Classification of Hamiltonians under Symmetry Constraints

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Abstract

It is a well-know fact that insulating quantum systems are given by a certain Hilbert space H and that the Hamiltonians, topological phases and energy states of this quantum system are given by gapped self-adjoint operators on H, the connected components of these operators and functionals on $B(H)$ respectively. In case a symmetry condition is imposed, the Hamiltonians must respect this symmetry. This restriction affects the remaining topological phases and states as well. In this context it is natural to investigate what quantum systems and ensuing observables, states and topological phases can arise from a Hilbert space H and a symmetry group G.

The investigation into this classification is made under the assumption that G is compact. In the end results are also given for the case that G contains a (non-compact) lattice symmetry $\mathbb{Z}^d$. The notion of symmetry relies on Wigner's theorem and is formulated using twisted group extensions as was proposed by D. Freed. This starting point leads to real representation theory, which in turn yields to some unexpected classification results.

The classification of the quantum systems (ways to implement the symmetry group) is achieved by means of elementary group cohomology. The classification of real representations by J. Dyson is subsequently used to classify the general form of the Hamiltonians. Finally, the states and topological phases are classified by irreducible subspaces of the representation of the symmetry group. In case the symmetry can be related to a Clifford module, a classification of the topological phases can be given using K-theory. This was proposed by A. Kitaev and is now a popular research area. A generalisation to symmetry groups containing a d-dimensional lattice symmetry can be achieved by parametrising the compact case over the d-dimensional torus.

Contents

Introduction

In the introduction we provide a quick sketch of the main problem in this thesis.

Assume that we are given a (classical) physical system with phase space consisting of the integers $\mathbb{N}$. States ω are probability distributions on the phase space. That is to say, we must assign to each number $n \in \mathbb{N}$ a probability $\omega(n) \in [0, 1]$. The number $\omega(n)$ indicates the probability to find a particle in phase n. Let $M(\mathbb{R})$ be the set of Borel measurable subsets of $\mathbb{R}$ and $\mathfrak{P}(\mathbb{N})$ the power set of $\mathbb{N}$. An observable is a way to assign outcomes to the phase space. Classically an observable would be a function

$$E : M(\mathbb{R}) \to \mathfrak{P}(\mathbb{N}) \tag{1}$$

assigning to each measurable subset U of $\mathbb{R}$ a subset of $\mathbb{N}$. The subset $E(U) \subset \mathbb{N}$ is interpreted as the part of the phase space that grants outcomes $U \subset \mathbb{R}$ under the observable in question. An observable should satisfy

$$E(\emptyset) = \emptyset \tag{2}$$
$$E(\mathbb{R}) = \mathbb{N} \tag{3}$$
$$E(\sqcup_{i=0}^{\infty} U_i) = \sqcup_{i=0}^{\infty} E(U_i). \tag{4}$$

The probability for an outcome of a measurement to be contained in U is given by

$$\omega(E(U)). \tag{5}$$

In the quantum-mechanical setting we replace $\mathbb{N}$ by a Hilbert space H with orthonormal basis $\{e_n \mid n \in \mathbb{N}\}$. A state ω is a probability distribution on some basis $\{e_i\}$ of H. We write this state as an operator

$$\mathfrak{r} = \sum_{i=1}^{\infty} p_i P_{e_i}, \tag{6}$$

where p_i is the probability of the basis-vector e_i and P_{e_i} is the orthonormal projection on e_i. In this way a state forms a probability measure on orthonormal subsets U of H.

$$U \to \mathrm{Tr}(\mathfrak{r} P_U), \tag{7}$$

with P_U the orthonormal projection on U. This time an observable E is a function

$$E : M(\mathbb{R}) \to P(H), \tag{8}$$

where $P(H)$ is the set of orthonormal projections on H. This map should satisfy

$$E(\emptyset) = 0 \tag{9}$$

$$E(\mathbb{R}) = \mathbb{I} \tag{10}$$

$$E(\sqcup_{i=1}^{\infty} U_i) = \oplus_{i=1}^{\infty} E(U_i) \text{ and for all } i \neq i' \; E(U_{i'}) \perp E(U_i). \tag{11}$$

The probability for the set of outcomes $U \subset \mathbb{R}$ is now

$$\sum_{i=1}^{\infty} \omega(e_i), \tag{12}$$

where e_i is a basis of $E(U)$. For short we can write this expression by $\omega(E(U))$.

Let a group G act on the observables

$$E \to g \cdot E. \tag{13}$$

At this point it is not yet clear what correct implementations of G actions on the set of observables should look like.

In this thesis we are interested in the observable associated to the energy, the so-called Hamiltonian. We will impose a symmetry condition on this observable E and require that

$$E = g \cdot E. \tag{14}$$

A quantum system will be a Hilbert space H, a symmetry group G and a way to impose a symmetry with underlying group G on the observables. Our mission is now to classify for a Hilbert space H and symmetry group G all possibilities we are presented with. Firstly we need to track down all possible ways in which we can implement a symmetry with underlying group G. Secondly we need to find all Hamiltonians satisfying eq. (14). Thirdly we investigate what states such a Hamiltonian can separate. Lastly, the connected components of the space of possible Hamiltonians are tracked down. The last part is done under the assumption that we are looking at an insulator and is useful in the context of topological insulators.

In chapter 1 we define the correct notion of symmetry (to be justified in chapter 2) and classify the possible quantum systems.

In chapter 2 we find that the maps E correspond to self-adjoint operators and that symmetries should act as unitary or anti-unitary operators. This will justify the definition of a quantum system. In the end we classify the general form of a Hamiltonian respecting the symmetry.

In chapter 3 we define quantum states. The set of states restricted to observables satisfying eq. (14) is studied and classified.

In chapter 4 we impose that the Hamiltonian is ‘gapped’. Under this assumption we treat two ways in which we can classify the connected components of the Hamiltonians.

The previous chapters deal with compact symmetry groups. In the last chapter we generalise to systems presenting a (non-compact) $\mathbb{Z}^d$ lattice symmetry.

Chapter 1

Quantum systems

In this chapter we start with introducing the definition of a quantum system. Next we classify the possibilities under a fixed symmetry group G and Hilbert space H. The justification of the definition is postponed to the next chapter.

1.1 Quantum systems

The definition of a quantum system relies on the notions of a graded group, a twisted extension [11] and an (anti)-unitary operator. We start out by defining these three notions.

First we define a grading of a topological group G.

Definition 1.1.1. *A ϕ-graded group is a topological group equipped with a continuous homomorphism*

$$\phi : G \to \mathbb{Z}/2\mathbb{Z}. \tag{1.1}$$

Elements $g \in G$ for which $\phi(g)$ is 1 or -1 are called even and odd respectively.

A twisted extension with respect to a grading ϕ is a small adaptation of the better known notion of a central extension.

Definition 1.1.2. *Let A be an Abelian topological group. A central extension τ of a topological group G by A is a short exact sequence*

$$e \longrightarrow A \xrightarrow[\iota]{} G^\tau \xrightarrow[\pi]{} G \longrightarrow e \tag{1.2}$$

of topological groups, where A lies in the centre of G^τ.

In other words, ι and π are continuous homomorphisms, which are open onto their image, that are injective and surjective respectively. Notice we abuse the notation a bit by saying $A \subset G^\tau$. Furthermore $\iota(A) = ker(\pi)$ and ι lies in the centre of G^τ.

In order to get a twisted extension we adapt the commutation relations in definition 1.1.2 a little.

Definition 1.1.3. *Let A be an Abelian group and G be a ϕ-graded group. A ϕ-twisted extension τ of a group G by A is a short exact sequence*

$$e \longrightarrow A \xrightarrow[\iota]{} G^\tau \xrightarrow[\pi]{} G \longrightarrow e \tag{1.3}$$

of topological groups. This time we require for the commutation relations

$$ag^\tau = g^\tau a^{\phi(\pi(g^\tau))}, \tag{1.4}$$

for $a \in A$. We usually abbreviate $\phi(\pi(g^\tau))$by writing $\phi(g^\tau)$.

Let $\mathbb{T}$ be the subgroup of complex numbers of norm one. Throughout this paper we will be interested in extensions of ϕ-graded Lie groups by $A = \mathbb{T}$ only. It turns out that such extensions are Lie groups as well.

Theorem 1.1.4. *Let G be a ϕ-graded Lie group, which is a semi-direct product of the connected component of the identity of G and a discrete subgroup group. Let A be an Abelian compact Lie group. If G^τ is a twisted extension of G by A, there is a unique way to equip G^τ with a manifold structure, for which π and ι in eq.* (1.2) *are smooth homomorphisms. G^τ is a Lie group with respect to this manifold structure.*

Proof. For the untwisted (central) see [42] theorem 16.02 which combines results from [29] and [16]. By lemma 1.2.15 taking extensions distributes over the semi-direct product. Since extensions of discrete groups by $\mathbb{T}$ can always be equipped with a manifold structure, namely taking the disjoint union of copies of $\mathbb{T}$, we only need to worry about twisted extensions of the component of the identity. Twisted extensions are due to the continuity of ϕ always central extensions. We can hence apply the untwisted case. □

Corollary 1.1.5. *Theorem 1.1.4 implies that we may restrict ourselves to twisted extensions of G that are Lie groups.*

Now we turn our attention to anti-unitary operators. The adjoint of a bounded anti-linear operator a is defined to be the unique operator satisfying

$$\langle a\psi, \phi\rangle = \overline{\langle \psi, a^*\phi\rangle}. \tag{1.5}$$

Definition 1.1.6. *An operator v is called anti-unitary if it is anti-linear and $vv^* = v^*v = \mathbb{I}$.*

There is a simple way to describe these anti-unitary operators. Let H be a separable Hilbert space and C be the anti-linear operation of complex conjugation defined by

$$C\psi = \overline{\psi}. \tag{1.6}$$

Lemma 1.1.7. *If the Hilbert space H is separable each anti-unitary operator v can be written as*

$$v = uC, \tag{1.7}$$

with u unitary.

Proof. Let v be an anit-unitary operator. In that case $u = vC$ is a unitary operator. Since $C^2 = \mathbb{I}$ we find $v = uC$ as desired. □

Write $\mathrm{Aut}_{QM}(H)$ for the group of unitary and anti-unitary operators. Equip this group with the following grading

$$\phi(u) = \begin{cases} 1 \text{ if } u \text{ is unitary,} \\ -1 \text{ if } u \text{ is anti-unitary.} \end{cases} \tag{1.8}$$

Equip this group with the strong operator topology.

Lastly we get to the notion of a twisted representation.

Definition 1.1.8. *Let G be a ϕ-graded group. A ϕ-twisted representation of G^τ is a continuous homomorphism*

$$\rho^\tau : G^\tau \to Aut_{QM}(H) \tag{1.9}$$

respecting the grading. Furthermore, we require $\rho^\tau(\lambda) = \lambda \mathbb{I}$ for all $\lambda \in \mathbb{T} \subset G^\tau$.

We are now ready to formally define what we mean by a quantum system.

Definition 1.1.9. *A quantum system is a tuple*

$$(H, G, \phi, \tau, \rho^\tau), \tag{1.10}$$

with H a separable complex Hilbert space, G a finite dimensional Lie group, ϕ a grading of G, τ a ϕ-twisted extension of G, and finally ρ^τ a ϕ-twisted representation of G^τ. In regard of theorem 1.1.4 we additionally require that G is the semi-direct product of the connected component of the identity and a discrete sub-group.

Non-compact Lie groups are generally quite hard to handle. Restricting to compact Lie groups is therefore usually necessary for the classification theorems to come.

The occurrence of a Hilbert space in the definition plays the role of the 'phasespace' in classical physics and should not come as a surprise to anyone who has encountered quantum physics before. The group G signifies the symmetry group of the system and should not raise too many eyebrows either. However, the extension τ and implementation of the symmetry in a possibly anti-unitary way might look unnatural. The justification of this is postponed to section 2.2.1. See also Appendix A for an example of an anti-unitary symmetry.

We yet need to say when quantum systems are equivalent or isomorphic.

Definition 1.1.10. *A twisted extension G^τ of G by A is isomorphic to an extension $G^{\tau'}$ of G' by A' if there exists a commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & G^\tau & \longrightarrow & G & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} & & \downarrow \\ 1 & \longrightarrow & A' & \longrightarrow & G^{\tau'} & \longrightarrow & G' & \longrightarrow & 1, \end{array} \tag{1.11}$$

where ψ_1, ψ_2 and ψ_3 are (Lie) group isomorphisms. We denote this by $G^\tau \cong G^{\tau'}$

The expression $G^\tau \cong G^{\tau'}$ is stronger then merely isomorphism of groups. In turn, two representations ρ^τ and $\rho^{\tau'}$ are equivalent whenever their domains G^τ and $G^{\tau'}$ are isomorphic and there exists a unitary operator u such that

$$\rho^\tau(g^\tau) = u\rho^{\tau'}(\psi(g^\tau))u^{-1}, \tag{1.12}$$

with ψ the given isomorphism between G^τ and $G^{\tau'}$.

Definition 1.1.11. *Two quantum systems $(H, G, \phi, \tau, \rho^\tau)$ and $(H', G', \phi', \tau', \rho'^\tau)$ are equivalent whenever $\rho^\tau \cong \rho'^\tau$, in particular $G'^\tau \cong G^\tau$ and $H \cong H'$.*

We now define what we mean by isomorphism of quantum systems. The definition may seem arbitrary, a justification follows in definition 2.3.4 below.

Definition 1.1.12. *Given two quantum systems* $(H, G, \phi, \tau, \rho^\tau)$ *and* $(H', G', \phi', \tau', \rho'^\tau)$*, let* A_{sa} *and* A'_{sa} *respectively be the set of bounded self-adjoint operators that intertwine* $\rho^\tau(G^\tau)$ *respectively* $\rho'^{\tau'}$*. We say that the two quantum systems are isomorphic whenever*

$$A_{sa} = uA'_{sa}u^* \tag{1.13}$$

for some unitary or anti-unitary operator $u : H \to H'$*.*

The next proposition makes sure that two equivalent quantum systems are isomorphic.

Proposition 1.1.13. *Whenever* $(H, G, \phi, \tau, \rho^\tau)$ *and* $(H', G', \phi', \tau', \rho^{\tau'})$ *are equivalent we have*

$$(H, G, \phi, \tau, \rho^\tau) \cong (H', G', \phi', \tau', \rho^{\tau'}). \tag{1.14}$$

Proof. It is sufficient to prove that the self-adjoint intertwiners A_{sa} and A'_{sa} of $\rho^\tau(G^\tau)$ and $\rho'^\tau(G'^\tau)$ are unitarily equivalent whenever ρ^τ and ρ'^τ are unitary equivalent with respect to the unitary operator u. To this end we show that the map

$$A_{sa} \to A'_{sa} \tag{1.15}$$

$$a \mapsto uau^{-1} \tag{1.16}$$

is well defined and bijective. To see that it is well defined we need to prove that $uA_{sa}u^{-1} \subset A'_{sa}$. That is to say, for each $a \in A$, uau^{-1} intertwines $\rho^{\tau'}(G^{\tau'})$. We verify this by calculating for $g^{\tau'} \in G^{\tau'}$,

$$\begin{aligned} uau^{-1}\rho^{\tau'}(g^{\tau'}) &= uau^{-1}u\rho^\tau(\phi(g^\tau))u^{-1} = ua\rho^\tau(\phi(g^\tau))u^{-1} \\ &= u\rho^\tau(\psi'(g^\tau))u^{-1}uau^{-1} = \rho^{\tau'}(g^{\tau'})uau^{-1}. \end{aligned} \tag{1.17}$$

Bijectivity of this map follows from bijectivity of u. □

Remark 1.1.14. *Mind that isomorphism is a strictly weaker notion then equivalence.*

An important question remaining is whether it is really important what twisted extension of G we choose to work with. Is it just a matter of bookkeeping, or can two distinct extensions lead to non-isomorphic quantum systems? The following example demonstrates that two distinct extensions generally lead to distinct quantum systems.

Example 1.1.15. *Let* $H = \mathbb{C}^2$ *and* $G = \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$*. Pick the extension* G^τ *to be the direct product* $\{\pm 1\} \times \mathbb{T}$*. Consider the following twisted representation*

$$\rho^\tau((\pm 1, \lambda))v = \begin{cases} v \text{ if } v \in L(e_1) \\ \pm v \text{ if } v \in L(e_2) \end{cases} \tag{1.18}$$

The self-adjoint elements commuting with this representation are the matrices for which e_1 *and* e_2 *are eigenvectors.*

Now take $G^{\tau'}$ *to be the extension that is the set* $\pm 1 \times \mathbb{T}$ *with multiplication defined by*

$$(\pm 1, \lambda) \cdot (1, \lambda') = (\pm 1, \lambda\lambda') \tag{1.19}$$

$$(\pm 1, \lambda) \cdot (-1, \lambda') = (\mp 1, \lambda^{-1}\lambda'). \tag{1.20}$$

The multiplication is associative since $\mathbb{T}$ *is commutative. Now consider the following representation of* $G^{\tau'}$*:*

$$(1, \lambda) \mapsto \lambda\mathbb{I} \tag{1.21}$$

$$(-1, \lambda) \mapsto C\lambda\mathbb{I}, \tag{1.22}$$

with C complex conjugation. The self-adjoint elements commuting with this representation consists of all self-adjoint real matrices.

The two distinct extensions therefore lead to distinct isomorphism classes of quantum systems.

We conclude that we cannot restrict ourselves to just one extension of G, but must study them all.

1.2 Classification of extensions

The aim in this section is to find all possible twisted extensions of a given symmetry group G and Hilbert space H. That is to say, given a group G and a Hilbert space H, we want to track down all possible gradings ϕ and twisted extensions τ.

A grading ϕ of the Lie group G is by definition an element of the set

$$\mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z}), \tag{1.23}$$

where we consider continuous homomorphisms only.

In the rest of this section we will track down all possible ϕ-twisted extensions G^τ of G for some fixed grading. The discussion consists of four parts. First we determine the twisted extensions of discrete, simply connected and connected groups, thereafter we combine these cases to obtain a result for more general Lie groups.

1.2.1 The discrete case

Let G be a discrete group and A be an Abelian Lie group. The ϕ-twisted extensions of G by A can be identified with an easily computable group. The construction of this group is a small modification of the usual procedure for the case of central extensions [43].

Define the set of cocycles $Z^2(G, A)$ (with respect to the chosen ϕ) to be all functions $c : G \times G \to A$ that satisfy

$$c(g, h)c(gh, z) = c(h, z)^{\phi(g)} c(g, hz) \tag{1.24}$$

$$c(g, e) = c(e, a) = e. \tag{1.25}$$

The set $Z^2(G, A)$ forms a group under pointwise multiplication. A cocycle $c \in Z^2(G, A)$ determines a twisted extensions of G by A. Write

$$G \times_c A \tag{1.26}$$

for the group that is $G \times A$ as a set and whose multiplication is defined by

$$(g, a) \cdot_c (g', a') = (gg', c(g, g')a^{\phi(g')}a'). \tag{1.27}$$

Group inversion is then defined by

$$(g, a)^{-1} = (g^{-1}, a^{-\phi(g^{-1})} c(g, g^{-1})^{-1}). \tag{1.28}$$

Equation (1.26) is a ϕ-twisted extension of G by A. It turns out that in fact all ϕ-twisted extensions of discrete groups are isomorphic to eq. (1.26) for some $c \in Z^2(G, A)$.

Lemma 1.2.1. *Let G be a discrete group and A be an Abelian Lie group. The map sending a cocycle $c \in Z^2(G, A)$ to*

$$G \times_c A \tag{1.29}$$

is a surjection onto the ϕ-twisted extensions of G by A.

Proof. Let G^τ be a ϕ-twisted extension of G. Fix once and for all a section $s : G \to G^\tau$ and define the function $c : G \times G \to A$ by

$$c(g, h) = s(g)s(h)s(gh)^{-1}. \tag{1.30}$$

This function c is chosen in such a way that for $g, h \in G$ and $a, a' \in A$

$$s(g)as(g')a' = s(gg')c(g, g')a^{\phi(g')}a'. \tag{1.31}$$

Associativity of multiplication implies eq. (1.24) and hence that $c \in Z^2(G, A)$. It is easy to see that the extension G^τ is isomorphic to the extension in eq. (1.29) for this particular cocycle. □

The next step is to find the possibilities up to isomorphism (recall eq. (1.11)). To this end, let $B^1(G, A)$ be the set of functions $k : G \times G \to A$ that can be written as

$$k(g, h) = b(g)^{-\phi(h)}b(h)b(gh)^{-1}, \tag{1.32}$$

for some map $b : G \to A$. It is easy to verify that $B^1(G, A)$ is a subgroup of $Z^2(G, A)$.

Lemma 1.2.2. *Let G be a discrete group and A an Abelian Lie group. For $c, c' \in Z^2(G, A)$*

$$G \times_c A \cong G \times_{c'} A \tag{1.33}$$

iff

$$c' = ck, \tag{1.34}$$

for some $k \in B^1(G, \mathbb{T})$.

Proof. First assume eq. (1.33). There exists an isomorphism $\psi : G \times_c A \to G \times_{c'} A$ fitting in the diagram eq. (1.11). This implies that

$$\psi(g, a) = (g, ab(g)), \tag{1.35}$$

for some function $b : G \to A$. We calculate

$$\begin{aligned}(gh, c'(g, h)) &= (g, e) \cdot_{c'} (h, e) = \psi(g, b(g)^{-1}) \cdot_{c'} \psi(h, b(h)^{-1}) \\ &= \psi\big((g, b(g)^{-1}) \cdot_c (h, b(h)^{-1})\big) = \psi\big((gh, b(g)^{-\phi(h)}b(h)^{-1}c(g, h)\big) \\ &= (gh, b(g)^{-\phi(h)}b(h)^{-1}c(g, h)b(gh))\end{aligned} \tag{1.36}$$

verifying that

$$c = c'k \tag{1.37}$$

for $k \in B^1(G, \mathbb{T})$.

On the other hand, each k in $B^1(G, \mathbb{T})$ for which $c = kc'$ provides a function b that defines an isomorphism ψ via

$$\psi(g, \lambda) = (g, b(g)a). \tag{1.38}$$

□

Applying the above two lemmas to the case of quantum systems at hand, we obtain the following result.

Corollary 1.2.3. *Let G be a discrete symmetry group. The twisted extensions up to isomorphism, are in bijective correspondence with*

$$Hom(G, \mathbb{Z}/2\mathbb{Z}) \times Z^2(G, \mathbb{T})/B^1(G, \mathbb{T}), \tag{1.39}$$

where the first part of the cartesian product fixes the grading ϕ and the second part determines the ϕ-twisted extension.

1.2.2 The simply connected case

We would like to find a similar identification in the case of twisted extensions of simply connected Lie groups. Note that twisted extensions of simply connected Lie groups are in fact just central extensions. The procedure for discrete subgroups is not suitable in this case, since the construction of central extensions using cocycles in $Z^2(G, A)$ does not guarantee a smooth multiplication. The correct procedure is to transit to extensions of Lie algebras. For a start we define what we mean by a central extensions of Lie algebras [44].

Definition 1.2.4. *A central extension of a Lie algebra $\mathfrak{g}$ by $\mathfrak{a}$ is a sequence of Lie algebras*

$$0 \longrightarrow \mathfrak{a} \xrightarrow[i]{} \mathfrak{g}^\tau \xrightarrow[\pi]{} \mathfrak{g} \longrightarrow 0, \tag{1.40}$$

where the arrows are Lie algebra homomorphisms, the map i is injective, the map π is surjective and $Im(i) = ker(\pi)$. Furthermore

$$[\mathfrak{a}, \mathfrak{g}^\tau] = 0. \tag{1.41}$$

Two central extensions are isomorphic whenever there exists a Lie algebra isomorphism L that fits in the following commuting diagram

$$\begin{array}{ccccccccc} & & & & \mathfrak{g}^{\tau'} & & & & \\ & & & \nearrow & \downarrow L & \searrow & & & \\ 0 & \longrightarrow & \mathfrak{a} & \longrightarrow & \mathfrak{g}^\tau & \longrightarrow & \mathfrak{g} & \longrightarrow & 0. \end{array} \tag{1.42}$$

The following lemma justifies the definition above.

Lemma 1.2.5. *Let G be a Lie group, A an Abelian Lie group and G^τ a central extension*

$$1 \longrightarrow A \xrightarrow[\iota]{} G^\tau \xrightarrow[\pi]{} G \longrightarrow 1. \tag{1.43}$$

Let in turn $\mathfrak{a}$, $\mathfrak{g}^\tau$ and $\mathfrak{g}$ be the Lie algebras of the respective Lie groups. The following sequence

$$0 \longrightarrow \mathfrak{a} \xrightarrow[T_e\iota]{} \mathfrak{g}^\tau \xrightarrow[T_e\pi]{} \mathfrak{g} \longrightarrow 0, \tag{1.44}$$

where T_e is the derivation at the unit, is a central extension of $\mathfrak{g}$ by $\mathfrak{a}$.

Furthermore two central extensions of Lie algebras obtained from isomorphic Lie group are isomorphic.

Proof. The map ι is an embedding of A in G^τ, hence $T_e\iota$ will be an embedding of $\mathfrak{a}$ in $\mathfrak{g}^\tau$. Secondly, since the Lie groups $Im\iota$ and $ker\pi$ are equal, their Lie algebras should be equal as well. Therefore $ImT_e\iota = kerT_e\pi$. Since G has dimension $dim(G^\tau) - dim(A)$ and the kernel of $T_e\pi$ has dimension $dim(A)$, it follows that $T_e\pi$ is surjective. Lastly, the fact that A is contained in the centre of G^τ implies that

$$[\mathfrak{a}, \mathfrak{g}^\tau] = 0. \tag{1.45}$$

This verifies all requirements in definition 1.2.4.

If two extension are isomorphic there exists an isomorphism ψ fitting in the diagram eq. (1.11). The derivative of this map grants the desired isomorphism between the central extensions of the Lie algebra. □

Central extensions of connected Lie groups determine central extensions of Lie algebras. The question remaining is to what extent the central extensions of Lie algebras determine central extensions of connected Lie groups. The question up to what extent a Lie algebra determines a Lie group is a well known result in Lie theory.

Proposition 1.2.6. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Up to isomorphism there exists one and only one simply connected Lie group with this Lie algebra.*

Any connected Lie group G with Lie algebra $\mathfrak{g}$ is isomorphic to

$$\tilde{G}/D, \tag{1.46}$$

where $\tilde{G}$ is the unique simply connected Lie group with Lie algebra $\mathfrak{g}$ and D a discrete subgroup in the centre of $\tilde{G}$. The subgroup D is determined up to isomorphism of $\tilde{G}$. That is to say

$$\tilde{G}/D \cong \tilde{G}/D' \tag{1.47}$$

iff there exists an automorphism $\psi : \tilde{G} \to \tilde{G}$ such that $\psi(D) = D'$.

Proof. See [47] corollary 1.21. □

This suggests the following definition.

Definition 1.2.7. *Let G be a Lie group. We call a simply connected group $\tilde{G}$ for which $\tilde{G}/D \cong G$, for some discrete subgroup in the centre of $\tilde{G}$, the universal cover of G.*

Consider the map assigning to each Lie group its Lie algebra. The above result shows that this map is bijective after restriction to the domain of simply connected Lie groups. It is now natural to wonder whether the restriction to simply connected groups makes the map in lemma 1.2.5, taking central extensions of Lie groups to central extensions of Lie algebras, bijective as well. In order to prove this we need the following result.

Proposition 1.2.8. *Let $\tilde{G}$ and $\tilde{H}$ be two simply connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. A Lie algebra homomorphism $L : \mathfrak{g} \to \mathfrak{h}$ induces a unique homomorphism $\tilde{\psi} : \tilde{G} \to \tilde{H}$ for which $T_e\tilde{\psi} = L$.*

Let G and H be Lie groups groups and let $\tilde{G}$ and $\tilde{H}$ be their respective universal covers. For each homomorphism $\psi : G \to H$ there exists a unique homomorphism $\tilde{\psi} : \tilde{G} \to \tilde{H}$, for which $T_e\tilde{\psi} = T_e\psi$. This particular map $\tilde{\psi}$ is the unique map that descents to ψ after taking the quotient of $\tilde{G}$ and $\tilde{H}$ by the discrete subgroups $D \subset \tilde{G}$ and$D' \subset \tilde{H}$ for which $G = \tilde{G}/D$ and $H = \tilde{H}/D'$.

Furthermore, if $\mathfrak{a} \subset \mathfrak{g}$ is an ideal then $\tilde{A}$ is a normal Lie sub-group of $\tilde{G}$ and the simply connected Lie group with Lie algebra $\mathfrak{g}/\mathfrak{a}$ is isomorphic to $\tilde{G}/\tilde{A}$.

Proof. See [39] chapter 8. □

The above proposition delivers the bijection we were after.

Lemma 1.2.9. *Let $\tilde{G}$ and $\tilde{A}$ be simply connected Lie groups with Lie algebra $\mathfrak{g}$ respectively $\mathfrak{a}$. The map taking isomorphism classes of central extensions $\tilde{G^\tau}$ of $\tilde{G}$ by $\tilde{A}$ to isomorphism classes of central extensions $\mathfrak{g}^\tau$ of $\mathfrak{g}$ by $\mathfrak{a}$, via the assignment in lemma 1.2.5 is bijective.*

Proof. First we note that a central extension $\tilde{G^\tau}$ of a simply connected group $\tilde{G}$ by a simply connected group $\tilde{A}$ is once again simply connected. This can be seen by using the standard long exact sequence of a fibration

$$\cdots \longrightarrow \pi_2(\tilde{G}) \longrightarrow \pi_1(\tilde{A}) \longrightarrow \pi_1(\tilde{G^\tau}) \longrightarrow \pi_1(\tilde{G}) \longrightarrow \pi_0(\tilde{A}) \longrightarrow \cdots, \tag{1.48}$$

where $\pi_1(\tilde{G^\tau})$ is squeezed between two trivial groups and hence trivial itself.

The assignment is injective by the first par the claim in proposition 1.2.6.

We now prove that the map is surjective. Let

$$0 \longrightarrow \mathfrak{a} \xrightarrow[i]{} \mathfrak{g}^\tau \xrightarrow[p]{} \mathfrak{g} \longrightarrow 0 \tag{1.49}$$

be some central extension of $\mathfrak{g}$ by $\mathfrak{a}$. By proposition 1.2.8 we obtain a sequence of simply connected Lie groups

$$e \longrightarrow \tilde{A} \xrightarrow[\iota]{} \tilde{G^\tau} \xrightarrow[\pi]{} \tilde{G} \longrightarrow e. \tag{1.50}$$

Since $\mathfrak{a} \subset \mathfrak{g}^\tau$ is an ideal we obtain by proposition 1.2.8 that $\tilde{A}$ is a normal Lie sub-group of $\tilde{G^\tau}$ and $\tilde{G} \cong \tilde{G^\tau}/\tilde{A}$. Pick ι' the inclusion of A in G^τ and π' to be the quotient of $\tilde{G^\tau}$ by $\tilde{A}$. The derivation of these maps ι' and π' at the unit is i respectively p and hence by the uniqueness in proposition 1.2.8 $\iota = \iota'$ and $\pi = \pi'$. Therefore $\tilde{G^\tau}$ is indeed a central extension of $\tilde{G}$ by $\tilde{A}$. □

We found that all central extensions of simply connected Lie groups by Abelian simply connected Lie groups are given by central extensions of their respective Lie algebras. In the case of quantum systems we are interested in extensions by $\mathbb{T}$, which is not simply connected. Luckily we can generalise the result.

Proposition 1.2.10. *Let G be a simply connected Lie group. The central extensions of G by $\mathbb{T}$ are in bijective correspondence with Lie algebra extensions of $\mathfrak{g}$ by $\mathbb{R}$. The bijection is given by composing the following two maps. First assign to an extension*

$$e \longrightarrow \mathbb{T} \xrightarrow[i]{} G^\tau \xrightarrow[\pi]{} G \longrightarrow e \tag{1.51}$$

the by proposition 1.2.8 unique extension

$$e \longrightarrow \mathbb{R} \xrightarrow[\tilde{i}]{} \tilde{G^\tau} \xrightarrow[\tilde{\pi}]{} G \longrightarrow e, \tag{1.52}$$

where $\mathbb{R}$ and $\tilde{G^\tau}$ are the universal covers of $\mathbb{T}$ and G^τ. To finish we assign to such a central extension of G by $\mathbb{R}$ a Lie algebra extension in the way of lemma 1.2.5.

Proof. Since Lie algebra extensions are in bijective correspondence with central extensions of simply connected Lie groups was already established it suffices to prove that bijectivty of the first map. the following map is bijective. We should check that eq. (1.52) is indeed a central extension. The sequence eq. (1.51) is a central extension of connected groups and hence their Lie algebras form by lemma 1.2.5 a central extension as well. By lemma 1.2.5 this implies that the sequence eq. (1.52) of simply connected Lie groups is a central extension.

Next we prove surjectivity of this mapping. Let $\tilde{G^\tau}$ be an extension of G by $\mathbb{R}$. That is to say, we have an exact sequence

$$e \longrightarrow \mathbb{R} \xrightarrow[\tilde{i}]{} \tilde{G^\tau} \xrightarrow[\tilde{\pi}]{} G \longrightarrow e. \tag{1.53}$$

Define $D = \tilde{i}(\mathbb{Z})$. Obviously $\tilde{\pi}(D) = e$ and $\tilde{i}^{-1}(D) = \mathbb{Z}$. For this reason the exact sequence descends to an exact sequence

$$e \longrightarrow \mathbb{R}/\mathbb{Z} \cong \mathbb{T} \xrightarrow[\tilde{i}]{} \tilde{G^\tau}/D \xrightarrow[\tilde{\pi}]{} G \longrightarrow e. \tag{1.54}$$

Hence $\tilde{G^\tau}/D$ is an extension of G by $\mathbb{T}$. The universal cover of this extension is $\tilde{G^\tau}$, proving surjectivity.

Lastly, we look at the injectivity of the map. Assume $\tilde{G^\tau}$ and $\tilde{G^{\tau}}'$ are isomorphic. In regard of eq. (1.11) the isomorphism above should satisfy

$$\tilde{i} = \tilde{\psi} \circ \tilde{i}'. \tag{1.55}$$

The isomorphism hence descends to an isomorphism

$$G^\tau \cong \tilde{G^\tau}/\tilde{i}(D) \cong \tilde{G^{\tau}}'/\tilde{i}(D') \cong G^{\tau'}. \tag{1.56}$$

□

In case G is not simply connected, a central extension of $\mathfrak{g}$ by $\mathbb{R}$ might not lift to a central extension of G by $\mathbb{T}$.

In order to find the central extensions of a simply connected Lie groups G by $\mathbb{T}$ we can by lemma 1.2.9 equivalently search for central extensions of the Lie algebras $\mathfrak{g}$ by $\mathbb{R}$. In case G is only connected we need to look for extension of $\mathfrak{g}$ by $\mathbb{R}$ such that $\mathrm{Ad}(D) = 0$. For this reason we now focus on finding the Lie algebra extensions. The central extension problem for Lie algebras can be dealt with in a similar way as we did in the discrete case [43].

Define for a Lie algebra $\mathfrak{g}$ and Abelian Lie algebra $\mathfrak{a}$ the set $Z^2(\mathfrak{g}, \mathfrak{a})$ of all bilinear functions $c : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ satisfying

$$c(X, Y) = -c(Y, X) \tag{1.57}$$

$$0 = c(X, [Y, Z]) + c(Z, [X, Y]) + c(Y, [Z, X]). \tag{1.58}$$

The set $Z^2(\mathfrak{g}, \mathfrak{a})$ forms a group under pointwise addition. For a cocycle $c \in Z^2(\mathfrak{g}, \mathfrak{a})$ we can define a Lie algebra

$$\mathfrak{g} \oplus_c \mathfrak{a} \tag{1.59}$$

by defining for the Lie bracket as

$$[X \oplus X', Y \oplus Y']_c = [X, Y]' \oplus c(X, Y), \tag{1.60}$$

where $[,]'$ is the Lie bracket of $\mathfrak{g}$.

Lemma 1.2.11. *All central extensions of* $\mathfrak{g}$ *by* $\mathfrak{a}$ *are isomorphic to eq.* (1.59) *for some* $c \in Z^2(\mathfrak{g}, \mathfrak{a})$.

Proof. Since we are dealing with a central extension

$$[\mathfrak{a}, \mathfrak{g}^\tau] = 0. \tag{1.61}$$

It follows that the Lie bracket is determined by a bracket

$$\mathfrak{g}^\tau/\mathfrak{a} \times \mathfrak{g}^\tau/\mathfrak{a} \to \mathfrak{g}^\tau. \tag{1.62}$$

Since $\mathfrak{g} \cong \mathfrak{g}^\tau/\mathfrak{a}$ this is to say the Lie algebra is defined once we know the bracket

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}^\tau. \tag{1.63}$$

This Lie bracket has for $X, Y \in \mathfrak{g}$ the general form

$$[X, Y] = [X, Y]' + c(X, Y), \tag{1.64}$$

where $[,]'$ is the bracket in $\mathfrak{g}$ and c is some bilinear function $c : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$. By the requirements of the Lie bracket we can derive that c satisfies eq. (1.57) and is hence a cocycle in $Z^2(\mathfrak{g}, \mathfrak{a})$. This proves that g^τ is isomorphic to eq. (1.59) for some $c \in Z^2(\mathfrak{g}, \mathfrak{a})$. □

Next we need to track down these possibilities up to isomorphism. Define a subgroup $B^1(\mathfrak{g}, \mathfrak{a})$ of $Z^2(\mathfrak{g}, \mathfrak{a})$ by selecting all elements satisfying

$$c(X, Y) = \omega([X, Y]) \tag{1.65}$$

for some linear function $\omega : \mathfrak{g} \to \mathfrak{a}$.

Lemma 1.2.12. *Two extensions* $\mathfrak{g} \oplus_c \mathfrak{a}$ *and* $\mathfrak{g} \oplus_{c'} \mathfrak{a}$ *of* $\mathfrak{g}$ *by* $\mathfrak{a}$ *are isomorphic iff*

$$c = c' + b \tag{1.66}$$

for some $b \in B^1(\mathfrak{g}, \mathfrak{a})$.

Proof. Assume that

$$\mathfrak{g} \oplus_c \mathfrak{a} \cong \mathfrak{g} \oplus_{c'} \mathfrak{a}. \tag{1.67}$$

Since the two Lie algebras are isomorphic there exists a bijective linear map $L : \mathfrak{g} \oplus_c \mathfrak{a} \to \mathfrak{g} \oplus_{c'} \mathfrak{a}$ respecting the Lie bracket. Since eq. (1.42) commutes we have for each $a \in \mathfrak{a}$

$$La = a. \tag{1.68}$$

For $X, Y \in \mathfrak{g}$, define

$$b(X, Y) = [X, Y] - L([X, Y]), \tag{1.69}$$

obviously $b \in B^1(\mathfrak{g}, \mathfrak{a})$. Furthermore, we find that

$$\begin{aligned} c(X, Y) - c'(X, Y) &= [X, Y]_c - [X, Y]_{c'} = [X, Y]_c + L\big([X, Y]_c\big) \\ &= [X, Y] + c(X, Y) - L\big([X, Y] - c(X, Y)\big) \\ &= [X, Y] - L[X, Y] = b(X, Y), \end{aligned} \tag{1.70}$$

where we used eq. (1.68) in the fourth step. This verifies eq. (1.66).

On the other hand, for each $b \in B^1(\mathfrak{g}, \mathfrak{a})$ we may define a linear function L by

$$L(Z) \begin{cases} [X, Y] + b(X, Y) \text{ if } Z \in [\mathfrak{g}^\tau, \mathfrak{g}^\tau] \ , \\ Z \text{ if } Z \in [\mathfrak{g}^\tau, \mathfrak{g}^\tau]^\perp. \end{cases} \tag{1.71}$$

It is a straightforward check that L is a Lie algebra isomorphism fitting in the diagram eq. (1.42). □

Wrapping it all together leads to the main result for central extensions of connected Lie groups.

Corollary 1.2.13. *Let G be a simply connected Lie group. The central extensions of G by $\mathbb{T}$ bijectively correspond with the group*

$$Z^2(\mathfrak{g}, \mathbb{R})/B^1(\mathfrak{g}, \mathbb{R}). \tag{1.72}$$

The correspondence is obtained by using the three bijective maps constructed in the above. Firstly, the bijective mapping between central extensions of G by $\mathbb{T}$ and central extensions of $\tilde{G}$ by $\mathbb{R}$ in proposition 1.2.10. Secondly, the bijective map in lemma 1.2.9 between central extensions of simply connected Lie groups and their Lie algebras. Lastly the bijective correspondence of central Lie algebra extensions and the group in eq. (1.72).

1.2.3 The connected case

In many examples the group G is merely connected and not simply connected. In this case we can look at an adaptation of the discrete case [43]. In the same spirit as before, define the group $Z^2_s(G, A)$ consisting of functions that satisfy eq. (1.57) and are smooth around the identity $e \in G$. Define $B^1(G, A)$ to be the subgroup of $Z^2(G, A)$ of elements for which eq. (1.32) holds.

Proposition 1.2.14. *The central extensions of a connected Lie group G by an Abelian Lie group A are in bijectively correspond with*

$$Z^2_s(G, A)/B^1_s(G, A). \tag{1.73}$$

via the assignment

$$c \to G \times_c A. \tag{1.74}$$

Proof. The proof of this proposition is a modification of the proof in lemma 1.2.1 and lemma 1.2.2. First we note that there exists a section $s : G \to G^\tau$ that is locally smooth around $e \in G$. This in turn implies that the cocycle c with respect to this s should be smooth around $e \in G$ as well. To finish we need to show whenever c is smooth around $e \in G$, that $G \times_c A$ is a smooth manifold with smooth multiplication.

Pick to this end a contractible neighbourhood V of e on which c is smooth. Now $V \times A$ allows us to construct a fundamental system of neighbourhoods turning G^τ into a manifold. The cocycle c is smooth and hence the multiplication will be smooth as well. □

As opposed to the other groups defined thus far, this group is in general quite hard to compute explicitly.

1.2.4 The general case

Thus far we obtained an identification of the ϕ-twisted extensions of discrete, simply connected and connected Lie groups by relatively easy computable groups. If we want to obtain an identification for general Lie groups G occurring in quantum systems we need to invoke the additional assumption made in the definition of a quantum system. A symmetry group in a quantum system is given by the following semi-direct product

$$G \cong G_e \times_\alpha (G/G_e), \tag{1.75}$$

where G_e is the connected component of $e \in G$ and α is a homomorphism

$$\alpha : G/G_e \to \mathrm{Aut}(G_e). \tag{1.76}$$

Recall that multiplication in a semi-direct product $G_e \times_\alpha (G/G_e)$ is defined by

$$(u, z) \cdot (u', z') = (u\alpha(z)(u'), zz'). \tag{1.77}$$

Since G_e is connected and G/G_e is discrete, we can through corollary 1.2.3 and corollary 1.2.13 find the respective ϕ-twisted extensions of these groups. From a ϕ-twisted extensions of G_e and G/G_e we can construct a ϕ-twisted extensions G^τ of G in the following way. Given a homomorphism

$$\tilde{\alpha} : G/G_e \to Aut(G_e^\tau) \tag{1.78}$$

that lifts the homomorphism α occurring in the quantum system. That is to say, for each $z \in G/G_e$ the diagram

$$\begin{array}{ccc} G_e^\tau & \xrightarrow[\pi]{} & G_e \\ {\scriptstyle \tilde{\alpha}(z)}\downarrow & & \downarrow{\scriptstyle \alpha(z)} \\ G_e^\tau & \xrightarrow[\pi]{} & G_e \end{array} \tag{1.79}$$

must commute. The quotient

$$G_e^\tau \times_{\tilde{\alpha}} (G/G_e)^\tau / \sim \text{, where } (u^\tau, z^\tau) \sim (\lambda u^\tau, \lambda z^\tau) \text{ for } \lambda \in \mathbb{T}, \tag{1.80}$$

with multiplication defined by

$$(u^\tau, z^\tau) \cdot (u'^\tau, z'^\tau) = (u^\tau \tilde{\alpha}(\pi(z^\tau))(u'^\tau), z^\tau z'^\tau), \tag{1.81}$$

is a ϕ-twisted extensions of G. As a matter of fact we obtain all ϕ-twisted extensions of G this way [11] page 29.

Lemma 1.2.15. *Let G be a semi-direct product $G_e \times_\alpha (G/G_e)$. Any extension of G by $\mathbb{T}$ is isomorphic to eq.* (1.80) *for some lift $\tilde{\alpha}$ of α.*

Proof. Let G^τ be a ϕ-twisted extension of G and π the quotient map $\pi : G^\tau \to G^\tau/\mathbb{T} \cong G$. Then G^τ is a semi-direct product of $\pi^{-1}(G_e) = G_e^\tau$ and $\pi^{-1}(G/G_e) = (G/G_e)^\tau$. The $\tilde{\alpha}'$ of this semi-direct product is

$$\tilde{\alpha}' : (G/G_e)^\tau \to Aut(G_e^\tau) \tag{1.82}$$

$$\tilde{\alpha}(z^\tau)(u^\tau) = z^\tau u^\tau z^{\tau -1}. \tag{1.83}$$

Since $\mathbb{T}$ commutes with G_e, the homomorphism $\tilde{\alpha}'$ only depends on $\pi(z^\tau)$. Therefore $\tilde{\alpha}'$ can be replaced by a homomorphism $\tilde{\alpha} : G/G_e \to Aut(G_e^\tau)$. We conclude that G^τ is isomorphic to eq. (1.80), with respect to this $\tilde{\alpha}$. □

In order to find all ϕ-twisted extensions we may proceed by finding all ϕ-twisted extensions of G_e and G/G_e plus all lifts $\tilde{\alpha}$ of α. The ϕ-twisted extensions of connected and discrete Lie groups were already taken care of in the previous two subsections.

Write $Lift(\alpha)$ for the set of homomorphisms $\tilde{\alpha}$ that lift α.

Theorem 1.2.16. *Let G be a symmetry group of some quantum system. If G_e is simply connected the twisted extensions of G are in bijcective correspondence with the set*

$$Hom(G/G_e, \mathbb{Z}/2\mathbb{Z}) \times Lift(\alpha) \\ \times Z^2(G/G_e, \mathbb{T})/B^1(G/G_e, \mathbb{T}) \times Z^2(\mathfrak{g}, \mathbb{Z})/B^1(\mathfrak{g}, \mathbb{R}). \tag{1.84}$$

If G_e is not simply connected the twisted extensions of G are given by

$$Hom(G/G_e, \mathbb{Z}/2\mathbb{Z}) \times Lift(\alpha) \\ \times Z^2(G/G_e, \mathbb{T})/B^1(G/G_e, \mathbb{T}) \times Z^2_s(G_e, \mathbb{T})/B^1_s(G_e, \mathbb{T}), \tag{1.85}$$

where we may have some redundancy due to the fact that distinct elements in $Lift(\alpha)$ might result in isomorphic groups.

Proof. The expression $Hom(G/G_e, \mathbb{Z}/2\mathbb{Z})$ gives us the gradings of G. By assumption we can write

$$G = G/G_e \times_\alpha G_e, \tag{1.86}$$

for some homomorphism $\alpha : G/G_e \to Aut(G_e)$. By lemma 1.2.15 we can simply search for twisted extensions of G_e respectively G/G_e and lifts of α. The group G/G_e is discrete and the twisted extension are therefore given by corollary 1.2.3. In case G_e is simply connected, the extensions are given by corollary 1.2.13. In case G_e is merely connected the extensions are given by proposition 1.2.14. Finally $Lift(\alpha)$ gives all possible lifts of α. □

1.2.5 Examples

For the sake of concreteness we treat two examples for finding twisted extensions using theorem 1.2.16.

Consider the group $\mathbb{Z}$. First we look for the number of gradings of this group. The grading ϕ is fixed whenever we know $\phi(1)$, so there are two gradings in all.

We will now calculate the group $Z^2(\mathbb{Z}, \mathbb{T})/B^1(\mathbb{Z}, \mathbb{T})$. We find by eq. (1.24) that for $c \in Z^2(\mathbb{Z}, \mathbb{T})$

$$c(z, z')c(0, z) = c(0, z' + z)c(z, z'). \tag{1.87}$$

that is to say $(0, z)$ is equal for each $z \in \mathbb{Z}$. Furthermore whenever we know $c(\pm 1, z)$ for all $z \in \mathbb{Z}$, we find all $c(z', z)$ for $z' \neq 0$ by ingratiatingly applying the equality

$$c(z' \pm 1, z) = c(1, z)^{\phi(z')} c(z', z \pm 1)c(z', \pm 1)^{-1}. \tag{1.88}$$

The equality above is again obtained from eq. (1.24). The observations above imply that two cocycles $c, c' \in Z^2(\mathbb{Z}, \mathbb{T})$ are equal iff

$$c(0, 0) = c'(0, 0) \text{ and } c(\pm 1, z) = c'(\pm 1, z), \text{ for all } z \in \mathbb{Z}. \tag{1.89}$$

For some $c \in Z^2(\mathbb{Z}, \mathbb{T})$ define the function $b : \mathbb{Z} \to \mathbb{T}$ by

$$\begin{aligned} b(0) &= c(0, 0) \\ b(1) &= c(1, 1) \\ b(-1) &= c(-1, 1) \\ b(z + 1) &= b(z - 1)b(z)c(1, z)^{-1} \text{ for } z \geq 2 \\ b(z - 1) &= b(z + 1)b(z)c(-1, z)^{-1} \text{ for } z \leq -2. \end{aligned} \tag{1.90}$$

This function is defined in such a way that eq. (1.89) holds for

$$c'(z,z') = b(z)b(z')b(z+z')^{-1} \in B^1(\mathbb{Z},\mathbb{T}). \tag{1.91}$$

Therefore, $c \in B^1(\mathbb{Z},\mathbb{T})$. This holds for any cocycle, hence $Z^2(\mathbb{Z},\mathbb{T}) = B^1(\mathbb{Z},\mathbb{T})$. This means there is only one ϕ-twisted extension.

There hence exist two twisted extensions of $\mathbb{Z}$: the first is simply the direct product $\mathbb{T}\times\mathbb{Z}$, whereas the other is the semi-direct product $\mathbb{T}\times_\phi\mathbb{Z}$ with group action defined by

$$(\lambda,z)(\lambda',z') = (\lambda\lambda'^{\phi(z)}, z+z'), \tag{1.92}$$

where $\phi(z)$ equals 1 if z is even and -1 if z is odd.

Now consider the Lie group $\mathbb{R}$. This group is simply connected so we only need to calculate $Z^2(\mathbb{R},\mathbb{R})/B^1(\mathbb{R},\mathbb{R})$. We start with determining $Z^2(\mathbb{R},\mathbb{R})$. The cocycle c must be bilinear and anti-symmetric. We obtain for $X, Y \in \mathbb{R}$

$$c(X,Y) = c(X,\lambda X) = \lambda c(X,X) = 0, \tag{1.93}$$

where $\lambda\in\mathbb{R}$ is chosen such that $\lambda X = Y$. We found that $Z^2(\mathbb{R},\mathbb{R})$ is trivial. There is hence is no need to bother with $B^1(\mathbb{R},\mathbb{R})$.

There is only one twisted extension of $\mathbb{T}$, namely the trivial extension

$$\mathbb{R}\times\mathbb{T}. \tag{1.94}$$

1.2.6 Group cohomology

It is helpful to see the groups $Z^2(G,A)/B^1(G,A)$, $Z^2(\mathfrak{g},\mathfrak{a})/B^1(\mathfrak{g},\mathfrak{a})$ and $Z^2_s(G,A)/B^1_s(G,A)$ in the context of a chain complex [32]. We use the standard chain complex of Lie algebras [43] and a slightly adapted version of the standard chain complex for discrete groups in order to make it fit for twisted extensions [44].

Definition 1.2.17. *A chain complex is a sequence of Abelian groups and homomorphisms*

$$0 \xrightarrow{\delta_1} C^1 \xrightarrow{\delta_2} C^2 \xrightarrow{\delta_3} C^3 \xrightarrow{\delta_4} \cdots, \tag{1.95}$$

where for each $n\in\mathbb{N}$, $\delta_{n+1}\circ\delta_n = 0$.

Define the n-th cohomology group to be

$$H^n(G,A) = ker(\delta_{n+1})/Im(\delta_n). \tag{1.96}$$

We start with the case of a discrete group G and an Abelian Lie group A. Write $C^n(G,A)$ for the Abelian group of functions

$$\sigma : \times^n G \to A \tag{1.97}$$

under pointwise multiplication. Secondly, define boundary maps by

$$\delta_{n+1} : C^n(G,A) \to C^{n+1}(G,A) \tag{1.98}$$

$$\delta_{n+1}(\sigma)(g_1,\cdots,g_{n+1}) = \sigma(g_2,\cdots,g_{n+1})^{\phi(g_1)}. \tag{1.99}$$

$$\Pi_{i=1}^n \sigma(g_1,\cdots,g_{i-1},g_ig_{i+1},\cdots,g_{n+1})^{(-1)^i}\sigma(g_1,\cdots,g_n)^{(-1)^{n+1}}. \tag{1.100}$$

A straightforward check gives that $\delta_{n+1} \circ \delta_n = 0$ for each $n \in \mathbb{N}$.

Define the chain complex $C_s(G, A)$ in the same way, only now additionally requiring that the maps in eq. (1.97) are smooth around $e \in G$.

Secondly we turn to the case of a Lie algebras. Let $\mathfrak{g}$ and $\mathfrak{a}$ be two Lie algebras. Write $C^n(\mathfrak{g}, \mathfrak{a})$ for the Abelian group of multi-linear and anti-symmetric functions

$$\sigma : \times^n \mathfrak{g} \to \mathfrak{a}, \tag{1.101}$$

under pointwise addition. Next, define the boundary maps

$$\delta_{n+1} : C^n(\mathfrak{g}, \mathfrak{a}) \to C^{n+1}(\mathfrak{g}, \mathfrak{a}) \tag{1.102}$$

$$\delta_{n+1}(\sigma)(X_1, \cdots, X_{n+1}) = \sum_{i<j} (-1)^{i+j} \sigma([X_j, X_i], X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_{j-1}, X_{j+1}, \cdots, X_{n+1}). \tag{1.103}$$

A straight forward check gives that $\delta_{n+1} \circ \delta_n = 0$ for each $n \in \mathbb{N}$.

Corollary 1.2.18. *From the above we can conclude*

$$Z^2(G, A)/B^1(G, A) = H^1(G, A) \tag{1.104}$$

$$Z^2(\mathfrak{g}, \mathfrak{a})/B^1(\mathfrak{g}, \mathfrak{a}) = H^1(\mathfrak{g}, \mathfrak{a}) \tag{1.105}$$

$$Z_s^2(G, A)/B_s^1(G, A) = H_s^1(G, A). \tag{1.106}$$

1.3 Pull-back construction

In this section we discuss a short cut for obtaining some twisted extensions, skipping the calculation of the cohomology groups. We do this by using the pull-back construction [11].

Definition 1.3.1. *Let A and G be Lie groups, $\gamma : G \to A$ a continuous homomorphism and $\pi : A^\tau \to A$ a twisted extension. Define the pull-back G^τ to be the group*

$$\{(g, a^\tau) \in G \times A^\tau \mid \gamma(g) = \pi(a^\tau)\}, \tag{1.107}$$

with multiplication defined in each component of the cartesian product by the multiplications in A^τ and G. Write γ^τ for the projection on the second component of the cartesian product.

As one can expect from the notation G^τ is a twisted extension of G.

Lemma 1.3.2. *For a continuous homomorphism of Lie groups $\gamma : G \to A$ and a twisted extension A^τ of A, the pull-back G^τ is a twisted extension of G.*

Proof. The direct product $G \times A^\tau$ is a Lie group. Since γ and π are continuous we see by taking limits that G^τ is a closed subgroup of $G \times A^\tau$. A well known result in Lie theory ([5] theorem 9.1) states that a closed subgroup G^τ is in fact a Lie subgroup.

Furthermore, the embedding $i' : G^\tau \to G \times A^\tau$ and the projection $\pi' : G \times A^\tau \to G$ are smooth. Define $\pi : G^\tau \to G$ as the composition $\pi' \circ i'$. The kernel of this map is obviously $\mathbb{T}$. We found that G^τ is a twisted extension of G as desired. □

We can visualize the pull-back construction by the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} e & \longrightarrow & \mathbb{T} & \dashrightarrow & G^\tau & \dashrightarrow & G & \longrightarrow & e \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma^\tau} & & \downarrow{\scriptstyle\gamma} & & \downarrow \\ e & \longrightarrow & \mathbb{T} & \longrightarrow & A^\tau & \longrightarrow & A & \longrightarrow & e \end{array}, \tag{1.108}$$

where the dashed arrows and G^τ are constructed from the rest of the diagram.

Lemma 1.3.3. *Let G and A be Lie groups and let $\gamma : G \to A$ be a surjective continuous homomorphism. The map assigning to a twisted extension A^τ of A its pull back G^τ is an injective mapping from the set of twisted extensions of A to twisted extensions of G.*

Proof. Suppose that two pull-backs G^τ and $G^{\tau'}$ are isomorphic. There exists an isomorphism ψ between these twisted extensions

$$\psi : \{(g, a^\tau) \in G \times A^\tau \mid \gamma(g) = \pi(a^\tau)\} \to \{(g, a^{\tau'}) \in G \times A^{\tau'} \mid \gamma(g) = \pi'(a^{\tau'})\}. \tag{1.109}$$

Since ψ is required to fit in the diagram eq. (1.11) and since the quotient map of both G^τ and $G^{\tau'}$ onto G is simply the projection on the first component, ψ must leave the first component in the cartesian product fixed. We may recast the isomorphism by an isomorphism

$$\psi : \{a^\tau \in A^\tau \mid \exists_{g\in G}\gamma(g) = \pi(a^\tau)\} \to \{a^{\tau'} \in A^{\tau'} \mid \exists_{g\in G}\gamma(g) = \pi'(a^{\tau'})\}. \tag{1.110}$$

Finally since γ is surjective this is simply an isomorphism

$$\psi : A^\tau \to A^{\tau'}. \tag{1.111}$$

This isomorphsim fits in the diagram eq. (1.11) and hence $A^\tau \cong A^{\tau'}$. This concludes the proof of injectivity of the mapping. ☐

Note that for each normal closed subgroup N of a Lie group G, the homomorphism $\gamma : G \to G/N$ is surjective. Lemma 1.3.3 grants that the pull-back of twisted extensions of G/N, are distinct twisted extensions of G. So for every normal subgroup $N \subset G$ we can search for twisted extensions of the simpler group G/N and by pulling them back obtain some twisted extensions of G in an easier way.

For a ϕ-graded group we can apply this procedure to the normal subgroup G_1 consisting of the elements for which $\phi(g) = 1$. There are two possibilities

$$G/G_1 = \{\pm 1\} \text{ or } G/G_1 = \{1\}. \tag{1.112}$$

In case that $G/G_1 = \{1\}$ there is only the trivial extension, namely $\mathbb{T}$. In case that $G/G_1 = \{\pm 1\}$, there are two possibilities. First assume that elements in the pre-image of -1 commute with $\mathbb{T}$. In this case there is only the trivial twisted extension

$$\{\pm 1\} \times \mathbb{T}. \tag{1.113}$$

Now assume that elements in the pre-image of -1 anti-commute with $\mathbb{T}$. In this case we are left with the extension

$$\{\pm 1\} \times' \mathbb{T}, \tag{1.114}$$

where the multiplication is defined by

$$(\pm 1, \lambda) \cdot (\pm 1, \lambda') = (1, \lambda^{\pm 1}\lambda') \text{ and } (\mp 1, \lambda) \cdot (\pm 1, \lambda') = (-1, \lambda^{\pm 1}\lambda'). \tag{1.115}$$

The pull-backs of these three twisted extensions provide three distinct twisted extensions of G. This is commonly referred to as the threefold way in solid state physics [11].

1.4 Classification of twisted representations

In the previous part of this chapter we found all possible twisted extensions of a Lie group G. Now we turn our attention to finding the twisted representations of these G^τ. In order to do this, we will translate the problem to orthogonal graded real representations.

First, we introduce two graded groups of importance in this context.

Definition 1.4.1. *Fix some Hilbert space H.*

Write $Aut_{QM}(H)$ for the group of unitary and anti-unitary operators on H, equipped with the grading in eq. (1.8).

Write $H_\mathbb{R}$ for the real linear span of some basis of H. We give the $\mathbb{R}$-linear algebra $B(H_\mathbb{R} \oplus H_\mathbb{R})$ a grading by decomposing

$$B(H_\mathbb{R} \oplus H_\mathbb{R}) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \right\} \; a, b \in B(H_\mathbb{R}). \tag{1.116}$$

Write $Aut(H_\mathbb{R} \oplus H_\mathbb{R})$ for the group consisting of homogeneous orthogonal operators in $B(H_\mathbb{R} \oplus H_\mathbb{R})$.

We would like to bring a twisted representation ρ^τ in $\mathrm{Aut}_{QM}(H)$ back to a representation ρ_{gr} of G^τ in $\mathrm{Aut}(H_\mathbb{R} \oplus H_\mathbb{R})$, which resembles the usual approach for group representations. To this end we will show that $\mathrm{Aut}_{QM}(H)$ and $\mathrm{Aut}(H_\mathbb{R} \oplus H_\mathbb{R})$ are isomorphic as graded groups. For the proof we need some groundwork. First note that

$$H \cong \mathbb{C} \otimes_\mathbb{R} H_\mathbb{R} = (1 \otimes_\mathbb{R} H_\mathbb{R}) \oplus (i \otimes_\mathbb{R} H_\mathbb{R}) \cong H_\mathbb{R} \oplus H_\mathbb{R}. \tag{1.117}$$

We can decompose each $\mathbb{C}$-linear map on H as

$$a = a_1 + ia_2, \tag{1.118}$$

for $a_1, a_2 : H_\mathbb{R} \to H_\mathbb{R}$. In the same way we can write each anti-linear operator as

$$b = aC = a_1 C + ia_2 C, \tag{1.119}$$

where C is complex conjugation and a is a linear operator with decomposition $a_1 + ia_2$ as before.

Proposition 1.4.2. *The graded group $Aut_{QM}(H)$ is isomorphic to the graded group $Aut(H_\mathbb{R} \oplus H_\mathbb{R})$. Using the notation above the isomorphism in question is given by sending each unitary operator a to*

$$\psi(a) = \psi(a_1 + ia_2) = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix} \tag{1.120}$$

and each anti-unitary operator aC to

$$\psi(aC) = \psi(a_1 C + ia_2 C) = \begin{pmatrix} a_1 & a_2 \\ a_2 & -a_1 \end{pmatrix}. \tag{1.121}$$

Proof. It is an easy verification that the maps in eq. (1.120) and eq. (1.121) form a graded algebra isomorphism between $B(H_\mathbb{R} \oplus H_\mathbb{R})$ and the algebra generated by the linear and anti-linear operators. To complete the proof we must show that each unitary or anti-unitary operator is indeed sent to an orthogonal element in $B(H_{\mathbb{R}} \oplus H_{\mathbb{R}})$. It is sufficient to prove that the map ψ respects the $*$-operation. For the linear case

$$\psi((a_1 + ia_2)^*) = \psi(a_1^* - ia_2^*) = \begin{pmatrix} a_1^* & a_2^* \\ -a_2^* & a_1^* \end{pmatrix} = \psi(a_1 + ia_2)^*. \tag{1.122}$$

For the anti-linear case

$$\psi(((a_1+ia_2)C)^*) = \psi(C(a_1^*-ia_2^*)) = \psi((a_1^*+ia_2^*)C) = \begin{pmatrix} a_1^* & a_2^* \\ a_2^* & -a_1^* \end{pmatrix} = \psi((a_1+ia_2)C)^*. \tag{1.123}$$

□

It follows that we can recast a ϕ-twisted representation ρ^τ of G^τ in $\mathrm{Aut}_{QM}(H)$ as a graded representation ρ^τ_{gr} in the algebra $\mathrm{Aut}(H_{\mathbb{R}} \oplus H_{\mathbb{R}})$ simply by setting

$$\rho^\tau_{gr}(g^\tau) = \psi(\rho^\tau(g^\tau)), \tag{1.124}$$

where ψ is the isomorphism in proposition 1.4.2. The map ϕ that indicated $\rho^\tau(g^\tau)$ to be unitary or anti-unitary now describes the commutation relation with the operator

$$J = \begin{pmatrix} \emptyset & -\mathbb{I} \\ \mathbb{I} & \emptyset \end{pmatrix} \tag{1.125}$$

that plays the role of the operator i. On $H_{\mathbb{R}} \oplus H_{\mathbb{R}}$ we have

$$\rho^\tau_{gr}(g^\tau)J = \phi(g^\tau)J\rho^\tau_{gr}(g^\tau). \tag{1.126}$$

To finish, we yet need to show that this assignment respects equivalence of representations.

Lemma 1.4.3. *Two twisted representations ρ^τ and ρ'^τ of G^τ are unitary equivalent iff the representations ρ^τ_{gr} and ρ'^τ_{gr} obtained via the map ψ in proposition 1.4.2 are orthogonally equivalent.*

Proof. Whenever ρ^τ and $\rho^{'\tau}$ are unitary equivalent there exists a unitary operator u such that $\rho^\tau = u\rho'^\tau u^{-1}$. This operator u will be sent to an orthogonal $o = \psi(u)$, providing an orthogonal equivalence between ρ^τ_{gr} and ρ'^τ_{gr}.

For the other way around, assume that o is an orthogonal equivalence between the graded orthogonal representations ρ^τ_{gr} and ρ'^τ_{gr}. Since ρ^τ and ρ'^τ send $\lambda \in \mathbb{T} \subset G^\tau$ to $\lambda\mathbb{I}$, we must have

$$\rho^\tau_{gr}(\sin(\theta) + i\cos(\theta)) = \rho'^\tau_{gr}(\sin(\theta) + i\cos(\theta)) = \begin{pmatrix} \sin(\theta)\mathbb{I} & -\cos(\theta)\mathbb{I} \\ \cos(\theta)\mathbb{I} & \sin(\theta)\mathbb{I} \end{pmatrix}. \tag{1.127}$$

The orthogonal operator o should therefore commute with these matrices. An easy calculation shows that o must be of degree zero with respect to the grading in definition 1.4.1. That is to say, it corresponds to a unitary u under the isomorphism ψ. This u now provides a unitary equivalence of ρ^τ and ρ'^τ. □

All this yields to the main result of this section.

Theorem 1.4.4. *Up to orthogonal equivalence, the ϕ-twisted representations of G^τ are in bijective correspondence with ϕ-graded orthogonal representations of G^τ in $B(H_{\mathbb{R}} \oplus H_{\mathbb{R}})$ satisfying*

$$\rho^\tau_{gr}(\lambda) = \rho^\tau_{gr}(sin(\theta) + icos(\theta)) = \begin{pmatrix} \sin(\theta)\mathbb{I} & -\cos(\theta)\mathbb{I} \\ \cos(\theta)\mathbb{I} & \sin(\theta)\mathbb{I} \end{pmatrix}, \tag{1.128}$$

for $\lambda \in \mathbb{T}$.

Proof. Equation (1.124) gives a bijection between these representations of G^τ in $\mathrm{Aut}_{QM}(H)$ and $\mathrm{Aut}(H_\mathbb{R} \oplus H_\mathbb{R})$. By lemma 1.4.3 this bijection respects equivalence. Since we specify that we want a graded orthogonal representation and since $\mathrm{Aut}(H_\mathbb{R} \oplus H_\mathbb{R})$ are precisely all homogeneous orthogonal elements we can equivalently represent the group in $B(H_\mathbb{R} \oplus H_\mathbb{R})$. □

This theorem together with theorem 1.2.16 finishes our aim to characterize all quantum systems with Hilbert space H and symmetry group G.

Chapter 2

Observables

In the introduction we defined observables as projection valued measures. Projection-valued measures are in bijective correspondence with self-adjoint operators. For this reason we focus in this chapter on self-adjoint operators on a Hilbert space. Later on, we turn our attention to the correct notion of symmetry. In the end we classify the general form of bounded observables that respect the symmetry.

2.1 Self-adjoint operators

Good candidates for the observables of a quantum system would be the bounded self-adjoint linear operators on the Hilbert space H that commute with $\rho^\tau(G^\tau)$. Many important observables however turn out to be unbounded. This forces us to allow unbounded operators as well. Such an unbounded self-adjoint operator a cannot be defined on the whole Hilbert space H. We therefore require the second best thing and consider possibly unbounded linear operators with a dense domain $D(a) \subset H$.

Definition 2.1.1. *Let H be a Hilbert space. An operator a is a linear function from a dense subspace $D(a)$ of H to H. Write the set of all operators as $O(H)$. Write $B(H)$ for the subset of bounded operators.*

Due to the fact that unbounded operators are only defined on a dense domain $D(a) \subset H$, you constantly need to keep track of this domain. Even the most basic definitions are hence a bit more subtle then you might expect.

Two operators are equal iff they have equal domains and coincide on that domain. In case $D(a) \cap D(b)$ is dense, define the operator $a + b$ on this domain by $(a + b)\psi = a\psi + b\psi$. In case $\{\psi \in D(b) \mid b\psi \in D(a)\}$ is dense, define ab on this domain by $(ab)\psi = a(b\psi)$. In case $\psi \in D(a)$, define $(\lambda a)\psi$ by $\lambda(a\psi)$. Lastly, we define the quite elaborate operation of taking adjoints, [38] definition 13.1.

Definition 2.1.2. *Let a be an operator and $D(a) \subset H$ its domain. The domain $D(a^*)$ of a^* is the linear space consisting of vectors ψ such that the functional*

$$D(a) \to \mathbb{C} \tag{2.1}$$
$$\phi \mapsto \langle a\phi, \psi\rangle, \tag{2.2}$$

is bounded. That is to say, $\langle a\phi, \psi\rangle \leq C \|\phi\|$ for som $C \in \mathbb{N}$.

In case $D(a^*)$ is dense we define a^* on this domain by sending ψ to the unique ψ' for which

$$\langle a\phi, \psi\rangle = \langle \phi, \psi'\rangle \tag{2.3}$$

for all $\phi \in D(a)$. This vector ψ' exists due to the Riesz representation theorem.

An operator a is called symmetric if $D(a) \subset D(a^*)$ and $a\psi = a^*\psi$ for all $\psi \in D(a)$. In case an operator a is symmetric and $D(a) = D(a^*)$, a is called self-adjoint. Write $O_{sa}(H)$ for the set of self-adjoint operators and $B_{sa}(H)$ for the set of self-adjoint operators in $B(H)$.

The subset $B(H)$ of $O(H)$ forms a C^*-algebra under the operations above.

In this thesis our interest lies in self-adjoint operators only. Note the annoying fact that a symmetric operator might not be self-adjoint simply because its domain was chosen 'too small'. To counter this we define so-called extensions of operators, [38] definition 13.1.

Definition 2.1.3. *Let a' be some operator on a Hilbert space H. An operator a for which $D(a') \subset D(a)$ and for which a and a' coincide on $D(a')$ is called an extension of a. Denote this by $a' \subset a$. In case a is self-adjoint it is called a self-adjoint extension. Write $Ext_{sa}(a')$ for the set of all self-adjoint extensions of a'.*

Note that only symmetric operators can have self-adjoint extensions. A symmetric operator may have none or multiple self-adjoint extension. We now take a moment to study extensions of symmetric operators. There is a canonical way to define an extension of an operator, [37] page 250.

Definition 2.1.4. *An operator is called closable if the closure of its graph $\overline{G(a)} = \overline{\{(\psi, a\psi) \mid \psi \in D(a)\}}$ is the graph of an operator. The operator whose graph is given by $\overline{G(a)}$ is called the closure of a. We denote a^- for the closure of a. An operator a for which a^- exists and coincides with a is called closed.*

Keep in mind that the closure of the graph of an operator does not need to be a graph of an operator. That is to say, not every operator has a closure. Also keep in mind that an operator might have other closed extensions besides its closure. A closed extension of an operator a is always an extension of a^- as well. That is to say, the closure of a is its minimal closed extension.

Lemma 2.1.5. *The adjoint of an operator is, if it exists, closed.*

Proof. Let a be any operator. We need to show that the graph $G(a^*)$ is closed. To this end, define the unitary operator

$$u : H \times H \to H \times H \tag{2.4}$$
$$u(\psi, \phi) = (-\psi, \phi). \tag{2.5}$$

We claim that $G(a^*) = V(G(a)^\perp)$. We have $(\phi_1, \phi_2) \in VG(a)^\perp$ iff for all $\psi \in D(a)$

$$\langle(\phi_1, \phi_2), V(\psi, a\psi)\rangle = 0. \tag{2.6}$$

Rewriting this equation delivers for all $\psi \in D(a)$

$$\langle \phi_1, \psi\rangle = \langle \phi_2, a\psi\rangle. \tag{2.7}$$

This is by definition iff $\phi_1 = a^*\phi_2$, validating the claim. Since taking the orthoplement of a subspace always results in a closed subspace this finishes the proof. □

In particular self-adjoint operators are closed. The following lemma characterizes closable operators, [37] page 253.

Lemma 2.1.6. *An operator a is closable iff the adjoint a^* exists. In this case $a^- = a^{**}$.*

Proof. Let a be an operator with adjoint a^*. Furthermore, let u be the unitary operator from lemma 2.1.5 and recall that $G(a^*) = u(G(a))^\perp$. Since $D(a) \subset D(a^{**})$ we find that a^{**} exists and hence in particular

$$\overline{G(a^{**})} = u(G(a^*))^\perp. \tag{2.8}$$

The following calculation now proves the claim

$$\overline{G(a)} = G(a)^{\perp\perp} = u^2(G(a)^{\perp\perp}) = u(uG(a)^\perp)^\perp = u(G(a^*))^\perp = G(a^{**}). \tag{2.9}$$

□

For symmetric operators we have $D(a) \subset D(a^*)$, this implies that all symmetric operators are closable.

As already mentioned an operator might have none, one or multiple self-adjoint extensions. In the special case that the closure of an operator is self-adjoint, the self-adjoint extension is unique, [37] page 253.

Lemma 2.1.7. *Let a be a closable operator. If a^- is self-adjoint, it is the only self-adjoint extension of a.*

Proof. Assume a is a closable operator, whose closure is self-adjoint. Let a' be a self-adjoint extension of a. On the one hand we have

$$a^- = a^{**} \subset a'^{**} = a'^- = a'. \tag{2.10}$$

On the other hand

$$a' = a'^* \subset a^* = a^{***} = a^{-*} = a^-. \tag{2.11}$$

We found $a' = a^-$, concluding the uniqueness. □

In particular we found that a self-adjoint operator a has only one self-adjoint extension, namely a itself. We now treat examples for the cases that there is one, multiple and no self-adjoint extension of a symmetric operator.

Example 2.1.8. *For the case that there exists a unique self-adjoint extension, we use the well-known position operator in quantum physics, [18] page 17. Take $H = L^2(\mathbb{R}, \mathbb{C})$. Let $D(X')$ be the subspace the compactly supported functions in H. Define the operator X' on this domain by*

$$(X'\psi)(x) = x\psi(x). \tag{2.12}$$

Now let X be the closure of X'. We claim that X is self-adjoint. First of all we have $\psi \in D(X)$ iff there exists a sequence $\{\psi_n\} \subset D(X')$ converging to ψ for which the limit $\lim_{n\to\infty} X'\psi_n$ exists. This is in turn iff for every $\psi_n \to \psi$

$$\langle \phi, \lim_{n\to\infty} X'\psi_n \rangle = \int_{\mathbb{R}} dx \overline{\phi(x)} \lim_{n\to\infty} x\psi_n(x) = \int_{\mathbb{R}} dx x \overline{\phi(x)} \psi(x) \tag{2.13}$$

exists and coincides for every $\phi \in H$. This holds true iff

$$\phi \to \int_{\mathbb{R}} dx \overline{(X\phi)(x)} \psi(x) = \int_{\mathbb{R}} dx x \overline{\phi(x)} \psi(x) \tag{2.14}$$

is bounded on $D(X)$. We obtain that $D(X) = D(X^*)$. Finally, $X = X^*$ on this domain since for $\psi, \phi \in D(X)$

$$\int_{\mathbb{R}} dx \overline{X\psi(x)}\phi(x) = \int_{\mathbb{R}} dx\ x\overline{\psi(x)}\phi(x) = \int_{\mathbb{R}} dx \overline{\psi(x)} X\phi(x). \tag{2.15}$$

Now we turn our attention to a symmetric operator that has multiple self-adjoint extensions, [40] chapter 1. Set $H = L^2([0,1], \mathbb{C})$ and let D be the domain of infinitely differentiable compsSactly supported functions on $(0,1)$ in H. Define the operator a on this dense domain D by

$$a\psi = -\frac{d^2}{dx^2}\psi. \tag{2.16}$$

By integration by parts one can show that a is symmetric. The operator has however multiple self-adjoint extensions. Define a the domains D_D and D_N of H as functions on $[0,1]$ that satisfy Dirichlet and Neumann boundary conditions respectively

$$D_D = \{\psi \in C^\infty[0,1] \mid \psi(0) = \psi(1) = 0\} \tag{2.17}$$

$$D_N = \{\psi \in C^\infty[0,1] \mid \frac{d}{dx}\psi(0) = \frac{d}{dx}\psi(1) = 0\}. \tag{2.18}$$

The closure of $-\frac{d^2}{dx^2}$ on these two domains give rise to two distinct self-adjoint operators both extending a.

Lastly, an example of a symmetric operator having no self-adjoint extensions. Consider a Hilbert space with orthonormal basis $e_1, e_2, \cdots$. Define an alternative basis $u_1, u_2, \cdots$ by

$$u_n = e_n - e_{n+1}. \tag{2.19}$$

Let D be the domain of all finite linear combinations of the set $\{u_n \mid n \in \mathbb{N}\}$. Define the operator a on this domain by

$$au_n = ie_n + ie_{n+1}. \tag{2.20}$$

This operator a is symmetric, but has no self-adjoint extensions. See [37] page 259 for a proof.

2.1.1 Spectral decomposition

The reason that we are interested in self-adjoint operators is their bijective correspondence with projection valued measures. Recall from the introduction that these projection-valued measures play the role of observables in quantum physics. This bijective relation is granted by the spectral theorem, which forms in many aspects the core of quantum physics. Before we can state this theorem we need some preliminary notation [38] definition 12.17.

Definition 2.1.9. *A projection valued measure E on a measure space X is a function from the measurable subsets of X to projections in $B(H)$. We require $E(X) = \mathbb{I}$, $E(\emptyset) = 0$, and for all $\psi, \phi \in H$ the function*

$$M(X) \to \mathbb{R} \tag{2.21}$$

$$U \mapsto \langle \psi, E(U)\phi \rangle \tag{2.22}$$

is a measure on X. We write

$$\int_X f(x) d\langle \psi, E(x)\phi \rangle \tag{2.23}$$

for the Riemann-Stieltjes integral of the function f on X with respect to this measure.

For X some subset of $\mathbb{C}$, f an integrable function $f : X \to \mathbb{C}$ and E a projection valued measure on X, we introduce the notation

$$a = \int_X f dE(\lambda), \tag{2.24}$$

for the operator with domain

$$D(a) = \{\psi \in H \mid \int_X |f(\lambda)|^2 d\langle\psi, E(\lambda)\psi\rangle < \infty\} \tag{2.25}$$

defined through

$$\langle\phi, a\psi\rangle = \int_X f(\lambda) d\langle\phi, E(\lambda)\psi\rangle. \tag{2.26}$$

Next we generalise the notion of a spectrum to the case of unbounded operators.

Definition 2.1.10. *The spectrum $\sigma(a)$ of an operator a consists of all values $\lambda \in \mathbb{C}$ for which there does not exists a bounded operator b such that*

$$\forall_{\psi \in H} \; (a - \lambda\mathbb{I})b\psi = \psi \tag{2.27}$$

$$\forall_{\psi \in D(a)} \; b(a - \lambda\mathbb{I})\psi = \psi. \tag{2.28}$$

For later use we additionally introduce the following notation:

$$\sigma_p(a) = \{\lambda \in \sigma(a) \mid \exists_{\psi \in H} \text{ s.t. } a\psi = \lambda\psi\} \tag{2.29}$$

$$\sigma_c(a) = \sigma(a) \backslash \sigma_p(a). \tag{2.30}$$

The following is the famous spectral theorem.

Theorem 2.1.11. *The self-adjoint operators $O_{sa}(H)$ are in bijective correspondence with projection valued measures on $\mathbb{R}$. The bijection is given by*

$$E \mapsto \int_{\mathbb{R}} \lambda dE(\lambda). \tag{2.31}$$

Furthermore, for all subsets $U \subset \mathbb{R}$ for which $U \cap \sigma(a) =$ we have $E(U) = 0$.

The same goes for the unitary operators $U(H)$, only this time we need integrate over the norm one complex numbers $\mathbb{T}$ instead of $\mathbb{R}$.

Proof. See [38] theorem 12.21 for the bounded case. See theorem 13.24 and theorem 13.30 for the unbounded case. □

Base on the introduction this suggests the following definition.

Definition 2.1.12. *Let H be a Hilbert space. We call an element in $O_{sa}(H)$ an observable.*

It is important to realise that self-adjoint operators do not necessarily have an orthonormal basis of eigenvectors. The decomposition in eq. (2.31) should be seen as a generalisation of this notion, where the projection-valued measure replaces the projections on eigenspaces. As you might guess from the spectral decomposition, an operator is bounded iff its spectrum is bounded.

Proposition 2.1.13. *A self-adjoint operator a is bounded iff its spectrum $\sigma(a)$ is bounded from above and below. In fact*

$$\|a\| = sup|\sigma(a)|. \tag{2.32}$$

Proof. Let a be a self-adjoint operator. In case a is bounded the spectral radius theorem [38] theorem 10.13 states

$$\sup|\sigma(a)| = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} . \tag{2.33}$$

Since

$$\|a\|^2 = \|aa^*\| = \|a^2\| \tag{2.34}$$

$\sigma(a)$ is bounded.

On the other hand assume that $\sigma(a)$ is bounded by C. We can estimate for ψ and ϕ two norm one vectors in H

$$\int_{\sigma(a)} \lambda d\langle\phi, E(\lambda)\psi\rangle \leq C. \tag{2.35}$$

Therefore a is bounded.

Equation (2.32) follows from the above discussion if we set $\|a\| = \infty$ whenever a is unbounded. □

Now we turn our attention to unitary group representations. In case of a commutative group G with a representation ρ we can decompose all $\rho(G)$ simultaneously. This is called the SNAG theorem.

Theorem 2.1.14. *For a unitary representation ρ of a Abelian group G there exists a projection valued measure on the set of characters $\hat{G}$ such that for each $g \in G$,*

$$\rho(g) = \int_{\hat{G}} \chi(g) dE(\chi). \tag{2.36}$$

Proof. See [30] page 160. □

Lastly we generalise the notion in finite dimension that whenever two self-adjoint operators a and b commute, an eigenspace of b contains eigenspaces of a or vice versa.

Definition 2.1.15. *Given two self-adjoint operators a and b that commute and some $\lambda \in \sigma(a)$. Let U_n be some sequence of open sets in $\mathbb{R}$ for which $\cap_n U_n = \{\lambda\}$ write, $V_n \subset \sigma(b)$ be the largest possible set such that*

$$E(V_n)H \subset E(U_n)H. \tag{2.37}$$

The set $K = \cap_n V_n \subset \sigma(b)$ is set belonging to $\lambda \in \sigma(a)$.

This notion will be used in section 5.3.1 in the context of band structure.

2.1.2 Topology on $O(H)$

We now want to equip $O(H)$ with a topology. We generalise the strong topology on $B(H)$ as follows.

Definition 2.1.16. *For $a_0 \in O(H)$, $\epsilon > 0$, and $\psi \in D(a_0)$, define the set*

$$U(a_0, \psi, \epsilon) = \{a \in O(H) \mid \psi \in D(a), \|(a - a_0)\psi\| < \epsilon\}. \tag{2.38}$$

Define the topology on $O(H)$ by taking the sets

$$B = \{U(a_0, \psi, \epsilon)\} \tag{2.39}$$

as a subbase.

The topology is very weak, but exhibits some interesting properties.

Lemma 2.1.17. *The operator a is an extension of a' iff a is contained in every open neighbourhood around a'. In particular the topology in definition 2.1.16 restricted to $O_{sa}(H)$ is T_1.*

Proof. In case a is contained in every open neighbourhood of a we find that for each $\psi \in D(a)$ and each $\epsilon > 0$ that $\psi \in D(a)$ and

$$\|a\psi - a'\psi\| < \epsilon. \tag{2.40}$$

In other words a is an extension of a'. The other way around is obvious.

If we recall that a self-adjoint operator has no self-adjoint extensions other then itself, the second part of the claim follows. □

For a and a' two elements in $O(H)$ the open sets used in the proof above may intersect in case $D(a) \neq D(a')$. Limits are therefore generally not uniquely defined. A limit is only well defined if we mention the domain as well.

Proposition 2.1.18. *Two operators a and a' are equal whenever $D(a) = D(a')$ and there exists a sequence converging to both a and a'.*

Proof. Let a_n be a sequence converging to a and a'. For each $\psi \in D(a) = D(a')$ there exists for all $\epsilon > 0$ an $N \in \mathbb{N}$ such that for all $n > N$

$$\|a\psi - a_n\psi\| < \frac{\epsilon}{2} \text{ and } \|a'\psi - a_n\psi\| < \frac{\epsilon}{2}. \tag{2.41}$$

Hence $a\psi = a'\psi$ for all $\psi \in D(a) = D(a')$. □

Due to this fact the Hausdorff property breaks down in a rather manageable fashion. Next we check that the topology inherited by $B(H)$ is indeed the strong topology.

Proposition 2.1.19. *The restriction of the topology in definition 2.1.16 to $B(H)$ coincides with the strong topology.*

Proof. It is easy to see that the subbase of the strong topology is contained in the subbase of the topology in definition 2.1.16. For the other way around we show that the set

$$U(a, \psi, \epsilon) \cap B(H) \tag{2.42}$$

is contained in the subbase of the strong topology. There exists a bounded operator a_0 for which $a\psi = a_0\psi$. It follows that.

$$U(a, \psi, \epsilon) \cap B(H) = U(a_0, \psi, \epsilon) \cap B(H). \tag{2.43}$$

The latter one is contained in the subbase of the strong topology. We conclude that the respective topologies coincide. □

The topology was chosen in such a way that $B(H)$ is a dense subset in $O(H)$. It is clear that each finite intersection of sets in the subbase of the topology in definition 2.1.16 contains a bounded self-adjoint operator. We can sharpen this even further.

Proposition 2.1.20. *For each $a \in O(H)$ there exists a sequence in $B(H)$ converging to it. In case $a \in O_{sa}(H)$ there exists a sequence in $B_{sa}(H)$ converging to it.*

Proof. Given an operator $a \in O(H)$. Pick a basis $e_1, e_2, \cdots$ of $D(a)$. The operator a_N for which

$$a_N\psi = \begin{cases} a\psi \text{ if } \psi \in L\{e_1, e_2, \cdots, e_N\} \\ 0 \text{ otherwise.} \end{cases} \tag{2.44}$$

is bounded. The sequence a_N converges to a.

For the second part of the claim define for each $n \in \mathbb{N}$ the following function

$$f_n : O_{sa}(H) \to B_{sa}(H) \tag{2.45}$$

$$f_n(a) = f_n\left(\int_{\sigma(a)} \lambda dE_a(\lambda)\right) = \int_{-n}^{n} \lambda dE_a(\lambda). \tag{2.46}$$

Let a be any element in O_{sa}. We claim that the sequence defined by $a_n = f_n(a)$ converges to a. To this end we need to prove that for each finite intersection of sets in eq. (2.38) containing a there exists an $N \in \mathbb{N}$ such that a_n is contained in the intersection for all $n > N$.

Let $a_1, \cdots, a_n \in O_{sa}(H)$, $\psi_1 \in D(a_1), \cdots \psi_n \in D(a_n)$ and $\epsilon_1, \cdots, \epsilon_n > 0$ such that

$$a \in U(a_i, \psi_i, \epsilon_i). \tag{2.47}$$

We now demonstrate that there indeed exists an $N \in \mathbb{N}$ such that a_n is contained in each $U(a_i, \psi_i, \epsilon_i)$ for all $n > N$.

Pick $\epsilon = Min(\epsilon_i - \|(a - a_i)\psi_i\|) > 0$. Now pick $N \in \mathbb{N}$ such that for all $n > N$ and all i

$$\left\| \int_{\sigma(a)\setminus[-N,N]} dE_a(\lambda)\psi_i \right\| < \epsilon. \tag{2.48}$$

This implies by construction that for all $n > N$ and all ψ_i

$$\|(a - a_n)\psi_i\| < \epsilon. \tag{2.49}$$

It follows that for all $n > N$

$$\|(a_i - a_n)\psi_i\| = \|(a_i - a)\psi_i + (a - a_n)\psi_i\| \leq \|(a_i - a)\psi_i\| + \|(a - a_n)\psi_i\| < \|(a_i - a)\psi_i\| + \epsilon \leq \epsilon_i. \tag{2.50}$$

It follows that there indeed exists a sequence of bounded operators converging to a. □

The space $O_{sa}(H)$ is not Hausdorff, so if we want to grasp $O_{sa}(H)$ using convergent sequences in $B_{sa}(H)$ we still need to do some work. We proceed as follows. Let $X \subset H$ be a dense subspace and let a_n be a sequence of bounded operators such that the limit of $a_n\psi$ exists for all $\psi \in X$. Furthermore we require for each sequence ψ_m in X for which $\lim_{m\to\infty} \lim_{n\to\infty} a_n\psi_m$ exists that

$$\lim_{m\to\infty} \lim_{n\to\infty} a_n\psi_m \in X \tag{2.51}$$

and that we can interchange limits

$$\lim_{n\to\infty} \lim_{m\to\infty} a_n\psi_m = \lim_{m\to\infty} \lim_{n\to\infty} a_n\psi_m. \tag{2.52}$$

Denote such a sequence with subspace X as a pair by

$$(a_n, X). \tag{2.53}$$

Two of these tuples (a_n, X) and (b_n, Y) are called equivalent whenever $X = Y$ and for all $\psi \in X = Y$

$$\lim_{n\to\infty} a_n\psi = \lim_{n\to\infty} b_n\psi. \tag{2.54}$$

These tuples now determine $O_{sa}(H)$.

Theorem 2.1.21. *The map assigning to each tuple (a_n, X), as defined above an operator with domain X acting on H by*

$$(a_n, X)\psi = \lim_{n\to\infty} a_n\psi \tag{2.55}$$

is a well defined bijection between the tuples (a_n, X) and $O_{sa}(H)$.

Proof. First we check that the map is well defined. That is to say, we need to check that (a_n, X) is indeed sent to a linear operator. For $\psi, \psi' \in X$

$$(a_n, X)\psi+\lambda(a_n, X)\psi' = \lim_{n\to\infty} a_n\psi+\lambda \lim_{n\to\infty} a_n\psi' = \lim_{n\to\infty} a_n(\psi+\lambda\psi') = (a_n, X)(\psi+\lambda'\psi). \tag{2.56}$$

Next we need to show that (a_n, X) is self-adjoint. The domain of $(a_n, X)^*$ is given by all $\psi \in H$ for which the functional

$$X \to \mathbb{C} \tag{2.57}$$

$$\phi \mapsto \langle \lim_{n\to\infty} a_n\phi, \psi\rangle \tag{2.58}$$

is bounded. Assume that $\psi \in D((a_n, X)^*)$. In that case the above functional is bounded and hence for each sequence ψ_m in X converging to ψ

$$\lim_{m\to\infty} \lim_{n\to\infty} \langle\phi, a_n\psi_m\rangle \tag{2.59}$$

is bounded. Therefore,

$$\lim_{m\to\infty} \lim_{n\to\infty} a_n\psi_m \tag{2.60}$$

exists. This implies by construction that $\lim_{m\to\infty} \psi_m$ is contained in X. Hence $\psi \in X$. On the other hand, if $\psi \in X$, the functional in eq. (2.57) is given by

$$\phi \to \langle\phi, \lim_{n\to\infty} a_n\psi\rangle \tag{2.61}$$

and therefore bounded. As desired $X = D((a_n, X)^*)$. Since the sequence a_n consists of self-adjoint operators it follows that (a_n, X) and $(a_n, X)^*$ coincide on this domain X.

We proceed to check surjectivity. By proposition 2.1.20 there exists for each $a \in O_{sa}(H)$ a sequence of bounded operators a_n such that eq. (2.55) holds on $X = D(a)$. We need to check that $D(a)$ satisfies the requirements of X stated in the text preceding this theorem. Let ψ_m be a sequence in $D(a)$ converging to ψ such that the limit

$$\lim_{n\to\infty} \lim_{m\to\infty} a_n\psi_m = \lim_{m\to\infty} a\psi_m \tag{2.62}$$

exists. Since self-adjoint operators are closed, the limit is contained in $D(a)$. We yet need to verify that we can interchange limits. Since $\psi \in D(a)$ we find

$$\lim_{n\to\infty} \lim_{m\to\infty} a_n\psi_m = \lim_{n\to\infty} a_n \lim_{m\to\infty} \psi_m = \lim_{n\to\infty} a_n\psi = a\psi = \lim_{m\to\infty} a\psi_m = \lim_{m\to\infty} \lim_{n\to\infty} a_n\psi_m. \tag{2.63}$$

Hence we can interchange the limits. It follows that we may pick $X = D(a)$.

Lastly we check that the map is injective. Assume (a_n, X) and (b_n, Y) are sent to the same operator a. This implies $X = Y$ and that

$$\lim_{n\to\infty} a_n\psi = \lim_{n\to\infty} b_n\psi \tag{2.64}$$

for all $\psi \in X = Y$. It follows that $(a_n, X) = (b_n, Y)$. □

One can compare this theorem with the Banach Steinhaus theorem. This theorem states that for there exists for each sequence of bounded operators a_n, for which $a_n\psi$ converges for all $\psi \in H$, a bounded operator a such that

$$\lim_{n\to\infty} a_n\psi = a\psi \text{ for all } \psi \in H. \tag{2.65}$$

Due to theorem 2.1.21 we can control the set $O_{sa}(H)$ by the easier to grasp set $B_{sa}(H)$. The following example stresses that we cannot omit the domain X in the tuple.

Example 2.1.22. *Consider the Hilbert space $L^2([0,1],\mathbb{C})$. Recall that $\{e^{inx} \mid n \in \mathbb{Z}\}$ forms a basis of this Hilbert space. Define a sequence a_N of bounded operators by linearly extending the map*

$$a_N e^{inx} = \begin{cases} -n^2 e^{inx} \textit{ if } n \leq N \\ 0 \textit{ if } n > N. \end{cases} \tag{2.66}$$

This sequence converges to $-\frac{d^2}{dx^2}$. As seen in the second part of example 2.1.8 this results in distinct self-adjoint operators for the domains D_D and D_N.

For a sequence a_n of bounded self-adjoint operators, there may exist two dense domains X and Y such that eq. (2.51) and eq. (2.52) hold for both (a_n, X) and (a_n, Y).

Remark 2.1.23. *In the rest of this paper we work with the topology in definition 2.1.16. Always keep in mind that this is simply the strong topology whenever it is restricted to $B_{sa}(H)$.*

2.2 Symmetry

This section delivers on the promise made in section 1.1 to justify definition 1.1.9 and discuss the correct notion of symmetry in quantum physics.

A symmetry is a way of shuffling an initial object that leaves all relevant structure intact. Recall from the introduction that everything is built starting from a Hilbert space H. We could wonder what happened if we shuffle the elements in H using a bijective map

$$s : H \to H. \tag{2.67}$$

Note that this map s is not yet required to satisfy any properties. Such a map induces a map on $O(H)$ as follows

$$S(a, D(a)) = (sas^{-1}, sD(a)). \tag{2.68}$$

The physically important objects are the self-adjoint operators. The relevant structure is hence operations on self-adjoint operators that result in self-adjoint operators. We should be careful at this point, since many operations result in symmetric operators that are not self-adjoint, but might however have self-adjoint extensions. For this reason we make the following definition.

Definition 2.2.1. *Let H be a Hilbert space and $s : H \to H$ a bijective map. Write $O_{sym}(H)$ for the set of symmetric operators on H, which have a self-adjoint extensions. The map*

$$S : O_{sym}(H) \to O_{sym}(H) \tag{2.69}$$
$$S(a, D(a)) = (sas^{-1}, sD(a)) \tag{2.70}$$

is called a quantum symmetry if for each double sequence $(a_{ij})_{i=1,j=1}^{N,M} \in O_{sym}(H)$ and each sequence λ_i in $\mathbb{R}$, for which $\sum_{i=1}^{N} \lambda_i \prod_{j=1}^{M} a_{ij} \in O_{sym}(H)$ the following sets coincide.

$$Ext_{sa}(S(\sum_{i=1}^{N} \prod_{j=1}^{M} a_{ij})) = Ext_{sa}(\sum_{i=1}^{N} \prod_{j=1}^{M} S(a_{ij})). \tag{2.71}$$

Write $Aut_{QM}(O_{sym}(H))$ for the set consisting of all symmetries.

Note that $\mathrm{Aut}_{QM}(O_{sym}(H))$ forms a group under composition of maps. The reason that we required the map S to respect composition, addition and real scalar multiplication, is that these operations are used in physics to built new observables from old.

Example 2.2.2. *Once again consider the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$. The momentum operator is the closure of the operator $i\frac{d}{dx}$ defined on compactly supported differentiable functions. The position operator is the closure of the operator X defined on compactly supported functions by $(X\psi)(x) = x\psi(x)$. The self-adjoint operator that is the closure of*

$$h = -\frac{d}{dx}^2 + X^2 \tag{2.72}$$

is the energy operator (Hamiltonian) of a harmonic oscillator. A quantum symmetry S should shuffle the self-adjoint elements in such a way that an extension of $S(h)$ is once again the Hamiltonian of a harmonic oscillator with respect to the new momentum operator $S(i\frac{d}{dx})$ and position operator $S(X)$.

$$S(h) = -S(\frac{d}{dx})^2 + S(X)^2 \tag{2.73}$$

As one might expect, a unitary operator $u : H \to H$ induces a quantum symmetry.

Lemma 2.2.3. *Let H be a Hilbert space and u be a unitary operator. The map*

$$S : O_{sym}(H) \to O_{sym}(H) \tag{2.74}$$
$$S(a, D(a)) = (uau^{-1}, uD(a)) \tag{2.75}$$

is a symmetry.

Proof. First we show that $(ab)u^* = a(bu^*)$ for all unitary operators u. The domains of these tow operators coincide since

$$D((ab)u^*) = \{\psi \in uD(b) \mid bu^*\psi \in D(a)\} = u\{\psi \in D(b) \mid b\psi \in D(a)\} = D(a(bu^*)). \tag{2.76}$$

For $\psi \in D((ab)u^*)$ we have $\psi = u\psi'$ for some $\psi' \in D(ab)$ and so

$$(ab)u^*\psi = (ab)u^*u\psi' = (ab)\psi' = a(b\psi') = a(bu^*\psi) = a(bu^*)\psi. \tag{2.77}$$

We conclude that

$$S(ab) = u(ab)u^* = (uau^*)(ubu^*) = S(a)S(b). \tag{2.78}$$

For the addition we verify

$$D(uau^* + u\lambda bu^*) = uD(a) \cap uD(b) = u(D(a) \cap D(b)) = u(D(a + \lambda b)). \tag{2.79}$$

It is easy to see that for $\psi \in u(D(a + \lambda b))$ $u(a + b)u^*\psi = (uau^* + u\lambda bu^*)\psi$. We may conclude

$$S(a + \lambda b) = S(a) + \lambda S(b). \tag{2.80}$$

Since

$$S^{-1}(U(a_0, \psi, \epsilon)) = U(ua_0u^*, u\psi, \epsilon), \tag{2.81}$$

S is continuous.

We found that $S(\sum_{i=1}^N \prod_{j=1}^M \lambda_{i,j} a_{ij})) = \sum_{i=1}^N \prod \lambda_{i,j} S(a_{ij})$ for all double sequences $(a_{i,j})_{i=1,j=1}^{N,M}$. In particular eq. (2.71) holds and hence that S is a symmetry. □

We want to track down the whole group $\mathrm{Aut}_{QM}(O_{sym}(H))$. The map $a \mapsto uau^*$ from the lemma above forms a good starting point. Consider the map

$$a \mapsto uau^* \tag{2.82}$$

for u anti-unitary. For the same reasons as for the unitary case this map is well defined and respects all operations aside from multiplication by $\lambda \in \mathbb{C}$. In case of multiplication by $\lambda \in \mathbb{C}$ we find

$$u(\lambda a)u^* = \overline{\lambda} uau^*. \tag{2.83}$$

However, since we only require that a symmetry must respect multiplication by real scalars, we find that the map in eq. (2.82) is a quantum symmetry as well!

It is left to show that the above possibilities are in fact all possibilities. This can be shown by restricting to $B_{sa}(H)$. We proceed by showing that the symmetries $\mathrm{Aut}_{QM}(O_{sym}(H))$ can be restricted to the bounded self-adjoint operators $B_{sa}(H)$.

Lemma 2.2.4. *A restriction of a quantum symmetry S to $B_{sa}(H)$ is a continuous bijective map from $B_{sa}(H)$ to $B_{sa}(H)$ that satisfies eq.* (2.71).

Proof. We need to prove that an operator a is bounded iff $S(a)$ is bounded. Recall from proposition 2.1.13 that a is bounded iff the spectrum of a self-adjoint extension is bounded. It is therefore sufficient to show that $\sigma(a) = \sigma(S(a))$.

Let a be any bounded self-adjoint operator. If a is self-adjoint $a - \lambda\mathbb{I}$ is self adjoint and its inverse (if it exists) as well. Therefore, if $\lambda \notin \sigma(a)$ we find

$$S(a - \lambda\mathbb{I})S((a - \lambda\mathbb{I})^{-1}) = S((a - \lambda\mathbb{I})(a - \lambda\mathbb{I})^{-1}) = S(\mathbb{I}) = \mathbb{I}. \tag{2.84}$$

Since $S(a) - \lambda\mathbb{I} = S(a - \lambda\mathbb{I})$ this is iff $\lambda \notin \sigma(S(a))$. We found that $\sigma(S(a)) = \sigma(a)$, proving the claim. □

A continuous bijection $S' : B_{sa}(H) \to B_{sa}(H)$ satisfying the conditions in eq. (2.71) is called a symmetry on $B_{sa}(H)$. By lemma 2.2.4, a restriction of a symmetry S of $O_{sym}(H)$ to $B_{sa}(H)$ delivers a symmetry S' on $B_{sa}(H)$. Let's first track down the symmetries S' on $B_{sa}(H)$. The seemingly liberal definition of symmetry pins the symmetries further down than one might expect.

Theorem 2.2.5. *The group of symmetries $Aut_{QM}(B_{sa}(H))$ is isomorphic to the group of unitary and anti-unitary operators $Aut_{QM}(H)$ modded out by $\mathbb{T}$.*

In other words, the following sequence is exact:

$$1 \to \mathbb{T} \to Aut_{QM}(H) \to Aut_{QM}(B_{sa}(H)) \to 1. \tag{2.85}$$

Proof. We start out by extending S in a $\mathbb{C}$-linear fashion to $B(H)$. This map will in particular be a complex linear map satisfying

$$S(ab + ba) = S(a)S(b) + S(b)S(a), \tag{2.86}$$
$$S(a^*) = S(a)^*. \tag{2.87}$$

Such a map is called a Jordan map. All Jordan maps have the following form:

$$S(a) = uau^*, \text{ with } u \text{ unitary}, \tag{2.88}$$
$$S(a) = ua^*u^*, \text{ with } u \text{ anti-unitary}. \tag{2.89}$$

It is an easy check that maps of the above form are symmetries on $B_{sa}(H)$. To finish the proof we note that eq. (2.88) and eq. (2.89) are uniquely fixed by the unitary or anti-unitary involved, up to a scalar $\lambda \in \mathbb{T}$.

The statement that all Jordan maps are of the above form is equivalent to Wigner's theorem, see [12] for a nice proof of Wigner's Theorem. □

We found that the maps

$$u : H \to H, \tag{2.90}$$

for u either unitary or anti-unitary are the only maps inducing symmetries.

Corollary 2.2.6. *All symmetries on $O_{sym}(H)$ are of the form*

$$(a, D(a)) \mapsto (uau^*, uD(a)), \tag{2.91}$$

with u either unitary or anti-unitary. In other words the following sequence of topological groups is exact

$$1 \to \mathbb{T} \to Aut_{QM}(H) \to Aut_{QM}(O_{sym}(H)) \to 1. \tag{2.92}$$

The anti-unitary symmetries are not just a phantom of the abstract theory, but really occur in physics.

Example 2.2.7. *Time reversal on $L^2(\mathbb{R}, \mathbb{C})$ must be implemented by complex conjugation. Appendix A demonstrates why this needs to be the case.*

2.2.1 Group actions

A symmetry group G should now act on $O_{sym}(H)$ via a strongly continuous group homomorphism

$$\rho : G \to \mathrm{Aut}_{QM}(O_{sym}(H)). \tag{2.93}$$

We would now like to represent G on H. This can be achieved by a lift ρ^l of the map ρ

$$\begin{array}{ccc} & & G \\ & \overset{\rho^l}{\swarrow} & \downarrow \rho \\ \mathrm{Aut}_{QM}(H) & \longrightarrow & \mathrm{Aut}_{QM}(O_{sym}(H)). \end{array} \tag{2.94}$$

Such a lift ρ^l is, however, in general not a homomorphism, let alone a continuous one. In order to get a homomorphism to $\mathrm{Aut}_{QM}(H)$, we extend the group G by $\mathbb{T}$ and look at twisted representations ρ^τ of this twisted extensions G^τ of G. In this manner we arrive at definition 1.1.9 of a quantum system.

Theorem 2.2.8. *Each quantum system $(H, G, \phi, \tau, \rho^\tau)$ induces a continuous homomorphism*

$$G : \rho(= \pi\rho^\tau\pi^{-1}) \to Aut_{QM}(O_{sym}(H)), \tag{2.95}$$

where π is the map $\pi : G^\tau \to G$. Note that $\rho = \pi\rho^\tau\pi^{-1}$ is well defined even though $\pi^{-1}(g)$ is not unique.

The continuous homomorphisms are induced in a surjective manner. That is to say for each continuous homomorphism $\rho : G \to Aut_{QM}(O_{sym}(H))$ there exists a quantum system $(H, G, \phi, \tau, \rho^\tau)$ for which $\rho = \pi\rho^\tau\pi^{-1}$.

Proof. To prove the first part of the claim we must prove the existence of the dotted arrow in the following commuting diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{T} & \xrightarrow{\iota} & G^\tau & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ \downarrow & & \downarrow & & {\scriptstyle\rho^\tau}\downarrow & & {\scriptstyle\rho}\,\vdots & & \downarrow \\ 1 & \longrightarrow & \mathbb{T} & \xrightarrow[\iota]{} & \mathrm{Aut}_{QM}(O_{sym}(H)) & \xrightarrow[\pi]{} & \mathrm{Aut}_{QM}(H) & \longrightarrow & 1. \end{array} \tag{2.96}$$

For the second part of the claim we need to define some twisted extension G^τ of G and prove the existence of the dotted arrows in the following commuting diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{T} & \overset{\iota}{\dashrightarrow} & G^\tau & \overset{\pi}{\dashrightarrow} & G & \longrightarrow & 1 \\ \downarrow & & \downarrow & & {\scriptstyle\rho^\tau}\,\vdots & & {\scriptstyle\rho}\downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{T} & \xrightarrow[\iota]{} & \mathrm{Aut}_{QM}(H) & \xrightarrow[\pi]{} & \mathrm{Aut}_{QM}(O_{sym}(H)) & \longrightarrow & 1. \end{array} \tag{2.97}$$

Recall definition 1.3.1 and define G^τ, ι, π and ρ^τ by the pull back of $\mathrm{Aut}_{QM}(H) \to \mathrm{Aut}_{QM}(O_{sym}(H))$. By lemma 1.3.2 we find that indeed G^τ is a twisted extension and ρ^τ a twisted representation. □

Remark 2.2.9. *This justifies definition 1.1.9. A quantum system is just a way of encoding a representation $\rho : G \to Aut_{QM}(O_{sym}(H))$. That is to say, a quantum system fixes a Hilbert space and a symmetry.*

2.3 Observables

Definition 2.3.1. *A subset $X \subset O_{sa}(H)$ is said to posses a certain symmetry S whenever for each $a \in X$*

$$S(a) = a. \tag{2.98}$$

This now results in a natural definition of the observables of a quantum system.

Definition 2.3.2. *The set of observables Obs of a quantum system $(H, G, \phi, \tau, \rho^\tau)$ is the maximal subset of $O_{sa}(H)$ possessing the symmetries $\rho^\tau(G^\tau)$. That is to say Obs conists of all closed operators that commute with $\rho^\tau(G^\tau)$.*

In this respect a quantum system is just a way of picking a set of self-adjoint operators respecting a certain symmetry. The terminology 'observable' is a bit misleading. All self-adjoint operators are in principle observables, but we choose to restrict to self-adjoint operators respecting the symmetry. From now on we refer only to self-adjoint operators that respect the symmetry as an observable. This makes the term observable respective to the quantum system in consideration. The set $O_{sa}(H)$ holds a special status. The set $O_{sa}(H)$ is the set of observables if no symmetry is present. In this respect $O_{sa}(H)$ is the set of the 'actual' observables.

As an example we treat two important observables in physics [18] page 17.

Example 2.3.3. *The most common Hilbert space of a quantum system is the already used $L^2(\mathbb{R}, \mathbb{C})$ with trivial symmetry group $\{e\}$.*

The position operator X in example 2.1.8 is the observable associated to the position of a particle. Using partial integration we can byy similar means as for X show that the closure P of $i\frac{d}{dx}$ defined on the domain of compactly supported differentiable functions is self-adjoint. This operator P is the observable associated with the momentum.

Now let the symmetry group $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ act on $L^2(\mathbb{R}, \mathbb{C})$ by

$$(\pm 1 \cdot \psi)(x) = \psi(\pm 1 \cdot x). \tag{2.99}$$

The position operator X is no longer an observable of such a quantum system as it does not commute with the group action. We should replace X with $|X|$ defined by $(|X|\psi)(x) = |x|\psi(x)$. This operator does intertwine the action and is hence a good position observable for this particular quantum system.

The correct notion of isomorphism is now clear.

Definition 2.3.4. *Two quantum systems with the same Hilbert space H and observables Obs respectively Obs' are isomorphic whenever there exists $S \in Aut_{QM}(O_{sym}(H))$ such that*

$$Obs = S(Obs'). \tag{2.100}$$

The inconvenience of observables is the lack of algebraic structure. This motivates the search for something presenting more algebraic structure.

Definition 2.3.5. *The algebra of observables $A \subset B(H)$ consists of all bounded operators that intertwine $\rho^\tau(G^\tau)$.*

Note that $A_{sa} = Obs \cap B(H)$. The algebra of observables is an $\mathbb{R}$-linear algebra closed under taking adjoints. The self-adjoint elements of the algebra of observables are now precisely the bounded observables. Note that since $\rho^\tau(G^\tau)$ may act anti-unitarily, the algebra A is in general real linear and not complex linear. In case $\rho^\tau(G^\tau)$ contains solely unitary operators, the algebra of observables A forms a C^*-algebra.

Note that the C^*-algebra $B(H)$ holds a special status as the algebra of observables in case no symmetry is present. The following lemma justifies why we may restrict to bounded observables.

Proposition 2.3.6. *Write A_{sa} for the self-adjoint elements in A. With respect to the topology in definition 2.1.16,*

$$\overline{A_{sa}} \cap \{a \in O_{sa}(H) \mid \rho^\tau(G^\tau)D(a) = D(a)\} = Obs. \tag{2.101}$$

In other words if we restrict to those operators whose domain is respected by $\rho^\tau(G^\tau)$ the closure of the bounded observables are the observables.

Proof. Assume $a \in \overline{A}_{sa}$ and $\rho^\tau(G^\tau)D(a) = D(a)$. For each ψ in $D(a)$ and each $n \in \mathbb{N}$ there exists an $a_n \in A$ contained in $U(a, \psi, \frac{1}{2n}) \cap U(a, \rho^\tau(g^\tau)\psi, \frac{1}{2n})$. This implies that for all $n \in \mathbb{N}$

$$\begin{aligned} \|a\rho^\tau(g^\tau)\psi - \rho^\tau(g^\tau)a\psi\| &= \|(a\rho^\tau(g^\tau)\psi - a_n\rho^\tau(g^\tau)\psi) - (a_n\rho^\tau(g^\tau)\psi - \rho^\tau(g^\tau)a\psi)\| \\ &\leq \|a\rho^\tau(g^\tau)\psi - a_n\rho^\tau(g^\tau)\psi\| + \|\rho^\tau(g^\tau)a_n\psi - \rho^\tau(g^\tau)a\psi\| \\ &\leq \|a\rho^\tau(g^\tau)\psi - a_n\rho^\tau(g^\tau)\psi\| + \|a\psi - a\psi\| < \frac{1}{n}. \end{aligned} \tag{2.102}$$

Hence for each $\psi \in D(a)$ we have $a\rho^\tau(g^\tau)\psi = \rho^\tau(g^\tau)a\psi$ and hence $a \in Obs$.

On the other hand, assume that $a \in Obs$. Recall f_n from eq. (2.45). The sequence of bounded self-adjoint operators $f_n(a)$ converges to a. The operator $f_n(a)$ has support $E_a(-n, n)H$. The subspace $E_a(-n, n)$ is invariant for $\rho^\tau(G^\tau)$ and a subspace of $D(a)$. Furthermore, $f_n(a)\psi = a\psi$ for all $\psi \in E_a(-n, n)$. Therefore,

$$f_n(a)\rho^\tau(g^\tau)\psi = f_n(a)\rho^\tau(g^\tau)E(-n,n)\psi = a\rho^\tau(g^\tau)E(-n,n)\psi = \rho^\tau(g^\tau)aE(-n,n)\psi = \rho^\tau(g^\tau)f_n(a)\psi. \tag{2.103}$$

therefore $f_n(a) \in A$. Since $f_n(a) \to a$, we conclude $a \in \overline{A}_{sa}$. □

In the sense of theorem 2.1.21 this means that all observables are given by pairs (a_n, X), where a_n is a sequence in A_{sa} and X is a dense subspace of H for which $\rho^\tau(g^\tau)X = X$ for all $g^\tau \in G^\tau$. We obtain that the observables are determined by A_{sa}. The notion of isomorphism is therefore also determined on the bounded observables only.

Corollary 2.3.7. *Given two quantum systems with observables Obs and Obs′ respectively and algebra of observables A and A′ respectively. We find*

$$Obs = Obs' \text{ iff } A_{sa} = A'_{sa}. \tag{2.104}$$

Furthermore, for some symmetry S

$$S(Obs) = Obs' \text{ iff } S(A_{sa}) = A'_{sa}. \tag{2.105}$$

Remark 2.3.8. *This at last justifies definition 1.1.12.*

There is a lot of theory for C^*-algebras, it is therefore natural to define a C^*-algebra lying close to the bounded observables $B(Obs)$.

Definition 2.3.9. *Write $\mathbb{C}(A)$ for the C^*-algebra generated by the bounded observables.*

Let's compare $\mathbb{C}(A)$ and A.

Lemma 2.3.10. *If $\rho^\tau(G^\tau)$ consists solely of unitary operators we find that A and $\mathbb{C}(A)$ coincide.*

Proof. Obviously the complex algebra generated by the bounded observables is a subset of the algebra of observables.

To prove the converse inclusion we decompose a bounded intertwiner into two observables

$$\frac{a + a^*}{2} - i\frac{ia - ia^*}{2} = a_1 - ia_2. \tag{2.106}$$

□

This lemma no longer holds if we include anti-unitary operators in the representation. Namely, i is contained in the complex algebra generated by the observables, but it is not contained in A, for it is not an intertwiner of an anti-unitary action. One could wonder whether we could save lemma 2.3.10 by taking the real algebra generated by the bounded observables instead. The following example shows that this surprisingly enough does not hold.

Example 2.3.11. *In section 2.4 we will see that the quaternions*

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}, \end{pmatrix} \; a, b \in \mathbb{C}, \tag{2.107}$$

can be the intertwiners of some twisted representation. The self-adjoint elements are

$$\begin{pmatrix} a & 0 \\ 0 & a, \end{pmatrix} \; a, b \in \mathbb{R}. \tag{2.108}$$

The real algebra generated by the self-adjoint elements is not equal to the whole algebra of the quaternions!

The algebra of observables is hence not be uniquely determined by the observables. This is quite a disappointing result.

2.4 Classification of the observables

In this last section we want to classify the representation of the symmetry(= algebra generated by $\rho^\tau(G^\tau)$) and its commutant (= algebra of observables) in case of quantum systems with a compact symmetry group. That is to say, we want to find the most general form of both the algebra generated by $\rho^\tau(G^\tau)$ and its commutant.

We look at the problem in a slightly broader context and later specialise to the case of quantum systems.

Remark 2.4.1. *In this section we assume G to be a graded compact Lie group. Furthermore, ρ is always assumed to be a real graded representation of G on $B(H_\mathbb{R} \oplus H_\mathbb{R})$, with respect to the grading in definition 1.4.1.*

Note that we drop the requirement that the extended symmetry group must contain $\mathbb{T}$ and that the twisted extension must send $\lambda \in \mathbb{T}$ to $\lambda\mathbb{I}$. In the end we will re-impose these conditions and see what possibilities in the classification are left.

We start by introducing some preliminary notation. A division algebra is an algebra in which every non-zero element is invertible. By Frobenius theorem there are, up to isomorphism, only three division algebras over the real numbers. Namely:

$$\mathbb{R} = L\{1\} \tag{2.109}$$

$$\mathbb{C} = L\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\} \tag{2.110}$$

$$\mathbb{H} = L\left\{\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}, \right. \tag{2.111}$$
$$\left. \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}\right\},$$

where L stands for taking the $\mathbb{R}$-linear span. The division algebras are represented as real matrices, since we study real representations ρ of G. For some given representation ρ, define the following three algebras:

$$C = L\{\rho(g) \in \rho(G) \mid J\rho(g) = \rho(g)J\} \tag{2.112}$$

$$B = L\{J, \rho(g) \in \rho(G) \mid J\rho(g) = \rho(g)J\} \tag{2.113}$$

$$D = L\{J, \rho(G)\}, \tag{2.114}$$

where J is the operator in eq. (1.125). Write C', B', D' for the respective commutants of the algebras above. Note that $C \subset B \subset D$. For the special case of quantum systems D' is the algebra of observables, since in this case $J \in \rho(G)$.

The first subsection is devoted to finding the general forms of D', that is, the algebra of observables. The second subsection is devoted to finding the general forms of D and its subalgebras B and C.

In both sections the strategy will be to decompose D into irreducible components.

Definition 2.4.2. *The group algebra A with underlying group G and representation ρ is the $\mathbb{R}$-linear span of $\rho(G)$. One may take infinite converging (under the strong topology) sums.*

It is not hard to show that a group algebra is a strongly closed involutive unital sub-algebra of $B(H)$. Since ρ is a homomorphism, A is automatically closed under multiplication. The algebra A is closed under taking adjoints as well since this corresponds with inverting in G. We found that A is indeed an algebra. Lastly we need to show that the algebra is closed. Since G is compact any sequence g_n has a limit point $g \in G$. The result can be generalised for infinite sums.

Definition 2.4.3. *A group algebra A is irreducible if the only invariant subspaces of H under A are $\{0\}$ and H.*

Lemma 2.4.4. *A group algebra is irreducible iff the underlying representation is irreducible.*

Proof. It suffices to prove that a linear subspace $U \subset H$ is invariant for A iff it is invariant for the underlying representation ρ. If U is invariant for A it is in particular invariant for $\rho(G) \subset A$. On the other hand if U is invariant for $\rho(G)$ it will be a closed subspace. Therefore u is invariant for its linear span as well. □

Write G_1 for the subgroup of elements $g \in G$ for which $\rho(g)$ commutes J. The algebras D, B and C are group algebras, with groups G, $\{J, G_1\}$ and G_1. In all of the three cases the underlying group is compact and hence the algebras can be decomposed into irreducible group algebras. These irreducible algebras will be finite dimensional and have even dimension. The strategy is now to assume that D is an irreducible algebra and later to take direct sumes of these irreducible algebras together in order to obtain the general result.

2.4.1 Classification of the algebra of observables

First we look for the algebra of observables under the assumption that D is irreducible. The following proposition, which is a generalisation of Schur's lemma, describes the commutant of irreducible algebras.

Proposition 2.4.5. *Let $E \subset B(H_{\mathbb{R}} \oplus H_{\mathbb{R}})$ be an irreducible algebra, with respect to some irreducible representation ρ of G. The commutant E' of E is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (see eq.* (2.109)*).*

Proof. We claim that the commutant E' is a division algebra. Let a be a non-zero element in E'. Both $ker(a)$ and $Im(a)$ are invariant subspaces with respect to ρ. Since the representation is irreducible we find either $Im(a) = \{0\}$ or $Im(a) = H_{\mathbb{R}} \oplus H_{\mathbb{R}}$. We assumed $a \neq 0$, hence the only possibility is that $Im(a) = H_{\mathbb{R}} \oplus H_{\mathbb{R}}$. By the same reasoning $ker(a) = \{0\}$. In other words, a is invertible. This proves that E' is a division algebra. By Frobenius theorem the only division algebras over the real numbers are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. □

We now introduce some notation in order to state Weyl's theorem. Let E be an algebra. Write $I_n \times E$ for the algebra consisting of $n \times n$ block diagonal matrices with the **same** operator $a \in E$ on the diagonal. Write E_n for the algebra of $n \times n$ matrices with possibly **distinct** operators $a \in E$ in the entries. In other words

$$I_n \times E = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a \end{pmatrix} \mid a \in E \right\} \tag{2.115}$$

$$E_n = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & & \\ a_{2,1} & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1,n} \\ & & a_{n,n-1} & a_{n,n} \end{pmatrix} \mid a_{i,j} \in E \right\}. \tag{2.116}$$

We are now ready to state Weyl's theorem.

Theorem 2.4.6. *Every group algebra A can be written in terms of irreducible algebras E^i*

$$A = \oplus_i I_{n_i} \times E^i_{m_i}, \tag{2.117}$$

where $m_i, n_i \in \mathbb{N}$. Write $E^{i'}$ for the commutant of E^i. Then

$$A' = \oplus_i I_{m_i} \times E^{i'}_{n_i}. \tag{2.118}$$

Proof. see [45] theorem 3.5B. □

Together with proposition 2.4.5 this theorem provides the general form of D'.

Corollary 2.4.7. *Combining theorem 2.4.6 with proposition 2.4.5 and using the isomorphism $I_n \times E_m \cong E_m$ we find the general expression of D' we were after. Every algebra of observables is isomorphic to direct sums of matrices over the division algebra $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Write $M_{n\times n}(\mathbb{R})$ for the $n \times n$ matrices over $\mathbb{R}$. The algebra of observables has the following form:

$$\oplus_{i=1}^{\infty} \left(\mathbb{R} \otimes_{\mathbb{R}} M_{n_i\times n_i}(\mathbb{R})\right) \oplus_{i=1}^{\infty} \left(\mathbb{C} \otimes_{\mathbb{R}} M_{n_i\times n_i}(\mathbb{R})\right) \oplus_{i=1}^{\infty} \left(\mathbb{H} \otimes_{\mathbb{R}} M_{n_i\times n_i}(\mathbb{R})\right). \tag{2.119}$$

In the spirit of [7] we find that all division algebras over the real numbers are treated on equal footing in quantum physics.

We are interested in the self-adjoint ellements of this set. For $\mathbb{R} \otimes_{\mathbb{R}} M_{n\times n}(\mathbb{R})$ these are given by the symmetric matrices. For $\oplus_{i=1}^{\infty}\mathbb{C} \otimes_{\mathbb{R}} M_{n\times n}(\mathbb{R})$ these are given by the union of the symmetric matrices times 1 and the skew symmetric matrices times i. For $\oplus_{i=1}^{\infty}\mathbb{H} \otimes_{\mathbb{R}} M_{n\times n}(\mathbb{R})$, these are given by the union of the symmetric matrices in $\mathbb{H}$ times the symmetric matrices in $M_{n\times n}(\mathbb{R})$ and the skew symmetric matrices in $\mathbb{H}$ times the skew symmetric matrices in $M_{n\times n}(\mathbb{R})$.

To make the above statement relevant we of course need to show that all of the three cases above indeed occur in some physical situation.

Example 2.4.8. *Time reversal on position space $L^2(\mathbb{R}^d, \mathbb{C})$ is simply complex conjugation. The observables can therefore be represented as real infinite matrices. The irreducible subspaces are one dimensional subspaces so the restriction of the interwiners(= observables) to these subspaces is clearly $\mathbb{R}$.*

In case of space inversion, the group $\{\pm 1\}$ acts on $L^2(\mathbb{R}^d, \mathbb{C})$ as follows

$$(\pm 1 \cdot \psi)(x) = \psi(\pm x) \tag{2.120}$$

. The irreducible subspaces are simply one dimensional subspaces and the intertwiners(= observables) are obviously given by multiplication by $\mathbb{C}$.

Lastly we look at time reversal symmetry in spin space ($= \mathbb{C}^2$). Spin rotation is generated by the famous Pauli spin matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.121}$$

Time reversal T must be anti-unitary, square to $-\mathbb{I}$ and flip the spin. There is only one option left:

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} C, \tag{2.122}$$

where C stands for complex conjugation. Under the map in proposition 1.4.2, this operator is send to

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{2.123}$$

A straightforward calculation shows that the the commutant of this matrix is $\mathbb{H}$ in eq. (2.109). *Under the map in proposition 1.4.2 these operators all correspond to $\mathbb{C}$-linear operators. Hence the algebra of observables is isomorphic to the quaternions.*

This implies that all three division algebras also occur in physics.

We proceed treating the right notion of equivalence on group algebras.

Definition 2.4.9. *Let $A_1, A_2 \subset B(H_\mathbb{R} \oplus H_\mathbb{R})$ be two group algebras with group G. The algebras A_1 and A_2 are equivalent if there exists a bijective linear operator N such that the map $a \to NaN^{-1}$ is a bijection between A_1 and A_2.*

Lemma 2.4.10. *The operator N in definition 2.4.9 can be chosen orthogonal.*

Proof. Let N be any operator providing an equivalence between two algebras A_1 and A_2. We can write N as a polar decomposition $N = OP = P'O$, with O an orthogonal operator and P, P' two positive operators. Let $o \in E_1$ be some orthogonal operator and write o' for NoN^{-1}. Now rewriting the equation $No = o'N$ using the polar decomposition of N grants

$$P'Oo = o'OP. \tag{2.124}$$

Using uniqueness of the orthogonal operator in the polar decomposition, we conclude $Oo = o'O$. We find that the map

$$o \to OoO^{-1} \tag{2.125}$$

is a bijection between the orthogonal operators of A_1 and A_2. Since A_1 and A_2 are generated by their orthogonal operators this map in fact provides a bijection between A_1 and A_2. Hence N can be replaced by the orthogonal operator O. □

Note that for a group G two irreducible group algebras are equivalent if and only if the representations are orthogonally equivalent. This justifies the definition.

2.4.2 Classification of the symmetry algebra

In this section we follow [10] to find the general forms of the algebras D, B and C in eq. (2.112). Again we start with assuming irreducibility of D and use theorem 2.4.6 in the end to obtain the general result. Recall that the dimension of irreducible subspaces is finite and even.

Remark 2.4.11. *In the following we will frequently make use of the fact that the algebras $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ in eq.* (2.109) *have dimension* $1, 2$ *and* 4 *respectively.*

By proposition 2.4.5 there are three possibilities for D' in the $2n \times 2n$ matrices, namely

$$D' \cong \mathbb{I}_{2n} \times \mathbb{R}, \tag{2.126}$$
$$D' \cong \mathbb{I}_{n} \times \mathbb{C}, \tag{2.127}$$
$$D' \cong \mathbb{I}_{\frac{1}{2}n} \times \mathbb{H}, \tag{2.128}$$

where we use the notation in eq. (2.115). In case n is not even the last possibility is discarded. The algebra $D \subset B(H)$ is closed and contains $\mathbb{I}$, hence by the bicommutant theorem (see Appendix B) $D = D''$. In the three cases we find

$$D \cong \mathbb{R}_{2n} \tag{2.129}$$
$$D \cong \mathbb{C}_{n} \tag{2.130}$$
$$D \cong \mathbb{H}_{\frac{1}{2}n}, \tag{2.131}$$

where we used that $\mathbb{R}' \cong \mathbb{R}$, $\mathbb{C}' = \mathbb{C}$ and $\mathbb{H}' = \mathbb{H}$, when seen as a sub-algebra of the algebra of matrices of size respectively 1×1, 2×2 and 4×4. Again in case n is not even, the last case is dismissed.

Next we turn our attention to the sub-algebras C and B of D. In the case of B our hand turns out to be rather forced.

Lemma 2.4.12. *The dimension of B is half of the dimension of D or $B = D$.*

Proof. The subalgebra B consists of the homogeneous elements in D with degree 0. If $B \neq D$ then there exists an element $g \in G$ for which $\rho(g)$ is homogeneous of degree -1. This operator $\rho(g)$ defines a linear map on D, by multiplication. Since the algebra is graded, this map sends the elements of degree 1 to elements of degree 1. Since $\rho(g^{-1})$ is the inverse of $\rho(g)$, this map is a linear bijection between the subspaces of the homogeneous elements degree 1 and those of degree -1. We may conclude the dimension of the homogeneous elements of degree 1 and -1 to be equal. Therefore B must be half the dimension of D. □

Remark 2.4.13. *From now on assume $D \neq B$.*

The dimension of D in the cases eq. (2.129), eq. (2.130) and eq. (2.131) is $4n^2$, $2n^2$ and n^2 respectively. This fixed the dimension of B there is only one possibility in each of the three cases. Namely:

$$B \cong \mathbb{C}_n, \tag{2.132}$$
$$B \cong \mathbb{C}_{\frac{1}{2}n} \oplus \mathbb{C}_{\frac{1}{2}n}, \tag{2.133}$$
$$B \cong I_2 \times \mathbb{C}_{\frac{1}{2}n}. \tag{2.134}$$

Therefore B is fixed once we know D.

Next we turn our attention to the possibilities for the algebra C.

Lemma 2.4.14. *The dimension of the algebra C is half the dimension of the algebra B or $C = B$.*

Proof. Assume $C \neq B$. In this case $J \notin C$. For a basis $\{a_i\}$ of C the set $\{Ja_i\}$ are linearly independent operators not contained in C. Since $C \oplus JC = B$ we find that C has half the dimension of B. □

There are two group subalgebras of $\mathbb{C}_n$ that have half its dimension. Namely:

$$C = I_2 \times \mathbb{R}_n \tag{2.135}$$
$$C = \mathbb{C}_n \tag{2.136}$$
$$C = \mathbb{H}_{\frac{1}{2}n}, \tag{2.137}$$

where the last is only possible if n is even. This grants three possibilities for C in the cases of eq. (2.132) and eq. (2.134). The case eq. (2.133) is a little more subtle, since this time B consists of two copies of $\mathbb{C}_n$. The following lemma restricts the possibilities in this last case.

Lemma 2.4.15. *Let B have the form eq.* (2.133)*. The sub-algebra C will be the direct sum of two equivalent algebras.*

Proof. Let $G_1 \subset G$ be the subgroup of even elements. The algebra C is the algebra generated by elements $\rho(g)$ that commute with J, that is to say the algebra generated by $\rho(G_1)$. If we restrict G to G_1, we can, by eq. (2.133), decompose the representation

$$\rho = \rho_1 \oplus \rho_2. \tag{2.138}$$

Since $D \neq B$ we can pick an uneven element $a \in G\backslash G_1$. Introduce a representation ρ' of G by setting

$$\rho' : g \to \rho(aga^{-1}). \tag{2.139}$$

Clearly,

$$\rho'(g) = \rho(a)\rho(g)\rho(a)^{-1} = \rho(a)\rho_1(g)\rho(a)^{-1} \oplus \rho(a)\rho_2(g)\rho(a)^{-1}. \tag{2.140}$$

On the other hand,

$$\rho'(g) = \rho(aga^{-1}) = \rho_1(aga^{-1}) \oplus \rho_2(aga^{-1}) = \rho_1'(g) \oplus \rho_2'(g). \tag{2.141}$$

We find that either

$$\rho_1' = \rho(a)\rho_1\rho(a^{-1}) \text{ and } \rho_2' = \rho(a)\rho_2\rho(a^{-1}), \tag{2.142}$$

or

$$\rho_1' = \rho(a)\rho_2\rho(a^{-1}) \text{ and } \rho_2' = \rho(a)\rho_1\rho(a^{-1}). \tag{2.143}$$

We now show that eq. (2.142) is not possible, leaving us with eq. (2.143). For the sake of contradiction assume that indeed

$$\rho_1(aua^{-1}) = \rho(a)\rho_1(u)\rho(a)^{-1}, \tag{2.144}$$

for all $u \in G_1$. In this case we find for each $a' \in G\backslash G_1$ and $u \in G_1$ that

$$\begin{aligned} \rho_1(a'ua'^{-1}) &= \rho_1(aa^{-1}a'ua'^{-1}aa^{-1}) = \rho(a)\rho_1((a^{-1}a')u(a'^{-1}a))\rho(a)^{-1} \\ &= \rho(a)\rho(a^{-1}a')\rho_1(u)\rho(a'^{-1}a)\rho(a)^{-1} = \rho(a')\rho_1(u)\rho(a'^{-1}), \end{aligned} \tag{2.145}$$

where, in the third equality, we used that $a^{-1}a'$ is unitary. So if eq. (2.144) holds for one $a \in G^\tau\backslash G_1$, it holds for all $a \in G\backslash G_1$. The operator $\rho_1(e)$ is now an intertwiner of ρ. First we check for all $a \in G\backslash G_1$

$$\rho_1(e)\rho(a) = \rho(a)\rho_1(e)\rho(a)^{-1}\rho(a) = \rho(a)\rho_1(e). \tag{2.146}$$

Likewise for $u \in G_1$,

$$\rho_1(e)\rho(u) = (\rho_1(e) \oplus 0)(\rho_1(u) \oplus \rho_2(u)) = (\rho_1(u) \oplus \rho_2(u))(\rho_1(e) \oplus 0) = \rho(u)\rho_1(e). \tag{2.147}$$

It follows that $\rho_1(e)$ projects onto a non-trivial invariant subspace V_1 with respect to ρ. This contradicts that D and hence ρ should be irreducible. Therefore eq. (2.143) must hold, this immediately implies that $\rho_1 \sim \rho_2$ which proves the claim. □

The lemma above provides only four possibilities for C if B is of the form (2.133). Namely, either C is $I_2 \times \mathbb{C}_n$, or C is two copies of either (2.135), (2.136), or (2.137).

For the case that $D \neq B$ we can label the possibilities by

$$\mathbb{RR}, \mathbb{RC}, \mathbb{RH}, \mathbb{CR}, \mathbb{CC}1, \mathbb{CC}2, \mathbb{CH}, \mathbb{HR}, \mathbb{HC}, \mathbb{HH}, \tag{2.148}$$

where the first symbol indicates whether the algebra D is of the form eq. (2.129), eq. (2.130) or eq. (2.131). This label fixes B as well. The second symbol indicates C to be of the form eq. (2.135), eq. (2.136) or eq. (2.137). For $\mathbb{CC}$ we need an additional symbol $1, 2$ to specify whether C is isomorphic to $\mathbb{C}_n \oplus \mathbb{C}_n$ (case 1) or $I_2 \times \mathbb{C}_n$ (case 2).

If we now assume $B = D$, the representation ρ of G commutes with J and may hence be interpreted as a complex representation. By Schur's lemma D' must be isomorphic to $I_n \times \mathbb{C}$, hence fixing $D = B \cong \mathbb{C}_n$. The algebra C is simply

$$D/J \cong \mathbb{R}_n. \tag{2.149}$$

Under the map in proposition 1.4.2 this is the case $\mathbb{CR}$.

Theorem 2.4.16. *We can classify irreducible group algebras by the symbols*

$$\mathbb{RR}, \mathbb{RC}, \mathbb{RH}, \mathbb{CR}, \mathbb{CC}1, \mathbb{CC}2, \mathbb{CH}, \mathbb{HR}, \mathbb{HC}, \mathbb{HH} \tag{2.150}$$

indicating the general form of D and its subalgebras B and C. These irreducible subalgebras make up all other algebras in the way of theorem 2.4.6.

Corollary 2.4.17. *In the case of quantum systems the group is a twisted extension of some symmetry group. We must have*

$$\rho^{\tau}(i) = J. \tag{2.151}$$

This implies that $B = C$. There are four possibilities in all, namely

$$\mathbb{RC}, \mathbb{CR}, \mathbb{CC}1, \mathbb{HC}. \tag{2.152}$$

Using theorem 2.4.6 we can generalize to non-irreducible group algebras D.

Again it remains to show that all of these possibilities occur.

Example 2.4.18. *All examples are given by Clifford algebras. (see Appendix D.) The inclusion*

Table 2.1: Examples for C and D.

Symbol	D	C
$\mathbb{R}\mathbb{C}$	$\mathrm{Cliff}^{0,8}$	$\mathrm{Cliff}^{0,7}$
$\mathbb{C}\mathbb{R}$	$\mathrm{Cliff}^{0,3}$	$\mathrm{Cliff}^{0,2}$
$\mathbb{C}\mathbb{C}1$	$\mathbb{C}\mathrm{liff}^{2}$	$\mathbb{C}\mathrm{liff}^{1}$
$\mathbb{H}\mathbb{C}$	$\mathrm{Cliff}^{0,4}$	$\mathrm{Cliff}^{0,3}$

$C \subset D$ is obtained by leaving out one negative generator from the set of generators in the Cliford algebra D. This negative generator corresponds with the operator J.

Chapter 3

States

In this chapter we define and classify the equivalence classes of states upon restriction to the observables of some quantum system. The approach is through the familiar notion of a state on a C^*-algebra. Later on we restrict to so-called normal states. Some attention is also paid to the physical meaning of the mathematical construction.

Remark 3.0.19. *In this chapter a vector $\psi \in H$ is always assumed to be a unit vector. We write e_ψ and e_U for the orthogonal projection on $\mathbb{C} \cdot \psi$ and on the closed subspace U respectively.*

3.1 States

Recall that $B_{sa}(H)$ is the set of bounded observables if no symmetry is present. It therefore makes sense to define states on $B_{sa}(H)$.

Definition 3.1.1. *A state on $B_{sa}(H)$ is a continuous $\mathbb{R}$-linear functional satisfying*

$$\omega(\mathbb{I}) = 1 \tag{3.1}$$
$$\omega(a^2) \geq 0. \tag{3.2}$$

Equip the states with the weak-$*$ topology.

Each operator in $B(H)$ can be written as a sum of two self-adjoint elements in the way of eq. (2.106). Therefore we can $\mathbb{C}$-linearly extend the states to $B(H)$. A sate is in this respect a bounded $\mathbb{C}$-linear functional on $B(H)$ such that

$$\omega(\mathbb{I}) = 1 \tag{3.3}$$
$$\omega(aa^*) \geq 0. \tag{3.4}$$

The equivalence relies on the fact, [34] theorem 2.2.5, that

$$\{aa^* \mid a \in B(H)\} = \{a^2 \mid a \in B_{sa}(H)\}. \tag{3.5}$$

A bounded state with the requirements in eq. (3.3) is the usual definition of a state on a C^*-algebra, [34] page 89.

In a quantum system we impose a symmetry constraint on $B_{sa}(H)$. The following definition accommodates this restriction.

Definition 3.1.2. *Recall that $A_{sa} \subset B(H)$ is the algebra of observables of some quantumsystem. Two states on $B(H)$ are equivalent with respect to this quantum system, whenever the restriction of these states to A_{sa} coincide.*

Write $\mathfrak{S}(A)$ for the set of equivalence classes of sates. Equip $\mathfrak{S}(A)$ with the quotient topology. We refer to an equivalence class in $\mathfrak{S}(A)$ as a phase.

Remark 3.1.3. *The phases of states of a certain quantum system should be thought of in terms of degeneracy. Under a symmetry $\rho^\tau(G^\tau)$ the observables $B(Obs)$ can separate fewer sates then $B_{sa}(H)$. Two states of $B(H)$ contained in the same phase cannot be distinguished by the observables of the quantum system in consideration.*

Now break the symmetry, that is, restrict to a subgroup $G'^\tau \subset G^\tau$. We find $\rho^\tau(G'^\tau) \subset \rho^\tau(G^\tau)$. The set of observalbes $B(Obs')$ will grow. Therefore, the number of states that can be separated on this domain grows as well. We obtain a finer partition into equivalence classes and with that more phases. In case we break all symmetries the observables are $B_{sa}(H)$, this set separates all states by definition,making the phases coincide with the states.

The states on $B(H)$ endowed with the weak-$*$ topology form a very manageable space.

Theorem 3.1.4. *The states on $B(H)$ form a convex set that is compact with respect to the weak-$*$ topology.*

Proof. It is elementary to show that the space is convex and closed. Applying the Banach Alaoglu theorem, [38] theorem 3.15, implies that the closed unit ball is in fact compact. □

Due to the convexity of the set we can identify extreme points and make the following definition, [34] page 89.

Definition 3.1.5. *The extreme points of the states on $B(H)$ are called pure states.*

The Krein-Milman theorem guarantees that the pure states completely determine the set of states.

Proposition 3.1.6. *The set states is the closure of the convex hull of the pure states.*

Proof. See [38] theorem 3.21 for a proof. □

Since the set $\mathfrak{S}(A)$ consists of equivalence classes of states we obtain the same results.

Corollary 3.1.7. *Let A be an algebra of observables of some quantum system. The set $\mathfrak{S}(A)$ forms a compact convex set. The closure of the convex hull of the extreme points in $\mathfrak{S}(A)$ equals $\mathfrak{S}(A)$.*

Note that a non-pure element in $B(H)$ may fall into a pure equivalence class in $\mathfrak{S}(A)$. Likewise, a pure element in $B(H)$ may represent a non-pure element in $\mathfrak{S}(A)$. However, in finite dimension each pure phase in $\mathfrak{S}(A)$ can be represented by a pure state.

Lemma 3.1.8. *Assume that the quantum system in question has a finite dimensional Hilbert space. If a phase is pure it can be represented by a pure state.*

Proof. Assume that the equivalence class of the phase in question would not contain a pure state. Pick any representant from the phase and write it as a convex combination of pure states. These pure state do not fall in the same phase and hence the phase in $\mathfrak{S}(A)$ is a convex combination of some other phases. It follows that the phase in $\mathfrak{S}(A)$ is not pure. □

We now select a special subset of the states on $B(H)$. A density operator is an operator satisfying

$$\mathrm{Tr}(\mathfrak{r}) = 1 \tag{3.6}$$
$$\mathfrak{r} \geq 0. \tag{3.7}$$

We use this notion to define a subset of the states.

Definition 3.1.9. *A state is called normal if there exists a density operator such that*

$$\omega_{\mathfrak{r}}(a) = Tr(\mathfrak{r}a) \tag{3.8}$$

for all $a \in B(H)$. A state is called a vector state whenever it can be written as

$$\omega_\psi(a) = \langle \psi, a\psi \rangle. \tag{3.9}$$

For H finite dimensional all states are normal. However if the dimension of the Hilbert space is infinite there exist non-normal states.

Remark 3.1.10. *When we refer to $\mathfrak{r}$ or ψ as a state we mean the associated functionals*

$$\omega_{\mathfrak{r}}(a) = Tr(\mathfrak{r}a) \text{ or } \omega_\psi(a) = \langle \psi, a\psi \rangle \tag{3.10}$$

respectively.

These normal sates form a subset of the states, which is once again convex. The pure states of this convex set are the vector states.

Lemma 3.1.11. *A state ω on $B(H)$ is both normal and an extreme point of the set of normal states iff ω is normal and an extreme point of the set of all states iff ω is a vector state.*

Proof. By construction a state is both normal and a pure state of the normal states iff it is a vector state e_ψ.

Assume that ω_ψ can be written as some convex combination

$$\omega_\psi = t\omega + (1-t)\omega'. \tag{3.11}$$

Evaluating this states on $e_\psi \in B(H)$, we find

$$\omega_\psi(e_\psi) = 1 = t\omega(e_\psi) + (1-t)\omega'(e_\psi). \tag{3.12}$$

This implies $\omega'(e_\psi) = \omega(e_\psi) = 1$. By [17] section 4, the evaluation of a non-normal state on a finite dimensional representation always yields 0. This now implies that ω and ω' are normal. Therefore, a vector state should be a convex combination of normal states. Since e_ψ is pure with respect to the normal states we find that $\omega_\psi = \omega = \omega'$. We conclude that a state is a vectors state iff it is normal and pure. □

A phase in $\mathfrak{S}(A)$ is normal if it contains a normal state. A phase is non-normal if it does not contain any normal states. Once again the above results hold for the phases as well.

Corollary 3.1.12. *The normal phases $N(\mathfrak{S})(A)$ form a convex set, whose extreme points can be represented by vector states. The extreme points of $N(\mathfrak{S}(A))$ are extreme points of the whole of $\mathfrak{S}(A)$.*

In case of normal pure states we can generalize lemma 3.1.8 to infinite dimension.

Lemma 3.1.13. *A phase is pure and normal if it contains a pure state. In particular this pure state is a vector state.*

Proof. Assume that a pure normal phase ω does not contain any pure states. The phase is assumed normal, hence we can pick a normal state ω contained in this equivalence class. To this state ω belongs a density operator that can be diagonalized. We can hence write ω as a convex combination of vector states. These vector states are by assumption not contained in the class. The phase ω is hence a convex combination of the phases belonging to the vector states. This implies that the state ω is not pure. We may conclude that a pure and normal phase contains a pure state. □

3.1.1 Probability measure

Outcomes should be real numbers. We should therefore establish a probability distribution on $\mathbb{R}$ based upon an observable a and a state ω. Before we can define the so called Born measure we need some ground work.

A state ω is sequentially strongly continuous if $\omega(a_n) \to \omega(a)$ whenever $a_n \to a$ in the strong topology. All states are bounded and hence continuous with respect to the norm topology. The stronger notion of sequentially strong continuity however does not hold in general.

Proposition 3.1.14. *A normal phase in $\mathfrak{S}(A)$ is sequentially strong continuous on A_{sa}.*

Proof. A normal phase in $\mathfrak{S}(A)$ can be represented by a normal state. Normal states are convex combinations of vector states, we hence first prove the statement for the vector states. This is easily proven since for $a_n \to a$ strongly we have $a_n\psi \to a\psi$ for all $\psi \in H$. In particular

$$\omega_\psi(a_n) = \langle \psi, a_n\psi\rangle \to \langle \psi, a\psi\rangle = \omega_\psi(a). \tag{3.13}$$

Now let ω be any normal state and let a_n again be a sequence converging strongly to a. We can write $\omega = \sum_{m=0}^{\infty} \lambda_m \omega_{\psi_m}$. Since a_n converges strongly the set $\{a_n\psi \mid \psi \in H\}$ is bounded for all $\psi \in H$. Recall the uniform boundedness principle [38] theorem 2.6 to state that there exists $C \in \mathbb{N}$ such that $\|a_n\| \leq C$ for all $n \in \mathbb{N}$. This implies that $|\omega_{\psi_m}(a_n)| \leq \|\omega_{\psi_m}\| \, \|a_n\| \leq C$. By using dominated convergence in the second step and eq. (3.13) in the third we obtain

$$\lim_{n\to\infty} \omega(a_n) = \lim_{n\to\infty} \sum_{m=0}^{\infty} \lambda_m \omega_{\psi_m}(a_n) = \sum_{m=0}^{\infty} \lim_{n\to\infty} \lambda_m \omega_{\psi_m}(a_n) = \sum_{m=0}^{\infty} \lambda_m \omega_{\psi_m}(a) = \omega(a). \tag{3.14}$$

□

Using the above we can define a probability measure on the spectrum of an observable in case of normal states.

Lemma 3.1.15. *Let ω be a representative of a normal phase in $\mathfrak{S}(A)$ of some quantum system. For $a \in Obs$ any (possibly unbounded) observable with projection valued measure E (see theorem 2.1.11), the function assigning to each measurable set $X \subset \sigma(a)$*

$$X \to \omega(E(X)) \tag{3.15}$$

is a probability measure on $\sigma(a) \subset \mathbb{R}$.

Proof. We need to check the three axioms for a probability measure.

First we verify that the measure of the total space is 1 and that the measure of the empty set is 0:

$$\omega(E(\sigma(a))) = \omega(\mathbb{I}) = 1 \tag{3.16}$$

$$\omega(E(\emptyset)) = \omega(0) = 0. \tag{3.17}$$

The measure is positive since ω is positive.

For $\{U_n\}$ a sequence of mutually disjoint measurable sets

$$\omega(E(\sqcup_{n=1}^{\infty} U_n)) = \omega(\sum_{n=1}^{\infty} E(U_n)) = \omega(\lim_{N\to\infty} \sum_{n=1}^{N} E(U_n)) = \lim_{N\to\infty} \omega(\sum_{n=1}^{N} E(U_n)) = \sum_{n=1}^{\infty} \omega(E(U_n)), \tag{3.18}$$

where we used in the third equality that ω is normal and hence sequentially strong continuous. □

Note that the proof of countable additivity relies on the state being normal. The above lemma demonstrates how the spectral theorem, the self-adjoint operators and the states neatly collaborate.

Remark 3.1.16. *Definition 3.1.17 justifies why we restricted observables to self-adjoint elements and states to functionals satisfying eq.* (3.1)*.*

The assumption that a is self-adjoint is equivalent to the assumption that a has a projection-valued measure and real spectrum. In turn, the requirements in eq. (3.1) *make sure that the expression eq.* (3.15) *is a probability measure.*

Definition 3.1.17. *Let a be an observable with projection valued measure E and let ω be a normal state. Write $\omega(E)$ for the probability measure on $\mathbb{R}$ in eq.* (3.15)*.*

We could wonder whether it is possible to find a probability measure for non-normal states as well. The following lemma shows that the naive extension of definition 3.1.17 does not work.

Lemma 3.1.18. *For a quantum system with compact group G and for each representative ω of a non-normal phase in $\mathfrak{S}(A)$, there exists an observable a, with projection-valued measure E such that $\omega(E)$ is not a probability measure on $\sigma(a)$.*

Proof. First we show that countable additivity breaks down in the case of a pure non-normal state. There exists a non-normal state hence the dimension of H will be infinite. Write $H = \oplus_{n=1}^{\infty} U_n$ for a choice of mutually perpendicular irreducible subspaces. Define the (unbounded) observable a' on the dense domain consisting of finite linear combinations of elements in the subspaces U_n by

$$a' = \sum_{n=1}^{\infty} n e_{U_n}. \tag{3.19}$$

Now let a be the closure of a'. The operator a is self-adjoint and commutes with $\rho^{\tau}(G^{\tau})$ by construction. We have $\sigma(a) = \mathbb{N}$. On the one hand

$$\omega((E)(\mathbb{N})) = \omega(E(\sigma(a))) = \omega(\mathbb{I}) = 1. \tag{3.20}$$

On the other hand, since U_n is finite dimensional we have by a result from Glimm [17] section 4, that $\omega(e_{U_n}) = 0$ for all $n \in \mathbb{N}$. This implies

$$\sum_{n=1}^{\infty} \omega(E)(n) = \sum_{n=1}^{\infty} \omega(e_{U_n}) = \sum_{n=1}^{\infty} 0 = 0. \tag{3.21}$$

If $\omega(E)$ would be a measure it should be countably additive

$$1 = \omega(E)(\mathbb{N}) = \omega(E)(\sqcup_{n=1}^{\infty} n) = \sum_{n=0}^{\infty} \omega(E)(n) = 0. \tag{3.22}$$

This proves the failure of the countable additivity in the case of non-normal pure states.

In case ω is non-normal it contains at least one non-normal state ω' in its convex decomposition. We find

$$\sum_{n=0}^{\infty} \omega(e_{U_n}) = \sum_{n=0}^{\infty} \left(\lambda' \omega'(e_{U_n}) + \sum_{m=1}^{\infty} \lambda_m \omega_m(e_{U_n}) \right) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \lambda_m \omega_m(e_{U_i}) \leq \sum_{m=1}^{\infty} \lambda_m \tag{3.23}$$

$$= 1 - \lambda' < 1. \tag{3.24}$$

Since once again

$$\omega(\sum_{n=1}^{\infty} e_{U_n}) = \omega(\mathbb{I}) = 1 \tag{3.25}$$

this yields a contradiction. □

Assume that the assignment of probability measures to sates is sequentially continuous, we proceed to show that a probability measure for a pure non-normal phase of states cannot be found.

Proposition 3.1.19. *The normal states are weak$-*$ dense in the states.*

Proof. The algebra $B(H)$ is a unital Banach algebra whose pre-dual is the unital Banach algebra consisting of the trace phase operators. This means that we can apply [31] theorem 2.2 and state that the normal states are dense in $B(H)$ with respect to the weak$-*$ topology. □

The above proposition also implies that $N(\mathfrak{S}(A))$ lies dense in $\mathfrak{S}(A)$. Recall that a sequence of measures μ_n converges in the strong topology if

$$\mu_n \to \mu \iff \forall_{X \subset \sigma(a)} \mu_n(X) \to \mu(X). \tag{3.26}$$

We now postulate that the assignment

$$\omega \to \mu_\omega, \tag{3.27}$$

must be at least sequentially continuous, where we look at the measures under the strong topology and at the states under the weak$-*$ topology. This condition now implies that no measure for a non-normal state ω on $\sigma(a)$ can be given.

Proposition 3.1.20. *If the map in eq.* (3.27) *is sequentially continuous and we assign to each normal phase the probability measure in definition 3.1.17 we cannot continuously extend the map to non-normal phases of $\mathfrak{S}(A)$.*

Proof. By proposition 3.1.19, the normal states lie dense in the states. So for each non-normal state ω there is a sequence of normal states ω_n converging to it in the weak$-*$ topology. We find that for each measurable $X \subset \sigma(a)$

$$\lim_{n\to\infty} \omega_n(E(X)) = \omega(E(X)), \tag{3.28}$$

whence

$$\mu_\omega(X) = \lim_{n\to\infty} \mu_{\omega_n}(X) = \lim_{n\to\infty} \omega_n(E(X)) = \omega(E(X)). \tag{3.29}$$

By lemma 3.1.18 the right hand side of this equation is not a measure for all elements in *Obs*. This proves that the assignment in definition 3.1.17 cannot be extended to non-normal states. □

However, to obtain a sensible physical theory a state should induce a probability measure on $\mathbb{R}$ for any $a \in Obs$. For this reason we throw out the non-normal phases in $\mathfrak{S}(A)$.

Definition 3.1.21. *A quantum state is a normal sate on $B(H)$: we restrict $\mathfrak{S}(A)$ to $N(\mathfrak{S}(A))$.*

Remark 3.1.22. *Note that the states with which we are left are precisely the sates we used in the introduction.*

Let E be the projection valued measure of a. By construction in definition 2.1.9,

$$\lim_{N\to\infty} \sum_{z=-N}^{N} \frac{z}{n} \cdot E([\frac{z}{n}, \frac{z+1}{n}))= a \tag{3.30}$$

for the topology in definition 2.1.16. In case ω is normal we find the following elegant expression for the expectation value of a with respect to ω

$$\begin{aligned}\mathbb{E}_\omega(a) &= \int_{\sigma(a)} \lambda \, d\omega(E(\lambda)) = \lim_{n\to\infty} \sum_{z=-\infty}^{\infty} \frac{z}{n} \cdot \omega(E([\frac{z}{n}, \frac{z+1}{n}))) \\ &= \omega(\lim_{n\to\infty} \sum_{z=-\infty}^{\infty} \frac{z}{n} \cdot E([\frac{z}{n}, \frac{z+1}{n})) = \omega(a),\end{aligned} \tag{3.31}$$

where we used for the second equality that the integral is a Riemann-Stieltjes integral and in the third and fourth equality that ω is sequentially strong continuous (by proposition 3.1.14). For unbounded operators we use the notation ∞ in case the integral is not defined. For the variance we find

$$\begin{aligned}\mathrm{Var}_\omega(a) &= \int_{\sigma(a)} \lambda^2 \, d\omega E(\lambda) - \mathbb{E}(a)^2 = \lim_{n\to\infty} \sum_{z=-\infty}^{\infty} (\frac{z}{n})^2 \cdot \omega(E([\frac{z}{n}, \frac{z+1}{n})) - \omega(a)^2 \\ &= \omega(a^2) - \omega(a)^2.\end{aligned} \tag{3.32}$$

In case of non-normal states we can still postulate that $\omega(a)$ is the expectation value and that $\omega(a^2) - \omega(a)^2$ is the variance. The only thing left to check is that the variance is indeed positive.

Lemma 3.1.23. *For each state ω and each observable a*

$$0 \leq \omega(a^2) - \omega(a)^2 \tag{3.33}$$

Proof. By eq. (3.1)

$$0 \leq \omega((a - \omega(a)\mathbb{I})^2) = \omega(a^2 + \omega(a)^2\mathbb{I} - 2a\omega(a)\mathbb{I}) = \omega(a^2) - \omega(a)^2. \tag{3.34}$$

□

However in order to get a sensible physical theory we need a probability measure on $\mathbb{R}$ and not only an expectation value.

3.1.2 State collapse

Thus far we achieved for each state ω and each observable a a probability measure on $\mathbb{R}$. At some point a measurement will be made and a certain subset $X \subset \sigma(a)$ will be realised. At this point a rather unexpected move enters the scene. When the observable a is measured and a subset of outcomes X is realised, the initial state will collapse to a certain other state. We use Lüders rule [28] page 2.

Definition 3.1.24. *Let a be an observable, ω a state and $X \subset \sigma(a)$ a measurable set for which $\omega(E(X)) \neq 0$. Define the X-collapse to be the state*

$$\omega'(a) = \frac{1}{\omega(E(X))}\omega(E(X)aE(X)). \tag{3.35}$$

It remains to show that the above state is well defined.

Lemma 3.1.25. *Equation* (3.35) *defines a state.*

Proof. We need to prove that eq. (3.1) holds. First we check

$$\omega'(\mathbb{I}) = \frac{1}{E(X)}\omega(E(X)\mathbb{I}E(X)) = \frac{1}{\omega(E(X))}\omega(E(X)^2) = 1. \tag{3.36}$$

Secondly, we check

$$\omega'(aa^*) = \frac{1}{\omega(E(X))}\omega(E(X)aa^*E(X)) = \frac{1}{\omega(E(X))}\omega(E(X)a)(E(X)a)^*) \geq 0. \tag{3.37}$$

We conclude that ω' is a state. □

Remark 3.1.26. *The above is the 'measuring is influencing' result from quantum physics. Whenever a measurement is made the state collapses accordingly.*

Let ω be a state and a an observable. Let ω' be the X-collapsed state of ω under a. An interesting observation is that the X-collapse of ω' is once again ω'

$$\frac{1}{\omega'(E(X))}\omega'(E(X)aE(X)) = \frac{1}{\omega(E(X)E(X)E(X))}\omega(E(X)^2aE(X)^2) \tag{3.38}$$

$$= \frac{1}{\omega(E(X))}\omega(E(X)aE(X)) = \omega'(a). \tag{3.39}$$

Physically this means that whenever we measure a for the second time it realises the previous result with certainty.

3.1.3 Continuous spectrum and point spectrum

The subset $\sigma_p(a) \subset \mathbb{R}$ has a special status.

Proposition 3.1.27. *Given an observable a, there exists a state ω for which the variance is* 0 *and the expectation value is λ_0 iff $\lambda_0 \in \sigma_p(a)$.*

Proof. First we establish the following fact. A probability measure μ on $\mathbb{R}$ is either a point measure at λ_0 or it contain an open subset U around λ_0 such that $\mu(U^c) > 0$. In order to prove this assume that there would not exists an open U around λ_0 such that $\mu(U^c) > 0$. In particular, we find $\mu((\lambda_0 - \frac{1}{n}, \lambda_0 + \frac{1}{n})^c) = 0$. This implies

$$\mu(\{\lambda_0\}^c) = \mu(\cup_{n=1}^{\infty}(\lambda_0 - \frac{1}{n}, \lambda_0 + \frac{1}{n})^c) \leq \sum_{n=0}^{\infty} \mu(\lambda_0 - \frac{1}{n}, \lambda_0 + \frac{1}{n})^c) = 0. \tag{3.40}$$

Hence $\mu(\{\lambda_0\}) = 1$. It follows that μ is a point measure at λ_0.

Applying the above result to the case at hand, we find that if $E(\omega)$ is not a point measure that there exists a neighbourhood U around λ_0 such that $E(\omega)(U^c) > 0$. Hence there exists $\epsilon > 0$ such that $[\lambda - \epsilon, \lambda + \epsilon] \cap U^c = \emptyset$. This means that

$$\int_{-\infty}^{\infty} (\lambda - \lambda_0)^2 dE\omega(\lambda) \geq \int_{U^c} (\lambda - \lambda_0)^2 dE\omega(\lambda) > \epsilon\omega(E(U^c)) > 0 \tag{3.41}$$

contradicting the zero-variance assumption. The measure $\omega(E)$ on $\sigma(a)$ should therefore be a point measure. It follows that

$$\omega(E(\lambda_0)) = 1. \tag{3.42}$$

This can only hold if $E(\lambda_0)H \neq 0$, implying that $\lambda_0 \in \sigma_p(a)$.

On the other hand assume that $\lambda_0 \in \sigma_p(a)$. Pick $\psi \in E(\lambda_0)H$. The vector state ω_ψ has zero variance under a and λ_0 as its expectation value. □

For $\lambda \in \sigma_c(a)$ we have $\omega(E(\lambda)) = 0$, we hence cannot speak of a λ-collapse. There is however a sequence of states for which the previous lemma holds in the limit.

Proposition 3.1.28. *Whenever $\lambda' \in \sigma_c(a)$ there is a sequence of states $\{\omega_n\}$ for which*

$$Var_n(a) \to 0 \text{ and } \mathbb{E}_n(a) \to \lambda'. \tag{3.43}$$

Proof. Write $a - \lambda'\mathbb{I}$ by means of the projection valued measure E of a,

$$a - \lambda'\mathbb{I} = \int_{\sigma(a)} (\lambda - \lambda')dE(\lambda). \tag{3.44}$$

This operator in non-invertible by definition of the spectrum. This means that $E(X) \neq 0$ for every open X that contains $\lambda' \in X$. Define the open balls $B(\frac{1}{n}, \lambda')$ of radius $\frac{1}{n}$ around λ'. And pick for every $n \in \mathbb{N}$ an unit vector $\psi_n \in E(B(\frac{1}{n}, \lambda'))H$. We claim that the sequence ω_n of vector states of ψ_n is the sequence we are after. First we check the expectation value

$$\mathbb{E}_n(a) = \langle\psi_n, a\psi_n\rangle = \int_{\sigma(a)} \lambda \; d\langle\psi_n, E(\lambda)\psi_n\rangle = \int_{B(\frac{1}{n},\lambda')} \lambda \; d\langle\psi_n, E(\lambda)\psi_n\rangle \tag{3.45}$$
$$\to \lambda'\mathbb{I}.$$

Next we calculate the variance,

$$\begin{aligned}\mathrm{Var}_n(a) &= \int_{\sigma(a)} \lambda^2 \, d\langle \psi_n, E(\lambda)\psi_n\rangle - \left(\int_{\sigma(a)} \lambda \, d\langle \psi_n, E(\lambda\psi_n\rangle \right)^2 \\ &= \int_{B(\frac{1}{n},\lambda')} \lambda^2 \, d\langle \psi_n, E(\lambda)\psi_n\rangle - \left(\int_{B(\frac{1}{n},\lambda')} \lambda \, d\langle \psi_n, a\psi_n\rangle \right)^2 \\ &\to= \lambda'^2 \mathbb{I} - (\lambda' \mathbb{I})^2 = 0.\end{aligned} \tag{3.46}$$

□

Example 3.1.29. *Let X be the position observable in example 2.3.3. There is no function in $L^2(\mathbb{R}, \mathbb{C})$ for which $X\psi = x\psi$. However, X has a clear meaning of determining a position of a vector state ψ. Whenever we concentrate ψ around x the observable X will tend to realise x with certainty.*

More explicitly, whenever we pick

$$\psi_n = \frac{1}{2\sqrt{n\pi}} e^{n(x-x')^2}, \tag{3.47}$$

for the states e_{ψ_n}, the observable X, will tend to result in x' with certainty .

3.2 Classification of the quantum phases

As was argued in the previous section, our interest lies in the normal states on $B(H)$. Given some quantum system with algebra of observables A, we should classify the phases $N(\mathfrak{S})(A)$. By the convexity of this space, we may restrict ourselves to classifying the extreme points.

Our main result is the classification of normal pure phases by irreducible subspaces of the twisted representation of the particular quantum system. The claim is inspired by the fact that the states for indistinguishable particles are given by the irreducible representations of S_n [4] chapter 1.

Before we state and prove the main claim, we repeat some material from chapter 2 and lay some ground work. Recall the following result from proposition 2.4.5. Let U be an irreducible subspace of the representation ρ^τ. The automorphisms on U that intertwine the representation are isomorphic to either of the following division algebras:

$$\mathbb{C}, \mathbb{R}, \mathbb{H}. \tag{3.48}$$

Definition 3.2.1. *Let $\mathbb{D}$ be one of the division algebras in eq.* (3.48) *and let ρ^τ be a twisted representation of some group. An irreducible subspace of ρ^τ whose intertwining automorphisms form the division algebra $\mathbb{D}$ is called a subspace of type $\mathbb{D}$.*

Since G is compact we can decompose H into finite dimensional mutually perpendicular irreducible subspaces. Fix in each closed irreducible subspace $U \subset H$ a unit vector ψ_U in such a way that whenever $\sum_i \lambda_i \psi_{U_i} = \psi_U$ we have $\lambda_i \geq 0$ for all i. Recall that the intertwiners from U to U' are isomorphic to some division algebra over the real numbers $\mathbb{D}$. There hence exists a unique operator

$$T_U^{U'} : H \to H \tag{3.49}$$

such that $T_U^{U'} U^\perp = 0$ and

$$\langle \psi_{U'}, T_U^{U'} \psi_U \rangle = 1. \tag{3.50}$$

Fix these operators once and for all. Define

$$A_{\mathbb{D}} = L_{\mathbb{R}} \{ T_U^{U'} \mid U \text{ type } \mathbb{D} \text{ and } \langle \psi_{U'}, T_U^{U'} \psi_U \rangle = 1 \}, \tag{3.51}$$

where $L_{\mathbb{R}}$ is the $\mathbb{R}$-linear span allowing infinite (converging) sums.

Lemma 3.2.2. *The set $A_{\mathbb{D}}$ forms a closed involutive $\mathbb{R}$-linear subalgebra of $B(H)$.*

Proof. First we check the involution

$$T_U^{U'*} = T_U^{U'-1} = T_{U'}^U \in A_{\mathbb{D}}. \tag{3.52}$$

Now look at the composition $T_{U_1}^{U_1'} T_{U_2}^{U_2'}$. This operator is an intertwiner from U_2 to U_1' and

$$\langle \psi_{U_1'}, T_{U_1}^{U_1'} T_{U_2}^{U_2'} \psi_{U_2} \rangle = \langle \psi_{U_1'}, T_{U_1}^{U_1'} (\lambda \psi_{U_1} + \phi) \rangle, \tag{3.53}$$

where $\lambda > 0$ and $\phi \in U_1^\perp$. We continue

$$\cdots = \lambda \langle \psi_{U_1'}, T_{U_1}^{U_1'} \psi_{U_1} \rangle = \lambda. \tag{3.54}$$

Therefore $T_{U_1}^{U_1'} T_{U_2}^{U_2'} = \lambda T_{U_2}^{U_1'}$ with $\lambda \in \mathbb{R}$. So $T_{U_1}^{U_1'} T_{U_2}^{U_2'} \in A_{\mathbb{D}}$.

Lastly, we check that $A_{\mathbb{D}}$ is closed. Observe that for all $T_U^{U'}$ we have

$$\langle \psi_{V'}, T_U^{U'} \psi_V \rangle > 0 \tag{3.55}$$

for all irreducible subspaces V and V'. A limit T of a sequence in the real linear span of the operators $T_U^{U'}$ therefore satisfies

$$\langle \psi_{V'}, T \psi_V \rangle \in \mathbb{R} \tag{3.56}$$

for all irreducible subspaces V and V'. Write $H = \oplus_i U_i$. If we restrict T to U_i the above equation implies that $T|_U = \lambda T_U^{U_i'}$ for $U_i' = TU_i$ and $\lambda \in \mathbb{R}$. Since $T = \sum_{i=1}^\infty T|_{U_i}$ it follows that T is in the real linear span of the operators $T_U^{U'}$. □

In case we take the $\mathbb{C}$-linear span we obtain a C^*-algebra. We can write the algebra of observables in the following way:

$$A = (\mathbb{R} \otimes_{\mathbb{R}} A_{\mathbb{R}}) \oplus (\mathbb{C} \otimes_{\mathbb{R}} A_{\mathbb{C}}) \oplus (\mathbb{H} \otimes_{\mathbb{R}} A_{\mathbb{H}}). \tag{3.57}$$

We are interested in the pure normal phases, we can search for these phases on the individual components of the direct sum.

$$P_N(\mathfrak{S}(A)) = P_N(\mathfrak{S}(\mathbb{R} \otimes_{\mathbb{R}} A_{\mathbb{R}})) \oplus P_N(\mathfrak{S}(\mathbb{C} \otimes_{\mathbb{R}} A_{\mathbb{C}})) \oplus P_N(\mathfrak{S}(\mathbb{H} \otimes_{\mathbb{R}} A_{\mathbb{H}})). \tag{3.58}$$

Theorem 3.2.3. *Let $(H, G, \phi, \tau, \rho^\tau)$ be a quantum system with compact group G. Let $\mathcal{U}_{\mathbb{R}}$, $\mathcal{U}_{\mathbb{C}}$ and $\mathcal{U}_{\mathbb{H}}$ be the sets consisting of all irreducible subspaces of ρ^τ of type $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ respectively. Write $S(\mathbb{H})$ for the vector states of $B(\mathbb{C}^2)$. There is a bijective map between*

$$\mathcal{U} = \mathcal{U}_{\mathbb{R}} \cup \mathcal{U}_{\mathbb{C}} \cup (S(\mathbb{H}) \times \mathcal{U}_{\mathbb{H}}) \tag{3.59}$$

and the pure normal phases of states $P_N(\mathfrak{S}(A))$. The bijection in question is given by

$$\begin{aligned} \mathcal{U}_{\mathbb{R}} &\to P_N(\mathfrak{S}(\mathbb{R}\otimes_{\mathbb{R}} A_{\mathbb{R}})) \\ \omega_U(\lambda\otimes_{\mathbb{R}} a) &= \lambda Tr(e_U a) \\ \mathcal{U}_{\mathbb{C}} &\to P_N(\mathfrak{S}(\mathbb{C}\otimes_{\mathbb{R}} A_{\mathbb{C}})) \\ \omega_U(\lambda\otimes_{\mathbb{R}} a) &= \lambda Tr(e_U a) \\ S(\mathbb{H})\times\mathcal{U}_{\mathbb{H}} &\to P_N(\mathfrak{S}(\mathbb{H}\otimes_{\mathbb{R}} A_{\mathbb{H}})) \\ \omega_{(\omega',U)}(\lambda\otimes_{\mathbb{R}} a) &= \omega'(\lambda)\cdot Tr(e_U a). \end{aligned} \tag{3.60}$$

Proof. The search for pure normal phases $P_N(\mathfrak{S}(A))$ distributes over the direct sum in eq. (3.57). We can therefore treat the terms $\mathbb{D}\otimes_{\mathbb{R}} A_{\mathbb{D}}$ individually.

Recall the operators $T_U^{U'}$ and vectors ψ_U that were fixed in the text preceding this theorem. Introduce the notation $H' = L_{\mathbb{C}}\{\psi_{U_i} \mid U \text{ type } \mathbb{D}\}$ and $H'_{\mathbb{R}} = L_{\mathbb{R}}\{\psi_{U_i} \mid U \text{ type } \mathbb{D}\}$. Write $H = \oplus_i U_i$ with respect to some choice of mutually perpendicular irreducible subspaces U_i. The C^*-algebra $\mathbb{C}\otimes_{\mathbb{C}} A_{\mathbb{D}} = L_{\mathbb{C}}\{T_U^{U'} \mid U \text{ type } \mathbb{D}\}$ is isomorphic to $B(H')$. We construct an isomorphism ϕ as follows. Given any operator $a\in A_{\mathbb{D}}$ we can decompose

$$a = \oplus_i \lambda_{U_i} T_{U_i}^{U_i'}, \tag{3.61}$$

with $T_{U_i}^{U_i'}$ uniquely fixed and $\lambda_{U_i}\in\mathbb{R}$. Now ϕ sends this operator to the operator $a'\in B(H')$ defined by

$$a'\psi_{U_i} = \lambda_{U_i}\psi_{U_i'}. \tag{3.62}$$

This mapping ϕ is an isomorphism.

We will now treat the three cases individually. In each of the cases we make use of the following fact. Since due to lemma 3.2.2 $\mathbb{C}\otimes_{\mathbb{C}}\mathbb{D}\otimes_{\mathbb{C}} A_{\mathbb{D}}$ is a C^*-algebra, we can equivalently look for phases of states on $\mathbb{C}\otimes_{\mathbb{C}}\mathbb{D}\otimes_{\mathbb{C}} A_{\mathbb{D}}$ upon restriction to $\mathbb{D}\otimes_{\mathbb{R}} A_{\mathbb{D}}$.

First we track down the normal pure phases on $\mathbb{R}\otimes_{\mathbb{R}} A_{\mathbb{R}} = A_{\mathbb{R}}$. Under the isomorphism ϕ we find that $A_{\mathbb{R}}\cong B(H'_{\mathbb{R}})\subset B(H')$. If searching for phases of $B(H)$ under restriction to $A_{\mathbb{R}}$ we can look for phases of $B(H')$ under restriction to $B(H'_{\mathbb{R}})$. The self-adjoint elements of $B(H'_{\mathbb{R}})$ are given by the symmetric operators $B_{sym}(H'_{\mathbb{R}})$. All vector states on $B(H')$ can be separated on the domain $B_{sym}(H'_{\mathbb{R}})$. This is not hard to see since if $e_\psi\neq e_{\psi'}$ these states can be separated by the operator $e_\psi\in B_{sym}(H'_{\mathbb{R}})$. We conclude that the pure normal phases are simply given by the pure normal states on $B(H')$. The pure normal states are therefore the vector states e_ψ. Under the isomorphism ϕ these states correspond with states e_U, where $U\subset H$ is an irreducible subspace of type $\mathbb{R}$.

Secondly we look at $\mathbb{C}\otimes_{\mathbb{R}} A_{\mathbb{C}}$. Since we are searching for $\mathbb{C}$-linear states, this is the same as restricting to $\mathbb{C}\otimes_{\mathbb{C}} A_{\mathbb{C}}\cong A_{\mathbb{C}}$. This algebra is under ϕ isomorphic to $B(H')$. The pure normal phases are therefore simply the pure normal states on $B(H')$. The normal pure states are as already established the vector states e_ψ. Under the isomorphism ϕ these states correspond with states e_U, where $U\subset H$ is an irreducible subspace of type $\mathbb{C}$.

Lastly we proceed with $\mathbb{H}\otimes_{\mathbb{R}} A_{\mathbb{H}}$. Represent the quaternions $\mathbb{H}$ in $B(\mathbb{C}^2)$ as follows:

$$\mathbb{H}\cong\left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}, \end{pmatrix} \mid\ a,b\in\mathbb{C}\right\}. \tag{3.63}$$

Using the above identification and the isomorphism ϕ, we find that $\mathbb{H} \otimes_{\mathbb{R}} A_{\mathbb{H}}$ is contained in the C^*-algebra $B(\mathbb{C}^2) \otimes_{\mathbb{R}} B(H')$. For this reason we can look for phases of $B(\mathbb{C}^2) \otimes_{\mathbb{C}} B(H')$ under restriction to $\mathbb{H} \otimes_{\mathbb{R}} B(H'_{\mathbb{R}})$. The self-adjoint elements of $\mathbb{H} \otimes_{\mathbb{R}} B(H'_{\mathbb{R}})$ are given by

$$(\mathbb{H}_{sa} \otimes_{\mathbb{R}} B_{sym}(H'_{\mathbb{R}})) \oplus (\mathbb{H}_{skew} \otimes_{\mathbb{R}} B_{skew}(H'_{\mathbb{R}})). \tag{3.64}$$

The normal pure states are the vector states $e_{\psi' \otimes \psi} = e_\psi \otimes e_{\psi'}$, where $\psi \in \mathbb{C}^2$ and $\psi' \in H'$. We claim that all these states can be separated by $(\mathbb{H} \otimes_{\mathbb{R}} B(H'_{\mathbb{R}}))_{sa}$.

We already proved that $B_{sym}(H'_{\mathbb{R}}) \subset B(H')$ separates all states $e_{\psi'}$ with $\psi' \in H'$. Next we look at $\mathbb{H}$. For the skew adjoint part of the quaternions we have

$$\mathbb{H}_{skew} = \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \; a, b \in \mathbb{R}. \tag{3.65}$$

Observe that $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{H}_{skew} = B(\mathbb{C}^2)$. Therefore, once we know a $\mathbb{C}$-linear function on $\mathbb{H}_{skew}$ we know it on the whole of $B(\mathbb{C}^2)$. We may conclude that all states e_ψ, with $\psi \in \mathbb{C}^2$, can be separated on $\mathbb{H}_{skew}$.

Given two distinct states $e_\psi \otimes e_{\psi'}$ and $e_\phi \otimes e_{\phi'}$ we have two cases. In case $\psi' \neq \phi'$ we can pick a symmetric operator a in $B(H')$ separating ψ and ϕ. This way $\mathbb{I} \otimes a$ will separate the two respective states. In case $\psi \neq \phi$ and $\psi' = \phi'$ we can pick any anti-symmetric operator a in $B(H')$ and an anti-symmetric operator b in $\mathbb{H}$ that separates ψ' and ϕ'. This way $a \otimes b$ will separate the two respective states.

The phases in $P_N(\mathfrak{S}(\mathbb{H} \otimes_{\mathbb{R}} A_{\mathbb{H}}))$ are therefore given by the vector states on $\mathbb{C}^2 \otimes_{\mathbb{C}} H'$. Under the isomorphism ϕ these are the states $e_\psi \otimes e_U$, where $U \subset H$ is of type $\mathbb{H}$ and $\psi \in \mathbb{C}^2$.

The case that H' is one dimensional is special. In this instance we are only left with restricting states on $B(\mathbb{C}^2)$ to $\mathbb{H}_{sa}$. Since

$$\mathbb{H}_{sa} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{R} \right\}, \tag{3.66}$$

there is only one phase and hence only one normal pure phase in this case. □

Example 3.2.4. *In case the symmetry group G is trivial we find $G^\tau = \mathbb{T}$ and therefore the irreducible subspaces are one-dimensional. The conclusion is that all normal pure states on the algebra of observables $B(H)$ are given by the one dimensional projections.*

In case of the n-particle Hilbert space $\otimes^n H$ on which the symmetric group S_n acts by permuting the entries. Examples of irreducible subspaces are the completely symmetric and completely anti-symmetric states

$$\sum_{\sigma \in S_n} \sigma \cdot \otimes_{i=1}^n \psi_i \text{ respectively } \sum_{\sigma \in S_n} (-1)^{|\sigma|} \sigma \cdot \otimes_{i=1}^n \psi_i \tag{3.67}$$

form irreducible subspaces. These pure states are the well-known Bosonic and fermionic states. Note, however, that there are more irreducible subspaces and hence more pure states! See also [4] chapter 1.

Recall time reversal on the spin space discussed in example 2.4.8. The irreducible subspaces are of quaternionic type and hence contain multiple pure states!

We continue by finding a nice physical property to distinguish the pure normal states. To this end we define the notion of non-degeneracy.

Definition 3.2.5. *Let a and a' be two observables with spectral measures E and E' respectively. Whenever $\sigma(a) \subset \sigma(a')$ and for every measurable $X \subset \sigma(a)$ there exists a measurable $Y \subset \sigma(a')$ with $Y \cap \sigma(a) = X$ such that*

$$E(Y) = E(X), \tag{3.68}$$

we say a' is finer then a.

A non-degenerate observable is an observable a for which the only finer observable is a itself.

A non-degenerate observable is in this way the maximal element of the partial ordering induced by being finer. The idea behind the definition of non-degeneracy is to expresses that a separates as many outcomes as possible. It is now natural to proceed with the following definition.

Definition 3.2.6. *A state is called a state of maximal certainty if there exists a non-degenerate observable for which the variance is zero.*

The next theorem now shows the special status of normal pure states in physics.

Theorem 3.2.7. *Given a quantum system with compact symmetry group, a quantum state ω on the algebra of observables of this quantum system is a state of maximal certainty iff it is a pure normal state.*

Proof. Let ω be a state of maximal certainty and let a be a non-degenerate observable having zero variance under ω. Recall proposition 3.1.27 to see that we must have $\lambda \in \sigma_p'(a)$ and $\omega = \omega_\psi$ for some $\psi \in H$. The vector ψ must be contained in an irreducible subspace. The vector state ω_ψ therefore coincides with one of the normal pure states found in theorem 3.2.3.

On the other hand, it is easy to see that the normal pure states found in theorem 3.2.3 all have zero variance when applied to the observable that is the projection on the irreducible subspace belonging to that particular normal pure state. □

Remark 3.2.8. *The above result justifies the special significance attributed to the normal pure states and the non-degenerate observables. A non-degenerate observable distinguishes a maximal number of outcomes and the normal pure states are the states that hold maximal information for a certain maximally refined operator. These normal pure states are, however, still probabilistic with respect to most other observables, the best you can get in quantum physics.*

Chapter 4

Topological phases

Having read the previous chapter one question pops up. Where is the mechanics in quantum mechanics? Until now a state does not evolve over time. The formalism thus far is locked up at $t = 0$. To this end we will introduce the Hamiltonian, which gives rise to a time evolution. In the last section two ways to classify the connected components of Hamiltonians are given.

Remark 4.0.9. *It is worth noting that the theory is already very exciting, even though it only describes a system trapped in $t = 0$. In classical mechanics the system at $t = 0$ is little more than fixing a point in phase space. In the quantum case the the measuring is influencing property kicks in and introduces some 'movement' due to the collapsing of states.*

4.1 Hamiltonians

Definition 4.1.1. *A Hamiltonian of a quantum system is an observable with spectrum bounded from below.*

We make a small remark on the seemingly arbitrary assumption on the spectrum.

Remark 4.1.2. *Recall from proposition 3.1.28 that for each $\lambda \in \sigma(h)$ there exists a sequence of states ω_n which after a measurement tends to result in λ with certainty. In case $\sigma(h)$ would not be bounded from below, there exist states with arbitrary low energy expectation value. By ever further lowering the quantum system in lower energy states, we could extract an infinite amount of energy from the system. This would violate the laws of thermodynamics.*

In order to complete the formalism we need to introduce a time evolution. For the sake of a good build-up we start with defining a notion lying close to that of time evolution.

Definition 4.1.3. *An evolution is a continuous (with respect to the topology in definition 2.1.16) homomorphism*

$$\rho : \mathbb{R} \to Aut_{QM}(O_{sym}(H)). \tag{4.1}$$

Given ρ the evolution of a state is given by

$$\omega(t)(a) = \omega(\rho(t)(a)). \tag{4.2}$$

In theorem 2.2.5 we saw that either of the two cases below must hold

$$\rho(t)(a) = uau^*,\ u \text{ unitary} \tag{4.3}$$

$$\rho(t)(a) = vav^*,\ v \text{ anti-unitary}. \tag{4.4}$$

Since $\mathbb{R}$ is connected the image of its representation must be connected and hence fall completely in the unitary operators.

Using Stone's Theorem, [33] section 2.3, the representation is uniquely given by some self-adjoint generally unbounded operator a via

$$t \to e^{-ita}. \tag{4.5}$$

We want to interpret a as the Hamiltonian. We arrive at the following definition of a time evolution.

Definition 4.1.4. *A time evolution of a quantum system is an evolution that is generated by a Hamiltonian h of this quantum system.*

As a last remark we prove a well know result in quantum physics, [18] page 115.

Proposition 4.1.5. *The expectation value of an observable a does not change over time under arbitrary states ω iff the observable commutes with h.*

Proof. A bounded observable a commutes with h iff it commutes with e^{-ith}. This in turn is the case iff for every ω

$$\omega(e^{ith}ae^{-ith}) = \omega(a). \tag{4.6}$$

In other words,

$$\omega(t)(a) = \omega(a), \tag{4.7}$$

proving the claim. □

We can obtain the same result result for unbounded observables for which there exists a sequence of bounded observables, that commute with e^{-ith}, converging to it.

4.2 Classification of topological phases

In this section we take a look at the topological phases of some selected subset $\mathfrak{H}$ of Hamiltonians of some quantum system.

Definition 4.2.1. *Select some subset $\mathfrak{H}$ of the Hamiltonians of a quantum system. Equip $\mathfrak{H}$ with the topology in definition 2.1.16. A continuous path is a continuous function*

$$[0,1] \to \mathfrak{H} \tag{4.8}$$

$$t \to h_t. \tag{4.9}$$

Two hamiltonians h and h' are connected whenever there exists a continuous path

$$\mathbb{R} \to \mathfrak{H} \tag{4.10}$$

such that $h_0 = h$ and $h_1 = h'$.

Being connected is an equivalence relation $\sim$, whose equivalence classes are called topological phases. Denote the set of topological phases by $\mathfrak{Tp}(\mathfrak{H})$.

Example 4.2.2. *If we take a look at the subset of Hamiltonians of some quantum system without imposing any additional requirements, there is only one topological phase. Namely, every Hamiltonian h is connected to 0 via the path*

$$h_t = th. \tag{4.11}$$

The result of there being only one topological phase is not very interesting. We should add additional conditions on the Hamiltonians in order to extract non-trivial results. In the following we restrict to Hamiltonians of a certain quantum system with compact symmetry group presenting a so called gap in its spectrum. First we classify topological phases by invariant subspaces. Second we use the procedure of Kitaev [26] to classify the topological phases in case a Hamiltonian can be looked upon as an extension of some Clifford module.

Gapped Hamiltonians of quantum systems with compact symmetry group can be interpreted as 0-dimensional insulators. The physics behind the problem is explained in section 5.3 later on.

4.2.1 Gapped Hamiltonians

Let $(H, G, \phi, \tau, \rho^\tau)$ be a quantum system with compact group G. Let $\mathfrak{H}$ be the space consisting of Hamiltonians of this quantum system for which there exists an $\epsilon > 0$ such that $\sigma(h) \cap [-\epsilon, \epsilon] = \emptyset$. A Hamiltonian $h \in \mathfrak{H}$ is called gapped.

We start by observing a promising simplification on the gapped Hamiltonians, [11] page 28.

Definition 4.2.3. *The spectral flattening of a Hamiltonian h is the operator*

$$h^s = \int_{\sigma(h)^+} dE(\lambda) - \int_{\sigma(h)^-} dE(\lambda) = h^+ - h^-, \tag{4.12}$$

where E is the projection valued measure of h and $\sigma^\pm(h) = \sigma(h) \cap \mathbb{R}^\pm$.

The spectral flattening of a Hamiltonian h^s is connected to the original Hamiltonian h.

Lemma 4.2.4. *For $h \in \mathfrak{H}$ the spectral flattening h^s is connected to h via the path*

$$h_t = \int_{\sigma(h)} \frac{\lambda}{|t(1-\lambda)+\lambda|} dE(\lambda), t \in [0,1]. \tag{4.13}$$

Proof. Fix a $\psi \in D(h_{t_0})$ and $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that

$$\left\| \int_{\mathbb{R}\setminus[-N,N]} \lambda dE(\lambda)\psi \right\| < \frac{\epsilon}{4}. \tag{4.14}$$

For this N there also exists $\delta > 0$ such that for all $t \in [t_0 - \delta, t_0 + \delta]$,

$$\begin{aligned} &\left\| \int_{[-N,N]} \left(\frac{\lambda}{|t_0(1-\lambda)+\lambda|} - \frac{\lambda}{|t(1-\lambda)+\lambda|} \right) dE(\lambda)\psi \right\| \\ &\leq 2N \|\psi\| \, |\frac{N}{|t_0(1-N)+N|} - \frac{N}{|t(1-N)+N|}| < \frac{\epsilon}{2}. \end{aligned} \tag{4.15}$$

It follows that for all h_t with $t \in [t_0 - \delta, t_0 + \delta]$

$$\|(h_{t_0} - h_t)\psi\| < \left\| \int_{[-N,N]} \left(\frac{\lambda}{|t_0(1-\lambda)+\lambda|} - \frac{\lambda}{|t(1-\lambda)+\lambda|} \right) dE(\lambda)\psi \right\| + 2 \cdot \frac{\epsilon}{4} < \epsilon. \tag{4.16}$$

Therefore, $h_t \in U(h_{t_0}, \psi, \epsilon)$. The same result can be reached for finite intersections of sets of the form $h_t \in U(h_{t_0}, \psi, \epsilon)$. In case om M such sets we take the apply the procedure above M times and pick the minimum of the M chosen δ and the maximum of the M chosen N. Since the sets $h_t \in U(h_{t_0}, \psi, \epsilon)$ form a basis for the topology, we may conclude that indeed $h_t \to h_{t_0}$. It follows that h_t is a continuous path. By construction it connects h with h^s. ☐

Note that in the lemma above we were able to deform the positive spectrum of the Hamiltonian to 1 and the negative spectrum to -1. We can however not deform the whole spectrum to, say, 1 because of the gap at 0.

Using the deformation in eq. (4.13) we even find the following lemma.

Lemma 4.2.5. *The topological phases of $\mathfrak{H}$ coincide with the topological phases of spectral flattened Hamiltonians in $\mathfrak{H}$.*

Proof. By lemma 4.2.4 each topological phase contains a spectral flattened Hamiltonian. To finish the proof we must show that whenever two spectral flattened Hamiltonians h_0 and h_1 are connected, there exists a path between h_0 and h_1 solely consisting of spectral flattened Hamiltonians. To this end assume that h_0 and h_1 are two spectral flattened Hamiltonians that are connected. Let h_t be any continuous path connecting the two Hamiltonians. We claim that h_t^s is a continuous path between h_0 and h_1 as well.

First, we prove that there exists an $\epsilon > 0$ such that $\sigma(h_t) \cap [-\epsilon, \epsilon] = \emptyset$ for all $t \in [0,1]$. For the sake of contradiction assume that this would not hold. There would exist a sequence $h_{t_n}\psi_m$ with ψ_m norm one for all $m \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \lim_{m\to\infty} \|h_{t_n}\psi_m\| = 0. \tag{4.17}$$

Since $[0,1]$ is compact there exists a $t' \in [0,1]$ such that $\lim_{m\to\infty} h_{t'}\psi_m = 0$. This implies that $0 \in \sigma(h_{t'})$, contradicting our assumption.

Since there exists an $\epsilon > 0$ such that $[-\epsilon, \epsilon]$ is not contained in the spectrum of any h_t, we find

$$\left\| \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} \lambda d(E_{t_0} - E_t)\psi \right\| = \|(h_{t_0} - h_t)\psi\| \to 0. \tag{4.18}$$

Since $|\lambda| \geq \epsilon$,

$$\left\|(h_{t_0}^s - h_t^s)\psi\right\| = \left\| \int_{\sigma(h)} d(E_{t_0} - E_t)\psi \right\| \leq \frac{1}{\epsilon} \|(h_{t_0} - h_t)\psi\| \to 0. \tag{4.19}$$

We conclude that h_t^s is a continuous path, which by construction connects h_0 and h_1. ☐

For this reason we may restrict ourselves to classifying topological phases of spectral flattened Hamiltonians under their usual strong topology. In other words, we need to find the topological phases of operators of the form

$$h^s = h^+ - h^-. \tag{4.20}$$

The self-adjoint operator h^s has a discrete spectrum and therefore h^+ and h^- are projections on the eigenspaces of h^s with eigenvalues 1 and -1 respectively. Since 0 is not contained in the spectrum of h^s, we find

$$Im(h^-)^\perp = Im(h^+). \tag{4.21}$$

The topological phase is therefore fixed once we know $Im(h^-)$. The operator h^s intertwines $\rho^\tau(G^\tau)$ iff $Im(h^-)$ is an invariant subspace V. There is now a nice identification of topological phases of Hamiltonians with equivalence of the corresponding invariant subspaces.

Lemma 4.2.6. *For a given extended symmetry group G^τ and representation ρ^τ, we have*

$$h^+ - h^- = h^s \sim h'^s = h'^+ - h'^- \tag{4.22}$$

iff

$$Im(h^-) \sim Im(h'^-) \tag{4.23}$$

as invariant subspaces of ρ^τ.

Proof. We can decompose $Im(h^-)$ into irreducible subspaces $\oplus_{i=1}^{\infty} U_i$. It therefore suffices to show that $e_U \sim e_{U'}$ iff $U \sim U'$.

Assume $U \sim U'$. In this case there exists an intertwiner $T : U \to U'$. The operator $(1-t)\mathbb{I}+tT$ is an intertwiner as well. Its image U_t is therefore an irreducible subspace equivalent to U. The function $t \to e_{U_t}$ is a continuous path between e_U and $e_{U'}$.

On the other hand, assume that e_U and $e_{U'}$ are connected via a continuous path of projections on invariant subspaces e_t. By the compactness of the group the subspace U is finite dimensional. Therefore, there exists a $t_1 > 0$ such that for all $t \in (0, t_1)$ and all $\psi \in U$ we have

$$\|(e_t - e_U)\psi\| < 1. \tag{4.24}$$

We clai that the map

$$e_U : Im(e_t) \to Im(e_U) \tag{4.25}$$

is injective for $t \in (0, t_1)$. Assume for the sake of contradiction that there would exist a $\psi \in Im(e_t)$ such that $e_U\psi = 0$. This implies that $\|(e_t - e_U)\psi\| = 1$. This contradicts eq. (4.24). By the same reasoning,

$$e_t : Im(e_U) \to Im(e_t), \tag{4.26}$$

is injective. We conclude that the dimensions of $Im(e_U)$ and $Im(e_t)$ are equal. The map in eq. (4.25) is therefore a bijective intertwiner. It follows that $U \sim Im(e_t)$ for all $t \in (0, t_1)$.

For $t \in [0, 1]$ define an open interval V_t in $[0, 1]$ such that for all $t, t' \in V_t$ we have

$$\|(e_t - e_{t'})\psi\| < 1. \tag{4.27}$$

These open sets $\{V_t \mid t \in [0, 1]\}$ cover $[0, 1]$ and since the space $[0, 1]$ is compact there is a finite sub-cover. A finite repetition of the previous argument now yields that $U \sim Im(e_t)$ for all $t \in [0, 1]$. In particular, $U \sim U'$. □

The topological phase is uniquely fixed by the equivalence class of the invariant subspace $Im(h^-)$. We obtain the following theorem as a result.

Theorem 4.2.7. *Label the irreducible subspaces of ρ^τ, up to equivalence, by $\{1, \cdots N\}$. Write m_j for the multiplicity of the irreducible subspace of ρ^τ belonging to $j \in \{1, \cdots, N\}$. The map*

$$\{(n_1, \cdots n_N) \in \mathbb{N}^N \mid n_j \leq m_j\} \to \mathfrak{Tp}(\mathfrak{H}) \tag{4.28}$$

$$(n_1, \cdots n_N) \mapsto \oplus_{j=1}^{N} \oplus_{i=1}^{n_j} (e_{U_i^\perp} - e_{U_i}) \tag{4.29}$$

is a bijection.

4.2.2 Hamiltonians by Clifford module extensions

In many physical cases the search for h is equivalent with searching for an extension of a Clifford module. The reader is advised to consult Appendix D at this point. First we define what we mean by an extension of a Clifford module.

Definition 4.2.8. *A positive extension of a Cliff^{p,q}-module N is a Cliff$^{p+1,q}$-module that is isomorphic to the original Clifford module N after restriction. Write Ext$^+$(Cliff^{p,q}, N) for the set of positive extensions of N.*

In other words, a positive extension of a representation ρ of Cliff^{p,q} is a representation ρ' of Cliff$^{p+1,q}$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathit{Cliff}^{\,p+1,q} & & \\ {\scriptstyle i}\uparrow & \searrow^{\rho'} & \\ \mathit{Cliff}^{\,p,q} & \xrightarrow[\rho]{} & B(N). \end{array} \tag{4.30}$$

In the same way a negative extension of a Cliff^{p,q}-module N is a Cliff$^{p,q+1}$-module that is isomorphic to the original Clifford module after restriction. Write Ext$^-$(Cliff^{p,q}, N) for the set of negative extensions of N.

Finally, an extension of a $\mathbb{C}$lif^{q}-module N is a $\mathbb{C}$lif^{q+1}-module that is isomorphic to the former after restriction. Write Ext(Cliff^{p,q}, N) for the set of extensions of N.

A Clifford module on H is generated by a set of positive and negative generators $e_1, \cdots, e_p\ e'_1, \cdots, e'_q$ on H. A positive extension is in this respect nothing more than a self-adjoint operator e_{p+1} that squares to $\mathbb{I}$ and commutes with $e_1, \cdots, e_p\ e'_1, \cdots, e'_q$. A negative extension is on the other hand a skew adjoint e_{q+1} that squares to $-\mathbb{I}$ and anti-commutes with the generators $e_1, \cdots, e_p\ e'_1, \cdots, e'_q$. We can therefore equip the set of positive or negative extensions with the strong operator topology.

Recall that we may restrict the discussion to spectral flattened Hamiltonians. That is to say, Hamiltonians for which $h^2 = \mathbb{I}$. Symmetries and generators of symmetries have a tendency to square to either plus or minus $\mathbb{I}$. In many situations the requirement of commuting with $\rho^\tau(G^\tau)$ is equivalent with commuting with some Clifford module. By replacing h by a suitable operator h' it might be possible to find anti-commutation relations between h' and the Clifford module. In this way searching for h is equivalent with searching an additional (positve or negative) generator of some Clifford module. We treat some examples.

Example 4.2.9.
- ***No symmetry.*** *In case there is no symmetry, there are no restrictions on h. All self-adjoint operators that square to $\mathbb{I}$ are spectral flattened Hamiltonians of this quantum system. In other words, h is a self-adjoint element squaring to $\mathbb{I}$ and anti-commuting with the generators of a $\mathbb{C}$liff$^{\,0}$-module. Note that the last statement is empty since $\mathbb{C}$liff has no generators. The operator h will therefore be a positive extension of this module, since $h^2 = \mathbb{I}$.*

- ***Time reversal.*** *Time reversal is anti-unitary (see Appendix A), skew-adjoint and squares to $-\mathbb{I}$. This time we need to use a real graded representation in the way we did in section 1.4 to get rid of the anti-unitarity. Recall the operator J in eq.* (1.125) *that indicated the number i. The representation of time reversal is hence a Cliff$^{\,0,1}$-module. The search for* *skew-adjoint matrices Jh squaring to $-\mathbb{I}$ is equivalent to the search for self-adjoint matrices h squaring to $\mathbb{I}$. The matirces Jh anti-commute with T iff T commutes with h. We should hence search for matrices Jh that are negative extensions of the Cliff ^{0,1}-module.*

- ***Spin rotation.*** *The spin rotations are generated by the well known Pauli spin matrices*

$$J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; J_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; J_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.31}$$

  Commuting with the spin rotations is equivalent with commuting with its generators, hence h is required to commute with J_1, J_2 and J_3. This is equivalent with requiring that it commutes with just J_1 and J_2. The operators J_1 and J_2 form a $\mathbb{C}$liff ^{2}-module. Instead of searching for Hamiltonians h that commute with J_1 and J_2 we can equivalently search for operators iJ_1J_2h that anti-commutes with J_1 and J_2. The operator iJ_1J_2h is self-adjoint, squares to $\mathbb{I}$ and anti-commutes with the generators J_1 and J_2. We are therefore searching for extensions of a $\mathbb{C}$liff ^{2}-module.

- ***Charge conservation and time reversal.*** *Now consider charge conservation together with time reversal. Charge conservation means a $U(1)$ invariance under $e^{i\theta Q}$, where Q is the self-adjoint charge operator that squares to $\mathbb{I}$ and anti-commutes with T. Commuting with this $U(1)$ group is equivalent with commuting with its self-adjoint generator Q. Hence h needs to commute with the algebra generated by T and Q, equivalently, the skew-adjoint operator Jh needs to anti-commute with the algebra generated by T and TQ. This is a negative extension of the Cliff ^{0,2}-module generated by TQ and T.*

- ***Spin rotation and time reversal*** *This time the Hamiltonian needs to commute with J_1, J_2 and T. The matrices J_1, J_2 and T anti-commute with each other. Furthermore, T is skew-adjoint and J_1 and J_2 are self-adjoint. Therefore J_1, J_2 and T are generators of a Cliff ^{0,3}-module. Replacing h for the skew-adjoint JTh is now makes the search of Hamiltonians into searching for negative extensions of this module.*

The above example illustrates that searching for a spectral flattened operator h commuting with $\rho^\tau(G^\tau)$ can (at times) be interpreted as searching for an extending positive or negative generator e_{p+1} of some Clifford module. There is, however, no fundamental reason why this would be the case, it just turns out to be possible in many cases [26]. We are interested in the connected components of $\mathfrak{H}$. For this reason we start tracking down the connected components of extensions of Clifford modules.

Searching for negative extensions can be reinterpreted as searching for positive extensions. This is due to the isomorphism, [26] equation 18,

$$\text{Cliff}^{\,p+2,q} \cong \text{Cliff}^{\,q,p} \otimes_{\mathbb{R}} M_{2\times 2}(\mathbb{R}), \tag{4.32}$$

given by the unique extension (see lemma D.0.22) of

$$e_j \to e_j \otimes_{\mathbb{R}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ for } j \leq p \tag{4.33}$$

$$e_j \to e_j \otimes_{\mathbb{R}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ for } j \leq q \tag{4.34}$$

$$e_{p+1} \to \mathbb{I} \otimes_{\mathbb{R}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.35}$$

$$e_{p+2} \to \mathbb{I} \otimes_{\mathbb{R}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.36}$$

We do not equip $M_{2\times 2}(\mathbb{R})$ with a grading, hence by lemma D.0.28 $M_{2\times 2}(\mathbb{R})$ is Morita equivalent with $\mathbb{R}$. The Clifford algebra Cliff ^{q,p} is therefore Morita equivalent with Cliff $^{q,p} \otimes_{\mathbb{R}} M_{2\times 2}(\mathbb{R})$. This implies that the above isomorphism provides a bijection between positive extensions of a Cliff $^{p+2,q}$-module N and negative extensions of a Cliff ^{q,p}-module N'.

It is therefore sufficient to search for connected components of positive extensions of some Clifford module N. Positive extensions of a Clifford module N can equivalently be reinterpreted as gradings of the Clifford module N, [3] proposition 4.19.

Lemma 4.2.10. *Gradings of Clifford modules are in bijective correspondence with positive extensions of Clifford modules.*

Proof. A grading η on a module N of Cliff ^{q,p} is a decomposition $N = H_1 \oplus H_{-1}$ in such a way that Cliff ^{p,q} acts in a graded way. We may define an operator $e_{p+1} : H \to H$ by setting

$$e_{p+1}\psi = \begin{cases} \psi \text{ iff } \psi \in H_1 \\ -\psi \text{ if } \psi \in H_{-1} \end{cases} \tag{4.37}$$

This self-adjoint operator e_{p+1} anti-commutes with the generators $\{e_1, \cdots, e_p, e'_1, \cdots, e'_q\}$ in the Cliff p, q module and squares to $\mathbb{I}$. Adding e_{p+1} to the set of generators therefore forms a Cliff $^{p+1,q}$ module.

For the other way around, let $e_{p+1} : H \to H$ be an operator extending the Cliff ^{p,q} module to a Cliff $^{p+1,q}$ module. Since e_{p+1} is self-adjoint and squares to $\mathbb{I}$, it decomposes the Hilbert space by its eigenspaces with eigenvalue 1 respectively -1. Since e_{p+1} anti-commutes with the odd generators of Cliff ^{p,q}, this decomposition makes the Cliff ^{p,q} module into a graded module. This provides an inverse construction and hence proves the bijectivity of the relation in question.

The proof for the complex goes in a similar fashion. □

The above lemma implies that we can equivalently search for connected components of gradings. In particular we can equip gradings of some Clifford module with the topology it inherits from the positive extensions.

Corollary 4.2.11. *Assume that the search for a Hamiltonian can be reformulated as a Clifford extension problem. The topological phases of $\mathfrak{H}$ are in this case in bijective relation with connected components of gradings of some Clifford module in the way of lemma 4.2.10.*

Let N be a finite dimensional module Clifford module. We are interested in the module $\oplus_{i=1}^{\infty} N$. The following monoid and group are of interest.

Definition 4.2.12. *Fix a finite dimensional Clifford module N. Write $M_N(*)$ for the monoid consisting of elements*

$$(\oplus_{i=1}^{n} N, \eta), \tag{4.38}$$

where η is a grading of $\oplus_{i=1}^{n} N$. Addition is defined by

$$(\oplus_{i=1}^{n} N, \eta) + (\oplus_{i=1}^{m} N, \eta') = (\oplus_{i=1}^{n+m} N, \eta \oplus \eta'). \tag{4.39}$$

Write $Grad_N(Cliff^{p,q})$ for the topological space consisting of elements

$$(\oplus_{i=1}^{n} N, \eta_1, \eta_2), \tag{4.40}$$

where η_1 and η_2 are gradings of $\oplus_{i=1}^{n} N$. The set $Grad_N(Cliff^{p,q})$ forms a commutative monoid under the operation

$$(N, \eta_1, \eta_2) \oplus (N', \eta'_1, \eta'_2) = (N \oplus N', \eta_1 \oplus \eta'_1, \eta_2 \oplus \eta'_2). \tag{4.41}$$

Now define $K_N(*)$ as the group freely generated by $Grad_N(Cliff^{p,q})$ up to the relation

$$(\oplus_{i=1}^{n} N, \eta_1, \eta_2) + (\oplus_{i=1}^{m} N, \eta_1', \eta_2') = (\oplus_{i=1}^{m+n} N, \eta_1 \oplus \eta_1', \eta_2 \oplus \eta_2') \tag{4.42}$$
$$(N, \eta_1, \eta_2) \sim e \text{ iff } \eta_1 \sim \eta_2. \tag{4.43}$$

The group $K_N(*)$ in some sense describes the difference classes ofthe monoid $M_N(*)$. The monoid $M_N(*)$ provides information on what physical behaviour we would expect on $\oplus_{i=1}^{\infty} N$, but it can be hard to calculate. The group $K_N(*)$, on the other hand, can be related to topological K-groups, [23] chapter 4. The trade off is that the group $K_N(*)$ is harder to interpret.

Definition 4.2.13. *Write $Grad(Cliff^{p,q})$ for the topological space consisting of tuples*

$$(N, \eta_1, \eta_2), \tag{4.44}$$

where N is a finite dimensional $Cliff^{p,q}$-module and both η_1 and η_2 are gradings of this module N. The set $Grad(Cliff^{p,q})$ forms a commutative monoid under the operation

$$(N, \eta_1, \eta_2) \oplus (N', \eta_1', \eta_2') = (N \oplus N', \eta_1 \oplus \eta_1', \eta_2 \oplus \eta_2'). \tag{4.45}$$

Write $\overline{KO}^{p,q}(*)$ for the group freely generated by the connected components of $Grad(Cliff^{p,q})$ up to the relations

$$f + g \sim f \oplus g \tag{4.46}$$
$$f \sim e \text{ if } \eta_1 \text{ and } \eta_2 \text{ are path conected.}$$

In the case of a complex Clifford algebra $\mathbb{C}liff^{\,q}$, likewise define $\overline{K}^q(*)$.

As the notation might suggest, these groups are isomorphic to topological K-groups. See Appendix C.

Theorem 4.2.14.

$$KO^{p-q}(*) \cong \overline{KO}^{p,q}(*) \text{ and } K^{-q}(*) \cong \overline{K}^q(*). \tag{4.47}$$

Proof. This is a special case of [24] theorem 4.22 and 5.12. The general theorem is valid for an arbitrary compact topological space, here reduced to a point, and will be stated in the next chapter. □

By construction we find the following identification.

Corollary 4.2.15. *The group $K_N(*)$ is a subgroup of some topological K-group.*

Proof. The group $K_N(*)$ is by construction a subgroup of either $\overline{KO}^{p,q}(*)$ or $\overline{K^{-q}}(*)$. These groups coincide with topological K-groups by theorem 4.2.14. □

In case N is the only irreducible representation of the Clifford algebra, the above statement specializes to isomorphism of the whole group (rather then just a subgroup). Some explicit calculations are made at the end in section 5.4.2.

Chapter 5

Quantum systems with lattice symmetry

5.1 Quantum systems with lattice symmetry

Many physical systems posses a translational symmetry. A common example is a band insulator which will be treated in section 5.3.

Definition 5.1.1. *A quantum system with a d-dimensional lattice symmetry is a quantum system $(H, G, \phi, \tau, \rho^\tau)$ that satisfies the following requirements.*

- *The Hilberspace H is given by*

$$L^2(\mathbb{R}^d, \mathbb{C}) \otimes W \cong L^2(\mathbb{R}^d, W), \tag{5.1}$$

where W is some finite dimensional complex Hilbert space.

- *The symmetry group G is a semi-direct product*

$$G' \times_\alpha \mathbb{Z}^d, \tag{5.2}$$

with respect to some compact Lie group G' and homomorphism $\alpha : G' \to \mathbb{Z}^d$. For the extension $G^\tau \cong G'^\tau \times_{\tilde{\alpha}} \mathbb{Z}^{d^\tau}$, we require $\mathbb{Z}^{d^\tau} \cong \mathbb{Z}^d \times \mathbb{T}$.

- *Finally, for $z^\tau \in \mathbb{Z}^\tau \subset G^\tau$*

$$(\rho^\tau(z^\tau)\psi)(x) = (\rho^\tau(\lambda z)\psi)(x) = \lambda\psi(x + z). \tag{5.3}$$

An automorphism on $\mathbb{Z}$ is fixed once we know the image of -1. There are therefore only two bijective homomorphism, -1 is sent to either 1 or -1. In turn, all automorphisms α on $\mathbb{Z}^d$ are given by

$$(z_1, \cdots, z_d) \to (\pm 1 z_1, \cdots, \pm 1 z_d) \tag{5.4}$$

it makes sense to write

$$\alpha(g')(z) = \alpha(g')z \tag{5.5}$$

for $\alpha : G' \to \times^d \{\pm 1\}^d$.

Note that for $d > 0$ the symmetry group G is no longer compact. The classification theorems in the previous chapters only hold for the special case of zero dimensional quantum systems. The question is now how to generalise to quantum systems with a lattice symmetry.

5.2 Classification of quantum systems with lattice symmetry

In this section we decompose a quantum system with d-dimensional lattice over $\mathbb{T}^d$. In this way we can get rid of the non-compact $\mathbb{Z}^d$ part of the group. This will allow us to extract the same results we had for compact groups ($d = 0$) only now parametrized by $\mathbb{T}^d$.

5.2.1 Equivariant bundles

Preliminary we define what we mean by a Hilbert bundle [32] section 9.

Definition 5.2.1. *A Hilbert bundle over topological space X is a topological space $\mathfrak{E}$ together with a continuous and surjective map $\pi : \mathfrak{E} \to X$ in which each pre-image $\mathfrak{E}_x$ of $x \in X$ has the structure of a Hilbert space. Furthermore, there exists for each $x_0 \in X$ an upen subset $U \subset X$ such that $\pi^{-1}(U)$ is homeomorphic to $H \times U$.*

Two Hilbert bundles $\mathfrak{E}$ and $\mathfrak{E}'$ are isomorphic if there exists a homeomorphism $\psi : \mathfrak{E} \to \mathfrak{E}'$ such that the restrictions to the fibres $\mathfrak{E}_x$

$$\psi : \mathfrak{E}_x \to \mathfrak{E}'_x \tag{5.6}$$

are unitary maps.

In the spirit of [11], we now define a slight variation on an equivariant Hilbert bundle.

Definition 5.2.2. *Let G^τ be a ϕ-twisted extension of a Lie group G and let X be a topological space with a continuous G action. Define a G^τ action on X by*

$$g^\tau \cdot x = \pi(g^\tau) \cdot x, \tag{5.7}$$

where π is the map $\pi : G^\tau \to G$. A ϕ-twisted G^τ-equivariant Hilbert bundle is a complex finite dimensional Hilbert bundle $p : \mathfrak{E} \to X$ equipped with a continuous G^τ action on $\mathfrak{E}$ such that the map

$$\cdot\, g^\tau : \mathfrak{E}_x \to \mathfrak{E}_{g^\tau \cdot x} \tag{5.8}$$

is unitary whenever $\phi(g^\tau) = 1$, and anti-unitary whenever $\phi(g^\tau) = -1$. Furthermore, if $\lambda \in \mathbb{T} \subset G^\tau$, then

$$\lambda \cdot x = \lambda x. \tag{5.9}$$

Write $Vect^{\phi}_{G^\tau}(X)$ for the set of all ϕ-twisted G^τ-equivariant Hilbert bundles over X. Notice that the action of G^τ on X is implied in the notation.

In the light of section 1.4 we can double the dimension and equivalently interpret a ϕ-twisted G^τ-equivariant Hilbert bundle as a real Hilbert bundle with ϕ-graded G^τ action. We yet need a notion of isomorphism.

Definition 5.2.3. *Two ϕ-twisted G^τ-equivariant Hilbert bundles $\mathfrak{E}$ and $\mathfrak{E}'$ are isomorphic iff there exists a homeomorphism $\psi : \mathfrak{E} \to \mathfrak{E}'$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathfrak{E}_x & \xrightarrow{\psi} & \mathfrak{E}'_x \\ \downarrow{\scriptstyle \cdot g^\tau} & & \downarrow{\scriptstyle \cdot' g^\tau} \\ \mathfrak{E}_{g\cdot x} & \xrightarrow{\psi} & \mathfrak{E}'_{g\cdot x} \end{array} \quad . \tag{5.10}$$

We can also define ϕ-twisted G^τ-equivariant bundles using continuous families of operators [11].

Definition 5.2.4. *A set of bounded operators $\{a_x : H \to H \mid x \in X\}$ is called a continuous family over a topological space X on H if the assignment*

$$x \to a_x \tag{5.11}$$

is continuous in the topology of section 2.1.2.

A set of twisted representations $\{\rho_x^\tau : H \to H \mid x \in X\}$ is called a continuous family over a topological space X if the assignment

$$(x, g) \to \rho_x^\tau(g) \tag{5.12}$$

is continuous in the topology defined in section 2.1.2, which in this instance is equivalent to the strong topology.

We arrive at an alternative definition of the ϕ-twisted G^τ-equivariant Hilbert bundles.

Definition 5.2.5. *Let G^τ be some ϕ-twisted extension of a Lie group G, let X be a compact topological space with a continuous G action and let H be a Hilbert space. A ϕ-twisted G^τ-equivariant Hilbert bundle over X is a continuous family of projections P_x over X on finite dimensional subspaces of H together with a continuous family of twisted representations ρ_x^τ on H, for which*

$$\rho_x^\tau(g^\tau) : ImP_x \to ImP_{g\cdot x}. \tag{5.13}$$

Two such ϕ-twisted G^τ-equivariant Hilbert bundles coincide whenever ρ_x^τ and $\rho_x'^\tau$ coincide on ImP_x.

Note that the action of ρ_x^τ outside the subspace $ImP_x \subset H$ should be neglected, since we are only interested in the action of G^τ on the bundle.

Proposition 5.2.6. *Definition 5.2.2 and definition 5.2.5 are equivalent. The bijection is given by assigning to each tuple (P_x, ρ_x^τ) a G^τ-equivariant Hilbert bundle $\mathfrak{E}$ in the sense of definition 5.2.2 as follows. Define the fibres of $\mathfrak{E}$ by*

$$\mathfrak{E}_x = P_x H. \tag{5.14}$$

The topology is induced by the embedding

$$\mathfrak{E} \subset H \times X. \tag{5.15}$$

Lastly, the G^τ-action on $\mathfrak{E}$ is defined through

$$g^\tau \cdot (x, v) = (g \cdot x, \rho_x^\tau(g^\tau)v). \tag{5.16}$$

Proof. We start of by showing that te map is well defined. This can be done in two steps. Firstly, we prove that the images of P_x induce a Hilbert bundle and secondly we show that the action of G^τ is indeed continuous.

In order to prove that $\mathfrak{E}$ is a Hilbert bundle, we need to show that there exists for each $x_0 \in X$ an open neighbourhood $U \subset X$ such that $\mathfrak{E}_U \cong U \times \mathfrak{E}_{x_0}$. The family P_x is continuous and projects on a finite dimensional subspaces of H. There hence exists a neighbourhood U of x_0 such that for all $x \in U$ and for all unit vectors $\psi \in H$

$$\|(P_x - P_{x_0})\psi\| < 1. \tag{5.17}$$

We claim that the map

$$P_{x_0} : P_x H \to P_{x_0} H \tag{5.18}$$

is injective. For the sake of contradiction assume $P_{x_0}\psi = 0$ for some norm one vector $\psi \in P_x H$. This would imply that $\|(P_x - P_{x_0})\psi\| = 1$, contradicting eq. (5.17). This means that eq. (5.18) is indeed injective. By the same reasoning, the map

$$P_x : P_{x_0} H \to P_x H \tag{5.19}$$

is injective. It follows that $\dim(P_x H) = \dim(P_{x_0} H)$. We conclude that eq. (5.18) is in fact bijective.

The bijectivity of eq. (5.18) implies that the map

$$P_{x_0} : \mathfrak{E}_U = U \times P_x H \to U \times P_{x_0} H \tag{5.20}$$

$$(x, \psi) \mapsto (x, P_{x_0}\psi) \tag{5.21}$$

is a continuous fibre-wise linear bijection. The map is surjective and therfore, by the open mapping theorem, open onto its image. If we pick a basis in H and rescale the image of each basis vector, the map will preserve the norm and with that the inner product. We hence found a local trivialisation $\mathfrak{E}_U \cong U \times P_{x_0} H$ proving that the family P_x indeed fixes a Hilbert bundle.

Secondly, we show that eq. (5.16) is indeed continuous. The statement is local, it hence suffices to prove that for each $x_0 \in X$ there exists an open neighbourhood such that the above map is continuous when restricted to E_U. We hence only need to prove the continuity of

$$(g^\tau, x, \psi) \to (g^\tau \cdot x, \rho_x^\tau(g^\tau)\psi), \tag{5.22}$$

which holds by definition.

The above construction can be reversed. For a ϕ-twisted G^τ-equivariant Hilbert bundle $\mathfrak{E}$ in the sense of definition 5.2.2 we can take the trivial bundle $\mathfrak{E} \oplus \mathfrak{E}^\perp \cong X \times H$. We must now define a continuous family of representations and projections on H. Define $\rho_x^\tau(g^\tau)$ on H by letting it act on $\mathfrak{E}_x$ as the multiplication by g^τ and letting it act on $\mathfrak{E}_x^\perp$ simply by $\mathbb{I}$. Define a continuous family of projections P_x by projecting on $\mathfrak{E}_x \subset H$. This construction results in a ϕ-twisted G^τ-equivariant Hilbert bundle in the sense of definition 5.2.5. □

Due to this equivalence we can write a ϕ-twisted G^τ-equivariant bundle over X both as $\mathfrak{E}$ and as (P_x, ρ_x^τ).

Example 5.2.7. *Let G be the trivial group. The extended symmetry group is $G^\tau = \mathbb{T}$. The group $\mathbb{T}$ is represented by scalar multiplication on the fibres. Of course, all continuous families of projections commute with this representation. The ϕ-twisted $\mathbb{T}$-equivariant bundles over X are simply all continuous families of projections over X. In the light of proposition 5.2.6, these are the complex Hilbert bundles over X.*

Using the viewpoint of continuous families to describe Hilbert bundles we can define deformations of such bundles.

Definition 5.2.8. *Let ρ_x^τ be a continuous family of representations over X. A deformation of twisted equivariant Hilbert bundles is a continuous family of projections $P_{x,t}$ over $X \times [0,1]$, such that each $t' \in [0,1]$ $(P_{x,t'}, \rho_x^\tau)$ is a ϕ-twisted G^τ-equivariant Hilbert bundle.*

Two twisted equivariant Hilbert bundles (P_x, ρ_x^τ) and (P'_x, ρ_x^τ) over X are called homotopic whenever there exists a deformation $(P_{x,t}, \rho^\tau)$ for which $(P_{x,0}, \rho_x^\tau) = (P_x, \rho_x^\tau)$ and $(P_{x,1}, \rho_x^\tau) = (P'_x, \rho_x^\tau)$.

In case we are dealing with a compact space X, deformation classes and isomorphism classes coincide.

Proposition 5.2.9. *Let X be a compact topological space. Two homotopic ϕ-twisted G^τ-equivariant Hilbert bundles are isomorphic.*

Proof. Let $(P_{x,t}, \rho_x^\tau)$ be a deformation between two ϕ-twisted G^τ-equivariant Hilbert bundles. For each $(x_0, t_0) \in X \times [0,1]$ there exists an open interval $V \subset [0,1]$ containing t_0 such that for all $t \in V$

$$\|P_{x,t} - P_{x_0,t_0}\| < 1. \tag{5.23}$$

Since X is compact this can be done uniformly. That is to say, for each $t_0 \in [0,1]$ there exists an open neighbourhood V of t_0, such that for all $x \in X$ and all $t' \in V$

$$\|P_{x,t'} - P_{x,t_0}\| < 1. \tag{5.24}$$

Both Hilbert bundles (P_{x,t_0}, ρ_x^τ) and $(P_{x,t'}, \rho_x^\tau)$ are a sub-bundle of a trivial Hilbert bundle $X \times H$. That is to say, we can write (x, v) for an element in the Hilbert bundle (P_{x,t_0}, ρ_x^τ), with $x \in X$ and $v \in ImP_{x,t_0} \subset H$. Now construct a map ψ between the ϕ-twisted G^τ-equivariant Hilbert bundles (P_{x,t_0}, ρ_x^τ) and $(P_{x,t'}, \rho_x^\tau)$ as follows

$$\psi(x, v) = (x, P_{x,t_0} v). \tag{5.25}$$

By the same reasoning as in proposition 5.2.6 this map is bijective. If we rescale the images of an orthonormal basis the map preserves the norm as well. By construction ψ respects the fibres and is linear within these fibres. Furthermore, P_{x,t_0} projects on an invariant subspace of ρ_x^τ, whence ψ respects the G^τ action.

It is left to show that ψ is a homeomorphism. We first show that the map is open. A basis for the topology of $\mathfrak{E}$ is given by sets of the form $U \times B_\epsilon(V')$, where U is an open set of X and

$$B_\epsilon(v') = \{v \in ImP_x \mid \|v - v'\| < \epsilon.\} \tag{5.26}$$

Due to the linearity of ψ we may without loss of generality take $v' = 0$. The map ψ preserves the norm and is surjective onto $\mathfrak{E}'$. Sets of the form $U \times B_\epsilon(0)$ are therefore sent to sets of this same form. In particular the map is open. The same goes for the inverse of ψ. We may conclude that ψ is a homeomorphism.

We found that all Hilbert bundles in $\{(P_{x,t}, \rho_x^\tau) \mid t \in V\}$ are isomorphic. The space $[0,1]$ is compact. A finite repetition of the argument hence yields that all Hilbert bundles in $\{(P_{x,t}, \rho_x^\tau) \mid t \in [0,1]\}$ are isomorphic. In particular $(P_{x,0})$ and $P_{x,1}$ are isomorphic as desired. □

In case of quantum systems we consider a fixed representation ρ^τ of G^τ on H.

Definition 5.2.10. *Let ρ^τ be some twisted representation of G^τ on H. Write $Vect^\phi_{G^\tau,\rho^\tau}(X) \subset Vect^\phi_{G^\tau}(X)$ for the set consisting of elements*

$$(P_x, \oplus_{i=1}^n \rho^\tau), \tag{5.27}$$

for some $n \in \mathbb{N}$.

This set forms a sub-monoid under taking direct sums

$$(P_x, \oplus_{i=1}^n \rho^\tau) + (P'_x, \oplus_{i=1}^m \rho^\tau) = (P_x \oplus P'_x, \oplus_{i=1}^{n+m} \rho^\tau). \tag{5.28}$$

Lastly we construct a group lying closet to this monoid. Recall the notation

$$[X, Y] \tag{5.29}$$

for the homotopy classes of maps from X to Y.

Definition 5.2.11. *Write Rep_{ρ^τ} for the space consisting of elements*

$$(\oplus_{i=1}^n \rho^\tau, P, P'), \tag{5.30}$$

where P and P' are projections on finite dimensional subspaces commuting $\oplus_{i=1}^n \rho^\tau$. This space forms a monoid under the operation $\oplus$ given by

$$(\oplus_{i=1}^n \rho^\tau, P, P') \oplus (\oplus_{i=1}^m \rho^\tau, E, E') = (\oplus_{i=1}^{n+m} \rho^\tau, P \oplus E, P' \oplus E'). \tag{5.31}$$

We say $(\oplus_{i=1}^n \rho^\tau, P, P')$ is trivial whenever $PH \sim PH'$ as invariant subspaces.

Define the group $K^\phi_{G^\tau,\rho^\tau}(X)$ to be the group freely generated by $[X, Rep_{\rho^\tau}]$ up to the relations

$$(f+g)(x) = f(x) \oplus g(x) \tag{5.32}$$

$$f \sim e \text{ iff } f(x) \text{ is trivial for some and hence all } x \in X. \tag{5.33}$$

In some sense $K^\phi_{G^\tau,\rho^\tau}(X)$ describes difference classes of $Vect^\phi_{G^\tau,\rho^\tau}(X)$.

Now look at a similar model. Let N be a finite dimensional Clifford module (see Appendix D). In the spirit of definition 5.2.10 define $M_N(X)$ to be the monoid with elements

$$(e_x, \oplus_{i=1}^n N), \tag{5.34}$$

where e_x is a continuous family of self-adjoint operators over X that square to $\mathbb{I}$ and that anti-commute the generators of the Clifford module $\oplus_{i=1}^n N$. The addition is defined by

$$(e_x, \oplus_{i=1}^n N) + (e'_x, \oplus_{i=1}^m N) = (e_x \oplus e'_x, \oplus_{i=1}^{n+m} N). \tag{5.35}$$

By lemma 4.2.10 the elements of $M_N(X)$ consist of continuous families of gradings of $\oplus_{i=1}^n N$ over X. We would again like to construct a group lying 'close' to this monoid. Recall the monoid $Grad_N(\mathrm{Cliff}^{p,q})$ in definition 4.2.12.

Definition 5.2.12. *Fix a finite-dimensional Clifford module N. Write $K_N(X)$ for the group freely generated by elements*

$$[X, Grad_N(Cliff^{p,q})], \tag{5.36}$$

up to the relations

$$(f+g)(x) = f(x) \oplus g(x) \tag{5.37}$$

$$f \sim e \text{ iff for } f(x) = (\oplus_{i=1}^n N, \eta_1, \eta_2) \text{ we have } \eta_1 \sim \eta_2. \tag{5.38}$$

In some sense $K_N(X)$ describes difference classes of $M_N(X)$. Compare the monoid $M_N(X)$ and the group $K_N(X)$ with definition 4.2.12.

5.2.2 Decomposition over $\mathbb{T}^d$

The Hilbert space $L^2(\mathbb{R}^d, W)$ can in some sense be decomposed over the torus [11] Appendix D. To make this precise we introduce the direct integral, which generalises the direct sum.

Definition 5.2.13. *Let X be a Borel space with measure μ and let $\mathcal{H}$ a function assigning to each $x \in X$ a Hilbert space $\mathcal{H}(x)$. The direct integral of $\mathcal{H}$ over X consists of functions*

$$f : X \to \cup_{x \in x} \mathcal{H}(x) \text{ such that } f(x) \in \mathcal{H}(x) \text{ and } \int_X \|f(x)\|^2 \, d\mu < \infty, \tag{5.39}$$

up to almost everywhere equality. This set of functions forms a Hilbert space under the inner-product

$$\langle f, g \rangle = \int_X \langle f(x), g(x) \rangle d\mu. \tag{5.40}$$

Now we apply this definition to the situation at hand. Pick $X = \mathbb{T}^d$ and $\forall_{\lambda \in \mathbb{T}^d}\ \mathcal{H}(\lambda) = L^2([0,1], W)$. In this respect $L^2(\mathbb{R}^d, W)$ is a direct integral of $L^2([0,1], W)$ over $\mathbb{T}^d$. The isomorphism F between these Hilbert spaces must take vectors in $L^2(\mathbb{R}^d, W)$ to functions f on $\mathbb{T}^d$ with values in $L^2([0,1], W)$. In particular $f = F\psi$ must have two entries, firstly an element $\lambda \in \mathbb{T}^d$ and secondly an element $x \in x \in [0,1]^d$.

Theorem 5.2.14. *The Hilbert space $L^2(\mathbb{R}^d, W)$ is isomorphic to the direct integral of $L^2([0,1], W)$ over $\mathbb{T}^d$.*

Recall that $\mathbb{T}^d$ is the topological space of characters of $\mathbb{Z}^d$. The character $\lambda \in \mathbb{T}^d$ is defined by

$$\lambda(z) = \lambda^z = \lambda_1^{z_1} \cdots \lambda_d^{z_d}. \tag{5.41}$$

Note that we can replace $\mathbb{R}^d/\mathbb{Z}^d$ by $[0,1)^d \subset \mathbb{R}^d$ as domain of integration.

The isomorphism F is given by:

$$(F\psi)(x, \lambda) = \sum_{z \in \mathbb{Z}^d} \lambda(z)^{-1} \rho^\tau(z) \psi(x). \tag{5.42}$$

Its inverse is

$$F^{-1}(f)(x) = \int_{\mathbb{T}^d} d\lambda \ \lambda(z')(f)(\lambda, x'), \tag{5.43}$$

where we pick $x' \in [0,1)^d$ and $z' \in \mathbb{Z}^d \subset \mathbb{Z}^{d\tau}$ in such a way that $x' + z' = x$.

Proof. This is the Bloch-Floquet theorem, [11] equation D.19. □

Remark 5.2.15. *From here on F denotes the isomorphism in theorem 5.2.14.*

For each $\psi \in L^2(\mathbb{R}^d, W)$ and $\lambda \in \mathbb{T}^d$

$$F\psi(\lambda, \cdot) = f(\lambda, \cdot) \in L^2([0,1), W). \tag{5.44}$$

That is to say, if we leave the second entry of $F\psi$ open we obtain a vector in $L^2([0,1), W$. In fact for each $\lambda \in \mathbb{T}^d$ we have the following isomorphism

$$\{(F\psi)(\lambda, \cdot) \mid \psi \in L^2(\mathbb{R}^d, W)\} \cong L^2([0,1), W). \tag{5.45}$$

This fact will frequently be used in the proceeding text.Now let's see how ρ^τ behaves on these spaces. Recall from definition 5.1.1 that G^τ is a semi-direct product $G'^\tau \times_\alpha \mathbb{Z}^{d^\tau}$. For $g \in G^\tau$ we have

$$gz = (\alpha(g)z)g. \tag{5.46}$$

We find

$$\begin{aligned} F(\rho^\tau(g)\psi)(\lambda,\cdot) &= \sum_{z\in\mathbb{Z}} \lambda(z)^{-1}\rho^\tau(z)\rho^\tau(g)\psi(\cdot) \\ &= \rho^\tau(g)\sum_{z\in\mathbb{Z}}\lambda(z)^{-1}\rho^\tau(z)^{\alpha(g)}\psi(\cdot) = \rho^\tau(g)((F\psi)(\lambda^{-\alpha(g)},\cdot)). \end{aligned} \tag{5.47}$$

This way, $\rho^\tau(g)$ defines a map

$$\rho^\tau_\lambda(g) : \{(F\psi)(\lambda,\cdot) \mid \psi \in L^2(\mathbb{R}^d, W)\} \to \{F\psi(\lambda^{-\alpha(g)},\cdot) \mid \psi \in L^2(\mathbb{R}^d, W)\} \tag{5.48}$$

by

$$(\rho^\tau(g)_\lambda(F\psi)(\lambda,\cdot))(x') = (F\rho^\tau(g)\psi)(\lambda^{\alpha(g)}, x'). \tag{5.49}$$

Keep in mind that this map does, in general, not need to be linear or satisfy any other properties. For $z' \in \mathbb{Z}^d$ we have

$$\begin{aligned} F(\rho^\tau(z')\psi)(\lambda,\cdot) &= \sum_{z\in\mathbb{Z}}\lambda(z)^{-1}\rho^\tau(z)\rho^\tau(z')\psi(\cdot) = \sum_{z\in\mathbb{Z}}\lambda(z''z'^{-1})^{-1}\rho^\tau(z'')\psi(\cdot) \\ &= \lambda(z')(F\psi)(\lambda,\cdot). \end{aligned} \tag{5.50}$$

The spaces $F(\rho^\tau(z')\psi)(\lambda,\cdot)$ occurring in the direct integral can hence be seen as 'eigenspaces' of the translations. The family $\{\rho^\tau_\lambda(g)\}$ determines $\rho^\tau(g)$.

Lemma 5.2.16. *Let ρ^τ be a twisted representation. Write $\rho^\tau(g)_\lambda$ for the family of maps over $\mathbb{T}^d$ associated to $\rho^\tau(g)$ via eq.* (5.49)*. We can recover $\rho^\tau(g)$ from this family by setting*

$$(\rho^\tau(g)\psi)(x) = \int_{\mathbb{T}^d} d\lambda\ \lambda(-z')(\rho^\tau(g)_\lambda F\psi(\lambda,\cdot))(x'), \tag{5.51}$$

for $x = x' + z'$.

Proof. The proof is given by the following computation

$$\begin{aligned} \int_{\mathbb{T}^d} d\lambda\ \lambda(-z')\left(\rho^\tau(g)_\lambda F\psi(\lambda,\cdot)\right)(x') &= \left(\int_{\mathbb{T}^d} d\lambda\ \lambda(-z')\sum_{z\in\mathbb{Z}^d}\lambda(-z)\rho^\tau(z)(\rho^\tau(g)\psi)\right)(x') \\ &= \int_{\mathbb{T}^d} d\lambda \sum_{z\in\mathbb{Z}^d}\lambda(-z'-z)\rho^\tau(z)(\rho^\tau(g)\psi)(x') \\ &= \left(\int_{\mathbb{T}^d} d\lambda\sum_{z\in\mathbb{Z}^d}\lambda(-z)\rho^\tau(z'+z)(\rho^\tau(g)\psi)\right)(x') \\ &= \left(\int_{\mathbb{T}^d} d\lambda\sum_{z\in\mathbb{Z}^d}\lambda(-z)\rho^\tau(z)\rho^\tau(z)\rho^\tau(g)\psi\right)(x') \\ &= (\rho^\tau(z')\rho^\tau(g)\psi)(x') = \rho^\tau(g)\psi(x). \end{aligned} \tag{5.52}$$

□

Inspired by this lemma, we call ρ^τ_λ the decomposition of ρ^τ over $\mathbb{T}^d$. Since $\rho^\tau_\lambda(z)$ was already fixed by sending $\psi(\lambda,\cdot)$ to $\lambda(z)\psi(\lambda,\cdot)$ we only need to worry about ρ^τ restricted to G'^τ. We now require the following.

Assumption 5.2.17. *For a quantum system $(H, G, \phi, \tau, \rho^\tau)$ with lattice symmetry, we require that the decomposition of ρ^τ over $\mathbb{T}^d$ is a continuous family of twisted representations ρ^τ_λ of G'^τ on $L^2([0,1], W)$. In this way G'^τ acts on $\mathbb{T}^d$ by $g' \cdot \lambda = \lambda^{\alpha(g')}$.*

See definition 5.2.4 for what is meant by a continuous family. Since G' is compact this decomposition will allow us under sufficiently strong conditions to re-derive the already found classification theorems for compact groups, only this time parametrized over $\mathbb{T}^d$.

5.2.3 Classification theorems

Quantum systems

Given a Hilbert space H and symmetry group $G \cong G' \times_\alpha \mathbb{Z}^d$, we wonder what quantum systems we can construct. Use the theory established in section 1.2 to find the possible twisted extensions of G'. Under assumption 5.2.17 all continuous families of twisted representations of these extensions G'^τ over $\mathbb{T}^d$ now determine the possible quantum systems.

Corollary 5.2.18. *A quantum system with d-dimensional lattice symmetry for which $G \cong G' \times_\alpha \mathbb{Z}^d$ is a continuous family of ϕ-twisted representations of G'^τ on $L^2([0,1], W)$ over $\mathbb{T}^d$.*

Observables

For the observables we proceed in the same spirit as we did for the operators $\rho^\tau(g^\tau)$. We would again like to decompose the operator over $\mathbb{T}^d$. An observable commutes with the translations, therefore

$$F(a\psi)(\lambda,\cdot) = \sum_{z\in\mathbb{Z}} \lambda(z)^{-1}\rho^\tau(z)a\psi(\cdot) = a\sum_{z\in\mathbb{Z}} \lambda(z)^{-1}\rho^\tau(z)\psi(\cdot) = a((f\psi)(\lambda,\cdot)). \tag{5.53}$$

This way a defines for each $\lambda \in \mathbb{T}^d$ a map a_λ on $L^2([0,1), W)$ by setting for each $x' \in [0,1)$

$$(a_\lambda(F\psi)(\lambda,\cdot))(x') = (Fa\psi)(\lambda, x'). \tag{5.54}$$

The family $\{a_\lambda\}$ determines a by the same reasoning as in lemma 5.2.16. In order to get things under control we make the following assumption.

Assumption 5.2.19. *Let a be a bounded observable. We can pick a decomposition a_λ of a that is a continuous family of operators over $\mathbb{T}^d$.*

This assumptions expresses that a should depend continuously on the momentum. Compare this assumption with assumption 5.3.2 below.

The assumption turns out to be sufficient for a bijection between continuous families of self-adjoint bounded operators on $L^2([0,1], W)$ over $\mathbb{T}^d$ and bounded observables a on H. First we consider the case without any additional symmetries besides lattice translation.

Proposition 5.2.20. *Given a quantum system with lattice symmetry for which $G'^\tau = e$. Assign to each continuous family of bounded self-adjoint operators $\{a_\lambda\}$ over $\mathbb{T}^d$ on $L^2([0,1], W)$ an operator a as follows. The operator a is the unique operator for which for each $x \in \mathbb{R}^d$*

$$(a\psi)(x) = \int_{\mathbb{T}^d} d\lambda\ \lambda(-z')(a_\lambda(F\psi)(\lambda,\cdot))(x'), \tag{5.55}$$

where we again write $x = x' + z'$, with $x' \in [0,1]^d$ and $z' \in \mathbb{Z}^d$ and where F is the isomorphism in theorem 5.2.14. The constructed mapping is a bijection between bounded observables of this quantum system and continuous families of bounded self-adjoint operators on $L^2([0,1], W)$ over $\mathbb{T}^d$.

Proof. By assumption 5.2.19 this map is surjective. We proceed by showing that it is injective as well. Let a_λ and a'_λ be two distinct continuous families. There exists a $\lambda_0 \in \mathbb{T}^d$ and $v \in L^2([0,1), W)$ such that $a_\lambda v \neq a'_\lambda v$. By continuity there exists an open neighbourhood $U \subset \mathbb{T}^d$ of λ_0 such that $\|(a_\lambda - a'_\lambda)v\| > 0$ for $\lambda \in U$. Pick ψ such that $F\psi \in L^2([0,1) \times \mathbb{T}^d, W)$ is the function

$$(F\psi)(x,\lambda) = \begin{cases} v(x) \text{ if } \lambda \in U, \\ 0 \text{ if } \lambda \notin U. \end{cases} \tag{5.56}$$

For this vector ψ we find

$$\|(a-a')\psi\| = \|F\left((a-a')\psi\right)\| = \|(a_\lambda - a'_\lambda)F\psi\| = \left(\int_{\mathbb{T}^d} d\lambda \int_{[0,1)} dx\ |(a_\lambda - a'_\lambda)(F\psi)(\lambda,x)|^2\right)^{\frac{1}{2}} \tag{5.57}$$

$$= \left(\int_{U\subset\mathbb{T}^d} d\lambda\ \|(a_\lambda - a'_\lambda)v\|^2\right)^{\frac{1}{2}} > 0.$$

This implies that the two distinct families a_λ and a'_λ are mapped to distinct operators.

Lastly we need to show that $\{a_\lambda\}$ is self-adjoint and bounded iff a is self-adjoint and bounded.

The operator a is self-adjoint iff $a = a^*$. Due to the injectivity this is iff $(a^*)_\lambda = a_\lambda$. Since $(a^*)_\lambda = (a_\lambda)^*$ we find that a is self-adjoint iff the family $\{a_\lambda\}$ is self-adjoint.

Next we need to show that an operator a is bounded iff a_λ is bounded for each $\lambda \in \mathbb{T}^d$. First assume that a_λ is bounded for each $\lambda \in \mathbb{T}^d$. We find

$$\|a\| = \sup_\psi \|a\psi\| = \sup_\psi \|F(a\psi)\| \leq \sup_\lambda \sup_{(F\psi)(\lambda,\cdot)} \|a_\lambda (F\psi)(\lambda,\cdot)\| = \mathrm{Sup}_{\lambda\in\mathbb{T}^d} \|a_\lambda\|. \tag{5.58}$$

We claim that right-hand side of this equation is bounded. Let λ_n be any sequence in $\mathbb{T}^d$. Since $\mathbb{T}^d$ is compact and since $a_\lambda\psi$ varies continuously over $\mathbb{T}^d$ we find that for all $\psi \in H$ the set $\{a_{\lambda_n}\psi\}$ is bounded. Therefore, by the uniform boundedness principle, we find that $\|a_{\lambda_n}\|$ is bounded.

On the other hand assume that a is bounded. For each $\lambda \in \mathbb{T}^d$ there exists a sequence of unit vectors $\{v_n\} \subset L^2(\mathbb{R}^d/\mathbb{Z}^d, W)$ such that

$$\|a_\lambda v_n\| \to \|a_\lambda\|. \tag{5.59}$$

Let U_n be a sequence of open sets such that $U_1 \supset U_2 \supset \cdots$ and $\cap_{n=1}^\infty U_n = \{\lambda\}$. Now define the sequence $\{\psi_n\}$ in $L^2(\mathbb{T}^d \times [0,1)^d, W)$ by

$$\psi_n(\lambda,x) = \begin{cases} |U_n|^{-1} v_n(x) \text{ if } \lambda \in U_n \\ 0 \text{ otherwise} \end{cases}, \tag{5.60}$$

where $|U_n| = \int_{U_n} d\lambda$. For this sequence of unit vectors we have

$$\begin{aligned} \|a\| \geq \lim_{n\to\infty} \left\|aF^{-1}\psi_n\right\| &= \lim_{n\to\infty} \left\| \int_{\mathbb{T}^d} \lambda(-z) a_\lambda \psi_n(\lambda, \cdot) d\lambda \right\| \\ &= \lim_{n\to\infty} \left\| \int_{U_n} \lambda(-z) a_\lambda \psi_n(\lambda, \cdot) d\lambda \right\| = \|a_\lambda\| \,. \end{aligned} \tag{5.61}$$

□

We found that we can uniquely decompose each observable a over $\mathbb{T}^d$. The requirement of commuting with ρ^τ translates to

$$\rho^\tau_\lambda(g'^\tau) a_\lambda \rho^\tau_\lambda(g'^\tau)^{-1} = a_{g'\cdot\lambda}. \tag{5.62}$$

We only need to worry about G'^τ since the family a_λ commutes with $\rho^\tau(\mathbb{Z}^d)$ by construction. The equation above provides that a_λ is fixed as soon as we know it on $\mathbb{T}^d/G'$. We can therefore replace $\mathbb{T}^d$ by $\mathbb{T}^d/G'$ and G' by the subgroup G'' consisting of elements stabilizing $\mathbb{T}^d$. The observables are now given by continuous families of self-adjoint operators a_λ over $\mathbb{T}^d/G'$ that commute with $\rho^\tau_\lambda(G''^\tau)$. The family ρ^τ_λ restricted to G''^τ respects the fibres, since G'' acts trivially on $\mathbb{T}^d$.

Assumption 5.2.21. *We assume that there exists a ϕ-twisted representation ρ of G''^τ such that $\rho^\tau_\lambda|_{G''^\tau} = \rho$ for all $\lambda \in \mathbb{T}^d/G'$.*

The following definition now makes sense.

Definition 5.2.22. *Given a quantum system with lattice symmetry with symmetry group $\mathbb{Z}^d \times G'$. Let $G'' \subset G'$ be the stabiliser of $\mathbb{T}^d$. Let ρ be the representation of G'' in assumption 5.2.21. We define the algebra of observables A of this quantum system a quantum system to be the algebra of bounded operators that intertwine ρ, where ρ is the representation in assumption 5.2.21.*

Corollary 5.2.23. *Observables are given by continuous maps*

$$s : \mathbb{T}^d/G' \to A_{sa}. \tag{5.63}$$

Since A is the algebra of observables of a quantum system with compact group G'', the results from section 2.4 can be applied to find the general form of this algebra.

States

For quantum systems with compact groups we saw in theorem 3.2.3 that (normal) pure states could be identified with irreducible subspaces of ρ^τ. A twisted representation of a quantum system with lattice does not have any irreducible subspaces. The spaces $L^2([0,1], W)$ in theorem 5.2.14 are as close as we can get to irreducible subspaces. It therefore makes sense to define the following.

Definition 5.2.24. *A quantum state of a quantum system with lattice symmetry is a functional on the observables of the following form*

$$a_\lambda \to \int_{\mathbb{T}^d} d\lambda \omega_\lambda(a_\lambda), \tag{5.64}$$

where each ω_λ is a normal state on $B(L^2([0,1], W))$ in such a way that

$$\int_{\mathbb{T}^d} d\lambda \omega_\lambda(\mathbb{I}_\lambda) = 1. \tag{5.65}$$

The normal pure states on $B(L^2([0,1],W))$ are called the pure states.

Note that the pure quantum states are not quantum states. They are called pure states since we can obtain all states by taking convex sums and integrals of the pure states. Since the group G' acting on $B(L^2([0,1],W))$ is compact, we can apply theorem 3.2.3 to classify all equivalence classes of normal states upon restriction the the algebra of observables.

Topological phases

Recall from lemma 4.2.5 that we may restrict to bounded Hamiltonians. The Hamiltonian is therefore a bounded observable and can be regarded as a continuous family h_λ over $\mathbb{T}^d$. We are interested in the topological phases of the Hamiltonians. We should therefore study how deformation classes of Hamiltonians relate to deformation classes of their decompositions.

Two Continuous families of bounded operators h_λ and h'_λ are homotopic whenever there exists a continuous family $h_{\lambda,t}$ over $\mathbb{T}^d \times [0,1]$ such that $h_{\lambda_0} = h_\lambda$ and $h_{\lambda,1} = h'_\lambda$.

Lemma 5.2.25. *Two Hamiltonians h and h' are connected iff their respective decompositions $\{h_\lambda\}$ and $\{h'_\lambda\}$ are homotopic.*

Proof. Assume that the families h_λ and h'_λ are homotopic via the path $h_{\lambda,t}$. Write $h_{t'}$ for the Hamiltonian with decomposition $h_{\lambda,t'}$. We find

$$\begin{aligned}
\lim_{t\to t'} h_t\psi(x) &= \left(\lim_{t\to t'} \int_{\mathbb{T}^d} \lambda(-z) h_{\lambda,t}(F\psi)(\lambda,\cdot)(x)d\lambda\right) \\
&= \left(\int_{\mathbb{T}^d} \lambda(-z) \lim_{t\to t'} h_{\lambda,t}(F\psi)(\lambda,\cdot)(x)d\lambda\right) \\
&= \left(\int_{\mathbb{T}^d} \lambda(-z) h_{\lambda,t'} F\psi(\lambda,\cdot)(x)d\lambda\right) = h_{t'}\psi(x).
\end{aligned} \tag{5.66}$$

The integral and limit can be swapped since $h_{\lambda,t}$ is a continuous family over a compact space and hence as already seen bounded. We found that h_t is a continuous path. By construction it connects h and h'.

For the other way around assume that h and h' are path connected via the continuous path h_t. Write $h_{\lambda,t}$ for the continuous decomposition of h_t. For the sake of contradiction assume that there would exists a vector $\psi \in L^2([0,1],W)$ such that

$$\lim_{n\to\infty} h_{\lambda,t_n} f \neq h_{\lambda,t} f \tag{5.67}$$

for some sequence t_n converging to t and some $\lambda \in \mathbb{T}^d$. Since the family is continuous we find that there exists an open neighbourhood U of λ such that the equation above holds. For the vector f in the direct integral of $L^2([0,1],w)$ over $\mathbb{T}^d, W)$ defined by

$$f(x,\lambda) = \begin{cases} \psi(x) \text{ if } \lambda \in U \\ 0 \text{ if } \lambda \notin U \end{cases}, \tag{5.68}$$

we find

$$\lim_{n\to\infty} h_{t_n} F^{-1} f \neq h_t F^{-1} f. \tag{5.69}$$

This contradicts the assumption. Therefore, $h_{\lambda,t}$ is a continuous path connecting h_λ and h'_λ. □

The lemma above grants that we can look at deformation classes of families h_λ instead of deformation classes of Hamiltonians h.

If we make no restrictive assumptions there is only one topological phase. The continuous path

$$th \tag{5.70}$$

connects any Hamiltonian h to 0. For non-trivial results we need to restrict to a subset $\mathfrak{H}$ of the Hamiltonians. In the following we will study the subset of Hamiltonians that have a gap in their spectrum. These kind of systems have a clear physical meaning on which we will dwell in the next section.

5.3 Insulators

In this intermezzo section we give a brief outline of insulators and their topological phases. The discussion will justify why we restrict to gapped Hamiltonians and provides some applications.

We start with defining what we mean by an insulator.

Definition 5.3.1. *An insulator is a quantum system with lattice symmetry together with some fixed Hamiltonian h, satisfying $[x-\epsilon, x+\epsilon] \cap \sigma(h) = \emptyset$ for some $x \in \mathbb{R}$ and $\epsilon > 0$.*

The property $[x-\epsilon, x+\epsilon] \cap \sigma(h) = \emptyset$ is referred to as h being gapped. Take $x = 0$ for the sake of simplicity. As we will see later on the gap gives rise to the insulating property of the material. The requirement that the Hamiltonian h commutes with the translations originates from the lattice symmetry of the crystal.

5.3.1 Band structure

Since we assume the Hamiltonian to commute with the lattice translations $\rho^\tau(\mathbb{Z}^d)$, we can in the way of definition 2.1.15 assign to each $\lambda \in \sigma(\rho^\tau(z))$ a set $U_\lambda \subset \sigma(h)$ belonging to it. Since $\mathbb{Z}^d$ is commutative, theorem 2.1.14 implies that this can be done simultaneously for all translations and that for each $z \in \mathbb{Z}^d$, the space $\sigma(\rho^\tau(z))$ is the space of characters of $\mathbb{Z}^d$. Recall that the space of characters of $\mathbb{Z}^d$ is $\mathbb{T}^d$.

Assumption 5.3.2. *For all $\lambda \in \mathbb{T}^d$ the space $U_\lambda \subset \mathbb{R}$ is discrete. There exists a sequence of continuous maps $f_1, f_2, f_3, \cdots$*

$$f_i : \mathbb{T}^d \to \sigma(h) \tag{5.71}$$

such that $\cup_{i=1}^{\infty} f_i(\lambda) = U_\lambda$ for all $\lambda \in \mathbb{T}^d$.

Physically, this assumption expresses that the energy depends continuously on the momentum.

Definition 5.3.3. *A function f_i in assumption 5.3.2 is called a band of the particular insulator in question.*

We obtain in this way a so-called band structure plot [27] page 161 of the insulator. This is a graph depicting all bands of an insulator. An example of such a plot is shown in the figure below. The plot is for $d = 1$ and shows two bands.

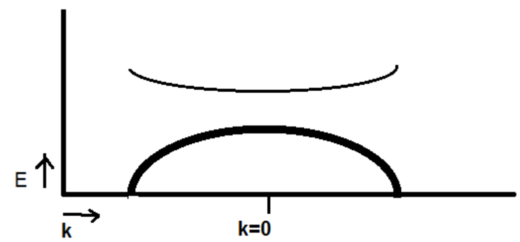

Figure 5.1: Schematic view of an insulator with two energy eigenstates for each k.

In the diagram we relabelled the characters $\mathbb{T}$ of $\mathbb{Z}$ by values $k \in [-\pi, \pi]$, in which the value k stands for the character $\lambda = e^{ik} \in \mathbb{T}^1$. Note that the points π and $-\pi$ are identified.

The band structure plot only describes the energy values in $\sigma(h)$ for each $\lambda \in \sigma(\rho^\tau(z)) = \mathbb{T}^d$. If we also want to keep track of what subspaces of H belong to the energy values (values in $\sigma(h)$), we arrive at the starting point of topological band structure. Recall from the previous section that we can write the Hamiltonian h as a continuous family h_λ of Hamiltonians on $L^2([0,1], W)$ over $\mathbb{T}^d$. We need the following lemma.

Lemma 5.3.4. *Let h be a Hamiltonian, h_λ its continuous decomposition and $f_1, f_2, \cdots$ its bands. We have*

$$\cup_{i=1}^{\infty} f_i(\lambda') = \sigma(h_{\lambda'}). \tag{5.72}$$

Proof. Set $f_i(\lambda') = E$. Since $\lambda' \in \sigma(\rho^\tau(z))$ and $E \in \sigma(h)$ and since E belongs to λ', there exists a sequence of unit vectors ψ_n such that both

$$\|h\psi_n - E\psi_n\| \to 0 \text{ and } \|(\rho(z)\psi_n - \lambda'\psi_n)\| \to 0. \tag{5.73}$$

Under the isomorphism F in theorem 5.2.14 the second part of this equation is equivalent with

$$\|(F\rho(z)\psi_n - \lambda' F\psi_n)\| \to 0. \tag{5.74}$$

Since

$$F\rho(z)\psi_n(\lambda, x) = \rho_\lambda F\psi(\lambda, \cdot)(x) = \lambda F\psi_n(\lambda, \cdot)(x) \tag{5.75}$$

it follows that $F\psi_n(\lambda, \cdot) \to 0$ for $\lambda \neq \lambda'$. That is to say, the sequence $F\psi_n$ is centred around $\lambda' \in \mathbb{T}^d$. The family h_λ is continuous and the limit therefore only depends on $h_{\lambda'}$. The firs part of eq. (5.73) now implies that

$$\|h_{\lambda'} F\psi_n(\lambda', \cdot) - E\psi(\lambda', \cdot)\| \to 0. \tag{5.76}$$

In other words $E \in \sigma(h_{\lambda'})$.

For the other way around, assume that $E \in \sigma(h_{\lambda'})$. Since $\sigma(h_{\lambda'})$ is discrete there exists a unit vector $v \in L^2([0,1], W)$ such that $h_{\lambda'}v = Ev$. Construct f_n in the direct integral of $L^2([0,1], W)$ over $\mathbb{T}^d$ as follows. Let U_n be a sequence of open neighbourhoods such that $\cap_{n=1}^{\infty} U_n = \{\lambda'\}$. Define a sequence $\{f_n\}$ by

$$f_n(\lambda, x) = \begin{cases} |U_n|^{-1} v_n(x) \text{ if } \lambda \in U_n \\ 0 \text{ otherwise} \end{cases}, \tag{5.77}$$

By construction we find for $\psi_n = F^{-1} f_n$ that $\|h\psi_n - E\psi_n\| \to 0$. We conclude that $E \in \sigma(h)$. This implies that $E \in \cup_{i=1}^{\infty} f_i(\lambda')$. □

Since assumption 5.3.2 implies that for each $\lambda' \in \mathbb{T}^d$ the spectrum of $h_{\lambda'}$ is discrete, we can speak of an eigenspace of $h_{\lambda'}$ belonging to a certain value $f_i(\lambda')$.

Definition 5.3.5. *Let h be the Hamiltonian of an insulator, h_λ its continuous decomposition and $f_1, f_2, \cdots$ its bands. Define a function g_i that assigns to each $\lambda' \in \mathbb{T}^d$ the eigenspace of $h_{\lambda'}$ belonging to $f_i(\lambda') \in \sigma(h_{\lambda'})$. This way we obtain a sequence of functions*

$$g_i : \mathbb{T}^d \to P(L^2([0,1], W)), \tag{5.78}$$

where we write $P(L^2([0,1], W))$ for the closed subspaces of $L^2([0,1], W)$. We call a function g_i a topological band.

The bands $g_1, g_2, \cdots$ deliver us the topological band structure of the Hamiltonian. Assume that each $g_i(\lambda)$ is finite dimensional. This way, a topologial band g_i indicates a family of finite-dimensional subspaces of $L^2([0,1], W)$ parametrized over $\mathbb{T}^d$. It will be shown in the next section that the continuity of the family h_λ implies that this family in fact indicates a sub-Hilbert bundle of the trivial bundle $\mathbb{T}^d \times L^2([0,1], W)$.

The band structure is a plot of a sequence of continuous functions $f_1, f_2, \cdots : \mathbb{T}^d \to \mathbb{R}$, whereas the topological band structure is a sequence of Hilbert bundles over $\mathbb{T}^d$. Informally, the band structure encodes the energy 'eigenvalues' and the topological ban structure the energy 'eigenspaces'.

5.3.2 The band gap

In this section we show that the gap in the spectrum is equivalent to the insulating property and discuss the physical origin of the gap. The arguments rely on intuïtion and are not mathematically precise.

Without loss of generality we may assume that the gap in the Hamiltonian is centred around 0. The Hamiltonian presenting a gap is equivalent to the requirement that for each band

$$\min_k |f(k)| > \epsilon. \tag{5.79}$$

This in turn is, as we will now demonstrate, equivalent with the insulating property.

First assume eq. (5.79) holds. A particle occupying a vector state of some E'_k is both an energy eigenvector and a translational eigenvector. Eigenvectors of the translations are of the form

$$\psi(x) = u(x)e^{ikx}, \tag{5.80}$$

with u a d-periodic function. This is called the Bloch condition [27] page 162. (Note that the argument becomes sketchy as this function is actually not L^2). This vector is a wave moving in the direction k. In order to see this write the time evolution (see chapter 4) of this eigenvector as follows

$$\psi(x,t) = u(x)e^{ikx}e^{-iht} = u(x)e^{ikx}e^{-iEt} = u(x)e^{ikx-iEt}. \tag{5.81}$$

This function moves in the direction k with velocity $\frac{E}{|k|}$. The last ingredient is that we are dealing with fermions and that therefore each state can be occupied by at most one particle. This consideration leads to the fact that bands can be completely filled. Fill all states with energy lower then 0 with electrons. Now all bands containing electrons are completely filled. There is an equal amount of electrons flowing from left to right as there is from right to left. This leaves us with no netto current. This in turn implies that for a current to flow, electrons must cross over to a higher energy band, for as this happens all momenta no longer cancel each other. For an electron to promote to a higher band it must however take a discrete energy jump. This means that a lot of energy must be added to the system for a notable current to flow. This makes the material an insulator.

On the other hand, if eq. (5.79) would not hold, one could change k by just adding an infinitesimal amount of energy, making the material a conductor. Hence the gap in the Hamiltonian is precisely the requirement of the system being insulating.

Finally, we heuristically explain the origin of these so called band gaps. For simplicity we look at a one-dimensional lattice, the argument for arbitrary dimension is similar. This deserves some attention since you would expect that the energy eigenvalues E vary continuously. We follow the standard approach [27] page 163 using Bragg reflection. Suppose a wave $e^{i(kx-Et)}$ goes through a lattice with periodicity p. We assume that the waves collide with the atoms on the lattice points and that these collisions are elastic. The reflected wave of e^{ikx-Et} does not lose energy if it collides and is therefore given by $e^{i(kx+Et)}$. A reflecting wave will occur only if it interferes constructively with the original wave. So the wavelength of the reflected wave must fit an integer number of times in p. That is to say that the Bragg condition

$$p \cdot k = n \cdot \pi \tag{5.82}$$

needs to hold. Now whenever k satisfies eq. (5.82), an energy eigenstate will constructively interfere with its reflected wave. For this reason such energy states always occur as a superposition, namely

$$\psi(x,t) = u(x)(e^{i(kx-Et)} + e^{i(kx+Et)}) \text{ or } u(x)(e^{i(kx-Et)} - e^{i(kx+Et)}). \tag{5.83}$$

These two standing waves cause different electron distributions in the lattice and hence have distinct potential energies. This causes a discontinuous step in the energy at this value of k. This explains why the spectrum of h will present a gap whenever eq. (5.82) is satisfied. The space of k-values that satisfy eq. (5.82) is called the Brillouin zone [27] page 33.

5.3.3 Topological phases

The above discussion motivates looking at the subspace $\mathfrak{H}$ of Hamiltonians of insulators. We could ask ourselves what topological properties this space has. A property that immediately has a clear meaning is the number of connected components. Two Hamiltonians contained in distinct connected components cannot be deformed to one another via a path, that is to say there is no deformation of one insulator to the other without closing the energy gap or breaking the symmetry. Being in the same connected component grants an equivalence relation motivating the following definition.

Definition 5.3.6. *The connected component of $\mathfrak{H}$ in which a certain insulator is contained is its topological phase.*

There are two possible reasons, see [9] abstract, for an insulator to have no deformation to some other insulator. This is captured in the following definition.

Definition 5.3.7. *A transition between two insulators in distinct topological phases is called symmetry protected if there is a deformation whenever we drop the symmetry constraint G'^{τ}. That is to say if we allow transitions over possibly non-G'^{τ}-invariant Hamiltonians. It is said to be topology protected otherwise.*

5.3.4 An application

In order to understand the physical significance of topological insulators let's turn our attention to an important application. For the sake of simplicity restrict to the one dimensional case.

For a compact interval K of $\mathbb{R}$, we first need to define a localized Hamiltonian h_K. For $K \subset \mathbb{R}$ write h'_K for the Hamiltonian h restricted to the subspace of compactly supported functions on U. Since

$$L^2(K, W) \cong L^2([0,1], W) \tag{5.84}$$

there is a unitary map ϕ between these spaces. Now define the localized Hamiltonian h_K to be

$$h_K = \phi h'_K \phi^{-1}. \tag{5.85}$$

The localized Hamiltonian is required to vary continuously, that is to say

$$h_{K_n} \to h_K, \text{ whenever } K_n \to K. \tag{5.86}$$

Now we turn our attention to the situation in the figure below, in which two materials (black and blue) are displayed. The red area is the area close to the edge of both materials. Assume that for each compact interval K outside the red area, the translational symmetry holds precisely enough to approximate the localized Hamiltonian h_K by a restriction of a Hamiltonian belonging to the idealised case of an infinite insulator. This idealised case is precisely what we have studied so far. Assume that the idealised Hamiltonians h_0 and h_1 in the respective materials are in distinct topological phases. Now let $\{h_{K_i} \mid i \in [0,1]\}$ be a set of localized Hamiltonians such that K_i continuously takes the interval K_0 contained in the blue area to K_1 contained in the black area. All the while assume that h_{K_i} remains compatible with the translation symmetry, that is, it remains a restriction of some idealised infinite dimensional insulator. The continuous path h_{K_i} this way defines a continuous path between the idealised h_0 and h_1. By definition of topological phases, the gap in the spectrum must close somewhere in the red area. The boundary between these two insulators therefore behaves like a conductor.

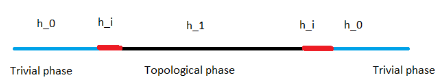

Figure 5.2: A topological insulator embedded in a trivial surrounding.

In case the transition between the topological phases is symmetry protected, the conducting edges can only be destroyed when the symmetry is broken. If the phase is topologically protected the conducting edges cannot be destroyed at all, unless the whole insulator undergoes a topological phase transition (and hence becomes conducting).

We should remark that the situation above is a bit more subtle. The assumption that all h_{U_i} commute with the translations might after all very well break down along the edges. That is to say the boundary between a trivial and topological insulator can be insulating just as well as a boundary between two trivial insulators can be conducting. However, it turns out that the topological and symmetry protection have a big impact, making distinct topological phases boundaries strongly leaning to conducting and the same topological phases boundaries strongly leaning to insulating [14] section 3.5.7.

This property of insulating edges has numerous applications, especially due to the fact that the boundaries can be made insulating or conducting at will by symmetry breaking.

5.4 Classification of topological phases of insulators

In this section we classify the topological phases of gapped Hamiltonians of quantum systems with lattice. We uphold assumption 5.2.19 and assumption 5.2.21. The first assumption allows us to continuously decompose the Hamiltonian over $\mathbb{T}^d/G'$ and look for connected components of these families instead. The second assumption makes sure that we can write ρ_λ^τ restricted to G''^τ by $\oplus_{i=1}^\infty \rho$ for all $\lambda \in \mathbb{T}^d$.

First we check that being gapped is inherited by the continuous family decomposing h.

Lemma 5.4.1. *Let h be a bounded Hamiltonian and h_λ be its continuous decomposition. Then we have $0 \notin \sigma(h)$ iff for all $\lambda \in \mathbb{T}^d$ $0 \notin \sigma(h_\lambda)$.*

Proof. If there exists a $\lambda \in \mathbb{T}^d$ for which $0 \in h_\lambda$, there exists a sequence of norm one vectors ψ_n in $L^2([0,1], W)$ such that $h_\lambda \psi_n \to 0$. Construct a sequence ψ_n' as in eq. (5.60). For this sequence we have $hf^{-1}\psi_n' \to 0$. This would imply that h has no bounded inverse and hence $0 \in \sigma(h)$. In other words $0 \notin \sigma(h)$ implies that $0 \notin \sigma(h_\lambda)$ for all $\lambda \in \mathbb{T}^d$.

On the other hand if $0 \notin \sigma(h_\lambda)$ for all $\lambda \in \mathbb{T}^d$ we find that all h_λ are invertible. The family h_λ^{-1} defines an inverse of h. Hence $0 \notin \sigma(h)$. □

Due to this lemma we can restrict ourselves to classifying continuous families of gapped bounded Hamiltonians on $L^2([0,1], W)$ commuting with $\oplus_{i=1}^\infty \rho$. We proceed by simplifying to spectral flattened Hamiltonians.

Lemma 5.4.2. *The topological phases of Hamiltonians are in bijective correspondence with connected components of continuous families of spectral flattened Hamiltonians.*

Proof. This follows by similar reasoning as was applied in lemma 4.2.4 and lemma 4.2.5. □

Remark 5.4.3. *In order to classify topological phases of gapped Hamiltonians of a quantum system with d-dimensional lattice symmetry, we can classify continuous families of spectral flattened Hamiltonians on $L^2([0,1], W)$ over $\mathbb{T}^d$ that commute $\oplus_{i=1}^\infty \rho$.*

The spectral flattened Hamiltonians are completely determined by the images of

$$h_\lambda^+ = \int_{\sigma(h_\lambda)^+} dE(\lambda) \text{ and } h_\lambda^- = \int_{\sigma(h_\lambda)^-} -dE(\lambda). \tag{5.87}$$

The operator h^+ is a projection and $Im(h) = H$, therefore

$$\mathrm{Im}(h_\lambda^+) = \overline{\mathrm{Im}(h_\lambda^+)} = \mathrm{Im}(h_\lambda^-)^\perp. \tag{5.88}$$

We are now close to an identification by Hilbert bundles.

Lemma 5.4.4. *The family h_λ^- forms a continuous family over $\mathbb{T}^d$.*

Proof. The spectrum of h_λ^- is just the point $\{-1\}$. We can therefore find a compact contour Γ enclosing $\sigma(h^-)$ that does not intersect $\sigma(h) = \{\pm 1\}$. Now use spectral calculus, [38] section 10.22, to restrict the operator

$$h_{\lambda_n}^- = h_{\lambda_n}|_{\sigma^-(h_\lambda)} = \int_\Gamma dx\ (h_{\lambda_n} - x\mathbb{I})^{-1}. \tag{5.89}$$

If $\lambda_n \to \lambda_0$ it follows that $(h_{\lambda_n} - x)^{-1} \to (h_{\lambda_0} - x)^{-1}$, since $(\lambda, x) \to (h_\lambda - x)^{-1}$ is continuous. The integral is over a compact space so in fact $h_{\lambda_n}^- \to h_{\lambda_0}^-$. □

Searching for Hamiltonians is equivalent to searching for continuous families of projections $\{h_\lambda^-\}$ that intertwine the twisted representations $\oplus_{i=1}^\infty \rho$. We need one last assumption.

Assumption 5.4.5. *$Im(h_\lambda^-)$ is finite dimensional.*

The operator h^- is the Hamiltonian restricted to the filled bands below the energy gap. This assumption is equivalent with assuming that there are finitely many electrons per atom.

Definition 5.2.5 is applicable. By proposition 5.2.9 the connected components coincide with isomorphism classes.

Corollary 5.4.6. *The topological phases of a quantum system are described by the monoid $Vect^\phi_{\rho,G''\tau}(\mathbb{T}^d/G')$ defined in definition 5.2.11. We can make this monoid into a group $K^\phi_{\rho,G''\tau}(\mathbb{T}^d/G')$ by the construction in definition 5.2.11.*

5.4.1 Classification by Clifford extensions

Now assume that we can translate the problem of finding Hamiltonians to finding positive extensions of some Clifford module N. See example 4.2.9 for situations in which this is indeed possible. In this case we are looking for continuous families of extensions of $\oplus_{i=1}^\infty N$ over $\mathbb{T}^d$. The monoid $M_N(\mathbb{T}^d)$ and the group $K_N(\mathbb{T}^d)$ in definition 5.2.12 describe this problem.

The monoid $M_N(\mathbb{T}^d)$ is usually difficult to compute, the group $K_N^\phi(\mathbb{T}^d)$ however is related to a topological K-group. In the following we describe this relation. First definition 4.2.13 and theorem 4.2.14 must be generalised.

Definition 5.4.7. *Write $Grad(Cliff^{p,q})$ for the topological space consisting of tuples*

$$(N, \eta_1, \eta_2), \tag{5.90}$$

where N is a finite dimensional $Cliff^{p,q}$-module and both η_1 and η_2 are gradings of this module N. For X a connected compact topological space write $[X, Grad(Cliff^{p,q})]$ for the homotopy classes of functions from X to $Grad(Cliff^{p,q})$. Introduce the notation $\oplus$ for

$$(f \oplus g)(x) = f(x) \oplus g(x) = (N, \eta_1, \eta_2) \oplus (N', \eta_1', \eta_2') = (N \oplus N', \eta_1 \oplus \eta_1', \eta_2 \oplus \eta_2'). \tag{5.91}$$

Furthermore, call f trivial if for one (and hence for all) $x \in X$,

$$f(x) = (N, \eta_1, \eta_2) \text{ with } \eta_1 \text{ and } \eta_2 \text{ path connected.} \tag{5.92}$$

Write $\overline{KO}^{p,q}(X)$ for the group freely generated by $[X, Grad(Cliff^{p,q})]$ up to the relations

$$\begin{aligned} f + g &\sim f \oplus g \\ f &\sim e \text{ iff } f \text{ is trivial.} \end{aligned} \tag{5.93}$$

In the case of a complex Clifford algebra $\mathbb{C}liff^q$ likewise define $\overline{K}^q(X)$.

As the notation suggests these groups are isomorphic to the topological K-groups.

Theorem 5.4.8.
$$KO^{p-q}(X) \cong \overline{KO}^{p,q}(X) \text{ and } K^{-q}(X) \cong \overline{K}^{q}(X). \tag{5.94}$$

Proof. See [24] theorem 4.22 and 5.12. □

By construction the topological K-groups contains $K_N^{\phi}(\mathbb{T}^d)$ as a subgroup.

Theorem 5.4.9. *The group $K_N(\mathbb{T}^d)$ isomorphic to a subgroup of some topological K-group.*

If N is the only irreducible representation of the Clifford algebra, the statement specialises to an isomorphism of the groups in question.

5.4.2 Some calculations

In this section we find the explicit results for the positive extensions of $\text{Cliff}^{0,q}$ or $\mathbb{C}\text{liff}^{q}$-modules in the physically relevant dimensions $d = 1, 2, 3$. Due to Bott periodicity we may in the real case restrict ourselves to $1 \geq q \leq 8$ and in the complex case to $q = 1, 2$.

Proposition 5.4.10. *There is one irreducible $\mathbb{C}\mathit{liff}^{q}$-module if $q = 1$ and there are two if $q = 2$. There is one irreducible $\mathit{Cliff}^{0,q}$-module if $q = 1, 2, 3, 5, 6, 7$ and there are two if $q = 4, 8$.*

Proof. See [3] proposition 5.4. □

The three Clifford algebras for which there exist two irreducible modules easily allow us to find the gradings(= positive extensions) of a module. This comes in handy, since in this case the group $K_N(\mathbb{T}^d)$ is not isomorphic to a topological K-group.

Proposition 5.4.11. *Positive extensions of irreducible $\mathit{Cliff}^{0,4}$, $\mathit{Cliff}^{0,8}$ and $\mathbb{C}\mathit{liff}^{2}$ modules have exactly two positive extensions.*

Proof. We prove it for the case of a $\text{Cliff}^{0,4}$-module N, the other cases are similar. Let e_1, e_2, e_3, e_4 be the odd generators of N. It is an easy check that

$$\eta = e_1 e_2 e_3 e_4 \tag{5.95}$$

is a grading. Let η' be another grading of the module. The operator η' anti-commutes with the generators e_i and therefore commutes with η. Hence

$$(\eta\eta')^2 = \mathbb{I}. \tag{5.96}$$

Furthermore, $\eta\eta'$ commute with all Clifford actions and is therefore an intertwiner. By lemma D.0.25,

$$\eta\eta' = \lambda\mathbb{I},\ \ \lambda \in \mathbb{H}, \mathbb{C}, \mathbb{R}. \tag{5.97}$$

Combining eq. (5.96) and eq. (5.97) implies $\lambda = \pm 1$ and therefore $\eta' = \pm\eta$. This means that there are exactly two gradings. □

The above proposition solves the extension problem for irreducible modules of the respective Clifford algebras. For a general Clifford modules of the Clifford algebras in question, the number of positive extensions are simply 2 times the number of irreducible sub-modules contained in the decomposition of N.

Now we turn our attention to the remaining cases. By proposition 5.4.10 there is in this case only one irreducible module of the pertinent Clifford algebra. This means, by theorem 5.4.9, that the group $K_N(\mathbb{T}^d)$ constructed in definition 4.2.13 is isomorphic to $K^{-q}(\mathbb{T}^d)$ in the complex case and to $KO^{-q}(\mathbb{T}^d)$ in the real case.

We would like to tabulate these topological K-groups in the physically relevant dimensions $d = 1, 2, 3$. In order to calculate these K-groups we use the homotopy equivalences

$$\mathbb{T}^1 \cong S^1, \tag{5.98}$$

$$\mathbb{T}^2 \cong S^1 \vee S^1 \vee S^2, \tag{5.99}$$

$$\mathbb{T}^3 \cong S^1 \vee S^1 \vee S^1 \vee S^2 \vee S^2 \vee S^2 \vee S^3. \tag{5.100}$$

Recall the notation $\vee$

$$X \vee Y \cong X \times Y/(x_0, Y_0), \tag{5.101}$$

for some $x_0 \in X$ and $uy_0 \in Y$. By excision the K groups distribute over these $\vee$. Furthermore,

$$\Sigma S^n \cong S^{n+1}, \tag{5.102}$$

where Σ indicates the reduced suspension (Appendix C). These two facts yield the following powerful equalities

$$K^{-q}(\vee_i S_i^{n_i}) \cong \oplus_i K^0(S_i^{n_i+q}) \text{ and } KO^{-q}(\vee_i S_i^{n_i}) \cong \oplus_i KO^0(S_i^{n_i+q}). \tag{5.103}$$

Combining the isomorphism in eq. (5.103) with the following table of the well known K and KO-groups of the spheres

Table 5.1: K- and KO-groups of S^d.

n	$KO^{-n}(S^0)$	$K^{-q}(S^0)$
1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$
2	$\mathbb{Z}/2\mathbb{Z}$	0
3	0	
4	$\mathbb{Z}$	
5	0	
6	0	
7	0	
8	$\mathbb{Z}$	

,

allows us to calculate all topological K and KO-groups over the d-dimensional torus. For the KO-groups we have the table

Table 5.2: KO-groups of $\mathbb{T}^d$.

n mod 8	$d = 1$	$d = 2$	$d = 3$
1	$\mathbb{Z}/2\mathbb{Z}$	$\oplus^3\mathbb{Z}/2\mathbb{Z}$	$\oplus^6\mathbb{Z}/2\mathbb{Z}$
2	$\mathbb{Z}/2\mathbb{Z}$	$\oplus^2\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z} \oplus^3 \mathbb{Z}/2\mathbb{Z}$
3	0	$\oplus^2\mathbb{Z}$	$\oplus^3\mathbb{Z}$
4	$\mathbb{Z}$	$\oplus^3\mathbb{Z}$	$\oplus^3\mathbb{Z}$
5	0	0	0
6	0	0	$\mathbb{Z}$
7	0	$\oplus^2\mathbb{Z}$	$\oplus^3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$
8	$\mathbb{Z}$	$\mathbb{Z} \oplus^2 \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z} \oplus^6 \mathbb{Z}/2\mathbb{Z}$

.

For the K-groups we have where we denote $\oplus^n A$ for $A \oplus A \cdots \oplus A$.

Table 5.3: K-groups of $\mathbb{T}^d$.

n mod 2	$d = 1$	$d = 2$	$d = 3$
1	0	$\otimes^2\mathbb{Z}$	$\oplus^4\mathbb{Z}$
2	$\mathbb{Z}$	$\mathbb{Z}$	$\oplus^3\mathbb{Z}$

,

Example 5.4.12. *Assume that $d = 1$, that the symmetry of the quantum system is given by a Cliff^{0,1}-module and that the Hamiltonian given by some positive extension of this module. The relevant group is isomorphic to the topological $KO^{-1}(\mathbb{T}^1)$ group. Using the table above this should be $\mathbb{Z}/2\mathbb{Z}$. Since there is by proposition 5.4.10 only one irreducible module we find $K^{-1}(\mathbb{T}^1) \cong K_N(\mathbb{T}^1)$. Having a closer look at $K_N(\mathbb{T}^1)$ in definition 5.4.7 provides that there are two continuous families of extensions for $\oplus_{i=1}^n N$ if n is odd and that there is only one continuous family of extension for $\oplus_{i=1}^n N$ if n is even. Taking $n \to \infty$ there are two topological phases for such an insulator.*

In case of dimension $d = 1$ and the search of topological phases is equivalent with searching extensions of $\mathbb{C}$liff^1. The relevant group is $K(\mathbb{T}) \cong 0$. There is hence only one grading of each module N. We find there to be only one topological phase for $n \to \infty$.

Conclusion

Now we answer the main question in this thesis. Namely the classification of quantum systems, observables (Hamiltonians), equivalence classes of states, and topological phases that occur under a phase space H and a symmetry group G.

- Quantum systems should be defined by tuples $(H, G, \phi, \tau, \rho^\tau)$, where H and G have the usual interpretation of the phase space and symmetry group. The correct implementation of G is encoded by ϕ, τ and ρ^τ. The observables (Hamiltonians) of such a system are the intertwiners of $\rho^\tau(G^\tau)$. These observables in turn give rise to the states, which are normal functionals on $B(H)$.
- The quantum systems with Hilbert space H and symmetry group G are given by the ϕ-twisted extensions G^τ of G and the twisted representations of G^τ. The extensions of G were tracked down in theorem 1.2.16, using cohomology groups. The twisted representations were characterized in section 1.4, using real representation theory.
- The algebra of observables determines, by proposition 2.1.20, the observables. The algebra of observables is in turn, by corollary 2.4.7, some direct sum of $n \times n$ matrices over either the real, the complex or the quaternion numbers. In this way we found the general form Hamiltonians can have under a symmetry. For the symmetry algebra there are four possibilities, all described in corollary 2.4.17.
- By the discussion in section 3.1.1 we may restrict ourselves to normal states, which are convex combination of pure normal states. The equivalence classes of pure normal states upon restriction to the observables were classified in theorem 3.2.7. The states could be identified with irreducible subspaces of the representation ρ^τ. The pure states are precisely the states having zero variance for some non-degenerate observable by theorem 3.2.7.
- In case that an irreducible subspace is of quaternionic type you need an addtional state on $\mathbb{H}$ to specify the state. Due to this real numbers, complex numbers and quaternion numbers can despite the result for the algebra of observables not be treated on equal footing as there are multiple $\mathbb{C}$-linear states on the quaternions as opposed to there being only one $\mathbb{H}$-linear state.
- There is only one topological phase in the algebra of observables. To obtain non-trivial results the condition of a gapped spectrum must be imposed. This assumption physically means restricting to Hamiltonians of insulators. We found a classification using irreducible subspaces in section 4.2.1. An identification using topological K-groups was laid out in corollary 4.2.15.
- In case of an insulator in dimension higher than zero the symmetry group contains non-compact lattice translations $\mathbb{Z}^d$. To counter the non-compactness of the symmetry group

we used theorem 5.2.14 to decompose H continuously over $\mathbb{T}^d$. Under sufficiently strong and heuristically justifiable assumptions we obtain the previous classification results for compact groups, only this time parametrized over $\mathbb{T}^d$.

Appendix

The appendices below contain additional information for the text in this thesis. There is a reference to the Appendix whenever relevant information can be found here.

Appendix A

Time reversal

It might look natural to discharge the anti-unitary operators as unphysical and only allow representations that map G^τ to the unitary operators. However, it turns out that some symmetries in physics are necessarily anti-unitary. The proposition in this section, for example, shows that time reversal needs to be implemented in an anti-unitary fashion. Besides this practical necessity it will be shown in section 3.2 that the anti-unitary operators allow quantum physics to treat all division algebras over the real numbers on equal footing, providing an aesthetic argument as well.

Proposition A.0.13. *Let $\rho : G \to Aut_{QM}(O_{sym}(H))$ be a symmetry of a certain quantum-system containing time translations and time reversal. Time reversal is an anti-unitary operation.*

Proof. Pick a twisted extension $\pi : G^\tau \to G$ and twisted representation $\rho^\tau : G^\tau \to \mathrm{Aut}_{QM}(H)$ such that the following diagram commutes.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{T} & \longrightarrow & G^\tau & \longrightarrow & G & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathrm{Aut}_{QM}(H) & \longrightarrow & \mathrm{Aut}_{QM}(O_{sym}(H)) & \longrightarrow & 1. \end{array} \tag{A.1}$$

Let $\mathbb{R} \cong U \subset G^\tau$ be the subgroup of time-translations and write by restricting the following diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \pi^{-1}(U) & \longrightarrow & U & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathrm{Aut}_{QM}(H) & \longrightarrow & \mathrm{Aut}_{QM}(O_{sym}(H)) & \longrightarrow & 1. \end{array} \tag{A.2}$$

Since $U \cong \mathbb{R}$ is a simply connected Lie group the extensions are given by the Lie algebra cohomology group

$$H^1(\mathbb{R}, \mathbb{R}) = \{1\}, \tag{A.3}$$

where we used the fact that $\mathbb{R}$ is the Lie algebra of $\mathbb{R}$. This calculation has been done explicitly in section 1.2.5.

It follows that there is only one central extension of $\mathbb{R}$, namely the trivial central extension

$$\pi^{-1}(U) \cong \mathbb{T} \times U. \tag{A.4}$$

Now define a section s by

$$s : U \to \pi^{-1}(U) \cong \mathbb{T} \times U \tag{A.5}$$

$$s(g) = (1, g). \tag{A.6}$$

Then $s(U) \subset G^\tau$ is a 1-dimensional Lie-subgroup of G^τ. Hence the restriction of ρ^τ to $s(U) \cong \mathbb{R}$ is a representation of $\mathbb{R}$ in the unitary operators. By the theorem of Stone the representation is given by

$$t \to e^{-ith}, \tag{A.7}$$

for some generally unbounded self-adjoint operator h. Now take a look at the element $g \in G^\tau$ signifying time reversal. Since e^{-ith} is a time translation we find

$$\rho^\tau(g) e^{-ith} \rho^\tau(g)^{-1} = e^{ith}, \tag{A.8}$$

which can only be the case if

$$\rho^\tau(g)(-ih)\rho^\tau(g^{-1}) = ih. \tag{A.9}$$

Tracking whether $\rho^\tau(g)$ is unitary or anti-unitary, via ϕ, gives:

$$- i\phi(g)\rho^\tau(g) h \rho^\tau(g^{-1}) = ih. \tag{A.10}$$

As h is the generator of time translations, it should be the Hamiltonian. If we demand that this operator is contained in the algebra of observables belonging to this quantum-system, it should intertwine the group action of this quantum-system, hence

$$- i\phi(g)h = ih. \tag{A.11}$$

We conclude (Assuming $h \neq 0$) that $\phi(g) = -1$ and hence that time reversal is anti-unitary. $\square$

Appendix B

Bicommutant theorem

The well-known bicommutant theorem [34] can be used in case of real Hilbert spaces as well. For the proof of the bicommutant theorem we need the notion of an orthogonal projection. In order to define this we need the following proposition [38].

Proposition B.0.14. *Let H be a real linear space with an innerproduct and V be a closed linear subspace of H. For each $\psi \in H$ the map*

$$V \to \mathbb{R} \tag{B.1}$$

$$\phi \to \|\psi - \phi\| \tag{B.2}$$

attains its minimum at a unique $\psi' \in V$.

Proof. This is Riesz lemma [38]. □

This proposition allows us to define an orthogonal projection on a closed subspace.

Definition B.0.15. *Let V be a closed linear subspace of a real Hilbert space H. The projection on V is a map that assigns to each $\psi \in H$ the unique element in V for which eq.* (B.1) *attains its minimum.*

For A a subalgebra of $B(H)$, recall the notation A' for the algebra of intertwiners of A.

Theorem B.0.16. *Let H be a real Hilbert space and let A be a unital subalgebra of $B(H)$ that is closed under taking adjoints. With respect to the strong closure of A we have*

$$\overline{A} = A''. \tag{B.3}$$

Proof. We take the same approach as in the complex case. The inclusion $A \subset A''$ is trivial. Since A'' is strongly closed we also find

$$\overline{A} \subset A''. \tag{B.4}$$

In order to prove the converse we first show that for each $\psi \in H$ and each $T \in A''$ there exists an $a \in A$ such that $a\psi = T\psi$. Let T be some element in A''. Let P be the projection on $\overline{A\psi}$. By lemma B.0.17 below P is contained in A'. The vector ψ is contained in $A\psi$, since $\mathbb{I} \in A$. It follows that

$$T\psi = TP\psi = PT\psi \in \overline{A\psi}. \tag{B.5}$$

Hence for each $\psi \in H$ there exists an operator $a \in A$ such that

$$a\psi = T\psi. \tag{B.6}$$

We can in fact get this result for an arbitrary (finite) number of vectors $\{\psi_i\}_{i=1}^n$. To prove this simply look at the Hilbert space $H' = \oplus_{i=1}^n H$ and the vector $\psi = \oplus_{i=1}^n \psi_i$. Use the notation in section 2.4.1 to replace T and A acting on H by $I_n \times T$ and $I_n \times A$ acting on $\oplus_{i=1}^n H$. By the same procedure we find an operator $a' \in I_n \times A$ such that

$$a'\psi = T\psi. \tag{B.7}$$

This implies that there exists an operator $a \in A$ for which

$$a\psi_i = T\psi_i \tag{B.8}$$

for all $\psi_i \in \{\psi_i\}_{i=1}^n$. In this way we can find a sequence in A converging point-wise to $T \in A''$. It follows that

$$A'' \subset \overline{A}. \tag{B.9}$$

□

In the proof we used the following lemma.

Lemma B.0.17. *The projection P commutes with A.*

Proof. Obviously $Pa = a$ for all $a \in A$. This implies

$$\langle \psi, aP\phi \rangle = \langle \psi, PaP\phi \rangle = \langle Pa^*P\psi, \phi \rangle = \langle a^*P\psi, \phi \rangle = \langle \psi, Pa\phi \rangle, \tag{B.10}$$

where we used that $a^* \in A$. Since this holds for all $\psi, \phi \in H$ we find $Pa = aP$ as desired. Hence $P \in A'$ □

Corollary B.0.18. *In case A is a unital strongly closed subalgebra of $B(H)$,*

$$A'' = A. \tag{B.11}$$

Appendix C

Group completion

There is a standard procedure to construct an Abelian group $K(M)$ for a commutative monoid M. This procedure is called group completion [23].

Definition C.0.19. *For M a commutative monoid the Grotendieck group $K(M)$ is the set*

$$M \times M \tag{C.1}$$

up to the relation

$$(m_1, m_2) \sim (m_1', m_2') \iff m_1 + m_2' + e = m_2 + m_1' + e \tag{C.2}$$

for some $e \in M$. The set $K(M)$ forms a commutative group under the operation

$$(m_1, m_2) + (m_1', m_2') = (m_1 + m_1', m_2 + m_2'). \tag{C.3}$$

This group satisfies the following universal property, that loosely speaking expresses that $K(M)$ is the Abelian group lying closest to the monoid M.

Proposition C.0.20. *For each Abelian group A and homomorphism $\psi : M \to A$ there is a unique homomorphism $\psi' : K(M) \to A$, such that the following diagram commutes*

$$\begin{array}{ccc} M & & \\ {\scriptstyle i}\downarrow & \searrow^{\forall_\psi} & \\ K(M) & \xrightarrow[\exists!_{\psi'}]{} & A \end{array} \quad . \tag{C.4}$$

Proof. The proof is left to the reader. □

We now apply this procedure to the specific case of Hilbert bundles. Write $Vect_{\mathbb{C}}(X)$ and $Vect_{\mathbb{R}}(X)$ for the real and complex vector bundles, up to isomorphism, over a topological space X up to the relation

$$\mathfrak{A} \sim e \text{ iff } \mathfrak{A} \text{ is trivial.} \tag{C.5}$$

These sets form commutative monoids under taking direct sums of the bundles. It is therefore natural to look at the group completion in definition C.0.19 of these monoids. We call these group completions $K^0(X)$ respectively $KO^0(X)$. In order to obtain a cohomology theory define additional groups [24] by setting

$$K^{-n}(X) = K^0(\Sigma^n X) \text{ and } KO^{-n}(X) = KO^0(\Sigma^n X), \tag{C.6}$$

where Σ is the reduced suspension defined by

$$\Sigma X = \big(X \times [0,1]\big)/\big((X,1) \cup (X,0) \cup (x_0,[0,1])\big) \text{ for a certain } x_0 \in X. \tag{C.7}$$

The topological K-groups form a cohomology theory. That is to say, the following requirements hold.

- Dimension:

$$K^0(\{x\}) = 0. \tag{C.8}$$

- Homotopy:
  If X and Y are homotopic

$$K^{-q}(X) \cong K^{-q}(Y), \tag{C.9}$$

  for all $q \in \mathbb{N}$.

- Additivity:
  For $X = \sqcup_{i=0}^{n} X_i$

$$K^{-q}(X) \cong \oplus_{i=0}^{n} K^{-q}(X_i) \tag{C.10}$$

- Exactness:
  For $i : Y \to X$ the inclusion and $p : X \to X/Y$ the quotient there exists a left long exact sequence

$$\cdots \xleftarrow{i^*} K^{-1}(Y) \xleftarrow{\delta} K^0(X) \xleftarrow{p^*} K^0(X/Y) \xleftarrow{i^*} K^0(Y). \tag{C.11}$$

We do not get into the proof of these properties here. A proof can be found in [24].

Appendix D

Clifford algebras

This Appendix provides a brief outline of Clifford algebras and their modules. The material is to supplement section 4.2.2 and section 5.4.

Definition D.0.21. *Let* $\{e_1, \cdots, e_d\}$ *be an orthonormal basis of* $\mathbb{R}^d$*. Define* $F(\mathbb{R}^d)$ *to be the real algebra freely generated by*

$$e_1, \cdots, e_d. \tag{D.1}$$

Define the real Clifford algebra $\mathit{Cliff}^{p,q}$ *to be* $F(\mathbb{R}^{p+q})$ *up to the relations*

$$e_i^2 = -\mathbb{I} \text{ for } i \leq p \tag{D.2}$$
$$e_i^2 = \mathbb{I} \text{ for } i > p. \tag{D.3}$$

The generators squaring to $\mathbb{I}$ *are called positive generators, likewise the generators squaring to* $-\mathbb{I}$ *are called negative generators.*

Clifford algebras are involutive algebras under the $*$ *operation defined by linear extension of the map*

$$e_i^* = e_i \text{ if } i \leq p \tag{D.4}$$
$$e_i^* = -e_i \text{ if } i > p \tag{D.5}$$
$$(\Pi_{i=1}^m e_i)^* = \Pi_{i=m}^1 e_i^*. \tag{D.6}$$

Optionally, equip a Clifford algebra with a grading ϕ *via*

$$\phi(\Pi_{i=1}^n e_i) = 1 \text{ if } n \text{ is even,} \tag{D.7}$$
$$\phi(\Pi_{i=1}^n e_i) = -1 \text{ if } n \text{ is odd.} \tag{D.8}$$

We refer to a Clifford algebra with this particular grading as a graded Clifford algebra.

The complex Clifford algebra $\mathbb{C}\mathit{liff}^{\,q}$ *is* $\mathbb{C} \otimes \mathit{Cliff}^{\,0,q}$.

Note that the complex Clifford algebra has only one index q since multiplication by i mixes positive and negative generators. Furthermore, note that the definition automatically implies for $l \neq k$

$$e_l e_k = -e_k e_l. \tag{D.9}$$

Lastly notice that the positive generators are self-adjoint, whereas the negative generators are skew-adjoint.

Lemma D.0.22. *For each (graded) algebra A and (graded) homomorphis $\psi : \mathbb{R}^{p+q} \to A$ satisfying*

$$\psi(e_i)^2 = \begin{cases} \mathbb{I} \text{ if } i \leq p \\ -\mathbb{I} \text{ if } i > p. \end{cases} \tag{D.10}$$

we obtain a commuting diagram

$$\begin{array}{ccc} \mathbb{R}^{p+q} & & \\ {\scriptstyle i}\downarrow & \searrow^{\psi} & \\ \mathit{Cliff}^{\,p,q} & \xrightarrow[\psi']{} & A, \end{array} \tag{D.11}$$

for a unique (graded) homomorphism ψ'. The homomorphism ψ' is a (graded) isomorphism iff the basis elements of $\mathit{Cliff}^{\,p,q}$ are bijectively send to a basis of A. The same holds for complex Clifford algebras.

Proof. The proof is left to the reader. □

We now calculate some Clifford algebras that will turn out to be all Clifford algebras up to so called Morita equivalence (defined in the next subsection). In order to do this we use the following lemma.

Lemma D.0.23.

$$\mathit{Cliff}^{\,0,p+2} \cong \mathit{Cliff}^{\,p,0} \otimes_{\mathbb{R}} \mathit{Cliff}^{\,0,2}. \tag{D.12}$$

Proof. Due to lemma D.0.22, the map

$$\text{Cliff}^{\,0,p+2} \to \text{Cliff}^{\,p,0} \otimes_{\mathbb{R}} \text{Cliff}^{\,0,2} \tag{D.13}$$

defined by

$$e_i \to \begin{cases} e_i \otimes_{\mathbb{R}} e_1 e_2 \text{ for } i > 2, \\ \mathbb{I} \otimes_{\mathbb{R}} e_i \text{ for } i \leq 2, \end{cases} \tag{D.14}$$

extends to an isomorphism. □

Using

$$\text{Cliff}^{\,0,1} \cong \mathbb{C} \quad \text{Cliff}^{\,1,0} \cong \mathbb{R} \oplus \mathbb{R} \tag{D.15}$$

$$\text{Cliff}^{\,0,2} \cong \mathbb{H} \;\text{Cliff}^{\,2,0} \cong M_{2\times 2}(\mathbb{R}), \tag{D.16}$$

and the isomorphisms [3] page 10

$$M_{n\times n}(\mathbb{D}) \cong \mathbb{D} \otimes_{\mathbb{R}} M_{n\times n}(\mathbb{R}) \tag{D.17}$$

$$M_{n\times n}(\mathbb{R}) \otimes M_{m\times m}(\mathbb{R}) \cong M_{nm\times nm}(\mathbb{R}), \tag{D.18}$$

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}, \tag{D.19}$$

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_{2\times 2}(\mathbb{C}), \tag{D.20}$$

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_{4\times 4}(\mathbb{R}), \tag{D.21}$$

together with iteratively applying lemma D.0.23 delivers us the following table, in which the grading of the algebras is left implicit.

Table D.1: Clifford algebras.

q,p	Cliff ^{0,q}	Cliff ^{p,0}	$\mathbb{C}$liff ^{q}
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C} \oplus \mathbb{C}$
2	$\mathbb{H}$	$M_{2\times 2}(\mathbb{R})$	$M_{2\times 2}(\mathbb{C})$
3	$\mathbb{H} \oplus \mathbb{H}$	$M_{2\times 2}(\mathbb{C})$	
4	$M_{2\times 2}(\mathbb{H})$	$M_{2\times 2}(\mathbb{H})$	
5	$M_{4\times 4}(\mathbb{C})$	$M_{2\times 2}(\mathbb{H}) \oplus M_{2\times 2}(\mathbb{H})$	
6	$M_{8\times 8}(\mathbb{R})$	$M_{4\times 4}(\mathbb{H})$	
7	$M_{8\times 8}(\mathbb{R}) \oplus M_{8\times 8}(\mathbb{R})$	$M_{8\times 8}(\mathbb{C})$	
8	$M_{16\times 16}(\mathbb{R})$	$M_{16\times 16}(\mathbb{R})$	

The complex Clifford algebras are simply calculated through $\mathbb{C}\text{liff}^q = \mathbb{C} \otimes \text{Cliff}^{0,q}$.

Clifford modules

In the following we are more interested in the (graded) representations of Clifford algebras in $B(H)$, with H some Hilbert space, than in the actual Clifford algebra itself. Such a representation is called a module.

Remark D.0.24. *From now on all modules are assumed finite dimensional.*

The representation theory of Clifford algebras has similarities to group representations. For example we can define irreducible Clifford modules as modules that do not posses a non-trivial invariant subspace under the given Clifford action.

Lemma D.0.25. *Every Clifford module can be decomposed into irreducible sub-modules. The algebra of intertwiners of an irreducible Clifford module is isomorphic to*

$$\{\lambda \mathbb{I} \mid \lambda \in \mathbb{D}\}, \tag{D.22}$$

where $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ in case of real Clifford algebras and $\mathbb{D} = \mathbb{C}$ in case of complex Clifford algebras.

Proof. It is an easy check that the orthoplement of a Clifford sub-module is a Clifford sub-module. Since the dimension is finite, this procedure of finding the smallest possible invariant sub-modules is finite. The second part of the claim follows by the exact same reasoning applied in the proof of proposition 2.4.5. □

A graded module M of a graded Clifford algebra is a module with a decomposition $M = M_1 \oplus M_{-1}$, in such a way that for each odd element e in the Clifford algebra

$$e : M_1 \to M_{-1} \text{ and } e : M_{-1} \to M_1. \tag{D.23}$$

The equation above automatically implies that even elements in the Clifford algebra respect M_1 and M_{-1}.

Example D.0.26. *Let A be an algebra and $M_{n\times n}(A)$ the $n \times n$ matrices over this algebra. We can equip $M_{n\times n}(A)$ with a $m < n$ grading as follows*

$$M_{n\times n}(A) = \begin{pmatrix} a & \emptyset \\ \emptyset & a' \end{pmatrix} \oplus \begin{pmatrix} \emptyset & b \\ b' & \emptyset \end{pmatrix}, \tag{D.24}$$

where $a \in M_{m\times m}(A), a' \in M_{(n-m)\times(n-m)}(A), b \in M_{m\times(n-m)}(A), b' \in M_{(n-m)\times m}(A)$*. Let* N *be an* A*-module and let* $M_{n\times n}(A)$ *act on* N^n *in the usual way. If we decompose*

$$N = N^m \oplus N^{n-m} \tag{D.25}$$

we obtain a graded $M_{n\times n}$ *module with respect to the* m*-grading.*

Since we are interested in the modules of the Clifford algebras and not so much in the Clifford algebras themselves, we should look at these algebras up to Morita equivalence [46].

Definition D.0.27. *For* A *an algebra write* M_A *for the category of (graded)* A*-modules. The objects of this category are modules of* A *and the morphisms are linear maps respecting the (graded)* A *action. Two (graded) algebras* A *and* A' *are (graded) Morita equivalent if there exist functors*

$$F : M_A \to M_{A'} \tag{D.26}$$

$$G : M_{A'} \to M_A \tag{D.27}$$

for which there exists a map η *assigning to each module* M *in* M_A *a (graded) isomorphism (not just morphism)* $f_M : M \to M'$ *in such a way that for each two objects* N N' *in the category* M_A *the following diagram commutes*

$$\begin{array}{ccc} FG(N) & \xrightarrow[FG(f)]{} & FG(N') \\ {\scriptstyle\eta(N)}\downarrow & & \downarrow{\scriptstyle\eta(N')} \\ N & \xrightarrow[f]{} & N'. \end{array} \tag{D.28}$$

Likewise we require there to be such a map η' *for* GF *and* M'_A*.*

Roughly speaking Morita equivalence expresses that we regard two algebras as equivalent if their representation theories are equivalent. Of course two isomorphic algebras are in particular Morita equivalent. Morita equivalence is a weakening of usual isomorphism. To illustrate the notion of Morita equivalence we take a look at the following Lemma.

Lemma D.0.28. *The* $n \times n$ *matrices over a certain algebra* A *with an* $m < n$ *grading as in example D.0.26 are Morita equivalent with the algebra* A*, with trivial grading.*

Proof. Let F be the functor taking an A-module N to a $M_{n\times n}(A)$ module N' by setting

$$F(N) = (N_1, \cdots, N_n) \tag{D.29}$$

and letting $M_{n\times n}(A)$ act on $F(N)$ in the usual way. For a A module morphism $f : N \to N'$ we set

$$F(f)F(N) = (f(N_1), \cdots, f(N_n)). \tag{D.30}$$

On the other hand for a $M_{n\times n}(A)$ module N' set the functor G

$$G(N') = N'_0 \tag{D.31}$$

and letting $a \in A$ act on this space in the way $m_{0,0}$ acts on N'_0.

It is easy to see that $GF(N) \cong N$. Now for a $M_{n\times n}(A)$ module $N' = (N'_0, \cdots N'_n)$ we must have that $N'_i \cong N'_j$. That is to say N'_0 fixes N'. Therefore again $FG(N') \cong N'$.

If we equip $M_{n\times n}(A)$ with the grading in example D.0.26, then for f an A-module homomorphism $F(f)$ is a graded $M_{n\times n}(A)$-module homomorphism. We may hence conclude that the respective algebras are graded Morita equivalent. ☐

Under Morita equivalence the Clifford algebras exhibit a periodicity, the so called Bott-periodicity [3].

Theorem D.0.29. *Up to graded Morita equivalence*

$$\mathit{Cliff}^{\,p,q} \sim \mathit{Cliff}^{\,p \ mod 8, q \ mod 8} \ \mathit{and}\ \mathbb{C}\mathit{liff}^{\,q} \sim \mathbb{C}\mathit{liff}^{\,q \ mod \ 2}. \tag{D.32}$$

Proof. First consider the real case. The Cliford algebras Cliff ^{0,8} and Cliff ^{8,0} are isomorphic to $M_{16\times 16}(\mathbb{R})$ with its 8-grading defined in example D.0.26. By lemma D.0.28 we find that both of these algebras are graded-Morita equivalent to $\mathbb{R}$. This means that Cliff ^{0,8} and Cliff ^{8,0} are Morita equivalent to Cliff ^{0,0} concluding the 8-periodicity in both the p and q index.

Next proceed with the complex case. We have

$$\mathbb{C}\mathrm{liff}^3 \cong \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cliff}^{\,0,3} \cong (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}) \oplus (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}) = M_{2\times 2}(\mathbb{C}) \oplus M_{2\times 2}(\mathbb{C}), \tag{D.33}$$

where we used table D.1 in the second step and eq. (D.20) in the last step. The algebra $M_{2\times 2}(\mathbb{R})$ is equipped with its 1 grading defined in example D.0.26. Apply lemma D.0.28 to continue the above equation by

$$\cdots \sim \mathbb{C} \oplus \mathbb{C} \sim \mathbb{C}\mathrm{liff}^{\,1}. \tag{D.34}$$

We may hence conclude the 2-periodicity. □

Note that if we would be looking up to Morita equivalence, omitting the grading, the real case would also be of period 2. This is the case since by lemma D.0.28

$$M_{2\times 2}(\mathbb{R}) \sim \mathbb{R}. \tag{D.35}$$

Our interest however lies in the graded case.

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