bohm–Bell setting) that violates (5.5) is highly nontrivial. Yurtsever (2001) gives a protocol for superluminal signaling if the outcome sequence is not 1-random, but his arguments are very hard to follow and the paper with complete details he announces has never appeared. His approach may be neither viable nor necessary, since such a protocol need only exist for outcome sequences violating (5.5), which is a *stronger* property than violating 1-randomness—since contrapositively 1-randomness implies all averaging properties like (5.5). In fact, further assumptions are necessary in order for Alice to have a chance (*sic*) to learn Bob’s settings b from her part $(x_n)_n$ of the (double) outcome sequence $(x_n, y_n)_n$; a complete lack of structure would also make any kind of signaling random. I know of only one such protocol (which, however, should suffice as a “proof of concept”): if the unlikely outcome sequence $(x_n, y_n)_n$ comes from *computable* functions $x_n = x(a, b, n)$ and $y_n = y(a, b, n)$ with $O(T^2)$ time bounds, then signaling is possible through a learning algorithm for computable functions, see Senno (2017), Theorem 3.2.7 and Bendersky *et al* (2017). This is the same result as in footnote 41; by Proposition 3.1.7 in Senno (2017), *mutatis mutandis* it applies both with and without hidden variables. This protocol enables Alice (or Bob) to signal after a *finite* (but alas unknown and arbitrarily large) number of measurements (this answers a speculation by Carlo Rovelli that finite-size corrections to the lack of randomness may conspire to prevent superluminal signaling).

A The Born rule revisited

Every argument on probabilities in quantum mechanics relies on the Born rule, which is so natural that it will probably (!) be with us forever. In order to explain its canonical mathematical origin, I use the C*-algebraic approach to quantum theory, which because of its ability to simultaneously talk about commutative and non-commutative algebras (and hence about classical and quantum probabilities) is especially useful in this context.⁵⁰ Indeed, the Born rule arises by restricting a quantum state ω on the non-commutative algebra $B(H)$, i.e. the algebra of all bounded operators on some Hilbert space H , to the commutative C*-algebra $C^*(a)$ generated by some self-adjoint operator $a = a^* \in B(H)$ and the unit operator 1_H (so perhaps $C^*(a, 1_H)$ would have been better notation).⁵¹

This view has been inspired by the Copenhagen Interpretation, in that the associated probabilities originate in the classical description of a measurement setting, as called for by Bohr. However, precisely because of this, my derivation also seems to undermine the Copenhagen claim of the irreducibility of randomness, which may now clearly be traced back to a voluntary loss of information in passing from an algebra to one of its subalgebras and ignoring the rest (see also §3). The Copenhagen Interpretation will be saved from this inconsistency by a mathematical property that has no classical analog.

As I see it, the Born rule is the quantum-mechanical counterpart of the following elementary construction in measure theory. Let X be a compact Hausdorff space,⁵² with associated commutative C*-algebra $A = C(X)$ of complex-valued continuous functions on X . Let $f \in A$ be real-valued and denote its (automatically closed) range in \mathbb{R} by $\sigma(f)$; the notation is justified by the fact that the spectrum of a self-adjoint operator in quantum theory is analogous to $\sigma(f)$ in every possible way. Let ω be a state on A , or, equivalently (by the Riesz representation theorem), a probability measure μ on X , related to ω by

$$\omega(f) = \int_X d\mu(x) f(x). \quad (\text{A.1})$$

The state ω then also defines a probability measure μ_f on the ‘spectrum’ $\sigma(f)$ through

$$\mu_f(\Delta) = \mu(f \in \Delta), \quad (\text{A.2})$$

where $\Delta \subseteq \sigma(f)$ and $f \in \Delta$ or $f^{-1}(\Delta)$ denotes the subset $\{x \in X \mid f(x) \in \Delta\}$ of X . Virtually all of probability theory is based on this construction, which I now rephrase. For any C*-algebra A with unit 1_A , let $C^*(a)$ be the smallest C*-algebra in A (under the same operations and norm) that contains a and the unit 1_A , as above for $A = B(H)$. If $a^* = a$, then $C^*(a)$ is the norm-closure of the set of all (finite) polynomials in a . Take $A = C(X)$ and $f \in C(X)$, assumed real-valued (since $f^* = f$). This gives an isomorphism

$$C(\sigma(f)) \xrightarrow{\cong} C^*(f); \quad (\text{A.3})$$

$$g \mapsto g \circ f, \quad (\text{A.4})$$

where $f \in C(X)$, $g \in C(\sigma(f))$, as follows from the Stone–Weierstrass Theorem.⁵³

⁵⁰See e.g. Haag (1992), Ruetsche (2011), or Landsman (2017) for the C*-algebraic approach. My discussion of the Born rule is partly taken from the latter, which in turn elaborates on Landsman (2009), but the corresponding ‘classical’ construction has been added and may be new; I find it very instructive.

⁵¹This is well defined, since $C^*(a)$ is just the intersection of all C*-algebras in A that contain a and 1_A .

⁵²With some modifications the construction easily generalizes to the locally compact case.

⁵³if $K \subset \mathbb{R}$ is compact, then polynomials on K are dense in $C(K)$ for the supremum-norm, i.e. the norm on $C(X)$. To prove that (A.3) - (A.4) is an isomorphism of C*-algebras one may therefore start with polynomial functions g (taking $K = \sigma(f)$) and finish with a continuity argument.

If C is a C^* -subalgebra of A with the same unit as A , then any state ω on A defines a state $\omega|_C$ on C . Thus $\omega|_{C^*(f)}$ is a state on $C^*(f)$. Furthermore, if $\varphi : B \rightarrow C$ is a unital homomorphism of unital C^* -algebras (i.e., a linear map preserving all operations as well as the unit—in our case φ is even an isomorphism), and if ω' is a state on C , then $\varphi_*\omega' = \omega' \circ \varphi$ is a state on B . Using (A.3) - (A.4), I apply this construction to

$$A = C(X), \quad B = C(\sigma(f)), \quad C = C^*(f); \quad (\text{A.5})$$

$$\omega \in S(C(X)), \quad \omega' = \omega|_{C^*(f)}, \quad \varphi(g) = g \circ f, \quad (\text{A.6})$$

where $S(A)$ is the state space of A , so that a state ω on $C(X)$ defines a state ω' on $C(\sigma(f))$. Once again using the Riesz representation theorem then turns ω' into a probability measure μ' on $\sigma(f)$; *and this is exactly the probability measure $\mu_f = \mu'$ defined in (A.2).*

Copying this reasoning for quantum mechanics *mutatis mutandis* immediately gives the Born measure. Instead of (A.5) - (A.6), for some given $a = a^* \in B(H)$ I now take

$$A = B(H), \quad B = C(\sigma(a)), \quad C = C^*(a); \quad (\text{A.7})$$

$$\omega \in S(B(H)), \quad \omega' = \omega|_{C^*(a)}, \quad \varphi(g) = g(a), \quad (\text{A.8})$$

where (A.3) - (A.4) is replaced by the continuous functional calculus (CFC), that is,

$$C(\sigma(a)) \xrightarrow{\cong} C^*(a); \quad (\text{A.9})$$

$$g \mapsto g(a). \quad (\text{A.10})$$

Here $\sigma(a)$ is the spectrum of a , defined as the set of $\lambda \in \mathbb{C}$ for which the operator $a - \lambda \cdot 1_H$ is *not* invertible in $B(H)$; if $\dim(H) < \infty$, then $\sigma(a)$ is the set of eigenvalues of a . In most applications the state ω is *normal*, i.e., given by a density operator ρ on H through

$$\omega(a) = \text{Tr}(\rho a). \quad (\text{A.11})$$

We then replay our record: a state $\omega \in S(B(H))$ restricts to a state $\omega|_{C^*(a)}$ on $C^*(a)$, which is mapped to a state on $C(\sigma(a))$ by the CFC, which state in turn is equivalent to a probability measure on $\sigma(a)$. The probability measure μ_a on $\sigma(a)$ obtained by the above construction is exactly the Born measure, which is more commonly defined as follows:

Theorem A.1 *Let H be a Hilbert space, let $a^* = a \in B(H)$, and let ω be a state on $B(H)$. There exists a unique probability measure μ_a on the spectrum $\sigma(a)$ such that*

$$\omega(g(a)) = \int_{\sigma(a)} d\mu_a(\lambda) g(\lambda), \quad g \in C(\sigma(a)). \quad (\text{A.12})$$

We now return to the issue raised in the introduction to this appendix: even short of hidden variables (see §3), to what extent is the probabilistic structure of quantum mechanics that is already inherent in the Born measure reducible to ignorance? From a naive perspective it is, for in the corresponding classical case the measure μ_f on $\sigma(f)$ does not determine the measure μ on X except when $C^*(f) = C(X)$ (which is certainly possible, take e.g. $X = [0, 1]$ and $f(x) = x$). However, in the quantum case it so happens that if a is *maximal* (i.e. its spectrum is nondegenerate, or, equivalently, $C^*(a)$ is a maximal commutative C^* -subalgebra of $B(H)$), then the measure μ_a on $\sigma(a)$ *does* determine the state ω , at least if ω is pure and normal, as in (A.11), *despite the fact that $C^*(a)$ is just a small part of $B(H)$.* So the Copenhagen Interpretation is walking a tightrope here, but it doesn't fall.⁵⁴

⁵⁴This discussion is closely related to the so-called Kadison–Singer conjecture in operator algebras, which however is nontrivial only for non-normal states. See Stevens (2016) and Landsman (2017), §2.6 and §4.3

The Born measure is a mathematical construction; what is its relationship to experiment? This relationship must be the source of the (alleged) randomness of quantum mechanics, for the Schrödinger equation is deterministic. We start by postulating, as usual, that $\mu_a(A)$ is the (single case) probability that measurement of the observable a in the state ω (that jointly give rise to the pertinent Born measure μ_a) gives a result $\lambda \in A \subset \sigma(a)$. What needs to be clarified in the above statement is the word ‘probability’.

Although I do not buy Lewis’s (1994) “Best Systems Account” (BSA) of laws of nature,⁵⁵ I do agree with his identification of single-case probabilities as numbers (consistent with the probability calculus as a whole) that *theory* assigns to events, upon which long-run frequencies provide *empirical evidence* for the theory in question,⁵⁶ but do not *define* probabilities.⁵⁷ The Born measure is a case in point: these probabilities are *theoretically given*, but have to be *empirically verified* by long runs of independent experiments. In other words, by the results reviewed below such experiments provide numbers whose role it is to test the Born rule as a hypothesis. This is justified by (4.3) in §4, which equates probabilities *computed* from long-run frequencies with the corresponding single-case probabilities that *define* the clause “with probability one” under which (4.3) holds, and without which clause the limit in (4.3) would be undefined. As explained in §§4-5, the relevance of (4.3) to quantum physics comes from Theorem 5.1, which I now prove.

Proof of Theorem 5.1. Let $a = a^* \in B(H)$, where H is a Hilbert space and $B(H)$ is the algebra of all bounded operators on H , and let $\sigma(a)$ be the spectrum of a . For simplicity (and since this is enough for our applications, where $H = \mathbb{C}^2$) I assume $\dim(H) < \infty$, so that $\sigma(a)$ simply consists of the eigenvalues λ_i of a (which may be degenerate). Let us first consider a *finite* number N of runs of an identical measurement of a . The first option in the theorem corresponds to a simultaneous measurement of the commuting operators

$$a_1 = a \otimes 1_H \otimes \cdots \otimes 1_H; \quad (\text{A.13})$$

...

$$a_N = 1_H \otimes \cdots \otimes 1_H \otimes a, \quad (\text{A.14})$$

all defined on the N -fold tensor product $H^N \equiv H^{\otimes N}$ of H with itself.⁵⁸ To put this in a broader perspective, consider *any* set $(a_1, \dots, a_N) \equiv \underline{a}$ of commuting operators on *any* Hilbert space K (of which (A.13) - (A.14) is obviously a special case with $K = H^N$). These operators have a *joint spectrum* $\sigma(\underline{a})$, whose elements are the *joint eigenvalues* $\underline{\lambda} = (\lambda_1, \dots, \lambda_N)$, defined by the property that there exists a nonzero joint eigenvector $\psi \in K$ such that $a_i\psi = \lambda_i\psi$ for all $i = 1, \dots, N$; clearly,

$$\sigma(\underline{a}) = \{\underline{\lambda} \in \sigma(a_1) \times \cdots \times \sigma(a_N) \mid e_{\underline{\lambda}} \equiv e_{\lambda_1}^{(1)} \cdots e_{\lambda_N}^{(N)} \neq 0\} \subseteq \sigma(a_1) \times \cdots \times \sigma(a_N), \quad (\text{A.15})$$

⁵⁵See Loewer (2004) and, concisely, Callender(2007) for summaries of Lewis’s BSA. As I understand it, Lewisian laws *describe* nature, but fail to *govern* or *guide* it in the way that the Law of Moses, carved in stone, is supposed to influence (decent) human behaviour. Now I agree with Loewer (2004) that it is hard to see how laws as mathematical expressions in physics books could govern anything, but I expect this problem to be solved in due course by an emergentist account in which lower-level behaviour—though never fundamental—gives rise to emergent laws at some higher substratum of reality. See also footnote 3.

⁵⁶Within the Lewisian ideology this is ultimately justified by his *Principal Principle*, which roughly speaking equates chance as subjective degree of belief (i.e. credence) with objective chance.

⁵⁷This, as is well known by now, is a dead end street (Hájek & Hitchcock, 2016). It should be mentioned though that the fascinating yet ultimately flawed attempts of von Mises to provide such a definition played a decisive role in the road towards algorithmic randomness (van Lambalgen, 1987).

⁵⁸This can even be replaced by a single measurement, see Landsman (2017), Corollary A.20.

where $e_{\lambda_i}^{(i)}$ is the spectral projection of a_i on the eigenspace for the eigenvalue $\lambda_i \in \sigma(a_i)$. Von Neumann's Born rule for the probability of finding $\underline{\lambda} \in \sigma(\underline{a})$ then simply reads

$$p_{\underline{a}}(\underline{\lambda}) = \omega(e_{\underline{\lambda}}), \quad (\text{A.16})$$

where ω is the state on $B(K)$ with respect to which the Born probability is defined. If $\dim(K) < \infty$, as I assume, we always have $\omega(a) = \text{Tr}(\rho a)$ for some density operator ρ , and for a general Hilbert space K this is the case iff the state ω is normal on $B(K)$. For (normal) pure states we have $\rho = |\psi\rangle\langle\psi|$ for some unit vector $\psi \in K$, in which case

$$p_{\underline{a}}(\underline{\lambda}) = \langle\psi, e_{\underline{\lambda}}\psi\rangle. \quad (\text{A.17})$$

The Born rule (A.16) follows from the same reasoning as the single-operator case explained in the run-up to Theorem A.1 (Landsman, 2017, §4.1):⁵⁹ we have an isomorphism

$$C^*(\underline{a}) \cong C(\sigma(\underline{a})) \quad (\text{A.18})$$

of (commutative) C^* -algebras, and under this isomorphism the restriction of the state ω , originally defined on $B(K)$, to $C^*(\underline{a})$ defines a probability measure $\mu_{\underline{a}}$ on the joint spectrum $\sigma(\underline{a})$, which is just the Born measure whose probabilities are given by (A.16). For (A.13) - (A.14) we have equality in (A.15); since in that case $\sigma(a_i) = \sigma(a)$, we obtain

$$\sigma(\underline{a}) = \sigma(a)^N, \quad (\text{A.19})$$

and, for all $\lambda_i \in \sigma(a)$ and states ω on $B(H^N)$, the Born rule (A.16) becomes

$$p_{\underline{a}}(\lambda_1, \dots, \lambda_N) = \omega(e_{\lambda_1} \otimes \dots \otimes e_{\lambda_N}). \quad (\text{A.20})$$

Now take a given state ω_1 on $B(H)$. Reflecting the idea that ω is the state on $B(H^N)$ in which N independent measurements of $a \in B(H)$ in the state ω_1 are carried out, choose

$$\omega = \omega_1^N, \quad (\text{A.21})$$

the state on $B(H^N)$ defined by linear extension of its action on elementary tensors:

$$\omega_1^N(b_1 \otimes \dots \otimes b_n) = \omega_1(b_1) \dots \omega_1(b_n). \quad (\text{A.22})$$

It follows that

$$\omega^N(e_{\lambda_1} \otimes \dots \otimes e_{\lambda_N}) = \omega_1(e_{\lambda_1}) \dots \omega_1(e_{\lambda_N}) = p_a(\lambda_1) \dots p_a(\lambda_N), \quad (\text{A.23})$$

so that the joint probability of the outcome $(\lambda_1, \dots, \lambda_N) \in \sigma(\underline{a})$ is simply

$$p_{\underline{a}}(\lambda_1, \dots, \lambda_N) = p_a(\lambda_1) \dots p_a(\lambda_N). \quad (\text{A.24})$$

Since these are precisely the probabilities for option 2 (i.e. the Bernoulli process), i.e.,

$$\mu_{\underline{a}} = \mu_a^N, \quad (\text{A.25})$$

this proves the claim for $N < \infty$.

⁵⁹Moreover, quite apart from the above derivation, the Born rule for multiple commuting operators is in fact a special case of the one for single operators (Landsman, 2017, §2.5).

To describe the limit $N \rightarrow \infty$, let B be any C*-algebra with unit 1_B ; below I take $B = B(H)$, $B = C^*(a)$, or $B = C(\sigma(a))$. We now take

$$A_N = B^{\otimes N}, \quad (\text{A.26})$$

the N -fold tensor product of B with itself.⁶⁰ The special cases above may be rewritten as

$$B(H)^{\otimes N} \cong B(H \otimes \cdots \otimes H); \quad (\text{A.27})$$

$$C^*(a)^{\otimes N} \cong C^*(a_1, \dots, a_N); \quad (\text{A.28})$$

$$C(\sigma(a))^{\otimes N} \cong C(\sigma(a) \times \cdots \times \sigma(a)), \quad (\text{A.29})$$

with N copies of H and $\sigma(a)$, respectively, and in (A.28) the a_i are given by (A.13) - (A.14).

We may then wonder if these algebras have a limit as $N \rightarrow \infty$. They do, but it is not unique and depends on the choice of observables, that is, of the infinite sequences $\mathbf{a} = (a_1, a_2, \dots)$, with $a_N \in A_N$, that are supposed to have a limit. One possibility is to take sequences \mathbf{a} for which there exists $M \in \mathbb{N}$ and $a_M \in A_M$ such that for each $N \geq M$,

$$a_N = a_M \otimes 1_B \cdots \otimes 1_B, \quad (\text{A.30})$$

with $N - M$ copies of the unit 1_B . On that choice, one obtains the infinite tensor product $B^{\otimes \infty}$, see Landsman (2017), §C.14. The limit of (A.27) in this sense is $B(H^{\otimes \infty})$, where $H^{\otimes \infty}$ is von Neumann's 'complete' infinite tensor product of Hilbert spaces,⁶¹ in which $C^*(a)^{\otimes \infty}$ is the C*-algebra generated by the operators (a_1, a_2, \dots) . The limit of (A.29) is

$$C(\sigma(a))^{\otimes \infty} \cong C(\sigma(a)^{\mathbb{N}}), \quad (\text{A.31})$$

where $\sigma(a)^{\mathbb{N}}$, which we previously saw as a measure space (as a special case of $X^{\mathbb{N}}$ for general compact Hausdorff spaces X), is now seen as a topological space with the product topology, in which it is compact.⁶² As in the finite case, we have an isomorphism

$$C^*(a)^{\otimes \infty} \cong C(\sigma(a))^{\otimes \infty}, \quad (\text{A.32})$$

and hence, on the given identifications,

$$C^*(a_1, a_2, \dots) \cong C(\sigma(a)^{\mathbb{N}}). \quad (\text{A.33})$$

It follows from the definition of the infinite tensor products used here that each state ω_1 on B defines a state ω_1^∞ on $B^{\otimes \infty}$. Take $B = B(H)$ and restrict ω_1^∞ , which *a priori* is a state on $B(H^{\otimes \infty})$, to its commutative C*-subalgebra $C^*(a_1, a_2, \dots)$. The isomorphism (A.33) then gives a probability measure $\mu_{\underline{a}}$ on the compact space $\sigma(a)^{\mathbb{N}}$, where the label \underline{a} now refers to the infinite set of commuting operators (a_1, a_2, \dots) on $H^{\otimes \infty}$. To compute this measure, I use (A.12) and the fact that by construction functions of the type

$$f(\lambda_1, \lambda_2, \dots) = f^{(N)}(\lambda_1, \dots, \lambda_N), \quad (\text{A.34})$$

where $N < \infty$ and $f^{(N)} \in C(\sigma(a)^{\mathbb{N}})$, are dense in $C(\sigma(a)^{\mathbb{N}})$ (with respect to the appropriate supremum-norm), and that in turn finite linear combinations of factorized functions $f^{(N)}(\lambda_1, \dots, \lambda_N) = f_1(\lambda_1) \cdots f_N(\lambda_N)$ are dense in $C(\sigma(a)^{\mathbb{N}})$. It follows from this that

$$\mu_{\underline{a}} = \mu_{\underline{a}}^\infty. \quad (\text{A.35})$$

Since this generalizes (A.25) to $N = \infty$, this finishes the proof of Theorem 5.1. \square

⁶⁰If B is infinite-dimensional, for technical reasons the so-called *projective* tensor product should be used.

⁶¹See Landsman (2017), §8.4 for this approach. The details are unnecessary here.

⁶²Cf. Tychonoff's Theorem. The associated Borel structure is the one defined in footnote 33 in §5.

B 1-Randomness

As we have seen, randomness (of measurement outcomes) in quantum mechanics was originally defined by Born as *indeterminism*. This is what hidden variable theories like 't Hooft's and Bohmian mechanics challenge, to which in turn Bell's Theorem and the Free Will Theorem (read in a specific way à la Cator & Landsman, 2014) provide obstacles. Indeterminism is a physical definition of randomness, which in mathematics is most closely matched by *incomputability*. However, there is a much deeper notion of randomness in mathematics, which is what at least in my view quantum mechanics should aspire to produce. This notion, now called *1-randomness*, has its roots in Hilbert's sixth problem, viz. the *Mathematical Treatment of the Axioms of Physics*,⁶³ which was taken up independently and very differently by von Mises (1919) and by Kolmogorov (1933).

Von Mises was a strict frequentist for whom probability was a derived concept, predicated on first having a good notion of a random sequence from which relative frequencies *defining* probability could be extracted. Kolmogorov, on the other hand, started from an axiomatic *a priori* notion of probability from which a suitable mathematical concept of randomness was subsequently to be extracted. Despite the resounding and continuing success of the first step, Kolmogorov's initial failure to achieve the follow-up led to his later notion of algorithmic randomness, which (subject to a technical improvement) was to become one of the three equivalent definitions of 1-randomness. In turn, von Mises's failure to adequately define random sequences eventually led to the other two.⁶⁴

Kolmogorov's problem, which was noticed already by Laplace and perhaps even earlier probabilists, was that, specializing to a 50-50 Bernoulli process for simplicity,⁶⁵ any binary string σ of length N has probability $P(\sigma) = 2^{-N}$ and any (infinite) binary sequence x has probability $P(x) = 0$, although say $\sigma = 0011010101110100$ looks much more random than $\sigma = 11111111111111$. In other words, their probabilities say little or nothing about the "randomness" of individual outcomes. Imposing statistical properties helps but is not enough to guarantee randomness. It is slightly easier to explain this in base 10, to which I therefore switch for a moment. If we call a sequence x *Borel normal* if each possible string σ in x has (asymptotic) frequency $10^{-|\sigma|}$ (so that each digit $0, \dots, 9$ occurs 10% of the time, each block 00 to 99 occurs 1% of the time, etc., then *Champernowne's number*

0123456789101112131415161718192021222324252629282930...

can be shown to be Borel normal. The decimal expansion of π is conjectured to be Borel normal, too (and has been empirically verified to be so in billions of decimals), but these numbers are hardly random: they are computable, which is an anti-pole to randomness.

Von Mises's problem was that his definition of a random sequence (called a *Kollektiv*), despite his great insights, simply did not work. Back to binary sequences, his notion of randomness was supposed to guarantee the existence of limiting relative frequencies (which in turn should define the corresponding single-case probabilities), but of course he understood that a sequence like 01010101... is not very random at all. His idea was that limiting relative frequencies should exist not only for the given sequence, but also for all its permutations defined by so-called *place-selection functions*, which he tried to find (in vain) by precluding successful gambling strategies on the digits of the sequence.

⁶³The investigations on the foundations of geometry suggest the problem to treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.' See Gorban (2018) for a recent survey.

⁶⁴See van Lambalgen (1987) or, for a lighter account, Diaconis & Skyrms (2018) for this history.

⁶⁵A *string* σ is finite row of bits, whereas a *sequence* x is an infinite one. The length of a string σ is $|\sigma|$.

Of the three equivalent definitions of 1-randomness already mentioned in §2, namely:

Incompressibility; *Patternlessness;* *Unpredictability,*

the first may be said to go back to Kolmogorov’s struggle, whereas the other two originate in later attempts by other mathematicians to improve the work of von Mises.⁶⁶

Although there isn’t a single “correct” mathematical notion of randomness, the notion of 1-randomness featured here stands out (and represents a consensus view) largely because it can be defined in these three equivalent yet fairly different ways, each of which realizes some basic intuition on randomness (of course defined through its obvious anti-pole!).

In what follows, these notions will be defined more precisely, followed by some of their consequences.⁶⁷ We assume basic familiarity with the notion of a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$, which may technically be defined through recursion theory or equivalently through Turing machines: *a function is computable if it can be computed by a computer.*

Incompressibility

The idea is that a string or sequence is random iff its shortest description is the sequence itself, but the notion of a description has to be made precise to avoid *Berry’s paradox*:

The Berry number is the smallest positive integer that cannot be described in less than eighteen words.

The paradox, then, is that on the one hand this number must exist, since only finitely many integers can be described in less than eighteen words and hence the set of such numbers must have a lower bound, while on the other hand Berry’s number cannot exist by its own definition. This is, of course, one of innumerable paradoxes of natural language, which, like the liar’s paradox, will be seen to lead to an incompleteness theorem once the notion of a description has been appropriately formalized in mathematics, as follows.⁶⁸

The *plain Kolmogorov complexity* $C(\sigma)$ of $\sigma \in \underline{2}^{\mathbb{N}}$ is defined as the length (in bits) of the shortest computer program (run on some fixed universal Turing machine U) that computes σ . The choice of U affects this definition only up to an σ -independent constant. For technical reasons (especially for defining the randomness of sequences) it is preferable to work with *prefix-free Turing machines* T , whose domain $D(T)$ consists of a prefix-free subset of $\underline{2}^*$, i.e., if $\sigma \in D(T)$ then $\sigma\tau \notin D(T)$ for any $\sigma, \tau \in \underline{2}^*$, where $\sigma\tau$ is the concatenation of σ and τ , as the notation suggests. This is also independent of U up to a constant, and defines the *prefix-free Kolmogorov complexity* $K(\sigma)$ as the length of the shortest program (run on some fixed universal prefix-free Turing machine U) that computes σ . For fixed $c \in \mathbb{N}$ we then say that $\sigma \in \underline{2}^*$ is *c-compressible* if $K(\sigma) < |\sigma| - c$; of course, this depends on U via $K(\sigma)$, but nonetheless a simple counting argument shows:

⁶⁶As mentioned and referenced in most literature in the next footnote, Kolmogorov’s work in the 1960s on the first definition was predated by R. Solomonoff and matched by independent later work of G. Chaitin. Key players around the other definitions were C.P. Schnorr and P. Martin-Löf, respectively.

⁶⁷In increasing order of technicality, readers interested in more detail are referred to Diaconis & Skyrms (2018, Chapter 8), Terwijn (2016) or Volchan (2002) at a popular level, then Grünwald & Vitányi (2008), Dasgupta (2011), Downey *et al* (2006), or Muchnik *et al* (1998), then Li & Vitányi (2008) or Calude (2010), and finally Downey & Hirschfeldt (2010). See also Baumeler *et al* (2017), Bendersky *et al* (2014), Calude (2004), Eagle (2019), Earman (2004), Kamminga (2019), Senno (2017), Svozil (1993, 2018), Wolf (2015), and Zurek (1989) for brief introductions to Kolmogorov randomness with applications to physics.

⁶⁸We write $\underline{2}^N$ for the set of all binary strings σ of length $|\sigma| = N \in \mathbb{N}$, and $\underline{2}^* = \cup_N \underline{2}^N$ for the set of all binary strings. Finally, $\underline{2}^{\mathbb{N}}$ denotes the set of all binary sequences x (which are infinite by convention).

At least $2^N - 2^{N-c+1} + 1$ strings σ of length $|\sigma| = N$ are *not* c -compressible.

Clearly, as σ grows and $c > 0$, these will form the overwhelming majority. Finally, σ is *Kolmogorov c -random* if it is not c -compressible, i.e., if $K(\sigma) \geq |\sigma| - c$, and *Kolmogorov random* if it is not c -compressible for any c , that is, if one even has $K(\sigma) \geq |\sigma|$. Since $K(\sigma)$ can at most be equal to the length of an efficient printing program plus the length of σ , this simply means that σ is Kolmogorov random if

$$K(\sigma) \approx |\sigma|, \tag{B.1}$$

where \approx means ‘up to a σ -independent constant’. Finally, if for a sequence $x \in \underline{2}^{\mathbb{N}}$ I write $x_{|N}$ for its initial segment of length N , then x is *Kolmogorov–Chaitin random* if

$$\lim_{N \rightarrow \infty} \frac{K(x_{|N})}{N} = 1. \tag{B.2}$$

Equivalently,⁶⁹ a sequence x is Kolmogorov–Chaitin random if there exists $c \in \mathbb{N}$ such that each truncation $x_{|N}$ satisfies $K(x_{|N}) \geq N - c$. We may therefore note that randomness of sequences, which I see as idealizations of finite strings, satisfies Earman’s principle (§4).

This definition of randomness of both finite strings and infinite sequences looks very appealing, but in a way it is self-defeating, since although it is defined in terms of computability, the complexity function K is not computable, and hence one cannot even determine algorithmically if a finite string σ is random (let alone an infinite one).⁷⁰

Moreover, Berry’s paradox strikes back through *Chaitin’s incompleteness Theorem*: For any consistent, sound, and sufficiently rich formal system F (containing basic arithmetic) there is a constant f such that the statement $K(\sigma) > f$ cannot be proved in F for any $\sigma \in \underline{2}^*$, although it is true for all (random) strings σ that satisfy $K(\sigma) \geq |\sigma| > f$ (and there are infinitely many such strings, as the above counting argument shows). To get a very rough idea of proof, let me just say that any proof in F of the sentence $K(\sigma) > f$ would identify σ and hence give a shorter description of σ than its complexity $K(\sigma)$ allows.

Chaitin’s Theorem gives a new and inexhaustible class of unprovable but true statements (which also lie in a very different class from those provided by Gödel himself):

For almost all random strings their randomness cannot be proved.⁷¹

To the extent that there are deep logico-philosophical truths, surely this is one! Compare:

‘It may be taken for granted that any attempt at defining disorder in a formal way will lead to a contradiction. This does not mean that the notion of disorder is contradictory. It is so, however, as soon as I try to formalize it.’⁷²

The satisfactory definition of 1-randomness proves this wrong, but the statement preceding it is a correct version of a similar intuition: *randomness is by its very nature elusive*.

⁶⁹See Calude (2010), Theorem 6.38 (attributed to Chaitin) for this equivalence.

⁷⁰To see this, identify $\underline{2}^* \cong \mathbb{N}$ (in a computable way of course) and define a function $L : \mathbb{N} \rightarrow \mathbb{N}$ by $L(n) = \min\{m \in \mathbb{N} \mid K(m) \geq n\}$, the shortest string m (seen as an element of \mathbb{N}) whose complexity $K(m)$ exceeds n , so that by construction $K(L(n)) \geq n$. If K were computable, then so would L be. Suppose T is the shortest prefix-free program that computes L . Since $K(L(n))$ is the length of shortest prefix-free program that computes $L(n)$, we have $K(L(n)) \leq |n| + |T|$, where $|n|$ is the length of σ under the above bijection $\sigma \leftrightarrow n$, so that $|n| \approx_{10} \log n$. Thus $n \leq K(L(n)) \leq |n| + |T|$, which cannot be true for large n .

⁷¹See Raatikainen (1998) for a detailed presentation of Chaitin’s incompleteness Theorem including a devastating critique of the far-reaching philosophical interpretation Chaitin himself gave of his theorem.

⁷²Statement by the influential German-Dutch mathematician and educator Hans Freudenthal from 1969, quoted in both van Lambalgen (1987), p. 8 and Terwijn (2016), p. 51.

Patternlessness

This is the most direct attack on the “paradox of probability”, which states that each individual sequence has probability zero despite the huge differences in their (true or apparent) “randomness”. We use the notation of §4, so that μ^∞ is the probability measure on the set $\underline{2}^{\mathbb{N}}$ of all binary sequences induced by the 50-50 measure on the outcome space $\{0, 1\}$ of a single fair coin flip (the discussion is easily adapted to other cases).

As opposed to single outcomes, a key role will be played by *tests*, i.e. (measurable) *sets* $T \subset \underline{2}^{\mathbb{N}}$ of outcomes for which $\mu^\infty(T) = 0$. Such a test defines a *property* that a sequence $x \in \underline{2}^{\mathbb{N}}$ may or may not have, namely membership of T . Conversely, and perhaps less circularly, one may start with a given pattern that x might have, which would make it appear less random if it had it, and express this pattern as a test T . Conceptually, tests contain outcomes that are supposed to be “atypical”, and randomness of $x \in \underline{2}^{\mathbb{N}}$ will be defined by the property of *not* belonging to any such test, making x “typical”. For example, the property $\lim_{N \rightarrow \infty} N^{-1} \sum_{n=1}^N x_n = 1/2$ is “typical” for a fair coin flip and indeed (by the strong law of large numbers) it holds with probability 1 (in that the set E of all x for which this limit exists and equals $1/2$ has $\mu^\infty(E) = 1$, so that its complement $T = \underline{2}^{\mathbb{N}} \setminus E$ has $\mu^\infty(T) = 0$). One has to proceed very carefully, though, since all singletons $T = \{x\}$ for $x \in \underline{2}^{\mathbb{N}}$ are to be excluded; indeed, these *all* have measure zero and yet, returning to the paradox of probability, some uncontroversially random sequence x (on whatever definition) would fail to be random by the criterion just proposed if all measure zero sets were included in the list of tests. Kolmogorov’s former postdoc Martin-Löf (1966) saw his way out of this dilemma, combining techniques from computability/recursion theory with some ideas from the intuitionistic/constructive mathematics of the great L.E.J. Brouwer:

1. One specializes to tests of the form $T = \bigcap_{n \in \mathbb{N}} U_n$, where $U_{n+1} \subseteq U_n$ and

$$\mu^\infty(U_n) \leq 2^{-n}, \tag{B.3}$$

which guarantees that $\mu^\infty(T) = 0$. Perhaps the simplest example is not the law of large numbers, for which see Li & Vitányi (2008), §2.4, Example 2.4.2, but the test where U_n consists of all sequences starting with n zeros; clearly, $\mu^\infty(U_n) = 2^{-n}$. In this case, the test $T = \{x_{\text{love}}\}$ *does* consist of a singleton $x_{\text{love}} = 000 \dots$ (zeros only).

2. Both the sets U_n and the map $n \mapsto U_n$ have to be computable in a suitable sense.⁷³
3. A sequence $x \in \underline{2}^{\mathbb{N}}$ is called *Martin-Löf random* if $x \notin T$ for any such test T .

If $x \notin T$, one says that x *passes* the test T . Note that the computability requirement implies that the set of all tests T satisfying these two criteria is countable, which fact by itself already shows what a huge cut in the set of all measure-zero sets has been achieved.⁷⁴

It should be clear that this definition makes no sense if the sample space is finite, but in order to adhere to Earman’s principle one could still check to what extent randomness of sequences is determined by their finite initial segments. This is indeed the case, as follows from the detailed structure of the admissible sets U_n above (see footnote 73).

⁷³First, each U_n has to be open in $\underline{2}^{\mathbb{N}}$, which means that $U_n = \bigcup_{\sigma \in V_n} N(\sigma)$, where $V_n \subset \underline{2}^*$ and $N(\sigma) = \{\sigma y \mid y \in \underline{2}^{\mathbb{N}}\}$ consists of all sequences $x = \sigma y$ that start with the given finite part σ . Second, each V_n must be countable, so that $V_n = \bigcup_m \sigma_{(n,m)}$, where each $\sigma_{(n,m)} \in \underline{2}^*$. Finally, there must be a single program enumerating the sets (V_n) , so that all in all one requires a (partial) computable function $\sigma : \mathbb{N} \times \mathbb{N} \rightarrow \underline{2}^*$ such that $U_n = \bigcup_m N(\sigma_{(n,m)})$ and hence $T = \bigcap_n \bigcup_m N(\sigma_{(n,m)})$. Such a set T is called *effective*, and if also (B.3) holds, then T is an *effective measure zero set* or *Martin-Löf test*.

⁷⁴There is even a *single* “universal” Martin-Löf test U such that $x \notin U$ iff x is Martin-Löf random.

Unpredictability

Unpredictability is formalized as the impossibility of a successful betting strategy on the successive digits of a random sequence $x = (x_1, x_2, \dots)$. Suppose a punter with initial capital $d(0)$ starts by betting $b(0)$ on $x_1 = 0$ and $b(1)$ on $x_1 = 1$, where $b(0) + b(1) = d(0)$, with fair payoff $d(x_1) = 2b(0)$ if $x_1 = 0$ and $d(x_1) = 2b(1)$ if $x_1 = 1$. Note that

$$d(0) + d(1) = 2(b(0) + b(1)) = 2d(0). \quad (\text{B.4})$$

This motivates the concept of a *martingale* as a function $d : \underline{2}^* \rightarrow [0, \infty)$ that satisfies

$$d(\sigma 0) + d(\sigma 1) = 2d(\sigma), \quad (\text{B.5})$$

for each $\sigma \in \underline{2}^*$. Both the betting strategy itself and the payoff (for given $x \in \underline{2}^{\mathbb{N}}$) can be reconstructed from d : the bet on the $N + 1$ 'th digit x_{N+1} is given by

$$b(x_{N+1} = 0) = \frac{1}{2}d(x_N 0); \quad (\text{B.6})$$

$$b(x_{N+1} = 1) = \frac{1}{2}d(x_N 1), \quad (\text{B.7})$$

and after N bets the punter owns $d(x|_N)$. A martingale d *succeeds* on $A \subset \underline{2}^{\mathbb{N}}$ if

$$\limsup_{N \rightarrow \infty} d(x|_N) = \infty \quad \text{for each } x \in A, \quad (\text{B.8})$$

in which case the punter beats the casino.⁷⁵ Our first impulse would now be to call $x \in \underline{2}^{\mathbb{N}}$ random if there exists no martingale that succeeds on $A = \{x\}$, but, as shown by Ville (1939), this property captures something seen before, crucially different from randomness:

Let $A \subset \underline{2}^{\mathbb{N}}$. Then $\mu^\infty(A) = 0$ if and only if there exists a martingale that succeeds on A .

Of course, A should be measurable. In particular, for any sequence $x \in \underline{2}^{\mathbb{N}}$ there exists a martingale that succeeds on x , and hence no sequence x would be random on the criterion that no martingale succeeds on it. Fortunately, the previous two definitions of randomness suggest that all that is missing is a suitable notion of *computability* for martingales. This notion was provided by Schnorr (1971): all that needs to be added is that the class of martingales d that succeed on A be *uniformly left computably enumerable*, in the sense that firstly each real number $d(\sigma)$ is the limit of a computable increasing sequence of rational numbers, and secondly that there is a single program that computes $d(\sigma)$ in that way.

Defining $x \in \underline{2}^{\mathbb{N}}$ to be *Schnorr random* if there exists no uniformly left computably enumerable martingale that succeeds on it,⁷⁶ one has the crowning theorem on 1-randomness:

Theorem B.1 *A sequence $x \in \underline{2}^{\mathbb{N}}$ is Kolmogorov–Chaitin random iff it is Martin–Löf random and iff it is Schnorr random (and these equivalent conditions define 1-randomness).*

Thus the criteria of incompressibility (Kolmogorov–Chaitin), patternlessness (Martin–Löf), and unpredictability (Schnorr) for randomness coincide, which is truly remarkable.⁷⁷

⁷⁵In practice someone beating the casino will go home with a finite amount of money. The fact that the right-hand side of (B.8) is infinite is the result of idealizing long strings by sequences: if the punter has a uniform winning strategy and places infinitely many bets, he will earn an infinite amount of money.

⁷⁶In the literature the term ‘Schnorr randomness’ is often used differently, namely to indicate that the martingales d in the above definition are merely computable, which yields a weaker notion of randomness. Also, note that Schnorr (1971) used so-called supermartingales in his definition, for which one has \leq instead of equality in (B.5), but martingales also work, cf. Downey & Hirschfeldt (2010), Theorem 6.3.4.

⁷⁷The equivalence between the criteria of Martin–Löf and of Kolmogorov–Chaitin was proved by Chaitin (cf. Calude, 2010, Theorem 6.35) and by Schnorr (1973). The equivalence between Martin–Löf and Schnorr is due to Schnorr (1971), Staz 5.3. See also Downey & Hirschfeldt (2010), Theorems 6.2.3 and 6.3.4.

Some reflections on 1-randomness

Any sound definition of randomness (for binary sequences) has to navigate between Scylla and Charybdis: if the definition is too weak (such as Borel normality), counterexamples will undermine it (such as Champernowne’s number), but if it is too strong, it will not hold almost surely in a 50-50 Bernoulli process.⁷⁸ In this respect 1-randomness does very well: see Theorem 4.1 and the theorem below (which lies at the basis of Corollary 4.2):

Theorem B.2 *Any 1-random sequence is Borel normal, incomputable, and contains any finite string infinitely often.*⁷⁹

In fact, since I sometimes use computability as a mathematical metaphor for determinism, it is worth mentioning that 1-random sequences are incomputable in a very strong sense: each 1-random sequence x is *immune* in the sense that neither the set $\{n \in \mathbb{N} \mid x_n = 1\}$ nor its complement $\{n \in \mathbb{N} \mid x_n = 0\}$ is computably enumerable (c.e.), or can even contain an infinite c.e. subset. Thus 1-randomness is far stronger than mere incomputability (or perhaps indeterminism), so that the numerous theorems in the literature that show, on the basis of either Bell’s Theorem or the Kochen–Specker Theorem, that measurement outcome sequences cannot be computable fall far short of the real goal of proving that they are 1-random; see Kamminga (2019) and references therein for a survey.

Furthermore, 1-randomness is a *tail property*: adding any finite segment to x , or deleting any finite part from it, or altering any finite number of digits in x , does not affect its property of being 1-random (Calude, 2010, Theorem 6.40).⁸⁰

Finally, it would be interesting to compare 1-random sequences with Brouwer’s *choice sequences* from intuitionistic mathematics, especially (in the light of our anti-polar concept of randomness of §2) the “lawless” ones.⁸¹ Von Mises (1919) mentioned choice sequences as an inspiration for his idea of a *Kollektiv* (van Lambalgen, 1996), but as far as I know little has been done with this idea in the modern setting. See also Gisin (2018).

⁷⁸This example is not essential for the theory, see Downey & Hirschfeldt (2010), §6.12. Even in the context of a 50-50 Bernoulli process, in the absence of theorems to that effect it is not easy to give examples of notions of randomness that are so strong that they fail to hold almost surely, but n -randomness may be a candidate. This notion is obtained by strengthening the requirements on the function σ mentioned in footnote 73; see Downey & Hirschfeldt (2010), §6.8, and n -randomness for $n > 1$ implies 1-randomness. But perhaps the following example works: some people might say that Chaitin’s Ω and its derivatives are the only genuinely random numbers (see e.g. Calude, 2010, §7.1); of course these have measure zero.

⁷⁹See the footnotes to Corollary 4.2 for the meaning of these terms, and Calude (2010), §6.4 for proofs.

⁸⁰Consistent with Earman’s principle, it follows from the Kolmogorov–Chaitin definition that even this tail property is ultimately determined by initial segments, just by larger ones than those removed.

⁸¹See Troelstra (1977) and references therein to the original literature. For a quick summary, quoted from Troelstra (1996): ‘A choice sequence in Brouwer’s sense is a sequence of natural numbers (to keep it simple), which is not a priori given by a law or recipe for the elements, but which is created step by step (...); the process will go on indefinitely, but it is never finished.’ He then goes on to discuss what he calls ‘a rather extreme variant of choice sequence, the so-called *lawless sequences*’: ‘Informally, we think of a lawless sequence of natural numbers as a process of choosing values in \mathbb{N} (...) under the a priori restriction that at any stage of the construction never more than an initial segment has been determined and that *no* restrictions have been imposed on future choices, [and that] there is a commitment to determine more and more values (so the sequence is infinite). (...) A lawless sequence may be compared to the sequence of casts of a die. There too, at any given stage in the generation of a sequence never more than an initial segment is known.’ See Hansen (2018) for a recent analysis emphasising the idea of lawlessness, which, in so far as the comparison with random sequences is appropriate, seems to lie on the side of *process randomness*, cf. footnote 25. On the other hand, 1-random sequences concern *product randomness*, and are not lawless at all (as shown for example by Theorem B.2 above). To further mark the difference, however lawless its *creation process*, no *finished* sequence can be totally lawless. For example, *Baudet’s Conjecture*, proved by van der Waerden (1921), implies that each binary sequence x satisfies certain arithmetic laws.

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