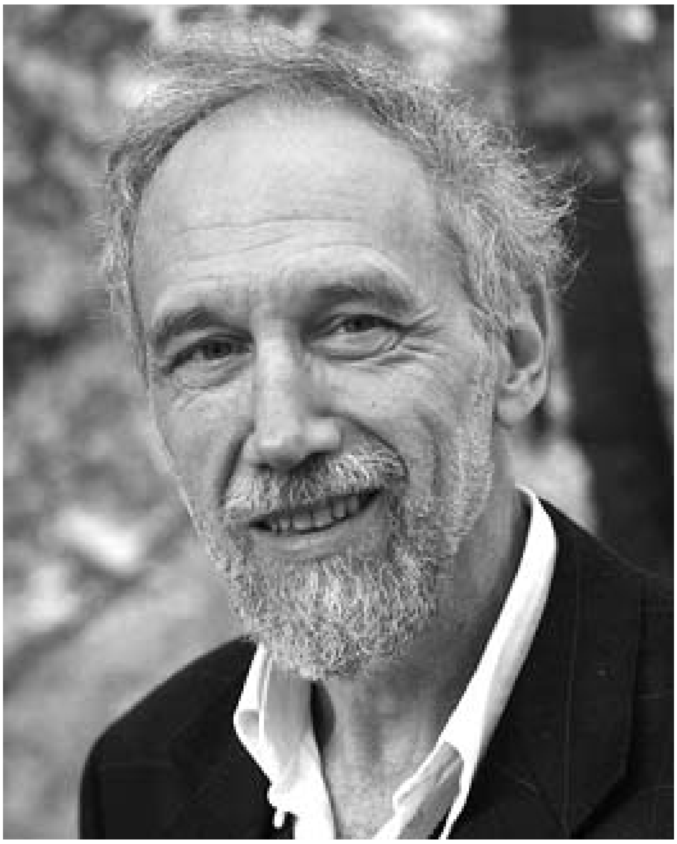
Commutative Spectral Triples

&

The Spectral Reconstruction Theorem



A Master Thesis by Richard Sanders

Department of Mathematical Physics
IMAPP
Faculty of Science
Radboud University Nijmegen

Abstract

Given a unital and commutative algebra $\mathcal{A}$ associated to a spectral triple, we show how a differentiable structure is constructed on the spectrum of such an algebra whenever the spectral triple satisfies eight so-called "axioms", in such a way that $\mathcal{A} \cong C^\infty(M)$. This construction is the celebrated "reconstruction theorem" of Alain Connes [14], [21]. We discuss two spin manifolds, the circle and the 4-sphere, and show how several key properties of these manifolds relate to mathematical concepts and constructions used in the reconstruction theorem. In addition, we review the theory of Fredholm modules, cyclic homology, and noncommutative integrals, which are used as tools in the reconstruction theorem.

Contents

Voorwoord (in Dutch)

Ik ben wel vaker optimistisch over hetgeen waar ik mee bezig ben. Zo ook over het bedenken, onderzoeken en schrijven van deze scriptie. Het heeft wat langer geduurd dan de mensen om mij heen graag gewild zouden hebben. Dit lijkt mij dan ook een geschikte plek om hen te bedanken voor hun geduld.
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Chapter 1

Introduction

This thesis revolves around the ideas and mathematical techniques forming collectively the theory of Noncommutative Geometry (NCG). This is a very broad subject with applications ranging from high energy physics to number theory [15]. This thesis focuses solely on the application of NCG to **Riemannian spin geometry**, i.e., the theory of differentiable manifolds equipped with a positive definite metric and a spin structure. This is a quite narrow focus. Therefore, in this introduction we seize the opportunity to discuss NCG and its applications in a broader context.
Last, we present the aim and a brief outline of this thesis.

The ancestor of NCG is the **Gelfand correspondence**, which states that the category of (locally) compact Hausdorff spaces is dual to the category of (unital) commutative C^*-algebras. See theorem 2.4, section 2.1, for the mathematical background. The main point of the Gelfand correspondence is that one can learn everything about a (locally) compact Hausdorff space by studying its associated commutative C^*-algebra, and vice versa.

From this point on, one can proceed in different ways. The first one, which forms the subject of this thesis, is to ask oneself which algebraic requirements one must add to a commutative C^*-algebra in order to define an object that is dual to more complex topological spaces, such as manifolds. A second direction is to depart from commutativity and study noncommutative C^*-algebras (and related objects) *as if they were* algebras of continuous functions over some "topological space".

Seen from the perspective of physics, both of these directions fit in a more universal framework, in which they might be usefully combined. Recall that the theory of General Relativity is formulated on non-compact, four-dimensional, Lorentzian manifolds whose metric satisfies the Einstein equations. Accordingly, there is an equivalence between forces (resulting

from gravitation) and the way space-time bends and curves. In other words, the theory is purely geometrical. On the other side of the spectrum, one has quantum theory, which is formulated in terms of (unbounded) operators acting on a Hilbert space representing the possible (pure) states of a system. Any C^*-algebra can be represented as a subalgebra of the algebra of bounded operators on some Hilbert space, showing that one of the basic building blocks of NCG has a natural place within quantum theory.
The set-up of the theory of NCG and the pertinent mathematical objects playing a central role in the theory suggest NCG might connect General Relativity and quantum theory in some overarching way.
We will get back to this idea at the end of the section: let us now first discuss the two approaches in some detail.

1.1 Dualization of manifolds

Any non-compact, Lorentzian manifold is, in particular, a locally compact Hausdorff space. To make these types of manifolds fit within the theory of NCG, one basically adds several key requirements and/or other mathematical objects to a commutative C^*-algebra and *presto*, one obtains the equivalent of the algebra of smooth functions over some manifold. This strategy might appear too straightforward to be of any use, but, in fact, it is precisely the one we will follow in chapters 3 to 5. However, we shall not find any dual description of non-compact Lorentzian manifolds, since dualization of these type of manifolds is currently beyond our grasp. There is much background information on what the algebraic equivalent of a non-compact, Lorentzian manifold might look like and what properties it should exhibit [41], [43], [51]. It is therefore not unimaginable that in the near future the process of dualizing Riemannian, compact manifolds as described in this thesis, can be extended to non-compact Lorentzian ones.
The dualization theorem in its current form [14] requires one additional property from a Riemannian, compact manifold: namely, that the manifold is **spin**. We shall now consecutively discuss the physical implications of the three "constraints" in question.

1.1.1 Riemannian metric

In short, the assumption that space-time is modeled by a Riemannian manifold violates experimental data, which shows that no object can move faster than the speed of light. However, especially in quantum field theory (both perturbative and constructive) the trick of "Wick rotation" effectively transforms a Lorentzian into a Euclidean (and hence Riemannian) space-time. Taking the latter as a starting point is easier and under appropriate assumptions one may eventually move back to the Lorentzian case. This procedure is less well understood in curved space-times, but it may well be possible that also in general Riemannian manifolds it may be used as a basis for relativistic (and hence Lorentzian) theories (cf. the work of Hawking on Euclidean path integrals for quantum gravity).

1.1.2 Compactness

There is a problem with assuming that a compact, four-dimensional manifold might be a good model for space-time. Though one could argue that there exists such a thing as the “beginning” and ”end” of time, compactness of Lorentzian manifold has a rather strong implication which appears un-physical. From [47]:

Theorem 1.1. *Using the Lorentzian metric, one can designate a specific time-like direction. When the Lorentzian manifold is compact, chronology is violated. I.e., there exists at least one closed, time-like curve.*

In more popular terms: on any compact, Lorentzian manifold, one can go backward and forward in time. Clearly, this does not correspond to any experimental result obtained so far.

1.1.3 Spin property

We shall have to add the requirement that the manifold is spin in order to dualize it. From a physical point of view, this is not such an undesirable extra requirement. It is a precondition for the spin-statistics theorem, which, among others, explains the stability of matter (through the Pauli exclusion principle) as well as Bose-Einstein condensation. Moreover, on a spin manifold one can globally define the Dirac equation, whose solutions include the positron. Lastly, the existence of particles with half-integer spin is an important building block of the Standard Model of high energy physics. In conclusion, in contrast to the previous assumption of Riemannianness and compactness, we shall not regard the demand that a manifold should be spin as an unwanted constraint which we somehow should get rid of at a later stage.

1.2 Noncommutative torus

We now cover an example of the second direction in NCG: we take the C^*-algebra corresponding to a known topological space, modify it to create a noncommutative C^*-algebra, and study its properties as if it were an abstract generalization of a topological space.
Recall that the ordinary torus, $\mathbb{T}^2$, is a compact Hausdorff space. Through the Gelfand correspondence (see chapter 2 for details), the torus is equivalently described by the set $C\left(\mathbb{T}^2\right)$ of continuous functions mapping from

the torus to the complex numbers. We shall describe $C(\mathbb{T}^2)$ in more detail. Define the two generators u and v:

$$u, v : \mathbb{T}^2 \to \mathbb{C}, \qquad u(x, y) = \exp(2\pi i x), \qquad v(x, y) = \exp(2\pi i y), \tag{1.1}$$

The generators satisfy the following relations

$$uv = vu, \tag{1.2}$$

$$u^*u = uu^* = v^*v = vv^* = 1. \tag{1.3}$$

The C^*-algebra $C(\mathbb{T}^2)$ can be described as the closure of the set of all finite linear combinations and all finite products of $1_{\mathbb{T}^2}, u, u^*, v$ and v^*. The closure is taken with respect to the supremum norm.
Equivalently, we can define $C(\mathbb{T}^2)$ as the C^*-algebra isomorphic to the universal C^*-algebra with two unitary generators u, v such that $uv = vu$ [21, 12.2].
The *noncommutative* torus is derived from the algebraic description of the ordinary torus. Take some irrational $\theta \in \mathbb{R}/\mathbb{Q}$. Let us define four elements U, U^*, V, V^* and an identity element 1 satisfying the following relations:

$$UV = \exp^{2\pi i\theta} VU, \tag{1.4}$$

$$U^*U = UU^* = V^*V = VV^* = 1. \tag{1.5}$$

We define the noncommutative torus $\overline{A}_\theta$ as the universal C^*-algebra generated by U and V. For a more explicit description of $\overline{A}_\theta$ (and a definition of a norm on A_θ such that the noncommutative torus is the closure of A_θ in that norm), see [46].
One can then proceed to study "classical" differential-geometric topics in the context of the noncommutative torus. Some examples are finitely generated projective modules over the noncommutative torus (i.e., the algebraic dual to complex vector bundles) [45], K-theory of the noncommutative torus (which classifies the abstract analogue to complex vector bundles) [33], and the theory of connections and the Yang-Mills equation [11].

1.3 NCG as "geometry of quantum mechanics"

As hinted at before, noncommutative geometry may tie together geometry and quantum mechanics. We will look at two examples in which NCG offers an interpretation of the underlying (noncommutative) geometry of some quantum system.
The first example, taken from [27, Ch. 1], directly relates to the noncommutative torus. Recall that the "classical" Hall effect is observed by applying a magnetic field perpendicular to a thin metal strip, through which electricity flows. Dependent on the direction of the flow of the electrons, a voltage

difference over the strip can be measured. The relationship is expressed as follows:

$$Ne\vec{E} + \vec{j} \wedge \vec{B} = 0, \tag{1.6}$$

where N is the number of charge carriers in the metal, e is their charge, $\vec{E}$ is the electric field, $\vec{j}$ is the current, and $\vec{B}$ is the magnetic field. The **Hall conductance**,

$$\sigma_H = \frac{Ne\delta}{|\vec{B}|}, \tag{1.7}$$

with δ the width of the strip, takes integer values at temperatures below 1K. This effect is known as the **integer quantum Hall effect** and is purely quantum mechanical. The effect can be fully described [24] by assuming that the Brillouin zone of the crystal structure of the metal can be modeled by a noncommutative torus. In particular, let e_1 and e_2 be generators of the 2-dimensional periodic lattice structure. As such, they are members of a group of translations. Assign to each of the generators a unitary transformation U, which acts on the electron wave functions by translating the wave function in a corresponding direction over the lattice. It turns out that the unitary transformations satisfy

$$U(e_1)U(e_2) = \exp(2\pi i\theta)U(e_2)U(e_1), \tag{1.8}$$

where θ is the magnetic flux through a fundamental domain of the lattice.

Our second and last example is the famous noncommutative description of the Standard Model of high energy physics [15, Ch. 1], [9]. The underlying space is assumed to be almost-noncommutative, i.e., the tensor product of the collection of smooth functions over ordinary compact, Riemannian spin manifold with a finite dimensional C^*-algebra [7]. At the moment, there are many advanced extensions of this model, incorporating Yang-Mills theory [3] and supersymmetry [54].

1.4 Outline of thesis and prerequisites

In chapter 2 we introduce commutative spectral triples by roughly outlining how one can associate a unital and commutative spectral triple to a compact spin manifold. We use the circle and the 4-sphere as leading examples. The rest of the thesis is devoted to showing in what manner a commutative spectral triple, satisfying eight so-called “axioms”, defines, in turn, a compact spin manifold. In chapter 3 we discuss the definition of a unital and commutative spectral triple, define the eight axioms and discuss these in the context of the spectral triple associated to the circle and the 4-sphere. In chapter 4 we show how to construct a compact manifold from a unital and commutative spectral triple. We finish in chapter 5 with proving that this manifold is a spin manifold.

It is recommended that the reader has an elementary understanding of the following subjects prior to reading this thesis:

- Differential geometry;
- Functional analysis;
- Algebraic topology.

Chapter 2

Introducing the spheres

The purpose of this chapter is to acquaint the reader with the **canonical spectral triple**. From now on, let us refer to a closed and compact Riemannian spin manifold simply as a **spin manifold**. It has been widely discussed [20], [21, Ch. II, III], [28], [56] that from any spin manifold one can construct a canonical spectral triple. The reader who is familiar with this construction can skip this chapter in its entirety, though it might be worthwhile to review the representation theory of the spin groups (theorems 2.25 and 2.38), Plymen's theorem on Spin^c-manifolds (theorem 2.46) and the globalization of the charge conjugation operator (theorem 2.48). The reasons to recall parts of the discussion on canonical spectral triples here are threefold.
First, we shall (mostly in chapter 3) meet various algebraic objects. When these objects are *commutative* (or, at least, are constructed using commutative algebras) they are equivalent to various (differential)geometric objects related to spin manifolds. That is, after all, the result of Connes' spin manifold theorem. It might therefore facilitate the discussion and improve the lucidity of the arguments to think of these algebraic objects as noncommutative generalizations of the notions developed in this chapter.
A second motivation stems from the fact that in the course of chapters 4 and 5 we will be faced with questions which potentially can be approached in many ways. The way we will actually proceed is by mimicking (in an algebraic context) what we would have done in the framework of a spin manifold. The canonical spectral triple will serve as a source of inspiration in that regard.
The last, auxiliary, purpose is to preliminarily introduce several concepts and results which will recur throughout the text.

We have chosen to introduce the canonical spectral triple in its most general form. Next to the general theory, we discuss two leading examples which we will encounter throughout the text. The first one is the 4-sphere.

The reason is that the 4-sphere is a sufficiently simple spin manifold which nonetheless is complicated enough for our purposes: it has a non-trivial tangent bundle and the four-dimensional spin groups have a rich structure. The second example is the circle. Its canonical spectral triples have quite straightforward descriptions. The simplicity of the circle allows us to make several constructions very explicit which would otherwise be too cumbersome to write down.

Our approach in describing the canonical spectral triple will be as follows. We start out with the primordial matter: the Hausdorff topological spaces and their algebraic equivalents, the commutative unital C^*-algebras. From that point on we will continue to add more properties and introduce more conditions, thereby constructing spin manifolds piece-by-piece.

2.1 Topology

Definition 2.1 (C^*-algebra)**.** *Let A be an algebra*[1] *over the complex numbers. We say that the algebra is a C^***-algebra** if it has the following properties:*

- *There is a map $^* : A \to A$ such that for all $\lambda, \mu \in \mathbb{C}$ and $a, b \in A$*
$$(a^*)^* = a, \qquad (a \cdot b)^* = b^* \cdot a^*, \qquad (\mu a + \lambda b)^* = \overline{\mu} a^* + \overline{\lambda} b^*,$$
where $\bar{\cdot}$ denotes complex conjugation.
- *There is a norm $\|\cdot\| : A \to \mathbb{R}^+$ satisfying*
$$\|a \cdot b\| \leq \|a\| \, \|b\| \,, \qquad \textbf{(submultiplicativity)}$$
$$\|a^* \cdot a\| = \|a\|^2 \,, \qquad (C^*\textbf{-property})$$
for all $a, b \in A$.
- *A is closed with respect to its norm (i.e., a Banach space).*

*We also call a^* the **adjoint** of a. Elements of a C^*-algebra that are equal to their adjoint are called **self-adjoint**. Elements that commute with their adjoint are dubbed **normal**. Another (normal) type of elements are the **unitary** ones. $u \in A$ is unitary if and only if*

$$u^* u = u u^* = 1 \tag{2.1}$$

We distinguish two important types of self-adjoint elements:

$$p = p^* = p^2; \qquad \textbf{(projection)} \tag{2.2}$$
$$F = F^*, \quad F^2 = 1. \qquad \textbf{(symmetry)} \tag{2.3}$$

[1] Throughout this thesis, all algebras are assumed to be associative.

*The **unital** C^*-algebras are C^*-algebras with a unit with respect to the algebra multiplication (this unit is unique).*
Morphisms of C^*-algebras are algebra morphisms that preserve involution, i.e., if $\varphi : A \to B$ is a morphism of two C^*-algebras, then in addition to $\mathbb{C}$-linearity:

$$\begin{aligned} &\varphi(a \cdot b) = \varphi(a)\varphi(b), \\ &\varphi(a^*) = \varphi(a)^* \qquad\qquad \forall a, b \in A. \end{aligned}$$

*These morphisms are called ***-morphisms**.*
Unital C^*-algebras and *-morphisms form a category. Commutative unital C^*-algebras and *-morphisms are a subcategory which we denote with $\boldsymbol{CCA}_1$.

In this thesis we focus on commutative and unital C^*-algebras. These arise naturally from compact topological spaces with the Hausdorff property. Let us outline the construction. Let X be a compact Hausdorff topological space and define

$$C(X) \equiv \{f : X \to \mathbb{C}; f \text{ is continuous}\}. \tag{2.4}$$

The multiplication in $C(X)$ is given by point-wise multiplication of functions, and the involution is defined by point-wise conjugation:

$$f^*(x) \equiv \overline{f(x)} \qquad \forall f \in C(X),\ \forall x \in X. \tag{2.5}$$

The algebra is unital with the indicator function of X, denoted by 1_X, as unit. The norm that makes $C(X)$ a C^*-algebra, dubbed the **supremum norm**, is defined as

$$\|f\|_\infty \equiv \sup_{x \in X} \{|f(x)|\}. \tag{2.6}$$

Together with continuous maps, compact Hausdorff spaces form a category denoted with **CH**. This leads to the following definition.

Definition 2.2 (The functor C)**.** *The functor $C : \boldsymbol{CH} \to \boldsymbol{CCA}_1$ maps each object X to the C^*-algebra $C(X)$ described above. Let $\varphi : X \to Y$ be a continuous map of compact Hausdorff spaces. Then $C(\varphi)$ defines a *-morphism $C(\varphi) : C(Y) \to C(X)$ by*

$$C(\varphi)(g) = g \circ \varphi. \tag{2.7}$$

*It is readily verified that C is a **contravariant** functor.*

The famous Gelfand theorem [25, Ch. 4] states that there are no other commutative and unital C^*-algebra than those constructed using a compact Hausdorff space. The theorem is formulated in terms of the so-called **spectrum** of a C^*-algebra. We want to define the spectrum on a wider class of algebras than just the C^*-algebras.

Definition 2.3 (Spectrum of an involutive algebra)**.** *Let* A *be an involutive algebra over the complex numbers. The* ***spectrum*** *of* A*, denoted by Spec* (A)*, is the collection of non-zero* **-morphisms from* A *to the complex numbers. An element of the spectrum of a* C^**-algebra is called a* ***character*** *(of* A*).*

Theorem 2.4 (Gelfand duality, compact version)**.** *For each object* $A \in \boldsymbol{CCA}_1$*, Spec* (A) *is a compact Hausdorff space in the so-called* ***Gelfand topology****, defined as the weakest topology making all functions* $\omega \mapsto \omega(a)$ *continuous, with* $\omega \in$ *Spec* A *and* $a \in A$*. Spec is extended to a contravariant functor from* $\boldsymbol{CCA}_1 \to \boldsymbol{CH}$ *in the following way. Let* $\varphi : A \to B$ *be a* **-morphism. Define*

$$\mathit{Spec}\ (\varphi) : \mathit{Spec}\ (B) \to \mathit{Spec}\ (A), \qquad \mathit{Spec}\ (\varphi)(\omega) = \omega \circ \varphi. \tag{2.8}$$

The map

$$\hat{\cdot} : A \to C(\mathit{Spec}\ A), \qquad \hat{a}(\omega) \equiv \omega(a) \qquad \textit{\textbf{(Gelfand transform)}} \tag{2.9}$$

is an isomorphism of C^**-algebras. The* ***evaluation map***

$$\varepsilon_\bullet : X \to \mathit{Spec}\ (C(X)), \qquad \varepsilon_x(f) = f(x) \tag{2.10}$$

is an isomorphism of compact Hausdorff topological spaces.
Denote with $1_{\boldsymbol{CCA}_1}, 1_{\boldsymbol{CH}}$ *the identity functor of the category of commutative and unital* C^**-algebras and the category of compact Hausdorff spaces, respectively. The functor* $1_{\boldsymbol{CCA}_1}$ *is naturally isomorphic to* $C \circ$ *Spec via the Gelfand transform and* $1_{\boldsymbol{CH}}$ *is naturally isomorphic to Spec* $\circ\, C$ *via the evaluation map.*
In other words, $\boldsymbol{CCA}_1$ *is (categorically)* ***dual*** *to* $\boldsymbol{CH}$*.*

Remark 2.5. *Throughout this thesis we shall often implicitly identify any element of a commutative* C^**-algebra with its Gelfand transform and any point of a compact Hausdorff space with its image under the evaluation map.*

Several topological properties of a compact Hausdorff space carry over to algebraic properties of the associated C^*-algebras. We list two of them here.

Lemma 2.6. *A compact space* X *is connected if and only if* $C(X)$ *contains no non-trivial projections.*

Proof. If X is not connected it has at least 2 connected components, say U and V. The indicator functions 1_U and 1_V are both projections in $C(X)$ differing from the unit of the algebra.
Conversely, assume $C(X)$ contains a projection p. Then p is a real function satisfying $p(x)^2 = p(x)$ for all $x \in X$, i.e., $p(x) \in \{0, 1\}$. Since p is continuous, $p^{-1}(\{0\}) \equiv U$ and $p^{-1}(\{1\}) \equiv V$ are both closed. But since $U \cap V = \emptyset$ and $U \cup V = X$, they are also both open. Moreover, $U \cap V = \emptyset$. If p is not equal to 1_X or p is not equal to the zero function both U and V are non-empty, showing X is not connected. □

Definition 2.7 (Separability)**.** *A C^*-algebra is said to be **separable** when it contains a countable subset that lies dense in the C^*-algebra.*

Quoting [21, Prop. 1.11] we have the following identification.

Lemma 2.8. *A commutative C^*-algebra A is separable if and only if Spec A is metrizable.*

Example 2.9 (S^1, S^4)**.** *For k equal to 1 or 4, we define*

$$S^k = \left\{ (x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1}; \sum_{i=1}^{k+1} x_i^2 = 1 \right\}. \tag{2.11}$$

Equipping the spheres with the topology induced by canonical imbedding $S^k \hookrightarrow \mathbb{R}^{k+1}$ shows that both spheres are compact and connected metrizable spaces with the Hausdorff property. Hence the algebras $C(S^1)$ and $C(S^4)$ are unital, commutative, and separable C^*-algebras without any non-trivial projections.

2.2 Differential structure

Let M be a compact and connected p-dimensional manifold. It is readily verified that the topology of M is Hausdorff. Metrizability follows for instance from Urysohn's metrization theorem [37, §34].
We define the collection of smooth functions on M by:

$$C^\infty(M) \equiv \{f : X \to \mathbb{C}; f \text{ is smooth}\}. \tag{2.12}$$

The collection of smooth functions on M has a useful property. We first state some definitions.

Definition 2.10 (Spectrum)**.** *Let $a \in \mathsf{A}$ with A an involutive and unital algebra over the complex numbers. The **spectrum** of a in A is the subset of $\mathbb{C}$ consisting of those $\lambda \in \mathbb{C}$ such that*

$$a - \lambda 1 \in \mathsf{A} \tag{2.13}$$

*is not invertible. We denote the spectrum of a with $Spec_{\mathsf{A}}(a) \subset \mathbb{C}$. The **spectral radius** of an element $a \in \mathsf{A}$ is given by*

$$r(a) \equiv \sup_{\lambda \in Spec_{\mathsf{A}} a} \{|\lambda|\} \in [0, \infty]. \tag{2.14}$$

For a unital and commutative C^*-algebra A:

$$\text{Spec}_A f = \{f(x); x \in \text{Spec } A\} \qquad \forall f \in A. \tag{2.15}$$

Definition 2.11 (Pre-C^*-algebra). *Let $\mathcal{A}$ be an involutive, dense and unital subalgebra of a C^*-algebra A. We say that $\mathcal{A}$ is a* ***pre-C^*-algebra*** *when it has the following properties.*

1. *$\mathcal{A}$ is a Fréchet algebra whose topology is finer than the one inherited by A (see appendix A for the definitions regarding Fréchet spaces);*

2. *Take some $a \in \mathcal{A}$ and let f be holomorphic on a open neighborhood U of $Spec_A a$. Let Γ be a Jordan curve in U that winds once around $Spec_A a$. Let*

$$f(a) \equiv \frac{1}{2\pi i} \oint_{\Gamma} f(z) \frac{1}{z1 - a} dz. \tag{2.16}$$

   Then $f(a) \in \mathcal{A}$ and

$$Spec_{\mathcal{A}} f(a) = f(Spec_{A}(a)). \tag{2.17}$$

   This property of $\mathcal{A}$ is called ***stability under holomorphic functional calculus****. Analogously, stability under smooth or continuous functional calculi can be defined.*

A general unital C^*-algebra is stable under continuous functional calculus with respect to its normal elements only. From [25, Thm. 4.4.5]:

Theorem 2.12. *Let A be a unital C^*-algebra and $a \in A$ a normal element. Each $f \in C\,(Spec_A a)$ defines an element $f(a) \in A$ such that*

$$\begin{aligned} f(Spec_A a) &= Spec_A f(a); \\ \|f(a)\| &= \|f\|_\infty \end{aligned} \tag{2.18}$$

This implies that commutative unital C^*-algebras are stable under continuous functional calculus. This is not such a surprise when we realize that a composition of two continuous functions is also a continuous function. The spectral radius and the norm of a C^*-algebra are closely related.

Theorem 2.13. *Let A be a unital C^*-algebra. For all $a \in A$,*

$$\|a\| = \sqrt{r(a^* a)}. \tag{2.19}$$

Proof. Define $b \equiv a^* a$. b is normal so we can apply (2.18) to the identity map on the spectrum of b in A. The statement now follows from the C^*-property of the norm. □

For a commutative unital C^*-algebra this relation follows directly from the definition of the norm of the C^*-algebra:

$$r(f) = \sup_{\lambda \in \{f(x); x \in X\}} \{|\lambda|\} = \sup \{|f(x)|; x \in X\} = \|f\|_\infty .$$

Before proceeding it is useful to establish some terminology regarding norms.

Definition 2.14. *Let U, V be normed vector spaces with norms $\|\cdot\|_U$ and $\|\cdot\|_V$, respectively. Let $A : U \to V$ be a linear map. Define the* ***operator norm*** *on A as*

$$\|A\|_{op} \equiv \sup_{\|u\|_U=1} \{\|Au\|_V\} \in [0, \infty]. \tag{2.20}$$

The operator norm is a norm on the collection of linear maps from U to V for which the expression on the right-hand side of (2.20) is finite. Those linear maps are called ***bounded****. When U and V are finite dimensional vector spaces we call the operator norm the* ***matrix norm*** *as well.*

Lemma 2.15. *The operator norm is submultiplicative for bounded operators.*

Proof. Let $A, B : U \to U$ be bounded linear maps relative to the norm $\|\cdot\|_U$ on U. Take some $u \in U$ and let $u' = \frac{u}{\|u\|_U}$.

$$\begin{aligned} &\|Au\|_U = \left\|Au'\right\|_U \|u\|_U \leq \|A\|_{\text{op}} \|u\|_U \quad \Rightarrow \\ &\|ABu\|_U \leq \|A\|_{\text{op}} \|B\|_{\text{op}} \|u\|_U , \end{aligned}$$

from which submultiplicativity follows. The general result for linear maps $A : U \to V$ and $B : V \to W$ between normed vector spaces U, V and W readily follows. □

Lemma 2.16. *Let M be a compact, finite-dimensional manifold. $C^\infty(M)$ is a pre-C^*-algebra.*

Proof. The fact that $C^\infty(M) \subset C(M)$ lies dense follows from the complex version of the Stone–Weierstrass theorem [17, V.§8].
According to lemma A.3 and corollary A.6 in appendix A we need to find a countable collection of submultiplicative semi-norms $\{p_k; k \in \mathbb{N}\}$ on $C^\infty(M)$ with the property that if $p_k(f) = 0$ for all k, then $f = 0$. By lemma B.3 in appendix B the set of linear differential operators on the tangent bundle of M forms a unital algebra over $C^\infty(M)$ with a countable base. Choose a base $\left\{P^k; k \in \mathbb{N}\right\}$ such that P^0 is the unit differential operator. For each k, $P^k f$ is a p-by-p matrix with partial derivatives of f of several orders as elements. These elements are bounded in the supremum norm since M is compact, so we can define a countable set of mappings $p_k : C^\infty(M) \to \mathbb{R}^+$ by

$$p_k(f) \equiv \left\|P^k f\right\|_{\text{op}}, \tag{2.21}$$

where the matrix norm is relative to the supremum norm. Take some $f \in C^\infty(M)$ such that $p_k(f) = 0$ for all k. This implies that

$$\|p_0(f)\| = \|f\|_\infty = 0 \quad \Rightarrow \quad f = 0.$$

So the p_k are, in fact, semi-norms.
These semi-norms generate a topology on $C^\infty(M)$ that is finer than the topology induced from that of $C(M)$. We show the algebra is closed in this topology. Take some Cauchy sequence $\{f_n\} \subset C^\infty(M)$. Using the fact that the P^k are a base of the algebra we infer that $\partial_{\vec{\alpha}} f_n$ is a Cauchy sequence with respect to the topology induced by the supremum norm for *each* multi-index $\vec{\alpha} \in \mathbb{N}^p$. The compactness of M then implies that $f_n \to f \in C^\infty(M)$.
The semi-norms p_k are not submultiplicative. However, by forming matrices of differential operators of different ranks in a clever way we can construct submultiplicative semi-norms out of the p_k. This method is illustrated in example 2.17 for the 1-sphere.
We now show that $C^\infty(M)$ is stable under holomorphic functional calculus. Take some $f \in C^\infty(M)$ and let ω be a holomorphic function defined on a neighborhood around $\mathrm{Spec}_{C(M)} f$. We wish to use Cauchy's integral formula to show that $\omega(f)$, as defined by (2.16), equals the composition $\omega \circ f$. For that we first need to show that ω is also a holomorphic function on some neighborhood of $\mathrm{Spec}_{C^\infty(M)} f$. In fact, we have a stronger result to our disposal:

$$\mathrm{Spec}_{C^\infty(X)} f = \mathrm{Spec}_{C(X)} f. \tag{2.22}$$

We shall prove this first. By definition Spec $_{C(X)} f \subseteq$ Spec $_{C^\infty(X)} f$. Assume there is a continuous function g which inverts $f - \lambda 1_X$, i.e., $\lambda \notin$ Spec $_{C^\infty(X)} f$. The function g is given by

$$g = \frac{1}{f - \lambda}.$$

Using (2.15) we see that f never attains the value λ. So g is a smooth function and $\lambda \notin \mathrm{Spec}_{C^\infty(X)} f$.
This implies that $\omega(f) = \omega \circ f$. Any holomorphic function can be uniquely identified with a (harmonic) complex valued smooth function, so $\omega \circ f \in C^\infty(M)$. Equality (2.17) then follows from equation (2.22), finishing the proof that $C^\infty(M)$ is a pre-C^*-algebra. □

Example 2.17 (S^1)**.** *All the aforementioned statements are valid for the 1-sphere as well. Let us focus on the construction of semi-norms on* $C^\infty\left(S^1\right)$*. The 1-sphere has a one-dimensional tangent bundle. Every first-order differential operator on the tangent bundle is locally of the form*

$$A^1(x)\frac{d}{dx},$$

for some coordinate function x *and some smooth function* $A^1 \in C^\infty\left(S^1\right)$*. A base of the algebra of differential operators is therefore generated by powers* *of* $D \equiv \frac{d}{dx}$. *Take some* $f \in C^\infty\left(S^1\right)$. *Naively defining the semi-norms* $\{p_k; k \in \mathbb{N}\}$ *by*

$$p_k(f) \equiv \left\| D^k f \right\|_\infty,$$

shows already for $k = 2$ *that we cannot expect the semi-norm to be submultiplicative. Hence we employ a little trick. Take the* $k+1$*-fold direct sum of the tangent bundle. Let*

$$\rho_k(f) : \bigoplus_{i=1}^{k+1} \Gamma^\infty\left(M, TS^1\right) \to \bigoplus_{i=1}^{k+1} \Gamma^\infty\left(M, TS^1\right);$$

$$\rho_k(f) \equiv \begin{pmatrix} f & Df & \cdots & \frac{D^k f}{k!} \\ 0 & f & \ddots & \vdots \\ \vdots & \ddots & f & Df \\ 0 & \cdots & 0 & f \end{pmatrix}, \qquad p_k(f) = \|\rho_k(f)\| \tag{2.23}$$

where the latter norm is defined in a four-step process. First, take a metric g *on the tangent bundle. Second, define the supremum norm on sections of the tangent space relative to the metric* g*, i.e.,* $\|\zeta\|_\infty \equiv \sup_{x \in M} \{g_x(\zeta_x, \zeta_z)\}$. *Third, extend the supremum norm to* $k+1$*-tuples of sections of the tangent bundle in the usual way, i.e., the norm is given by*

$$(\zeta_1, \ldots, \zeta_{k+1}) \mapsto \sqrt{\|\zeta_1\|_\infty^2 + \ldots + \|\zeta_{k+1}\|_\infty^2}. \tag{2.24}$$

Lastly, define the norm in equation (2.23) as the matrix norm relative to the latter norm (recall that $C^\infty\left(S^1\right)$ *is represented on sections of the tangent bundle by multiplication operators). By lemma 2.15, the matrix norm is submultiplicative, so we just need to show that* ρ_k *is an algebra morphism for each* k.

We now compare $\rho_k(f \cdot g)$ *with* $\rho_k(f) \circ \rho_k(g)$.

$$[\rho_k(f \cdot g)]_{i,j} = \frac{D^{i-j}(fg)}{(i-j)!} = \frac{1}{(i-j)!} \sum_{n=0}^{i-j} \binom{i-j}{n} D^{i-j-n}(f) D^n(g) =$$

$$\sum_{n=0}^{i-j} \frac{1}{n!(i-j-n)!} D^{i-j-n}(f) D^n(g).$$

$$[\rho_k(f) \circ \rho_k(g)]_{i,j} = \sum_{n=1}^{k+1} [\rho_k(f)]_{i,n} [\rho_k(g)]_{n,j} =$$

$$\sum_{n=1}^{k+1} \frac{D^{i-n}(f)}{(i-n)!} \frac{D^{n-j}(g)}{(n-j)!} \overset{\text{(upper triangularity)}}{=} \sum_{n=j}^{i} \frac{D^{i-n}(f)}{(i-n)!} \frac{D^{n-j}(g)}{(n-j)!} =$$

$$\sum_{n'=0}^{i-j} \frac{D^{i-j-n'}}{(i-j-n')!} \frac{D^{n'}}{n'!} = \frac{1}{(i-j)!} \sum_{n'=0}^{i-j} \binom{i-j}{n'} D^{i-j-n'}(f) D^{n'}(g).$$

Both expressions are equal, so $C^\infty\left(S^1\right)$ is a Fréchet algebra.

2.3 Clifford algebras

Let us take a short sidestep to discuss some preliminaries regarding the Clifford algebras and spin groups.

Definition 2.18 (Inner products)**.** *Let U be a real vector space. A* ***symmetric bilinear form*** *on U is a bilinear map*

$$g : U \times U \to \mathbb{R}, \qquad g(u,v) = g(v,u) \quad \forall u, v \in U.$$

The form is said to be ***nondegenerate*** *when the matrix representation of g, seen as a map from U itself, is an invertible map. The map is* ***positive definite*** *if $g(u,u) \geq 0$ for all $u \in U$. When nondegeneracy and positive definiteness is satisfied we say that g is an* ***inner product****.*
Let V be a complex vector space. A ***sesquilinear form*** *is a map $h : V \times V \to \mathbb{C}$ such that for all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{C}$*

$$h(u, \lambda v + \mu w) = \lambda h(u,v) + \mu h(u,w), \qquad h(u,v) = \overline{h(v,u)}.$$

Again, we say that h is nondegenerate when its matrix representation, seen as a map from V to itself, is invertible. Positive definiteness is defined again as the requirement $h(v,v) \geq 0$ for all $v \in V$. Sesquilinear, positive definite and nondegenerate forms are called ***Hermitian inner products****.*
Inner products are closely related to ***quadratic forms****. Let W be either a real or complex vector space. A quadratic form is a map $q : W \to \mathbb{R}$ defined by*

$$q(w) \equiv w^T A w, \tag{2.25}$$

with $A : W \to W$ a linear map. A quadratic form q is said to be nondegenerate and positive definite when $q(w) \geq 0$ always and $q(w) = 0$ implies $w = 0$ respectively.
A nondegenerate and positive definite quadratic form q defines a (Hermitian) inner product by

$$(v, w) \mapsto \frac{1}{2}\left[q(v+w) - q(v) - q(w)\right].$$

Real finite-dimensional vector spaces equipped with inner products form a category whose arrows are given by linear maps that preserve the inner product:

$$A : U \to V, \qquad g(Au, Av) = g(u,v) \qquad \forall u, v \in U$$

which is equivalent to stating that

$$A^T A = AA^T = 1$$

We denote the category of real finite-dimensional vector spaces equipped with a inner product by $\boldsymbol{FVec}^F_{\mathbb{R}}$*. The category of finite-dimensional complex vector spaces equipped with a Hermitian inner product is defined analogously, with the unitary linear maps as arrows. The latter category is denoted by* $\boldsymbol{FVec}^F_{\mathbb{C}}$*.*

Definition 2.19 (Clifford algebras)**.** *Take some p-dimensional vector space V in* $\boldsymbol{FVec}^F_{\mathbb{R}}$ *with inner product g. The* ***tensor algebra*** *of V is given by*

$$\mathcal{T}(V) \equiv \bigoplus_{k=0}^{\infty} V^{\otimes k} \tag{2.26}$$

with $V^{\otimes 0} \equiv \mathbb{R}$*. The* ***pure*** *elements (of order k) of the tensor algebra are the elements of* $V^{\otimes k}$ *under the canonical imbedding in the tensor algebra. Note that this construction also goes through for complex vector spaces, this time with* $V^{\otimes 0} \equiv \mathbb{C}$*.*
Let $\mathcal{I}_g$ *be the two-sided ideal of the tensor algebra generated by the set*

$$\{u \otimes v + v \otimes u - 2g(u, v)1; u, v \in V\}. \tag{2.27}$$

We then define the real and the complex ***Clifford algebras*** *by*

$$C\ell_g(V) \equiv \mathcal{T}(V)/\mathcal{I}_g; \tag{2.28}$$

$$\mathbb{C}\ell_g(V) \equiv C\ell_g(V) \otimes_{\mathbb{R}} \mathbb{C}. \tag{2.29}$$

The pure elements of a Clifford algebra are defined similar to the pure elements of the tensor algebra. When we refer to "the" Clifford algebra in the rest of the text, the complex version is implied, unless stated otherwise.
Note that the construction of the Clifford algebra also goes through for degenerate bilinear and sesquilinear forms. Using a completely degenerate bilinear or sesquilinear form h, we see that in the real as well as the complex case case

$$C\ell_h(V) \equiv \Lambda^{\bullet}(V). \qquad (\textbf{\textit{exterior algebra}}) \tag{2.30}$$

The product in the Clifford algebra is denoted with a dot instead of the tensor symbol. Take some base $\{e_1, \ldots, e_p\}$ *of V. Then every element in the (complex) Clifford algebra is a real (complex) linear combination of elements of the form*

$$e_{i_1} \cdots e_{i_k}, \qquad i_1 < \ldots < i_k, \qquad k \leq p. \tag{2.31}$$

Remark 2.20. *The Clifford algebra can also be defined on* $\boldsymbol{FVec}^F_{\mathbb{C}}$ *right away. Take some* $(V,g) \in \boldsymbol{FVec}^F_{\mathbb{R}}$. *Define:*

$$\begin{aligned} &W \equiv V \otimes_{\mathbb{R}} \mathbb{C}, \qquad h : W \times W \to \mathbb{C}, \\ &h(u,v) \equiv g(\Re u, \Re v) + g(\Im u, \Im v) + ig(\Re u, \Im v) - ig(\Im u, \Re v) \quad \forall u,v \in W. \end{aligned} \tag{2.32}$$

Then $(W,h) \in \boldsymbol{FVec}^F_{\mathbb{C}}$ *and*

$$\mathbb{C}\ell_h(W) = \mathbb{C}\ell_g(V).$$

So both definitions of the Clifford algebra are compatible with respect to complexification of real vector spaces equipped with an inner product.

From [31, Prop. 1.1]:

Lemma 2.21. *Let* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$. *The Clifford algebra construction defines a functor*

$$\mathbb{C}\ell : \boldsymbol{FVec}^F_{\mathbb{F}} \to \boldsymbol{Alg}_{\mathbb{C}} \tag{2.33}$$

where the category on the right-hand side is the category of algebras over the complex numbers with algebra morphisms as arrows.

Proof. The proof follows readily from the following two definitions.
On the objects of the category we define $\mathbb{C}\ell(V,g) \equiv \mathbb{C}\ell_g(V)$. For $f : (V,g) \to (W,g')$ an arrow in $\mathbf{FVec}^F_{\mathbb{F}}$, the action of $\mathbb{C}\ell(f)$ on pure elements $v_1 \cdots v_k \in \mathbb{C}\ell_g(V)$ is given by

$$\mathbb{C}\ell(f)(v_1 \cdots v_k) = f(v_1) \cdots f(v_k), \tag{2.34}$$

and then extended by linearity to the whole of $\mathbb{C}\ell_g(V)$. □

Corollary 2.22. *Up to isomorphism of vector spaces equipped with an inner product, the Clifford algebra only depends on the dimension of the vector space on which the Clifford algebra is constructed and the signature of the inner product* g *used to construct the Clifford algebra.*

Proof. Use Sylvester's theorem [53], which shows that objects of the same dimension and signature in $\mathbf{FVec}^F_{\mathbb{F}}$ are isomorphic for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The result then follows from the functorial property of $\mathbb{C}\ell$. □

From now on, assuming g is a (Hermitian) inner product, we shall only index Clifford algebras by their dimension and write

$$C\ell_p \quad \& \quad \mathbb{C}\ell_p \tag{2.35}$$

for the Clifford algebras defined on a p-dimensional vector space.

It is now time for a few examples of Clifford algebras.

Example 2.23 ($p = 1$)**.** *Let V be a one-dimensional real vector space, i.e., we identify V with $\mathbb{R}$. We define the following inner product on V:*

$$g(u, v) \equiv uv \qquad u, v \in V.$$

The tensor algebra of V is given by:

$$\mathcal{T}(V) = \mathbb{R} \cdot 1 \oplus \mathbb{R} \oplus (\mathbb{R} \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R}) \oplus \dots \tag{2.36}$$

Let us denote the base element of the second term of the above sum (the term corresponding to $V^{\otimes 1}$) with e_1. Note that the base of the first term in the above equation (the "scalar" part of the tensor algebra) is given by the unit $1 \in \mathbb{R}$. In the Clifford algebra $C\ell_1$, these base elements satisfy the following relations:

$$1^2 = 1, \qquad e_1 \cdot 1 = 1 \cdot e_1 = e_1, \qquad e_1^2 = g(e_1, e_1) = 1. \tag{2.37}$$

This implies that the Clifford algebra is two-dimensional and generated by the base $\{1, e_1\}$ over $\mathbb{R}$. So

$$C\ell_1 \cong \mathbb{R} \oplus \mathbb{R} \quad \Rightarrow \quad \mathbb{C}\ell_1 \cong \mathbb{C} \oplus \mathbb{C}. \tag{2.38}$$

For $k \in \mathbb{N}$, denote by

$$M_k(\mathbb{F}) \tag{2.39}$$

the $k \times k$ matrix algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The previous example can be recast in the form

$$\mathbb{C}\ell_1 \cong M_1(\mathbb{C}) \oplus M_1(\mathbb{C}). \tag{2.40}$$

In general, every simple matrix algebra has (up to equivalence) only one irreducible representation [30]. Every matrix algebra over $\mathbb{F}$ is simple, so we see that $\mathbb{C}\ell_1$ has two inequivalent irreps.

Example 2.24 ($p = 4$)**.** *Let $\{e_1, \dots, e_4\} \subset \mathbb{R}^4$ be an orthonormal base. Define the* ***gamma matrices*** *$\{\gamma^1, \dots, \gamma^4\} \subset M_4(\mathbb{C})$ as the images:*

$$e_1 \mapsto \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad e_2 \mapsto \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$e_3 \mapsto \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad e_4 \mapsto \gamma^4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}.$$

Denote with I_4 the identity in $M_4(\mathbb{C})$. By straightforward evaluation it follows that

$$\{\gamma^i, \gamma^j\} = \delta_{i,j} I_4, \qquad (\gamma^i)^* = \gamma^i \qquad \forall i, j.$$

The gamma matrices are therefore a faithful complex representation of the real Clifford algebra $C\ell_4$. By complexifying $\mathbb{R}^4$, the gamma matrices also define a complex representation of $\mathbb{C}\ell_4$. One can verify by hand that the complex linear span of the algebra generated by the gamma matrices coincides with $M_4(\mathbb{C})$, so

$$\mathbb{C}\ell_4 \cong M_4(\mathbb{C}). \tag{2.41}$$

Up to equivalence, the four-dimensional Clifford algebra therefore only has one irreducible representation.

It is no coincidence that the complex representations of the Clifford algebras in the previous two examples had such nice properties. From [31, I.4]:

Theorem 2.25. *When p is even the Clifford algebra $\mathbb{C}\ell_p$ has, up to equivalence, only one complex irrep. We denote the corresponding representation space by Δ and the representation by c. This representation is also referred to as the* ***Clifford representation****.*
When p is odd there are two inequivalent complex irreducible representations of $\mathbb{C}\ell_p$. The two representations are denoted by $c^\pm$ and the pertinent representation spaces are denoted by $\Delta^\pm$. When it is not relevant to distinguish between the odd and the even case, we shall call these representations collectively Clifford representation and denote them (ambiguously) by (c, Δ).
More explicitly, the above is a consequence of the following isomorphisms:

$$\mathbb{C}\ell_p \cong \begin{cases} M_{2^k}(\mathbb{C}) & p = 2k \\ M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C}) & p = 2k+1 \end{cases} \tag{2.42}$$

There is a distinguished element lying in every Clifford algebra called the **volume element** or the **chirality element**.

Definition 2.26. *Let $k = \lfloor \frac{p}{2} \rfloor$ and take some positively oriented base $\{e_1, \ldots, e_p\}$ of $\mathbb{C}^p$. The chirality element is given by*

$$\gamma \equiv (-i)^k e_1 \cdots e_p. \tag{2.43}$$

This expression is invariant under a choice of positively oriented base [21, Def. 5.2]. We ambiguously identify γ with its image under the Clifford representation.

The Clifford algebra comes equipped with many interesting automorphisms.

Definition 2.27. *The automorphism $v \mapsto -v$ defined on $\mathbb{C}^p$ extends by functoriality to the* ***grading operator*** *in $\mathbb{C}\ell_p$. We denote the grading operator by χ.*
A second automorphism is the anti-linear ***conjugation****. It is defined on pure elements as*

$$\overline{u_1 \cdots u_k} \equiv \overline{u}_1 \cdots \overline{u_k}, \tag{2.44}$$

and then extended by linearity.
A third map is the anti-automorphism $! : \mathbb{C}\ell_p \to \mathbb{C}\ell_p$ defined as:

$$!(u_1 \cdots u_k) \equiv u_k \cdots u_1 \tag{2.45}$$

on pure elements and then extended by linearity.
Composing the conjugation with the map ! defines an involution $^* : \mathbb{C}\ell_p \to \mathbb{C}\ell_p$. The ***charge conjugation*** *$\kappa : \mathbb{C}\ell_p \to \mathbb{C}\ell_p$ is the composition of the grading operator and the conjugation.*

The grading operator defines a $\mathbb{Z}_2$-grading on $\mathbb{C}\ell_p$. We denote the $\pm$-eigenspaces of the grading operator by $\mathbb{C}\ell_p^{\pm}$. From [31, I.3]:

Lemma 2.28. *For $p > 0$:*

$$\mathbb{C}\ell_{p-1} \cong \mathbb{C}\ell_p^{+}. \tag{2.46}$$

We use the previous result to construct an algebra out of the Clifford algebra, which has only one irreducible complex representation. Also this representation will be called the Clifford representation whenever no ambiguity can arise.

Definition 2.29.

$$\mathbb{C}\ell_p^{(+)} \equiv \begin{cases} \mathbb{C}\ell_p & p \text{ even} \\ \mathbb{C}\ell_p^{+} & p \text{ odd} \end{cases} \tag{2.47}$$

It is readily verified that also $\mathbb{C}\ell_p^{(+)}$ defines a functor from $\mathbf{FVec}_{\mathbb{F}}^{F}$ to $\mathbf{Alg}_{\mathbb{C}}$, for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

The grading operator and the chirality element are closely related.

Lemma 2.30. *Let $\mathbb{C}\ell_p$ be the p-dimensional complex Clifford algebra. Let Δ be the representation space of the Clifford algebra associated with the Clifford representation. The chirality element is a symmetry. The adjoint action of the chirality element on Δ, given by*

$$Ad_\gamma(v) = c(\gamma) v c(\gamma)^{-1} \qquad \forall v \in \Delta, \tag{2.48}$$

equals the action of the grading operator when p is even. In that case we say that the chirality operator ***implements*** *the grading operator.*

Proof. For any choice of a positively oriented orthonormal base $\{e_1, \ldots, e_p\}$ of Δ, we have

$$\gamma^* = i^{\lfloor \frac{p}{2} \rfloor} e_p \cdots e_1 = i^{\lfloor \frac{p}{2} \rfloor} (-1)^{\sum_{i=1}^{p-1} (p-i)} e_1 \cdots e_p = \\ i^{\lfloor \frac{p}{2} \rfloor} (-1)^{\frac{p(p-1)}{2}} e_1 \cdots e_p.$$

By distinguishing the individual cases for $p = l \mod 4$ one can verify that γ is self-adjoint. From this follows that

$$\gamma^2 = \gamma^* \gamma = i^{\lfloor \frac{p}{2} \rfloor} (-i)^{\lfloor \frac{p}{2} \rfloor} e_p \cdots e_1 \cdot e_1 \cdots e_p = 1.$$

We evaluate the adjoint action on base elements of Δ:

$$\mathrm{Ad}_\gamma(c(e_i)) = c(\gamma) c(e_i) c(\gamma) = c(\gamma^2)(-1)^{p-1} c(e_i) = (-1)^{p-1} c(e_i).$$

So on base elements, Ad_γ implements the grading operator when the dimension is even. We can extend the action to the whole Clifford algebra by noting that the adjoint action is linear and satisfies

$$\mathrm{Ad}_\gamma(c(e_{i_1} \cdots e_{i_k})) = \mathrm{Ad}_\gamma(c(e_{i_1})) \cdots \mathrm{Ad}_\gamma(c(e_{i_k})).$$

In conclusion, Ad_γ is equal to the grading operator when p is even. Note that for p odd the adjoint action of the chirality element on Δ is just the identity map in Δ. □

There is also a map on Δ implementing the charge conjugation.

Lemma 2.31. *Let Δ be the representation space for the Clifford algebra $\mathbb{C}\ell_p^{(+)}$. For every Hermitian inner product h on Δ we can find an anti-linear map $C : \Delta \to \Delta$ with the properties that for all $\psi, \varphi \in \Delta$ and all $a \in \mathbb{C}\ell_p^{(+)}$:*

$$h(C\psi, C\varphi) = h(\varphi, \psi); \qquad \textbf{\textit{(anti-unitarity)}} \tag{2.49}$$

$$Ad_C(c(a)) = c(\kappa(a)); \tag{2.50}$$

$$C^2 = \begin{cases} 1 & p = 0, 1, 6, 7 \mod 8 \\ -1 & \textit{otherwise} \end{cases} \tag{2.51}$$

$$C\gamma = \begin{cases} \gamma C & p = 0, 3, 4, 7 \mod 8 \\ -\gamma C & \textit{otherwise} \end{cases} \tag{2.52}$$

We say that C is the ***charge conjugation operator****.*

Proof. The Clifford representation defines a $\mathbb{C}\ell_p^{(+)}$-$\mathbb{C}$-bimodule structure on Δ. We first shall show that the set $\Delta^\sharp$, defined as

$$\Delta^\sharp \equiv \{\langle\psi|, \psi \in \Delta, \langle\psi| : \Delta \to \mathbb{C}, \langle\psi|(\varphi) = h(\psi, \varphi)\}, \tag{2.53}$$

is a $\mathbb{C}\ell_p^{(+)}$-$\mathbb{C}$-bimodule as well. After we have established this, we will define the charge conjugation operator and subsequently show it has the required properties.
Take some $a, b \in \mathbb{C}\ell_p^{(+)}$ and $\psi \in \Delta$. From C.4 and C.5 we know that $\Delta^\sharp$ is a right-$\mathbb{C}$-module as well. A left-action of of $\mathbb{C}\ell_p^{(+)}$ on $\Delta^\sharp$ is defined by:

$$a \cdot \langle\psi| \equiv \langle\psi| \circ \kappa(a) = \langle\psi| \circ \chi(a!). \tag{2.54}$$

This action turns $\Delta^\sharp$ into a left-$\mathbb{C}\ell_p^{(+)}$-module:

$$a \cdot b \cdot \langle\psi| = a \cdot \langle\psi| \circ \chi(!b) = \langle\psi| \circ \chi(!b) \circ \chi(!a) =$$
$$\langle\psi| \circ \chi(!b!a) = \langle\psi| \circ \chi(!(ab)) = (ab) \cdot \langle\psi|.$$

Take some $\lambda \in \mathbb{C}$. Noting that $\mathbb{C}$ is a commutative C^*-algebra, the right-$\mathbb{C}$-module structure on $\Delta^\sharp$ is given by:

$$\langle\psi| \cdot \lambda \equiv \langle\psi\overline{\lambda}|.$$

One can show there exists a $\mathbb{C}\ell_p^{(+)}$-$\mathbb{C}$-bimodule isomorphism $T : \Delta^\sharp \to \Delta$, see [21, H. 5.1]. We can define $C : \Delta \to \Delta$ as follows:

$$C\psi \equiv T(\langle\psi|). \tag{2.55}$$

We shall verify C satisfies properties (2.49) and (2.50) and we will show that C is anti-linear. The other two properties of the charge conjugation operator, namely the value of its square and the commutation relations with γ, are verified in [21, H. 9.5].
Take some $\lambda \in \mathbb{C}$ and $\psi \in \Delta$:

$$C(\psi\lambda) = T(\langle\psi\lambda|) = T(\langle\psi|\overline{\lambda}) = T(\langle\psi|)\overline{\lambda} = C(\psi)\overline{\lambda},$$

since T is a $\mathbb{C}$-module isomorphism. Hence we have shown that C is an anti-linear map.
We now demonstrate anti-unitarity.

$$h(C\psi, C\varphi) = h(T(\langle\psi|), T(\langle\varphi|)) = [\langle\psi|, \langle\varphi|]_{\mathbb{C}}\,,$$

where $[\cdot,\cdot]_{\mathbb{C}}$ is the $\mathbb{C}$-valued inner product on $\Delta^\sharp$ and by noting that T is preserves the inner product in the sense of definition C.3. By construction (see C.5 for this construction), the inner product $[\langle\psi|, \langle\varphi|]_{\mathbb{C}}$ is equal to:

$$h(\varphi, \psi),$$

so that,

$$h(C\psi, C\varphi) = h(\varphi, \psi),$$

as required.
To verify that the charge conjugation operator implements charge conjugation $\kappa : \mathbb{C}\ell_p^{(+)} \to \mathbb{C}\ell_p^{(+)}$, we seek an alternative expression for $\langle a\psi|$ with $a \in \mathbb{C}\ell_p^{(+)}$.
Last, by the definition of $\Delta^\sharp$ we see that $\langle a\psi| = \langle\psi| \circ a^* = \chi(\overline{a}) \cdot \langle\psi|$. Here we use the fact that $\chi^2 = 1$. This implies:

$$\begin{aligned} &Ca\psi = T(\langle a\psi|) = T(\chi(\overline{a})\langle\psi|) = \chi(\overline{a})T\langle\psi| = \chi(\overline{a})C\psi, \quad \Rightarrow \\ &CaC^{-1}\psi = \chi(\overline{a})\psi = \kappa(a)\psi. \end{aligned}$$

Apart from verifying (2.51) and (2.51), which are proven in [21, H. 9.5], we see that the C thus constructed is a charge conjugation operator. □

Remark 2.32. *Note that when Δ is equipped with a charge conjugation operator, we can find a $\mathbb{C}\ell_p^{(+)}$-$\mathbb{C}$-bimodule isomorphism $T : \Delta^\sharp \to \Delta$: the identification*

$$T(\langle\psi|) \equiv C(\psi) \tag{2.56}$$

does the trick.

Example 2.33. *For the cases $p = 1$ and $p = 4$ we give an explicit expression for the charge conjugation operators.*

- *$p = 1$. We restrict ourselves to the irreducible representation $c : \mathbb{C}\ell_1 \to End_{\mathbb{C}}(\Delta^+) \cong \mathbb{C}$ given by*

$$c(\lambda, \lambda')\mu = \lambda\mu. \tag{2.57}$$

  Let h be an inner product on $\mathbb{C}$ defined by $h(\lambda, \mu) = \overline{\lambda}\mu$. We can therefore choose C to be the complex conjugation operator. The chirality element is represented on $\mathbb{C}$ by the identity operator, showing that C has all the required properties.

- *$p = 4$. For all $a \in \mathbb{C}\ell_4$ we can define the charge conjugation operator $C : \Delta \to \Delta$ with $\Delta \cong \mathbb{C}^4$ in this case as follows:*

$$C(a) \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \circ \overline{a}. \tag{2.58}$$

  The representation of the chirality element is given by the matrix given by

$$[\gamma]_{i,j} = \begin{cases} \delta_{i,i} & i = 1, 2 \\ -\delta_{i,i} & i = 3, 4 \end{cases} \tag{2.59}$$

  Verifying that C has the required properties now follows from straightforward calculations.

The additional maps on $\mathbb{C}\ell_4$ allow us to endow this algebra with more structure. For example:

Lemma 2.34. *$\mathbb{C}\ell_p$ is a finite dimensional C^*-algebra.*

Proof. There are several ways to proceed. The fastest approach would be to use the operator norm on Δ, so that we can define a norm on the Clifford algebra by using the Clifford representation.
However, in the next section we will need a norm on $\mathbb{C}\ell_p$ that makes no reference to a representation of the Clifford algebra on an *external* Hilbert space. We therefore take this longer route.
There is a *vector space* isomorphism $\mathbb{C}\ell_p \cong \Lambda^\bullet \mathbb{C}^p$, given by the map defined on pure elements as

$$u_{i_1} \cdots u_{i_k} \mapsto u_{i_1} \wedge \cdots \wedge u_{i_k}, \tag{2.60}$$

and then extended by linearity. Let h be a Hermitian inner product on $\mathbb{C}^p$. We borrow the expression of the point-wise inner product of a p- and a q-form on a manifold, given on pure elements by

$$\langle u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_q \rangle = \delta_{p,q} \det \left[h(u_i, v_j) \right] \tag{2.61}$$

to define, via the above vector space isomorphism, a Hermitian inner product on $\mathbb{C}\ell_p$. This inner product turns $\mathbb{C}\ell_p$ into a finite-dimensional Hilbert space. We define the norm on $\mathbb{C}\ell_p$ as the operator norm relative to this norm on the Hilbert space. □

The Clifford algebra contains two groups that are of great interest to us.

Definition 2.35. *The sets $Spin^c_p, Spin_p \subset \mathbb{C}\ell_p$, defined by*

$$Spin^c_p \equiv \{u_1 \cdots u_{2k} \in \mathbb{C}\ell_p; u_i \in \mathbb{C}^p, |u_i| = 1\}; \tag{2.62}$$

$$Spin_p \equiv \left\{u \in Spin^c_p; \kappa(u) = u\right\}, \tag{2.63}$$

*are groups, with the product in the Clifford algebra as the group operation. They are collectively called the **spin groups**. The group $Spin_p$ is sometimes referred to as the "real" spin group, from its invariance under charge conjugation. $Spin^c_p$ is referred to as the **charged** spin group.*

Lemma 2.36.

$$Spin^c_p \cong \left(Spin_p \times U(1)\right) / \mathbb{Z}_2 \tag{2.64}$$

Proof. The group morphism

$$f : u \times \lambda \mapsto \lambda u, \qquad f : \mathrm{Spin}_p \times U(1) \to \mathrm{Spin}^c_p \tag{2.65}$$

has the group $\{(1,1),(1,-1)\} \cong \mathbb{Z}_2$ as its kernel. Now take some $u_1 \cdots u_{2k} \in \text{Spin}^c_p$. By definition, we can find a $\lambda_i \in U(1)$ and a unit vector $w_i \in \mathbb{R}^p$ such that $u_i = \lambda_i w_i$. The identification

$$f(w_1 \cdots w_{2k}, \lambda_1 \cdots \lambda_{2k}) = u_1 \cdots u_{2k} \tag{2.66}$$

shows f is surjective. The statement now follows from the First Isomorphism Theorem (for groups). □

Example 2.37 $(p = 1, 4)$**.** *For $p = 1$ we have the isomorphisms*

$$Spin^c_1 \cong U(1); \tag{2.67}$$
$$Spin_1 \cong \mathbb{Z}_2. \tag{2.68}$$

$Spin^c_1$ and $Spin_1$ are both contained in the even part of the Clifford algebra $\mathbb{C}\ell_1$. The reduction of the Clifford representation to the spin groups is the reduction of the irrep used in example 2.33. This reduction is an irreducible complex representation of the spin groups.
For the case $p = 4$, the real spin group takes a familiar form. There are two sets of matrices: the "top" matrices

$$\begin{aligned} w_0^{\uparrow} &\equiv \frac{1}{2}\left(I_4 + \gamma\right), & w_1^{\uparrow} &\equiv \frac{1}{2}\left(\gamma^1\gamma^2 - \gamma^3\gamma^4\right), \\ w_2^{\uparrow} &\equiv \frac{1}{2}\left(\gamma^1\gamma^3 + \gamma^2\gamma^4\right), & w_3^{\uparrow} &\equiv \frac{1}{2}\left(\gamma^2\gamma^3 - \gamma^1\gamma^4\right), \end{aligned} \tag{2.69}$$

and the "down" matrices

$$\begin{aligned} w_0^{\downarrow} &= \frac{1}{2}\left(I_4 - \gamma\right), & w_1^{\downarrow} &\equiv \frac{1}{2}\left(\gamma^1\gamma^2 + \gamma^3\gamma^4\right), \\ w_2^{\downarrow} &\equiv \frac{1}{2}\left(\gamma^1\gamma^3 - \gamma^2\gamma^4\right), & w_3^{\downarrow} &\equiv \frac{1}{2}\left(\gamma^2\gamma^3 + \gamma^1\gamma^4\right), \end{aligned} \tag{2.70}$$

which generate $Spin_4$ in the following way:

$$Spin_4 = \left\{ \sum_i \alpha_i w_i^{\uparrow} + \beta_i w_i^{\downarrow}; \alpha_i, \beta_i \in \mathbb{R}, \sum_i \alpha_i^2 = \sum_i \beta_i^2 \right\}. \tag{2.71}$$

Note that we have the following relations:

$$\begin{aligned} & w_i^{\uparrow} \cdot w_j^{\downarrow} = 0 \qquad \forall i, j; \\ & \left(w_0^{\uparrow}\right)^2 = \left(w_0^{\downarrow}\right)^2 = 1; \\ & \left(w_i^{\uparrow}\right)^2 = \left(w_i^{\downarrow}\right)^2 = -1 \qquad i \in \{1,2,3\}. \end{aligned} \tag{2.72}$$

Recall that $SU(2) = \left\{ A \in M_2(\mathbb{C}); A = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, |\alpha|^2 + |\beta|^2 = 1 \right\}$. *Writing this out in terms of real coefficients also shows the diffeomorphism between* $SU(2)$ *and* S^3*:*

$$SU(2) = \left\{ A \in M_2(\mathbb{C}); A = \begin{pmatrix} a_0 + ia_1 & -a_2 + ia_3 \\ a_2 + ia_3 & a_0 - ia_1 \end{pmatrix}, \sum_i a_i^2 = 1 \right\} =$$

$$\left\{ a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + a_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \vec{a} \in S^3 \right\} \equiv$$

$$\left\{ a_0 1 + a_1 w_1 + a_2 w_2 + a_3 w_3, \vec{a} \in S^3 \right\}.$$

So the relations (2.72) imply that the map

$$\sum_i \alpha_i w_i^{\uparrow} + \beta_i w_i^{\downarrow} \mapsto \left(\sum_i \alpha_i w_i, \sum_i \beta_i w_i \right) \tag{2.73}$$

gives an isomorphism $Spin_4 \cong SU(2) \times SU(2)$*. One copy of* $SU(2)$ *is contained in the upper left corner of* $M_4(\mathbb{C})$*, and the other resides in the lower right. By general considerations [10, Ch. VI] one sees that the representation of* $SU(2)$ *on* $\mathbb{C}^2$ *obtained this way is irreducible. The restriction of* c *to* $Spin_4$ *therefore induces two inequivalent irreducible representations of* $Spin_4$*.*

The spin groups are both contained in $\mathbb{C}\ell_p^{(+)}$, which has a unique irreducible representation. The previous example suggests that whenever p is odd, the Clifford representation reduces to one irrep of the spin groups, while for p even the restriction of Clifford representation decomposes into two irreps. This is a general phenomenon [31, Prop. I.5.15]:

Theorem 2.38. *When* p *is odd, restriction of Clifford representation to the spin groups yields an irreducible representation of these groups. When* p *is even, the restriction of the Clifford representation to the spin groups decomposes into two inequivalent irreps.*

Finally, we note an important relationship between the spin groups and the special orthogonal groups. Namely, Spin_p^c is a **2-to-1 cover** of $SO_p \times U(1)$, i.e., there is a Lie group morphism $\Lambda : \mathrm{Spin}_4^p \to SO_p \times U(1)$ [31, App. D] such that

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_p^c \xrightarrow{\Lambda} SO_p \times U(1) \longrightarrow 1 \tag{2.74}$$

is a short exact sequence of Lie groups.
Furthermore, Spin_p is a 2-to-1 cover of SO_p [31, Thm. 2.9], i.e.,

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_p \xrightarrow{\Lambda'} SO_p \longrightarrow 1 \tag{2.75}$$

is a SES of Lie groups. Note that we have not yet explicitly shown that Spin_p and Spin_p^c are Lie groups. However, SO_p and its product with $U(1)$ are, so by a theorem on universal covering groups [6, Ch. 13] also Spin_p and Spin_p^c are Lie groups.

2.4 Spin manifolds

The purpose of this section is to translate the results on Clifford algebras and the spin groups of section 2.3 to a global setting. Unless stated otherwise, let M be a compact, connected, and oriented manifold of dimension p. Recall that a differentiable structure on M is an equivalence class of atlases. Two atlases in the same differentiable structure are equivalent when they are smoothly connected. We shall denote the tangent space of M by TM and the cotangent space by T^*M.

Let us recall a basic concept in differential geometry, namely the global version of the inner products we encountered in the previous section.

Definition 2.39 (Metric)**.** *A **Riemannian metric** is a smooth section*

$$g \in \Gamma^{\infty}(M, T^*M \otimes T^*M), \tag{2.76}$$

such that for each $x \in M$, $g_x(\cdot, \cdot)$ is an inner product on T_xM.
By complexifying the sections we obtain a smooth complex-valued section

$$h \in \Gamma^{\infty}_{\mathbb{C}}\left(M, T^*M \otimes T^*M\right), \tag{2.77}$$

*which is a fiber-wise Hermitian inner product defined on fibers by equation (2.32). We say h is an **Hermitian metric**.*
These concepts readily extend to any vector bundle. Just as there is a fiber-wise correspondence between (complex) quadratic forms and (Hermitian) inner products, every (Hermitian) metric yields a (complex) quadratic form on a vector bundle.
It is a basic result in differential geometry [19] that any real vector bundle admits a Riemannian metric and that every complex vector bundle admits a Hermitian metric.

Let us briefly recall what is meant with principal fiber bundles and their morphisms.

Definition 2.40 (Principal fiber bundles)**.** *The quadruplet* $(\mathfrak{P}, \pi, M, G)$ *is a* ***principal fiber bundle***[2] *when:*

- *G is a Lie group;*
- *$\mathfrak{P}$ is a manifold equipped with a free right-action of G;*
- *$\pi : \mathfrak{P} \to M$ is a smooth map;*
- *$\mathfrak{P}/G$ is a manifold and there is a diffeomorphism $\varphi : \mathfrak{P}/G \to M$ such that $\pi = \varphi \circ \rho$, where $\rho : \mathfrak{P} \to \mathfrak{P}/G$ is the canonical map.*

For each $p \in M$, $\pi^{-1}(\{p\})$ is called the ***fiber*** *of the bundle. Freeness of the action of G implies that each fiber is isomorphic to G. G is also referred to as the* ***structure group*** *of the bundle. M is said to be the* ***base manifold*** *and $\mathfrak{Q}$ is referred to as the* ***total space*** *of the bundle.*
Let $(\mathfrak{Q}, \pi', M, H)$ be a second principal fiber bundle. A morphism of principal fiber bundles is a smooth map $f : \mathfrak{P} \to \mathfrak{Q}$ and a Lie group morphism $\Lambda : G \to H$ with the following properties.

- *$\pi' \circ f = \pi$;*
- *For all $p \in \mathfrak{P}$ and for all $g \in G$, $f(p \cdot g) = f(p) \cdot \Lambda(g)$.*

A **Spin**^{c} structure is a global version of equation (2.74).

Definition 2.41 (Spin^c structure)**.** *Let $\mathfrak{P}$ be the special orthogonal frame bundle associated to M. The direct product of this principal bundle with a trivial principal fiber bundle with $U(1)$ as structure group is again a principal fiber bundle. We say that M has a Spin^c structure when there is a principal fiber bundle $\mathfrak{Q}$ with Spin^c_p as structure group such that there is a short exact sequence of principal fiber bundle morphisms:*

$$\{1\} \times M \longrightarrow \mathfrak{O} \longrightarrow \mathfrak{Q} \longrightarrow \mathfrak{P} \times U(1) \longrightarrow \{1\} \times M \ , \tag{2.78}$$

where $\mathfrak{O}$ is some principal fiber bundle on M having $\mathbb{Z}_2$ as its structure group.

We will now describe two equivalent ways to verify whether M has a Spin^c structure, and if so, whether the Spin^c structure is unique. After briefly reviewing the equivalence of both methods, we finish with the definition of a Spin structure and the criteria for its existence.

[2]Note that we explicitly do not require the principal fiber bundles to be locally trivial because we will not make explicit use of this property. Therefore, the definition of principal fiber bundles is more condensed. However, in all the cases we shall encounter the involved principal fiber bundles will be locally trivial. This property is highly useful to deduce further results regarding the construction of algebra bundles out of vector bundles and the associated bundle construction. The interested reader is referred to the first few chapters of [50].

2.4.1 The classical method

The first criterion for the existence of a Spin^c structure is a differential geometric one [31], [4].
In definition D.13, appendix D, we show how to construct a cochain complex taking values in any Abelian group G from a chain complex. Let $\{C_i(M); i \in \mathbb{N}\}$ be the homology groups of M.[3] We furthermore write $C^i(M;G) \equiv \mathrm{Hom}(C_i(M), G)$. Quoting from [23, 3.E], an application of the zig-zag lemma, cf. lemma E.4 in appendix E, yields:

Lemma 2.42. *The SES of Abelian groups*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \longrightarrow 0$$

defines a long exact sequence

$$\cdots \longrightarrow H^i(M;\mathbb{Z}) \longrightarrow H^i(M;\mathbb{Z}) \longrightarrow \tag{2.79}$$

$$H^i(M;\mathbb{Z}_2) \xrightarrow{\beta} H^{i+1}(M;\mathbb{Z}) \longrightarrow \cdots$$

where β is called the ***Bockstein morphism****.*

TM, and in fact every vector bundle on M, defines a set of elements $\left\{w_i(TM) \in H^i(M;\mathbb{Z}_2)\right\}$, called the **Stiefel-Whitney classes**. See [34, §4] for a detailed treatment. When $p \geq 2$ it turns out, see [31, Ch.2, App. D], that M supports a Spin^c structure if and only if

$$W_3\left(TM\right) \equiv \beta\left(w_2(TM)\right) \in H^3(M;\mathbb{Z}) \tag{2.80}$$

is equal to zero. The different Spin^c structures, if they exist, are in bijective correspondence with the elements in

$$2H^2(M;\mathbb{Z}) \oplus H^1\left(M;\mathbb{Z}_2\right). \tag{2.81}$$

Example 2.43 (S^4)**.** *The 4-sphere is homotopic to a 4-simplex, so*

$$H_1\left(S^4;\mathbb{Z}\right) = H_2\left(S^4;\mathbb{Z}\right) = H_3\left(S^4;\mathbb{Z}\right) = 0,$$
$$H_0\left(S^4;\mathbb{Z}\right) \cong H_4\left(S^4;\mathbb{Z}\right) \cong \mathbb{Z}.$$

Using Poincaré duality, we see that $H^1\left(S^4\right) = H^2\left(S^4\right) = H^3\left(S^4\right) = 0$, so the 4-sphere has a unique Spin^c structure.

[3] As a consequence of the Handle Theorem [48, Ch. 6], every compact manifold is a finite CW complex. Therefore, the homology of a CW complex is meant here. Several manifolds (such as the circle and the 4-sphere) are triangularizable, i.e., homotopically equivalent to a simplicial complex. In those cases also the simplicial homology is meant.

The Spin^c structure of the circle is more interesting. In the example below, we explicitly use the classical method to examine the constituents of the Spin^c structure. Later, when discussing the noncommutative variant of definition 2.41, we use a different method to identify the Spin^c structure on the circle.

Example 2.44. *Since $H^1\left(S^1;\mathbb{Z}_2\right) \cong \mathbb{Z}_2$ and $H^2(M;\mathbb{Z}) = 0$, the circle has two inequivalent Spin^c structures. Recall that Spin$^c_1 \cong U(1)$. We identify the two Spin^c structures as follows. The first structure is defined by the trivial principal fiber bundle*

$$\mathfrak{Q} \equiv S^1 \times U(1). \tag{2.82}$$

Let $\mathfrak{P}$ be the special orthogonal frame bundle on M. Since $SO_1 \equiv \{e\}$, $\mathfrak{P}$ is the trivial bundle $M \times \{e\}$. The principal fiber bundle map

$$f : \mathfrak{Q} \to \mathfrak{P} \times U(1) \cong M \times U(1) \tag{2.83}$$

of equation (2.78) is given by

$$f(\exp(2\pi i\theta), p) = f(\exp(4\pi i\theta), p) \qquad \theta \in [0,1), p \in S^1. \tag{2.84}$$

Note that on fibers $\ker f = \mathbb{Z}_2$, as it should, so the principal fiber bundle $\mathfrak{O}$ in equation (2.78) is given by:

$$\mathfrak{O} \equiv \mathbb{Z}_2 \times M. \tag{2.85}$$

The second Spin^c structure on the circle is the "twisted" version. We first describe the total space of the bundle $\mathfrak{Q}$ in equation (2.78).
Take the space $U(1) \times [0,1]$. We identify the edges of this tube as follows:

$$(\exp(2\pi i\theta), 0) \sim (\exp(2\pi i\theta + \pi i), 1)\,. \tag{2.86}$$

The quotient space, which we shall denote with $\mathfrak{Q}$, is a structure similar to the Möbius strip, which happens to be a vector bundle associated to the principal bundle also described here. This space is a principal fiber bundle with $U(1)$ as structure group and the circle as base space. The map $f : \mathfrak{Q} \to \mathfrak{P} \times U(1)$ in equation (2.78) is given by

$$f([\exp(2\pi i\theta), p]) = \exp(2\pi i\theta, p), \qquad \theta \in [0,1), p \in S^1. \tag{2.87}$$

Its kernel, on fibers, is again the group $\mathbb{Z}_2$. Instead of a direct product, the principal fiber bundle $\mathfrak{O}$ in equation (2.78) is given by the edge of the Möbius strip.

2.4.2 Plymen's criterion

A second way of describing a Spin^c structure is an algebraic approach described in the 1986 article of Plymen [39]. Before introducing it, we define what is meant by a C^*-algebra bundle, which is basically a specific type of infinite-dimensional vector bundle.

Definition 2.45 (Bundle of C^*-algebras)**.** *Let A be a C^*-algebra. A bundle of C^*-algebras $\mathfrak{B}$ over a manifold M is a topological space with the following properties:*

- *There is a surjective map $\pi : \mathfrak{B} \to M$ called the **projection** of the bundle;*
- *There is a cover $\{U_i; i \in I\}$ of M and a set of homeomorphisms $\varphi_i : \pi^{-1}(U_i) \to U_i \times A$ called the **coordinate functions**;*
- *For each $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, the map*

$$\varphi_i \circ \varphi_j^{-1} : U_i \cap U_i \times A \to U_i \cap U_j \times A \tag{2.88}$$

  is a diffeomorphism of the submanifold $U_i \cap U_j$ in one variable and an automorphism of A in the other.

*We also refer to $\mathfrak{B}$ as the **total space** of the bundle. If the C^*-algebra involved is finite dimensional, it makes sense to equip $\mathfrak{B}$ with a manifold structure and demand that the projection and the coordinate functions are smooth maps.*

Recall [34, §3] that any functor $T : \mathbf{FVec}_{\mathbb{F}} \to \mathbf{FVec}_{\mathbb{F}}$ can be used to construct a new vector bundle from (co)tangent space, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The fibers of the newly created vector bundle are equal to the image of the fibers of (co)tangent space under the functor T. In particular, the Clifford algebra $\mathbb{C}\ell_p^{(+)}$ can be described by a functor (see definition 2.29). By using a Riemannian metric on cotangent space, we can construct a C^*-algebra bundle whose fibers are the C^*-algebras $\mathbb{C}\ell_p^{(+)}$, where p is the dimension of the manifold. See [21, Ch. 9.1] for further details. In a nutshell, the construction of this C^*-algebra bundle is done by composing the functor $\mathbb{C}\ell_p^{(+)}$ with the coordinate functions of the cotangent bundle as to form the coordinate functions of the C^*-algebra bundle. So the role of V in $\mathbb{C}\ell(V, g)$ is played by the T_x^*M with $x \in M$.
This C^*-algebra bundle is called the **Clifford bundle** and is essentially unique, given the manifold M. We denote this bundle by

$$\mathbb{C}\ell^{(+)}(M). \tag{2.89}$$

We furthermore denote the set of continuous and smooth sections of the Clifford bundle by

$$B(M) \equiv \Gamma\left(M, \mathbb{C}\ell^{(+)}(M)\right), \tag{2.90}$$

$$\mathcal{B}(M) \equiv \Gamma^{\infty}\left(M, \mathbb{C}\ell^{(+)}(M)\right)., \tag{2.91}$$

respectively. We can equip $\mathcal{B}(M)$ and $B(M)$ with a norm as follows. Let $\|\cdot\|$ be the norm as defined in lemma 2.34. The association

$$b \mapsto \sup_{x \in M} \|b(x)\| \tag{2.92}$$

then defines a norm on $\mathcal{B}(M)$ and $B(M)$. By standard arguments, one verifies that the closure of $\mathcal{B}(M)$ in this norm equals $B(M)$, i.e., $\mathcal{B}(M)$ lies dense in $B(M)$.

Plymen's criterion now states:

Theorem 2.46. *M has a Spin^c structure if and only if $B(M)$ is Morita equivalent to the C^*-algebra of continuous functions on M.*

We refer to appendix C for definitions regarding Hilbert modules and Morita equivalence. If a manifold has a Spin^c structure, we denote the equivalence bimodule implementing the Morita equivalence between $B(M)$ and $C(M)$ with S. We shall also refer to S as the **continuous spinors**. More on this nomenclature later.
Note that in NCG, theorem 2.46 is taken to be the definition of a Spin^c structure of a manifold.

2.4.3 Spin structures

Similar to a Spin^c structure, we can also define a Spin structure on a manifold M.

Definition 2.47 (Spin structure)**.** *Let $\mathfrak{Q}$ be the special orthogonal frame bundle associated to a manifold M. M is equipped with a **Spin structure** if and only if there is a principal fiber bundle $\mathfrak{P}$ having Spin$_p$ as its structure group, such that the following SES of principal fiber bundle maps exists:*

$$\{1\} \times M \longrightarrow \mathfrak{O} \longrightarrow \mathfrak{P} \longrightarrow \mathfrak{Q} \longrightarrow \{1\} \times M\ , \tag{2.93}$$

where $\mathfrak{O}$ is a principal fiber bundle having $\mathbb{Z}_2$ as its structure group.

The classical criterion to the existence of SES (2.93) is the condition that $w_2(TM) = 0$ [31, Ch. 2, Thm. 1.7]. An equivalent result is given by the following theorem, [21, Thm. 9.6], which is a global version of lemma 2.31.

Theorem 2.48. *One has $w_2(TM) = 0$, i.e., M admits a Spin structure if and only if there is a bijective anti-linear map $C : S \to S$ such that for all $\psi, \varphi \in S$, $f \in C(M)$ and $b \in B(M)$:*

$$C(\psi f) = C(\psi)\overline{f}; \tag{2.94}$$
$$C(b\psi) = \kappa(b)C\psi; \tag{2.95}$$
$$g(C\varphi, C\psi) = g(\psi, \varphi), \tag{2.96}$$

where g is a Hermitian metric on S.

Sketch of proof: The collection of "dual" sections

$$S^\sharp \equiv \mathrm{Hom}_{C(M)}(S, \mathbb{C}) \tag{2.97}$$

is a $B(M)$-$C(M)$-bimodule, with the actions of $B(M)$ and $C(M)$ defined point-wise by the actions of $\mathbb{C}\ell_p^{(+)}$ and $\mathbb{C}$ on $\Delta^\sharp$, as in lemma 2.31. Identical to the proof of that lemma, a global charge conjugation operator exists if and only if there exists a $B(M)$-$C(M)$-bimodule isomorphism

$$S \cong S^\sharp. \tag{2.98}$$

Such a bimodule isomorphism always exists between the fibers of the spinor bundle and its dual, as illustrated in lemma 2.31. The obstruction to the existence of a global bimodule isomorphism is the element $w_2(TM) \in H^2(M; \mathbb{Z}_2)$, see [21, Thm. 9.6] and the rest of [21, Ch. 9]. In other words, such an isomorphism exists if and only if the second Stiefel-Whitney class $w_2(TM)$ is equal to zero.

□

Note that the grading operator χ extends globally to an operator with the same name. Just as in the local case, χ grades the spinor bundle globally. To summarize, the moral of the story is [55]:

$$\text{Spin}^c \text{ structure} + \text{global charge conjugation} = \text{Spin structure}.$$

2.5 Dirac operators

Let M be a closed and connected Riemannian spin manifold of dimension p. We also refer to M as a **spin manifold**. For simplicity of exposition, throughout this section, assume that we have fixed a Spin structure on M. We denote the spinor bundle associated to the Spin structure by $\mathfrak{S}$.
The continuous and smooth functions on M are denoted by $A = C(M)$ and $\mathcal{A} = C^\infty(M)$, respectively. We shall furthermore use the following

notations:

$$S \equiv \Gamma(M, \mathfrak{S}); \qquad \textbf{(continuous spinors)} \tag{2.99}$$

$$\mathcal{S} \equiv \Gamma^{\infty}(M, \mathfrak{S}); \qquad \textbf{(smooth spinors)} \tag{2.100}$$

$$c : B(M) \to \Gamma(M, \mathrm{End}_A(\mathfrak{S})), \qquad \textbf{(Clifford multiplication)} \tag{2.101}$$

where the latter are isomorphic as C^*-algebras.

Definition 2.49 (Spin connection)**.** *By the Serre-Swan theorem [52] (see [28] for the smooth version), $\mathcal{S} \cong p\mathcal{A}^n$ for some projection $p \in M_n(\mathcal{A})$. Hence we can associate to the spinor bundle a generalized Levi-Civita connection (see appendix F for definitions) called the* ***spin connection****:*

$$\nabla^{\mathfrak{S}} : \mathcal{S} \to \mathcal{S} \otimes_{\mathcal{A}} \Omega^1(M), \tag{2.102}$$

defined by

$$\nabla^{\mathfrak{S}} \equiv pd, \tag{2.103}$$

where d is the exterior derivative.

We are now ready for our main definition, the **canonical Dirac operator**.

Definition 2.50 (Canonical Dirac operator)**.** *Combining the spin connection with the Clifford multiplication c, we obtain a complex-linear operator $D\!\!\!/ : \mathcal{S} \to \mathcal{S}$ given by*

$$D\!\!\!/ = -i(\hat{c} \circ \nabla^S), \qquad \hat{c}(v \otimes \psi) \equiv c(v)(\psi) \quad \forall \psi \in \mathcal{S}. \tag{2.104}$$

There is a canonical representation of the algebra of smooth, complex-valued functions on M.

Definition 2.51. *Let $\mathcal{H} \equiv L^2(M, \mathfrak{S})$ be the Hilbert space of square integrable spinors on M. There is a representation $\rho : C^{\infty}(M) \to B(\mathcal{H})$ given by*

$$(\rho(f)\psi)(x) = f(x)\psi(x) \tag{2.105}$$

for all $x \in M$, $f \in C^{\infty}(M)$ and $\psi \in \mathcal{H}$. Whenever this representation will be implied we simply write $f\psi$ instead of $\rho(f)\psi$.

For the cases $M = S^1$ or $M = S^4$, let us calculate the local expression of the Dirac operator using equation [21, (9.20)] and evaluate some of its properties.

Example 2.52 (S^1)**.** *We choose the Spin structure on S^1 corresponding to the trivial spinor bundle, i.e., $\mathcal{S} \cong C^\infty\left(S^1\right)$, so the projection associated to the set of smooth spinors is the unit in $C^\infty\left(S^1\right)$. Relative to a coordinate x, the action of the spin connection on spinors is locally given by*

$$\nabla^{\mathfrak{S}}\psi \equiv d\psi = \frac{d\psi}{dx}dx. \qquad \forall \psi \in \mathcal{S}. \tag{2.106}$$

$c(d\psi)$ is the identity operator on $\mathbb{C}$, so the action of the Dirac operator is locally given by

$$\not{D}\psi = -i\frac{d}{dx}\psi. \tag{2.107}$$

We see that the Dirac operator is an elliptic first-order differential operator. It symbol is given by the identity operator on $\mathbb{C}$, which coincides with the action of $c(dx)$. Using the expression of the charge conjugation operator and the chirality element we have found in example 2.33, we establish the following commutation relations.

$$\begin{aligned} &\left\{C, \not{D}\right\} = 0; \\ &\left[\chi, \not{D}\right] = 0. \end{aligned} \tag{2.108}$$

Take some $f \in C^\infty\left(S^1\right)$ and some square integrable spinor ψ. The commutator $[\not{D}, f]$, seen as a multiplication operator on the collection of smooth spinors, satisfies the following property:

$$[\not{D}, f]\psi = -i\frac{d}{dx}(f\psi) + if\frac{d}{dx}\psi = -i\left(\frac{d}{dx}f\right)\psi. \tag{2.109}$$

This implies that for all $f \in C^\infty\left(S^1\right)$:

$$[\not{D}, f] = -ic(df) = -i\sigma^{\not{D}}(\cdot, df). \tag{2.110}$$

Take some Hermitian metric g on the circle. This yields the following $\mathbb{C}$-valued inner product on $\mathcal{S}$:

$$\langle\psi, \varphi\rangle \equiv \int_{S^1} g(\psi, \varphi)\nu. \tag{2.111}$$

The Dirac operator $\not{D}$ is symmetric with respect to this inner product, since the circle has no boundary. Due to the compactness of S^1, the Dirac operator has a unique self-adjoint extension to the Hilbert space of square *integrable spinors on S^1 with respect to the inner product (2.111) [40, VIII.2]. Note furthermore that the C^1-spinors are dense in this Hilbert space. Defining:*

$$\not{D}^{-1}\psi \equiv 0 \qquad \forall \psi \in \ker \not{D}, \tag{2.112}$$

we obtain a well-defined inverse of the Dirac operator given by

$$\not{D}^{-1}\psi(x) = i \int_0^1 1_{[0,x]}\psi(y)dy, \tag{2.113}$$

where we have identified $C^\infty(S^1)$ with $\{f \in C^\infty([0,1]); f(0) = f(1)\}$. The inverse of the Dirac operator is an example of a Fredholm integral operator and, as such, is a compact operator.

We know highlight the differences of the Dirac operator on the circle associated to twisted spinor bundle versus the trivial one. As announced, this time we describe the Spin^c structure in light of theorem 2.46, instead of definition definition 2.41, which is more geometric in nature.

Example 2.53. *Instead of the more geometric description given in example 2.44, we shall regard the spinor bundle S' as a $B\left(S^1\right)$-$C\left(S^1\right)$ equivalence bimodule. Define*

$$S' \equiv \{\psi : [0, 2\pi) \to \mathbb{C}, \psi \textit{ continuous}, \psi(x) = -\psi(2\pi + x)\}. \tag{2.114}$$

We shall first show that S' is a right-$C\left(S^1\right)$-Hilbert module. The action of any $f \in C\left(S^1\right)$ on S' is simply given point-wise (right-)multiplication. The inner product $\langle \cdot, \cdot \rangle : S' \times S' \to C\left(S^1\right)$ of the Hilbert module is defined using any metric g on the circle:

$$\langle \psi, \varphi \rangle(x) \equiv g_x(\psi(x), \varphi(x)) \qquad \forall x \in S^1. \tag{2.115}$$

It is rather straightforward to verify that with this inner product, S' is indeed a right-$C\left(S^1\right)$-Hilbert module.
We now show $End^0_{C(S^1)}\left(S'\right) \cong B\left(S^1\right)$. S' is a one-dimensional bundle. The compact $C\left(S^1\right)$-endomorphisms of S' are therefore given by the collection of continuous sections of the complex trivial, 1-dimensional bundle over S^1, acting on S' by point-wise left-multiplication.
Recall that the Clifford bundle is constructed using the cotangent bundle. The fibers of the Clifford bundle are the Clifford algebra of the fibers of the cotangent bundle. In case of the circle, the cotangent bundle is a trivial, 1-dimensional bundle. As we have seen in example 2.23, the Clifford algebra of $V \cong \mathbb{R}$ is given by

$$\mathbb{C}\ell_1 \cong \mathbb{C} \oplus \mathbb{C}.$$

This implies that

$$\mathbb{C}\ell_1^{(+)} \cong \mathbb{C}.$$

Hence, $B\left(S^1\right)$ *is given by the continuous sections of the trivial bundle* $S^1 \times \mathbb{C}$. *By defining the action of* $B\left(S^1\right)$ *on* S' *by point-wise left-multiplication, we see that* $End^0_{C(S^1)}(S') \cong B\left(S^1\right)$. *As a consequence,* S' *is the imprimitivity bimodule implementing the Morita equivalence between* $C\left(S^1\right)$ *and* $B\left(S^1\right)$*, and hence called the collection of continuous spinors.*
Locally, the Dirac operator of the twisted Spin structure is given by

$$D \equiv -i\frac{d}{dx}, \tag{2.116}$$

for the local coordinate x. *This Dirac operator has the same properties and inverse as the one associated to the trivial Spin structure (say, the ordinary Dirac operator). However, a crucial difference lies within its spectrum. Whereas the eigenfunctions and the eigenfunctions of the ordinary Dirac operator are given by the sequences*

$$\left\{e^{2\pi inx} \in C\left(S^1\right), n \in \mathbb{Z}\right\}, \qquad \{2\pi in, n \in \mathbb{Z}\}, \tag{2.117}$$

respectively, the eigenfunctions and eigenvalues of the twisted Dirac operator are given by:

$$\left\{e^{2\pi i(n+1/2)x} C\left(S^1\right), n \in \mathbb{Z}\right\}, \qquad \{2\pi i(n+1/2), n \in \mathbb{Z}\}. \tag{2.118}$$

Example 2.54 (S^4)**.** *For the 4-sphere we only have one Spin structure to our disposal. Let us calculate the local expression of the Dirac operator using equation [21, (9.20)]. Namely, in terms of a local parametrization by spherical coordinates* $(\theta_1, \theta_2, \theta_3, \theta_4) \in [0, 2\pi)^3 \times [0, \pi) \equiv U$*, the Dirac operator is given by the expression*

$$\begin{aligned} \not{D} = -i &\begin{pmatrix} 0 & 0 & f_3\partial_3 - if_4\partial_4 & f_1\partial_1 - if_2\partial_2 \\ 0 & 0 & f_1\partial_1 + if_2\partial_2 & -f_3\partial_3 - if_4\partial_4 \\ f_3\partial_3 + if_4\partial_4 & f_1\partial_1 - if_2\partial_2 & 0 & 0 \\ f_1\partial_1 + if_2\partial_2 & -f_3\partial_3 + if_4\partial_4 & 0 & 0 \end{pmatrix} \\ &- i\sum_{i=1}^{4} g_i\gamma^i \equiv -i\left(\not{D}^+ + \not{D}^-\right) - i\sum_{i=1}^{4} g_i\gamma^i. \end{aligned} \tag{2.119}$$

Note that

$$\left(D^+\right)^* = D^-. \tag{2.120}$$

Here, the functions f_i, g_i are real smooth functions given locally by the following expressions. Note that, with the exception of g_4, they are non-zero on U.

$$
\begin{aligned}
f_1 &= 1, \\
f_2 &= \frac{1}{\sin\theta_1}, \\
f_3 &= \frac{1}{\sin\theta_1 \sin\theta_2}, \\
f_4 &= \frac{1}{\sin\theta_1 \sin\theta_2 \sin\theta_3}, \\
g_1 &= -\frac{1}{4}\left[3\cot\theta_1 + 2\frac{\cot\theta_2}{\sin\theta_1} + \frac{\cot\theta_3}{\sin\theta_1\sin\theta_2} + \frac{\cot\theta_3}{\sin\theta_1\sin\theta_2\sin\theta_3} - \right. \\
&\quad \frac{\cot\theta_1 + \cos\theta_1\sin\theta_1}{\sin\theta_1} - \frac{\cot\theta_1 + \cos\theta_1\sin\theta_1\sin^2\theta_2}{\sin\theta_1\cos\theta_1} - \\
&\quad \left.\frac{\cos\theta_1\sin\theta_1\sin^2\theta_2\sin^2\theta_3 + \cot\theta_1}{\sin\theta_1\sin\theta_2\sin\theta_3}\right], \\
g_2 &= \frac{1}{4}\left[\frac{\cot\theta_2 + \cos\theta_2\sin\theta_2}{\sin\theta_1\sin\theta_2} + \frac{\cot\theta_2 + \cos\theta_2\sin\theta_2\sin^2\theta_3}{\sin\theta_1\sin\theta_2\sin\theta_3}\right], \\
g_3 &= \frac{1}{4}\frac{\cot\theta_3 + \cos\theta_3\sin\theta_3}{\sin\theta_1\sin\theta_2\sin\theta_3}, \\
g_4 &= 0.
\end{aligned}
$$

The Dirac operator on the 4-sphere satisfies the same commutation relations with the chirality element and the charge conjugation as the Dirac operator on the circle. The principal symbol of the Dirac operator is locally given by

$$
\sigma^{\not{D}}(x,\cdot) = \begin{pmatrix} 0 & 0 & f_3 - if_4 & f_1 - if_2 \\ 0 & 0 & f_1 + if_2 & -f_3 - if_4 \\ f_3 + if_4 & f_1 - if_2 & 0 & 0 \\ f_1 + if_2 & -f_3 + if_4 & 0 & 0 \end{pmatrix} \Rightarrow \tag{2.121}
$$
$$
\left[\sigma^{\not{D}}\right]^{-1}(x,\cdot) = \frac{1}{f_1^2 + f_2^2 + f_3^2 + f_4^2}\sigma^{\not{D}}(x,\cdot).
$$

Again, we see that the Dirac operator is elliptic and that, as a multiplication operator on the smooth spinors:

$$
[\not{D}, f](x) = -ic(df)(x) = -i\sigma^{\not{D}}(x, df(x)), \tag{2.122}
$$

for all $f \in C^\infty(S^4)$.

Instead of verifying that the Dirac operator on the 4-sphere has the same functional analytic properties as we have described for the example of the

1-sphere, we quote the general theorem that establishes these properties for any Dirac operator defined on any spin manifold without boundary. Note that the aforementioned examples might suggest that the commutation relations (2.108) always hold. It turns out, however, that the commutation relations are dependent on the dimension of the manifold. From [20, Ch. 3] and [21, 9.5], we now have

Theorem 2.55. *Let M be a boundaryless spin manifold. Choose a Spin structure on M and fix a Hermitian metric g on the spinor bundle associated with the Spin structure. Let $\not{D}$ be the canonical Dirac operator defined with respect to that Spin structure. Let $\mathcal{H}$ be the Hilbert space of square-integrable spinors relative to g. We have the following results:*

- *$\not{D}$ is a closed and essentially self-adjoint (therefore self-adjoint) operator on $\mathcal{H}$ with dense domain;*
- *$\not{D}$ is elliptic and, as a multiplication operator on $\mathcal{H}$, we have the identity:*

$$[\not{D}, f] = -ic(df) = -i\sigma^{\not{D}}(\cdot, df) \tag{2.123}$$

  for all $f \in C^\infty(M)$;

- *We define the inverse of the Dirac operator to be zero on the kernel of $\not{D}$. This inverse is a compact operator on $\mathcal{H}$;*
- *The (anti)-commutation relations of $\not{D}$ with C and γ are given by*

$$C\not{D} = \begin{cases} \not{D}C & p = 0,2,3,4,6,7 \\ -\not{D}C & p = 1,5 \end{cases} \tag{2.124}$$

$$\gamma\not{D} = \begin{cases} \not{D}\gamma & p \textit{ odd} \\ -\not{D}\gamma & p \textit{ even} \end{cases} \tag{2.125}$$

Chapter 3

Spectral triples

In this chapter we define the basic element of post-1995 noncommutative geometry,[1] namely the **spectral triple**, and define several additional properties which are elevated to the status of axioms. It will turn out that for any *commutative* spectral triple satisfying all of these (eight) axioms, the involutive algebra involved is isomorphic to $C^\infty(M)$, the algebra of smooth functions on a closed and connected Riemannian spin manifold. This is Connes' celebrated **reconstruction theorem** as presented in [14] and [21, Ch. 11]. The proof basically consists of two steps. The first step is to show that the spectrum of the algebra is a compact connected manifold. This is taken up in chapter 4. The second part, discussed in chapter 5, shows that the manifold is orientable, has no boundary, and is equipped with a spin structure.

The algebra $\mathcal{A}$, which in the commutative case turns out to be equal to $C^\infty(M)$, is defined, in the spirit of the Gelfand-Naimark-Segal construction [25, Th. 4.5.2], through a representation on a given Hilbert space $\mathcal{H}$. To define a spectral triple, we add to the doublet $(\mathcal{A}, \mathcal{H})$ a third mathematical object, the **abstract Dirac operator** D, which is an (unbounded) operator on (some domain within) the Hilbert space. Together, these three objects form the building blocks of a spectral triple. We shall start with a definition of the spectral triple, and explain its relation to the concepts introduced in chapter 2. Subsequently, we shall lay out the additional axioms of the spectral triple and motivate them.

The axioms are grouped as follows. In the next section we will present the axioms that are related to the properties of the abstract Dirac operator and the way the Dirac operator interacts with the other two objects of the spectral triple. In section 3.2 we focus on several bounded operators on $\mathcal{H}$. The last two axioms, discussed in section 3.3, equip $\mathcal{H}$ with additional

[1] In 1995 Connes published the article "Noncommutative and reality" [13], which contains an early exposition on spectral triples and their relations with spin geometry and the Standard Model of high energy physics.

structure. The discussion will rely on the existence of an "integral" $⨍ : \mathcal{A} \to \mathbb{C}$. In section 3.3 we will therefore define this integral and show it has the required properties, except for positive definiteness. One uses the fact that $⨍$ is positive definite in the last step of the reconstruction theorem to show that the manifold constructed in that theorem is a spin manifold. It therefore seems appropriate to postpone the proof of positive definiteness to chapter 5.

3.1 Spectral triples

Definition 3.1 (Spectral triple)**.** *A triple $(\mathcal{A}, D, \mathcal{H})$ is called a **spectral triple** when:*

- *$\mathcal{H}$ is a separable Hilbert space;*
- *$\mathcal{A}$ is a unital and involutive algebra that has a faithful *-representation π on $\mathcal{H}$;*
- *D is an unbounded, self-adjoint operator $D : \text{dom } D \to \mathcal{H}$ such that $(D - i)^{-1}$ is compact and dom D is dense in $\mathcal{H}$. Note that due to self-adjointness, D is also a closed operator. D is also referred to as an **abstract Dirac operator**, or just as a **Dirac operator**.;*
- *For all $a \in \mathcal{A}$, $[D, a] \in B(\mathcal{H})$.*

*The spectral triple is called **commutative** when $\mathcal{A}$ is (as an algebra). We furthermore define D^{-1} as zero on the kernel of D and as the inverse of D on $(\ker D)^{\perp}$.*
Denote the inner product on the Hilbert space by

$$(\cdot, \cdot)_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \tag{3.1}$$

and let

$$\|\xi\|^2_{\mathcal{H}} \equiv (\xi, \xi)_{\mathcal{H}}. \tag{3.2}$$

Last, we define:

$$\mathcal{H}^{\infty} \equiv \bigcap_{i=1}^{\infty} \text{dom } D^i. \tag{3.3}$$

Remark 3.2. *When the action of $\mathcal{A}$ on $\mathcal{H}$ is implied we shall simply write $a\xi$ instead of $\pi(a)\xi$.*
The map

$$a \mapsto \|\pi(a)\|_{op} \equiv \sup_{\|\xi\|_{\mathcal{H}}=1} \{\|a\xi\|_{\mathcal{H}}\} \tag{3.4}$$

defines a norm on $\mathcal{A}$, denoted by $\|a\|$. Its completion is a commutative unital C^*-algebra, denoted by A. By Gelfand's theorem [25, 4.4.3] $A \cong C(Spec\ A)$, with Spec A a compact Hausdorff space.

Let us outline how the spin manifold S^1 gives rise to a commutative spectral triple.

Example 3.3 (S^1)**.** *The commutative, unital and involutive algebra $\mathcal{A} \equiv C^\infty\left(S^1\right)$ lies dense in the C^*-algebra of continuous functions on S^1. For the Spin structure on S^1 we choose the trivial principal fiber bundle $S^1 \times Spin_1 \cong S^1 \times \mathbb{Z}_2$, so that the spinor bundle $\mathfrak{S}$ is isomorphic to $S^1 \times \mathbb{C}$. The set of square-integrable sections of the spinor bundle $\mathcal{H} \equiv L^2(M, \mathfrak{S})$ is a Hilbert space on which $\mathcal{A}$ has a faithful representation given by*

$$(f \cdot \psi)(x) = f(x)\psi(x) \qquad \forall f \in \mathcal{A},\ \forall \psi \in \mathcal{H}. \tag{3.5}$$

The abstract Dirac operator $D : dom\ D \subset \mathcal{H} \to \mathcal{H}$ is given by the canonical Dirac operator of the Spin structure on S^1, which is locally given by $\not{D} \equiv -i\frac{d}{dx}$ on a coordinate chart (x, U) of S^1. As we have seen in the previous chapter, this operator satisfies all the properties in definition 3.1. The domain of the Dirac operator is the Hilbert space of the square-integrable spinors on S^1. Using the local description of the Dirac operator one sees that $\mathcal{H}^\infty$ is given by the collection of smooth spinors $\Gamma^\infty(M, \mathfrak{S})$.

We return to the general case. Just as smooth spinors are dense in the Hilbert space of square-integrable ones, the definition of the spectral triple has the following implication:

Lemma 3.4. *$\mathcal{H}^\infty$ is dense in $\mathcal{H}$.*

Before stating the first three axioms of a commutative spectral triple, let us quote a result regarding the modulus of the Dirac operator. Though not all unbounded operators admit a polar decomposition, the closed ones do. From [40, Th. VIII.32]:

Theorem 3.5 (Polar decomposition)**.** *There is a unique positive (and hence self-adjoint operator) $|D| : dom\ D \to \mathcal{H}$ and a partial isometry U on $\mathcal{H}$ with initial space $(\ker D)^\perp$, final space $\overline{ran\ D}$, and $\ker|D| = \ker D$, such that*

$$D = U|D|. \tag{3.6}$$

Axiom 1 (Finite summability)**.** *$|D|^{-1}$ is a compact operator. Its eigenvalues form a sequence $\left\{s_n\left(|D|^{-1}\right) \in \mathbb{R}^+, n \in \mathbb{N}\right\}$. By compactness, for large n this is a decreasing sequence. We demand that there is a $p \in \mathbb{N}$ such that the n-th eigenvalue in that sequence is of order $\mathcal{O}\left(n^{-\frac{1}{p}}\right)$ as $n \to \infty$. The smallest such p is called the **dimension** of the spectral triple.*

We illustrate this axiom by analyzing the eigenvalues of $|\not{D}|^{-1}$, where $\not{D}$ is a Dirac on the circle.

Example 3.6 (S^1)**.** *As seen in the previous chapter, D^{-1} is given by*

$$D^{-1}\psi(x) \equiv i \int_0^1 1_{[0,x]}\psi(y)dy \tag{3.7}$$

when ψ is not in the kernel of D, and D^{-1} is zero otherwise (i.e., on constants). D^{-1} is a compact operator. Since $\mathcal{H}$ is isomorphic (as a Hilbert space) to the set of square integrable functions on the circle, a base for $\mathcal{H}$ is given by the set

$$\left\{e_n : S^1 \to \mathbb{C}; e_n(x) = \exp(2\pi i n x), n \in \mathbb{Z}\right\}. \tag{3.8}$$

The eigenvalues of D^{-1} with respect to this base form the sequence

$$\left\{\frac{1}{2\pi n}; n \in \mathbb{Z}/\{0\}\right\},$$

implying the eigenvalues of $|D|^{-1}$ are given by the sequence

$$\left\{\frac{1}{2\pi n}; n \in \mathbb{N}/\{0\}\right\} \tag{3.9}$$

with each of the eigenvalues having a multiplicity of 2. The n-th eigenvalue is of order $n^{-\frac{1}{1}} = n^{-1}$, so that the dimension of the spectral triple $\left(C^\infty\left(S^1\right), \not{D}, L^2\left(S^1, \mathfrak{S}\right)\right)$ equals the dimension of the circle, namely $p = 1$.

For a spin manifold M, the spectral triple is given by $\left(C^\infty(M), \not{D}, L^2(M, \mathfrak{S})\right)$. Any spin manifold M satisfies axiom 1 with $p = \dim M$. This follows from the fact that $\not{D}$ is the formal square root of the spinor Laplacian modulo a multiple of the scalar curvature. This is the Lichnerowitz formula [21, Th. 9.16]. The behavior of the eigenvalues of $|\not{D}|^{-1}$ then follows from the behavior of the eigenvalues of the spinor Laplacian, which are similar to those of the scalar Laplacian [55, Ch. 3.5] and are given by Weyl's theorem.
Relationship (2.123) implies that the commutator of the canonical Dirac operator with any smooth function on the spin manifold is a section of the endomorphism bundle of the spinor bundle. Such a section is pointwise $\mathbb{C}$-linear and globally $C^\infty(M)$-linear. This implies that

$$[[\not{D}, f], g] = -i[c(df), g] = 0. \tag{3.10}$$

We will require this property to hold for noncommutative spectral triples as well:

Axiom 2 (First-order)**.** *for all* $a, b \in \mathcal{A}$*,* $[[D, a], b] = 0$*.*

The last axiom of this section is a very strong assumption regarding the functional analytic properties of $\mathcal{A}$ and several related algebras. It will play an important role in the definition of the noncommutative integral in section 3.3 and in the first part of the reconstruction theorem in chapter 4. In essence, it tells us which operators on $\mathcal{H}$ we can regard as being "smooth". Provided that $\mathcal{H}^\infty$ is a (right) $\mathcal{A}$-module (which we will require by imposing axiom 7):

Axiom 3 (Strong regularity)**.** *Let*

$$\delta(T) \equiv [|D|, T], \tag{3.11}$$

with

$$\text{dom } \delta = \{T \in B(\mathcal{H}); T \, \text{dom } |D| \subset \text{dom } |D|, \delta(T) \in B(\mathcal{H})\}. \tag{3.12}$$

Then $\mathcal{A}$*,* $[D, \mathcal{A}]$ *and* $\text{End}_{\mathcal{A}}(\mathcal{H}^\infty)$ *are in*

$$B^\infty(\mathcal{H}) \equiv \bigcap_{i=1}^{\infty} \text{dom } \delta^i. \tag{3.13}$$

Here δ^0 *is the identity map, with dom* $\delta^0 = B(\mathcal{H})$*.*

The predicate "strong" is used to set the above property apart from (ordinary) **regularity**, which is defined as follows.

Definition 3.7 (Regularity)**.** *A spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is said to be* ***regular*** *when* $\mathcal{A}$ *and* $[D, \mathcal{A}]$ *lie in* $B^\infty(\mathcal{H})$*.*

3.2 Orientation

In this section we focus on several bounded operators on the Hilbert space $\mathcal{H}$. One of those is the **grading operator**, which we have encountered in chapter 2 in its local form as χ in definition 2.27, and in its global form at the end of section 2.4. The 4^{th} axiom will establish a connection between the grading operator and the **noncommutative volume element**, which is an element of the top Hochschild homology group $C_p(\mathcal{A})$ of $\mathcal{A}$. See appendix D for more background information regarding de Rham cohomology and Hochschild homology.
The purpose of the noncommutative volume element is twofold. Firstly, as shall be proved in chapter 5, the volume element ensures that the manifold

is orientated. We shall briefly discuss this result (in the context of the spin manifold S^4) below. The second purpose, which we shall discuss in chapter 4, is that the volume element encodes part of the data used to reconstruct the coordinate functions of the manifold associated to a commutative spectral triple.
Let us start with stating a condition of orientability of a manifold.
From [19, Ch. 2.8]:

Lemma 3.8. *A p-dimensional manifold M is **orientable** if and only if there is a non-zero section of $\Omega^p_{\mathbb{R}}(M)$. Such a non-zero section is also referred to as a **volume form**.*

Example 3.9 (S^4)**.** *Take some coordinate chart (U, x), $U \subset S^4$, such that the cotangent bundle is locally trivial on U. Let $\{dx^1, \ldots, dx^4\}$ be a local base of the cotangent bundle on U. By construction, the Clifford bundle is also trivial on U. According to definition 2.26 and example 2.33, the local section corresponding to the chirality element can be represented by*

$$-dx^1 \cdot dx^2 \cdot dx^3 \cdot dx^4. \tag{3.14}$$

As vector bundles, $\Lambda^\bullet_{\mathbb{C}}\left(S^4\right) \cong \mathbb{C}\ell_4\left(S^4\right)$: let $p \in S^4$ and let $\pi, \tilde{\pi}$ be the projection of the complexified exterior bundle and the projection of the Clifford bundle, respectively. Take some $\left(p, dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) \in \pi^{-1}(\{p\})$ and $\left(p, dx^{i_1} \cdots dx^{i_k}\right) \in \tilde{\pi}^{-1}(\{p\})$ with $k \leq 4$ and $i_1 < \ldots < i_k$. Point-wise, on pure elements in the fiber of $\Lambda^\bullet_{\mathbb{C}}\left(S^4\right)$, the bundle isomorphism is given by

$$(p, dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \mapsto (p, dx^{i_1} \cdots dx^{i_k}), \tag{3.15}$$

and is extended to the whole fiber by $\mathbb{C}$-linearity.
Combining this isomorphism with lemma 3.8, we can conclude that the condition for S^4 to be orientable is equivalent to the condition that there is a section $\omega \in B(M)$ for which $\omega(p)$ is equal to the chirality element γ of $\mathbb{C}\ell_4$, for each $p \in S^4$.
Using the Clifford multiplication c, we can represent ω as a bounded operator on the Hilbert space of square integrable spinors $\mathcal{H}$:

$$(c(\omega)\psi)(p) = -c\left(dx^1\right) \circ c\left(dx^2\right) \circ c\left(dx^3\right) \circ c\left(dx^4\right)(\psi(p)) = \tag{3.16}$$
$$-\gamma^1 \circ \gamma^2 \circ \gamma^3 \circ \gamma^4(\psi(p)), \tag{3.17}$$

$\forall p \in U$, $\forall \psi \in \mathcal{H} = L^2\left(S^4, \mathfrak{S}\right)$. For a suitable choice of base, the action of ω locally corresponds to the grading operator:

$$(c(\omega)\psi)(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_1(p) \\ \psi_2(p) \\ \psi_3(p) \\ \psi_4(p) \end{pmatrix}. \tag{3.18}$$

Since S^4 has a Spin structure, the Clifford multiplication c is a bundle map, so the identification of ω with the grading operator carries over to a global level.
The grading operator can also be expressed using the identity (2.123). We (locally) obtain:

$$c(\omega) = -i^4[\not{D}, x^1][\not{D}, x^2][\not{D}, x^3][\not{D}, x^4] \in B(\mathcal{H}). \tag{3.19}$$

To make a link with Hochschild homology, we use the anti-symmetrization map $\mathbb{A}_4$, as defined by equation (D.19), appendix D, to locally identify an anti-symmetric Hoschchild 4-chain with the volume form on M. Locally,

$$\mathbb{A}_4\left(dx^1 \wedge \cdots \wedge dx^4\right) = \frac{1}{4!}\sum_{\sigma \in S_4} \epsilon(\sigma) x^{\sigma(1)} \otimes \cdots \otimes x^{\sigma(4)}. \tag{3.20}$$

Let $\mathcal{A} = C^\infty\left(S^4\right)$. Define a map $\pi_{\not{D}} : C_i(\mathcal{A}) \to B(\mathcal{H})$ by

$$\pi_{\not{D}}(f_0 \otimes \cdots \otimes f_i) = f_0[\not{D}, f_1]\cdots[\not{D}, f_i], \tag{3.21}$$

and extend $\pi_{\not{D}}$ by $\mathbb{C}$-linearity. Locally:

$$\pi_{\not{D}} \circ \mathbb{A}_4\left(dx^1 \wedge \cdots \wedge dx^4\right) = \frac{1}{4!}\sum_{\sigma \in S_4} \epsilon(\sigma)[\not{D}, x^1]\cdots[\not{D}, x^4] =$$

$$(-i)^4 \frac{1}{4!}\sum_{\beta \in S_4} \epsilon(\sigma) c(dx^{\sigma(1)})\cdots c(dx^{\sigma(4)}) = -c(\omega),$$

due to the fact that the base elements anti-commute under the Clifford representation.
These identifications also hold on a global level. Instead of taking the base of cotangent space on a single chart, one should use a partition of unity ψ_α subordinate to some atlas $\{(U_\alpha, x_\alpha)\}$ and apply the previous formulas to the volume form

$$\sum_\alpha \psi_\alpha dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^4 \in \Omega^4(M). \tag{3.22}$$

The corresponding element $C_4(\mathcal{A})$ is then given by

$$\frac{1}{4!}\sum_\alpha \sum_{\beta \in S_4} \epsilon(\beta)\psi_\alpha \otimes x_\alpha^{\beta(1)} \otimes \cdots \otimes x_\alpha^{\beta(4)}. \tag{3.23}$$

The previous example indicates that finding a suitable element in the top homology chain group $C_p(\mathcal{A})$, where $\mathcal{A}$ is the involutive algebra of a commutative spectral triple with dimension p, might be sufficient to show that the manifold we construct from the spectral triple is orientable. Note, furthermore, that in the previous example this element in the Hochschild

chain group consisted of sums of tensor products consisting of *all* coordinate functions. Part of the difficulty of the reconstruction theorem is obtaining these coordinate functions.
Before moving on, we need to generalize a map used in the last example.

Definition 3.10. *Let π_D be as defined in equation (3.21). For pure elements in the i-th Hochschild chain group:*

$$\pi_D : C_i(\mathcal{A}) \to B(\mathcal{H}), \qquad \pi_D(a_0 \otimes \cdots \otimes a_i) \equiv a_0[D, a_1] \cdots [D, a_i]. \quad (3.24)$$

Subsequently, π_D is extended to the whole chain group by $\mathbb{C}$-linearity. Note that axiom 3 guarantees that the right-hand side of this equation lies in $B(\mathcal{H})$.

For commutative spectral triples, we have an object equivalent to the grading operator. We relate it to an element in the top Hochschild homology class.

Axiom 4 (Orientation)**.** *Let $(\mathcal{A}, D, \mathcal{H})$ be a commutative spectral triple satisfying axioms 1 to 3.*
There is an anti-symmetric Hochschild p-cycle $\boldsymbol{c} \in Z_p(\mathcal{A})$ such that $\chi \equiv \pi_D(\boldsymbol{c})$ satisfies:

- $\chi^* = \chi$*;*
- $\chi^2 = 1$*;*
- *if p is even, then $\chi D = -D\chi$; otherwise $\chi = 1$.*

This $\boldsymbol{c}$ *is called the* ***Hochschild orientation cycle*** *or the* ***noncommutative volume form****.*
In what follows we shall denote the expansion of $\boldsymbol{c}$ in $C_p(\mathcal{A})$ by:

$$\sum_{\alpha} \sum_{\beta \in S_p} c_\alpha^0 \otimes c_\alpha^{\beta(1)} \otimes \cdots \otimes c_\alpha^{\beta(p)}, \quad (3.25)$$

which we shall often abbreviate to

$$\boldsymbol{c} = \sum_{\alpha} c_\alpha^0 \otimes c_\alpha^1 \otimes \cdots \otimes c_\alpha^p, \quad (3.26)$$

for α in some finite index set.

The second bounded operator we want to define on the Hilbert space $\mathcal{H}$ is the charge conjugation. There is no need to adjust the definition of the global charge conjugation in theorem 2.48.

Axiom 5 (Reality)**.** *There is an anti-unitary operator $C : \mathcal{H} \to \mathcal{H}$ with $C^2 = \pm 1$, $CD = \pm DC$ and $C\chi = \pm \chi C$, where the sign depends on the dimension and is given in table 3.1. Moreover, $a = Ca^*C^{-1}$. This operator C is called the* ***reality operator****.*

Table 3.1: Values of C^2 and its (anti)-commutation relations with D and χ, relative to the dimension p of the manifold.

$p \mod 8$	0	1	2	3	4	5	6	7
$C^2 = \pm 1$	+	+	-	-	-	-	+	+
$CD = \pm DC$	+	-	+	+	+	-	+	+
$C\chi = \pm \chi C$	+	+	-	+	+	+	-	+

The purpose of charge conjugation is to turn a Spin^c structure on a manifold into a Spin structure. It is therefore no surprise that the operator will play no role in this chapter and the chapter to come. It will enter the stage again in chapter 5 when we will prove that the manifold we construct from the commutative spectral triple is, in fact, a spin manifold.
The next axiom will guarantee that the manifold we construct from the commutative spectral triple is connected.

Axiom 6 (Irreducibility)**.** *There is no non-trivial projection on $\mathcal{H}$ that commutes with $\pi(\mathcal{A}), D, C$ and χ.*

Remark 3.11. *Axiom 6 is not strictly necessary. If there were such a projection, then $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ in the C^*-algebraic sense, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ as Hilbert spaces, and $D = D_1 \oplus D_2, \chi = \chi_1 \oplus \chi_2, C = C_1 \oplus C_2$ as operators. It is easily checked that for $i = 1, 2$, $(\mathcal{A}_i, \mathcal{H}_i, D_i, \chi_i, C_i)$ are two spectral triples satisfying axioms 1 - 5 and 7-8. The compactness of Spec A guarantees that by repeating this process one would eventually end up with a finite set of irreducible spectral triples satisfying all the axioms.*
We try to avoid cluttered and awkward notation as much as possible, and therefore we impose irreducibility right from the start.

3.3 The noncommutative integral

The purpose of this section is to establish an integral on $\mathcal{A}$ in the sense of a trace, which we will denote by $⨍ : \mathcal{A} \to \mathbb{C}$. We will also refer to this trace as the **noncommutative integral**, since the definition will turn out to be valid for noncommutative spectral triples as well. We cover the noncommutative integral now since we need to define it first before we are able to formulate the last two axioms.
Let us start by generalizing the usual definition of a trace a bit. See appendix C for definitions regarding positivity (in relation to involutive algebras).

Definition 3.12 (Trace)**.** *Let* A, B *be involutive algebras. A trace* $Tr : \mathsf{A} \to \mathsf{B}$ *has the following properties:*

1. $Tr(ab) = Tr(ba)$ *for any two* $a, b \in \mathsf{A}$*;*

2. Tr *is* $\mathbb{C}$*-linear;*

3. Tr *is a* ***positive*** *map, i.e., when* $a \in \mathsf{A}^+$ *then* $Tr(a) \in \mathsf{B}^+$*;*

4. Tr *is* ***non-degenerate****: if* $Tr(a) = 0$ *for any* $a \in \mathsf{A}^+$ *then* $a = 0$*.*

Together, conditions (3) and (4) are also called ***positive definiteness****.*

As stated earlier, in this section we focus on the definition of the integral and we show that it satisfies properties (1) to (3). We shall postpone the proof that $⨍$ satisfies (4) to chapter 5.
Before delving into a general formulation of the noncommutative integral we will take a look at integration of smooth functions on the circle in order to get more insight on how to proceed in the general case. The ordinary Riemannian integral on a manifold [19, Ch. 3] makes ample reference to the underlying space and its atlas. Those concepts are contrary to the philosophy of NCG, which tries to depart from spaces and their points as much as possible. Instead, we focus on algebraic and functional-analytic concepts, which rely heavily on the properties of compact operators. We will therefore first state some definitions on compact operators before proceeding.

Definition 3.13. *Let* $\mathcal{H}$ *be a infinite-dimensional, separable Hilbert space with inner product* $\langle \cdot, \cdot \rangle$ *and* $T \in K(\mathcal{H})$ *a compact operator. If* T *is self-adjoint, we can apply a special case of the* ***spectral theorem****: the eigenfunctions* $\{e_i; i \in \mathbb{N}\}$ *of* T *form an orthonormal base of* $\mathcal{H}$*. Denote the set of eigenvalues of* T *by* $\{\lambda_i(T); Te_i = \lambda_i(T)e_i\}$*. Relative to this base, there is a unique expansion*

$$T = \sum_i \lambda_i(T) p_i \tag{3.27}$$

where $\lambda_i \neq \lambda_j$ *for* $i \neq j$*,* $\lim_{i\to\infty} |\lambda_i| = 0$*, and* $p_i \in B(\mathcal{H})$ *is the finite-dimensional projection such that* $p_i = \sum_{i, \lambda_j = \lambda_i} \langle e_i, \cdot \rangle$*.*
The dimension of $p_i\mathcal{H} = Tr(p_i)$ *is called the* ***multiplicity*** *of the eigenvalue* λ_i*.*
From now on, we shall always assume that we have rearranged the orthonormal base of $\mathcal{H}$ *corresponding with* T *in such a way that* $\{|\lambda_n(T)|\}$ *is a decreasing series.*

For general compact operators $S \in K(\mathcal{H})$, we have the following unique expansion:

$$S = \sum_j s_j(S)\langle e_j, \cdot\rangle \tilde{e}_j, \tag{3.28}$$

where $\{e_j\}, \{\tilde{e}_j\}$ are two orthonormal bases of $\mathcal{H}$, and the set $\{s_j(S)\}$ is the collection of eigenvalues of $\sqrt{S^*S} \equiv |S|$. They, too, have finite multiplicity (except, possibly, the case $s_j = 0$). The $\{s_j(S)\}$ are also referred to as the ***singular values*** *or* ***characteristic values*** *of S. Just as we did with the eigenvalues we shall arrange them as a decreasing series.*
For any $T \in K(\mathcal{H})$, define the partial sum

$$\sigma_N(T) \equiv \sum_{n=0}^{N-1} s_n(T).$$

It turns out that for compact operators T a description of $\sigma_N(T)$ as a series is too limited for our purposes. We need to extend $\sigma_\bullet(T)$ to a function on the real line. For each $T \in K(\mathcal{H})$ define the function $\sigma_\lambda(T) : \mathbb{R}^+ \to \mathbb{R}^+$ by interpolation as follows:

$$\sigma_\lambda(T) = (1-t)\sigma_n(T) + t\sigma_{n+1}(T) \qquad \lambda = n + t, n = \lfloor \lambda \rfloor. \tag{3.29}$$

As announced, we now discuss an example.

Example 3.14 (S^1)**.** *As we have seen in the previous chapter, S^1 admits two Spin structures. As in example 3.3, we are interested in the one associated to a trivial spinor bundle. Let us identify both the smooth functions on the 1-sphere and the smooth spinors with the collection*

$$\{f : [0,1] \subset \mathbb{R} \to \mathbb{C}; f(0) = f(1), f \text{ smooth}\}. \tag{3.30}$$

Recall that we have the following expressions for the Dirac operator and its inverse (in terms of the local coordinate x):

$$D\!\!\!/ = -i\frac{d}{dx}, \tag{3.31}$$

$$D\!\!\!/^{-1}\psi(x) = \begin{cases} i\int_0^1 1_{[0,x]}\psi(y)dy & \psi \notin \ker D\!\!\!/ \\ 0 & \psi \in \ker D\!\!\!/ \end{cases} \tag{3.32}$$

for all smooth spinors ψ. The eigenvalues of the compact positive operator $|D\!\!\!/|^{-1}$ are given by the sequence (3.9):

$$\left\{\frac{1}{2\pi n}; n \in \mathbb{N}/\{0\}\right\}, \tag{3.33}$$

where each eigenvalue has multiplicity 2. The eigenfunctions of $|\not{D}|^{-1}$*, given by the sequence*

$$\left\{e_n; e_n(x) = e^{2\pi inx}, n \in \mathbb{Z}/\{0\}\right\}, \tag{3.34}$$

form a base of the Hilbert space of square-integrable spinors $\mathcal{H}$ *modulo the constant ones, which is isomorphic to the Hilbert space of non-constant square-integrable functions on the circle.*
Let $f \in C^\infty\left(S^1\right)$ *be a positive function. Before arriving at an alternative expression of the integral of* f*, we shall first take a look at the operator*

$$f|\not{D}|^{-s}, \qquad s \in \mathbb{C},\ \Re s > 1. \tag{3.35}$$

The eigenvalues of $|\not{D}|^{-s}$ *are summable. Since* f *is bounded, we see that the operator* $f|\not{D}|^{-s}$ *is trace-class. Therefore, the value of the trace is independent of the base used to calculate the trace. Hence, we evaluate* $Tr\left(f|\not{D}|^{-s}\right)$ *using the base* $\{e_n\}$ *described above.*

$$Tr\left(f|\not{D}|^{-s}\right) = \sum_{n\in\mathbb{Z}/\{0\}} \langle e_n, f|\not{D}|^{-s} e_n\rangle = \sum_{n\in\mathbb{Z}/\{0\}} \frac{1}{(2\pi|n|)^s}\langle e_n, f e_n\rangle. \tag{3.36}$$

Using the fact that Fourier expansion

$$f = \sum_{k\in\mathbb{Z}} a_k e^{2\pi ikx}$$

converges uniformly on the circle, we see that (3.36) is equal to:

$$\frac{2a_0}{(2\pi)^s}\sum_{n\in\mathbb{N}/\{0\}}\frac{1}{n^s} = \frac{2}{(2\pi)^s}a_0\zeta(s),$$

where $\zeta(s)$ *is the Riemann zeta function.*
We now derive the alternative expression of the integral. Let again be f *a positive smooth function on the circle and let* $\{e'_n\}$ *be some orthogonal base of* $\mathcal{H}$*. The expression*

$$\sum_{n=-N,n\neq 0}^{N} \langle e'_n, f|\not{D}|^{-1}e'_n\rangle$$

is $\log N$*-divergent. Therefore (disregarding convergence and independence of based used), we symbolically define*

$$Tr^+(f|\not{D}|^{-1}) \equiv \lim_{N\to\infty}\frac{1}{\log N}\sum_{n=-N,n\neq 0}^{N}\langle e'_n, f|\not{D}|^{-1}e'_n\rangle. \tag{3.37}$$

It turns out that [42, Thm. 12, Prop. 15] (3.37) does converge, is independent of base used and equal to:

$$\lim_{s \uparrow 1^+} (s-1)\, Tr \left(f |\not{D}|^{-s} \right). \tag{3.38}$$

As we have seen before, the last expression equals a constant times the coefficient a_0 times the value of the residue of the zeta function at $s = 1$, which equals 1. Therefore, we have (up to a constant):

$$Tr^+ \left(f |\not{D}^{-1} \right) = a_0 = \int f \nu. \tag{3.39}$$

This result is readily extended to any smooth function g on the circle by noting that any given $g \in C^\infty \left(S^1 \right)$ can be written as the sum of four positive functions.
Last, we note the following. We know that $f|D|^{-1} \in K(\mathcal{H})$. Hence, we can also see Tr^+ as a map

$$Tr^+ : dom\ Tr^+ \subset K(\mathcal{H}) \to \mathbb{C}.$$

The above example will serve as a model for defining integration on general (both commutative and noncommutative) spectral triples. Let us list some key properties of the procedure.

1. We needed a compact operator $|\not{D}|^{-1}$ on a Hilbert space that carries a representation of $\mathcal{A} = C^\infty \left(S^1 \right)$. The existence of such a Hilbert space will turn out to be necessary in general. Not surprisingly (for a spectral triple), $\mathcal{H}$ fulfills this role.

2. We defined a map $\mathrm{Tr}^+ : \mathrm{dom}\ \mathrm{Tr}^+ \subset K(\mathcal{H}) \to \mathbb{C}$;

3. We found an operator $|\not{D}|^{-1}$ lying in the domain of Tr^+. Notably, for each $a \in \mathcal{A}$, $a|\not{D}|^{-1} \in \mathrm{dom}\ \mathrm{Tr}^+$. It will turn out that the domain of Tr^+ is quite rich in structure and that, judging from the calculations done in previous example, the singular values of any compact operator will determine whether that operator lies in the domain of Tr^+.

4. The compact operator $|\not{D}|^{-1} \in K(\mathcal{H})$ is such that for $M = S^1$, for each $f \in C^\infty(M)$, we have

$$\int_M f \nu = \mathrm{Tr}^+ \left(f |\not{D}|^{-1} \right). \tag{3.40}$$

Mimicking this procedure, our goal will be to find a formulation of the integral for spectral triples such that for commutative spin geometries equation (3.40) holds. As suggested, we use a certain subclass of the compact operators on $\mathcal{H}$ as a domain for a map Tr^+, whose definition shall be our first concern. Secondly, we will show that $|D|^{-p} \in \mathrm{dom}\ \mathrm{Tr}^+$, with p the dimension of the spectral triple $(\mathcal{A}, D, \mathcal{H})$. We then wrap up with a discussion on Connes' trace theorem, which establishes (3.40) for a general canonical spectral triple associated to a spin manifold.

3.3.1 The Dixmier trace

Let $\mathcal{H}$ be the Hilbert space of definition 3.1, with the spectral triple satisfying axioms 1 to 6.

The definition of the integral in example 3.14 heavily relied on the behavior of the eigenvalues of the compact operator $|\not{D}|^{-1}$, as the eigenvalues go to infinity. With respect to that behavior, $|\not{D}|^{-1}$ is in a sense neither "too small" nor "too large". It is not "too large" because it is a compact operator. If we would order its eigenvalues to form a decreasing series in $\mathbb{R}^+$, that series would converge to zero, in such a way that for each $\epsilon > 0$ there is a finite-dimensional $E \subset \mathcal{H}$ such that $\|T|_{E^\perp}\| \leq \epsilon$.

On the other hand, $|\not{D}|^{-1}$ is not "small enough" to be trace-class: indeed, the limit

$$\lim_{N\to\infty} \sum_{n=1}^{N} \langle e_n, |\not{D}|^{-1} e_n \rangle = C \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n}, \tag{3.41}$$

with C some constant, does not exist.

The intermediate size we are looking for is a so-called **infinitesimal (operator) of order 1**. We say that the compact operator T is an **infinitesimal of order** α when α is the smallest real number such that the singular values of T are of order $\mathcal{O}(n^{-\alpha})$ as $n \to \infty$. In example 3.14, $|\not{D}|^{-1}$ is an infinitesimal of order 1.

An infinitesimal of order 1 has at most logarithmically divergent singular values. The collection of compact operators with that property is called the **Dixmier ideal**:

$$\mathcal{L}^{1+}(\mathcal{H}) \equiv \left\{ T \in K(\mathcal{H}); \sup_{N \geq 3} \left\{ \frac{\sigma_N(T)}{\log N} \equiv \gamma_N(T) \right\} < \infty \right\}. \tag{3.42}$$

Like the name suggests, the Dixmier ideal is a two-sided ideal in the algebra of bounded operators on $\mathcal{H}$. See [21, Ch. 7.C] for more information on ideals of compact operators.

It would be tempting to define $\mathrm{Tr}^+ : \mathcal{L}^{1+}(\mathcal{H}) \to \mathbb{C}$ as

$$\mathrm{Tr}^+(T) \equiv \lim_{N\to\infty} \gamma_N(T) \qquad \forall\, T \in \mathcal{L}^{1+}(\mathcal{H}). \tag{3.43}$$

However, the property that $\gamma_N(T)$ is bounded is not strong enough to guarantee the existence of this limit. Luckily, this defect can be fixed in an ingenious way. In [18], Dixmier defined a whole array of traces on $\mathcal{L}^{1+}(\mathcal{H})$, which are related to the aforementioned limit in a special case. These traces, called **Dixmier traces**, all spring from the so-called **Cesàro average** of the function $\frac{\sigma_\lambda(T)}{\log\lambda}$, defined as:

$$\tau_\lambda(T) \equiv \frac{1}{\log\lambda}\int_3^\lambda \frac{\sigma_x(T)}{x\log x}dx \qquad \lambda \in [3,\infty). \tag{3.44}$$

Note that $\tau_\lambda(\cdot)$ is not a trace. It turns out [16, Lemma A.4], however, that for all *positive* $T, S \in \mathcal{L}^{1+}(\mathcal{H})$,

$$\tau_\lambda(T) + \tau_\lambda(S) - \tau_\lambda(T+S) = \mathcal{O}\left(\frac{\log(\log\lambda)}{\log\lambda}\right) \qquad \lambda\to\infty. \tag{3.45}$$

The next step is therefore to get rid of the right-hand side of equation (3.45).

Lemma 3.15. *Let $C_b([3,\infty))$ be the C^*-algebra of bounded and continuous functions on $[3,\infty)$. Form the unital C^*-algebra*

$$B_\infty \equiv C_b([3,\infty))/C_0([3,\infty)). \tag{3.46}$$

The association $\lambda \to \tau_\lambda(T)$ defines a unique element $\tau(T) \in B_\infty$ such that $\tau(T+S) = \tau(T) + \tau(S)$ for all $S, T \in \left(\mathcal{L}^{1+}(\mathcal{H})\right)^+$.

Proof. The fact that $T \in \mathcal{L}^{1+}(\mathcal{H})$ implies that $\tau_\lambda(T)$ is bounded. We are finished when we are able to show that the function $\lambda \to \frac{\log(\log\lambda)}{\log\lambda} \in C_0([3,\infty))$. This can be shown by ordinary calculus □

The map τ is almost a trace:

Lemma 3.16. *Restricted to positive elements in the Dixmier ideal, τ is a trace.*

Proof. Let T, S be positive elements of $\mathcal{L}^{1+}(\mathcal{H})$.

- $\tau(S+T) = \tau(S) + \tau(T)$ by lemma 3.15;
- For any $\alpha \in \mathbb{C}$ with $\alpha \geq 0$, $\tau(\alpha S) = \alpha\tau(S)$, which follows from the definition of singular values;
- $\tau(S) \geq 0$, since the integrand of the Cesàro mean (3.44) is positive;
- $\tau(ST) = \tau(TS)$. By result A.2 in [16, App. A] we have the following property, which is notably stronger than what is required for τ to be a trace on $\mathcal{L}^{1+}(\mathcal{H})$:

$$\tau(AT) = \tau(TA) \qquad \forall T \in \mathcal{L}^{1+}(\mathcal{H}),\ \forall A \in B(\mathcal{H}). \tag{3.47}$$

□

Theorem 3.17. *τ extends to a trace on $\mathcal{L}^{1+}(\mathcal{H})$.*

Proof. From [21, 7.C] we know that $\mathcal{L}^{1+}(\mathcal{H})$ is linearly generated by its positive operators in the following way. For some self-adjoint T in the Dixmier ideal we can find two projections P_+ and P_- such that:

$$P_\pm = \frac{1}{2}(1 \pm F) \qquad \text{with } F \text{ a symmetry;}$$
$$P_+|T|P_+, P_-|T|P_- \geq 0;$$
$$T = P_+|T|P_+ - P_-|T|P_-.$$

We define τ on $\mathcal{L}^{+1}(\mathcal{H})$ in two stages. For self-adjoint T, let:

$$\tau(T) \equiv \tau(P_+|T|P_+) - \tau(P_-|T|P_-). \tag{3.48}$$

For general T in the Dixmier ideal, use the usual decomposition into self-adjoint operators to define:

$$\tau(T) \equiv \frac{1}{2}\tau\left(T + T^*\right) + \frac{i}{2}\tau\left(iT - iT^*\right). \tag{3.49}$$

This shows that the domain of τ includes the Dixmier ideal. We verify that τ has the properties of a trace.
By construction, τ is $\mathbb{C}$-linear on the Dixmier ideal and is a positive function. Assume $\tau(T) = 0$ for some $T \in \mathcal{L}^{1+}(\mathcal{H})$. In general, $\tau_\lambda(T)$ has non-zero values only at $\lambda \to \infty$, implying that the contributions of the eigenvalues of T to $\tau_\lambda(T)$ come from the $\sigma_x(T)$ for very large x. Since the singular values of T, and thus the $\sigma_x(T)$, are actually converging to zero for large x, this situation will not occur. Hence all the singular values of T have value zero and $T = 0$. In conclusion, τ is a non-degenerate (faithful) map.
Now, again use the equality (3.47) to show that τ is a trace on $\mathcal{L}^{1+}(\mathcal{H})$. □

To define a trace from the Dixmier ideal to the complex numbers, all one has to do is to compose τ with some function from B_∞ to the complex numbers that does not spoil the properties of the trace. If we would demand that this function takes value 1 on the unit of B_∞, we see that a minimum requirement is that the function is a **state** on B_∞. There are plenty of states on B_∞: for example, each character $\omega \in \Delta B_\infty$ is a state (since $\omega(a) = \omega(b^*b) = \omega(b)^*\omega(b) \geq 0$ for each positive a). By the Gelfand correspondence there are as many characters as there are points in the spectrum of B_∞, the latter being nonempty.
So we arrive at the following definition.

Definition 3.18 (Dixmier traces)**.** *There is a collection of traces on the Dixmier ideal, called the* ***Dixmier traces****, defined by*

$$Tr_\omega(T) \equiv \omega(\tau(T)) \qquad \forall\, T \in \mathcal{L}^{1+}(\mathcal{H}), \forall\, \omega \in S(B_\infty), \tag{3.50}$$

with $S(B_\infty)$ defined as the collection of states on $B_\infty = C_b([3,\infty))/C_0([3,\infty))$.

The last step is to remove the unwanted reference to states on B_∞. The essential result is given by the following lemma.

Lemma 3.19. *The Dixmier traces of an operator $T \in \mathcal{L}^{1+}(\mathcal{H})$ are equal for each choice of state ω on B_∞ if and only if the limit in equation (3.43) exists and is finite. In that case*

$$Tr_\omega(T) = \lim_{N\to\infty} \gamma_N(T) \equiv Tr^+(T) \qquad \forall\, \omega \in S(B_\infty). \tag{3.51}$$

Proof. From the definition of the Cesàro-mean it directly follows that $\lim_{\lambda\to\infty} \tau_\lambda(T)$ exists iff $\gamma_N(T)$ has a limit.
Let us denote the limit $\lim_{\lambda\to\infty} \tau_\lambda(T)$ by C. Then the equivalence class of τ_λ in B_∞ is a constant function: $\lambda \mapsto \tau_\lambda(T) - C \in C_0([3,\infty))$, so $[\tau_\lambda(T)] = C \in B_\infty$. As a result, $\omega\,[\tau_\lambda(T)] = C$ for each state.
Conversely, assume that $\lim_{\lambda\to\infty} \tau_\lambda(T)$ does not exist. This implies that for each real positive number z there are $x, y \in [3,\infty)$ such that $x, y > z$ and $[\tau_\lambda(T)](x) \neq [\tau_\lambda(T)](y)$. Since B_∞ is a commutative C^*-algebra, each character is mapped to an evaluation map under the Gelfand transform. Hence we are able to find two characters $\omega_x, \omega_y \in S(B_\infty)$ such that

$$\omega_x[\tau_\lambda(T)] = [\tau_\lambda(T)](x) \neq [\tau_\lambda(T)](y) = \omega_y[\tau_\lambda(T)].$$

□

The **measurable** operators on $\mathcal{H}$ are those elements T in the Dixmier ideal such that $\mathrm{Tr}^+(T)$ exists. For measurable operators we refer to Tr^+ as the Dixmier trace.
The Dixmier trace gives meaning to the concept of "size" we talked about earlier: the trace maps the "small" infinitesimals of order higher than 1 to zero. Measurable infinitesimals of order 1 have the right size for the Dixmier trace, since the trace can take any value on them. "Large" operators, i.e., the infinitesimals of order lower than 1, are not in the domain of the Dixmier trace.
We finish this section by describing an important property of the Dixmier trace, which is a special case of the so-called noncommutative **Hölder-inequality**. Quoting [21, Prop. 7.16]:

Lemma 3.20. *Let $T \in \mathcal{L}^{1+}(\mathcal{H})$ and $S \in B(\mathcal{H})$. For any Dixmier trace Tr_ω (as well as Tr^+) we have*

$$Tr_\omega(|TS|) \leq \|S\|\; Tr_\omega(|T|). \tag{3.52}$$

3.3.2 Measurability of $|D|^{-p}$

Let p be the dimension of the spectral triple $(\mathcal{A}, D, \mathcal{H})$.
Since the Dixmier ideal is a two-sided ideal in $B(\mathcal{H})$ and $\mathcal{A}$ has a representation on $B(\mathcal{H})$, we can, mimicking example 3.14, look for a compact operator K such that aK is measurable for all $a \in \mathcal{A}$. Axiom 1 suggests that a good candidate for K is $|D|^{-p}$. Our next step shall therefore be to show that

$$a|D|^{-p} \tag{3.53}$$

is **measurable** for each $a \in \mathcal{A}$. The easy part lies in showing that $a|D|^{-p} \in \mathcal{L}^{1+}(\mathcal{H})$, using axiom 1.
The compactness of $|D|^{-1}$ implies that of $|D|^{-p}$. Axiom 1 then implies that $\sigma_n(|D|^{-p}) = \sum_{i=0}^{n-1} s_n(|D|^{-p}) \sim C_1 + C_2(n-1) \cdot \frac{1}{n}$, for some positive constants C_1, C_2 as $n \to \infty$. Then:

$$\sup_{n \geq 3} \left\{ \frac{\sigma_n(a|D|^{-p})}{\log n} \right\} \leq \|a\| \sup_{n \geq 3} \left\{ \frac{\sigma_n(|D|^{-p})}{\log n} \right\} < \infty,$$

by the previous remark.
This leaves us with checking that $\mathrm{Tr}_\omega(a|D|^{-p})$ is independent of the state ω used. The author is unaware of a direct proof showing $\mathrm{Tr}_\omega(a|D|^{-p}) = \mathrm{Tr}_{\omega'}(a|D|^{-p})$ for two different states or that the limit

$$\lim_{N \to \infty} \gamma_N \left(a|D|^{-p}\right)$$

exists, so we take the option of deriving a different expression for the Dixmier trace that is not dependent on the states of B_∞. We shall, in fact, show that $\mathrm{Tr}_\omega(a|D|^{-p})$ is, up to a constant, equal to the action of the Chern-Connes character on $a\mathbf{c}$. The latter action has no reference whatsoever to characters in B_∞. See appendix G for more background information.
We first do some preparatory work. Quoting from [21], theorem 10.20:

Theorem 3.21. *If the p-dimensional spectral triple $(\mathcal{A}, D, \mathcal{H})$ satisfies axiom 1, then $Tr_\omega(\cdot|D|^{-p})$ is a hypertrace on $\mathcal{A}$. That is, for each $T \in B(\mathcal{H})$*

$$Tr_\omega(Ta|D|^{-p}) = Tr_\omega(aT|D|^{-p}).$$

Let us rewrite the integral in a more suitable form, using axiom 4. Take some $\omega \in S(B_\infty)$ and $a \in \mathcal{A}$.

$$\mathrm{Tr}_\omega \left(a|D|^{-p}\right) = \mathrm{Tr}_\omega \left(\chi^2 a|D|^{-p}\right) \overset{\text{hypertrace}}{=} \mathrm{Tr}_\omega \left(\chi a \chi |D|^{-p}\right) =$$
$$\mathrm{Tr}_\omega \left(\chi a \sum_\alpha c_\alpha^0 \left[D, c_\alpha^1\right] \cdots \left[D, c_\alpha^p\right] |D|^{-p}\right),$$

where the c^i_α are the elements of $\mathcal{A}$ as defined in axiom 4, i.e.,

$$\mathbf{c} = \sum_\alpha c^0_\alpha \otimes c^1_\alpha \otimes \cdots \otimes c^p_\alpha.$$

This justifies the following definition.

Definition 3.22 (ϕ^D_ω)**.** *Let $\omega \in S(B_\infty)$. Define a map*

$$\phi^D_\omega : \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{p+1 \ times} \to \mathbb{C};$$
$$\phi^D_\omega \equiv \lambda_p Tr_\omega \left(\chi a_0 [D, a_1] \cdots [D, a_p] |D|^{-p}\right), \tag{3.54}$$

where λ_p is the same constant as defined in equation (G.27), appendix G.

This leads to the following tidy expression:

$$\mathrm{Tr}_\omega \left(a|D|^{-p}\right) = \lambda_p^{-1} \phi_\omega(a\mathbf{c}). \tag{3.55}$$

ϕ^D_ω is linear in each of its entries, so the map is a Hochschild cochain. It is precisely this cochain which coincides with the Chern-Connes character on $a\mathbf{c}$. In order to define the Chern-Connes character we need to show that the commutative spectral triple defines a Fredholm module that satisfies the requirements of theorem G.6, appendix G. The proof is mainly taken from [8].
The first step is to prove some results regarding several operator-valued functions of D.

Lemma 3.23. *Let $(\mathcal{A}, D, \mathcal{H})$ be a spectral triple satisfying axiom 1. Then for all μ in the resolvent of D:*

$$\frac{1}{D^2 - \mu} \in K(\mathcal{H}). \tag{3.56}$$

If $\lambda \in (-\infty, 0)$ then

$$\frac{1}{\sqrt{D^2 - \lambda}} \in K(\mathcal{H}); \tag{3.57}$$
$$\left\| \frac{1}{D^2 - \lambda} \right\| \leq \frac{1}{-\lambda}; \tag{3.58}$$
$$\left\| \frac{D}{\sqrt{D^2 - \lambda}} \right\| \leq 1; \tag{3.59}$$
$$\left\| \frac{D}{D^2 - \lambda} \right\| \leq \frac{1}{2\sqrt{-\lambda} - \lambda}. \tag{3.60}$$

Proof. The first statement follows from the resolvent equation and the fact that $D^{-2} = |D|^{-2}$ is compact and $\frac{1}{D^2-\lambda}$ is bounded:

$$\frac{1}{D^2} - \frac{1}{D^2-\mu} = \frac{\mu}{D^2(D^2-\mu)} \in K(\mathcal{H}) \quad \Rightarrow \tag{3.61}$$

$$\frac{1}{D^2-\mu} = \frac{1}{D^2} - \frac{\mu}{D^2(D^2-\mu)} \in K(\mathcal{H}). \tag{3.62}$$

We now prove the other four identities, starting with relation (3.57).
D^2 is a positive operator so its spectrum is contained in $\mathbb{R}^+$, implying that for $\lambda \in (-\infty, 0)$ and $\xi \in \text{dom } D^2$:

$$\langle (D^2-\lambda)\xi, \xi\rangle_{\mathcal{H}} = \langle D^2\xi, \xi\rangle_{\mathcal{H}} - \lambda\langle \xi, \xi\rangle_{\mathcal{H}} \geq 0. \tag{3.63}$$

So $D^2 - \lambda$ is a positive operator. The operator $\frac{1}{\sqrt{D^2-\lambda}}$ is the square root of a positive compact operator and is therefore a positive compact operator itself.
The function $f : x \mapsto \frac{x}{\sqrt{x^2-\lambda}}$ is bounded by 1, since λ is negative. Using the spectral calculus for unbounded self-adjoint operators on $\frac{D}{\sqrt{D^2-\lambda}}$, we see that

$$\left\| \frac{D}{\sqrt{D^2-\lambda}} \right\| \leq 1. \tag{3.64}$$

Applying the same trick to the functions $g_1 : x \mapsto \frac{1}{x^2-\lambda}$ and $g_2 : x \mapsto \frac{x}{x^2-\lambda}$ proves the third and the fifth statement. □

The next step is to use this lemma to show that a commutative spectral triple, together with several of the aforementioned axioms, defines a Fredholm module.

Theorem 3.24. *Assume a spectral triple $(\mathcal{A}, D, \mathcal{H})$ satisfies axioms 1 and 3. If p is even, the spectral triple defines an even Fredholm module $(\mathcal{A}, F, \mathcal{H})$. When p is odd the Fredholm module is odd.*

Proof. Assume $\ker D = \{0\}$. We first prove the theory based on this assumption. At the end of the proof, we generalize the result.
We define

$$F \equiv D|D|^{-1}. \tag{3.65}$$

F is a symmetry by construction. We now show that $[F, a]$ is a compact operator for all $a \in \mathcal{A}$. We first rewrite the commutator as

$$[F, a] = [D, a]|D|^{-1} + D[|D|^{-1}, a]. \tag{3.66}$$

The starting point is an operator-valued integral representation of the bounded and positive operator $|D|^{-1}$. From [38]:

$$|D|^{-1} = \left(|D|^{-2}\right)^{\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}} \left(1 + \lambda D^{-2}\right)^{-1} D^{-2} d\lambda =$$

$$\frac{1}{\pi}\int_0^\infty \left(D^2+\lambda\right)^{-1} \frac{d\lambda}{\sqrt{\lambda}}. \tag{3.67}$$

Applying this to (3.66), we obtain:

$$[F,a] = \frac{[D,a]}{\pi}\int_0^\infty \left(D^2+\lambda\right)^{-1}\frac{d\lambda}{\sqrt{\lambda}} + \frac{D}{\pi}\left[\int_0^\infty \left(D^2+\lambda\right)^{-1}\frac{d\lambda}{\sqrt{\lambda}}, a\right].$$

Using (3.58) of lemma 3.23, we see that (3.67) converges in the operator norm on $\mathcal{H}$. As a consequence, for all bounded operators $A \in B(\mathcal{H})$:

$$A|D|^{-1} = \frac{1}{\pi}\int_0^\infty A\left(D^2+\lambda\right)^{-1}\frac{d\lambda}{\sqrt{\lambda}}. \tag{3.68}$$

So the expression for $[F,a]$ is identical to:

$$[F,a] = \frac{1}{\pi}\int_0^\infty [D,a]\left(D^2+\lambda\right)^{-1}\frac{d\lambda}{\sqrt{\lambda}} + \frac{D}{\pi}\int_0^\infty \left[\left(D^2+\lambda\right)^{-1}, a\right]\frac{d\lambda}{\sqrt{\lambda}}. \tag{3.69}$$

Let us focus on the last term of this expression. Define for the moment

$$f(D,a) \equiv \left[\left(D^2+\lambda\right)^{-1}, a\right]. \tag{3.70}$$

We wish to show that

$$\frac{D}{\pi}\int_0^\infty f(D,a)\frac{d\lambda}{\sqrt{\lambda}} = \frac{1}{\pi}\int_0^\infty Df(D,a)\frac{d\lambda}{\sqrt{\lambda}}. \tag{3.71}$$

Note the due to the fact that D is a closed operator on $\mathcal{H}$, we can pull D inside the integral "point-wise", i.e., for all $\xi \in \operatorname{dom} D$:

$$\left(\frac{D}{\pi}\int_0^\infty f(D,a)\frac{d\lambda}{\sqrt{\lambda}}\right)\xi = \frac{1}{\pi}\int_0^\infty Df(D,a)\xi\frac{d\lambda}{\sqrt{\lambda}}. \tag{3.72}$$

We obtain a stronger result as follows. By strong regularity and the polar decomposition, a dom $D^2 \subset$ dom D^2. So:

$$f(D,a) = \frac{1}{D^2+\lambda} a \frac{D^2+\lambda}{D^2+\lambda} - \frac{D^2+\lambda}{D^2+\lambda} a \frac{1}{D^2+\lambda} =$$
$$-\frac{1}{D^2+\lambda}[D^2,a]\frac{1}{D^2+\lambda} = -\frac{D}{D^2+\lambda}[D,a]\frac{1}{D^2+\lambda} -$$
$$\frac{1}{D^2+\lambda}[D,a]\frac{D}{D^2+\lambda} \quad \Rightarrow$$
$$\|Df(D,a)\| \leq \left\|\frac{D^2}{D^2+\lambda}[D,a]\frac{1}{D^2+\lambda}\right\| + \left\|\frac{D}{D^2+\lambda}[D,a]\frac{D}{D^2+\lambda}\right\| \leq$$
$$\|[D,a]\|\left(\frac{1}{\lambda}+\frac{1}{4\lambda^3}\right),$$

due to lemma 3.23. Hence the integral (3.72) converges uniformly and the identity (3.71) holds. So we finally obtain:

$$[F,a] =$$
$$\frac{1}{\pi}\int_0^\infty \left[\frac{[D,a]}{D^2+\lambda} - \frac{D^2}{D^2+\lambda}[D,a]\frac{1}{D^2+\lambda} - \frac{D}{D^2+\lambda}[D,a]\frac{D}{D^2+\lambda}\right]\frac{d\lambda}{\sqrt{\lambda}}. \quad (3.73)$$

All terms are compact operators: for the last term, use

$$\frac{D}{D^2+\lambda} = \frac{D}{\sqrt{D^2+\lambda}}\frac{1}{\sqrt{D^2+\lambda}}.$$

So $[F,a] \in K(\mathcal{H})$.
When p is even, $\chi = \pi_D(\mathbf{c})$ yields a $\mathbb{Z}_2$-grading of $\mathcal{H}$ which commutes with the action of $\mathcal{A}$ and anti-commutes with F so $(\mathcal{A},F,\mathcal{H})$ is an even Fredholm module. When p is odd this grading is effectively lost and $(\mathcal{A},F,\mathcal{H})$ is odd as a Fredholm module.
We now move to the general case, where $\ker D \neq \{0\}$. Any Hilbert space $\mathcal{H}$ of the general spectral triple $(\mathcal{A},D,\mathcal{H})$ can be expressed as

$$\mathcal{H} = \mathcal{H}' \oplus \ker D, \quad (3.74)$$

where the kernel of the restricted map $D : \text{dom } D \subset \mathcal{H}' \to \mathcal{H}'$ is zero.
We construct a Fredholm cycle on the spectral triple $(\mathcal{A},D,\mathcal{H})$ in the following way. From [12, Ch. 4], we gather that the kernel of the Dirac operator is finite-dimensional. So without any issues, we can construct a Fredholm module $(\mathcal{A},F',\mathcal{H}')$ as done above. Assume this Fredholm module is odd.
We define the Fredholm module $(\mathcal{A},F,D)$ by letting $\mathcal{A}$ act on $\mathcal{H}$ as

$$a(\zeta,\eta) = (a\zeta,0) \in \mathcal{H}, \qquad \zeta \in \mathcal{H}', \eta \in \ker D. \quad (3.75)$$

The partial symmetry F is then defined as

$$F \equiv F' \oplus 1_{\ker D}, \tag{3.76}$$

where $1_{\ker D} : \mathcal{H} \to \mathcal{H}$ is the identity operator on $\ker D$ and zero everywhere else.
In the other case, namely, $(\mathcal{A}, F', \mathcal{H}')$ is an odd Fredholm module, see the last paragraphs [12, 4.2.γ] for an explicit construction of a Fredholm module on $(\mathcal{A}, D, \mathcal{H})$. □

In order to construct the Chern-Connes character on $(\mathcal{A}, F, \mathcal{H})$, we need a result that links the dimension of the spectral triple with the Schatten class of the commutator of F with any element of the algebra $\mathcal{A}$. For reference, see the pre-requisites of theorem G.6.

Theorem 3.25. *The dimension of the spectral triple is the smallest integer p such that*

$$[F, a] \in \mathcal{L}^{p+1}(\mathcal{H}). \tag{3.77}$$

Proof. Take $b \in \mathcal{A}$ to be skew-adjoint, i.e., $b^* = -b$. Then both $[D, b]$ and $[F, b]$ are bounded self-adjoint operators. Using the order structure of self-adjoint operators on $B(\mathcal{H})$ we see that

$$\begin{aligned} &- \|[D, b]\| \leq [D, b] \leq \|[D, b]\| \quad \Rightarrow \\ &- \|[D, b]\| \, |D|^{-1} \leq [F, b] \leq \|[D, b]\| \, |D|^{-1}, \end{aligned}$$

since by the proof of the previous theorem:

$$[F, a] = \left[D|D|^{-1}, a\right] = \frac{1}{\pi} \int_0^\infty \frac{[D, a]}{D^2 + \lambda} \frac{d\lambda}{\sqrt{\lambda}}.$$

This implies that relative to a base that diagonalizes $|D|^{-1}$, the singular values $s_k([F, a])$ satisfy:

$$\begin{aligned} &s_k([F, a]) \leq \|[D, a]\| \, s_k\left(|D|^{-1}\right) \quad \Rightarrow \\ &s_k([F, a])^{p+1} \leq \|[D, a]\|^p \, s_k\left(|D|^{-1}\right)^{p+1} = \mathcal{O}\left(n^{-\frac{p+1}{p}}\right). \end{aligned}$$

We know from ordinary calculus that the sequence n^{-1} is not summable, whereas $n^{-\frac{p+1}{p}}$ is, so indeed the dimension of the spectral triple is the smallest integer such that $[F, a]$ lies in the Schatten $p + 1$-class. □

All the preconditions of theorem G.6 in appendix G are met. Hence the Chern-Connes character is well-defined on $C_n(\mathcal{A})$, i.e., there is a cyclic Hochschild cocycle

$$\begin{aligned} &\tau_F^p \in C^p(\mathcal{A}); \\ &\tau_F^p(a_0, \ldots, a_p) \equiv \lambda_p \mathrm{Tr} \, \left(\chi F \, [F, a_1] \cdots [F, a_p]\right), \end{aligned}$$

where Tr is the ordinary trace as can be defined on trace class operators in $\mathcal{L}^1(\mathcal{H})$.
This leads to the final result of this section.

Theorem 3.26. *For any $a \in \mathcal{A}$*

$$\phi_\omega^D(a\boldsymbol{c}) = \tau_F^p(a\boldsymbol{c}). \tag{3.78}$$

Proof. Connes' character formula [12, IV.2.γ] states that for a regular spectral triple satisfying axioms 1, 2, and 4, the Chern-Connes character and ϕ_ω^D coincide on cycles. By virtue of commutativity of the algebra $\mathcal{A}$, also $a\mathbf{c}$ is a cycle. We can see this by direct evaluation:

$$b(a\mathbf{c}) = \sum_\alpha \sum_{i=0}^{p-1} (-1)^i a c_\alpha^0 \otimes \cdots \otimes c_\alpha^i a_\alpha^{i+1} \otimes \cdots \otimes c_\alpha^p +$$
$$\sum_\alpha (-1)^p c_\alpha^p a c_\alpha^0 \otimes c_\alpha^1 \otimes \cdots \otimes c_\alpha^{p-1} = ab(\mathbf{c}) \overset{\text{Ax4}}{=} 0.$$

□

3.3.3 Defining the noncommutative integral

Taking stock, for any commutative spectral triple satisfying axioms 1 to 6, we have defined a map

$$⨍ : \mathcal{A} \to \mathbb{C};$$
$$⨍ a \equiv \mathrm{Tr}^+\left(a|D|^{-p}\right) = \mathrm{Tr}_\omega(a|D|^{-p}) = \lambda_p^{-1}\phi_\omega^D(a\mathbf{c}) \quad \forall \omega \in S(B_\infty). \tag{3.79}$$

This map is called the **noncommutative integral**. As announced, it satisfies three-quarters of the properties of a trace on $\mathcal{A}$. As a result of theorem 3.17:

Corollary 3.27. *The noncommutative integral satisfies properties (1) to (3) of definition 3.12.*

We verify property (4), that is, the positivity of the integral, in chapter 5. The expression for the integral on S^1, as derived in example 3.14, coincides with the noncommutative integral. This is no coincidence. Up to a scaling factor, which depends on the dimension of the spin manifold, the trace coincides with the ordinary integral on a spin manifold. As a corollary of Connes' trace theorem [21, Corr. 7.21] (see the last paragraph of [21, Ch. 9.4]) we see that for any spin manifold M and all $a \in C^\infty(M)$:

$$⨍ a = \int_M a\nu. \tag{3.80}$$

In chapter 5, after we have established that the spectrum of the C^*-algebra A is a manifold in chapter 4, we will turn this relation upside-down to define a volume form and investigate its properties. In the process, we will find that the noncommutative integral defines, up to a constant, the ordinary integral over the manifold.

3.3.4 Axioms 7 and 8

The preceding theory enables us to state the last two axioms.

Axiom 7 (Finiteness)**.** *$\mathcal{H}^\infty$ is a finitely generated projective right-$\mathcal{A}$-module. Write p as the projection in $M_n(\mathcal{A})$ such that*

$$\mathcal{H}^\infty \cong p\mathcal{A}^n \tag{3.81}$$

as $\mathcal{A}$-modules.

Remark 3.28. *It turns out that a projection $p \in M_n(\mathcal{A})$ defined by $\mathcal{H}^\infty \cong p\mathcal{A}^n$ is unique up to the adjoint action of a unitary element in $M_N(\mathcal{A})$ for some N. In other words, if $\mathcal{H}^\infty \cong q\mathcal{A}^k$ for some projection $q \in M_k(\mathcal{A})$, then there is a unitary element such that $p = uqu^*$. See lemma F.1 in appendix F for more background information.*

For spin manifolds, axiom 7 is just an application of the Serre-Swan theorem for smooth vector bundles [52], which states that:

$$\mathcal{H}^\infty = \Gamma^\infty(M, \mathfrak{S}).$$

Note that, given axiom 7, $\mathcal{H}^\infty$ has a useful structure.

Lemma 3.29. *$\mathcal{H}^\infty$ is a right-$\mathcal{A}$-pre-Hilbert module.*

Proof. The right-action of $\mathcal{A}$ on $\mathcal{H}^\infty$ is given by

$$(a_1, \dots, a_n) \cdot a = (a_1 \cdot a, \dots, a_n \cdot a). \tag{3.82}$$

Since $\mathcal{A}$ is commutative, $\mathcal{H}^\infty$ is actually a symmetric $\mathcal{A}$-bimodule.
We now need to find an $\mathcal{A}$-valued inner product on $\mathcal{H}^\infty$. Take any basis $\{e_i\} \subset \mathcal{A}^n$ and define $(\xi, \eta)_\mathcal{A} \equiv \sum_{i,j} a_i^* p_{ij} b_j$ where the a_i, b_j and the p_{ij}'s are the components of ξ, η and p in terms of the chosen basis, respectively. This definition would not be proper when it is depends on a choice of base of $\mathcal{A}^n$. Let $\{e_i'\} \subset \mathcal{A}^n$ be another one, connected via a unitary $U \in M_n(\mathcal{A})$. Then

$$(\xi, \eta) = \sum_{i,j} a_i^* p_{ij} b_j = \sum_{i,j,k,l,m,n} (U_{ik} a_k)^* U_{il} p_{lm} U_{mj}^* U_{jn} b_n =$$
$$\sum_{i,j,k,l,m,n} a_k^* U_{ki}^* U_{il} p_{lm} U_{mj}^* U_{jn} b_n = \sum_{k,l,m,n} a_k^* \delta_{kl} a_k^* p_{lm} \delta_{mn} b_n = \sum_{k,m} a_k^* p_{km} b_m.$$

The $\mathcal{A}$-sesquilinearity property is evident from the definition, so all we need to check is positive definiteness. Writing $(\xi, \xi)_{\mathcal{A}}$ in a base where p_{ij} is diagonal, we see that the inner product is a sum of $a_i^* a_i$, which by definition is strictly positive, provided the a_i's are non-zero for all i. □

Let $\mathcal{S}$ be the collection of smooth sections of a spinor bundle on a spin manifold M. There are two ways to describe an inner product on $\mathcal{S}$. The first is the inner product inherited from the Hilbert space of square integrable spinors. In the framework of a spectral triple, this inner product should correspond to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The second one is by using the right-$C^\infty(M)$-pre-Hilbert module structure and then integrating the $C^\infty(M)$-valued inner product over the whole space. The latter inner product should correspond to a noncommutative integral of the form

$$⨍ \langle \cdot, \cdot \rangle_{\mathcal{A}}. \tag{3.83}$$

For any commutative spectral triple, as it stands, $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ are unrelated. The following axiom links both inner products.

Axiom 8 (Absolute continuity)**.** *Let $\eta, \xi \in \mathcal{H}^\infty$ and $a \in \mathcal{A}$. Then:*

$$\langle \xi, \eta \rangle_{\mathcal{H}} = ⨍ \langle \xi, \eta \rangle_{\mathcal{A}}.$$

From now on we shall assume that any commutative spectral triple we encounter satisfies the previous 8 axioms.
As a final remark for this chapter, note that any commutative spin geometry satisfies axioms 1 - 5 and 7 - 8 in general, and 6 in particular, when the manifold is connected. This is thoroughly discussed in [21, Ch. 9, 10 & 11.1].

Chapter 4

A differentiable structure on Spec $\mathcal{A}$

In this chapter we outline the construction of a manifold from a commutative spectral triple satisfying all of the previously introduced 8 axioms. This construction is the proof to **Connes' reconstruction theorem** [14]. Furthermore, we shall show that such a manifold arising from a commutative spectral triple is compact, connected and has dimension equal to the dimension of the spectral triple.
In section 4.1, we prove the reconstruction theorem, given several key assumptions. Sections 4.2 to 4.5 are dedicated to showing how these assumptions follow from the definition of a commutative spectral triple satisfying the 8 axioms introduced in the previous chapter.

4.1 Connes' reconstruction theorem

Note that in this thesis, we have implicitly identified an element of (a subset of) a C^*-algebra with its Gelfand transform, and also identified the elements of the spectrum with the points in a topological space. We shall now explicitly make this identification to elucidate the proof of the reconstruction theorem.
When constructing a manifold from a commutative spectral triple, one of the first few questions to arise is what topological space will be admissible to define a differentiable structure on. The answer, as it will turn out, is that the spectrum of the commutative C^*-algebra $A = \overline{\mathcal{A}}$ can be used, which (by Gelfand duality) is a compact Hausdorff space. The first assumption relates the spectrum of $\mathcal{A}$ with that of A.

Assumption 1.

$$\mathit{Spec}\ A = \mathit{Spec}\ \mathcal{A}. \tag{4.1}$$

The following two assumptions deal with local homeomorphisms mapping from Spec $\mathcal{A}$ to $\mathbb{R}^p$, which will eventually act as coordinate functions for the differentiable structure on Spec $\mathcal{A}$.
Spec A is a compact Hausdorff space because A has a unit. Suppose that assumption 1 holds. We see that the spectrum Spec $\mathcal{A}$ is a compact Hausdorff space as well.

Assumption 2. *Let p be the dimension of the spectral triple $(A, D, \mathcal{H})$. There exists a cover $\{N_\alpha; \alpha \in I\}$ of Spec $\mathcal{A}$ such that for each character $\chi \in$ Spec $\mathcal{A}$, there is a neighborhood N_α with the following properties. There are p self-adjoint elements $\left\{x_\alpha^i, i \in \{1, \ldots, p\}\right\} \subset \mathcal{A}$ such that the map*

$$\hat{x}_\alpha : \text{Spec } \mathcal{A} \to \mathbb{R}^p, \qquad \hat{x}_\alpha \equiv \left(\hat{x}_\alpha^1, \ldots, \hat{x}_\alpha^p\right) \tag{4.2}$$

defines a local homeomorphism of N_α with some open open set in $\mathbb{R}^p$.

As already mentioned in chapter 3, we shall see in the next sections that the x_α^i derive from the elements $c_\alpha^i \in \mathcal{A}$ that occur in the expansion of the noncommutative volume element:

$$\mathbf{c} = \sum_\alpha c_\alpha^0 \otimes c_\alpha^1 \otimes \cdots \otimes c_\alpha^p \in C_p(\mathcal{A}), \tag{4.3}$$

for α in some finite index set (that is not necessarily equal to I).

Assumption 3. *Suppose that the first two assumptions hold. There is a smooth family of maps $\{\tau_t; t \in \mathbb{R}^p\}$ with:*

$$\tau_\bullet : \mathbb{R}^p \to Aut(\mathcal{A}), \tag{4.4}$$

and $\tau_0 = id_\mathcal{A}$, satisfying:

1. *For each $\chi \in$ Spec $\mathcal{A}$ and all $a \in \mathcal{A}$, the function $\chi \circ \tau_\bullet : \mathbb{R}^p \to$ Spec $\mathcal{A}$, defined by*

$$\chi \circ \tau_t(a) = \chi\left(\tau_t(a)\right) \tag{4.5}$$

   for all $t \in \mathbb{R}^p$, is a homeomorphism of a neighborhood of $0 \in \mathbb{R}^p$ with a neighborhood of $\chi \in$ Spec $\mathcal{A}$;

2. *For each $\alpha \in I$ (where I is the index set of the cover introduced in assumption 2), the map*

$$\hat{x}_\alpha \circ \chi \circ \tau_\bullet : \mathbb{R}^p \to \mathbb{R}^p \tag{4.6}$$

   is a local diffeomorphism.

Next, we shall require $\mathcal{A}$ to be a stable under smooth functional calculus (refer to definition 2.11 in section 2.2 for the terminology):

Assumption 4. *Let $\{a_1, \ldots, a_n\}$ be a set of self-adjoint elements of $\mathcal{A}$ and denote their joint spectrum with $\Delta \subset \mathbb{R}^n$. Take $f : \mathbb{R}^n \to \mathbb{C}$ a smooth function defined in a neighborhood of Δ. Define $f(a_1, \ldots, a_n)$ as the image of $f|_\Delta$ in the C^*-algebra generated by $\{1, a_1, \ldots, a_n\}$ under the Gelfand isomorphism.*[1] *Then $f(a_1, \ldots, a_n) \in \mathcal{A}$.*

Last, we require that there is a "smooth" partition of unity of Spec $\mathcal{A}$.

Assumption 5. *Subordinate to any given cover $\{U_\beta; \beta \in J\}$ of Spec $\mathcal{A}$, there is a partition of unity $\{\psi_\beta\} \subset \mathcal{A}$.*

We now prove the reconstruction theorem.

Theorem 4.1. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a commutative spectral triple satisfying axioms 1 to 8 and hence assumptions 1 to 5 (which follow from axioms 1 to 8).*
There is an involutive algebra isomorphism $\mathcal{A} \cong C^\infty(M)$, where M is a compact p-dimensional manifold.

Proof. The proof consists of several steps. In the first one, we show that Spec $\mathcal{A}$ is a compact, p-dimensional manifold. In the second, we show that locally, $\mathcal{A}|_V \cong C^\infty(V)$ for some $V \subset$ Spec $\mathcal{A}$. Last, we show we can extend this result globally.

1. According to assumption 1, Spec $\mathcal{A}$ = Spec A, so that Spec $\mathcal{A}$ is isomorphic to a compact Hausdorff space. The pertinent homeomorphism is implemented through the evaluation map. We start with showing that Spec $\mathcal{A}$ is a manifold.
   Let $\{N_\alpha; \alpha \in I\}$ be a cover of Spec $\mathcal{A}$ such that $\hat{x}_\alpha$ is a local homeomorphism of N_α with some open set in $\mathbb{R}^p$. Take some $\chi \in$ Spec $\mathcal{A}$. Pick α such that N_α is a neighborhood of χ. Here, we have used assumption 2, which verifies the existence of these objects. Now choose N_β to be a second neighborhood of χ such that x_β is a local homeomorphism of N_β with some open set in $\mathbb{R}^p$. We shall modify the tentative charts $\{(\hat{x}_\alpha, N_\alpha); \alpha \in I\}$ so that we can apply assumption 3.
   We first translate the functions $\hat{x}_\alpha$ and $\hat{x}_\beta$ such that:
   $$\hat{x}_\alpha(\chi) = \hat{x}_\beta(\chi) = 0.$$
   Second, we shrink N_α and N_β such that both 1. and 2. of assumption 3 hold. We claim that $\{(\hat{x}_\alpha, N_\alpha), \alpha \in I\}$ is an atlas of Spec $\mathcal{A}$. Namely, we show that on the overlap $N_\alpha \cap N_\beta$ the expression
   $$\hat{x}_\alpha \circ \hat{x}_\beta^{-1} : x_\beta(N_\alpha \cap N_\beta) \subset \mathbb{R}^p \to x_\alpha(N_\alpha \cap N_\beta) \subset \mathbb{R}^p \qquad (4.7)$$

[1] Note that we have used the fact that A is stable under continuous functional calculus.

is a smooth map. On the modified domains, the map $\chi \circ \tau_\bullet$ is a homeomorphism. We can rewrite the above expression as:

$$\hat{x}_\alpha \circ \hat{x}_\beta^{-1} = \hat{x}_\alpha \circ \chi \circ \tau_\bullet \circ (\chi \circ \tau_\bullet)^{-1} \circ \hat{x}_\beta^{-1} =$$
$$\hat{x}_\alpha \circ \chi \circ \tau_\bullet \circ (\hat{x}_\beta \circ \chi \circ \tau_\bullet)^{-1} .$$

The latter, then, is a smooth map due to point 2. of assumption 3. The above holds for each $\chi \in \text{Spec}\ \mathcal{A}$ and, after modifying the coordinate functions and coordinate neighborhoods in the way as described above, we see that $\{(\hat{x}_\alpha, N_\alpha), \alpha \in I\}$ is an atlas for Spec $\mathcal{A}$.

2. We already know that the Gelfand transform $\hat{\ } : \mathcal{A} \to C(\text{Spec}\ \mathcal{A})$ is an involutive algebra morphism. Hence it is our goal to show the Gelfand transform implements a bijective correspondence of $\mathcal{A}$ with $C^\infty(\text{Spec}\ \mathcal{A})$. As announced, we first realize this result locally.
Take some $\chi \in \text{Spec}\ \mathcal{A}$ and choose a neighborhood V of χ with compact closure that is small enough to be fully contained within a coordinate neighborhood and on which the assumptions 2 and 3 hold relative to an index $\alpha \in I$.
Take some $f \in C^\infty(x_\alpha(V))$. This implies that $f : \hat{x}_\alpha(V) \to \mathbb{C}$ is smooth. Now $\hat{x}_\alpha(V)$ is the joint spectrum of the self-adjoint elements $x_\alpha = (x_\alpha^1, \ldots, x_\alpha^p)$, so we can apply the smooth functional calculus, assumption 4, to f:

$$f(x_\alpha^1, \ldots, x_\alpha^p) \equiv f(x_\alpha) \in \mathcal{A}. \tag{4.8}$$

Now take some $a \in \mathcal{A}|_V$. We show that $\hat{a} \circ \hat{x}_\alpha^{-1} : \mathbb{R}^p \to \mathbb{C}$ is a smooth function.

$$\hat{a} \circ \hat{x}_\alpha^{-1} = \hat{a} \circ \chi \circ \tau_\bullet \circ (\chi \circ \tau_\bullet)^{-1} \circ x_\alpha^{-1} = \chi \circ \tau_\bullet(a) \circ (x_\alpha \circ \chi \circ \tau_\bullet)^{-1} .$$

By assumption 3, the part in round brackets, $x_\alpha \circ \chi \circ \tau_\bullet$, is smooth. The same assumptions states that $\tau_t(a)$ is a smooth function $\mathbb{R}^p \to \mathcal{A}$, where smoothness is relative to the supremum-norm of $\mathcal{A}$ (see also appendix A). Composing $\tau_t(a)$ with a character does not spoil this smoothness property, hence the latter expression is a smooth map. Hence $\hat{a} \circ \chi \circ \tau_\bullet$ is smooth, so that $\hat{a} \in C^\infty(V)$.

The above two results show that if $f : x_\alpha(V) \to \mathbb{C}$ is smooth, then $f(x_\alpha) \in \mathcal{A}|_V$, and when $a \in \mathcal{A}|_V$, then $\hat{a} \circ x_\alpha^{-1} : x_\alpha(V) \to \mathbb{C}$ is smooth. Hence we want to verify whether

$$\widehat{f(x_\alpha)} \circ \hat{x}_\alpha^{-1} \stackrel{?}{=} f, \tag{4.9}$$
$$\left(\hat{a} \circ \hat{x}_\alpha^{-1}\right)(x_\alpha) \stackrel{?}{=} a. \tag{4.10}$$

We first show that $f(x_\alpha) = f \circ x_\alpha$. Take some $\kappa \in$ Spec $\mathcal{A}$. As in the continuous functional calculus, $f(x_\alpha)$ is defined as the norm-limit of a series in $\mathcal{A}$. Since any character in Spec $A =$ Spec $\mathcal{A}$ is continuous (see lemma 4.10), we see that $\kappa(f(x_\alpha)) = f(\kappa(x_\alpha))$. This holds for all κ in the spectrum, hence $\widehat{f(x_\alpha)}$ coincides with $f \circ \hat{x}_\alpha$. This implies

$$\widehat{f(x_\alpha)} \circ \hat{x}_\alpha^{-1} = f \circ \hat{x}_\alpha \circ \hat{x}_\alpha^{-1} = f,$$

which shows that equation (4.9) holds. Relation (4.10) readily follows from this (bearing in mind that the Gelfand transform is an isomorphism):

$$\widehat{\left(\hat{a} \circ \hat{x}_\alpha^{-1}\right)(x_\alpha)} = \hat{a} \circ \hat{x}_\alpha \circ x_\alpha^{-1} = \hat{a} \qquad \Leftrightarrow \qquad \left(\hat{a} \circ \hat{x}_\alpha^{-1}\right)(x_\alpha) = a.$$

In conclusion, we have the following local relationship:

$$a \in \mathcal{A}|_V \quad \Leftrightarrow \quad \hat{a} \circ \hat{x}_\alpha^{-1} \in C^\infty(V), \tag{4.11}$$

where the association is a bijective correspondence.
The above constructions do not depend on χ or its neighborhood V, so we can state that for each $\kappa \in$ Spec $\mathcal{A}$ we can find a neighborhood V such that (4.11) holds.

3. By assumption 5, relative to any open cover $\{U_\alpha\}$ of Spec $\mathcal{A}$ we can find a "smooth" partition of unity $\{\psi_\alpha\} \subset \mathcal{A}$ such that any $a \in \mathcal{A}$ can be written as

$$a = \sum_\alpha a\psi_\alpha, \tag{4.12}$$

with $\operatorname{supp} \psi_\alpha \subset U_\alpha$, i.e., $a\psi_\alpha \in \mathcal{A}|_{U_\alpha}$. Using compactness of the spectrum we can make sure that the we can find a finite partition of unity relative to a cover for which each of the elements is "small" enough to satisfy the conditions of assumptions 2 and 3. Hence, using the linearity of the Gelfand transform and the result (4.11), we see that:

$$a \in \mathcal{A} \quad \Leftrightarrow \quad \hat{a} \in C^\infty(\text{Spec } \mathcal{A}), \tag{4.13}$$

where the association is a bijective correspondence. Recall that the Gelfand transform is an involutive algebra morphism. Hence (4.13) implements an isomorphism

$$\mathcal{A} \cong C^\infty(\text{Spec } \mathcal{A}). \tag{4.14}$$

□

In the remainder of this chapter we will validate assumptions 1 to 5. We will do this in the following order:

1. In the next section, it will be shown that Spec $\mathcal{A}$ = Spec A (assumption 1);
2. In sections 4.3 and 4.4 we will validate assumptions 2 and 3;
3. The results on the smooth functional calculus and a smooth partition of unity (assumptions 4 and 5) will be discussed in section 4.5.

4.2 Topology of $\mathcal{A}$ and $\mathcal{H}^\infty$

The main result in this subsection is the proof that the character spaces of $\mathcal{A}$ and its closure A are equal. At the end of this section we will treat the topology of $\mathcal{H}^\infty$ in depth.
Before delving into the main results of this section, we first introduce the concept of the **bicommutant** of an algebra. We then cover the definition of **weak operator topology** and **strong operator topology**. We will then relate these concepts by the **bicommutant theorem**.

Definition 4.2 (Bicommutant)**.** *Let $\mathcal{M}$ be a unital and involutive subalgebra of $B(\mathcal{H})$, with $\mathcal{H}$ a Hilbert space. The **commutant** $\mathcal{M}'$ of $\mathcal{M}$ relative to $B(\mathcal{H})$ is an involutive algebra defined by*

$$\mathcal{M}' \equiv \{a \in B(\mathcal{H}), am = ma \ \forall m \in \mathcal{M}\} . \tag{4.15}$$

*The **bicommutant** $\mathcal{M}''$ is the commutant of the commutant of $\mathcal{M}$ relative to $B(\mathcal{H})$.*

The following lemma can be proven using elementary algebra.

Lemma 4.3. *Let $\mathcal{M}$ be as in definition 4.2. Relative to $B(\mathcal{H})$ we have:*

$$\mathcal{M}' = \mathcal{M}''', \tag{4.16}$$

$$\mathcal{M}'' = \mathcal{M}''''. \tag{4.17}$$

Definition 4.4 (Weak and strong operator topology)**.** *Let $\mathcal{M}$ be a unital and involutive subalgebra of $B(\mathcal{H})$. Let $\{T_i; i \in I\}$ be a **net** in $\mathcal{M}$, that is, is a function from a directed set I to $\mathcal{M}$.*
*The weak operator topology of $\mathcal{M}$, or **WOT**, is the weakest topology of $\mathcal{M}$ such that $\langle \eta, T_i\xi\rangle$ converges for all $\eta, \xi \in \mathcal{H}$ and any net in $\mathcal{M}$.*
*The strong operator topology, or **SOT**, is the weakest topology such that $\langle T_i\xi, T_i\xi\rangle$ is continuous for any $\xi \in \mathcal{H}$ and any net in $\mathcal{M}$.*

We now quote the bicommutant theorem. From [25, Thm. 5.3.1]:

Theorem 4.5 (Bicommutant theorem)**.** *Let $\mathcal{M}$ be a unital and involutive subalgebra of $B(\mathcal{H})$. The following three statements are equivalent:*

- $\mathcal{M} = \mathcal{M}''$.
- *$\mathcal{M}$ is closed under the weak operator topology.*
- *$\mathcal{M}$ is closed under the strong operator topology.*

In what follows, we shall need the following result:

Lemma 4.6. *$\mathcal{A}''$ is a C^*-algebra.*

Proof. Via a faithful representation of $\mathcal{A}$ on $B(\mathcal{H})$ (see definition 3.1) we can identify $\mathcal{A}$ with a unital and involutive subalgebra of $B(\mathcal{H})$. This implies that $\mathcal{A}''$ is an unital and involutive subalgebra of $B(\mathcal{H})$. We equip $\mathcal{A}''$ with the operator norm of $B(\mathcal{H})$. Since $\mathcal{A}'' \subset B(\mathcal{H})$, this norm is submultiplicative and satisfies the C^*-property for elements of $\mathcal{A}''$.
Since $\mathcal{A}'' = \mathcal{A}''''$, by lemma 4.3, $\mathcal{A}''$ is closed under both the WOT and the SOT. It follows [25, Ch. 5] that $\mathcal{A}''$ is closed with respect to the operator norm on $B(\mathcal{H})$ as well.
In conclusion, $\mathcal{A}''$ is a C^*-algebra whose norm is given by the operator norm on $B(\mathcal{H})$. □

The last preparatory step is to present an important result. We will often use the following lemma, proven in [14, 2.1]. The proof of the lemma makes integral use of axioms 2, 3, 7, and 8. We state the lemma

Lemma 4.7 (Workhorse lemma)**.** *For any $a \in \mathcal{A}''$, the following conditions are equivalent:*

1. *$a \in \mathcal{A}$;*
2. *$[D, a]$ is bounded and both a and $[D, a]$ belong to the domain of δ^m, for any integer m;*
3. *a belongs to the domain of δ^m, for any integer m;*
4. *$a\mathcal{H}^\infty \subset \mathcal{H}^\infty$.*

Note that

Corollary 4.8. *$\delta^m(a)\xi \in \mathcal{H}^\infty$ for all m and $\xi \in \mathcal{H}^\infty$.*

Proof. For $m = 0$ this is true by the workhorse lemma. Let the claim be correct for m and take $\xi \in \mathcal{H}^\infty$. Then

$$\delta(\delta^m(a))\xi = |D|\delta^m(a)\xi - \delta^m(a)|D|\xi.$$

By definition of $\mathcal{H}^\infty$ and the induction assumption the first term on the right-hand side lies in $\mathcal{H}^\infty$. Hence $|D|\xi \in \mathcal{H}^\infty$ as well, so $\delta^{m+1}(a)\xi \in \mathcal{H}^\infty$. This concludes the proof. □

In theorem 4.9 we will show that $\mathcal{A}$ is a pre-C^*-algebra. When reading the proof of the theorem, the following picture might help. The top row consists of algebras, of which A, $\mathcal{A}''$ and $B(\mathcal{H})$ are C^*-algebras. The bottom row denotes the topology in which the algebra is closed. The seminorms $\{p_k; k \in \mathbb{N}\}$ (which turn out to make $\mathcal{A}$ a Fréchet space) are defined by equation (4.22).

$$
\begin{array}{ccccccc}
\mathcal{A} & \subset & A & \subset & \mathcal{A}'' & \subset & B(\mathcal{H}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\{p_k; k \in \mathbb{N}\} & & \|a\| \equiv \|\pi(a)\|_{\text{op}} & & \text{WOT} & & \|a\|_{\text{op}}
\end{array}
$$

Theorem 4.9. *$\mathcal{A}$ is a pre-C^*-algebra.*

Proof. We first prove that $\mathcal{A}$ is stable under holomorphic functional calculus. We then proceed showing that $\mathcal{A}$ is a Fréchet-space with topology finer than A.
Recall that any C^*-algebra is stable under holomorphic functional calculus [25, Thm 3.3.5], so $f(a)$ is well-defined in $\mathcal{A}''$ for $a \in \mathcal{A}$ and any holomorphic function f defined on an open set U around $\text{Spec}_A a$.
f has a power series expansion which converges to $f(a) \in \mathcal{A}''$ in the SOT and WOT, i.e.,

$$f(a) = \lim_{N\to\infty} \sum_{k=0}^{N} c_k a^k \equiv \lim_{N\to\infty} f_N(a), \qquad a \in \text{dom}\,\delta. \tag{4.18}$$

Take some $N, M, K \in \mathbb{N}$ with $N > M > K$.

$$\|(\delta f_N - \delta f_M)\xi\| \leq \sum_{k=M+1}^{N} |c_k| \left\| \delta \left(a^k\right) \xi \right\|. \tag{4.19}$$

Using the identity

$$\delta\left(a^k\right) = \sum_{i=1}^{k} a^{k-i}\delta(a)a^{i-1},$$

we see that the right-hand side of (4.19) is dominated by

$$\sum_{k=M+1}^{N} \sum_{i=1}^{k} |c_k| \left\| a^{k-1}\delta(a)a^i\xi \right\| \leq \|\delta(a)\|_{\text{op}} \|\xi\| \sum_{k=M+1}^{N} k|c_k| \|a\|^{k-1}. \tag{4.20}$$

For every $z \in U$ the derivative of f is given by

$$f'(z) = \sum_{k=1}^{\infty} kc_k z^{k-1}.$$

The derivative f', as a continuous function on A, has a compact spectrum [25, Ch. 4]. The radius of convergence of f' is therefore strictly larger than $\|a\|$. This implies that the left-hand side of equation (4.20) goes to zero as the integer K goes to infinity. Hence $\delta f_N(a)$ is a Cauchy sequence in $\mathcal{A}''$, thus converge. The closedness of δ (which follows from that of D and the definition of the domain of δ) implies that the sequence converges to $\delta f(a)$.
For higher powers of δ, the explicit expression for $\delta^l(a^k)$ can quickly become quite intractable. However, all of these higher powers satisfy the inequality

$$\sum_{k=M+1}^{N} |c_k| \left\| \delta^l(a^k)\xi \right\| \leq \|\xi\| \sum_{k=M+1}^{N} \frac{k!}{(k-l-1)!} |c_k| \left\| \delta^l(a) \right\|_{op} \|a\|^{k-l}, \quad (4.21)$$

provided that the integer M is large enough. As a complex function, every derivative of any power of f has a radius of convergence larger than $\|a\|$. This implies that $\delta^l f(a)$ exists in $\mathcal{A}''$ for all $l \in \mathbb{N}$, from which it follows that $f(a) \in B^\infty(\mathcal{H})$. Then $f(a) \in \mathcal{A}$, by the workhorse lemma. This finishes the proof that $\mathcal{A}$ is stable under holomorphic functional calculus.
In the spirit of lemma A.3, appendix A, we shall find a countable family of seminorms which define the Fréchet-topology on $\mathcal{A}$. Just as in example 2.17, defining the semi-norms as $\left\| D^k a \right\|_{\text{op}}$ for $k \in \mathbb{N}$ is fruitless. However, the map δ is a differential operators, just like $\not{D}$ in example 2.17. We proceed along identical ways as we have done in case of the circle and define a submultiplicative norm as follows:

$$\rho_k(a) \equiv \begin{pmatrix} a & \delta(a) & \cdots & \frac{\delta^k(a)}{k!} \\ 0 & a & \ddots & \vdots \\ \vdots & \ddots & a & \delta(a) \\ 0 & \cdots & 0 & a \end{pmatrix}, \qquad p_k(a) \equiv \|\rho_k(a)\|_{B(\mathcal{H}^{k+1})}, \quad (4.22)$$

with $\mathcal{H}^{k+1} = \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{k+1\text{-times}}$.
By definition, $\rho_0(a) \equiv a$. The norm on the right-hand side is the operator norm on the Hilbert space $\mathcal{H}^{k+1}$.
The ρ_k have the following properties: $\rho_k(a+b) = \rho_k(a) + \rho_k(b)$ and $\rho_k(\lambda a) = \lambda \rho_k(a)$ for all $\lambda \in \mathbb{C}$ and $a, b \in \mathcal{A}$. Moreover, if $\rho_k(a) = 0$ for all k then $p_0(a) = \|a\| = 0$, showing $a = 0$. So, for each k, ρ_k is a semi-norm. We show these seminorms are submultiplicative and that $\mathcal{A}$ is complete in the topology defined by this family of seminorms.
As we already have verified in example 2.17, $[\rho_k(ab)]_{i,j} = [\rho_k(a)\rho_k(b)]_{i,j}$ for all i, j and k, so we see that ρ_k is a representation of $\mathcal{A}$ on $\mathcal{H}^{k+1}$. The Banach-property of the norm $\|\cdot\|_{B(\mathcal{H}^{k+1})}$ then shows p_k is submultiplicative.

Let $\{a_n\}$ be a Cauchy sequence in $\mathcal{A}$ converging in the topology generated by the seminorms. Then $a_n \to a \in A$, so certainly $a \in \mathcal{A}''$.
The closedness of δ^k shows that $\delta^k(a_n) \to \delta^k(a)$ in $B(\mathcal{H})$, so $a \in \mathcal{A}$ by lemma 4.7. From this, we conclude that $\rho_k(a_n) \to \rho_k(a)$, showing that $a_n \to a$ in the topology generated by the seminorms, as required.
Lastly, recall that the sets $B_\epsilon(a) = \{b \in \mathcal{A}; p_k(a-b) < \epsilon \ \forall k\}$ form a base of the Fréchet-topology. Since $p_0(a) = \|a\|$, this topology is necessarily finer than the one induced by the representation of $\mathcal{A}$ on $\mathcal{H}$.
In conclusion, $\mathcal{A}$ is a pre-C^*-algebra. □

Lemma 4.10. *Spec $\mathcal{A}$ = Spec A.*

Proof. By restriction, any character $\chi \in$ Spec A is a *-morphism from $\mathcal{A}$ to $\mathbb{C}$. So Spec $A \subseteq$ Spec $\mathcal{A}$.
Conversely, if each $\chi \in$ Spec $\mathcal{A}$ turns out to be continuous, then χ can be uniquely extended to a morphism $\tilde{\chi} \in$ Spec A, finishing the proof. We now validate that claim.
From [25, Cor. 1.2.5] we know that χ, as a linear functional on A, is continuous if and only if its kernel is closed. From [25, Cor. 3.2.4] we see that χ's kernel is a maximal two-sided ideal in A. Hence, its kernel is closed and χ is continuous.

□

The Fréchet-algebra structure of $\mathcal{A}$ induces a topology on $p\mathcal{A}^n$. We are now able to extend the algebra isomorphism $\mathcal{H}^\infty \cong p\mathcal{A}^n$ of axiom 7 to an isomorphism of topological spaces as well.

Lemma 4.11. *As topological spaces, $p\mathcal{A}^n \cong \mathcal{H}^\infty$.*

Proof. Choose a set of generators $\{\xi_1, \dots, \xi_n\} \subset \mathcal{H}^\infty$. By axiom 7, the map

$$f : (a_1, \dots, a_n) \mapsto \sum_i \xi_i a_i, \tag{4.23}$$

from $\mathcal{A}^n$ to $\mathcal{H}^\infty$, is linear, surjective, and injective when restricted to $p\mathcal{A}^n$. If we can prove that both $p\mathcal{A}^n$ and $\mathcal{H}^\infty$ are Fréchet-spaces and that this map is continuous in the pertinent Fréchet-topologies, we can apply the open mapping theorem (theorem A.7 in appendix A) to show this map is open, thereby finishing the proof.
Equip $\mathcal{A}^n$ with the n-fold product of the Fréchet-topology on $\mathcal{A}$ and take some sequence $\{\vec{a}_n\} \subset p\mathcal{A}^n$ converging to $\vec{a} \in \mathcal{A}^n$. This implies $p_k(a_n^i - a^i) \to 0$ for large n, for all $i \in \{1, \dots, n\}$ and all k. Then:

$$p_k\left((pa)^i - a_n^i\right) = p_k\left(\sum_j p_{ji}a^i - p_{ji}a_n^i - (a^i - a_n^i)\right) \leq$$

$$\sum_j p_k(p_{ji}(a^i - a_n^i)) + p_k(a^i - a_n^i) \leq \sum_j p_k(p_{ji})p_k(a^i - a_n^i) + p_k(a^i - a_n^i).$$

Thus $p\vec{a} = \vec{a}$, showing $\vec{a} \in p\mathcal{A}^n$. Combine this with lemma A.2, appendix A, to conclude that $p\mathcal{A}^n$ is a Fréchet-space as well.
Let $\{r_k : \mathcal{H}^\infty \to \mathbb{R}^+; k \in \mathbb{N}\}$ with $r_k(\xi) \equiv \left\|D^k\xi\right\|_{\mathcal{H}}$. These are all semi-norms and, if $\left\|D^k\xi\right\|_{\mathcal{H}} = 0$ for all k, then certainly $\|\xi\|_{\mathcal{H}} = 0 \Rightarrow \xi = 0$. Now let $\{\xi_n\} \subset \mathcal{H}^\infty$ be a Cauchy-sequence, i.e. $\left\|D^k(\xi_n - \xi_m)\right\|_{\mathcal{H}} < \epsilon$ for all k and $n, m > N$. The completeness of $\mathcal{H}$ guarantees that $D^k\xi_n \to \xi^{(k)} \in \mathcal{H}$ for all k. Since D and all its power are closed, we see that $\xi^{(k)}$ equals $D^k\xi$. Hence $\mathcal{H}^\infty$ is a Fréchet-space as well.
From lemma 3.5 we gather that
$\|D\xi\|_{\mathcal{H}} = (U|D|\xi, U|D|\xi) \le \|U\|_{\mathrm{op}} \||D|\xi\|_{\mathcal{H}}$, where $D = U|D|$ is the polar decomposition of D. By definition $U|_{\mathrm{ran}|D|}$ is an isometry so $|D|U|D| = |D|^2$. This implies that $\left\|D^k\xi\right\|$ is dominated by $\|U\| \left\||D|^k\xi\right\|$. Let $C \equiv \|U\|_{\mathrm{op}}$, then

$$r_k(f(\vec{a})) \le \sum_i \left\|D^k a_i\xi_i\right\|_{\mathcal{H}} \le C \sum_i \left\||D|^k a_i\xi_j\right\|_{\mathcal{H}}.$$

We now proceed with proving that

$$|D|^k a\xi = \sum_{n=0}^{k} \binom{n}{k} \delta^n(a)|D|^{k-n}\xi \qquad \forall\xi \in \mathcal{H}^\infty, \forall a \in \mathcal{A}.$$

By the definition of $\delta(a)$, this is correct for $k = 0$. By corollary 4.8

$$\begin{aligned} |D|^{k+1}a\xi = \sum_{n=0}^{k} \binom{k}{n} |D|\delta^n(a)|D|^{k-n}\xi = \\ \sum_{n=0}^{k} \binom{k}{n} \left[\delta^{n+1}(a) + \delta^n(a)|D|\right] |D|^{k-n} = \\ \sum_{n=0}^{k+1} \left[\binom{k}{n-1} + \binom{k}{n}\right] \delta^n(a)|D|^{k+1-n}, \end{aligned} \tag{4.24}$$

by shifting $n + 1 \mapsto n$ in the first term of (4.24) and noting that by definition $\binom{k}{k+1} = \binom{k}{-1} = 0$. The property $\binom{k}{n-1} + \binom{k}{n} = \binom{k+1}{n}$ then proves the statement. Hence:

$$r_k(f(\vec{a}) \le \sum_{n,i} C \binom{k}{n} \left\|\delta^n(a_i)|D|^{k-n}\xi_i\right\|_{\mathcal{H}} \le \sum_{n,i} C \binom{k}{n} \|\delta^n(a_i)\|_{op} \left\||D|^{k-n}\xi_i\right\|_{\mathcal{H}},$$

using the boundedness of $\delta(a)$. Evaluating the definition of $p_k(a_i)$, i.e., equation (4.22), directly shows that if $p_k(a_i) \to 0$, then also $\|\delta^n(a_i)\| \to 0$ for all $n \le k$. Hence f is continuous, as required. □

Lemma 4.12. *$End_{\mathcal{A}}(\mathcal{H}^\infty) \cong pM_n(\mathcal{A})p$ as algebras.*

Proof. Take some $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}^\infty)$ and let $f : p\mathcal{A}^n \to \mathcal{H}^\infty$ be an $\mathcal{A}$-module isomorphism, whose existence was postulated by axiom 7. Let $T' = f^{-1} \circ T \circ f$ then $T' \in M_n(\mathcal{A})$. By point-wise evaluation we obtain $pT' = T'p = T'$, so T' lies actually in $pM_n(\mathcal{A})p$. So $T \mapsto T'$ is now an invertible morphism, yielding the desired isomorphism. □

Corollary 4.13.

$$End_{\mathcal{A}}\left(\mathcal{H}^\infty\right) \cong pM_n(\mathcal{A})p \tag{4.25}$$

as topological spaces.

Proof. This follows from lemma 4.11.
The topology on $pM_n(\mathcal{A})p$ is defined by the set of seminorms

$$P_k(A) \equiv \sup\left\{ \sum_{i=1}^{n} p_k\left[(A\vec{a})^i\right]; \vec{a} = (a_1, \ldots, a_n), \sum_{j=0}^{\infty}\sum_{i=1}^{n} p_j(a^i) = 1 \right\}. \tag{4.26}$$

Analogously for $\mathrm{End}_{\mathcal{A}}(\mathcal{H}^\infty)$ we define:

$$R_k(T) \equiv \sup\left\{ r_k(T\xi); \sum_{j=1}^{\infty} r_j(\xi) = 1 \right\}. \tag{4.27}$$

The required isomorphism readily follows from these definitions. □

Remark 4.14. *The previous material can be used to show that A is norm-separable by proving that $\mathcal{A}$ is separable in its Fréchet-topology, which is done in [14, Prop. 2.3 (2)].*

4.3 Local openness

As announced, the coordinate charts of Spec $\mathcal{A}$ will be derived from the noncommutative volume element

$$\mathbf{c} = \sum_\alpha c_\alpha^0 \otimes c_\alpha^1 \otimes \cdots \otimes c_\alpha^p.$$

Let us start defining the function $s_\alpha : \mathrm{Spec}\,\mathcal{A} \to \mathbb{R}^p$ by:

$$s_\alpha \equiv \left(c_\alpha^1, \ldots, c_\alpha^p\right).$$

Though the s_α are closely related to the x_α of assumption 2, they are not identical. In the next section we show how s_α and x_α are related. Here, in

this section, we prove that the s_α are locally open. That is, we show that we can find an open cover $\{U_\alpha\} \subset \text{Spec } \mathcal{A}$ such that $s_\alpha|_{U_\alpha}$ is open for all α. As we proceed we find and define the rest of the functions of assumptions 2 and 3.
We start with constructing an open cover $\{U_\alpha\}$ of Spec $\mathcal{A}$ in the next subsection. In subsection 4.3.2 we shall construct p derivations (i.e., noncommutative vector fields) δ_j of $\mathcal{A}$. It will turn out that for all $\chi \in U_\alpha$, we have:

$$\chi \det\left(\delta_j\left(c_\alpha^k\right)\right) \neq 0.$$

This expression, reminiscent of the condition that the determinant of the Jacobian is non-zero, is then retro-actively used to define a (local) function:

$$h_\chi : \mathbb{R}^p \to \text{Spec } \mathcal{A},$$

with the property that the Jacobian of the function

$$\left(c_\alpha^1 \circ h_\chi, \ldots, c_\alpha^p \circ h_\chi\right) : \mathbb{R}^p \to \mathbb{R}^p$$

does not vanish. We then apply the implicit function theorem to derive the local openness of the s_α on U_α.

4.3.1 A cover of Spec $\mathcal{A}$

Recall that the Hochschild orientation cycle $\mathbf{c}$ is anti-symmetric. Its expansion in $C_p(\mathcal{A})$ can therefore be written as

$$\sum_\alpha \sum_{\beta \in S(p)} \epsilon(\beta) c_\alpha^0 \otimes c_\alpha^{\beta(1)} \otimes \cdots \otimes c_\alpha^{\beta(p)}, \tag{4.28}$$

with $c_\alpha^0 \neq 0$.
Its image under the map π_D, as defined in axiom 4, is given by:

$$\chi = \sum_\alpha \sum_{\beta \in S(p)} \epsilon(\beta) c_\alpha^0 \left[D, c_\alpha^{\beta(1)}\right] \cdots \left[D, c_\alpha^{\beta(p)}\right]. \tag{4.29}$$

By definition of $\mathcal{H}^\infty$, the workhorse lemma and axiom 2, we see that $[D, a] \in \text{End}_\mathcal{A}(\mathcal{H}^\infty)$ for each $a \in \mathcal{A}$. This implies that the whole right-hand side of (4.29) is an element of $\text{End}_\mathcal{A}(\mathcal{H}^\infty)$, showing χ is as well. This implies that for both sides of the following equation:

$$1_{\text{End}_\mathcal{A}(\mathcal{H}^\infty)} = \sum_\alpha c_\alpha^0 \sum_{\beta \in S(p)} \epsilon(\beta) \chi \left[D, c_\alpha^{\beta(1)}\right] \cdots \left[D, c_\alpha^{\beta(p)}\right] \tag{4.30}$$

we can take the trace in $\mathrm{End}_{\mathcal{A}}(\mathcal{H}^\infty)$. Let us define:

$$\rho_\alpha \equiv (-1)^{p+p^2}\mathrm{Tr}\left(\sum_{\beta\in S(p)} \epsilon(\beta)\chi\left[D, a_\alpha^{\beta(1)}\right]\cdots\left[D, a_\alpha^{\beta(p)}\right]\right),$$

where the factor $(-1)^{p+p^2}$ ensures ρ_α is self-adjoint. The map $\rho_\alpha : \mathrm{Spec}\ \mathcal{A} \to \mathbb{C}$ is therefore both real and continuous. This show the sets

$$U_\alpha = \{\chi \in \mathrm{Spec}\ \mathcal{A}; \rho_\alpha(\chi) \neq 0\} \tag{4.31}$$

are open. They cover Spec $\mathcal{A}$ if we can prove that there is always at least one α such that $\rho_\alpha(\kappa) \neq 0$ for each $\kappa \in \mathrm{Spec}\ \mathcal{A}$.

By the Serre–Swan theorem [52], axiom 7 tells us that $\mathcal{H}^\infty$ is a dense subring of $C(\mathrm{Spec}\ \mathcal{A}, \mathfrak{X})$ where $\mathfrak{X}$ is some continuous vector field. The trace of the left-hand side of (4.30) equals the function that adds to each $\chi \in \mathrm{Spec}\ \mathcal{A}$ the dimension of $\mathfrak{X}$ at that point. There is at least one connected component of Spec $\mathcal{A}$ on which this dimension is not zero, otherwise $\mathcal{H}^\infty$ would be trivial. But in fact, by using axiom 6, we can show that there is only one connected component.

Lemma 4.15. *Spec $\mathcal{A}$ is connected.*

Proof. Assume for a second that Spec $\mathcal{A}$ is not connected (say it has disconnected pieces U and V). Then the indicator function of U, 1_U, is a non-trivial projector in $B(\mathcal{H})$ commuting with $\mathcal{A}$. Moreover,

$$[C, 1_U] = C1_U - 1_UC = 1_U^*C - 1_UC = 0.$$

$\pi_D(\mathbf{c})1_U = 1_U\pi_D(\mathbf{c})$, since $[D, a]$ satisfies axiom 2. Lastly,

$$\begin{aligned} &1_U[D, 1_U] = [D, 1_U]1_U \quad\Leftrightarrow\quad 1_UD1_U - D1_U = D1_U - 1_UD1_U \quad\Leftrightarrow \\ &21_UD1_U = D1_U + 1_UD \quad\Rightarrow 1_UD1_U = 1_UD \quad\&\quad 1_UD1_U = D1_U. \end{aligned}$$

We conclude also $[D, 1_U] = 0$, thereby contradicting axiom 6. Hence Spec $\mathcal{A}$ is connected. □

Returning to the discussion about the open cover of Spec $\mathcal{A}$, we see that $\dim\mathfrak{X} = \sum_\alpha c_\alpha^0\rho_\alpha$, implying for each $\chi \in \mathrm{Spec}\ \mathcal{A}$ there is always at least one ρ_α such that $\rho_\alpha(\chi) \neq 0$. In conclusion, $\{U_\alpha\}$ is a cover of Spec $\mathcal{A}$.

4.3.2 Derivations of $\mathcal{A}$

To eventually formulate the implicit function theorem we need to find several derivations on $\mathcal{A}$. When $\mathcal{A} = C^\infty(M)$ for any spin manifold M, we do not need to look very far: the smooth vector fields and the Dirac operator provide us with multiple derivations. So far, for the commutative spectral triple we do not have such powerful objects. Luckily, it turns out that the commutator of D with any $a \in \mathcal{A}$ has a sufficiently rich structure.

Lemma 4.16. *$[D,a]$ equals a finite sum $\sum_i \delta_i(a)\gamma^i$, where the γ^i lie in $End_{\mathcal{A}}(\mathcal{H}^\infty)$ and the δ_i are *-derivations of $\mathcal{A}$, which are continuous in the Fréchet-topology.*

Proof. In the previous section we have seen that $[D,a] \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}^\infty)$.
Given the standard base $\{v_i\}$ of $\mathcal{A}^n$ (i.e., elements consisting of a 1 on the i-th position and zero elsewhere) we can define $e_i = pv_i$, with p as in 7, to find a base of $\mathcal{H}^\infty$. The coefficients of $[D,a]$ in the base $\{e_{ij} \equiv e_i \otimes e_j\} \subset \mathrm{End}_{\mathcal{A}}(\mathcal{H}^\infty)$ are given by:

$$(e_i, [D,a]e_j)_{\mathcal{A}} \equiv \delta'_{i,j}(a). \tag{4.32}$$

We check this is a derivation. $\mathbb{C}$-linearity is obvious from the definition of the inner product. We have:

$$\begin{aligned}&(e_i,[D,ab]e_j)_{\mathcal{A}} = (e_i,[D,a]be_j)_{\mathcal{A}} + (e_i,a[D,b]e_j)_{\mathcal{A}} = \\ &(e_i,b[D,a]e_j)_{\mathcal{A}} + a\delta'_{i,j}(b) = b\delta'_{i,j}(a) + a\delta'_{i,j}(b) = \delta'_{i,j}(a)b + a\delta'_{i,j}(b).\end{aligned}$$

However, $\delta_{i,j}$ is not a *-derivation, as can been seen by explicit verification:

$$\delta'_{i,j}(a^*) = (e_i,[a^*,D]e_j) = (e_i(aD - Da), e_j) = -\overline{(e_i,[D,a]e_j)}.$$

This is fixed by taking the linear combination

$$\delta_{i,j}(a) \equiv \frac{1}{4}(e_i,[D,a]e_j) + \frac{1}{4}(e_j,[D,a]e_i) + \frac{i}{4}(e_i,[D,a]e_j) + \frac{i}{4}(e_j,[D,a]e_i),$$

which is an involutive map.
By changing the base of $\mathrm{End}_{\mathcal{A}}(\mathcal{H}^\infty)$ by $e_{ij} \mapsto \frac{1}{4}(1+i)e_{ij} + \frac{1}{4}(1+i)e_{ji}$ we can see directly that $[D,a]$ expanded in this base is a normal matrix. It can be diagonalized and written as:

$$[D,a] = \sum_i \delta_i(a)\gamma^i, \tag{4.33}$$

for some elements $\gamma^i \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}^\infty)$.
This finishes the first part. We now show δ_i is continuous.
The map $a \mapsto (e_i,[D,a]e_i)$ is the composition of the following maps:

$$\begin{aligned}&f_1 : \mathcal{A} \to \mathrm{End}_{\mathcal{A}}(\mathcal{H}^\infty), && f_1(a) = [D,a],\\ &f_2 : \mathrm{End}_{\mathcal{A}}(\mathcal{H}^\infty) \to \mathcal{H}^\infty, && f_2(T) = Te_i,\\ &f_3 : \mathcal{H}^\infty \times \mathcal{H}^\infty \to \mathcal{A}, && f_3(\eta,\xi) = (\eta,\xi)_{\mathcal{A}}.\end{aligned}$$

By lemma 4.11 and corollary 4.13, for the purpose of this proof we can replace $\mathcal{H}^\infty$ by $p\mathcal{A}^n$ and $\mathrm{End}_{\mathcal{A}}(\mathcal{H}^\infty)$ by $pM_n(\mathcal{A})p$. We consecutively show that the f_i are continuous.

1. Let us define the following mappings:
$$p'_k : \mathcal{A} \to \mathbb{R}^+, \qquad\qquad p'_k(a) = p_k([D, a]). \tag{4.34}$$
These are seminorms as well. Since $[D, \cdot]$, just like δ, is closed, $\mathcal{A}$ is complete with respect to the seminorms p'_k. Let us denote the original Fréchet-topology by $(\mathcal{A}, \mathcal{T})$. We create a new Fréchet-topology on $\mathcal{A}$ by using the norms p_k and p'_k. Denote this one with $(\mathcal{A}, \mathcal{T}')$. The identity map from $(\mathcal{A}, \mathcal{T}')$ to $(\mathcal{A}, \mathcal{T})$ is linear, bijective and continuous by construction. Apply the Open Mapping theorem for Fréchet-spaces, theorem A.7, to see that the two topological spaces are homeomorphic. This implies that f_1 is continuous.

2. By definition of the topology on $\mathrm{End}_{\mathcal{A}}(\mathcal{H}^\infty)$, the function f_2 is continuous.

3. Take some $a, b, c \in p\mathcal{A}^n$ such that for all $i \in \{1, \ldots, n\}$ and all k we have that $p_k(a^i - b^i) < \delta$. Then:
$$p_l\left(\langle a - b, c\rangle_{\mathcal{A}}\right) \leq \sum_{i,j=1}^{n} p_l\left[((a^i)^* - (b^i)^*)p_{ij}c_j\right]$$
for all l due to the submultiplicativity of the seminorms. Combine this with the fact that for the C^*-norm on A we have $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$ to see that the above inequality implies continuity in the first variable of the inner product. Analogously, it can be shown that the $\mathcal{A}$-valued inner product is continuous in the second variable, proving the joint continuity of f_3.

□

The above results yield the following property of the derivations δ_i.

Theorem 4.17. *Let U_α be the cover of Spec $\mathcal{A}$ defined in section 4.3.1. For each $\chi \in U_\alpha$, one has $\chi \det(\delta_j(a^k_\alpha)) \neq 0$.*

Proof. We have (omitting the index α):
$$\sum_{\beta \in S(p)} \epsilon(\beta)\left[D, c^{\beta(1)}\right] \cdots \left[D, c^{\beta(p)}\right] =$$
$$\sum_{\beta, j_1, \ldots, j_p} \epsilon(\beta)\delta_{j_1}\left(c^{\beta(1)}\right)\gamma^{j_1} \cdots \delta_{j_p}\left(c^{\beta(p)}\right)\gamma^{j_p}. \tag{4.35}$$

Let us evaluate equation (4.35) for fixed j_1 to j_p. Assume there are k and i such that $j_k = j_i = n$. Then
$$\sum_{\beta \in S(p)} \epsilon(\beta)\delta_{j_1}\left(c^{\beta(1)}\right) \cdots \delta_n\left(c^{\beta(k)}\right) \cdots \delta_n\left(c^{\beta(i)}\right) \cdots \delta_{j_p}\left(c^{\beta(p)}\right)\gamma^{j_1} \cdots \gamma^{j_p} = 0,$$

since the δ_j commute. This implies the sum in equation (4.35) is over the $j_1 \neq j_2 \neq \ldots \neq j_p$ and the equation can be rewritten as:

$$\sum_{\beta,\alpha \in S(p)} \epsilon(\beta)\delta_{\alpha(1)}(c^{\beta(1)}) \cdots \delta_{\alpha(p)}(c^{\beta(p)})\gamma^{\alpha(1)} \cdots \gamma^{\alpha(p)}.$$

The commutativity of the δ_j's simplifies this sum to

$$\sum_{\alpha \in S(p)} \det\left(\delta_j\left(c^k\right)\right)\gamma^{\alpha(1)} \cdots \gamma^{\alpha(p)}. \qquad (4.36)$$

Recall that up to a constant and up to composition by χ, equation (4.36) equals ρ_α (when inserting back the indices α, that is). Now ρ_α does not vanish on U_α, so neither does $\chi \det\left(\delta_j\left(a_\alpha^k\right)\right)$ for all $\chi \in U_\alpha$.

□

4.3.3 The maps h_χ

Take some $\chi \in U_\alpha$, where the U_α form the cover of Spec $\mathcal{A}$ as defined in subsection 4.3.1.
Theorem 4.17 gives us a hint about how to define a local inverse h_χ for s_α. We are looking for a function $h_\chi : \mathbb{R}^p \to$ Spec $\mathcal{A}$ satisfying

$$\left.\frac{\partial}{\partial t^j} c_\alpha^k \circ h_\chi\right|_{t=0} = \chi\delta_j(c_\alpha^k). \qquad (4.37)$$

The first step in solving the above equation is an analysis of the following differential equation. See appendix A for more background regarding Fréchet-derivatives.

Theorem 4.18. *For each derivation δ_j as defined in lemma (4.16), the expression*

$$\frac{d}{dt}\sigma_t^j(a) = \delta_j(\sigma_t^j(a)) \qquad (\sigma_0 = id_{\mathcal{A}}) \qquad \forall a \in \mathcal{A} \qquad (4.38)$$

has a unique solution of class C^1. Let $\sigma_t \equiv \sigma_{t_1}^1 \circ \cdots \circ \sigma_{t_p}^p$. Then

1. *$\sigma_t \in Aut(\mathcal{A})$;*

2. *$\sigma_t(a)$ is of class C^∞ for all $a \in \mathcal{A}$.*

Proof. In [14, §5,§6] it is proven that (4.38) not only has the required unique solution, but also that $\sigma_t^j : \mathcal{A} \to \mathcal{A}$ is continuous. The fact that such a solution exists depends on a certain property of the derivation δ_j. This property of derivations of $\mathcal{A}$ is called **expability**. Note that the proof depends on the spectral triple being *strongly* regular instead of just regular. See section 3.1 for definitions.
Let us now verify 1. and 2. of theorem 4.18.

1. Uniqueness of the solution of (4.38) implies that any two maps $s, r : \mathbb{R} \times \mathcal{A} \to \mathcal{A}$ satisfying (4.38) are equal. So, for instance, we can show $\mathbb{C}$-linearity of the solution:

$$\frac{d}{dt}\sigma_t^j(\lambda a) = \delta_j(\sigma_t^j(\lambda a)), \qquad \sigma_0^j(\lambda a) = \lambda a \qquad \&$$
$$\frac{d}{dt}\lambda\sigma_t^j(a) \overset{(*)}{=} \lambda\frac{d}{dt}\sigma_t^j(a) = \delta_j(\lambda\sigma_t^j(a)), \qquad \lambda\sigma_0(a) = \lambda a,$$

where at $(*)$ we have used the linearity of the Fréchet derivative (see lemma A.9). Hence $\sigma_t^j(\lambda a) = \lambda\sigma_t^j(a)$. In the same manner we can exploit the fact that δ_j is a *-derivation and that the Fréchet-derivative is $\mathbb{C}$-linear to see that

$$\sigma_t^j(a+b) = \sigma_t^j(a) + \sigma_t^j(b), \tag{4.39}$$
$$\sigma_t^j(ab) = \sigma_t^j(a)\sigma_t^j(b), \tag{4.40}$$
$$\sigma_t^j(a^*) = \sigma_t^j(a)^*. \tag{4.41}$$

Hence $\sigma_t^j \in \mathrm{Aut}(\mathcal{A})$ and, as a composition of automorphisms, also σ_t is an automorphisms of $\mathcal{A}$.

2. Assume for now that $\delta_j(\sigma_t^j(a))$ is of class C^1 with derivative $\delta_j^2(\sigma_t^j(a))$. By comparing the derivatives

$$\frac{d}{dt}\delta_j(\sigma_t^j(a)) = \delta_j \circ \delta_j(\sigma_t^j(a)) \quad \& \tag{4.42}$$
$$\frac{d}{dt}\sigma_t^j(\delta_j(a)) = \delta_j\sigma_t^j(\delta_j(a)), \tag{4.43}$$

which for $t = 0$ give the same result on the boundary, and using uniqueness we see that $\delta_j(\sigma_t^j(a)) = \sigma_t^j(\delta_j(a))$. This not only shows $\sigma_t^j(a)$ is of class C^∞ but also proves that all its higher derivatives $\frac{d^k}{dt^k}\sigma_t^j$ are continuous maps from $\mathcal{A}$ to itself.
We now verify that $\delta_j(\sigma_t^j(a))$ is of class C^1. Indeed,

$$p_k\left(\frac{1}{\epsilon}\left[\delta_j(\sigma_{t+\epsilon s}^j(a)) - \delta_j(\sigma_t(a))\right] - \delta_j^2(\sigma_t^j(a))\right) =$$
$$p_k\left(\delta_j\left(\frac{1}{\epsilon}\left(\sigma_{t+\epsilon s}^j(a) - \sigma_t^j(a)\right) - \delta_j(\sigma_t^j(a))\right)\right). \tag{4.44}$$

As the argument of δ_j in (4.44) goes to zero in the Fréchet-topology (which it does by differentiability of $\sigma_t^j(a)$), the continuity of δ_j in that same topology dictates that also (4.44) goes to zero. The limit exists, and formally we write

$$\frac{d}{dt}\delta_j(\sigma_t^j(a)) \equiv D(t,s). \tag{4.45}$$

Continuity in the first argument follows from continuity of $\sigma_t(a)$ as a map $\mathbb{R} \to \mathcal{A}$ and continuity of the δ_j. This leaves us with:

$$p_k(D(t,s_1) - D(t,s_2)) = p_k\left(\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\delta_j\left(\sigma^j_{t+\epsilon s_1}(a) - \sigma^j_{t+\epsilon s_2}(a)\right)\right). \tag{4.46}$$

By comparing again $\frac{d}{dt_1}\sigma^j_{t_1}(\sigma^j_{t_2}(a)) = \delta_j(\sigma^j_{t_1}(\sigma^j_{t_2}(a)))$ and $\frac{d}{dt}\sigma^j_t(a) = \delta_j(\sigma^j_t(a))$, which coincide for $t = t_1 + t_2 = 0$, we can rewrite (4.46) as

$$p_k\left(\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\delta_j\sigma^j_t\left(\sigma^j_{\epsilon s_1}(a) - \sigma^j_{\epsilon s_2}(a)\right)\right).$$

The continuity of σ^j_t shows the right hand side of (4.46) goes to zero as $s_1 \to s_2$.
By lemma A.9 in appendix A, the composition of smooth functions is smooth, so $\sigma_t(a)$ is of class C^∞.

□

We now define h_χ as follows:

Definition 4.19. *Fix some α and take $\chi \in U_\alpha$, where $\{U_\alpha\} \subset$ Spec $\mathcal{A}$ is the open cover constructed in subsection 4.3.1. Define the map $h_\chi : \mathbb{R}^p \to$ Spec $\mathcal{A}$ by:*

$$(h_\chi)_t \equiv \chi \circ \sigma_t \qquad t \in \mathbb{R}^p, \tag{4.47}$$

$$a \circ h_\chi \equiv \chi \circ \sigma_\bullet(a) \qquad \forall a \in \mathcal{A}. \tag{4.48}$$

Corollary 4.20. *$h_\chi : \mathbb{R}^p \to$ Spec $\mathcal{A}$ is a continuous function, $c^k_\alpha \circ h_\chi$ is a smooth function at $t = 0$, and*

$$\frac{\partial}{\partial t^j}c^k_\alpha \circ (h_\chi)_t\bigg|_{t=0} = \chi\left(\delta_j(c^k_\alpha)\right). \tag{4.49}$$

Proof. The topology of Spec $\mathcal{A}$ is given by point wise convergence. Recall that in corollary 4.10 we have shown that any character is a continuous function. This implies $\chi \circ \sigma_\bullet(a) : \mathbb{R}^p \to \mathbb{C}$ is continuous for each $a \in \mathcal{A}$. The characters are linear by definition and the Fréchet-topology on $\mathcal{A}$ is finer than the one induced by the representation on $\mathcal{H}$. Therefore, we can use the same method as in the proof of theorem 4.18.2 to show $\chi \circ \sigma(a)$ is a smooth function. Since σ is an automorphism of $\mathcal{A}$, $c^k_\alpha \circ \chi \circ \sigma$ is a real smooth function. Let $t = (t_1, \ldots, t_p)$. Using the chain rule of lemma A.9 in appendix A, we see that

$$\frac{\partial}{\partial t^j}c^k_\alpha \circ (h_\chi)_t\bigg|_{t=0} = \chi\,\frac{\partial}{\partial t^j}\sigma^1_{t_1} \circ \cdots \circ \sigma^p_{t_p}(c^k_\alpha)\bigg|_{t=0} =$$
$$\chi\left(\sigma^1_0 \circ \cdots \circ \sigma^{j-1}_0 \circ \delta_j \circ \sigma^{j+1}_0 \circ \cdots \sigma^p_0(c^k_\alpha)\right) = \chi(\delta_j(c^k_\alpha)).$$

□

4.3.4 Local openness of s_α

Combining equation (4.49) with result 4.17, we see that for all $\chi \in U_\alpha$:

$$\frac{\partial}{\partial t^j} c_\alpha^k \circ h_\chi \Big|_{t=0} \neq 0.$$

As a result of the implicit function theorem, this implies that the function $s_\alpha \circ h_\chi$ is a local diffeomorphism of some neighborhood of zero in $\mathbb{R}^p$ with $s_\alpha(\chi) \in \mathbb{R}^p$. Using this fact, we show:

Lemma 4.21. *$s_\alpha : U_\alpha \to \mathbb{R}^p$ is an open map (in the relative topology on U_α, i.e., s_α : Spec $\mathcal{A} \to \mathbb{R}^p$ is locally open).*

Proof. Take some open subset $V \subset U_\alpha$. By corollary 4.20, $h_\chi^{-1}(V)$ is open. So

$$s_\alpha \circ h_\chi \circ h_\chi^{-1}(V) = s_\alpha(V) \tag{4.50}$$

is open as well. □

4.4 Injectivity of s_α

In short, we have gathered from the preceding section that there exists an open cover $\{U_\alpha\} \subset$ Spec $\mathcal{A}$ such that:

- $s_\alpha : U_\alpha \to \mathbb{R}^p$ is open;
- For each $\chi \in U_\alpha$ there is an open $V \subset \mathbb{R}^p$, with $0 \in V$, and a neighborhood $V' \subset s_\alpha(U_\alpha)$ of $s_\alpha(\chi)$, such that $s_\alpha \circ \chi \circ \sigma : V \to V'$ is a diffeomorphism.
- The map $h_\chi : \mathbb{R}^p \to$ Spec $\mathcal{A}$, defined by

$$(h_\chi)_t \equiv \chi \circ \sigma_t, \tag{4.51}$$

  is continuous.

From [14, Thm. 10.3] we gather:

Lemma 4.22. *Take any open subset $V \subset U_\alpha$ such that $\overline{V} \subset U_\alpha$. There is a dense and open $Y \subset s_\alpha(V)$ such that for each $\chi \in s_\alpha^{-1}(Y) \cap V$ there is a neighborhood N with $s_\alpha|_N$ injective.*

This lemma paves the way for the next important result:

Theorem 4.23. *Assumptions 2 and 3 of theorem 4.1, as stated in subsection 4.1, hold true for a commutative spectral triple.*

Proof. We use the same notation as in lemma 4.22. Take $\chi \in \text{Spec}\,\mathcal{A}$. Let d be any metric on the spectrum of $\mathcal{A}$. Due to the fact that Spec $\mathcal{A}$ is a compact Hausdorff space, by Urysohn's metrization theorem, such a metric exists [37]. There is an α such that $\chi \in U_\alpha$. Let ϵ be small enough such that

$$B_\epsilon(\chi) \equiv \{\kappa \in \text{Spec}\,\mathcal{A}; d(\chi,\kappa) < \epsilon\} \subset U_\alpha.$$

Then $\overline{B_\epsilon(\chi)} \subset U_\alpha$ as well. We could be done very quickly if it happened that, in applying lemma 4.22 it would be guaranteed that $s_\alpha(\chi) \in Y$. This is unfortunately not the case, so we need to shift around in the spectrum of $\mathcal{A}$ a bit until we hit a point in Y.
Since $\chi \circ \sigma$ is continuous, we can find a small neighborhood (say W) of $0 \in \mathbb{R}^p$, contained in V, such that W is mapped to $B_\epsilon(\chi)$. Then $s_\alpha \circ \chi \circ \sigma$ maps W bijectively to an open set around $s_\alpha(\chi)$ which should intersect Y by virtue of the density of Y. Let therefore $t_0 \in V \subset \mathbb{R}^p$ be such that $s_\alpha \circ \chi \circ \sigma_{t_0} \in Y$ (which, again, is possible because Y is dense). Define $\chi_0 \equiv \chi \circ \sigma_{t_0}$. By lemma 4.22 there is an open neighborhood N_{χ_0} of χ_0 such that $s_\alpha|_{N_{\chi_0} \cap U_\alpha}$ is a homeomorphism. Let:

$$x_\alpha^k \equiv \sigma_{t_0}(c_\alpha^k) \qquad , \qquad \tau_t = \sigma_{t_0+t} \circ \sigma_{t_0}^{-1}. \tag{4.52}$$

The remainder of this proof is devoted to showing that this shift in $\mathcal{A}$ does not spoil the homeo- and isomorphism properties of the shifted maps x_α and τ_t. Indeed:

1. $x_\alpha \equiv (x_\alpha^1, \ldots, x_\alpha^p)$ is a homeomorphism of some neighborhood of χ with a neighborhood of $x_\alpha(\chi)$.
   Proof. Let $N_\chi \equiv \{\kappa \circ \sigma_{t_0}^{-1}; \kappa \in N_{\chi_0}\}$. As remarked earlier, σ_{t_0} is a homeomorphism of $\mathcal{A}$, so N_χ is an open neighborhood of χ. By construction, $x_\alpha|_{N_\chi \cap U_\alpha}$ is a homeomorphism with some open neighborhood of $x_\alpha(\chi)$.

2. $x_\alpha \circ \chi \circ \tau_t$ is a diffeomorphism from an open neighborhood of 0 to an open neighborhood of $x_\alpha(\chi)$.
   Proof. $x_\alpha \circ \chi \circ \tau_t = s_\alpha \circ \chi \circ \sigma_{t+t_0}$. The right-hand side is differentiable as a composition of a differentiable function with a translation in $\mathbb{R}^p$. Since $t_0 \in V$ by construction, also $x_\alpha \circ \chi \circ \tau_t$ has the same local diffeomorphism property as $s_\alpha \circ \chi \circ \tau_t$.

3. $\chi \circ \tau_t$ is a homeomorphism of an open set around $0 \in \mathbb{R}^p$ with some neighborhood of χ in Spec $\mathcal{A}$.
   Proof. This follows directly from the previous two statements.

This finishes the verification of assumptions 2 and 3 of subsection 4.1.

□

Due to compactness of Spec $\mathcal{A}$, a finite number of the N_χ cover U_α. Denote the index set of these χ by $\mathcal{X}_\alpha$. Then the collection $\{(x_\alpha, N_\chi); \chi \in \mathcal{X}_\alpha\}$ is a good candidate for an atlas on Spec $\mathcal{A}$. Re-arranging these index sets into a single index α with index set I we get the notation $\{(x_\alpha, N_\alpha), \alpha in I\}$ as used in theorem 4.1.

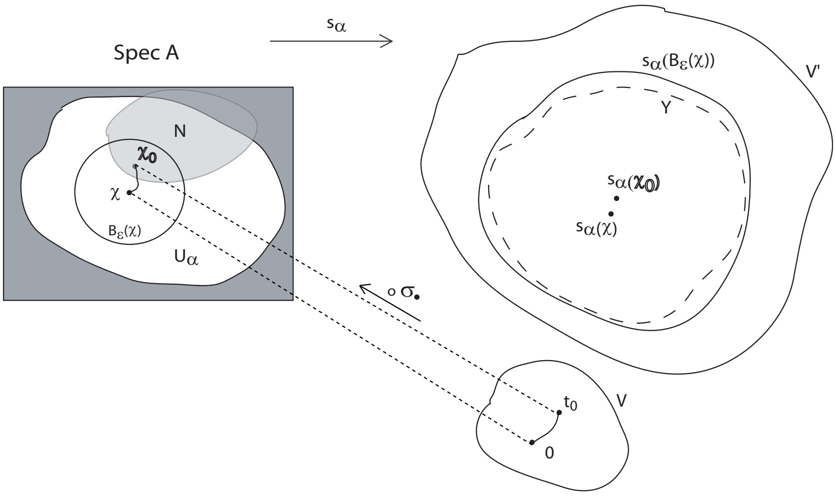

Figure 4.1: The relevant open sets and maps illustrating the properties of the maps s_α and $\chi \circ \sigma$, as defined in theorem 4.23. Note that, due to the continuity of σ, we can adjust χ_0 such that $t_0 \in V$.

4.5 Smoothness properties of $\mathcal{A}$

We have already seen that $\mathcal{A}$ is stable under holomorphic functional calculus. Holomorphic functions can be seen as a special type of differentiable functions (i.e., the harmonic ones) and therefore stability under smooth functional calculus is a much stronger statement. We only need to apply smooth functional calculus within the context of theorem 4.1, so it suffices to show the stability of $\mathcal{A}$ under smooth functional calculus of self-adjoint elements of $\mathcal{A}$ only.

Theorem 4.24 (Smooth functional calculus)**.** *Let $\{a_1, \ldots, a_n\}$ be a set of self-adjoint elements of $\mathcal{A}$ and denote their joint spectrum by $\Delta \subset \mathbb{R}^n$. Take $f : \mathbb{R}^n \to \mathbb{C}$ a smooth function defined in a neighborhood of Δ. Define $f(a_1, \ldots, a_n)$ as the image of $f|_\Delta$ in the C^*-algebra generated by $\{1, a_1, \ldots, a_n\}$ under the Gelfand isomorphism. Then $f(a_1, \ldots, a_n) \in \mathcal{A}$.*

Proof. We start with finding a different expression for $f : \mathbb{R}^n \to \mathbb{R}$ using Fourier analysis.
Note that the joint spectrum Δ is compact. By using a suitable smooth bump function, f can be restricted to a function in $C_c^\infty(\mathbb{R}^n, \mathbb{R})$ such that its restriction equals f on Δ. Take some $(y_1, \ldots, y_n) \in \mathbb{R}^n$, so $\exp(is_j y_j)^* = \exp(-is_j y_j)$ for any real $s_j \in \mathbb{R}$, with j running from 1 to n. We express $f(y_1, \ldots, y_n)$ using the Fourier transform:

$$f(y_1, \ldots, y_n) = \frac{1}{(2\pi)^n} \int \hat{f}(s_1, \ldots, s_n) \exp\left(i \sum_{j=1}^{n} s_j y_j\right) ds_1 \cdots ds_n, \quad (4.53)$$

where $\hat{f}$ is a Schwartz-function of class $\mathcal{J}$.[2]
Now take a set of self-adjoint $\{a_1, \ldots, a_n\} \subset \mathcal{A}$. Since f is a continuous function, the continuous functional calculus of $\mathcal{A}''$ implies that $f(a_1, \ldots, a_n) \in \mathcal{A}''$. We will use the workhorse lemma to show that $f(a_1, \ldots, a_n) \in \mathcal{A}$ by showing that $f(a_1, \ldots, a_n) \in B^\infty(\mathcal{H})$. This is similar to the strategy used in theorem 4.9.
The first step is to rewrite $f(a_1, \ldots, a_n) \in \mathcal{A}''$ by a "Fourier transform", that is, we apply the continuous functional calculus of $\mathcal{A}''$ to the right-hand side of equation (4.53). Using the fact that the a_i are self-adjoint, so $\exp(is_j a_j)^* = \exp(-is_j a_j)$, we obtain $f(a_1, \ldots, a_n) \in \mathcal{A}''$, where $f(a_1, \ldots, a_n)$ is defined by (4.53).
As in theorem 4.9, we use the workhorse lemma and proceed by showing that $f(a_1, \ldots, a_n) \in B^\infty(\mathcal{H})$. First, note that by the result of the aforementioned theorem,

$$\exp\left(i \sum_{j=1}^{n} s_j a_j\right) \in \text{dom } \delta^k \qquad \forall k \in \mathbb{N}.$$

We simplify the exposition a bit by first setting $n = 1$. We prove:

$$\left\|\delta^k \exp(isa)\right\| \leq C_k |s|^k \qquad \forall k \quad \& \quad |s| \to \infty. \quad (4.54)$$

Let us derive an expression for $k = 1$. In $B(\mathcal{H})$, the operator

$$e^{ista} \delta(a) e^{isa(t-1)}$$

is bounded and dominated by $\|\delta(a)\|_{\text{op}}$. This allows us to integrate the

[2] A smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is **Schwartz** when $\|f\|_{\vec{\alpha},\vec{\beta}} \equiv \sup_{y \in \mathbb{R}^n} |y^{\vec{\alpha}} \partial_{\vec{\beta}} f(x)| < \infty$ for each multi-index $\vec{\alpha}, \vec{\beta}$.

function as a function of t:

$$is\int_0^1 \exp(isat)\delta(a)\exp(isa(t-1))dt =$$
$$\sum_{n,k=0}^{\infty} \frac{(is)^{n+k+1}}{n!k!} t^n(1-t)^k a^n\delta(a)a^k dt =$$
$$\sum_{n,k=0}^{\infty} \frac{(is)^{n+k+1}}{(n+1)!(k-1)!(n+k+1)} a^n\delta(a)a^k dt.$$

We rearrange the sum in the following way: first, we sum over all $n+k=m-1$, and then sum over all $m>0$. There is double-counting involved by the amount equal to the number of ways we can choose n out of $m-1$. Correcting for double counting, the above equation equals:

$$\sum_{m=1}^{\infty} \frac{(is)^m}{m} \sum_{n=1}^{m-1} \binom{m-1}{n}^{-1} \frac{1}{n!}\frac{1}{((m-1)-n)!} a^{n-1}\delta(a)a^{m-n} =$$
$$\sum_{m=1}^{\infty} \frac{(is)^m}{m\cdot(m-1)!} \sum_{n=1}^{m-1} a^{n-1}\delta(a)a^{m-n}.$$

According to the results in theorem 4.9, this expression equals $\delta\exp(isa)$. Hence

$$\left\|\delta^1\exp(isa)\right\|_{\mathrm{op}} = |s|\left\|\int_0^1 \exp(isat)\delta(a)\exp(isa(t-1))dt\right\|_{\mathrm{op}}$$
$$\leq |s|\int_0^1 \|\delta(a)\|_{\mathrm{op}}\,dt \equiv \tilde{C}_1|s|.$$

The general expression for $\delta^k(\exp(isa))$, which is somewhat more involved but derived in the same manner, is given by equation (18) of [14]. Let $\beta_t(T)=\exp(itsa)T\exp(-itsa)$ for any real t and operator $T\in B(\mathcal{H})$. Then:

$$\frac{1}{k!}\delta^k\left[\exp(isa)\right]\exp(-isa) =$$
$$\sum_{\alpha_j\in\mathbb{N}^+;\sum\alpha_j=k} i^l s^l \int_{\sigma_l} \beta_{u_1}\left(\frac{\delta^{\alpha_1}(a)}{\alpha_1!}\right)\cdots\beta_{u_l}\left(\frac{\delta^{\alpha_l}(a)}{\alpha_l!}\right)d\vec{u}, \tag{4.55}$$

where the integral is over the standard l-simplex σ_l: $\sum_{i=1}^{l} u_i=1$, $u_i>0$ and $u_1\leq\ldots\leq u_l$. Hence

$$\left\|\delta^k\exp(isa)\right\|_{\mathrm{op}} \leq k! \sum_{\alpha_j\in\mathbb{N}^+;\sum\alpha_j=k} |s|^l \left\|\delta(a)^{\alpha_1}\right\|_{\mathrm{op}}\cdots\left\|\delta(a)^{\alpha_l}\right\|_{\mathrm{op}}.$$

For large $|s|$, the highest-order term of $|s|^l$ is the dominant one, so we obtain

$$\left\|\delta^k \exp(isa)\right\|_{\text{op}} \leq S_k|s|^k \left\|\delta(a)^k\right\|_{\text{op}} \equiv \tilde{C}_k|s|^k \qquad |s| \to \infty. \tag{4.56}$$

Therefore, in general:

$$\left\|\delta^k \exp\left(i\sum_{j=1}^{n} s_j a_j\right)\right\|_{\text{op}} =$$

$$\left\|\sum_{k_1=0}^{k}\binom{k}{k_1}\delta^{k_1}\exp(is_1a_1)\sum_{k_2=0}^{k-k_1}\binom{k-k_1}{k_2}\delta^{k_2}\exp(is_2a_2)\cdots\right.$$

$$\left.\sum_{k_{n-1}}^{k-\sum_{j=1}^{n-2}k_j}\binom{k-\sum_{j=1}^{n-2}k_j}{k_{n-1}}\delta^{k_{n-1}}\exp(is_{n-1}a_{n-1})\delta^{k-\sum_{j=1}^{n-1}k_j}\exp(is_na_n)\right\|_{\text{op}}.$$

This expression is the norm of a finite sum of terms which, for large $|s_j|$, are dominated by bounds of the form of equation (4.56). Note that for each term the powers of δ are never larger than k. Hence we can find a constant C_k such that:

$$\left\|\delta^k\left(i\sum_{j=1}^{n} s_j a_j\right)\right\|_{op} \leq C_k|s|^{n\cdot k} \qquad |s| \to \infty. \tag{4.57}$$

Since δ and its powers are closed operators we have:

$$\delta^k f(a_1,\ldots,a_n) = \frac{1}{(2\pi)^n}\int \hat{f}(s_1,\ldots,s_n)\delta^k \exp\left(i\sum_j s_j a_j\right) ds_1\cdots ds_n$$

where the right-hand side exists in $B(\mathcal{H})$ because $\hat{f}$ is a Schwarz-function, using the inequality of equation (4.57). I.e., $f \in \mathcal{A}'' \cap B^\infty(\mathcal{H}) = \mathcal{A}$, as required. □

The last property of $\mathcal{A}$ that we need to prove is that it admits a "smooth" partition of unity. As a compact space, Spec $\mathcal{A}$ admits a continuous partition of unity with respect to some open cover. If Spec $\mathcal{A} = M$, for some manifold M, what we then could do is use the continuous partition of unity in the following way. Let $\{U_\alpha\}$ be an open cover of Spec $\mathcal{A}$. Let 1_M be a partition of unity such that

$$1 = \sum_\alpha \psi_\alpha, \tag{4.58}$$

with supp $\psi_\alpha \subset U_\alpha$ for each α. Now, multiply each of the ψ_α with smoothing operators such that the newly created (smooth) partition of unity lies arbitrarily close to the old continuous one in the supremum norm on $C(M)$. One could imagine that a continuous partition would involve several pillbox type of functions (with slightly inclined edges). Operating on these pillboxes with smoothing operators basically means that we smoothen out the offset at the lid of the pillbox. For the case of a commutative spectral triple, such a constructive proof does not lie within comfortable reach. We therefore move to the framework of algebraic K-theory by noting that each continuous partition of unity determines a one-dimensional bundle on Spec A. Since $K_0(A) \cong K_0(\mathcal{A})$ (see theorem F.4 in appendix F) we can reconstruct a partition of unity on $\mathcal{A}$ from the vector bundle associated to the projection in the one-dimensional continuous bundle. Roughly, one takes the element $[p] \in K_0(A)$ representing the continuous partition of unity and uses a continuous path of idempotents in $M_N(\mathcal{A})$ (for large $N \in \mathbb{N}$) to construct an element $[p'](=[p]) \in K_0(\mathcal{A})$ such that

$$\left\|p - p'\right\|_{\text{op}} < \epsilon \tag{4.59}$$

for any $\epsilon > 0$. Then, one uses p' to construct a partition of unity of $\mathcal{A}$. In [44, Lem 2.10] and [55], these steps are described in detail. We quote the result here:

Theorem 4.25 (A smooth partition of unity)**.** *Let $\{U_\alpha\}$ be any finite open cover of Spec $\mathcal{A}$. There exist $\psi_\alpha \in \mathcal{A}$ with supp $\psi_\alpha \subset U_\alpha$, $\psi_\alpha \in [0,1]$ and $\sum_\alpha \psi_\alpha(\chi) = 1$ for each $\chi \in$ Spec $\mathcal{A}$.*

In summary, we have done the following in this chapter. For a commutative spectral triple satisfying the eight axioms first discussed in chapter 3, we have shown in the first section that the spectrum of $\mathcal{A}$ is a manifold, given five assumptions. In the preceding four sections, we then showed that for a commutative spectral triple satisfies the eight axioms, the assumptions are correct. That finalizes the proof of Connes' reconstruction theorem.

Chapter 5

Spin structure on Spec $\mathcal{A}$

In the previous chapter we have established that, given axioms 1 to 4, 6 and 7, the pre-C^*-algebra $\mathcal{A}$ equals the algebra of smooth functions on some compact and connected manifold M of dimension p. In this chapter we prove three additional properties, which are all related to the topology of this manifold. We start showing in section 5.1 that M has no boundary and is orientable. Also, the deferred proof of the positive definiteness of the noncommutative integral will be given in this section. The last part, section 5.2, is devoted to showing that the manifold has a Spin structure. In what follows, we shall denote the manifold as constructed in the previous chapter by M. Whenever we refer to "the" or "this" manifold, M is implied.

5.1 Closedness and orientability

Lemma 5.1. *M has no boundary: $\partial M = \emptyset$.*

Proof. Assume M has at least one boundary point z. In that case, there is a chart (x, U) around z such that $x(U)$ lies in the half-plane $H^p \equiv \{(y_1, \ldots, y_p) \in \mathbb{R}^p; y_p \geq 0\}$ with $x(z) \in \partial H^p$. In that case, according to the results in section 4.4, there is another chart $(x_\alpha, N_\alpha \cap U)$ with $x_\alpha(N_\alpha \cap U) \subset H^p$ and an open neighborhood of $x_\alpha(z) = x(z)$. With respect to the topology on $\mathbb{R}^p$, however, the point $x(z)$ cannot have any open neighborhoods that are contained in H^p, so this is a contradiction. Therefore, ∂M is empty. □

The remarks at the beginning of section 3.2 show that if we want to prove that the manifold is orientable we might as well construct a section of $\Omega^p_{\mathbb{R}}(M)$ that is non-zero everywhere. Specifically, from [19, Ch. 2.8]:

Lemma 5.2. *A p-dimensional manifold M is orientable if and only if there is a section $\nu \in \Omega^p_{\mathbb{R}}(M)$ such that $\nu(x) \neq 0$ for all $x \in M$ (i.e., ν is a* ***volume form****).*

In section 3.3 we omitted the proof that the noncommutative integral

$$\fint : \mathcal{A} \to \mathbb{C} \tag{5.1}$$

is a faithful non-degenerate map. The volume form and the integral are related by the usual procedure of integration on any oriented manifold. It is therefore no big surprise that in the next theorem we can demonstrate the existence of a volume form and the positive definiteness of the integral in one go. The map ϕ_ω^D, as defined in equation (3.54), is pivotal to the argument. We therefore first list some properties of this map. Recall that the Hochschild cochains are defined as

$$C^p(\mathcal{A}) \equiv \mathrm{Hom}\left(A^{\otimes(p+1)}, \mathbb{C}\right). \tag{5.2}$$

Refer to appendix D and E for notation and other background information regarding cyclic cohomology.

Lemma 5.3. *The map $\phi_\omega^D \in C^p(\mathcal{A})$ has the following properties:*

1. *ϕ_ω^D is a Hochschild cocycle;*

2. *Let $A_n : C^n(\mathcal{A}) \to C^n(\mathcal{A})$ be the anti-symmetrization map for Hochschild cochains, defined by*

$$A_n\varphi(f_0, \ldots, f_n) = \frac{1}{n!} \sum_{\beta \in S_n} \epsilon(\beta)\varphi\left(f_0, f_{\beta(1)}, \ldots, f_{\beta(n)}\right). \tag{5.3}$$

Then $A_p\varphi_\omega^D$ is a Hochschild cocycle, too, and in Hochschild cohomology we have the equality

$$\left[\phi_\omega^D\right] = \left[A_p\phi_\omega^D\right]. \tag{5.4}$$

3.

$$B\left[\phi_\omega^D\right] = 0. \tag{5.5}$$

Proof.

1. This is verified by direct evaluation:

$$
\begin{aligned}
&b\phi_\omega^D(f_0,\ldots,f_{p+1}) = \\
&\mathrm{Tr}_\omega\left(\chi f_0 f_1[D,f_2]\cdots[D,f_{p+1}]|D|^{-p}\right) + \\
&\mathrm{Tr}_\omega\left(\chi\sum_{i=1}^{p}(-1)^i f_0[D,f_1]\cdots[D,f_i f_{i+1}]\cdots[D,f_{p+1}]|D|^{-p}\right) + \\
&(-1)^{p+1}\mathrm{Tr}_\omega\left(\chi f_0 f_{p+1}[D,a_1]\cdots[D,f_p]|D|^{-p}\right) = \\
&\mathrm{Tr}_\omega\left(\chi f_0 f_1[D,f_2]\cdots[D,f_{p+1}]|D|^{-p}\right) + \\
&\mathrm{Tr}_\omega\left(\chi\sum_{i=1}^{p}(-1)^i f_0 f_i[D,f_1]\cdots\widehat{[D,f_i]}\cdots[D,f_{p+1}]|D|^{-p}\right) + \qquad (5.6)\\
&\mathrm{Tr}_\omega\left(\chi\sum_{i=1}^{p}(-1)^i f_0 f_{i+1}[D,f_1]\cdots\widehat{[D,f_{i+1}]}\cdots[D,f_{p+1}]|D|^{-p}\right) + \\
&(-1)^{p+1}\mathrm{Tr}_\omega\left(\chi f_0 f_{p+1}[D,f_1]\cdots[D,f_p]|D|^{-p}\right),
\end{aligned}
$$

   where the notation $\widehat{\cdot}$ expresses omission of that particular term. Shifting $i \mapsto i+1$ in term (5.6) shows the whole expression cancels in pairs, hence $b\phi_\omega^D = 0$ and $\phi_\omega^D \in Z^p(\mathcal{A})$.

2. See corollary D.20, appendix D.

3. Define $\varphi \in Z^p(\mathcal{A})$ as

$$\varphi \equiv \phi_\omega^D - \tau_F^p. \qquad (5.7)$$

   The map $\psi : C_{p-1}(\mathcal{A}) \to \mathbb{C}$, defined by

$$\psi(a) = \begin{cases} \varphi(c) & a = bc \text{ for } c \in C_p(\mathcal{A}) \\ 0 & \text{otherwise} \end{cases}, \qquad (5.8)$$

   is well defined since $d\psi$ and φ coincide on $Z_p(\mathcal{A})$ by Connes' character theorem, see theorem 3.26 in section 3.3. Since $B_{p-1}(\mathcal{A}) \subset C_{p-1}(\mathcal{A})$ is a linear subspace, the map ψ lies in $C^{p-1}(\mathcal{A})$. Therefore,

$$\phi_\omega^D - \tau_F^p = d\psi, \qquad \left[\phi_\omega^D\right] = \left[\tau_F^p\right] \in HH^p(\mathcal{A}). \qquad (5.9)$$

   Note that τ_F^p is cyclic, see lemma G.7 in appendix G, and $Bb = -bB$ from corollary E.13 in appendix E. So

$$
\begin{aligned}
&B\left[\phi_\omega^D\right] = B\left[\tau_F^p\right], \quad B\tau_F^p = Ns'(1-\lambda)\tau_F^p = 0 \qquad \Rightarrow \\
&B\left[\phi_\omega^D\right] = 0. \qquad (5.10)
\end{aligned}
$$

□

We are now ready to continue with the main result of this section.

Theorem 5.4. *There is a non-degenerate volume form $\nu \in \Omega^p_{\mathbb{R}}(M)$ such that*

$$⨍ f = \int_M f\nu \tag{5.11}$$

for all $f \in \mathcal{A}$.

Proof. Recall from section 3.3 that

$$⨍ f = \phi^D_\omega(f\mathbf{c}) = \mathrm{Tr}_\omega\left(f|D|^{-p}\right) \qquad \forall f \in \mathcal{A},\ \forall\omega \in S(B_\infty). \tag{5.12}$$

We write $\mathbf{c} \in Z_p(\mathcal{A})$ as

$$\mathbf{c} = \sum_\alpha c^0_\alpha \otimes c^1_\alpha \otimes \cdots \otimes c^p_\alpha. \tag{5.13}$$

Then $\left[\phi^D_\omega\right] = \left[A_p\phi^D_\omega\right]$ by the previous lemma. According to the identification $(D.39)$, we can find a p-current $C_{\phi^D_\omega} \equiv C \in \Omega^{dR}_p(M)$ such that for all $f \in \mathcal{A}$,

$$⨍ f = \int_C f \sum_\alpha c^0_\alpha dc^1_\alpha \wedge \cdots \wedge dc^p_\alpha. \tag{5.14}$$

We shall show that the p-form

$$\nu \equiv \sum_\alpha c^0_\alpha dc^1_\alpha \wedge \cdots \wedge dc^p_\alpha \tag{5.15}$$

defined by the previous equation is the volume form we are looking for. The properties of ν are derived using the properties of the Dixmier trace and the associated cocycle ϕ^D_ω. From the discussion in [21], pages 494 to 498, we gather that if f is a real and positive function then

$$\mathrm{Tr}_\omega(f|D|^{-p}) = 0 \quad \Rightarrow \quad f = 0. \tag{5.16}$$

Together with the earlier results, this finally establishes that the noncommutative integral is a trace, see also definition 3.12 in section 3.3. By lemma 5.3 and theorem E.7, appendix D, we see that $C \in \Omega^{dR}_p(M)$ is a closed current. The positive definiteness of the trace then shows C is a non-zero closed current. Since the p-boundaries in $\Omega^{\mathrm{dR}}_p(M)$ are empty, the equivalence class of C in $H^{\mathrm{dR}}_p(M)$ is non-zero.

Applying Poincaré-duality for general (i.e., potentially unoriented) connected and closed manifolds (see for instance [23, Ch. 3.3] or [5, §7]) we see that either M is oriented and $H_p^{\mathrm{dR}}(M) \neq 0$ or M is unoriented and $H_p^{\mathrm{dR}}(M) = 0$. The latter case does not apply here. Hence M is oriented and C is a non-negative multiple of the fundamental class, i.e., $[C] = t[M]$ for $t > 0$. We can therefore write:

$$⨍ f = t \int\limits_M f \sum_\alpha c_\alpha^0 dc_\alpha^1 \wedge \cdots \wedge dc_\alpha^p. \tag{5.17}$$

This leaves us with showing that $\nu(x) \neq 0$ for all $x \in M$. Assume there is a $q \in M$ with $\nu(q) = 0$. Take some local chart (U, x) around q and tentatively define f as a real and positive function that is non-zero on some neighborhood around q.
Then:

$$\left| \int\limits_M f dx^1 \wedge \cdots \wedge dx^p \right| = t^{-1} \left| A_p \phi_\omega^D \left(f, x^1, \ldots, x^p \right) \right| \leq \frac{1}{tp!} \sum_{\beta \in S_p} \left| \operatorname{Tr}_\omega \left(\chi f \left[D, x^{\beta(1)} \right] \cdots \left[D, x^{\beta(p)} \right] |D|^{-p} \right) \right|. \tag{5.18}$$

Fix some $\beta \in S_p$. Since the x^i are real, f, χ, and $[D, x^i]$ are self-adjoint for any i. We therefore have a self-adjoint $T \in \mathcal{L}^{1+}(\mathcal{H})$, defined by

$$T \equiv \chi f \left[D, x^{\beta(1)} \right] \cdots \left[D, x^{\beta(p)} \right] |D|^{-p}. \tag{5.19}$$

By theorem 3.17 we know that we can decompose T as

$$T = T_+ - T_-;$$
$$T_\pm = P_\pm |T| P_\pm, \qquad P_\pm = \frac{1}{2}(1 \pm F),$$

with $T_\pm \geq 0$ in the Dixmier ideal and F a symmetry. From this we infer that:

$$P_\pm P_\mp = 0 \quad \Rightarrow$$
$$|T|^2 = T^2 = T_+^2 - T_+ T_- - T_- T_+ + T_-^2 = T_+^2 + T_-^2 = (T_+ + T_-)^2 \quad \Rightarrow$$
$$|T| = T_+ + T_-.$$

Using the positivity of the integral we derive

$$\operatorname{Tr}_\omega(T) = \operatorname{Tr}_\omega(T_+) - \operatorname{Tr}_\omega(T_-) \qquad \Rightarrow$$
$$|\operatorname{Tr}_\omega(T)| \leq |\operatorname{Tr}_\omega(T_+)| + |\operatorname{Tr}_\omega(T_-)| = \operatorname{Tr}_\omega(T_+) + \operatorname{Tr}_\omega(T_-) = \operatorname{Tr}_\omega(|T|).$$

The right-hand side of equation (5.18) is therefore dominated by

$$\frac{1}{tp!}\sum_{\beta\in S_p}\mathrm{Tr}_\omega\left(f\left|\left[D,x^{\beta(1)}\right]\right|\cdots\left|\left[D,x^{\beta(p)}\right]\right||D|^{-p}\right), \tag{5.20}$$

where in the latter expression we have used the fact that $\chi^* = \chi$ and $\chi^2 = 1$, so $|\chi| = \sqrt{\chi^*\chi} = 1$. We are now set to apply the special case of the noncommutative Hölder-inequality, i.e., lemma 3.20, repeatedly to (5.20). This yields:

$$\mathrm{Tr}_\omega\left(\left|\left[D,x^{\beta(1)}\right]\right|\cdots\left|\left[D,x^{\beta(p)}\right]\right|f|D|^{-p}\right)\leq\prod_{i=1}^{p}\left\|[D,x^i]\right\|\mathrm{Tr}_\omega(f|D|^{-p}).$$

Plugging this into inequality (5.18) we see that

$$\left|\int_M f dx^1\wedge\cdots\wedge dx^p\right|\leq t^{-1}\prod_{i=1}^{p}\left\|[D,x^i]\right\|\mathrm{Tr}_\omega(f|D|^{-p}) =$$

$$t^{-1}\prod_{i=1}^{p}\left\|[D,x^i]\right\|\lambda_p^{-1}\phi_\omega^D(f\mathbf{c}) = \prod_{i=1}^{p}\left\|[D,x^i]\right\|\int_M f\sum_\alpha c_\alpha^0 dc_\alpha^1\wedge\cdots\wedge dc_\alpha^p.$$

The next step is to rewrite ν around q.
Let (N_α, x_α) be a chart around q, where the chart is constructed via the method of chapter 4. According to definition (4.52) we can locally find an automorphism of $\mathcal{A}$, say G, such that

$$x_\alpha^k \equiv G\circ c_\alpha^k \qquad \forall k\in\{1,\ldots,p\}. \tag{5.21}$$

As a corollary of Milnor's exercise, theorem [34, p.11], the automorphisms of the algebra of smooth functions of a compact manifold are in bijective correspondence with diffeomorphisms of that manifold. See [35] for more background information.
This correspondence is implemented via the pullback construction, i.e., we can find a diffeomorphism ψ of M such that

$$x_\alpha^k = G\circ c_\alpha^k = c_\alpha^k\circ\psi \qquad \forall k\in\{1,\ldots,p\}. \tag{5.22}$$

Relative to differentiation on the chart (U,x), we denote respectively the Jacobian of x_α in the point $\psi^{-1}(q)$ by $J_{\psi^{-1}(q)}(x_\alpha)$, the Jacobian of c_α in the point q by $J_q(c_\alpha)$ and the Jacobian of ψ in $\psi^{-1}(q)$ by $J_{\psi^{-1}(q)}(\psi)$.
Using the chain rule

$$\begin{aligned}&J_{\psi^{-1}(q)}(x_\alpha) = J_q(c_\alpha)\cdot J_{\psi^{-1}(q)}(\psi),\\&0\neq\det J_{\psi^{-1}(q)}(x_\alpha) = \det J_q(c_\alpha)\cdot\det J_{\psi^{-1}(q)}(\psi).\end{aligned}$$

This implies $\det J_q(c_\alpha) \neq 0$. So:

$$\left(dc_\alpha^1 \wedge \cdots \wedge dc_\alpha^p\right)(q) = \det \left[\left. \frac{\partial c_\alpha^j}{\partial x^i} \right|_q \right] \left(dx^1 \wedge \cdots \wedge dx^p\right)(q) \neq 0.$$

We can now rewrite the expression for ν using the common coordinate functions:

$$\nu = \sum_\alpha c_\alpha^0 \det \left[\frac{\partial c_\alpha^j}{\partial x^i} \right] dx^1 \wedge \cdots \wedge dx^p \equiv g dx^1 \wedge \cdots \wedge dx^p, \tag{5.23}$$

with g a smooth function such that $g(q) = 0$. By continuity of g, there is some compact $K \subset U$, with $q \in K$, such that

$$|g(x)| < \left(\prod_{i=1}^p \| [D, x^i] \| \right)^{-1}. \tag{5.24}$$

We now specify f as follows:

- $f = 1$ on K;
- $f = 0$ on $M \backslash U$;
- $f \in [0, 1]$ on U;

Functions with the same properties as f exist in abundance on any compact topological space, see [37, Ch. 4].
In summary, there is an $\epsilon > 0$ such that

$$\mathrm{Vol}(K) + \epsilon =$$
$$\left| \int_M f dx^1 \wedge \cdots \wedge dx^p \right| \leq \prod_{i=1}^p \| [D, x^i] \| \int_M f g dx^1 \wedge \cdots \wedge dx^p \leq$$
$$\prod_{i=1}^p \| [D, x^i] \| \max_{x \in K} \{ |f(x) g(x)| \} \cdot \mathrm{Vol}(K) < \mathrm{Vol}(K).$$

This is a contradiction, so ν is non-zero everywhere. This concludes the proof of theorem 5.4.

□

5.2 Spin structure

We now proceed showing that the manifold has a Spin^c structure. Due to Plymen's theorem, see section 2.4, we need to show that there is a right-$C(M)$-Hilbert module that implements a Morita equivalence between $C(M)$ and the Clifford bundle $B^{(+)}(M)$. In the light of axiom 7 it is not surprising that the spinor space $\mathcal{H}$ will fulfill this role. We will take the following steps. First we show that $\mathcal{H}$ is a right-$C(M)$-Hilbert module. Then we will construct the Clifford bundle. Next, we shall prove that both C^*-algebras are Morita equivalent. We finish showing there is a global charge conjugation operator, thus establishing the fact that M has a Spin structure.

Lemma 5.5. *$\mathcal{H}$ is a right-A-Hilbert module.*

Proof. Since $\mathcal{H}^\infty$ is a finitely generated projective $\mathcal{A}$-module we have a canonical action of $\mathcal{A}$ on $\mathcal{H}^\infty$. We extend this action to a right-action of A on $\mathcal{H}$ the following way. Let $a \in A$ and $\xi \in \mathcal{H}$. There are sequences $\{a_n\} \subset \mathcal{A}$ and $\{\xi_n\} \subset \mathcal{H}^\infty$ such that

$$\|a_n - a\| \to 0 \qquad \|\xi_n - \xi\|_{\mathcal{H}} \to 0.$$

Let $n, m > N$ for some $N \in \mathbb{N}$. Then:

$$\|\xi_n a_n - \xi_m a_m\|_{\mathcal{H}} \leq \|\xi_n - \xi_m\|_{\mathcal{H}} \|a_n\| + \|a_n - a_m\| \|\xi_m\|_{\mathcal{H}} .$$

Therefore $\{x_n a_n\}$ is a Cauchy sequence in $\mathcal{H}$. Its limit defines the right-A-module structure on $\mathcal{H}$ by setting

$$\xi \cdot a \equiv \lim_{n\to\infty} \xi_n a_n. \tag{5.25}$$

The spectrum of A is a metrizable space, since it is a compact manifold. The algebra A is therefore a separable algebra. The preliminary requirements of definition C.3 are now met. We consecutively prove points (1) to (3) of that definition.
In lemma 3.29, we have already constructed a nondegenerate $\mathcal{A}$-valued inner product

$$\langle\cdot,\cdot\rangle_{\mathcal{A}} : \mathcal{H}^\infty \times \mathcal{H}^\infty \to \mathcal{A}, \qquad \langle\xi_1, \xi_2\rangle_{\mathcal{A}} \equiv \sum_{i,j} a_i^* p_{ij} b_j, \tag{5.26}$$

with a_i and b_j the components of ξ_1 and ξ_2 under the module morphism $\mathcal{H}^\infty \hookrightarrow \mathcal{A}^n$. The p_{ij} are the components of a projection $p \in M_n(\mathcal{A})$. Noting that any projection $p \in M_n(\mathcal{A})$ is also a projection in $M_n(A)$, we can, using the same limiting procedure as before, extend this inner product to an A-valued inner product on $\mathcal{H}$ by defining

$$\langle\xi, \eta\rangle_A \equiv \lim_{n\to\infty} \langle\xi_n, \eta_n\rangle_{\mathcal{A}} \quad \text{with} \quad \xi_n \to \xi \quad \& \quad \eta_n \to \eta. \tag{5.27}$$

In corollary 4.11 we have shown that the module isomorphism

$$\mathcal{H}^\infty \cong p\mathcal{A}^n \tag{5.28}$$

is a homeomorphism of Fréchet-spaces. Both Fréchet-spaces are generated by semi-norms which include $\|\cdot\|_{\mathcal{H}}$ for $\mathcal{H}^\infty$ and the product norm based on $\|\cdot\|$ for $p\mathcal{A}^n$. This implies that the module isomorphism is continuous with respect to the norm on $\mathcal{H}^\infty$ as a subspace of $\mathcal{H}$ and with respect to the norm of $p\mathcal{A}^n$ induced by the product norm on pA^n. "Closing" (5.28) in these norms then yields

$$\mathcal{H} \cong pA^n \tag{5.29}$$

as topological spaces. The product norm on pA^n is equivalent to the norm $\|\langle\cdot,\cdot\rangle_A\|$, hence $\mathcal{H}$ is complete with respect to the latter norm.
Take some positive real $f \in A$. Thanks to the result in lemma 3.29, we see that the value of $\langle\xi,\eta\rangle_A$ does not change as we transform the base in A^n. Let U be a base transformation such that p is a diagonal matrix with only zero's and the unit of A on the diagonal. For convenience, assume the top-left element in the matrix representation of p is non-zero. In this base we see that the element

$$a \equiv (\sqrt{f}, 0, \ldots, 0) \tag{5.30}$$

satisfies $pa = a$ and $\sum_{i,j=1}^{n} (a^i)^* p_{ij} a^j = f$. Moving back to $\mathcal{H}$ by $U^{-1}a \mapsto \xi \in \mathcal{H}$ shows

$$\langle\xi,\xi\rangle_A = f. \tag{5.31}$$

We can construct any positive and real element of A in this manner. A is $\mathbb{C}$-linearly generated by the positive and real functions so $\mathcal{H}$ is full. □

Just as in the case of the 4-sphere, constructing the Clifford bundle boils down to defining a metric on the (co)tangent space. We will therefore first explore an alternative method for defining a Riemannian metric on the 4-sphere and then apply this construction to the more general case of a commutative spectral triple.
Take some section $\alpha \in \Omega^1_{\mathbb{R}}(S^4)$. Using the local expression of the principal symbol (2.121), it follows that

$$\sigma^{\not{D}}(x,\alpha_x)^2 = \sum_{i=1}^{4} f_i^2 \left(\alpha_x^i\right)^2 \cdot 1_{\mathcal{S}_x}, \tag{5.32}$$

where the components of α_x are given on the same base used for the matrix representation of the principal symbol. Recall that the f_i are all real functions. So this equation defines a non-degenerate quadratic real form on the fibers of the cotangent bundle in the following way:

$$\alpha_x \mapsto g_x(\alpha_x,\alpha_x) \equiv \sigma^{\not{D}}(x,\alpha_x)^2. \tag{5.33}$$

From the discussion of the end of appendix B we know that the principal symbol is invariant under coordinate changes. The 4-sphere is orientable, so we can take a section of the algebra bundle $\mathrm{End}_{C^\infty(S^4)}(\mathcal{S})$ such that the section equals (some positive multiple of) $1_{\mathcal{S}_x}$ everywhere. This implies that (5.33) defines a Riemannian metric on the cotangent bundle.
We can go even further and define a Hermitian metric on the complexified cotangent bundle of the 4-sphere. Take some $\alpha \in \Omega^1\left(S^4\right)$. By formula (2.122) we know that

$$\begin{aligned} &\sigma^{\not{D}}(x,\alpha_x)^* = \sigma^{\not{D}}(x,\alpha_x^*) \quad \Rightarrow \\ &\sigma^{\not{D}}(x,\alpha_x)^*\sigma^{\not{D}}(x,\alpha_x) = \sum_{i=1}^{4} f_i^2 \left|\alpha_x^i\right|^2 \cdot 1_{\mathcal{S}_x}. \end{aligned} \tag{5.34}$$

Hence we can construct the Hermitian product from the following complex, positive and non-degenerate quadratic form:

$$\alpha_x \mapsto g_x(\alpha_x,\alpha_x) \equiv \sigma^{\not{D}}(x,\alpha_x)^*\sigma^{\not{D}}(x,\alpha_x). \tag{5.35}$$

From the Serre–Swan theorem, we know that $\mathcal{H}^\infty$ consists of the collection of smooth sections of some smooth vector bundle $\mathcal{S}$ on M. This implies that $D : \mathcal{H}^\infty \to \mathcal{H}^\infty$ is a linear partial differential operator of order 1. Hence D has a principal symbol and we can try to copy the aforementioned procedure to construct a metric on T^*M. Consecutively, we will therefore:

1. find a useful representation of the principal symbol of D;
2. show that for real $\alpha \in \Omega^1_{\mathbb{R}}(M)$, the principal symbol generates a subalgebra of $\mathrm{End}(\mathcal{S}_x)$ on which the square of the principal symbol has a multiplicative action;
3. prove that this action is non-degenerate.

Similar to equation (2.122) for the case of the 4-sphere:

Lemma 5.6. *For all $f \in \mathcal{A}$:*

$$\sigma^D(x, df_x) = -i[D,f](x). \tag{5.36}$$

Proof. This is an application of identity (B.15), appendix B, i.e.:

$$\sigma^D(x,df_x) = \lim_{t\to\infty} \frac{\exp(itf(x))D\left(\exp(-itf(x))\right)}{t}. \tag{5.37}$$

Both the numerator and the denominator go to infinity as $t \to \infty$. The fraction of the derivatives of both functions is given by:

$$\begin{aligned} &-if(x)\exp(-itf(x))D(x)\exp(itf(x)) + \\ &\exp(-itf(x))D(x)if(x)\exp(itf(x)) \\ &= -i\exp(-itf(x))\left[D,f\right](x)\exp(itf(x)). \end{aligned}$$

Since $[D, f]$ commutes with any power f^k we see that the above expression equals $-i[D, f](x)$. Hence we can apply de L'Hopital's theorem to prove the result. □

With some foresight, we now define a map $c : \Omega^1(M) \to \mathrm{End}(\mathcal{S})$:

$$\sigma^D(x, \alpha_x) \equiv c(\alpha)(x). \tag{5.38}$$

Let $\alpha \in \Omega^1(M)$ and take some chart around $x \in M$. With respect to that chart we can find an $f \in \mathcal{A}$ such that $df = \alpha$. Just as in the case of $M = S^4$, the map c preserves involution:

$$c(\alpha^*)(x) = -i[D, \overline{f}](x) = (-i[D, f](x))^* = c(\alpha)(x)^*. \tag{5.39}$$

This calculation is independent of the chart, since the principal symbol, and thereby the map c, are invariant under coordinate change.
From this it follows that in $\mathrm{End}(\mathcal{S}_x)$ the elements

$$\{c(\alpha)(x); \alpha \in \Omega^1(M)\} \tag{5.40}$$

generate an involutive subalgebra $B'_x \subset \mathrm{End}(\mathcal{S}_x) \cong M_K(\mathbb{C})$, for some $K \in \mathbb{N}$.

Lemma 5.7. *For all $x \in M$ and any $\beta \in \Omega^1(M)$, $c(\beta)^2(x)$ acts as a multiplication operator in B'_x.*

Proof. Take some $\alpha \in \Omega^1(M)$. Choose a local coordinate chart around x and relative to that chart express $\alpha = df$ and $\beta = dg$ for certain smooth functions $f, g \in \mathcal{A}$. If we can prove that

$$\left[c(dg)^2(x), c(df)(x)\right] = 0 \tag{5.41}$$

for all $x \in M$ we are finished: by invariance of the principal symbol under coordinate transformations the previous equality holds for all choice of charts.
Using the symbol calculus of differential operators, see appendix B, we see that

$$c(\beta)^2(x) = \sigma^{D^2}(x, \beta_x). \tag{5.42}$$

But, by axiom 2, $c(df) = -i[D, f]$, which is a rank zero differential operator. It also has a principal symbol, and we can apply the principal symbol calculus once again to see that

$$[c(dg)^2(x), c(df)(x)] = -i\sigma^{[D^2,[D,f(x)]]}(x, dg_x). \tag{5.43}$$

Note that the operator D^2 is of rank 2 whilst the operator $[D, f]$ is of rank 0, so $[D^2, [D, f]]$ is of rank 2. Once again using the symbol calculus, we show that this principal symbol is zero. I.e., we will prove

$$\sigma^{[D^2,[D,f(x)]]}(x, dg(x)) = \lim_{t\to\infty} \frac{1}{t^2} \left[\exp(-itg(x))[D^2, [D, f]](x) \exp(itg(x))\right] = 0. \tag{5.44}$$

Let us focus on the numerator:

$$\begin{aligned}&\left\|\exp(-itg)[D^2[D, f]] \exp(itg)\right\| = \left\|[D^2[D, f]] \exp(itg)\right\| = \\ &\left\|(|D|\delta([D, f]) + \delta([D, f])|D|)(x) \exp(itg)\right\|,\end{aligned} \tag{5.45}$$

where the last equality can be verified by writing out the expression for $[D^2, [D, f]]$, noting that $D = D^*$ and using the axiom of strong regularity, i.e., axiom 3. The latter equation is therefore dominated by

$$\left\||D|\delta([D, f]) \exp(itg)\right\| + \left\|\delta([D, f])\right\| \left\||D| \exp(itg)\right\|. \tag{5.46}$$

Also $\delta([D, f]) \in \operatorname{dom} \delta$, so

$$|D|\delta([D, f]) = \delta^2([D, f]) + \delta([D, f])|D|,$$

which implies that (5.46) is bounded by

$$\left\|\delta^2([D, f])\right\| + 2\left\|\delta([D, f])\right\| \left\||D| \exp(itg)\right\|.$$

Note that by axiom 2,

$$\langle |D| \exp(itg)\xi, |D| \exp(itg)\xi\rangle_{\mathcal{H}} = \langle \xi, \exp(-itg)D^2 \exp(itg)\xi\rangle_{\mathcal{H}} \tag{5.47}$$

is of order $\mathcal{O}(t^2)$. Hence $\||D| \exp(itg)\|$ is of order $\mathcal{O}(t)$ so that

$$\left\|\delta^2([D, f])\right\| + 2\left\|\delta([D, f])\right\| \left\||D| \exp(itg)\right\| \tag{5.48}$$

is of order $\mathcal{O}(t)$, too. This in turn implies that (5.44) is zero. This result is valid for all $x \in M$, so the proof of lemma 5.7 is finished. □

In general,

$$B'_x \cong M_{k_1}(\mathbb{C}) \oplus \ldots \oplus M_{k_n}(\mathbb{C}) \subset M_K(\mathbb{C}) \cong \operatorname{End}(\mathcal{S}_x). \tag{5.49}$$

Let p_i be the projection in $M_K(\mathbb{C})$ projecting on the i-th summand of the above expansion of B'_x. By the previous lemma, the action of $c(\beta)^2$ can be decomposed as

$$c(\beta)^2 \equiv g_x(\beta_x, \beta_x)^1 p_1 + g_x(\beta_x, \beta_x)^2 p_2 + \ldots + g_x(\beta_x, \beta_x)^n p_n, \tag{5.50}$$

in which each of the $g_x(\cdot,\cdot)^i$ is a (potentially degenerate) Hermitian form. Assume $c(\beta)^2$ is non-degenerate and $n > 1$. Since the principal symbol is a continuous map, not only does $\mathrm{End}(\mathcal{S})$ decompose globally into a direct sum of submodules, but also the abstract Dirac operator then decomposes into a direct sum of abstract Dirac operators. This would violate axiom 6, hence the situation that $c(\beta)^2$ is non-degenerate *and* $n > 1$ cannot occur. Furthermore,

Lemma 5.8. $p_1 = 1_{\mathcal{S}_x}$.

Proof. Assume the converse, i.e. $n > 1$ in expression (5.50). According to the conclusion of the previous discussion, $c(\beta)^2$ is degenerate, i.e., there must be at least one point $x \in M$ such that $c(\beta)^2(x) = 0$. Let dx^i be a base of T_x^*M chosen in such a way that dx^p lies on the same ray as β, i.e.,

$$c(dx^p)^2(x) = c(dx^p)(x) = 0,$$

which implies that for all $\beta \in S_p$:

$$\begin{aligned} &c\left(dx^{\beta(1)}\right)(x)\cdots c\left(dx^{\beta(p)}\right)(x) = 0 \qquad \Rightarrow \\ &(-i)^p \chi(x)\left[D, x^1\right]\cdots[D, x^p] = 0. \end{aligned}$$

This result is invariant under base transformations of T_x^*M; any such transformation would only multiply previous expression with a factor equal to the determinant of the transformation. In the proof of theorem 5.4 we have seen that $\{dc_\alpha^1, \ldots, dc_\alpha^p\}$ forms a local base for each α. Hence

$$\sum_\alpha \chi(x) c_\alpha^0 \left[D, c_\alpha^1(x)\right]\cdots\left[D, c_\alpha^p(x)\right] = 0. \tag{5.51}$$

Thereby, via the identification

$$\int_M \nu = \phi_\omega^D(\mathbf{c})$$

of theorem 5.4, this implies that we have found a point $x \in M$ at which $\nu(x) = 0$, what is contrary to the result of theorem 5.4. □

We are now ready for the next important result.

Theorem 5.9. *M has a Spin^c structure.*

Proof. We generate the Clifford bundle in a similar manner as in section 2.4, by using the Hermitian metric on the cotangent bundle, defined as

$$g_x(\alpha_x, \beta_x) = c(\alpha)(x)^* c(\beta)(x). \tag{5.52}$$

Denote the collection of smooth sections of the Clifford bundle by $\mathcal{B}^{(+)}(M)$ and call the collection of continuous sections $B^{(+)}(M)$.
First, assume that p is even. Define a map

$$\tilde{c} : \mathcal{B}^{(+)}(M) \to \mathrm{End}_{\mathcal{A}}(\mathcal{H}^{\infty}) = \Gamma^{\infty}(M, \mathrm{End}_{\mathcal{A}}(\mathcal{S})) \tag{5.53}$$

in the following way. For each $p \in M$, choose a coordinate chart (U, x) around p. The Clifford algebra is locally linearly generated by the pure elements of the form

$$dx^{i_1} \cdots dx^{i_k}, \tag{5.54}$$

with $k \leq p$ and $i_1 < \ldots < i_k$ by convention. Let

$$\tilde{c}\left(dx^{i_1}(p) \cdots dx^{i_k}(p)\right) \equiv c\left(dx^{i_1}\right)(p) \cdots c\left(dx^{i_k}\right)(p), \tag{5.55}$$

and extend $\tilde{c}$ by linearity.
Again, using the transformation properties of the principal symbol, we see that $\tilde{c}$ extends globally to a map of algebra bundles. Fiber-wise, this map is an isomorphism of algebras due to the result of the last theorem and the definition of the Hermitian metric. Hence $\tilde{c}$ implements an isomorphism between the respective algebra bundles. By taking the C^*-closure of the algebra bundles we can extend $\tilde{c}$ to an isomorphism

$$B^{(+)}(M) \cong \mathrm{End}_A(\mathcal{H}). \tag{5.56}$$

Note that the fibers of $\mathcal{S}$ are finite-dimensional. This implies that

$$B^{(+)}(M) \cong \mathrm{End}_A^0(\mathcal{H}). \tag{5.57}$$

If p is odd, then the fibers of $B^{(+)}(M)$ are actually generated by the even elements of the form

$$dx^{i_1} \cdots dx^{i_{2k}}. \tag{5.58}$$

This changes nothing to the result of lemma 5.8, so relation (5.57) still holds. In summary, $\mathcal{H}$ is a $C(M)$-$B^{(+)}(M)$ imprimitivity bimodule. Apply now Plymen's theorem 2.46 in section 2 to see that M has a Spin^c structure. $\square$

Before showing that M has a Spin structure, we briefly look at some properties of the A-valued inner product on $\mathcal{H}$.

Lemma 5.10. *Let $\langle \cdot, \cdot \rangle' : \mathcal{H} \times \mathcal{H} \to A$ be an A-valued inner product satisfying*

$$\langle \xi, \eta \rangle_{\mathcal{H}} = ⨍ \langle \xi, \eta \rangle'. \tag{5.59}$$

Then $\langle \cdot, \cdot \rangle' = \langle \cdot, \cdot \rangle_A$.

Proof. For any $f \in A$,

$$\langle \xi, f\eta \rangle_{\mathcal{H}} - \langle \xi, f\eta \rangle_{\mathcal{H}} = \fint f \left[\langle \xi, \eta \rangle_A - \langle \xi, \eta \rangle'_A \right] = 0. \tag{5.60}$$

Choose $f = [\langle \xi, \eta \rangle_A - \langle \xi, \eta \rangle'_A]^*$ and use the non-degeneracy of the noncommutative integral to see that for each $\xi, \eta \in \mathcal{H}$ the different A-valued inner products coincide.

□

Theorem 5.11. *M has a Spin structure.*

Proof. We verify that the map $C : \mathcal{H} \to \mathcal{H}$ from axiom 5 satisfies the properties of theorem 2.48. Recall that C is anti-unitary with respect to the $\mathbb{C}$-valued inner product on $\mathcal{H}$. Take some $\xi, \eta \in \mathcal{H}$, $f \in A$ and $b \in B^{(+)}(M)$.

1. The condition $C(\xi f) = C(\xi)\overline{f}$ is a direct consequence of the anti-linearity of C and the fact that A acts on $\mathcal{H}$ by multiplication.

2. We examine how C acts on the generators of $B^{(+)}(M)$. C is a bounded operator. Hence, we can safely exchange the action of C with the operator valued limit in equation (B.15) of appendix B, i.e.,

$$C\tilde{c}(df)C^{-1} = \lim_{t\to\infty} \frac{1}{t} C \exp(itf) D \exp(-itf) C^{-1} =$$
$$\lim_{t\to\infty} \frac{1}{t} \exp(-it\overline{f}) CDC^{-1} \exp(it\overline{f}).$$

   The commutation relations of D with C vary with the dimension p mod 8 (see table 3.1). Hence we examine the various possibilities. First, let p be even, $p = 3 \mod 8$ or $p = 7 \mod 8$. Then $[D, C] = 0$. This leads to:

$$C\tilde{c}(df)C^{-1} = \lim_{t\to\infty} \exp(-it\overline{f}) D \exp(it\overline{f}) = \tilde{c}(-d\overline{f})) = \tilde{c}(\kappa(df)).$$

   Second, if $p = 1 \mod 8$ or $p = 5 \mod 8$, then $\{D, C\} = 0$. For odd p, the fibers of the Clifford bundle are generated by the even part. So

$$C\tilde{c}(df \cdot df)C^{-1} = C\tilde{c}(df)c(dg)C^{-1} = (-1)^2 \tilde{c}(\kappa(df))\tilde{c}(\kappa(dg)) = \tilde{c}(\kappa(df \cdot dg)).$$

3. Let us define an A-valued bilinear map on $\mathcal{H} \times \mathcal{H}$ by

$$\langle \xi, \eta \rangle'_A \equiv \langle C\eta, C\xi \rangle_A.$$

   We want to apply lemma 5.10 to relate this new inner product to $\langle \cdot, \cdot \rangle_A$. We therefore show how it defines an A-valued inner product on $\mathcal{H}$.

Take some $f \in C(M)$. We compute

$$\begin{aligned}
&\langle \xi, f\eta \rangle'_A = \langle Cf\eta, C\xi \rangle_A = \langle f^* C\eta, C\xi \rangle_A = f\langle C\eta, C\xi \rangle_A = f\langle \xi, \eta \rangle'_A; \\
&\langle \eta, \xi \rangle'_A = \langle C\xi, C\eta \rangle_A = \langle C\eta, C\xi \rangle^*_A = \big(\langle \xi, \eta \rangle'_A\big)^* \quad \Rightarrow \\
&\langle \xi, \xi \rangle'_A \geq 0.
\end{aligned}$$

If $\langle \xi, \xi \rangle'_A = 0$, then:

$$0 = \int\!\!\!\!\!- \langle \xi, \xi \rangle'_A = \int\!\!\!\!\!- \langle C\xi, C\xi \rangle_A = \langle C\xi, C\xi \rangle_A = \langle \xi, \xi \rangle_A,$$

which implies ξ must be zero. In summary, we know that $\langle \cdot, \cdot \rangle'_A$ is an A-valued inner product on $\mathcal{H}$ that satisfies the condition in lemma 5.10, so

$$\langle \xi, \eta \rangle'_A = \langle C\eta, C\xi \rangle_A = \langle \xi, \eta \rangle_A. \tag{5.61}$$

It then follows from theorem 2.48 in section 2 that M has a Spin structure.

We are now finished with the Reconstruction Theorem. Given a commutative spectral triple $(\mathcal{A}, D, \mathcal{H})$ of dimension p satisfying axioms 1 to 8, we have constructed a closed and connected p-dimensional spin manifold M such that $\mathcal{A} \cong C^\infty(M)$, D equals the Dirac operator $\not{D}$ and $\mathcal{H}$ is the Hilbert space of square integrable spinors relative to a given Spin structure on M.

□

Appendix A

Fréchet spaces

Pre-C^*-algebras, as introduced in section 2.2, play a pivotal role in the reconstruction theorem (chapter 4). An important example of a pre-C^*-algebra is the (involutive) algebra of smooth, complex valued functions on a manifold. Pre-C^*-algebras are **Fréchet spaces**. In this appendix we define Fréchet spaces and discuss several of their properties.

Definition A.1 (Fréchet space)**.** *A* ***topological vector space*** *V is a complex vector space with a topology such that addition and complex (scalar) multiplication are continuous. A topological vector space that is complete in the topology determined by a translation invariant metric is called a Fréchet space.*

We quote without proof:

Lemma A.2. *Let U be a closed subspace of the Fréchet space V. Then U is a Fréchet space in its own right.*

Lemma A.3. *Let V be a complex vector space and let $\{p_k : V \to \mathbb{R}^+\}$ be a countable set of seminorms, i.e., the p_k satisfy all properties of a norm except the requirement $p_k(v) = 0 \Rightarrow v = 0$. Assume furthermore that $\{v \in V; p_k(v) = 0 \ \forall k\} = \{0\}$. We can equip V with a topology defined by a translation invariant metric (in other words, V is a Fréchet space, except that it is not necessarily complete).*

Proof. Let

$$d(x, y) \equiv \sum_{k=0}^{\infty} 2^{-k} \frac{p_k(x-y)}{1 + p_k(x-y)} \qquad \forall x, y \in V.$$

From the definition we directly infer that $d(x, y) = d(y, x), d(x, x) = 0$ and $d(x + z, y + z) = d(x, y)$ for all $z \in V$. If $d(x, y) = 0$ then $p_k(x - y) = 0$ for all k, so $x = y$. So in order to see that d is a translation invariant metric

we only need to show that the metric satisfies the triangle inequality, which is a routine exercise.
Since $d(\lambda a, \lambda b) = |\lambda| d(a, b)$, multiplication with $\mathbb{C}$ is continuous. Equip $V \times V$ with the product topology. Continuity of addition now follows directly from the triangle inequality. □

Remark A.4. *The topology generated by the metric d in the previous lemma is equivalent to the topology generated by the open sets $B_\epsilon(v) = \{w \in V; p_k(v - w) < \epsilon \ \forall k\}$.*

Definition A.5 (Fréchet algebra). *A Fréchet algebra is both an algebra over the complex numbers and a Fréchet space, with the property that multiplication is continuous.*

Corollary A.6. *Let V be as in lemma A.3. Assume furthermore that V is an algebra over the complex numbers and that the seminorms are submultiplicative, i.e.,*

$$p_k(v \cdot w) \leq p_k(v) p_k(w) \qquad \forall v, w \in V, \ \forall k \in \mathbb{N}. \tag{A.1}$$

Then V is a Fréchet algebra in every respect, except for completeness.

Proof. Take $v_1, w_1, v_2, w_2 \in V$.

$$p_k(v_1 w_1 - v_2 w_2) \leq p_k(v_1(w_1 - w_2)) + p_k(w_2(v_1 - v_2)) \leq$$
$$p_k(v_1) p_k(w_1 - w_2) + p_k(w_2) p_k(v_1 - v_2).$$

Which goes to zero as $p_k(v_1 - v_2) \to 0$ and $p_k(w_1 - w_2) \to 0$. □

Theorem A.7 (Open mapping theorem). *Let $f : V \to W$ be a linear, continuous and surjective map between two Fréchet spaces. Then f is open.*

Proof. The Baire category theorem [38, 2.22] shows that any complete metric space, hence any Fréchet space, is a Baire space. From that point on the standard proof, as in [38, 2.24] for example, of the open mapping theorem goes through for Fréchet spaces as well. □

Definition A.8 (Fréchet derivative). *Let $f : V \to W$ be a continuous map between two Fréchet algebras V and W. We say that f is* ***differentiable****, or of* ***class*** *C^1, if there is a jointly continuous function $Df : V \times V \to W$ satisfying*

$$Df(x, t) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}(f(x + \epsilon t) - f(x)).$$

Df is also called the ***directional derivative*** *or the* ***Fréchet derivative*** *of f. The higher derivatives $D^k f(x, t_1, \ldots, t_k)$ are obtained by differentiating $D^{k-1}(x, t_1, \ldots, t_k)$. If f is a function such that all of its higher derivatives exist and are jointly continuous functions in all of their arguments, then f is said to be* ***smooth****, or of* ***class*** *C^∞.*

The set of real numbers $\mathbb{R}$ is also a Fréchet algebra. We therefore expect that the usual rules of calculus are valid for maps between Fréchet algebras as well.

Lemma A.9 (Calculus)**.** *Let U, V, and W be Fréchet algebras. $f_1, f_2 : U \to V$ and $g : V \to W$ are functions of class C^k and $\lambda, \mu \in \mathbb{C}$. Then:*

- *$\lambda f_1 + \mu f_2$ is of class C^k;*
- *$g \circ f_1$ is of class C^k. Its first derivative is given by $Df_1(x,t) \cdot Dg(f_1(x,t),t)$;*
- *$f_1 \cdot f_2$ is of class C^k with $D(f_1 \cdot f_2)(x,t) = f_1(x)Df_2(x,t) + Df_1(x,t)f_2(x)$.*

Proof. See [22, I.3] for the proofs. □

Appendix B

Differential operators

In this appendix we introduce **linear differential operators** and their associated **symbols**. An example of a linear differential operator is the Dirac operator, first introduced in chapter 2.
A specific type of symbol, the **principal symbol**, is used at the end of chapter 5 to prove that the manifold associated to a commutative spectral triple is spin. In the proof, several results about principal symbols are used. These results are quoted at the end of this appendix.
To avoid unnecessarily complicated notation, let us first introduce the multi-indices.

Definition B.1. *A* ***multi-index*** $\vec{\alpha}$ *is a set of natural numbers* $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$*. The rank of the multi-index is the sum of its components:* $|\vec{\alpha}| = \sum_{i=1}^{k} \alpha_i$*.*
The multi-index is used in two different ways.

- *The multi-index is an abbreviated notation for a label. For example, take two vector spaces* V, W *and let* $A : V^{\otimes k} \to W$ *be a multilinear map. Let* $\{v_1, \ldots, v_k\} \in V$ *and denote with* v_i^j *the* j*-th component of the* i*-th vector relative to a base for* V*. Let* $\{e_n\}$ *be a base for* W*. The action of* A *on* $v_1 \otimes \cdots \otimes v_k$*, given by*

$$A(v_1, \ldots, v_k) = \sum_{\alpha_1, \ldots, \alpha_k}^{\dim V} \sum_{n=1}^{\dim W} A^n_{\alpha_1 \cdots \alpha_k} v_1^{\alpha_1} \cdots v_k^{\alpha_k} e_n \qquad \text{(B.1)}$$

  can be abbreviated to

$$\sum_{j_1, \ldots, j_k}^{\dim V} \sum_{n=1}^{\dim W} A^n_{\vec{\alpha}} v_1^{\alpha_1} \cdots v_k^{\alpha_k} e_n. \qquad \text{(B.2)}$$

- *We use the multi-index as a shortened way to express repeated multiplication of several elements in a semigroup* G*. Take some*

$\{g_1, \ldots, g_k\} \subset G$. Define:

$$g^{\vec{\alpha}} \equiv g_1^{\alpha_1} \cdots g_k^{\alpha_k}. \tag{B.3}$$

We can use this to condense equation (B.2) even further:

$$A(v_1, \ldots, v_k) = \sum_{\vec{\alpha}}^{\dim V} \sum_{n=1}^{\dim W} A^n_{\vec{\alpha}} v^{\vec{\alpha}} e_n. \tag{B.4}$$

Another important example is the expression of higher-order derivatives. Relative to a coordinate chart (U, x) of some k-dimensional manifold define $\partial_i \equiv \frac{\partial}{\partial x^i}$. We then define:

$$\partial_{\vec{\alpha}} = \frac{\partial^{\alpha_1}}{\partial\,(x^1)^{\alpha_1}} \cdots \frac{\partial^{\alpha_k}}{\partial\,(x^k)^{\alpha_k}}. \tag{B.5}$$

In what follows, let M be compact k-dimensional manifold and $\mathfrak{X}$ a smooth real or complex n-dimensional vector bundle over M with projection π. We shall use the following notation for the sections of the vector bundle:

$$\Gamma(M, \mathfrak{X}) \equiv \{f : M \to \mathfrak{X}; \pi \circ f(x) = x, f \text{ is continuous}\}; \tag{B.6}$$

$$\Gamma^\infty(M, \mathfrak{X}) \equiv \{f : M \to \mathfrak{X}; \pi \circ f(x) = x, f \text{ is smooth}\}. \tag{B.7}$$

Definition B.2 (Linear differential operator)**.** *Assume that $\mathfrak{X}$ is a real vector bundle. A **linear differential operator of order (or rank)** m is an $\mathbb{R}$-linear map*

$$P : \Gamma^\infty(M, \mathfrak{X}) \to \Gamma^\infty(M, \mathfrak{X}) \tag{B.8}$$

with the following property. For each $p \in M$ there is a set of local coordinates $\left(x^1, \ldots, x^k\right)$ defined on some neighborhood U of p and a homeomorphism

$$\mathfrak{X}|_U \cong U \times \mathbb{R}^n, \tag{B.9}$$

such that P is given by the expression

$$P(x) = \sum_{|\vec{\alpha}| \le m} A^{\vec{\alpha}}(x) \partial_{\vec{\alpha}} \qquad \forall x \in U, \tag{B.10}$$

*with $A^{\vec{\alpha}}$ an n-by-n matrix whose elements are smooth functions on M. We also require that there is at least one $\vec{\beta}$ with $|\vec{\beta}| = m$ such that $A^{\vec{\beta}} \neq 0$. This definition extends readily to the case of complex vector bundles. Take some $\zeta_p \in T^*_p M$ and let $\left\{\zeta^j_p, j \in \{1, \ldots, k\}\right\}$ be the collection of its components relative to the base defined by the chart (U, x). We define the **symbols** of P as follows.*

$$p^P(p, \zeta_p) \equiv \sum_{j=0}^{m} p^P_{m-j}(x, \zeta_p), \qquad p^P_{m-j}(x, \zeta_p) \equiv \sum_{|\vec{\alpha}| = m-j} A^{\vec{\alpha}} \zeta_p^{\vec{\alpha}}. \tag{B.11}$$

The leading term is called the ***principal symbol:***

$$\sigma^P(p,\zeta_p) \equiv \sum_{|\vec{\alpha}|=m} A^{\vec{\alpha}}\zeta_p^{\vec{\alpha}}. \tag{B.12}$$

Lemma B.3. *The collection of differential operators on $\mathfrak{X}$ forms a unital algebra over $C^\infty(M)$ with a countable base.*

Proof. According to smooth version of the Serre–Swan theorem [52], [28, 4.2] $\Gamma^\infty(M,\mathfrak{X}) \cong p\mathcal{A}^n$ with

$$\mathcal{A} \equiv \begin{cases} C^\infty(M) & \text{when } \mathfrak{X} \text{ is a complex vector bundle} \\ C^\infty_{\mathbb{R}}(M) & \text{when } \mathfrak{X} \text{ is a real vector bundle} \end{cases} \tag{B.13}$$

and p an idempotent in $M_n(\mathcal{A})$. Note that the $\mathcal{A}$-linear sum and the composition of two linear differential operators on $\mathfrak{X}$ is again a linear differential operator. The collection of linear differential operators on $\mathfrak{X}$ is therefore a subalgebra of all $\mathcal{A}$-linear maps $pM_n(\mathcal{A}) \to pM_n(\mathcal{A})$, implying this algebra has a countable base. Note that the unit operator is also a differential operator, hence the collection of linear differential operators on $\mathfrak{X}$ is a unital algebra. □

We quote a list of definitions and results regarding principal symbols. We use the same notation as in the previous definitions.

- Denote with End$\mathfrak{X}$ the endomorphism bundle of $\mathfrak{X}$. Its fibers are the linear maps from the fibers of $\mathfrak{X}$ to itself. From [31, III.§1]:

  $$\begin{aligned} &\sigma^p \in \Gamma^\infty\left(T^*M, \mathrm{End}\mathfrak{X}\right); \\ &\sigma^p(x,\zeta_x): \mathfrak{X}_x \to \mathfrak{X}_x \qquad \forall x \in M,\ \forall \zeta \in T^*M, \end{aligned} \tag{B.14}$$

  so the principal symbol is invariant under a change of coordinates. If $\sigma^p(x,\zeta_x)$ is invertible for all $x \in M$ and for each covector which is non-zero everywhere then the differential operator is said to be **elliptic**.

- The principal symbol of P is given by the following handy formula [21, Prop. 7.27]. If $f \in C^\infty(M)$ locally satisfies $df = \zeta$ then:

  $$\sigma^P(x,\zeta_x) = \lim_{t\to\infty} t^{-m}\exp(-itf(x))P(x)\left[\exp(itf(x))\right] \tag{B.15}$$

  where m is the order of P.

- Let Q also be a differential operator of order m defined on the smooth sections of $\mathfrak{X}$. Define $R \equiv Q \circ P$. By [31, Prop. 1.7]

  $$\sigma^R = \sigma^Q \circ \sigma^P. \tag{B.16}$$

  This is the **principal symbol calculus**.

Appendix C

Morita equivalence

In this appendix we introduce **modules** and specify further to **Hilbert modules**. Modules are generalizations of vector spaces; Hilbert modules are generalizations of Hilbert spaces. We will see that an important example of a Hilbert module are the smooth vector fields of a manifold M. Next, we discuss **compact endomorphisms** of a Hilbert module. These are generalizations of the set of compact operators on a Hilbert space. Last, we state the definition of **Morita equivalence**.
We shall need the concept of Hilbert modules and Morita equivalence in chapters 2 and 5. In these chapters, we need will need to use Plymen's theorem. See section 2.4.2 for more background information. The theorem states that a manifold is spin if and only if a certain algebraic condition is met. This algebraic condition is formulated in terms of Morita equivalence.

Definition C.1 (Positivity). *Let* A *be an involutive algebra over the complex numbers. We say that an element* $a \in \mathsf{A}$ *is* ***positive*** *when there is an element* $b \in \mathsf{A}$ *such that*

$$a = b^* b. \tag{C.1}$$

We denote the set of positive elements of A *with* A^+*.*

Let A be a C^*-algebra. From [25, Ch. 4.2] we gather that the relation "$\leq$" on A defined by

$$a \leq b \quad \Leftrightarrow \quad b - a \in A^+ \qquad \Rightarrow \qquad A^+ = \{a \in A; 0 \leq a\} \tag{C.2}$$

defines a linear partial ordering on the positive elements. Before moving on we briefly review the definitions regarding modules, which are generalizations of vector spaces.

Definition C.2 (Modules)**.** *Let* $(V, +)$ *be an Abelian group and* A *an (associative) algebra over the complex numbers. In what follows, let* $a, b \in \mathsf{A}$ *and* $v, w \in V$*.*
V *is a* ***left-*A*-module*** *when there exists an action* $\cdot : \mathsf{A} \times V \to V$ *such that*

$$a \cdot (v + w) = a \cdot v + a \cdot w, \tag{C.3}$$
$$(a + b) \cdot v = a \cdot v + b \cdot v, \tag{C.4}$$
$$(ab) \cdot v = a \cdot (b \cdot v), \tag{C.5}$$
$$1 \cdot v = v. \tag{C.6}$$

A ***right-*A*-module*** V *satisfies similar properties with relation to an action* $\cdot' : V \times \mathsf{A} \to V$*.*
An A*-bimodule is both a right-*A*-module and a left-*A*-module satisfying*

$$(a \cdot v) \cdot' b = a \cdot (v \cdot' b). \tag{C.7}$$

An A*-bimodule is called* ***symmetric*** *when*

$$a \cdot v = v \cdot' a. \tag{C.8}$$

A ***module morphism*** *for two left-*A*-modules* V *and* W *is an* A*-linear map* $\varphi : V \to W$*:*

$$\varphi(a \cdot v + b \cdot w) = a \cdot \varphi(v) + b \cdot \varphi(w). \tag{C.9}$$

Module morphisms for right-A*-modules and* A*-bimodules are defined analogously.*
Let B *be a second involutive, complex algebra. An* A*-*B*-bimodule is both a left-*A*-module and a right-*B*-module such that multiplication from the left with* A *is compatible with multiplication from the right with* B*. An* A*-*B*-bimodule morphism is both a left-*A*-module morphism and a right-*B*-module morphism.*

We generalize the concept of a Hilbert space.

Definition C.3 (Hilbert modules)**.** *Let* A *be a* C^**-algebra with norm* $\|\cdot\|$*. Let* $\mathcal{G}$ *be a right-*A*-module. We say that* $\mathcal{G}$ *is a* ***right-*A*-Hilbert module*** *when:*

1. *there is a* $\mathbb{C}$*-sesquilinear map* $\langle \cdot, \cdot \rangle_A : \mathcal{G} \times \mathcal{G} \to A$ *such that for all* $\psi, \varphi \in \mathcal{G}$ *and* $a \in A$*:*

$$\langle \psi, \varphi a \rangle_A = \langle \psi, \varphi \rangle_A a; \tag{C.10}$$
$$\langle \psi, \varphi \rangle_A^* = \langle \varphi, \psi \rangle_A; \tag{C.11}$$
$$\langle \psi, \psi \rangle_A \in A^+; \tag{C.12}$$
$$\langle \psi, \psi \rangle_A = 0 \qquad \Rightarrow \qquad \psi = 0. \tag{C.13}$$

2. $\mathcal{G}$ is complete with respect to the norm defined by

$$\sqrt{\|\langle\psi,\psi\rangle_A\|}. \tag{C.14}$$

*3. $\mathcal{G}$ is **full**, i.e., the closure of the linear span of*

$$\{\langle\psi,\varphi\rangle_A;\psi,\varphi\in\mathcal{G}\} \tag{C.15}$$

is equal to A.

Note this is just the definition of a Hilbert space when we choose A to be equal to the complex numbers. A left-A-Hilbert module is defined analogously with the difference that we now demand

$$\langle a\psi,\varphi\rangle_A = a\langle\psi,\varphi\rangle \qquad \forall a\in A,\ \forall\psi,\varphi\in\mathcal{G}. \tag{C.16}$$

Morphisms of Hilbert modules are (left- and right-) module morphisms, continuous maps and preserve the inner product in the following sense. Let $T:\mathcal{G}\to\mathcal{G}'$ be a morphism of right-A-Hilbert modules. For all $\psi,\varphi\in\mathcal{G}$,

$$\langle T\psi,T\varphi\rangle_A = \langle\psi,\varphi\rangle'_A, \tag{C.17}$$

where $\langle\cdot,\cdot\rangle_A$ is the A-valued inner product on $\mathcal{G}$ and $\langle\cdot,\cdot\rangle'_A$ is the A-valued inner product on $\mathcal{G}'$.

We now show how one can construct Hilbert modules out of existing ones.

Definition C.4. *Let $\mathcal{G}$ be a right-A-Hilbert module with A-valued inner product $\langle\cdot,\cdot\rangle_A$. We define*

$$\{\langle\psi|,\psi\in\mathcal{G};\langle\psi|:\mathcal{G}\to A,\langle\psi|(\varphi)\equiv\langle\psi,\varphi\rangle_A\} \tag{C.18}$$

in the operator norm relative to the norm on A. We also write

$$\mathcal{G}^\sharp \equiv Hom^0_A(\mathcal{G},A) \tag{C.19}$$

There is a canonical right-A-Hilbert module structure on $\mathcal{G}^\sharp$, though we shall not present this one. Rather, we discuss an alternative structure that is used in the proof of lemma 2.31.

Lemma C.5. *Assume that A is commutative. $\mathcal{G}^\sharp$ is a right-A-Hilbert module.*

Proof. We use the same definitions and notations as in definition C.4. Take some $a\in A$ and $\psi,\varphi\in\mathcal{G}$. The right-action of A on $\mathcal{G}^\sharp$ is given by:

$$\langle\psi|\cdot a\equiv\langle\psi a^*|, \tag{C.20}$$

and the A-valued inner product $[\cdot,\cdot] : \mathcal{G}^\sharp \times \mathcal{G}^\sharp \to A$ is defined by:

$$[\langle\psi|, \langle\varphi|]_A = \langle\varphi, \psi\rangle_A. \tag{C.21}$$

We check that $\mathcal{G}^\sharp$ satisfies the properties of a right-A-Hilbert module:

$$[\langle\psi|, \langle\varphi| \cdot a]_A = [\langle\psi, \langle\varphi a^*|]_A = \langle\varphi a^*, \psi\rangle_A = \\ \langle\psi, \varphi a^*\rangle_A^* = (\langle\psi\varphi\rangle_A a^*)^* = a\langle\psi, \varphi\rangle_A.$$

Using the fact that A is commutative, we see that the last term equals:

$$\langle\psi, \varphi\rangle_A a = [\langle\psi|, \langle\varphi|]_A \, a,$$

so that

$$[\langle\psi|, \langle\varphi a|]_A = [\langle\psi|, \langle\varphi|]_A \, a,$$

as required.
We continue to verify the other properties:

$$[\langle\psi|, \langle\varphi|]_A^* = \langle\varphi, \psi\rangle_A^* = \langle\psi, \varphi\rangle_A = [\langle\varphi|, \langle\psi|]_A \,; \\ [\langle\psi|, \langle\psi|]_A = \langle\psi, \psi\rangle_A \in A^+; \\ 0 = [\langle\psi|, \langle\psi|]_A = \langle\psi, \psi\rangle_A \quad \Rightarrow \quad \psi = 0.$$

So the left-action and the A-valued inner product on $\mathcal{G}^\sharp$ satisfy the first set of properties of definition C.3. Completeness and fullness of $\mathcal{G}^\sharp$ follow directly from that of $\mathcal{G}$.

□

Example C.6. *Let A be a C^*-algebra. By equipping A with the inner product*

$$(a, b) \in A \times A \to A \qquad (a, b) \mapsto a^*b, \tag{C.22}$$

we see that A is a right-A-Hilbert module.

We can take the generalization of Hilbert spaces to Hilbert modules one step further.

Definition C.7 (Compact endomorphisms)**.** *Let $\mathcal{G}$ be a right-A-Hilbert module. The **adjointable** morphisms are the continuous morphisms $T : \mathcal{G} \to \mathcal{G}$ that come with an associated linear morphism $T^* : \mathcal{G} \to \mathcal{G}$ such that*

$$\langle T^*\psi, \varphi\rangle_A = \langle\psi, T\varphi\rangle_A \qquad \forall\psi, \varphi \in \mathcal{G}. \tag{C.23}$$

*Their collection forms the so-called **endomorphisms** of $\mathcal{G}$, denoted by $End_A(\mathcal{G})$.*

For any two $\psi, \varphi \in \mathcal{G}$ define the map $|\psi\rangle\langle\varphi| : \mathcal{G} \to A$ by

$$|\psi\rangle\langle\varphi|(\phi) \equiv \psi\langle\varphi, \phi\rangle_A, \quad \forall \phi \in \mathcal{G}. \tag{C.24}$$

These maps are examples of adjointable automorphisms of $\mathcal{G}$. The norm-closure of their linear span is called the algebra of ***compact endomorphisms****, denoted by $End^0_A(\mathcal{G})$. The collection of compact endomorphisms is a C^*-algebra. It forms an ideal in the set of endomorphisms. See [28, A.4] for more background.*

Consider the algebra of smooth functions on some compact manifold M. The collection of smooth vector fields $\Gamma^\infty(M, TM)$ forms a $C^\infty(M)$-bimodule. We are able to define a $C^\infty(M)$-valued inner product on the sections by using any metric g:

$$g(\zeta, \eta) \in C^\infty(M); \tag{C.25}$$

$$g(\zeta, \eta)(x) \equiv g_x(\zeta_x, \eta_x) \qquad \forall \zeta, \eta \in \Gamma^\infty(M, TM). \tag{C.26}$$

However, the collection of smooth sections is not a Hilbert module since $C^\infty(M)$ is not a C^*-algebra and requirements C.3.2 and C.3.3 are not satisfied. We therefore define a weaker notion.

Definition C.8 (Pre-Hilbert modules)**.** *Let $\mathcal{A}' \subset A$ be an involutive algebra lying dense in the C^*-algebra A.*
$\mathcal{G}$ is a ***right-$\mathcal{A}'$-pre-Hilbert module*** *when $\mathcal{G}$ is a right-$\mathcal{A}'$-module satisfying all the properties of a right-$\mathcal{A}'$-Hilbert module except for C.3.2 and C.3.3. Analogously, we can define left-$\mathcal{A}'$-pre-Hilbert modules.*

The second important definition defines an equivalence relation on C^*-algebras.

Definition C.9 ((Strong) Morita equivalence)**.** *Let A and B be two unital C^*-algebras. We say that A is* ***(strongly) Morita equivalent*** *to B, denoted by $A \overset{M}{\sim} B$, when there is a right-A-Hilbert module $\mathcal{G}$ such that $B \cong End^0_A(\mathcal{G})$. We call the Hilbert module $\mathcal{G}$ in question a A-B-**imprimitivity bimodule**.*

It is not apparent from the definition that this defines an equivalence relation, but it actually is. There are several other equivalent ways to define strong Morita-equivalence from which the symmetry and the transitivity of the relation are readily verified. See for instance [28, A.4] or [21, 4.5].

Appendix D

de Rham cohomology and the HKRC-theorem

In this appendix we define the algebraic analogue of the de Rham cochain groups, which, as a result of the **Hochschild-Kostant-Rosen-Connes (HKRC) theorem**, turns out to be the **Hochschild homology groups** of the algebra $C^\infty(M)$. More specifically, we shall show the de Rham cochain groups and the Hochschild homology groups are isomorphic as **graded algebras** and **symmetric $C^\infty(M)$-bimodules**.
The material discussed in chapters 3 to 5 relies heavily on the theory developed in this appendix.
Before delving into the theory let us make some preliminary definitions.

Definition D.1 (Graded differential algebras)**.** *Let $\left\{V^k; k \in \mathbb{N}, V^0 \cong \mathsf{A}\right\}$ be a collection of left-A-modules. Form their direct sum*

$$V^\bullet \equiv \bigoplus_{k=0}^{\infty} V^k. \qquad \text{(D.1)}$$

We identify V^k with the inclusion $V^k \hookrightarrow V^\bullet$. $V^\bullet$ becomes a left-A-module by letting A act on each of the components V^k.
*We say that a left-A-module $V^\bullet$ is a **graded differential algebra** when:*

1. *For each $k \in \mathbb{N}$, there is a $\mathbb{C}$-linear map $d : V^k \to V^{k+1}$ such that $d^2 = 0$, i.e., $(d, V^\bullet)$ is **cochain complex**;*

2. *$V^\bullet$ is a **graded algebra**: there is a multiplication $\cdot$ defined on $V^\bullet$ such that $v_k \cdot v_l \in V^{k+l}$ when $v_k \in V^k$ and $v_l \in V^l$. We shall furthermore require that, with respect to this multiplication, $V^\bullet$ is an associative algebra and that the algebra is **graded commutative**, i.e.*

$$v_k \cdot v_l = (-1)^{k+l} v_l \cdot v_k. \qquad \text{(D.2)}$$

3. *d is an **odd derivation** with respect to this multiplication: for any $v_k \in V^k$ and $v \in V^\bullet$*

$$d(v_k \cdot v) = d(v_k) \cdot v + (-1)^k v_k \cdot d(v). \tag{D.3}$$

We denote the graded differential algebra by $(d, V^\bullet)$, to make clear which map plays the role of the coboundary.
A morphism of graded differential complexes $(d, V^\bullet)$ and $(d', W^\bullet)$ is a collection of A-module morphisms $\{\varphi_k : V^k \to W^k; k \in \mathbb{N}\}$ such that:

1. *φ is a morphisms of cochain complexes, i.e., $\varphi_{k+1} \circ d = d' \circ \varphi_k$ for all k;*

2. *$\varphi_{k+l}(v_k \cdot k_l) = \varphi_k(v_k) \cdot \varphi_l(v_l)$ for all $a_k \in V^k$ and $a_l \in V^l$.*

The definition extends naturally to the case when the differential algebras are right-A-modules or A-bimodules.
These definitions also apply to the case when $(\partial, V_\bullet)$ is a chain complex, i.e., ∂ is degree-lowering of order 1. Now instead, for morphisms of graded differential algebras we demand that

$$\varphi_{k-1} \circ \partial = \partial' \circ \varphi_k. \tag{D.4}$$

For the rest of this section, assume M to be an oriented and compact manifold and A to be a unital and commutative algebra over the complex numbers.[1] We reserve the special notation $\mathcal{A}$ for the algebra $C^\infty(M)$.
We shall denote the de Rham cochains of order k with $\Omega^k_{dR}(M)$ and the whole cochain complex with $\Omega^\bullet_{dR}(M)$. Note that $\Omega^\bullet_{dR}(M)$ is a symmetric $C^\infty(M)$-bimodule. By definition $\Omega^0_{dR}(M) = C^\infty(M)$. Together with the exterior derivative d the pair $(d, \Omega^\bullet_{dR}(M))$ forms a graded differential algebra. The latter statements follows from the fact that the wedge product turns $\Omega^\bullet_{dR}(M)$ into a graded commutative algebra and that the exterior derivative d is odd with respect to this wedge product [19, Ch. 2].
In this section we shall first take a look at the Kähler differentials. To this algebra we can associate a graded differential algebra which will form the bridge between the Hochschild homology and the de Rham cohomology. This correspondence, expressed by the HKRC-theorem, is discussed next. Subsequently we shall take a look at the dual theory of Hochschild homology, to so-called **Hochschild cohomology**.
Let us pose the following universal question: amongst the symmetric A-bimodules $\mathcal{E}$ recipient of a derivation $D : \mathsf{A} \to \mathcal{E}$, is there a "largest" one? More precisely, for each pair $(D, \mathcal{E})$, is there a (necessarily unique) pair $(d, \Omega^1_{\text{ab}}(\mathsf{A}))$ and module morphism $\psi_D : \Omega^1_{\text{ab}}(\mathsf{A}) \to \mathcal{E}$ such that $\psi_D \circ d = D$?

[1] Note that, in general, much of the theory in this appendix can be applied to noncommutative algebras as well. See [32, Ch. 1] for more information.

Lemma D.2. *Let* $\Omega^1(\mathsf{A}) = \ker\{m : \mathsf{A}\otimes\mathsf{A} \to \mathsf{A}; m(a\otimes b) = ab\}$*. Then* $\Omega^1_{ab}(\mathsf{A}) \equiv \Omega^1(\mathsf{A})/(\Omega^1(\mathsf{A}))^2$ *is the solution to the aforementioned universal question. This solution is called the bimodule of* ***Kähler differentials****.*

Proof. Define the derivation $d : \mathsf{A} \to \mathsf{A}\otimes\mathsf{A}$ by $d(a) = 1\otimes a - a\otimes 1$. Any element of $\Omega^1(\mathsf{A})$ is of the form $\sum_i a_i \otimes b_i - a_i b_i \otimes 1 = \sum_i a_i(1\otimes b_i - b_i \otimes 1)$. Hence we can rewrite that element as $\sum_i a_i db_i$. Since

$$
\begin{aligned}
&adb - dba = a\otimes b - ab\otimes 1 - 1\otimes ab + b\otimes a = \\
&-(1\otimes a - a\otimes 1)\cdot(1\otimes b - b\otimes 1),
\end{aligned}
$$

$\Omega^1_{\mathrm{ab}}(\mathsf{A})$ is a symmetric A-bimodule.
Now let $(D,\mathcal{E})$ be any other symmetric A-bimodule such that $D : \mathsf{A}\to\mathcal{E}$ is a derivation. Define the map $\psi_D : \Omega^1(\mathsf{A}) \to \mathcal{E}$ by $\psi_D(adb) = aDb$ and extend by linearity. This map descends to $\Omega^1_{\mathrm{ab}}(\mathsf{A})$. For all $a_1 da_2$, $b_1 db_2 \in \Omega^1(\mathsf{A})$:

$$
\begin{aligned}
&\psi_D(a_1 da_2 \cdot b_1 db_2) = \psi_D(a_1 b_1 \otimes a_2 b_2) - \psi_D(a_1 a_2 b_1 \otimes b_2) + \\
&\psi_D(a_1 a_2 b_1 b_2 \otimes 1) - \psi_D(a_1 b_1 b_2 \otimes a_2) = \\
&a_1 b_1 \left[D(a_2 b_2) - a_2 Db_2 - b_2 Da_2\right] + a_1 a_2 b_1 b_2 D(1) = 0.
\end{aligned}
$$

Hence ψ_D is a module morphism. □

We will now proceed to construct a graded differential algebra from $\Omega^1_{\mathrm{ab}}(\mathsf{A})$. Take $\Omega^\bullet_{\mathrm{ab}}(\mathsf{A})$ as the anti-symmetrization of $\Omega^1_{\mathrm{ab}}(\mathsf{A})$, i.e., the tensor algebra factored out by the ideal generated by the elements

$$\left\{da\otimes da, da \in \Omega^1_{\mathrm{ab}}(\mathsf{A})\right\}.$$

By definition, $\Omega^0_{\mathrm{ab}}(\mathsf{A}) \cong \mathsf{A}$.
Then $\Omega^\bullet_{\mathrm{ab}}(\mathsf{A})$ is a symmetric A-bimodule by virtue of the following definitions:

$$a\cdot(a_0 da_1 \wedge\cdots\wedge da_n) \equiv a a_0 da_1 \wedge\cdots\wedge da_n, \tag{D.5}$$
$$(a_0 da_1 \wedge\cdots\wedge da_n)\cdot a \equiv a a_0 da_1 \wedge\cdots\wedge da_n. \tag{D.6}$$

Define

$$
\begin{aligned}
d : \Omega^n_{\mathrm{ab}}(\mathsf{A}) &\to \Omega^{n+1}_{\mathrm{ab}}(\mathsf{A}), \\
d(a_0 da_1\wedge\cdots\wedge da_n) &= da_0 \wedge da_1\wedge\cdots\wedge da_n.
\end{aligned}
$$

On constants $\lambda\in\mathbb{C}$, $d(\lambda) = 1\otimes\lambda - \lambda\otimes 1 = \lambda(1\otimes 1 - 1\otimes 1) = 0$, from which $d^2 = 0$. This implies that $(d, \Omega^\bullet_{\mathrm{ab}}(\mathsf{A}))$ is a cochain complex.

Defining the product as

$$(a_0 \wedge da_1 \wedge \cdots \wedge da_n) \cdot (b_0 \wedge db_1 \wedge \cdots \wedge db_m) \equiv \\ a_0 b_0 da_1 \wedge \cdots \wedge da_n \wedge db_1 \wedge \cdots \wedge db_m, \tag{D.7}$$

the complex turns into a graded differential complex in which the multiplication is graded commutative.
The usefulness of the exterior algebra of the Kähler differentials is expressed through the fact that they are a natural extension of the de Rham cochains.
Throughout this appendix, let $\mathcal{A} \equiv C^\infty(M)$, for M a compact manifold without boundary.

Theorem D.3. *As symmetric $\mathcal{A}$-bimodules and graded differential algebras*

$$\Omega^\bullet_{ab}(\mathcal{A}) \cong \Omega^\bullet_{dR}(M). \tag{D.8}$$

Proof. We are finished when we are able to show that

$$\Omega^1_{\mathrm{dR}}(M) \cong \Omega^1_{\mathrm{ab}}(\mathcal{A}). \tag{D.9}$$

The statement then follows from the fact that both the de Rham complex and $\Omega^\bullet_{\mathrm{ab}}(\mathcal{A})$ are defined via the same exterior algebra construction.
The result of lemma D.2 can be restated as $\mathrm{Der}(\mathcal{A}, \mathcal{E}) \cong \mathrm{Hom}_{\mathcal{A}}(\Omega^1_{\mathrm{ab}}(\mathcal{A}), \mathcal{E})$.
Take $\mathcal{E} = \mathcal{A}$, which becomes a symmetric bimodule by defining the action of $\mathcal{A}$ on itself by multiplication.
The derivations of $\mathcal{A}$, $\mathrm{Der}(\mathcal{A}, \mathcal{A})$, are the vector fields on M, which we denote by $\mathcal{X}_1(M)$. By definition,

$$\Omega^1_{\mathrm{dR}}(M) \equiv \mathrm{Hom}_{\mathcal{A}}(\mathcal{X}_1(M), \mathcal{A}) \cong \\ \mathrm{Hom}_{\mathcal{A}}(\mathrm{Der}(\mathcal{A}, \mathcal{A}), \mathcal{A}) = \mathrm{Hom}_{\mathcal{A}}\left(\mathrm{Hom}_{\mathcal{A}}\left(\Omega^1_{\mathrm{ab}}(\mathcal{A}), \mathcal{A}\right), \mathcal{A}\right) \cong \Omega^1_{\mathrm{ab}}(\mathcal{A}),$$

where the last isomorphism is due to the fact that $\Omega^1_{\mathrm{ab}}(\mathcal{A})$ is finite-dimensional.

□

We shall now describe the (homological!) equivalent of $\Omega^\bullet_{\mathrm{ab}}(\mathsf{A})$.

Definition D.4 (Hochschild chain groups)**.** *Let $C_n(\mathsf{A}) \equiv \mathsf{A} \otimes \mathsf{A}^{\otimes n}$ for $n \geq 0$ and $C_n(\mathsf{A}) = 0$ otherwise. We define the **boundary map** $b : C_n(\mathsf{A}) \to C_{n-1}(\mathsf{A})$ on pure elements as:*

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \equiv \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + \\ (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}, \tag{D.10}$$

and extend b by linearity. Let $C_\bullet(\mathsf{A}) \equiv \bigoplus_{n=0}^{\infty} C_n(\mathsf{A})$.
We define

$$Z_n(\mathsf{A}) \equiv \ker b : C_n(\mathsf{A}) \to C_{n-1}(\mathsf{A}); \qquad \textbf{\textit{(cycles)}} \tag{D.11}$$
$$B_n(\mathsf{A}) \equiv b\left(C_{n+1}(\mathsf{A})\right). \qquad \textbf{\textit{(boundaries)}} \tag{D.12}$$

Writing $b \circ b$ out in full for arbitrary elements in any $C_n(\mathsf{A})$ shows their composition is zero. We can therefore form the ***Hochschild homology groups***

$$HH_n(\mathsf{A}) \equiv Z_n(\mathsf{A})/B_n(\mathsf{A}). \tag{D.13}$$

The collection of Hochschild homology groups, denoted with $HH_\bullet(\mathsf{A})$, is an Abelian group due to the linearity of the boundary map. For $a \in \mathsf{A}$ the actions

$$a \cdot (a_0 \otimes a_1 \otimes \cdots \otimes a_n) \equiv aa_0 \otimes a_1 \otimes \cdots \otimes a_n, \tag{D.14}$$
$$(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \cdot a \equiv a_0 a \otimes a_1 \otimes \cdots \otimes a_n \tag{D.15}$$

turn $HH_\bullet(\mathsf{A})$ into a symmetric A-bimodule.
There is a distinguished product on $HH_\bullet(\mathsf{A})$, called the **shuffle product**.

Definition D.5 ((p,q)-shuffles)**.** *Let S_n be the permutation group of n elements. If $p + q = n$, an element $\sigma \in S_n$ is called a (p, q)**-shuffle** when*

$$\sigma(1) < \sigma(2) < \ldots < \sigma(p) \qquad \& \qquad \sigma(p+1) < \sigma(p+2) < \ldots < \sigma(p+q). \tag{D.16}$$

The collection of (p, q) shuffles is denoted with $S_n^{(p,q)}$.

Lemma D.6. *The shuffle product $sh_{p,q} : C_p(\mathsf{A}) \otimes C_q(\mathsf{A}) \to C_{p+q}(\mathsf{A})$, defined by*

$$sh_{p,q} : (a_0 \otimes a_1 \otimes \cdots \otimes a_p) \times (a'_0 \otimes a_{p+1} \otimes a_{p+2} \otimes \cdots \otimes a_{p+q}) \mapsto \sum_{\sigma \in S_n^{(p,q)}} \epsilon(\sigma) a_0 a'_0 \otimes a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \cdots \otimes a_{\sigma(p+q)} \tag{D.17}$$

makes $(b, C_\bullet(\mathsf{A}))$ a graded differential algebra. We also denote the shuffle product by

$$sh_{p,q}(a, b) \equiv a \times b. \tag{D.18}$$

Proof. See [32, Ch. 4.2] and [29, Ch. X.12] for the proofs that b is odd with respect to the shuffle product and that $C_\bullet(\mathsf{A})$ is an associative algebra.

The permutation implementing the interchange $a_p \times a_q \to a_q \times a_p$, given by

$$\begin{aligned} i &\mapsto i+q \qquad 1 \le i \le p \\ i &\mapsto i-p \qquad p+1 \le i \le p+q, \end{aligned}$$

is a (p,q)-shuffle with sign

$$(-1)^{(q-1)p} \cdot (-1)^{(p-1)q} = (-1)^{2pq-p-q} = (-1)^{p+q},$$

showing $C_\bullet(\mathsf{A})$ is graded commutative. □

Corollary D.7. *$HH_\bullet(\mathsf{A})$ is a graded commutative algebra under the shuffle product.*

Proof. If $a_k \in Z_k(\mathsf{A})$ and $a_l \in Z_l(\mathsf{A})$ then, by the preceding lemma,

$$b(a_k \times a_l) = b(a_k) \times a_l + (-1)^k a_k \times b(a_l) = 0 \Rightarrow a_k \times a_l \in Z_{k+l}(\mathsf{A}).$$

Let $a_{k+1} \in C_{k+1}(\mathsf{A})$ and $a_{l+1} \in C_{l+1}(\mathsf{A})$.

$$\begin{aligned} &(a_k + ba_{k+1}) \times (a_l + ba_{l+1}) = \\ &a_k \times a_l + b(a_{k+1} \times a_l) + (-1)^{k+1} a_{k+1} \times ba_l + (-1)^k b(a_k \times a_{l+1}) + \\ &(-1)^{k+1} ba_k \times a_{l+1} + (-1)^{k+1} b(ba_{k+1} \times a_{l+1}) + (-1)^k b^2 a_{k+1} \times a_{l+1} = \\ &a_k \times a_l + b\left(a_{k+1} \times a_l + (-1)^k a_k \times a_{l+1} + (-1)^{k+1} ba_{k+1} \times a_{l+1}\right), \end{aligned}$$

which implies that $[a_k] \times [a_l] = [a_k \times a_l]$ in $HH_{k+l}(\mathsf{A})$. □

In summary, we have established that $(b, C_\bullet(\mathsf{A}))$ is a graded differential algebra and a symmetric A-bimodule and that $HH_\bullet(\mathsf{A})$ is a graded commutative algebra and a symmetric A-bimodule. We now relate the Hochschild homology groups to the de Rham cohomology.

Definition D.8. *Define $\mathbb{A}_n : \Omega^n_{ab}(\mathsf{A}) \to C_n(\mathsf{A})$, the so-called* ***anti-symmetrization operator****, by*

$$\mathbb{A}_n(ada_1 \wedge \cdots \wedge da_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}. \qquad \text{(D.19)}$$

Lemma D.9. *The anti-symmetrization operator maps any element in $\Omega^n_{ab}(\mathsf{A})$ to a cycle in $C_n(\mathsf{A})$.*

Proof. We let b act on the right hand side of (D.19). Omitting the constant $\frac{1}{n!}$:

$$\begin{aligned} &\sum_{\sigma \in S_n} b\left(a_0 \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}\right) = \\ &\sum_{\sigma \in S_n, j=1}^{n-1} \epsilon(\sigma) \left[(-1)^j a_0 \otimes \cdots \otimes a_{\sigma(j)} a_{\sigma(j+1)} \otimes \cdots \otimes a_{\sigma(n)} + \right. \qquad \text{(D.20)} \\ &\left. (-1)^n a_0 a_{\sigma(n)} \otimes a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n-1)} + a_0 a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \cdots \otimes a_{\sigma(n)}\right]. \end{aligned}$$

Let us start with evaluating the last two terms in equation (D.20). They differ by the permutation $i \mapsto i+1$, which has $(-1)^{n-1}$ as sign. Therefore the last two terms cancel. For fixed j, the first term in (D.20) contains identical terms for half of the permutations. Since A is commutative, also this terms equates to zero. This implies $\mathbb{A}_n\left(\Omega^n_{\mathrm{ab}}(\mathsf{A})\right) \subset Z_n(\mathsf{A})$. $\square$

Does $\mathbb{A}_n$ also descend to the homological level? That would be the case if from the equation $\mathbb{A}_n(c) = ba$ for each $a \in C_{n+1}(\mathsf{A})$ it would follow that $c = 0$. We establish this as follows. Let $\varepsilon_n : Z_n(\mathsf{A}) \to \Omega^n_{\mathrm{ab}}(\mathsf{A})$ be the map

$$\varepsilon_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 da_1 \wedge \cdots \wedge da_n. \tag{D.21}$$

We have the identity

$$\varepsilon_n \circ \mathbb{A}_n = \mathrm{id}_{\Omega^n_{\mathrm{ab}}(\mathsf{A})}. \tag{D.22}$$

We evaluate $\varepsilon_{n-1}(ba)$ for $a \in C_n(\mathsf{A})$:

$$\varepsilon_{n-1}(ba) = a_0 a_1 da_2 \wedge \cdots \wedge da_n + \tag{D.23}$$

$$\sum_{j=1}^{n-1} (-1)^j a_0 da_1 \wedge \cdots \wedge d(a_j a_{j+1}) \wedge \cdots \wedge da_n + \tag{D.24}$$

$$(-1)^n a_n a_0 da_1 \wedge \cdots \wedge da_{n-1}.$$

Let us examine this sum term-by-term:

$$\begin{aligned}
&- a_0 a_1 da_2 \wedge da_3 \wedge \cdots \wedge da_n - a_0 a_2 da_1 \wedge \cdots \wedge da_n && (j=1)\\
&+ a_0 a_2 da_1 \wedge da_3 \wedge \cdots \wedge da_n + a_0 a_3 da_1 \wedge da_2 \wedge \cdots \wedge da_n && (j=2)\\
&\vdots\\
&(-1)^{n-1} a_0 a_{n-1} da_1 \wedge \cdots \wedge da_{n-2} \wedge da_n +\\
&(-1)^{n-1} a_0 a_n da_1 \wedge \cdots \wedge da_{n-1} && (j=n-1)
\end{aligned}$$

The above calculations illustrate that pair-wise the terms in (D.23) vanish. In summary:

Lemma D.10.

$$\varepsilon_n : HH_n(\mathsf{A}) \to \Omega^n_{ab}(\mathsf{A})) \tag{D.25}$$

is a morphisms of A*-modules such that*

$$HH_n(\mathsf{A}) \cong \Omega^n_{ab}(\mathsf{A}) \oplus \ker \varepsilon_n. \tag{D.26}$$

For arbitrary unital and commutative algebras over the complex numbers, $\ker \varepsilon_n \neq 0$. However, there is a very important case in which the kernel does vanish. From [32, Ch.4 & App. E]:

Theorem D.11 (Hochschild-Kostant-Rosenberg-Connes theorem)**.** *For* $\mathcal{A} = C^\infty(M)$*, with* M *a compact, boundaryless manifold:*

$$HH_n(\mathcal{A}) \cong \Omega^n_{ab}(\mathcal{A}) \tag{D.27}$$

as symmetric $\mathcal{A}$*-bimodules with* $\mathbb{A}_n$ *implementing the* $\mathcal{A}$*-module isomorphism.*
Moreover:

$$HH_\bullet(\mathcal{A}) \cong \Omega^\bullet_{ab}(\mathcal{A}) \tag{D.28}$$

as graded commutative algebras.
We will refer to this as the ***HKRC-theorem****.*

Using the additional structure of a graded algebra on the Hochschild homology groups, we can go a step further.

Corollary D.12. *Denote with* $\mathbb{A} \equiv \{\mathbb{A}_n; n \in \mathbb{N}\}$*. This collection of* $\mathcal{A}$*-module isomorphisms is a grading preserving algebra isomorphism*

$$HH_\bullet(\mathcal{A}) \cong \Omega^\bullet_{ab}(\mathcal{A}). \tag{D.29}$$

Proof. Grading is preserved as a result of the HKRC-theorem. We check the algebra morphism property of the map. Let $a_p \equiv a_0 da_1 \wedge \cdots \wedge da_p \in \Omega^p_{\text{ab}}(\mathcal{A})$ and $a_q \equiv a_0' da_{p+1} \wedge \cdots \wedge da_{p+q} \in \Omega^q_{\text{ab}}(\mathcal{A})$. Let $n = p + q$.

$$\begin{aligned} &sh_{p,q}\left(\mathbb{A}_p(a_p), \mathbb{A}_q(a_q)\right) = \frac{1}{p!q!} sh_{p,q} \left(\sum_{\sigma' \in S_p} \epsilon(\sigma') a_0 a_{\sigma'(1)} \otimes \cdots \otimes a_{\sigma'(p)}, \right. \\ &\left. \sum_{\sigma'' \in S_q} \epsilon(\sigma'') a_0' a_{\sigma''(p+1)} \otimes \cdots \otimes a_{\sigma''(p+q)} \right) = \\ &\frac{1}{p!q!} \sum_{\tau' \in S_p \times S_q, \tau \in S_n^{(p,q)}} \epsilon(\tau \circ \tau') a_0 a_0' a_{\tau(\tau'(1))} \otimes \cdots \otimes a_{\tau(\tau'(p+q))}. \end{aligned} \tag{D.30}$$

We shall need to show the following two results.

1. S_n and $S_n^{(p,q)} \circ S_p \times S_q$ are identical as sets.

   Proof. By definition, $S_n^{(p,q)} \circ S_p \times S_q \subseteq S_n$.
   Take some $\sigma \in S_n$. Let $\alpha \in S_p$ such that α orders the set $\{\sigma(1), \ldots, \sigma(p)\}$, i.e., $\sigma(\alpha^{-1}(1)) < \sigma(\alpha^{-1}(2)) < \ldots < \sigma(\alpha^{-1}(p))$.
   Define analogously $\beta \in S_q$ such that β orders $\{\sigma(p+1), \ldots, \sigma(p+q)\}$. Let

   $$\kappa \equiv \left\{ \begin{array}{cc} \alpha(i) & 0 \leq i \leq p \\ \beta(i-p)+p & p+1 \leq i \leq p+q \end{array} \right. .$$

Implying $\kappa \in S_p \times S_q$. The permutation $\tau \equiv \sigma \circ \kappa^{-1}$ satisfies $\tau \circ \kappa = \sigma$. By definition, $\tau \in S_n^{(p,q)}$, showing $S_n \subseteq S_n^{(p,q)} \circ S_p \times S_q$.

□

2. For each $\sigma \in S_n$ there are $\binom{n}{p}$ combinations $\tau \in S_n^{(p,q)}$ and $\tau' \in S_p \times S_q$ such that $\sigma = \tau \circ \tau'$.

   Proof. As shown in the previous proof, the expression for τ depends completely on the choice for τ'. Since τ is a (p, q)-shuffle, the identity $\sigma = \tau \circ \tau'$ is in fact determined by the images $\{\tau'(1), \ldots, \tau'(p)\}$. There are exactly $\binom{n}{p}$ ways to choose the first p values of τ' out of $p + q = n$ elements. □

   The equality

$$sh_{p,q}(\mathbb{A}_p(a_p), \mathbb{A}_q(a_q)) = \mathbb{A}_{p+q}(a_p \wedge a_q) \tag{D.31}$$

   now follows directly from the two results obtained above.

□

Before introducing the Hochschild cochain complex, we first describe a general method to create cochain complexes from a chain complex.

Definition D.13. *Let $\mathcal{C}$ be a chain complex with boundary operator ∂ and chain groups C_n freely generated over some Abelian group. Let G be any Abelian group. Define C^n as the dual space of C_n with respect to G, i.e.,*

$$C^n = Hom(C^n, G) \equiv \{\varphi : C^n \to G, \varphi \text{ is a group morphism}\}. \tag{D.32}$$

Let $d : C^n \to C^{n+1}$ be the map defined by

$$(d\psi)(c) \equiv \psi(\partial c) \qquad \forall \psi \in C^{n-1},\ \forall c \in C_{n+1}. \tag{D.33}$$

Then d defines a coboundary map which turns $(d, C^\bullet)$ into a cochain complex. De cohomology groups of this cochain complex are defined as the quotient of $Z^n \equiv \ker d : C^n \to C^{n+1}$ and $B^n \equiv imd : C^{n-1} \to C^n$.
The other way around a cochain complex defines a chain complex. However, note that in general:

$$Hom_G(H_n(\mathcal{C}, G)) \ncong H^n(\mathcal{C}, G). \tag{D.34}$$

Example D.14 (de Rham currents)**.** *The collection of* ***de Rham currents*** *of degree k are the continuous* $\mathbb{C}$*-linear maps* $\Omega^k_{dR}(M) \to \mathbb{C}$. *Their collection is denoted with* $\Omega_k^{dR}(M)$. *We represent the action of a k-current C on* $\omega_k \in \Omega^k_{dR}(M)$ *by*

$$\int_C \omega_k. \tag{D.35}$$

The boundary map is denoted by

$$\int_{\partial C} \omega_{k-1} \equiv \int_C d\omega_{k-1} \qquad \omega_{k-1} \in \Omega^{k-1}_{dR}(M). \tag{D.36}$$

When a current lies in the kernel of the boundary map, we say it is ***closed****.*

Also the exterior algebra of Kähler differentials has a dual formulation.

Definition D.15.

$$\Omega_1^{ab}(A) \equiv Hom_{\mathbb{C}}(\Omega^1_{ab}(A), \mathbb{C}). \tag{D.37}$$

Let $\Omega_\bullet^{ab}(A)$ *be the anti-symmetrization of* $\Omega_1^{ab}(A)$. *It is a graded differential algebra.*

Definition D.16 (Hochschild cohomology)**.** *The* ***Hochschild cochains*** *of* A *are defined as the cochain complex consisting of the chain groups:*

$$C^k(\mathsf{A}) \equiv Hom\left(\mathsf{A}^{\otimes k+1}, \mathbb{C}\right) \qquad \forall k \in \mathbb{N}. \tag{D.38}$$

The coboundary operator is also denoted by b and is defined as $(b\phi)(a) = \phi(ba)$ *for* $a \in C_k(\mathsf{A})$ *and* $\phi \in C^{k-1}(\mathsf{A})$. $(b, C^\bullet(\mathsf{A}))$ *is a cochain complex. We call its cohomology the* ***Hochschild cohomology****.*

Theorem D.17 (HKRC-theorem, dual version)**.** *Take some* $\phi \in C^k(\mathcal{A})$. *The association* $\phi \mapsto C_\phi$ *defined as*

$$\int_{C_\phi} a_0 da_1 \wedge \cdots \wedge da_k \equiv \frac{1}{k!} \sum_{\beta \in S_k} \epsilon(\beta)\phi(a_0, a_{\beta(1)}, \ldots, a_{\beta(k)}) \tag{D.39}$$

yields a k-current $C_\phi \in \Omega_k^{ab}(\mathcal{A})$. *This map induces an isomorphism of symmetric* $\mathcal{A}$*-bimodules*

$$HH^k(\mathcal{A}) \to \Omega_k^{ab}(\mathcal{A}) \tag{D.40}$$

and an isomorphism of graded algebras

$$HH^\bullet(\mathcal{A}) \cong \Omega_\bullet^{ab}(\mathcal{A}). \tag{D.41}$$

Proof. By definition, C_ϕ is a linear map over $\Omega^k_{dR}(M)$. Similar to our proof that $\mathbb{A}_n$ descends to the homological level, this map induces a map on the cohomological level. The HKRC-theorem then shows this map is an isomorphism of symmetric $\mathcal{A}$-bimodules. The algebra isomorphism now follows from corollary D.12. □

Remark D.18. *We denote the inverse map of $\phi \mapsto C_\phi$ by $C \mapsto \phi_C$.*

There is also an anti-symmetrization map within the Hochschild cochains.

Definition D.19. *The anti-symmetrization map is denoted by $A_n : C^n(\mathsf{A}) \to C^n(\mathsf{A})$ and is defined by*

$$A_n\varphi(a_0, \ldots, a_n) = \frac{1}{n!} \sum_{\beta \in S_n} \epsilon(\beta)\varphi\left(a_0, a_{\beta(1)}, \ldots, a_{\beta(n)}\right). \tag{D.42}$$

Corollary D.20. *Let $\varphi \in Z^n(\mathcal{A})$:*

$$[A_n\varphi] = [\varphi] \in HH^n(\mathcal{A}). \tag{D.43}$$

Proof. $b\varphi = 0$, so in particular for every term of (D.42) the action of the coboundary operator equals zero:

$$b\varphi\left(a_0, a_{\beta(1)}, \ldots, a_{\beta(p)}\right) = 0 \qquad \forall \beta \in S_n \quad \Rightarrow$$
$$A_n\varphi \in Z^n(\mathcal{A}).$$

To show that the difference of φ and $A_p\varphi$ is zero in Hochschild cohomology we use the dual version of the HKRC-theorem.

$$\int_{C_{A_n\varphi}} a_0 da_1 \wedge \cdots \wedge da_p = \frac{1}{n!} \sum_{\beta \in S_n} \epsilon(\beta) A_n\varphi\left(a_0, a_{\beta(1)}, \ldots, a_{\beta(n)}\right) =$$
$$\frac{1}{n!}\frac{1}{n!} \sum_{\alpha,\beta \in S_n} \epsilon(\alpha)\epsilon(\beta)\varphi\left(a_0, a_{\alpha(\beta(1))}, \ldots, a_{\alpha(\beta(n))}\right) =$$
$$\frac{1}{n!} \sum_{\alpha \in S_n} \epsilon(\alpha)\varphi\left(a_0, a_{\alpha(1)}, \ldots, a_{\alpha(n)}\right) = \int_{C_\varphi} a_0 da_1 \wedge \cdots \wedge da_p.$$

This hold for all choices $\{a_0, \ldots, a_p\}$, so $C_{A_n\varphi} = C_\varphi \in \Omega^{dR}_p(M)$, implying $[\varphi] = [A_n\varphi] \in HH^n(\mathcal{A})$. □

Appendix E

Cyclic cohomology

In what follows, let A be a commutative, associative and unital algebra over the complex numbers and let M be a compact manifold.
Now that we have made the step of casting the de Rham (co)chains into an algebraic mold we are naturally interested in taking the next step: what is a proper algebraic description of the de Rham (co)homology groups? We shall concern ourselves solely with the de Rham homology groups, and in particular with the direct sum of the even and the odd ones:

$$H^{\mathrm{dR}}_{\mathrm{even}}(M) \equiv \bigoplus_k H^{\mathrm{dR}}_{2k}(M); \qquad H^{\mathrm{dR}}_{\mathrm{odd}}(M) \equiv \bigoplus_k H^{\mathrm{dR}}_{2k+1}(M).$$

These groups are Abelian with respect to the same addition as in the de Rham homology.
It would be tempting to formulate the algebraic equivalent of the de Rham homology by finding a suitable boundary operator on the Hochschild cohomology groups and showing the cohomology it yields to be identical to the de Rham homology. Though we are able to cover some ground with the operator

$$B : C^k(\mathsf{A}) \to C^{k-1}(\mathsf{A}), \tag{E.1}$$

to be defined in theorem E.7, this operator, however, does not do the trick (see [32, Ch. 2.3]).
The author is not aware of any (co)boundary operator on the Hochschild (co)homology groups which does.
We are interested in cyclic (co)homology since it enables us to define a noncommutative equivalent to the Chern character isomorphism, its domain and its range. In the commutative case, the Chern character map happens to take its values in the even and odd de Rham cohomology groups. The noncommutative equivalent to the Chern character isomorphism is used in chapter 3, where we will discuss the noncommutative integral.

Before delving into an algebraic description of the even and odd de Rham homology groups we will first discuss some general results of homological algebra.

E.1 Algebraic preliminaries

Definition E.1 (Bicomplex)**.** *A collection of left-*A*-modules* $\{C^{p,q}; C_{p,q} = 0\ \forall p < 0\ \&\ \forall q < 0\}$ *is a* ***bicomplex****, or a* ***double complex****, when there are two boundary operators*

$$d_h^p : C^{p,q} \to C^{p+1,q}, \qquad \text{the } \textbf{horizontal} \text{ coboundary operator}, \tag{E.2}$$

$$d_v^q : C^{p,q} \to C^{p,q+1}, \qquad \text{the } \textbf{vertical} \text{ coboundary operator}, \tag{E.3}$$

such that

$$d_h^{p+1} \circ d_h^p = d_v^{q+1} \circ d_v^q = \left\{d_h^p, d_v^q\right\} = 0. \tag{E.4}$$

As customary, we drop the notation of the index for the horizontal and vertical boundary operators when their actions are identical for each pair (p,q)*. Write* $C^{\bullet\bullet} = \bigoplus_{p,q} C^{p,q}$*.*
This definition translates naturally to the case when the $C^{p,q}$ *are right-*A*-modules or* A*-bimodules.*

A bicomplex contains an infinite but countable number of cochain complexes which can be found by respectively fixing p or q in the bicomplex. There is also third type of cochain complex, supported on the diagonal.

Definition E.2 (Total complex)**.** *For any bicomplex* $C^{\bullet\bullet}$ *define* $(TotC^{\bullet\bullet})^n = \bigoplus_{p+q=n} C^{p,q}$*, the n-th cochain group of the* ***total complex*** $(d_h + d_v, Tot(C^{\bullet\bullet}))$*.*

We will need the following useful lemma about the cohomology of the total complex. From [5, Ch. 14], [32, Ch. 1.0]:

Lemma E.3. *Let* $\{H_p(C^{\bullet,q}); q \in \mathbb{N}\}$ *be the collection of horizontal cohomology groups. Assume that for all* $q \neq 0$ *they are zero. Set* $K^n = H_0(C^{n,\bullet})$*. Then*

$$(TotC^{\bullet\bullet})^n = H^n(K^\bullet, d_v). \tag{E.5}$$

Let us quote a famous theorem [36, §24] about chain maps.

Lemma E.4 (Zig-zag lemma)**.** *Let* $(d, \mathcal{C})$, $(d', \mathcal{D})$ *and* $(d'', \mathcal{E})$ *be chain complexes. Denote the p-th homology group of the complexes by* $H_p(\mathcal{C}), H_p(\mathcal{D})$ *and* $H_p(\mathcal{E})$, *respectively. Assume there is a short exact sequence of chain complexes*

$$0 \longrightarrow \mathcal{C} \xrightarrow{\psi} \mathcal{D} \xrightarrow{\varphi} \mathcal{E} \longrightarrow 0 . \tag{E.6}$$

Then for each $p \in \mathbb{N}$ *there is a long exact sequence*

$$\cdots \longrightarrow H_p(\mathcal{C}) \xrightarrow{\psi_*} H_p(\mathcal{D}) \xrightarrow{\varphi_*} H_p(\mathcal{E}) \xrightarrow{d'_*} H_{p-1}(\mathcal{C}) \longrightarrow \cdots \tag{E.7}$$

with ψ_*, φ_* *are the equivalence classes of* ψ *and* φ *in respectively* $H_p(\mathcal{D})$ *and* $H_p(\mathcal{E})$, *and* d'_* *is induced by the boundary operator* d'. *Here* d'_* *is called the* ***connecting morphism****.*
There is also a zig-zag lemma for cochain complexes. In that case, the degree-lowering map d'_* *in equation (E.7) is replaced by a degree-raising map of order 1 and all arrows of equation (E.7) are reversed.*

E.2 The map B

As announced before, we do not have a completely algebraic formulation of the de Rham (co)homology groups. However, we can find a map whose action coincides with the boundary operator in the de Rham (co)homology. We discuss it apart from the theory of cyclic (co)homology in this separate section.

Definition E.5. *Let* $\widetilde{B} : C^k(\mathsf{A}) \to C^{k-1}(\mathsf{A})$ *be the map*

$$\widetilde{B}\phi(a_0, \ldots, a_{k-1}) = \phi(1, a_0, \ldots, a_{k-1}) - (-1)^k \phi(a_0, \ldots, a_{k-1}, 1).$$

Lemma E.6. *Under the HKRC isomorphism (D.27), (D.28)*
$\partial : \Omega_k^{dR}(M) \to \Omega_{k-1}^{dR}(M)$ *corresponds to*
$\widetilde{B} : HH^k(C^\infty(M)) \to HH^{k-1}(C^\infty(M))$.

Proof. Take $C \in \Omega_k^{dR}(M)$. Then

$$\widetilde{B}\phi_C(a_0, \ldots, a_{k-1}) = \phi_C(1, a_0, \ldots, a_{k-1}) = \int_C da_0 \wedge \cdots \wedge da_{k-1} =$$

$$\int_{\partial C} a_0 da_1 \wedge \cdots da_{k-1} = \phi_{\partial C}(a_0, \ldots, a_{k-1}). \tag{E.8}$$

□

However, from the skew-symmetry of the integral and the fact that any element of the form $\varphi(a_0, \ldots, 1, \ldots, a_n)$ maps to 0 (due to the fact that $d(1) = 0$), we see that we are able to add several more terms to $\widetilde{B}$ without spoiling property (E.8). Let us therefore extend $\tilde{B}$:

Theorem E.7. *For $A = C^\infty(M)$, define $B : C^k(A) \to C^{k-1}(A)$ by:*

$$B\varphi(a_0, \ldots, a_k) = \sum_{j=0}^{k-1} (-1)^{j(k-1)} \varphi(1, a_j, \ldots, a_{k-1}, a_0, \ldots, a_{j-1}) +$$
$$(-1)^{(j-1)(k-1)} \varphi(a_j, \ldots, a_{k-1}, a_0, \ldots, a_{j-1}, 1).$$

Then ∂ corresponds to $\frac{1}{k}B$.

Proof. Just as in the previous lemma, this can be seen by writing out $B\varphi_C$ under the correspondence $\varphi_C \leftrightarrow C_\varphi$. The last term in the expression of $B\varphi_C$ is zero (see the introduction to theorem E.7). Let us calculate the remainder:

$$\sum_{j=0}^{k-1} (-1)^{j(k-1)} \varphi(1, a_j, \ldots, a_{k-1}, a_0, \ldots, a_{j-1}) =$$
$$\sum_{j=0}^{k-1} (-1)^{j(k-1)} \int_{C_\varphi} da_j \wedge \cdots \wedge da_{k-1} \wedge da_0 \wedge \cdots \wedge da_{j-1} =$$
$$k \int_{C_\varphi} da_0 \wedge \cdots \wedge da_{k-1} = k\varphi_{\partial C}(a_0, \ldots, a_{k-1}),$$

since $j(k-1) + j(k-j) = 2jk - j(j+1)$, which is even for all combinations j and k. □

E.3 Cyclic cohomology

We now delve into the theory of cyclic cohomology and show how cyclic cohomology relates to the even and the odd de Rham homology groups. The setup of this section is as follows. First, we define cyclic cohomology using a certain bicomplex. Second, we define cyclic cohomology as a subcomplex of the Hochschild chain complex. We then show both definitions are equal. The reason for this approach is that in the last part of this section, in which we show how cyclic cohomology relates to the de Rham homology, we use techniques which are best explained using one definition, and several techniques which are best explained using the other. We start by defining cyclic cohomology through the so-called **cyclic bicomplex**.

Definition E.8 (Maps in cyclic cohomology)**.** *The building blocks of the bicomplex are given by the Hochschild cochains groups.*

$$C^{p,q}(\mathsf{A}) \equiv C^q(\mathsf{A}). \tag{E.9}$$

Before defining the complex in full we introduce several maps between Hochschild cochain groups. Let $\lambda, N : C^n(\mathsf{A}) \to C^n(\mathsf{A})$ *be given by*

$$\lambda\varphi(a_0, \ldots, a_n) \equiv (-1)^n \varphi(a_n, a_0, \ldots, a_{n-1}) \qquad \textit{(\textbf{cyclic permuter})}; \tag{E.10}$$

$$N \equiv \sum_{i=0}^{n} \lambda^i \qquad \textit{(\textbf{cyclic skewsymmetrizer})}. \tag{E.11}$$

Let furthermore $b' : C^n(\mathsf{A}) \to C^{n+1}(\mathsf{A})$ *be the* ***truncated Hochschild boundary map****, which equals* $b : C^n(\mathsf{A}) \to C^{n+1}(\mathsf{A})$ *without the last term:*

$$b'\varphi(a_0, \ldots, a_{n+1}) \equiv \sum_{i=0}^{n} (-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}), \tag{E.12}$$

$$r \equiv b - b'. \tag{E.13}$$

And lastly, we define the maps $s, s', B : C^{n+1}(\mathsf{A}) \to C^n(\mathsf{A})$ *for* $n > 0$ *by*

$$s\varphi(a_0, \ldots, a_n) \equiv \varphi(1, a_0, \ldots, a_n) \qquad (\textit{\textbf{1}}^{\textit{\textbf{st}}} \textit{\textbf{ degeneracy operator}}); \tag{E.14}$$

$$s'\varphi(a_0, \ldots, a_n) \equiv (-1)^n \varphi(a_0, \ldots, a_n, 1) \qquad (\textit{\textbf{2}}^{\textit{\textbf{nd}}} \textit{\textbf{ degeneracy operator}}); \tag{E.15}$$

$$B \equiv N s'(1 - \lambda) \qquad (\textit{\textbf{Connes' boundary map}}). \tag{E.16}$$

For $n = 0$ *we put* $s = s' = 0$.
Connes' boundary map B *is the same operator as the one corresponding to* $(n+1)\partial$ *in the special case that the algebra* A *is equal to* $C^\infty(M)$*: in theorem E.7 it was explained that composing* $\widetilde{B} \equiv s'(1 - \lambda)$ *with any cyclic permutation (or even the sum of all possible cyclic permutations) does not change the value the integral takes due to the skew-symmetry.*

Let us relate the cyclic permuter, the truncated boundary map and r.

Lemma E.9. *The maps* $b, b' : C^n(\mathsf{A}) \to C^{n+1}(\mathsf{A})$ *equal to*

$$b' = \sum_{i=0}^{n} \lambda^i r \lambda^{-i-1}; \tag{E.17}$$

$$b = \sum_{i=0}^{n+1} \lambda^i r \lambda^{-i-1}. \tag{E.18}$$

Proof.

$$(-1)^i\varphi(a_0,\dots,a_ia_{i+1},\dots,a_{n+1}) = (-1)^{i+ni}\lambda^i\varphi(a_ia_{i+1},\dots,a_{i-1}) =$$
$$(-1)^{i+ni+n+1}\lambda^i r\varphi(a_{i+1},\dots,a_{n+1},a_0,\dots,a_i) =$$
$$(-1)^{i+ni+n+1-(i+1)(n+1)}\lambda^i r\lambda^{-i-1}\varphi(a_0,\dots,a_{n+1}).$$

Since $i+ni+n+1-in-i-n+1=2$, the sign equals $+1$.
The last statement follows from the observation that $\lambda^{n+1}=1$ in $C^{n+1}(\mathsf{A})$ and $\lambda^n=1$ in $C^n(\mathsf{A})$. □

We now shows how $C^{p,q}(\mathsf{A}) \equiv C^q(\mathsf{A})$ defines a bicomplex.

Lemma E.10. *The diagram*

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \\
C^2(\mathsf{A}) & \xrightarrow{1-\lambda} & C^2(\mathsf{A}) & \xrightarrow{N} & C^2(\mathsf{A}) & \xrightarrow{1-\lambda} & C^2(\mathsf{A}) & \xrightarrow{N} & \cdots \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \\
C^1(\mathsf{A}) & \xrightarrow{1-\lambda} & C^1(\mathsf{A}) & \xrightarrow{N} & C^1(\mathsf{A}) & \xrightarrow{1-\lambda} & C^1(\mathsf{A}) & \xrightarrow{N} & \cdots \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \\
C^0(\mathsf{A}) & \xrightarrow{1-\lambda} & C^0(\mathsf{A}) & \xrightarrow{N} & C^0(\mathsf{A}) & \xrightarrow{1-\lambda} & C^0(\mathsf{A}) & \xrightarrow{N} & \cdots
\end{array} \tag{E.19}$$

*defines a bicomplex $C^{\bullet\bullet}(\mathsf{A})$, called the **Tsygan double complex** or the **cyclic bicomplex**.*

Proof. We realize that $\lambda N = N$, showing the rows of the diagram form a cochain complex. A careful calculation of $(b')^2$ shows it equals zero, so together with the fact that $b^2=0$,a we now have shown that the columns are cochain complexes. This leaves us with showing the diagram is anti-commutative, i.e. $bN = Nb'$ and $(1-\lambda)b = b'(1-\lambda)$. We now make use of lemma E.9.

$$b'\lambda - \lambda b' = \sum_{i=0}^{n} \lambda^i r\lambda^{-i} - \lambda^{i+1}r\lambda^{-i-1} =$$
$$\sum_{i=0}^{n} \lambda^i r\lambda^{-i} - \sum_{i=1}^{n+1} \lambda^i r\lambda^{-i} = r - \lambda^{n+1}r\lambda^{-n-1}.$$

In $C^n(\mathsf{A})$, $\lambda^{n+1}=\lambda$. In $C^{n+1}(\mathsf{A})$, $\lambda^{-n-1}=1$. Hence the latter equals

$$b'\lambda - \lambda b' = (1-\lambda)r \qquad \Leftrightarrow \tag{E.20}$$
$$b'(\lambda-1) - (\lambda-1)b' = (1-\lambda)r \qquad \Leftrightarrow \qquad b'(1-\lambda) = (1-\lambda)b.$$

Finally, using lemma E.9 as well,

$$bN = \sum_{i=0}^{n+1} \lambda^i r \lambda^{i-1} N = \sum_{i=0}^{n+1} \lambda^i r N = NrN =$$
$$Nr \sum_{i=0}^{n} \lambda^i = N \sum_{i=0}^{n} \lambda^{-j-1} r \lambda^j = Nb',$$

bearing in mind that $N\lambda^j = N$ for each j.

□

An important property of the cyclic bicomplex is that its odd columns have trivial cohomology.

Lemma E.11. *The cochain complex* $\left\{\left(-b', C^{n+1}(\mathsf{A})\right); n \in \mathbb{N}\right\}$ *is acyclic.*

Proof. Let us start remarking that

$$\left(-b'\right) \circ (-s)\varphi(a_0, \ldots, a_n) = \sum_{i=0}^{n+1} (-1)^i \varphi(1, a_0, \ldots, a_i a_{i+1}, \ldots, a_n) =$$
$$\varphi(a_0, \ldots, a_n) + \sum_{i=0}^{n} (-1)^{i+1} \varphi(1, a_0, \ldots, a_i a_{i+1}, \ldots, a_n) =$$
$$\left(1 - (-s) \circ \left(-b'\right)\right) \varphi(a_0, \ldots, a_n) \qquad \Rightarrow$$
$$b' \circ s + s \circ b' = 1.$$

This shows that the map $-s$ is a chain homotopy between the identity and the zero map on the cochain complex, showing the cohomology of the complex is zero. □

Corollary E.12.

$$1 - \lambda = b's'(1 - \lambda) + s'(1 - \lambda)b. \tag{E.21}$$

Proof. Using lemma E.10:

$$b'(1 - \lambda) = (1 - \lambda)b \quad \Rightarrow$$
$$b'\widetilde{B} + \widetilde{B}b = b's'(1 - \lambda) + s'b'(1 - \lambda).$$

Using the same strategy as in lemma E.11 we see that $b's' + s'b' = 1$. □

Corollary E.13.

$$Bb + bB = 0. \tag{E.22}$$

Proof. Applying the previous corollary:

$$0 = N(1-\lambda) = Nb's'(1-\lambda) + Ns'(1-\lambda)b = bB + Bb. \tag{E.23}$$

□

The total complex $\mathrm{Tot}^n(C^{\bullet\bullet}(\mathsf{A})) \equiv \bigoplus_{i=0}^{n} C^i(\mathsf{A})$ is a cochain complex, whose cohomology is called **cyclic cohomology**. The n-th cohomology group is denoted by $HC^n(\mathsf{A})$.
The reason for defining cyclic cohomology in this way (there are other ways, as we shall see shortly) is that we can apply the zig-zag lemma to get a crucial result. From [21, H10]:

Theorem E.14 (Connes' long exact sequence)**.** *There are maps* $I_* : HC^n(\mathsf{A}) \to HH^n(\mathsf{A})$, $B_* : HH^n(\mathsf{A}) \to HC^{n-1}(\mathsf{A})$ *and* $S_* : HC^{n-1}(\mathsf{A}) \to HC^{n+1}(\mathsf{A})$ *such that*

$$\cdots \longrightarrow HC^n(\mathsf{A}) \xrightarrow{I_*} HH^n(\mathsf{A}) \xrightarrow{B_*} HC^{n-1}(\mathsf{A}) \xrightarrow{S_*}$$

$$HC^{n+1}(\mathsf{A}) \xrightarrow{I_*} HH^{n+1}(\mathsf{A}) \longrightarrow \cdots \tag{E.24}$$

Sketch of proof. Define the map $S : \mathrm{Tot}^n(C^{\bullet\bullet}(\mathsf{A})) \to \mathrm{Tot}^{n+2}(C^{\bullet\bullet}(\mathsf{A}))$, called the **periodicity morphism**, on pure elements as

$$S(\varphi_0 \oplus \varphi_1 \oplus \ldots, \varphi_n) \equiv (\varphi_0 \oplus \ldots \oplus \varphi_n \oplus 0 \oplus 0) \in \mathrm{Tot}^{n+2}(C^{\bullet\bullet}(\mathsf{A})), \tag{E.25}$$

and extend by linearity. From the periodicity of the cyclic bicomplex one can infer that S is a cochain map.
The map which takes S's image as its kernel is the map $I : \mathrm{Tot}^{n+2}(C^{\bullet\bullet}(\mathsf{A})) \to C^{n+1,0}(\mathsf{A}) \oplus C^{n+2,1}(\mathsf{A})$ defined as

$$I(\varphi_0 \oplus \ldots \oplus \varphi_{n+2}) = \varphi_{n+1} \oplus \varphi_{n+2}. \tag{E.26}$$

Were we implicitly have assumed that n is odd. When n is even, the image of I lies in $C^{n+1,1}(\mathsf{A}) \oplus C^{n+2,0}(\mathsf{A})$. Assume n is odd for the moment. $\{C^{n+1,0} \oplus C^{n+2,1}\}$ is the direct sum of two cochain complexes. The first has b as its coboundary map, the second one has $-b'$ as coboundary map. It can be shown that I is a cochain map as well [21, 10.1]. Hence we have obtained a short exact sequence between the following three cochain complexes. The first two are the cochain groups $\{\mathrm{Tot}^n(C^{\bullet\bullet}(\mathsf{A}))\}$ and

$\{\mathrm{Tot}^{n+2}(C^{\bullet\bullet}(\mathsf{A}))\}$, both with the cyclic cohomology as cohomology. The third is the cochain complex $\{b \oplus -b', C^{\bullet,0}(\mathsf{A}) \oplus C^{\bullet,1}(\mathsf{A}); n \in \mathbb{N}\}$, by lemma E.11, has the Hochschild cohomology of A as its cohomology. In other words, we have obtained the following SES of cochain complexes:

$$0 \longrightarrow CC^{\bullet\bullet} \xrightarrow{S} CC^{\bullet\bullet} \xrightarrow{I} CC^{\bullet 0} \oplus CC^{\bullet 1} \longrightarrow 0 \tag{E.27}$$

The required result now follows from the zig-zag lemma.

□

We shall describe the morphisms in Connes' exact sequence (E.24) in more detail. In what follows, we shall identify I with I_* and S with S_*.
First, the notation suggests that the connecting morphism B_* equals Connes' boundary map extended to the domain of the Hochschild cohomology groups. That fact is verified in [21, Ch. 10.1].
Second, we can find an alternative description of I. For that we define cyclic cohomology differently. In this approach, we regard the cyclic cochains as a submodule of the Hochschild cohomology groups.

Definition E.15. *The collection of* ***cyclic cochains*** $C^n_\lambda(\mathsf{A}) \subset C_n(\mathsf{A})$ *is the submodule defined by* $C_\lambda(\mathsf{A}) \equiv \ker(1-\lambda)$. *Explicit calculation shows that*

$$[b, \lambda] = 0, \tag{E.28}$$

showing $(b, C^n_\lambda(\mathsf{A}))$ *is a cochain complex. Its cohomology is also called* ***cyclic cohomology****. Let* $Z^n_\lambda(\mathsf{A})$ *and* $B^n_\lambda(\mathsf{A})$ *be the* ***cyclic cocycles*** *and the* ***cyclic boundaries****, respectively. Denote the inclusion map by* $\imath : C^n_\lambda(\mathsf{A}) \to C^n(\mathsf{A})$.
To emphasize the distinction between elements in the cyclic cohomology and elements in the ordinary Hochschild cohomology, we shall write $[\varphi]_\lambda \in HC^n(\mathsf{A})$.

This prompts us to verify the following:

Lemma E.16. *The cohomology of the total complex of the cyclic bicomplex equals the cohomology of the cyclic cochains.*

Proof. Append the cyclic bicomplex (E.19) with the cyclic cochain complex as follows:

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow b & & \uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \\
C^2_\lambda(\mathsf{A}) & \xrightarrow{\imath} & C^2(\mathsf{A}) & \xrightarrow{1-\lambda} & C^2(\mathsf{A}) & \xrightarrow{N} & C^2(\mathsf{A}) & \xrightarrow{1-\lambda} & C^2(\mathsf{A}) & \xrightarrow{N} & \cdots \\
\uparrow b & & \uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \\
C^1_\lambda(\mathsf{A}) & \xrightarrow{\imath} & C^1(\mathsf{A}) & \xrightarrow{1-\lambda} & C^1(\mathsf{A}) & \xrightarrow{N} & C^1(\mathsf{A}) & \xrightarrow{1-\lambda} & C^1(\mathsf{A}) & \xrightarrow{N} & \cdots \\
\uparrow b & & \uparrow b & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \\
C^0_\lambda(\mathsf{A}) & \xrightarrow{\imath} & C^0(\mathsf{A}) & \xrightarrow{1-\lambda} & C^0(\mathsf{A}) & \xrightarrow{N} & C^0(\mathsf{A}) & \xrightarrow{1-\lambda} & C^0(\mathsf{A}) & \xrightarrow{N} & \cdots
\end{array} \tag{E.29}$$

We wish to apply lemma E.3 to this new bicomplex. To that effect we show that the rows of the above diagram are acyclic cochain complexes. Define two maps $h, h' : C^n(\mathsf{A}) \to C^n(\mathsf{A})$ by:

$$h \equiv -\frac{1}{n+1}\sum_{i=1}^{n} i\lambda^i \quad \& \quad h' \equiv \frac{1}{n+1}\mathrm{id}. \tag{E.30}$$

These maps satisfy:

$$\begin{aligned}
h'N + (1-\lambda)h &= \frac{1}{n+1}\sum_{i=0}^{n}\lambda^i - \frac{1}{n+1}\sum_{i=1}^{n} i\lambda^i + \frac{1}{n+1}\sum_{i=1}^{n} i\lambda^{i+1} \\
&= \frac{1}{n+1}\mathrm{id} - \frac{1}{n+1}\sum_{i=1}^{n}(i-1)\lambda^i - (i-1)\lambda^i + n\lambda^{n+1} = \mathrm{id}.
\end{aligned}$$

Since h and h' commute with λ, also

$$Nh' + h(1-\lambda) = \mathrm{id}.$$

Therefore, h and h' ensure the identity map of $C^n(\mathsf{A})$ is homotopic to the zero map, hence the p-th cohomology groups of the horizontal cochain complexes are all zero for $p > 0$. Then apply lemma E.3 to see that the cohomology of the total complex is equal to the cohomology of the first vertical cochain complex. □

Remark E.17. *The explicit isomorphism of lemma E.16 is simply given by*

$$\varphi_n \in C^n_\lambda(\mathsf{A}) \mapsto 0 \oplus 0 \oplus \ldots \oplus \varphi_n \in \mathit{Tot}^n(CC(\mathsf{A})). \tag{E.31}$$

See [32, Thm. 2.1.5].

Corollary E.18. *On the cohomological level $I = \imath$, where I is the morphism in Connes' exact sequence (E.24) and $\imath$ is the inclusion of cyclic cochains into Hochschild cochains.*

We now demonstrate the main result. The relevance of the cyclic cohomology groups is that in the case $\mathcal{A} = C^\infty(M)$, for some compact manifold M, we get a very useful relationship between the cyclic cohomology groups and the de Rham homology groups. For all $k \in \mathbb{N}$

Theorem E.19.

$$HC^k(\mathcal{A}) \cong Z_k^{dR}(M) \bigoplus_{i;k-2i\in\{0,1\}} H_{k-2i}^{dR}(M). \tag{E.32}$$

Proof. Take some $[\varphi]_\lambda \in HC^k(\mathcal{A})$. We need to infer that the image of $\varphi \in Z_\lambda^k(\mathcal{A})$ under the map $\varphi \to C_\varphi$ is closed in the de Rham homology. Here B plays a decisive role:

$$B\varphi(a_0, \ldots, a_{k-1}) = \sum_{j=0}^{k-1} (-1)^{j(k-1)} \varphi(1, a_j, \ldots, a_{k-1}, a_0, \ldots, a_{j-1}) +$$
$$(-1)^{(j-1)(k-1)} \varphi(a_j, \ldots, a_{k-1}, a_0, \ldots, a_{j-1}, 1) =$$
$$\sum_{j=0}^{k-1} (-1)^{j(k-1)} \varphi(1, a_j, \ldots, a_{k-1}, a_0, \ldots, a_{j-1}) +$$
$$(-1)^{(j-1)(k-1)+k} \varphi(1, a_j, \ldots, a_{k-1}, a_0, \ldots, a_{j-1}).$$

The difference $j(k-1) - (j-1)(k-1) - k = k - 1 - k = -1$, so the terms in the sum cancel pairwise. Hence $B\varphi = 0$ on cyclic cochains, so according to theorem E.7, C_φ is a cyclic de Rham current.
According to the dual version of the HKRC-theorem, theorem D.17:

$$\varphi \sim \frac{1}{k!} \sum_{\beta \in S_k} \epsilon(\beta) \varphi(a_0, a_{\beta(1)}, \ldots, a_{\beta(k)}) = \mathbb{A}_k(\varphi) \in HH^k(\mathcal{A}).$$

A logical step is to calculate their difference in cyclic cohomology, since it does not necessarily follow that the two terms are also cyclic-cohomologous. Hence we shall verify first that $\mathbb{A}_k\varphi$ is cyclic as well.

$$\lambda\mathbb{A}_k\varphi(a_0,\ldots,a_k) = \frac{1}{k!}\sum_{\beta\in S_k}\epsilon(\beta)(-1)^k\varphi(a_{\beta(k)},a_0,\ldots,a_{\beta(k-1)}) =$$
$$\frac{(-1)^k}{k!}\sum_{\beta\in S_k}\epsilon(\beta)\int\limits_{C_\varphi} a_{\beta(k)}da_0\wedge\cdots\wedge da_{\beta(k-1)} =$$
$$\frac{(-1)^k}{k!}\sum_{\beta\in S_k}\epsilon(\beta)\int\limits_{C_\varphi} d(a_{\beta(k)}a_0)\wedge\cdots\wedge da_{\beta(k-1)} -$$
$$\frac{(-1)^k}{k!}\sum_{\beta\in S_k}\epsilon(\beta)a_0da_\beta(k)\wedge\cdots\wedge da_{\beta(k-1)}.$$

The first term in the last expression is zero, since $\partial C_\varphi = 0$. In the second term, moving the $da_{\beta(k)}$ behind $da_{\beta(k-1)}$ requires $k-1$ interchanges, thus adding an extra term $(-1)^{k-1}$. This compensates for the factor $-(-1)^k$ and we see that the above expression equals $\mathbb{A}_k\varphi$, showing $\mathbb{A}_k\varphi$ is cyclic. The above can be neatly summarized by the expression

$$I[\varphi - \mathbb{A}_k\varphi]_\lambda = 0.$$

According to Connes' periodicity theorem, theorem E.14, there is a $[\psi]_\lambda \in HC_\lambda^{k-2}(\mathcal{A})$ such that $S([\psi]_\lambda) \sim \varphi - \mathbb{A}_k\varphi \in HC^k(\mathcal{A})$. We now repeat the process for ψ, implying $\psi - \mathbb{A}_{k-2}\varphi \cong S([\psi']_\lambda)$, so

$$\begin{aligned}&\varphi = \mathbb{A}_k\varphi + S([\psi]_\lambda) \mod B_\lambda^k(\mathcal{A}) =\\ &\mathbb{A}_k\varphi + S([\mathbb{A}_{k-2}\varphi + S([\psi']_\lambda)]) \mod B_\lambda^k(\mathcal{A}) =\\ &\mathbb{A}_k\varphi + S\Big(\mathbb{A}_{k-2}\varphi \mod B_\lambda^{k-2}(\mathcal{A})) + S^2([\psi']_\lambda)\Big).\end{aligned}$$

Now $\ker S = \mathrm{im}B$, so the the formula reduces to

$$\varphi = \mathbb{A}_k\varphi + S\left(\mathbb{A}_{k-2}\varphi\right) + S^2\left([\psi']_\lambda\right).$$

By repeating this procedure we arrive at the following expression:

$$\varphi = \sum_{j=0;k-2j\in\{0,1\}} S^j\left(\mathbb{A}_{k-2j}\varphi\right) \mod B_\lambda^k(\mathcal{A}).$$

$\mathbb{A}_{k-2j}\varphi$ maps to a closed current in $\Omega_{k-2j}^{\mathrm{dR}}(M)$, just as φ does. The mapping is unique up to a boundary $B_\lambda^{k-2j}(\mathcal{A})$, which corresponds to a boundary $\partial\alpha \in \Omega_{k-2j+1}^{\mathrm{dR}}(M)$. Hence every $[\mathbb{A}_{k-2j}\varphi]_\lambda$ uniquely defines an element in the de Rham homology. Denote the current obtained this way by $C_{\varphi_{k-2j}}$ with $C_{\varphi_k} = C_\varphi$. This induces a map

$$\omega_k : HC^k(\mathcal{A}) \to Z_k^{\mathrm{dR}}(M) \bigoplus_{i=1}^{k-2i\in\{0,1\}} H_{k-2i}^{\mathrm{dR}}(M),$$

defined as:

$$[\varphi]_\lambda = \sum_{j=0}^{k-2j\in\{0,1\}} \left[S^j\left(\mathbb{A}_{k-2j}\varphi\right)\right]_\lambda \mapsto C_\varphi \oplus \bigoplus_{j=1}^{k-2j\in\{0,1\}} \left[C_{\varphi_{k-2j}}\right],$$

by remarking that $C_{\mathbb{A}_k\varphi} = C_\varphi$, as we have proven earlier.
Here ω_k is injective by construction. To prove surjectivity, take some $[C_\xi] \in H^{\mathrm{dR}}_{k-2j}(M)$. Then $C_\xi \mapsto \xi \in HH^{k-2j}(\mathcal{A})$ by the HKRC-theorem, so $[S^j(\xi)] \in HH^k(\mathcal{A})$. The fact that C_ξ is closed shows ξ is cyclic (this is the same computation we have done above), hence $[S^j(\xi)]_\lambda \in HC^k(\mathcal{A})$, since S takes cyclic cochains to cyclic cochains by the periodicity theorem.
Applying the same construction to $[S^j(\xi)]_\lambda$ as to $[\psi]_\lambda$, we see that the image of $[S^j(\xi)]_\lambda$ under ω_k has no components in the homology groups $HH^{\mathrm{dR}}_{k-2i}(M)$ for $i < j$ other than $C_\xi \in HH^{\mathrm{dR}}_{k-2j}(M)$ (for the reason that $\ker S = \mathrm{im} B$). The terms in the lower-dimensional homology groups are zero as well, since ξ is a closed current. This shows that every element in the right hand side of equation (E.32) has a unique pre-image in $HC^k(\mathcal{A})$. □

Let us set $\#k$ to be zero when k is even, and one when k is odd.
With the help of the periodicity morphism, the above can be restated in a more elegant way. Note that the set of even cyclic cohomology groups and the set of odd cyclic cohomology groups form two distinct directed systems, with powers of S as morphisms connecting the directed set of Abelian groups. Using the definition of S (E.25) together with the isomorphism of theorem E.19 we see that

$$C_{\varphi_k} + [C_{\varphi_{k-2}}] + \ldots + [C_{\varphi_{\#k}}] \mapsto 0 + [C_{\varphi_k}] + [C_{\varphi_{k-2}}] + \ldots + [C_{\varphi_{\#k}}], \quad \text{(E.33)}$$

under the action of S. This leads us to our final conclusion.

Corollary E.20. *Let $\mathcal{A} = C^\infty(M)$. Define the two **periodic cyclic cohomology groups** as:*

$$HP^0(\mathcal{A}) = \lim_{\to} HC^{2k}(\mathcal{A}) \qquad \& \qquad HP^1(\mathcal{A}) = \lim_{\to} HC^{2k+1}(\mathcal{A}). \quad \text{(E.34)}$$

Then

$$HP^0(\mathcal{A}) \cong H^{dR}_{even}(M) \qquad \& \qquad HP^1(\mathcal{A}) \cong H^{dR}_{odd}, \quad \text{(E.35)}$$

where the isomorphism is an isomorphism of algebras over the complex numbers.

Appendix F

K-theory

In this appendix we briefly review the elementary properties of topological and C^*-algebraic K-theory, and quote some important results. This material is taken from [1], [2] and [21]. Next, we define what is meant by connections and curvature, and finish with giving the relationship between C^*-algebraic K-theory and the odd and the even de Rham cohomology groups. Connections are used to define the Dirac operator in chapter 2.
Let X be a compact Hausdorff space.[1] To X we can associate a set of Abelian groups in such a way that this association only depends on the topology of X. Namely, let $\mathfrak{X}, \mathfrak{Y}$ be two continuous complex vector bundles on X. We define an equivalence relation on the set of continuous complex vector bundles by saying that $\mathfrak{X} \sim \mathfrak{Y}$ if and only if $\mathfrak{X}$ and $\mathfrak{Y}$ are isomorphic as vector bundles.
The **first topological K-group** of X, $K^0(X)$, is obtained as follows. $K^0(X)$ is the Abelian group with one generator for every equivalence class of continuous complex vector bundles. The group operation is defined as follows:

$$[\mathfrak{X}] + [\mathfrak{Y}] \equiv [\mathfrak{X} \oplus \mathfrak{Y}] \,, \tag{F.1}$$

where the sum on the right-hand side of (F.1) is the Whitney sum of vector bundles.
The association $K^0 : X \mapsto K^0(X)$ has the following properties:

- The tensor product of two vector bundles determines a ring structure on $K^0(X)$;
- K^0 is a contravariant functor from the category of Hausdorff and compact topological spaces to the category of Abelian groups;
- If $X \overset{h}{\sim} Y$ are two homotopic compact Hausdorff spaces, then $K_0(X) \cong K_0(Y)$.

[1] Note that most of the following definitions also apply to more general topological spaces as well.

The Serre–Swan theorem [52] states that for each complex and continuous vector bundle $\mathfrak{X}$ over a fixed base space X, we can uniquely associate a finitely generated projective $C(X)$-module.[2] Let $A \equiv C(X)$. The Serre–Swan theorem then implies $\Gamma(X, \mathfrak{X})$ is isomorphic to pA^n, with p a projection in the matrix algebra $M_n(A)$, for some integer n. Can we fully characterize the vector bundle by the projection p? Let us make some observations. Each projection $p \in M_n(A)$ defines a projection $p' \in M_{n+1}(A)$ by

$$p' = \begin{pmatrix} p & & 0 \\ & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in M_{n+1}(A). \tag{F.2}$$

We want to associate both p and p' to $\mathfrak{X}$, since we want to regard them as inherently identical. It therefore make sense to define an equivalence relation on the set of k-times-k A-valued matrices by declaring $A \sim B$ if and only if B contains A in the upper-left corner and has zeroes everywhere else (or vice versa). Furthermore, the choice of p should not depend on the isomorphism $\Gamma(X, \mathfrak{X}) \cong pA^n$. In fact:

Lemma F.1. *Let $\varphi : pA^n \to qA^k$ be a module morphism, with q a projection in $M_k(A)$. Then there is a unitary $u \in M_{\max\{n,k\}}(A)$ such that*

$$p = uqu^*. \tag{F.3}$$

Proof. Let m be the dimension of pA^n. We choose a base in $M_n(A)$ such that p is represented by the diagonal matrix with elements

$$[p]_{ij} = \begin{cases} 1\delta_{ij} & i \leq m \\ 0 & m < i \end{cases}. \tag{F.4}$$

Give or take a few rows and columns containing only zero's, q can be represented in $M_k(A)$ in an identical manner. Otherwise, pA^n and qA^k could never be isomorphic. Now take $m = \max\{n, k\}$, and imbed pA^n and qA^k in A^m. There are two unitary matrices $U, V \in M_m(A)$ that respectively diagonalize p and q in the previous manner. The diagonal form of p and q exactly coincides on $M_m(A)$, so

$$UpU^* = VqV^* \qquad \Leftrightarrow \qquad p = U^*VqV^*U. \tag{F.5}$$

Now set u equal to U^*V, so that $p = uqu^*$. $\square$

[2]Note that the converse, i.e., that we can uniquely associate a complex and continuous vector bundle to a finitely generated projective $C(X)$-module is, in general, not true. The converse does hold by mapping the modules to a certain equivalence class of vector bundles. See [52] for more detail.

These observations motivate the construction of the C^***-algebraic K-groups**.

Definition F.2 (K_0)**.** *Let A be a unital (not necessarily commutative) C^*-algebra. Form the set $M_*(A) = \{M_n(A); n \in \mathbb{N}^+\}$. Then $M_*(A)$ is ordered by inclusion of $M_m(A) \hookrightarrow M_n(A)$ by putting $M_m(A)$ in the upper-left corner of $M_n(A)$ (assume $m \leq n$). It forms a directed system of complex algebras. The category of complex algebras has direct limits, so we can define*

$$M(A) \equiv \lim_{\rightarrow} M_n(A). \tag{F.6}$$

This simply consists of arbitrarily large matrices with entries in A, of which only a finite number of entries are nonzero.
Let $U(A)$ be the set of unitary elements of $M(A)$ and let $U(A)$ act on $M(A)$ by conjugation. Define $K_0(A)$ as the group freely generated by equivalence classes $[p]$ of projectors in $M(A)/U(A)$ (i.e., for two projectors $p, q \in M(A)$, $p \sim q$ if and only if $p = uqu^*$ for some $u \in U(A)$), together with the relation

$$[p] + [q] \equiv [p \oplus q]. \tag{F.7}$$

Note there is also an equivalent formulation of the equivalence relation defined above. We say $p \sim q$ when there is a path of projections $\{e_t \in M(A)\}$ such that $e_0 = p$ and $e_1 = q$.

Before continuing we define what is meant with saying that two unital C^*-algebras A and B are **homotopic**.

Definition F.3 (Homotopic C^*-algebras)**.** *Let $\varphi, \psi : A \to B$ be morphisms of C^*-algebras. They are said to be **homotopic morphisms** when there is a collection of morphism $\{\Phi_t : A \to B; t \in [0,1]\}$ with the following properties:*

- $\Phi_0 = \varphi$;
- $\Phi_1 = \psi$;
- *The map $t \mapsto \Phi_t(a)$ is in*

$$C([0,1], B) \equiv \{f : [0,1] \to B, f \text{ continuous}\}. \tag{F.8}$$

We write $\varphi \overset{h}{\sim} \psi$.
Now, A is homotopic to B when there are morphisms $\alpha : A \to B$, $\beta : B \to A$ such that $\alpha \circ \beta \overset{h}{\sim} 1_A$ and $\beta \circ \alpha \overset{h}{\sim} 1_B$.

We now continue with discussing the properties of the association $A \mapsto K_0(A)$. Not surprisingly, K_0 has virtually the same properties as the topological K-functor.

- The direct sum of two projections defines an Abelian group structure on $K_0(A)$;
- The tensor product of two projections turns $K_0(A)$ into a ring;
- $K_0(A)$ is a functor from the category of unital C^*-algebras to the category of rings;
- If $A \sim B$ are homotopic as C^*-algebras, then $K_0(A) = K_0(B)$;
- $K_0(C(X)) \cong K^0(X)$.

From the properties of the topological K-groups it is apparent that the association depends solely on the way the continuous vector bundles twist around the topological space. One could therefore expect that it would make no difference if we would consider *smooth* vector bundles instead. This turns out to be the case. Topological K-theory extends trivially to equivalence classes of isomorphic smooth vector bundles, which yield the same K_0-groups. Algebraic K-theory is also applicable to pre-C^*-algebras, of which $C^\infty(M)$, where M is a smooth and compact manifold, is an example. See section 2.2 for the definitions. The main result, quoted from [21, Ch. 3.8], is as expected:

Theorem F.4. *Let $\mathcal{A}$ be a pre-C^*-algebra and let $A \equiv \overline{\mathcal{A}}$ be the (unique) C^*-algebraic closure of $\mathcal{A}$. Then*

$$K_0(\mathcal{A}) \cong K_0(A). \tag{F.9}$$

The analogous result in topological K-theory now follows from the identification $K_0\left(C^\infty(M)\right) \cong K^0(M)$, or may be proven directly.
Next, we discuss a concept related to projections in matrix algebras of pre-C^*-algebras.

Definition F.5 (Connections)**.** *Let $\mathcal{A} \equiv C^\infty(M)$ for some compact manifold M, and fix a projection $p \in M_N(\mathcal{A})$. A* ***connection*** *is a $\mathbb{C}$-linear map defined as follows:*

$$\nabla : \mathcal{A} \to \Omega^1_{ab} \quad (N = 1), \nabla : p\mathcal{A}^N \to p\mathcal{A}^N \otimes_{\mathcal{A}} \Omega^1_{ab}(\mathcal{A}) \quad (N \neq 1), \tag{F.10}$$

such that for all $s \in p\mathcal{A}^N$ and $a \in \mathcal{A}$ we have

$$\nabla(sa) = (\nabla s)a + s \otimes a. \tag{F.11}$$

A connection ∇ is said to be ***Hermitian*** *if for all $s, t \in p\mathcal{A}^N$*

$$(s, \nabla t)_{\mathcal{A}} + (\nabla s, t)_{\mathcal{A}} = d(s, t)_{\mathcal{A}}$$

where $(\cdot, \cdot)_{\mathcal{A}}$ is the $\mathcal{A}$-valued sesquilinear form that makes $p\mathcal{A}^N$ a right-$\mathcal{A}$-pre-Hilbert module and d is the de Rham coboundary (or the exterior derivative, if you wish). See appendix D for more background information.

The familiar fundamental theorem of Riemannian geometry takes a very tangible form in algebraic language. As a result of [21, 8.14]:

Lemma F.6. *Let $\mathfrak{X}$ be a complex smooth vector bundle over M such that $\Gamma^\infty(M, \mathfrak{X}) \cong p\mathcal{A}^n$ for some projection $p \in M_n(\mathcal{A})$. Then every Hermitian connection on $\mathfrak{X}$ is locally of the form*

$$\nabla s = pds + \alpha s$$

for any smooth section s of $\mathfrak{X}$ and skew-adjoint $\alpha \in pM_n(\Omega^1_{ab}(\mathcal{A}))p$. When $\mathfrak{X} = TM$ is the tangent bundle, the Levi-Civita connection ∇^g is obtained by setting $\alpha = 0$ (i.e. the connection is ***torsion-free****).*

This prompts the following generalization.

Definition F.7 (Generalized Levi-Civita connection)**.** *Let $\mathfrak{X}$ be a complex vector bundle over M dual to $p\mathcal{A}^n$ for some projection in $M_n(\mathcal{A})$, with $\mathcal{A} = C^\infty(M)$. The* ***generalized Levi-Civita connection*** *is the connection that is locally given on sections s of $\mathfrak{X}$ by*

$$\nabla^{\mathfrak{X}} s \equiv pds. \tag{F.12}$$

Let $\mathcal{A} = C^\infty(M)$ with M some compact manifold. The classical **Chern character** [21, Ch. 8.4] implements isomorphisms

$$K_0(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^{\text{even}}_{\text{dR}}(M), \tag{F.13}$$

$$K_1(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^{\text{odd}}_{\text{dR}}(M). \tag{F.14}$$

These isomorphisms are roughly constructed as follows. One takes a projection $p \in M_n(\mathcal{A})$. This projection defines a generalized Levi-Civita connection. One can then show that the map

$$\text{ch} : M_n(\mathcal{A}) \to \Omega^\bullet_{\text{dR}}(M), \tag{F.15}$$

$$\text{ch}(p) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} \text{Tr}\left[p(dp)^{2k}\right], \tag{F.16}$$

with d the exterior derivative, defines a de Rham cochain and is stable under both types of equivalence classes in the definition of the algebraic K-groups (i.e., the direct limit and the equivalence up to adjoint action with a unitary operator).
In the next section we will encounter a noncommutative variant of the Chern character.

Appendix G

Fredholm modules and cycles

In this appendix we discuss **Fredholm modules** and the **cycles** that arise from these. Next, we discuss the **Chern–Connes character**, which links algebraic K-theory with cyclic cohomology, similar to the classical Chern character (see appendix F). In chapter 3 we show that any commutative spectral triple defines a Fredholm module. The fact that this Fredholm module, in turn, defines a cycle is pivotal to the construction of the noncommutative integral. Part of the theory of this appendix is also relevant to the material developed in chapter 5.
We start with defining Fredholm modules.

Definition G.1 (Fredholm modules)**.** *Let A be an involutive and associative complex algebra and let $\sigma : \mathsf{A} \to B(\mathcal{H})$ be an involutive representation of A on some separable Hilbert space $\mathcal{H}$. Let $F : \mathcal{H} \to \mathcal{H}$ be an operator such that $F = F^*$, $F^2 = 1$ (i.e. F is a* ***symmetry****) and*

$$[F, \sigma(a)] \in K(\mathcal{H}) \qquad \forall\ a \in \mathsf{A}, \tag{G.1}$$

where $K(\mathcal{H})$ is the C^*-algebra of compact operators on $\mathcal{H}$. We call such a triplet $(\mathsf{A}, \mathcal{H}, F)$ an ***odd Fredholm module****.*
An ***even Fredholm module*** *consists of the same data with an additional* $\mathbb{Z}_2$***-grading*** *γ on $\mathcal{H}$, that is, a map*

$$\gamma \in B(\mathcal{H}), \quad \gamma = \gamma^* \quad \& \quad \gamma^2 = 1, \tag{G.2}$$

which splits $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ into the $\pm$-eigenspaces of γ. We demand that the grading commutes with the action of A and anti-commutes with the symmetry F.

Definition G.2 (Cycles)**.** *Let* A *be an involutive and associative algebra over the complex numbers. A* ***cycle over*** A *of dimension* n *is a triplet* $\left(\Omega^\bullet, d, \int\right)$*, where:*

1. $\Omega^\bullet \equiv \bigoplus_{k=0}^{n} \Omega^k$ *is a graded differential algebra over* A *with coboundary operator* $d : \Omega^l \to \Omega^{l+1}$ *(see definition D.1 in appendix D);*
2. *we have an algebra morphism from* Ω^0 *to* A*;*
3. $\int$ *is a* $\mathbb{C}$*-linear map* $\int : \Omega^\bullet \to \mathbb{C}$ *such that:*

$$\int w_k = 0 \qquad k < n; \tag{G.3}$$

$$\int w_k w_l = (-1)^{kl} \int w_l w_k; \tag{G.4}$$

$$\int dw_{n-1} = 0. \tag{G.5}$$

We also call $\int$ *an* ***abstract integral****.*

The archetypal example is, of course, the de Rham cochain complex, which even satisfies $\Omega^0 \cong C^\infty(M)(= \mathsf{A})$. A much larger class of cycles is obtained through Fredholm modules. Before showing how we can associate a cycle to (a subclass of) the Fredholm modules, we first need some more information regarding the **Schatten** p**-classes**.

Definition G.3 (Schatten p-class)**.** *Let* $\mathcal{H}$ *be a separable Hilbert space. Take some* $p \in [1, \infty)$ *and define the Schatten* p*-class as*

$$\mathcal{L}^p(\mathcal{H}) \equiv \left\{ T \in K(\mathcal{H}); \sum_{k=0}^{\infty} s_k(T)^p < \infty \right\} \tag{G.6}$$

where the $s_k(T)$ *are the* ***singular values*** *of* T*, defined as the eigenvalues of* $|T|$ *obtained through the expansion*

$$|T| = \sum_k s_k(T) \langle e_k, \cdot \rangle e_k \tag{G.7}$$

for some base $\{e_k\}$ *of* $\mathcal{H}$*. Note that* $s_k(T) \geq 0$ *for all* k*, since* $|T|$ *is positive.*
We call $\mathcal{L}^1(\mathcal{H})$ *the collection of* ***trace class*** *operators on the Hilbert space* $\mathcal{H}$*. These have the property that for each base* $\{e_k\}$ *of the Hilbert space the map*

$$Tr : \mathcal{L}^1(\mathcal{H}) \to \mathbb{C};$$

$$Tr(T) \equiv \sum_{k=0}^{\infty} \langle e_k, T e_k \rangle \tag{G.8}$$

converges and does not depend on the base chosen. This defines a ***trace*** *on the trace class operators (also see definition 3.12).*

We quote several important results about Schatten p-classes, see [49, Ch.1, Ch.2] for more background information.
For all $p \in [1, \infty)$ the Schatten p-class is an ideal in the algebra of bounded operators of the Hilbert space on which the class is defined.
The association

$$\|\cdot\|_p : \mathcal{L}^p(\mathcal{H}) \to [0, \infty);$$
$$\|T\|_p \equiv \left(\sum_{k=0}^{\infty} s_k(T)^p \right)^{\frac{1}{p}} \tag{G.9}$$

defines a norm on the Schatten p-class. Note that this norm can also be used to define the Schatten p-class:

$$T \in \mathcal{L}^p \quad \Leftrightarrow \quad \|T\|_p < \infty. \tag{G.10}$$

Let $r, p, q \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and take $T \in \mathcal{L}^p(\mathcal{H})$, $S \in \mathcal{L}^q(\mathcal{H})$. Then $TS \in \mathcal{L}^r(\mathcal{H})$ and

$$\|TS\|_r \leq \|T\|_p \|S\|_q . \tag{G.11}$$

This inequality is called the **Hölder inequality** (for Schatten classes). It has two important corollaries.

Corollary G.4. *If $r = 1$ then TS is trace class and the map Tr satisfies*

$$Tr(TS) = Tr(ST). \tag{G.12}$$

By repeated application of the Hölder inequality:

Corollary G.5. *For all $i \in \{1, \ldots, n\}$, let $T_i \in \mathcal{L}^n(\mathcal{H})$. The product*

$$T_1 \cdots T_k \tag{G.13}$$

lies in $\mathcal{L}^{\frac{n}{k}}(\mathcal{H})$. In particular, $T_1 \cdots T_n$ is trace class.

Equation (G.12) is compatible with the fact that Tr defines a trace on $\mathcal{L}^1(\mathcal{H})$: if $S, T \in \mathcal{L}^1(\mathcal{H})$ then $TS \in \mathcal{L}^{\frac{1}{2}}(\mathcal{H}) \subset \mathcal{L}^1(\mathcal{H})$.

We now show how a Fredholm module (with some additional requirements) defines a cycle.

Theorem G.6. *Let* $(\mathsf{A}, \mathcal{H}, F)$ *be a Fredholm module with the following properties.*

- *For all* $a \in \mathsf{A}$
$$[F, \sigma(a)] \in \mathcal{L}^{n+1}(\mathcal{H}). \tag{G.14}$$
Moreover, this n *is the smallest integer such that the relation (G.14) holds for all* $a \in \mathsf{A}$.

- *The integer* n *in equation (G.14) is odd when the Fredholm module is odd, and is even when the Fredholm module is even.*

If the above holds, the Fredholm module defines an n*-dimensional cycle with* $\Omega^0 = \mathsf{A}$.

Proof. Let us start by denoting the representation of A on $B(\mathcal{H})$ by $\sigma(a)\xi \equiv a\xi$, for any $\xi \in \mathcal{H}$ and $a \in \mathsf{A}$.
We first construct a graded differential algebra.
For $k > 0$, Ω^k is defined as the $\mathbb{C}$-linear span of operators of the form

$$a_0[F, a_1] \cdots [F, a_k] \in K(\mathcal{H}), \qquad a_i \in \mathsf{A}. \tag{G.15}$$

For $k = 0$, we define $\Omega^0 \equiv \mathsf{A}$.
The left-A-module structure of Ω^k is given by

$$a \cdot \sum_i a_0^i \left[F, a_1^i\right] \cdots \left[F, a_k^i\right] \equiv \sum_i a a_0^i \left[F, a_1^i\right] \cdots \left[F, a_k^i\right]. \tag{G.16}$$

We then define

$$\Omega^\bullet \equiv \bigoplus_{k=0}^{n} \Omega^k, \tag{G.17}$$

as well as a map $\cdot : \Omega^\bullet \times \Omega^\bullet \to K(\mathcal{H})$, as follows:

$$\begin{aligned} &\omega_k = a_0 \left[F, a_1\right] \cdots \left[F, a_k\right] \in \Omega^k; \\ &\omega_l = b_0 \left[F, b_1\right] \cdots \left[F, b_l\right] \in \Omega^l; \\ &\omega_k \cdot \omega_l \equiv a_0 \left[F, a_1\right] \cdots \left[F, a_k\right] b_0 \left[F, b_1\right] \cdots \left[F, b_l\right]. \end{aligned} \tag{G.18}$$

The product is then extended by $\mathbb{C}$-linearity to the whole of $\Omega^\bullet$. We shall verify that (G.18) lies in Ω^{k+l}. If $k + l > n$, then (G.18) is satisfied by definition. For the other case we examine the action of A on ω_k from the right. We claim that for $a_{k+1} \in \mathsf{A}$:

$$a_0\,[F,a_1]\cdots[F,a_k]\,a_{k+1} =$$
$$\sum_{j=1}^{k}(-1)^{k-j}a_0\,[F,a_1]\cdots[F,a_ja_{j+1}]\cdots[F,a_{k+1}]+$$
$$(-1)^k a_0a_1\,[F,a_2]\cdots[F,a_{k+1}] \in \Omega^k.$$

Let $k=0$. Then the formula trivially holds. Assume the formula holds for k. Then:

$$a_0\,([F,a_1]\cdots[F,a_k])\,[F,a_{k+1}]\,a_{k+2} =$$
$$a_0\,([F,a_1]\cdots[F,a_k])\,([F,a_{k+1}a_{k+2}]-a_{k+1}\,[F,a_{k+2}]) =$$
$$a_0\,[F,a_1]\cdots[F,a_k]\,[F,a_{k+1}a_{k+2}]+$$
$$\sum_{j=1}^{k}(-1)^{k-j}a_0\,[F,a_1]\cdots[F,a_ja_{j+1}]\cdots[F,a_{k+1}]\,[F,a_{k+2}]+$$
$$(-1)^k a_0a_1\,[F,a_2]\cdots[F,a_{k+1}]\,[F,a_{k+2}] =$$
$$\sum_{j=1}^{k+1}(-1)^{k-j}a_0\,[F,a_1]\cdots[F,a_{k+2}]+$$
$$(-1)^k a_0a_1\,[F,a_2]\cdots[F,a_{k+1}]\,[F,a_{k+2}]\,.$$

This implies that $\omega_k \cdot \omega_l \in \Omega^{k+l}$. Since $\mathsf{A} = \Omega^0$, the product is clearly compatible with the A-module structure on Ω^k. Associativity of the product in $\Omega^\bullet$ follows from the associativity of composition of operators on $\mathcal{H}$. We can now safely conclude that $\Omega^\bullet$ is a graded algebra.
The next step is to define a coboundary operator $d:\Omega^k \to \Omega^{k+1}$. Take some $\omega_k \in \Omega^k$ and define a map

$$d\omega \equiv F\omega - (-1)^k\omega F. \tag{G.19}$$

Extend d by linearity to the whole of $\Omega^\bullet$. We now show that $d(\omega_k)$ has the correct range. First note that:

$$F[F,a] = F^2a - FaF = a - FaF = aF^2 - FaF = -[F,a]F. \tag{G.20}$$

So,

$$d(\omega_k) = Fa_0\,[F,a_1]\cdots[F,a_k] - (-1)^k\omega_kF =$$
$$[F,a_0]\,[F,a_1]\cdots[F,a_k] + a_0F\,[F,a_1]\cdots[F,a_k] - (-1)^k\omega_kF =$$
$$[F,a_0]\,[F,a_1]\cdots[F,a_k] \in \Omega^{k+1}.$$

Next up is to show that $d \circ d = 0$, namely,

$$d(d\omega_k) = F\,[F, a_0]\,[F, a_1] \cdots [F, a_k] - (-1)^k\,[F, a_0]\,[F, a_1] \cdots [F, a_k]\,F = 0$$

by (G.20).
Now we verify that d is an odd derivation. Take $\omega_k, \omega_l \in \Omega^k, \Omega^l$ as defined in equation (G.18).

$$\begin{aligned}
&d(\omega_k \cdot \omega_l) = F\omega_k\omega_l - (-1)^{k+l}\omega_k\omega_l F = \\
&d(\omega_k) \cdot \omega_l + (-1)^k\omega_k F\omega_l + (-1)^{k+l}(-1)^l\omega_k d(\omega_l) - (-1)^{k+l}(-1)^l\omega_k F\omega_l = \\
&d(\omega_k) \cdot \omega_l + (-1)^k\omega_k d(\omega_l).
\end{aligned}$$

By linearity, the required condition

$$d(\omega_k \cdot \omega) = d(\omega_k) \cdot \omega + (-1)^k\omega_k d(\omega)$$

follows for all $\omega \in \Omega^\bullet$ and $\omega_k \in \Omega^k$.
This concludes the proof that $(d, \Omega^\bullet)$ is a graded differential algebra.
The last step is to define an abstract integral and show it has all the required properties. We distinguish two cases.

- Let n be odd. Take some $\{a_0, \ldots, a_n\} \subset \mathsf{A}$ and define

  $$\begin{aligned}
  &\omega_n \equiv a_0\,[F, a_1] \cdots [F, a_n] \quad \Rightarrow \\
  &d\omega_n = F\omega_n + \omega_n F = [F, a_0]\,[F, a_1] \cdots [F, a_n] \in \mathcal{L}^1(\mathcal{H})
  \end{aligned}$$

  by lemma G.5. Since $\|\cdot\|_1$ is a norm, this results extends to any $\omega \in \Omega^n$. Recall that all Schatten p-classes are ideals in $B(\mathcal{H})$, so $Fd\omega \in \mathcal{L}^1(\mathcal{H})$. We define the abstract integral as

  $$\int \omega \equiv \begin{cases} \frac{1}{2}\mathrm{Tr}\,(Fd\omega) & \omega \in \Omega^n \\ 0 & \text{otherwise} \end{cases} . \tag{G.21}$$

- Let n be even. Using the same arguments as before, $\gamma Fd\omega \in \mathcal{L}^1(\mathcal{H})$ for any $\omega \in \Omega^n$, so we define

  $$\int \omega \equiv \begin{cases} \frac{1}{2}\mathrm{Tr}\,(\gamma Fd\omega) & \omega \in \Omega^n \\ 0 & \text{otherwise} \end{cases} . \tag{G.22}$$

Since $d^2 = 0$, the abstract integral vanishes on $d\omega$ for $\omega \in \Omega^{n-1}$, regardless of the value of n. We finish the proof by examining the behavior of the abstract integral on products.

- Assume n is odd and take some $\omega_k, \omega_l \in \Omega^k, \Omega^l$ such that $k + l = n$. Then

  $$\int \omega_k\omega_l = \frac{1}{2}\mathrm{Tr}\,(Fd(\omega_k\omega_l)) = \frac{1}{2}\mathrm{Tr}\,(Fd\omega_k \cdot \omega_l) + \frac{(-1)^k}{2}\mathrm{Tr}\,(F\omega_k d\omega_l)\,. \tag{G.23}$$

By corollary G.5 and the fact that the Schatten classes are ideals, we see that

$$\begin{aligned}
&Fd\omega_k \in \mathcal{L}^{\frac{n+1}{k+1}}(\mathcal{H});\\
&\omega_l \in \mathcal{L}^{\frac{n+1}{l}}(\mathcal{H});\\
&F\omega_k \in \mathcal{L}^{\frac{n+1}{k}}(\mathcal{H});\\
&d\omega_l \in \mathcal{L}^{\frac{n+1}{l+1}}(\mathcal{H}) \quad \Rightarrow\\
&Fd\omega_k\omega_l, F\omega_k d\omega_l \in \mathcal{L}^1(\mathcal{H}).
\end{aligned}$$

We can therefore apply the identity (G.12) and use the tracial property of the map Tr. Equation (G.23) then equals

$$\frac{1}{2}\mathrm{Tr}\,(\omega_l F d\omega_k) + \frac{(-1)^k}{2}\mathrm{Tr}\,(d\omega_l F\omega_k) =$$
$$\frac{(-1)^{l+1}}{2}\mathrm{Tr}\,(d\omega_l \cdot d\omega_k) + \frac{(-1)^l}{2}\mathrm{Tr}\,(F\omega_l d\omega_k) + \frac{(-1)^{k+l+1}}{2}\mathrm{Tr}\,(Fd\omega_l\omega_k)\,.$$

Note that $k+l+1 = n+1$, which is an even number. So the right-hand side equals

$$\frac{(-1)^l}{2}\mathrm{Tr}\,(d\omega_l \cdot d\omega_k) + \frac{1}{2}\mathrm{Tr}\,(Fd(\omega_l \cdot \omega_k)) = 0 + (-1)^{kl}\int \omega_l \cdot \omega_k,$$

since either k or l is even and the trace is zero on Ω^{n+1}.

- Assume n is even. Using the same notation as before, one has

$$\int \omega_k\omega_l = \frac{1}{2}\mathrm{Tr}\,(\gamma F d\omega_k \cdot \omega_l) + \frac{(-1)^k}{2}\mathrm{Tr}\,(\gamma F\omega_k d\omega_l) =$$
$$\frac{1}{2}\mathrm{Tr}\,(\omega_l\gamma F d\omega_k) + \frac{(-1)^k}{2}\mathrm{Tr}\,(d\omega_l \cdot \gamma F\omega_k)\,. \tag{G.24}$$

The grading γ anti-commutes with F and commutes with the action of any $a \in \mathsf{A}$. Hence, for all $a \in \mathsf{A}$:

$$\gamma[F,a] = \gamma F a - \gamma a F = -Fa\gamma + aF\gamma = -[F,a]\gamma.$$

We use this relation in equation (G.24) to obtain:

$$\frac{(-1)^l}{2}\mathrm{Tr}\,(\gamma\omega_l F d\omega_k) + \frac{(-1)^{k+l+1}}{2}\mathrm{Tr}\,(\gamma d\omega_l F\omega_k) =$$
$$\frac{(-1)^{l+l+1}}{2}\mathrm{Tr}\,(\gamma d\omega_l \cdot d\omega_k) + \frac{(-1)^{l+l}}{2}\mathrm{Tr}\,(\gamma F\omega_l d\omega_k) +$$
$$\frac{(-1)^l}{2}\mathrm{Tr}\,(\gamma F d\omega_l \cdot \omega_k) =$$
$$\frac{(-1)^l}{2}\mathrm{Tr}\,\left(\gamma F\left[d\omega_l \cdot \omega_k + (-1)^l\omega_l d\omega_k\right]\right) = (-1)^l\int \omega_l \cdot \omega_k.$$

If l is odd, k is odd and kl is odd. If l is even, so are k and kl. Hence the last expression equals

$$(-1)^{kl} \int \omega_l \cdot \omega_k.$$

This finishes the proof that the Fredholm module $(\mathsf{A}, \mathcal{H}, F)$, under the additional assumptions, defines a cycle $\left(\Omega^\bullet, d, \int\right)$. □

Now take a Fredholm module $(\mathsf{A}, \mathcal{H}, F)$ that satisfies the prerequisites of theorem G.6, so that it defines an n-dimensional cycle. Assume, furthermore, that A is unital and commutative. According to the definitions in appendix D, we can construct a Hochschild chain complex from A. It is easily verified that the abstract integral

$$\int : \underbrace{\mathsf{A} \times \cdots \times \mathsf{A}}_{n \text{ times}} \to \mathbb{C} \tag{G.25}$$

is a multilinear map, i.e. the abstract integral is a Hochschild cochain. For these particular Fredholm modules we rename the abstract integral. Take some $\omega_n \in \Omega^n$ such that $\omega_n = a_0 [F, a_1] \cdots [F, a_n]$. Then

$$\tau_F^n(a_0, a_1, \ldots, a_n) \equiv 2\lambda_n \int \omega_n = \lambda_n \mathrm{Tr}\, (\gamma F [F, a_0] \cdots [F, a_n])\,; \tag{G.26}$$

$$\lambda_n \equiv \frac{\Gamma\left(\frac{n}{2}+1\right)}{2n!}. \tag{G.27}$$

In the two definitions above, γ is equal to the unit in $B(\mathcal{H})$ when the Fredholm module is odd and equals the $\mathbb{Z}_2$-grading of $\mathcal{H}$ when the module is even. The map τ_F^n is then extended by linearity to the whole of Ω^n. We call the cochain τ_F^n the **Chern-Connes character** of the Fredholm module $(\mathsf{A}, \mathcal{H}, F)$.

Lemma G.7. *Any Chern-Connes character is a cyclic cocycle.*

Proof. Take some $a_0, \ldots, a_n \in \mathsf{A}$ and let $a \equiv a_0 \otimes \cdots \otimes a_n \in C_n(\mathsf{A})$. Then

$$b\tau_F^p(a) = \tau_F^p \left(\sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + \right.$$
$$\left. (-1)^n a_0 a_n \otimes a_1 \otimes \cdots \otimes a_{n-1} \right) = 2\lambda_n \int \left(\sum_{i=0}^{n} \omega_{n-1}^i \right),$$

with

$$\omega_{n-1}^i \equiv \begin{cases} a_0 [F, a_0] \ldots [F, a_i a_{i+1}] \cdots [F, a_n] & 0 \le i < n \\ [F, a_0 a_n] \cdots [F, a_{n-1}] & i = n \end{cases}.$$

So $\omega^i_{n-1} \in \Omega^{n-1}$ for all i. The linearity of the Chern-Connes character and the properties of the abstract integral show that $b\tau^n_F = 0$ on the n-th Hochschild chain group. To see that this cocycle is cyclic, we examine the action of the cyclic permuter on the the cocycle. Let a again be as defined before. Then

$$\lambda\tau^n_F(a) = (-1)^n \lambda_n \mathrm{Tr}\, (\gamma F\, [F, a_n]\, [F, a_0] \cdots [F, a_{n-1}])\, .$$

Using relation (G.20), this becomes

$$(-1)^{n+1} \lambda_n \mathrm{Tr}\, (\gamma\, [F, a_n]\, F\, [F, a_0] \cdots [F, a_{n-1}])\, .$$

If the dimension of the Fredholm module is odd, then $\gamma = 1$. Since $n + 1$ is even, the above equation equals

$$\lambda_n \mathrm{Tr}\, ([F, a_n]\, \gamma F\, [F, a_0] \cdots [F, a_{n-1}]) = \tau^n_F(a),$$

by the tracial property of Tr. If the dimension of the Fredholm module is odd, γ anti-commutes with $[F, a_n]$, and we obtain the same result.
Let $\mathcal{A} = C^\infty(M)$. As announced, the Chern-Connes character resembles the classical Chern character. In fact, the so-called **finitely summable** Fredholm modules define an extraordinary cohomology theory denoted by $\mathfrak{K}^i(\mathcal{A})$ for $i = \{0, 1\}$. This cohomology theory is dual to algebraic K-theory through the so-called **analytic index map**. See [26, Ch. 4.2] for more background.
The Chern-Connes character then implements the following isomorphisms:

$$\tau_\bullet : \mathfrak{K}^0(\mathcal{A}) \to HP^0(\mathcal{A}) \cong H^{\mathrm{dR}}_{\mathrm{even}}(M), \tag{G.28}$$

$$\tau_\bullet : \mathfrak{K}^1(\mathcal{A}) \to HP^1(\mathcal{A}) \cong H^{\mathrm{dR}}_{\mathrm{odd}}(M). \tag{G.29}$$

□

List of Symbols

$(\cdot,\cdot)_{\mathcal{H}}$	Inner product on $\mathcal{H}$, eqn. (3.0)
1_X	Indicator function of X
$[\cdot]_{i,j}$	Matrix element on i-th row and j-th column
$[\cdot]_\lambda$	Equivalence class of cyclic cohomological elements in Hochschild cohomology, p. 139
$⨍$	Noncommutative integral, eqn. (3.68)
$\hat{\cdot}$	Gelfand transform, eqn. (2.8)
$\int_C$	Action of a de Rham current C, eqn. (D.34)
$\int$	Abstract integral, p. 152
$\|\cdot\|_p$	Norm on Schatten p-class, eqn. (G.8)
$\|\cdot\|_\infty$	Supremum norm, eqn. (2.5)
$\|\cdot\|_{\mathcal{H}}$	Norm on $\mathcal{H}$, eqn. (3.1)
$\|\cdot\|_{\mathrm{op}}$	Operator norm, eqn. (2.19)
$\bar{\cdot}$	Complex conjugation
$\sim$	Equivalence relationship
$\times$	Shuffle product, eqn. (D.16)
*	Involution map defined on an involutive algebra, p. 8
A	1) The C^*-algebraic closure of $\mathcal{A}$, 2) A general C^*-algebra, p. 41
$\mathcal{A}$	1) Involutive algebra of a spectral triple, 2) general pre-C^*-algebra, p. 40
A	Involutive and associative algebra over the complex numbers

A^+	Set of positive elements of an involutive algebra A, p. 115
$\vec{\alpha}$	Multi-index, p. 111
Ad	Adjoint action
A_n	Anti-symmetrization map for Hochschild cochains, eqn. (D.41)
$\mathbb{A}_n$	Anti-symmetrization operator, eqn. (D.18)
B	Connes' boundary map, eqn. (E.8)
b'	Truncated Hochschild boundary map, eqn. (E.11)
B_∞	C^*-algebra, eqn. (3.37)
$\mathcal{B}(M)$	Smooth sections of the Clifford bundle associated to the manifold M, eqn. (2.92)
$B(\mathcal{H})$	Collection of bounded operators of the Hilbert space $\mathcal{H}$
$B(M)$	Continuous sections of the Clifford bundle associated to the manifold M, eqn. (2.91)
$B^\infty(\mathcal{H})$	Collection of "smooth" operators on $\mathcal{H}$, eqn. (3.11)
c	Clifford multiplication (as representation of the Clifford algebra), p. 20
c	Clifford multiplication (as map between sections of bundles), p. 34
C	Reality operator, p. 46
CH	Category of compact topological spaces with the Hausdorff property and continuous maps
$C(X)$	Collection of complex-valued continuous functions on a topological space X
$C^\infty(M)$	Collection of complex-valued smooth functions on a manifold, eqn. (2.11)
$C\ell_p, \mathbb{C}\ell_p$	p-dimensional real and complex Clifford algebras, p. 17
$C^\infty_{\mathbb{R}}(M)$	Collection of smooth $\mathbb{R}$-valued functions on a manifold
$\mathbb{C}\ell^{(+)}(M)$	Clifford bundle associated to a manifold M, p. 32
$C^k(\mathsf{A})$	k-th Hochschild cochain group, eqn. (D.37)

$C^{\bullet\bullet}(\mathsf{A})$	Cyclic bicomplex of an algebra A, p. 136
$C_n(\mathsf{A})$	n-th Hochschild chain group of the algebra A, p. 122
$C^n_\lambda(\mathsf{A})$	n-th cyclic cochain group of the algebra A, p. 139
$\mathbf{CCA}_1$	Category of commutative and unital C^*-algebras and *-morphisms, p. 9
$\mathbf{c}$	Noncommutative volume form, p. 46
D	Dirac operator of a spectral triple, p. 40
$\not{D}$	Canonical Dirac operator, eqn. (2.105)
Δ	Representation space(s) for the Clifford algebra, p. 20
δ_i	Derivations on $\mathcal{A}$, p. 78
$\varepsilon_\bullet$	Evaluation map, eqn. (2.9)
ε_n	Quasi-inverse of the anti-symmetrizaton map, eqn. (D.20)
$\mathbf{FVec}^F_\mathbb{R}$	Category of finite dimensional, real vector spaces equipped with an inner product, p. 16
$\mathbf{FVec}^F_\mathbb{C}$	Category of finite dimensional, complex vector spaces equipped with a Hermitian inner product, p. 16
φ^D_ω	Hochschild cochain of $\mathcal{A}$ which coincides with the Chern-Connes character on cycles, eqn. (3.45)
ϕ_C	The image of a de Rham current C in Hochschild cohomology, p. 129
$\Gamma(M, \mathfrak{X})$	Collection of continuous sections of a vector bundle, eqn. (B.5)
$\Gamma^\infty(M, \mathfrak{X})$	Collection of smooth sections of a smooth vector bundle, eqn. (B.6)
$\mathcal{H}$	1) Hilbert space of a spectral triple, 2) General Hilbert space, p. 40
$\mathcal{H}^\infty$	Intersection of domain of all powers of D, eqn. (3.2)
h_χ	, p. 82
$HC^n(\mathsf{A})$	n-th cyclic cohomology group of an algebra A, p. 137
$HH_n(\mathsf{A})$	n-th Hochschild homology group of the algebra A, eqn. (D.12)

$HH_\bullet(\mathsf{A})$	Hochschild chain complex of the algebra A, p. 123
$K(\mathcal{H})$	Collection of compact operators on the Hilbert space $\mathcal{H}$, p. 49
$K_i(A)$	i-th algebraic K-group of a (pre-)C^*-algebra A, p. 147
λ	Cyclic permuter, eqn. (E.9)
λ_n	Constant depending on the integer n, eqn. (G.26)
$\mathcal{L}^{1+}(\mathcal{H})$	Dixmier ideal, eqn. (3.33)
$\mathcal{L}^1(\mathcal{H})$	Collection of trace class operators on the Hilbert space $\mathcal{H}$, p. 152
$\mathcal{L}^p(\mathcal{H})$	Schatten p-class defined on the Hilbert space $\mathcal{H}$, eqn. (G.5)
$M_k(A)$	Matrix algebra over an algebra A
N	Cyclic skewsymmetrizer, eqn. (E.10)
N_α	Charts of the differentiable structure on Spec $\mathcal{A}$, p. 84
∇	Connection, p. 148
$\nabla^{\mathfrak{S}}$	Spin connection, eqn. (2.103)
$\nabla^{\mathfrak{X}}$	Generalized Levi-Civita connection associated to a complex vector bundle $\mathfrak{X}$, eqn. (F.11)
ν	Volume form, p. 35
$\Omega^p(M)$	The p-forms on a manifold M
$\Omega_1^{\mathrm{ab}}(\mathsf{A})$	Dual Kähler differentials of an algebra A, eqn. (D.36)
$\Omega^k_{\mathrm{dR}}(M)$	The k-th de Rham cochain group, p. 120
$\Omega^\bullet_{\mathrm{ab}}(M)$	The exterior algebra of the Kähler differentials of an algebra A, p. 121
$\Omega^\bullet_{\mathrm{dR}}(M)$	The de Rham cochain complex, p. 120
$\Omega_\bullet^{\mathrm{ab}}(\mathsf{A})$	The exterior algebra of the dual Kähler differentials of an algebra A, p. 128
$\Omega^p_{\mathbb{R}}(M)$	The real p-forms on a manifold M
$\Omega^1_{\mathrm{ab}}(\mathsf{A})$	Kähler differentials of an algebra A, p. 121
p^P	Symbol of linear differential operator P, eqn. (B.10)

π_D	Representation of the Hochschild chain groups on the algebra of bounded operators of $\mathcal{H}$, p. 46
$\mathbb{R}^+$	The positive real numbers including zero
$r(a)$	Spectral radius of an element a in an involutive algebra, eqn. (2.13)
S	1) Set of continuous sections of the spinor bundle, eqn. (2.100)
S	2) Periodicity morphism, eqn. (E.24)
$\mathcal{S}$	Set of smooth sections of the spinor bundle, eqn. (2.101)
s	First degeneracy operator, eqn. (E.13)
s'	Second degeneracy operator, eqn. (E.14)
S^k	k-sphere, p. 11
S_k	Symmetric group of degree k
$S_n^{(p,q)}$	Collection of (p,q)-shuffles, p. 123
s_α	Maps related to coordinate charts of manifold associated to commutative spectral triple, eqn. (4.30)
SO_p	p-th special orthogonal group
Spec (A)	The spectrum of an involutive algebra A over the complex numbers, p. 10
$\mathrm{Spec}_{\mathsf{A}}(a)$	Spectrum of an element of a involutive unital algebra, p. 11
Spin_p^c	Charged spin group, eqn. (2.63)
Spin_p	Real spin group, eqn. (2.64)
σ^P	Principal symbol of a linear differential operator P, eqn. (B.11)
σ_t	Automorphism of $\mathcal{A}$, p. 80
$\mathcal{T}(V)$	Tensor algebra of a finite dimensional vector space V, eqn. (2.27)
τ_F^n	Chern–Connes character, eqn. (G.25)
$\tau_\lambda(T)$	Cesàro average of the function $\frac{\sigma_\lambda(T)}{\log\lambda}$, with T a compact operator, eqn. (3.35)

$\tau(T)$	Equivalence class of the Cesàro average of T in B_∞, p. 51
TM	Tangent space of a manifold M
T^*M	Cotangent space of a manifold M
$\mathrm{Tot}\,(C^{\bullet\bullet})$	The total (co)homology of a bicomplex $C^{\bullet\bullet}$, p. 132
$\mathrm{Tr}^{\,+}(T)$	Dixmier trace of a measurable operator T, eqn. (3.42)
$\mathrm{Tr}_{\,\omega}(T)$	Dixmier trace of compact operator T, relative to state ω on B_∞, eqn. (3.41)
Tr	Trace, p. 47
U_α	Elements of a cover of Spec $\mathcal{A}$, p. 76
$U(1)$	First unitary group
x_α	Coordinate functions of the differentiable structure on Spec $\mathcal{A}$, p. 84

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