Group-crossed extensions of representation categories in algebraic quantum field theory

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To my parents,
Bea & Farooq Sheikh,
with love and gratitude

Contents

Preface

The content of this thesis is the result of four years of research. I am very grateful that during these four years I received a lot of support from several people.

First of all I want to thank my supervisor Michael Müger. In the beginning of the project his role mainly consisted of guiding me through the literature on algebraic quantum field theory and category theory. In a later stage we often had these brainstorm sessions in which we were writing our ideas on the blackboard in his office. I can remember very well that during these sessions I was always impressed by the expertise of Michael and by the fact that he could explain things to me so clearly. In general, I was also very happy with the fact that Michael gave me a lot of freedom to explore different directions of research myself, without asking me for updates every week. For me this was really the best way to work and it gave me the feeling that my supervisor trusted me. Thank you very much, Michael!

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Although I often decided to work from home during these four years, whenever I was in the office at the university there was always a good atmosphere because of all the nice colleagues I had. I especially want to thank Noud Aldenhoven, Franscesca Arici, Peter Badea, Jord Boeijink, Wouter Cames van Batenburg, Giovanni Caviglia, Johan Commelin, Henk Don, Gert Heckman, Jasper van Heugten, Roberta Iseppi, Erik Koelink, Rutger Kuyper, Klaas Landsman, Bert Lindenhovius, Astrid Linssen, Jie Liu, Milan Lopuhaä, Ioan Marcut, Ben Moonen, Michael Müger, Joost Nuiten, Greta Oliemeulen, Sander Rieken, Frank Roumen, Maarten Solleveld, Bernd Souvignier, Abel Stern, Ruben Stienstra, Walter van Suijlenkom, Henrique Tavares, Julius Witte, Sander Wolters and Florian Zeiser. I also want to thank all students who attended my classes on operator algebras, analysis 2, topology, Fourier analysis, complex functions and introduction to mathematical physics, because we had a good time together.

It is safe to say that I would have never studied mathematics if I did not have any people showing me the beauty of mathematics. For this reason I want to show my deep gratitude to Joop van der Vaart, my high school teacher in mathematics, who showed me that mathematics can be really fun. Thank you so much, Joop! When I studied mathematics at the University of Amsterdam, there were several good lecturers who inspired me, such as Robbert Dijkgraaf, Gerard Helminck, Tom Koornwinder, Eric Opdam, Henk Pijls, Hessel Posthuma, Jasper Stokman, Erik Verlinde and Jan Wiegerinck. They strongly influenced my direction of specialization within mathematics during my study, which eventually led to me applying for this PhD project.

My parents, Bea and Farooq Sheikh, have always been there for me in my life, also during this project. They are the most loving and caring people I know and they have learned me to work hard in order to achieve my goals, mainly by giving me the right example. As the youngest child I have the privilege that I always had my brother Haroon Sheikh and my sister Zahira Ramkhelawan-Sheikh who could show me around in life. They have learned me many things, both on the intellectual level as on the level of general life lessons. Together, the five of us have always been a warm and loving family, in which I was able to grow up and become who I wanted to be. Thank you so much, my fellow Sheikhs! Although she is not my family in the biological sense, at this place I also want to thank Roos van den Berg-Pham. We grew up together like brother and sister when we were little kids living in the same street, and today our friendship is still as strong as it was back then.

In the last years our family has extended a lot. I also want to thank all new members of our family for all their support and love during this project, such as Rinish Ramkhelawan, the two little kids Qaish and Zahrya Ramkhelawan, Dino Suhonic, Sharmila Moennalal, Charini Ramkhelawan, Veejay Jawalapersad, Rajsnie Soemeer, Washand Ramlagan and Elise van den Bos. Unfortunately, we also lost a dear family member during this period, namely Navin Narain. I remember very well that I was working on the content of Subsection 4.10.2 of this thesis when I heard the sad news that Navin had died, so for that reason the content of that subsection will always remind me of him.

In high school I developed a very close friendship with Remy Alidarso, Joris van den Berg, Marijn Delisse, Swayambhu Djwalapersad, Alexander Schmitz, Just Schornagel and Casper Stubbé. Today we are still best friends, although we do not see each other on a daily base like we did back in high school. I always enjoy our funny moments together, as well as the more intellectual conversations we have. Thanks guys!

During this PhD project there were several moments when things were not going as smoothly as I wished. At these moments it was just nice to hit the gym at the end of the day and lose all my frustrations from work there. This was not only established by training hard, but also by the fact that there are some really great people training in my gym who always make sure that there is a nice atmosphere in which we can laugh a lot. I especially want to mention Ryan Abdoelkarim, Evert Djais and Frank Schouten, because these three guys have also become my friends outside of the gym.

After submitting the first version of the manuscript of this thesis, I soon began working for the insurance company Nationale Nederlanden. After a few months I received all feedback from Michael and Klaas and I had to make some changes to the manuscript. To accomplish this, I took quite some days off from work, and I am very grateful that this was possible at that moment. During this period I often chose to go to the office to work on the manuscript, so that I could work within the nice atmosphere that is always present there. This made things much more pleasant to me and for that reason I want to thank all my colleagues at Nationale Nederlanden.

Chapter 1

Introduction

It is somewhat challenging to already explain the main research problem in the introduction when, in order to fully understand the problem, several definitions, notational conventions and proven results are needed that are perhaps not known to most mathematicians. For this reason we have decided to formulate our research problem here simply by using all these needed results and terminology without any further explanation. In the chapters following this introduction we will precisely state everything we need, so the reader is not expected to already understand everything that is mentioned in this introduction. Since this research project was motivated by a problem in algebraic quantum field theory, which is mathematical formalization of a physical theory, we will begin with a brief discussion of quantum physics.

1.1 Physical background

In both classical physics and quantum physics, a physical system is described in terms of observables and states. Observables are the physical quantities of the system that can be measured by an observer. Typical examples of observables are the energy of the physical system or the position and velocity coordinates of the particles that constitute the system. Roughly speaking, the state of a system is a characterization of the condition of the physical system and is often expressed in terms of the values of certain observables. In the most standard mathematical description of quantum theory, one assigns to each physical system a Hilbert space H. The observables corresponding to the system are then represented by self-adjoint linear operators acting on H and the states corresponding to the system are represented by density operators on H, i.e. positive trace-class operators with trace equal to 1. Suppose that at a certain time the system is in a state that is described by a density operator ρ and suppose that we are interested in the probability of finding a number in the interval $[a, b]$ when we measure a certain observable that is represented by the self-adjoint operator A. The procedure of calculating this probability is as follows. Because A is a self-adjoint operator on H, the spectral theorem allows us to write it as $A = \int_{\mathbb{R}} x dE_A(x)$, where E_A is the spectral measure of A. Then the probability $p(\rho, A; [a, b])$ of finding a number in the interval $[a, b]$ when measuring the observable A while the state of the system is ρ is given by $p(\rho, A; [a, b]) = \mathrm{Tr}(\rho E_A([a, b]))$.

For example, if the system consists of a single non-relativistic particle with spin 0 moving through 3-dimensional space $\mathbb{R}^3$, the corresponding Hilbert space (in the so-called position representation) is $H = L^2(\mathbb{R}^3)$ and the operators X_j and P_j corresponding to the position and momentum coordinates of the particle, respectively, are given by

$$(X_j\psi)(x) = x_j\psi(x) \qquad \text{and} \qquad (P_j\psi)(x) = \frac{\hbar}{i}\frac{\partial\psi}{\partial x_j}(x)$$

for $\psi \in L^2(\mathbb{R}^3)$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $j \in \{1, 2, 3\}$. Other observables such as energy and angular momentum can be expressed in terms of the observables X_j and P_j. The operators X_j and P_j satisfy the commutation relations

$$[X_j, X_k] = 0, \qquad [P_j, P_k] = 0, \qquad [X_j, P_k] = i\hbar\delta_{jk} \cdot 1_H.$$

In fact, given these commutation relations, the particular choice of the Hilbert space H and of the operators X_j and P_j is in a certain sense unique according to the Stone-von Neumann theorem; for details we refer to Theorem 6.4 in chapter IV of [88]. In other words, in this case the algebraic properties of these operators completely determine the Hilbert space H.

In more complicated systems, for instance those appearing in quantum field theory, there is no analogue to the Stone-von Neumann theorem and hence there is no preferred Hilbert space corresponding to a system. One solution to this problem, as proposed by Haag and Kastler in their famous paper [41], is to assign to each physical system a C^*-algebra $\mathcal{A}$, called the algebra of (bounded) observables, rather than a Hilbert space with concrete operators acting on it. In order to implement the spacetime structure into this algebra $\mathcal{A}$, one assigns to each bounded region O of spacetime a subalgebra $\mathcal{A}(O)$ of $\mathcal{A}$ which is interpreted as the algebra of observables that can be measured in the region O. In this way we obtain an assignment $O \mapsto \mathcal{A}(O)$ that assigns to each bounded region O of spacetime an algebra $\mathcal{A}(O)$. This approach to quantum theory is called algebraic quantum field theory (AQFT).

1.2 Motivation for the project

Despite the rather physical motivation for algebraic quantum field theory, one may consider it as an area of mathematics rather than as an area of physics, because it is formulated in terms of precise mathematical axioms and was developed as an effort to make quantum field theory (as a physical theory) mathematically rigorous. For a mathematician there is of course no reason to stick with this physical origin of AQFT and therefore we can just as well alter the axioms and make them a bit more general. For instance, instead of only considering assignments $O \mapsto \mathcal{A}(O)$ where the sets O are regions in spacetime, we can consider any assignment $i \mapsto \mathcal{A}(i)$ where i is an element of a set with certain ordering properties. In our investigation we will take this set to be the real line $\mathbb{R}$ and we will study assignments $I \mapsto \mathcal{A}(I)$ where to each interval $I \subset \mathbb{R}$ we assign an algebra $\mathcal{A}(I)$. All these algebras $\mathcal{A}(I)$ are subalgebras of one big algebra $\mathcal{A}$.

A mathematically interesting aspect of an AQFT $\mathcal{A}$ is the study of its representations. From a physical point of view, it is also very convenient to consider representations of $\mathcal{A}$ on some Hilbert space. Namely, after a representation has been chosen, we are back in the situation again where we can consider density operators and we can carry out physically meaningful computations as explained in the preceding section. In the representation theory of an AQFT there is a special class of representations, namely the class of Doplicher-Haag-Roberts representations, or DHR representations for short. We will not provide the defining conditions of such representations here, nor will we explain why the class of DHR representations was considered in the first place. For this we simply refer to the original papers [18], [19], [20] and [21]. From these papers we also know that the DHR representations of an AQFT $\mathcal{A}$ can be obtained from the so-called localized transportable endomorphisms of $\mathcal{A}$, also called the DHR endomorphisms of $\mathcal{A}$. The subclass of DHR endomorphisms that have finite statistics forms a braided tensor category $\mathrm{Loc}_f(\mathcal{A})$ with certain additional properties. We thus have

$$\text{Algebraic quantum field theory } \mathcal{A} \quad \rightsquigarrow \quad \text{Braided tensor category } \mathrm{Loc}_f(\mathcal{A}).$$

We will also assume that there is a group G that acts on $\mathcal{A}$ in such a way that each $\mathcal{A}(I)$ is mapped onto itself under this action.[1] This G-action on $\mathcal{A}$ gives rise to a G-action on the braided tensor category

[1]Actually this group action is not just a purely mathematical extra ingredient to the framework. In algebraic quantum field theory (without any generalizations made by mathematicians) one also considers local fields that are acted upon by a so-called gauge group, giving rise to field algebras $\mathcal{F}(O)$. The algebra $\mathcal{F}(O)^G$ of fixed points under this G-action is then considered as the local algebra of observables corresponding to O.

$\mathrm{Loc}_f(\mathcal{A})$, which thus becomes a braided G-category. As shown in Müger's paper [78], in the presence of such a G-action we can define a class of left/right G-localized endomorphisms which is more general than the class of DHR endomorphisms. This more general class forms a braided G-crossed category $G - \mathrm{Loc}_f^{L/R}(\mathcal{A})$ that contains $\mathrm{Loc}_f(\mathcal{A})$ as a full braided subcategory. More precisely, $\mathrm{Loc}_f(\mathcal{A})$ is the full subcategory of $G - \mathrm{Loc}_f^{L/R}(\mathcal{A})$ determined by the objects that have degree e, where $e \in G$ denotes the identity element. Thus

$$\text{Group } G \text{ acting on the AQFT } \mathcal{A} \quad \rightsquigarrow \quad \text{Braided } G\text{-crossed category } G - \mathrm{Loc}_f^{L/R}(\mathcal{A})$$

and we have a full inclusion

$$\mathrm{Loc}_f(\mathcal{A}) \subset G - \mathrm{Loc}_f^{L/R}(\mathcal{A})$$

of a braided G-category in a braided G-crossed category.

At the beginning of this research project we were particularly interested in the following example. Suppose that we are given some AQFT $\mathcal{A}$ on the real line $\mathbb{R}$. For any natural number N we can consider the N-fold tensor product $\mathcal{A}^{\otimes N}$, which is again an AQFT on $\mathbb{R}$. The corresponding assignment is simply given by $I \mapsto \mathcal{A}(I)^{\otimes N}$. The category $\mathrm{Loc}_f(\mathcal{A}^{\otimes N})$ of DHR endomorphisms is equivalent to the N-fold enriched product $\mathrm{Loc}_f(\mathcal{A})^{\boxtimes N}$ of the category $\mathrm{Loc}_f(\mathcal{A})$. On the N-fold tensor product $\mathcal{A}^{\otimes N}$ the symmetric group S_N acts in the obvious way, i.e. by permutation of the N factors. We are thus in the situation above where we have an AQFT $\mathcal{A}^{\otimes N}$ with an S_N-action. In particular, we have the full inclusion

$$\mathrm{Loc}_f(\mathcal{A})^{\boxtimes N} \subset S_N - \mathrm{Loc}^{L/R}(\mathcal{A}^{\otimes N}).$$

In this case, the group action on the AQFT is rather special in the sense that the group S_N only permutes the N factors of $\mathcal{A}^{\otimes N}$ and does not involve any details about $\mathcal{A}$. For this reason it seemed reasonable to assume that $S_N - \mathrm{Loc}_f^{L/R}(\mathcal{A}^{\otimes N})$ is determined up to equivalence by $\mathrm{Loc}_f(\mathcal{A})$ and N alone. Thus we expected that if $\mathcal{A}$ and $\mathcal{B}$ are two AQFTs, then we have the implication

$$\mathrm{Loc}_f(\mathcal{A}) \simeq \mathrm{Loc}_f(\mathcal{B}) \qquad \Rightarrow \qquad S_N - \mathrm{Loc}_f^{L/R}(\mathcal{A}^{\otimes N}) \simeq S_N - \mathrm{Loc}^{L/R}(\mathcal{B}^{\otimes N}) \tag{1.2.1}$$

for all N. From the assumption that this implication should be true, we were led to believe that we might be able to construct $S_N - \mathrm{Loc}_f^{L/R}(\mathcal{A}^{\otimes N})$ categorically (up to equivalence) from $\mathrm{Loc}_f(\mathcal{A})$ and the number N, i.e. that there exists a categorical construction

$$(N, \mathrm{Loc}_f(\mathcal{A})) \quad \rightsquigarrow \quad S_N - \mathrm{Loc}_f^{L/R}(\mathcal{A}^{\otimes N}) \tag{1.2.2}$$

for each $N \in \mathbb{Z}_{\geq 2}$. The search for such a construction was the original starting point for our project. However, a more recent result of Bischoff in his note [7] implies that the implication (1.2.1) above is false, as we will now briefly explain. In the case where $\mathcal{A}$ is an AQFT on $\mathbb{R}$ that arises from a holomorphic completely rational chiral conformal quantum field theory, it is known that for each $q \in S_N$ the braided S_N-crossed category $S_N - \mathrm{Loc}_f^{L/R}(\mathcal{A}^{\otimes N})$ contains precisely one equivalence class of irreducible objects of degree q and that the tensor structure of $S_N - \mathrm{Loc}_f^{L/R}(\mathcal{A}^{\otimes N})$ is therefore determined by a 3-cocycle $\omega_{\mathcal{A},N}$ on S_N with values in the circle group $S^1 \subset \mathbb{C}$. Up to equivalence, it is determined by the cohomology class $[\omega_{\mathcal{A},N}] \in H^3(S_N, S^1)$. If the implication (1.2.1) above is true, then the collection $\{[\omega_{\mathcal{A},N}] : N \in \mathbb{Z}_{\geq 2}\}$ has to be independent of the chosen holomorphic model $\mathcal{A}$. Hence for each N the cohomology class $[\omega_{\mathcal{A},N}]$ has to be trivial, because in Theorem 2 of [31] it is shown that there exist holomorphic models $\mathcal{A}$ for which $[\omega_{\mathcal{A},N}]$ is trivial for all N (namely those holomorphic models for which the central charge c is a multiple of 24). In [7] Bischoff has given a counterexample to this statement, namely he has given a holomorphic model $\mathcal{A}$ for which $[\omega_{\mathcal{A},3}]$ is non-trivial. Knowing now that our conjectured implication (1.2.1) was actually wrong, we also know now that the categorical construction (1.2.2) above does not exist and therefore also that our original approach was doomed to fail. Although we were not aware of this at the beginning of the project, we nevertheless decided to focus on a somewhat different problem derived from this original problem, which we will now explain.

Note that the search for the construction in (1.2.2), which unfortunately does not exist as we know now, meant that we were trying to extend the braided S_N-category $\mathrm{Loc}_f(\mathcal{A})^{\boxtimes N}$ to the braided S_N-crossed category $S_N - \mathrm{Loc}_f^{L/R}(\mathcal{A}^{\otimes N})$. Formulated somewhat more abstractly, we were trying to extend a braided G-category $\mathcal{C}$ to a braided G-crossed category $\mathcal{D} \supset \mathcal{C}$ such that the full subcategory of $\mathcal{D}$ determined by the objects with degree e coincides with $\mathcal{C}$. Such extensions are also called braided G-crossed extensions of a braided G-category. A more general approach to our original problem is thus to define a construction of a braided G-crossed extension $\mathcal{D}$ of a braided G-category $\mathcal{C}$ and at the time we hoped that we could show that this construction gives us $S_N - \mathrm{Loc}^{L/R}(\mathcal{A}^{\otimes N})$ (up to equivalence) out of $\mathrm{Loc}_f(\mathcal{A}^{\otimes N}) \simeq \mathrm{Loc}_f(\mathcal{A})^{\boxtimes N}$. The search for such an abstract categorical construction became the new starting point for our research project. Since several of our non-trivial categorical results in Chapter 4 were strongly motivated by particular observations from AQFT, we have also included a chapter on AQFT. Furthermore, it could be the case that under certain more special conditions on the AQFT $\mathcal{A}$ it is still possible to construct $S_N - \mathrm{Loc}^{L/R}(\mathcal{A}^{\otimes N})$ categorically from $\mathrm{Loc}_f(\mathcal{A})$ and that our particular construction has some relevance in such cases.

1.3 Outline of the thesis

This thesis consists of five chapters, including the present introduction. Each of these chapters consists of several sections, some of which are further subdivided into subsections. At the beginning of each chapter we will explicitly announce which results of that chapter are new. We will now give a brief overview of the content of each chapter.

Chapter 2 will be about category theory and has two main purposes. The first purpose is to introduce the definitions and results that are needed in order to fully understand the problem of extending G-categories and in order to understand the categorical aspects of AQFT. This will make the following chapters run more smoothly in the sense that we do not need to introduce many concepts from category theory in those chapters, which might be considered as being distracting. The second purpose of the chapter will be to prove several lemmas that will be needed to prove some of our more involved theorems in later chapters.

Chapter 3 discusses AQFT and plays a major role in the thesis for several reasons. As explained before, our original research problem concerned a particular categorical construction in the representation theory of AQFTs, and our new research problem originated from generalizing the idea of such a construction. For that reason a proper understanding of (certain aspects of) AQFT is essential in understanding the motivation for our research project. Besides this motivational role, the content of the chapter was also the starting point for some of our main results in Chapter 4. For instance, we would have never conjectured the content of Theorem 4.10.7 without our considerations in Subsection 3.2.2. Chapter 3 consists of two sections. The first section is devoted to the theory of operator algebras, because these form the main ingredient for AQFTs. Besides the basic facts about operator algebras, this section also includes C^*-tensor categories, the crossed product of a BTC^* with a symmetric tensor subcategory, as well as some elements from the theory of type III subfactors. In the second section we will introduce the axioms of an AQFT on $\mathbb{R}$ that also carries a group action, as well as the notion of left/right group-localized endomorphisms of such an AQFT, and we will consider some results[2] concerning the categories that can be constructed from these endomorphisms. At the very end of Chapter 3 we will give our motivation for the main construction that will be carried out in Chapter 4.

[2]Besides our own results, we will also spend some time on reviewing the main results of Müger's paper [78], which is a paper that has played a prominent role in our research. Since the approach in [78] was not suited for our purposes, we decided to reconsider the content of that paper from a somewhat different point of view. This not only meant that we had to adjust or generalize certain statements in that paper, but also that we had to give alternative proofs for some of these statements because the proofs in [78] did not fit within our approach. So although some results in Chapter 3 were already proven in [78], we have included their proofs in Chapter 3 because our proofs are different from the ones in [78] and cannot be found elsewhere in the literature.

In Chapter 4 we will first introduce the construction of the G-crossed Drinfeld center $Z_G(\mathcal{C})$ of a G-category $\mathcal{C}$. Our motivation for this construction was already given at the end of Chapter 3. We will not only construct $Z_G(\mathcal{C})$ for strict G-categories $\mathcal{C}$, but also for non-strict ones. For the non-strict case we will also prove that $Z_G(\mathcal{C}) \simeq Z_G(\mathcal{C}')$ whenever $\mathcal{C} \simeq \mathcal{C}'$.[3] We will then prove that $Z_G(\mathcal{C})$ is equivalent to a certain relative Drinfeld center and also that it is equivalent to a certain category of bimodule functors. Next we will show that certain nice properties of $\mathcal{C}$, such as the property of having retracts, direct sums and duals, as well as the property of being semisimple, are inherited by $Z_G(\mathcal{C})$. We will then consider a concrete example of the construction of $Z_G(\mathcal{C})$ that also has some relevance for AQFT. After that we will prove a result about $Z_G(\mathcal{C})$ for the case where $\mathcal{C}$ is a G-spherical fusion category. We end the chapter by discussing the situation where $\mathcal{C}$ has a braiding, in which case $Z_G(\mathcal{C})$ turns out to have some nice internal structure which we were able to unravel because of our knowledge of AQFT.

Finally, in Chapter 5 we will glance back at all previous chapters and see what we have learned. We will also give a suggestion for another possible approach to the main problem.

1.4 List of results

As indicated in the overview above, at the beginning of each chapter we will announce which results in that chapter are new. However, we will also sum them up here.

- *Theorem 2.6.10 in Subsection 2.6.3.* This theorem will be used in Chapter 4 to prove the statements in Section 4.5, but it is also interesting in its own right because it characterizes a certain class of functors of $\mathcal{C}$-bimodule categories on a tensor category $\mathcal{C}$ when $\mathcal{C}$ is also equipped with some non-trivial structure of a $\mathcal{C}$-bimodule category.
- *Theorem 2.8.24 in Subsection 2.8.5.* This theorem introduces the mirror image of a braided G-crossed category and will be used to characterize the categorical relation between left and right G-localized endomorphisms of quantum field theories on $\mathbb{R}$ in Subsection 3.2.2, which in turn formed the main inspiration for our results in Subsection 4.10.1.
- *The content of Section 2.9.* However, the constructions in this section are straightforward generalizations of the ones in [74].
- *Lemma 3.2.14 in Subsection 3.2.3.* This lemma is essential in proving Theorem 3.2.20, but it can also be a useful result in AQFT by itself.
- *The proof of Theorem 3.2.20 in Subsection 3.2.3.* This theorem was proven by Müger in his paper [78], but we will provide an alternative proof, based on our Lemma 3.2.14.
- *The content of Subsection 3.2.2.* In this subsection we will investigate the relation between the categories that arise from left and right G-localized endomorphisms of an AQFT. These results were leading for our results in Subsection 4.10.1 and also formed the motivation for the introduction of the mirror image of a braided G-crossed category in Subsection 2.8.5.
- *Theorem 4.2.1 in Section 4.2.* In this theorem we construct the G-crossed Drinfeld center $Z_G(\mathcal{C})$ of a strict G-category. However, as also mentioned at the beginning of Chapter 4, at the final stage of our research project we found out that this construction had already been carried out by Barvels in [6].
- *Theorem 4.3.3 in Subsection 4.3.1.* In this theorem we construct the G-crossed Drinfeld center $Z_G(\mathcal{C})$ of a non-strict G-category. Since this construction is rather involved, we have shifted the details to Appendix A.
- *Theorem 4.3.4 in Subsection 4.3.2.* This theorem states that equivalent (non-strict) G-categories $\mathcal{C}$ and $\mathcal{C}'$ give rise to equivalent G-crossed Drinfeld centers $Z_G(\mathcal{C})$ and $Z_G(\mathcal{C}')$. The proof is rather long and technical and has been shifted to Appendix B.
- *Theorem 4.4.7 in Section 4.4.* This theorem states that the G-crossed Drinfeld center is equivalent (in the sense of braided G-crossed categories) to a certain relative Drinfeld center.

[3] The proofs in the non-strict case are very long and technical. For this reason we have decided not to include them in the main body of the thesis, but in the appendices.

- *Theorem 4.5.4 in Subsection 4.5.2.* This theorem provides us with a braided G-crossed structure on a certain category of functors of bimodule categories. This theorem is also an important result that is needed for Theorem 4.5.5.
- *Theorem 4.5.5 in Subsection 4.5.3.* This theorem states that the G-crossed Drinfeld center of a G-category is equivalent to the braided G-crossed category that was constructed in Theorem 4.5.4.
- *The content of Sections 4.6 and 4.7.* Here we demonstrate that $Z_G(\mathcal{C})$ inherits several nice properties from $\mathcal{C}$. The results in these two sections are new, but certainly not original because they are straightforward generalizations of the results in [75].
- *Theorem 4.8.2 in Section 4.8.* This theorem describes the structure of the G-crossed Drinfeld center of a certain special kind of G-category.
- *The content of Section 4.9.* Here we eventually prove that $Z_G(\mathcal{C})$ has full G-spectrum if $\mathcal{C}$ is a G-spherical fusion category over a quadratically closed field and satisfies $\dim(\mathcal{C}) \neq 0$. The methods used in this section are straightforward generalizations of the methods used in [75].
- *Proposition 4.10.3 in Subsection 4.10.1.* This proposition shows that we can define an alternative braided G-crossed structure on $Z_G(\mathcal{C})$ in case the G-category $\mathcal{C}$ is braided. When $Z_G(\mathcal{C})$ is equipped with this alternative structure, we denote it by $Z_G^\star(\mathcal{C})$.
- *Theorem 4.10.4 in Subsection 4.10.1.* This theorem proves the equivalence between $Z_G^\star(\mathcal{C})$ and $Z_G(\mathcal{C})^\bullet$ in case the G-category $\mathcal{C}$ is braided, where the latter denotes the mirror image of $Z_G(\mathcal{C})$ as introduced in Theorem 2.8.24.
- *Proposition 4.10.6 in Subsection 4.10.1.* In this proposition we construct an equivalence $\dagger : Z_G^\star(\mathcal{C}) \to Z_G(\mathcal{C})$ of braided G-crossed categories.
- *Theorem 4.10.7 in Subsection 4.10.1.* This is our main result concerning the internal structure of $Z_G(\mathcal{C})$ in case the G-category $\mathcal{C}$ is braided. The content of the theorem was inspired by observation of the categories that arise in AQFT.
- *Theorem 4.10.12 in Subsection 4.10.2.* This theorem provides a first step in finding braided G-crossed extensions of a braided G-category $\mathcal{C}$ within $Z_G(\mathcal{C})$. In Corollary 4.10.13 this is applied to modular tensor G-categories to obtain more satisfactory results.

Chapter 2

Category theory

In this chapter on category theory we will introduce most of the results on categories that will be needed later. An exception to this is the notion of C^*-categories, which will be introduced in the next chapter, after we have defined C^*-algebras. Much of the content of this chapter is already known, but there are some new results. The most important of these is Theorem 2.6.10 in Subsection 2.6.3, which will be used in Chapter 4 to prove the statements in Section 4.5. Another important result in this chapter is Theorem 2.8.24, which will be used to characterize the categorical relation between left and right G-localized endomorphisms of quantum field theories on $\mathbb{R}$ in Subsection 3.2.2, but it will also be used in Subsection 4.10.1. Another result in this chapter that is new, but certainly not original because it is a straightforward generalization of known results[1], concerns the construction of 2-categories from a collection of Frobenius algebras in Subsection 2.9.3. Perhaps Proposition 2.8.23 is also new, but it is not very deep.

There are many good texts on category theory, the standard reference being [68]. In this chapter we have often used [28], [48] and [81].

2.1 Categories, functors and natural transformations

Although it would be reasonable to expect the reader to have some basic knowledge of category theory, we have decided to begin our discussion by giving the definition of a category. The main reason for this is that it allows us to properly introduce all our notation and terminology concerning categories, which might not be standard to some readers. Our definition of a category uses the notation as in [48].

Definition 2.1.1 A *category* $\mathcal{C}$ consists of the following data:

- a class $\mathrm{Obj}(\mathcal{C})$ whose elements are called the *objects* of the category;
- a class $\mathrm{Hom}(\mathcal{C})$ whose elements are called the *morphisms* of the category;
- an *identity* map $\mathrm{id} : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C})$, denoted $V \mapsto \mathrm{id}_V$;
- a *source* map $s : \mathrm{Hom}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$, denoted $f \mapsto s(f)$;
- a *target* map $b : \mathrm{Hom}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$, denoted $f \mapsto b(f)$;
- a *composition* $\circ : \mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C})$, denoted $(f, g) \mapsto g \circ f$, where

$$\mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Hom}(\mathcal{C}) := \{(f, g) \in \mathrm{Hom}(\mathcal{C}) \times \mathrm{Hom}(\mathcal{C}) : b(f) = s(g)\}$$

  denotes the class of composable[2] morphisms.

This collection of data is required to satisfy the following three conditions:

(1) for any object $V \in \mathrm{Obj}(\mathcal{C})$ we have $s(\mathrm{id}_V) = b(\mathrm{id}_V) = V$;

[1] Namely, in [74] this construction was carried out for one single Frobenius algebra.

[2] If $(f, g) \in \mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Obj}(\mathcal{C})} \mathrm{Hom}(\mathcal{C})$ we say that f and g are composable.

(2) for any morphism $f \in \mathrm{Hom}(\mathcal{C})$ we have $\mathrm{id}_{b(f)} \circ f = f \circ \mathrm{id}_{s(f)} = f$;
(3) for any morphisms f, g, h satisfying $b(f) = s(g)$ and $b(g) = s(h)$ we have $(h \circ g) \circ f = h \circ (g \circ f)$.

The source and target maps are not always mentioned explicitly when defining any particular category, but these maps can be convenient sometimes, such as in the definition of the opposite category below. Instead, one often uses some different notation that we will introduce now. Instead of writing $V \in \mathrm{Obj}(\mathcal{C})$ to denote that V is an object in the category $\mathcal{C}$, we will simply write $V \in \mathcal{C}$. When $V, W \in \mathcal{C}$, we will use the customary notation

$$\mathrm{Hom}_{\mathcal{C}}(V, W) := \{f \in \mathrm{Hom}(\mathcal{C}) : s(f) = V \text{ and } b(f) = W\}$$

and we will say that any $f \in \mathrm{Hom}_{\mathcal{C}}(V, W)$ is a *morphism from V to W*. We will also write $\mathrm{End}_{\mathcal{C}}(V) := \mathrm{Hom}_{\mathcal{C}}(V, V)$ and any $f \in \mathrm{End}_{\mathcal{C}}(V)$ will be called an *endomorphism of V*. If $f \in \mathrm{Hom}_{\mathcal{C}}(V, W)$ and if it is clear from the context that f is a morphism in the category $\mathcal{C}$, we will sometimes simply write $f : V \to W$. If there exists an *isomorphism* $f : V \to W$, i.e. a morphism $f : V \to W$ for which there exists a morphism $f^{-1} : W \to V$ such that $f^{-1} \circ f = \mathrm{id}_V$ and $f \circ f^{-1} = \mathrm{id}_W$, then we will write $V \cong W$ and we will say that V and W are isomorphic objects, which obviously defines an equivalence relation on the objects of the category. If $V \in \mathcal{C}$, then a morphism $f \in \mathrm{End}_{\mathcal{C}}(V)$ is called an *idempotent* if it satisfies $f^2 := f \circ f = f$.

A category $\mathcal{C}$ is called *discrete* if the only morphisms in $\mathcal{C}$ are the identity morphisms. Thus, for any $V \in \mathcal{C}$ we have $\mathrm{End}_{\mathcal{C}}(V) = \{\mathrm{id}_V\}$ and if $V, W \in \mathcal{C}$ with $V \neq W$ then $\mathrm{Hom}_{\mathcal{C}}(V, W) = \emptyset$.

Example 2.1.2 We will now show how we can use given categories to construct new ones.

(1) If $\mathcal{C}$ is a category, it is easy to see that we obtain a category $\mathcal{C}^{\mathrm{op}}$ by defining $\mathrm{Obj}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Obj}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(V, W) := \mathrm{Hom}_{\mathcal{C}}(W, V)$, $\mathrm{id}_V^{\mathrm{op}} := \mathrm{id}_V$, $s^{\mathrm{op}}(f) := b(f)$, $b^{\mathrm{op}}(f) := s(f)$ and $g \circ^{\mathrm{op}} f := f \circ g$ for any $V, W \in \mathrm{Obj}(\mathcal{C}^{\mathrm{op}})$ and $f, g \in \mathrm{Hom}(\mathcal{C}^{\mathrm{op}})$, where of course g is such that $s^{\mathrm{op}}(g) = b^{\mathrm{op}}(f)$. The category $\mathcal{C}^{\mathrm{op}}$ is called the *opposite category* of $\mathcal{C}$.
(2) If $\{\mathcal{C}_i\}_{i=1,\ldots,n}$ is a collection of categories, then we obtain a category $\mathcal{C}_1 \times \ldots \times \mathcal{C}_n$ by defining $\mathrm{Obj}(\mathcal{C}_1 \times \ldots \times \mathcal{C}_n) = \mathrm{Obj}(\mathcal{C}_1) \times \ldots \times \mathrm{Obj}(\mathcal{C}_n)$, $\mathrm{Hom}_{\mathcal{C}_1 \times \ldots \times \mathcal{C}_n}((V_1, \ldots, V_n), (W_1, \ldots, W_n)) = \mathrm{Hom}_{\mathcal{C}_1}(V_1, W_1) \times \ldots \times \mathrm{Hom}_{\mathcal{C}_n}(V_n, W_n)$, $\mathrm{id}_{(V_1,\ldots,V_n)} = (\mathrm{id}_{V_1}, \ldots, \mathrm{id}_{V_n})$ and componentwise composition.

Definition 2.1.3 A category $\mathcal{C}$ is called *locally small* if $\mathrm{Hom}_{\mathcal{C}}(V, W)$ is a set for any two objects $V, W \in \mathcal{C}$. A category $\mathcal{C}$ is called *small* if it is locally small and if $\mathrm{Obj}(\mathcal{C})$ is a set.

> In order to prevent any potential technicalities, we will always assume in the rest of this thesis that all our categories are small.

If $\{\mathcal{C}_\alpha\}_{\alpha \in A}$ is a collection of categories, then we define the category

$$\bigsqcup_{\alpha \in A} \mathcal{C}_\alpha \tag{2.1.1}$$

as follows. Its set of objects is the disjoint union $\bigsqcup_{\alpha \in A} \mathrm{Obj}(\mathcal{C}_\alpha)$ of sets. If $(V, \alpha_1), (W, \alpha_2) \in \bigsqcup_{\alpha \in A} \mathrm{Obj}(\mathcal{C}_\alpha)$, then we define

$$\mathrm{Hom}_{\bigsqcup_{\alpha \in A} \mathcal{C}_\alpha}((V, \alpha_1), (W, \alpha_2)) = \begin{cases} \mathrm{Hom}_{\mathcal{C}_\alpha}(V, W) & \text{if } \alpha_1 = \alpha_2 = \alpha \\ \emptyset & \text{if } \alpha_1 \neq \alpha_2. \end{cases}$$

with the obvious composition. We will call the resulting category the *disjoint union* of the categories $\{\mathcal{C}_\alpha\}$.

Definition 2.1.4 A *subcategory* $\mathcal{C}$ of a category $\mathcal{D}$ consists of a subset $\mathrm{Obj}(\mathcal{C}) \subset \mathrm{Obj}(\mathcal{D})$ and of a subset $\mathrm{Hom}(\mathcal{C}) \subset \mathrm{Hom}(\mathcal{D})$ that are stable under the identity, source, target and composition maps in $\mathcal{D}$. We say that $\mathcal{C}$ is a *full subcategory* if $\mathrm{Hom}_{\mathcal{C}}(V, W) = \mathrm{Hom}_{\mathcal{D}}(V, W)$ for all $V, W \in \mathcal{C}$. A full subcategory $\mathcal{C}$ of $\mathcal{D}$ is called *skeletal* if it contains precisely one object of each equivalence class of isomorphic objects. We say that $\mathcal{C}$ is a *replete subcategory* if $V \in \mathcal{C}$ implies that $W \in \mathcal{C}$ for all $W \in \mathcal{D}$ with $W \cong V$.

Let $\mathcal{D}$ be a category and let $\mathcal{S} \subset \mathrm{Obj}(\mathcal{D})$ be a subset of objects. Then we get a subcategory $\mathcal{C}$ of $\mathcal{D}$ by taking $\mathrm{Obj}(\mathcal{C}) := \mathcal{S}$ and $\mathrm{Hom}_{\mathcal{C}}(V,W) = \mathrm{Hom}_{\mathcal{D}}(V,W)$ for all $V, W \in \mathrm{Obj}(\mathcal{C})$. Note that this is a full subcategory. We will often refer to it by saying that it is *the full subcategory of $\mathcal{D}$ determined by the objects in the set $\mathcal{S}$*. Note that if $\mathcal{S}$ contains precisely one object of each equivalence class of isomorphic objects, then the full subcategory of $\mathcal{D}$ determined by the objects in $\mathcal{S}$ is skeletal. Consequently, every category has a full skeletal subcategory.

Definition 2.1.5 Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor*[3] $F : \mathcal{C} \to \mathcal{D}$ consists of a map $F : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ and of a map $F : \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{D})$ such that

- for any $V \in \mathrm{Obj}(\mathcal{C})$ we have $F(\mathrm{id}_V) = \mathrm{id}_{F(V)}$;
- for any $f \in \mathrm{Hom}(\mathcal{C})$ we have $s(F(f)) = F(s(f))$ and $b(F(f)) = F(b(f))$;
- if $f, g \in \mathrm{Hom}(\mathcal{C})$ with $b(f) = s(g)$, we have $F(g \circ f) = F(g) \circ F(f)$.

Example 2.1.6 There are several important examples of functors:

(1) If $\mathcal{C}$ is a category then we denote by $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ the identity functor, which maps all objects and morphisms onto themselves.

(2) If we have categories $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ then we write $G \circ F : \mathcal{C} \to \mathcal{E}$ to denote the composition of F and G, which is easily seen to be a functor again.

(3) If we have categories $\{\mathcal{C}_i, \mathcal{D}_i\}_{i=1,\ldots,n}$ and functors $F_i : \mathcal{C}_i \to \mathcal{D}_i$, we write

$$F_1 \times \ldots \times F_n : \mathcal{C}_1 \times \ldots \times \mathcal{C}_n \to \mathcal{D}_1 \times \ldots \times \mathcal{D}_n$$

to denote the functor determined by $(V_1, \ldots, V_n) \mapsto (F_1(V_1), \ldots, F_n(V_n))$ and $(f_1, \ldots, f_n) \mapsto (F_1(f_1), \ldots, F_n(f_n))$.

(4) If $\mathcal{D}$ is a category and $\mathcal{C} \subset \mathcal{D}$ is a subcategory, then we denote by $\mathscr{I} : \mathcal{C} \to \mathcal{D}$ the inclusion functor.

Definition 2.1.7 Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

(1) Then F is called *faithful*, respectively *full*, if for any two objects $U, V \in \mathcal{C}$ the map

$$F : \mathrm{Hom}_{\mathcal{C}}(U,V) \to \mathrm{Hom}_{\mathcal{D}}(F(U), F(V))$$

is injective, respectively surjective. If it is both, we say that F is *fully faithful*.

(2) If for each $W \in \mathcal{D}$ there exists a $V \in \mathcal{C}$ such that $F(V) \cong W$, then F is called *essentially surjective*.

If $\mathcal{D}$ is a category and $\mathcal{C} \subset \mathcal{D}$ is a subcategory, then clearly the inclusion functor $\mathscr{I} : \mathcal{C} \to \mathcal{D}$ is always faithful. Note that the inclusion functor is full if and only if $\mathcal{C}$ is a full subcategory of $\mathcal{D}$.

Definition 2.1.8 Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* φ from F to G, denoted $\varphi : F \to G$, is a family $\{\varphi_U : F(U) \to G(U)\}_{U \in \mathcal{C}}$ of morphisms in $\mathcal{D}$ such that, for any $f \in \mathrm{Hom}_{\mathcal{C}}(U,V)$ the square

$$\begin{array}{ccc} F(U) & \xrightarrow{\varphi_U} & G(U) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(V) & \xrightarrow{\varphi_V} & G(V) \end{array}$$

commutes. If each φ_U is an isomorphism, then we say that $\varphi : F \to G$ is a *natural isomorphism*, and in this case we will call F and G *equivalent functors*. We will write $\mathrm{Nat}(F,G)$ to denote the set of all natural transformations from F to G, and we will write $\mathrm{Aut}(F)$ to denote the set of all natural isomorphisms from F to itself (i.e. the natural automorphisms of F).

[3]More precisely, this is actually called a covariant functor, in contrast to the notion of contravariant functors. However, we will not consider contravariant functors, so whenever we speak of a functor, we will always mean a covariant functor.

Remark 2.1.9 (1) If $F : \mathcal{C} \to \mathcal{D}$ is a functor, we will write $\mathrm{id}_F : F \to F$ to denote the natural automorphism given by the family $\{(\mathrm{id}_F)_U\}_{U\in\mathcal{C}}$ with $(\mathrm{id}_F)_U := \mathrm{id}_{F(U)}$ for all $U \in \mathcal{C}$.
(2) If $\varphi : F \to G$ is a natural isomorphism, then the family $\{\varphi_U^{-1} : G(U) \to F(U)\}_{U\in\mathcal{C}}$ defines a natural isomorphism $\varphi^{-1} : G \to F$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors and let $\varphi : F \to G$ and $\psi : G \to H$ be natural transformations. Then the family

$$\{(\psi \circ \varphi)_U : F(U) \to H(U)\}_{U\in\mathcal{C}}$$

defined by $(\psi \circ \varphi)_U := \psi_U \circ \varphi_U$ defines a natural transformation $\psi \circ \varphi : F \to H$. As a consequence, we obtain a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, where the objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are the functors from $\mathcal{C}$ to $\mathcal{D}$, the identity morphism of $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is given by id_F as defined in Remark 2.1.9 above, the morphisms are given by $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) = \mathrm{Nat}(F, G)$ and the composition is the composition of natural transformations as we have just defined. We will write $\mathrm{End}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}, \mathcal{C})$.

Definition 2.1.10 If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence* from $\mathcal{C}$ to $\mathcal{D}$ if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\varphi : \mathrm{id}_{\mathcal{D}} \to F \circ G$ and $\psi : G \circ F \to \mathrm{id}_{\mathcal{C}}$. In this case we will say that $\mathcal{C}$ is equivalent[4] to $\mathcal{D}$ and write $\mathcal{C} \simeq \mathcal{D}$. We will write $\mathrm{Aut}(\mathcal{C})$ to denote the full subcategory of $\mathrm{End}(\mathcal{C})$ determined by the objects that are an equivalence from $\mathcal{C}$ to itself.

The following well-known lemma is often convenient if one wants to prove that a given functor is an equivalence. A proof of this lemma can be found in [48].

Lemma 2.1.11 *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is fully faithful and essentially surjective.*

As a direct application of this lemma, if $\mathcal{C}$ is a category and if $\mathcal{S} \subset \mathcal{C}$ is a full skeletal subcategory, then the inclusion functor establishes an equivalence $\mathcal{S} \simeq \mathcal{C}$.

2.2 Tensor categories

The central concept in this chapter is the notion of a tensor category. Tensor categories are often called monoidal categories, because they share some properties with monoids, which are sets with an associative product operation and a unit element with respect to this product. Before we give the definition of a tensor category we will first consider in some detail what it means to have a tensor product on a category.

Definition 2.2.1 Let $\mathcal{C}$ be a category. Then a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is called a *tensor product.*

Suppose that $\otimes$ is a tensor product on $\mathcal{C}$. For $U, V \in \mathcal{C}$ we will write $U \otimes V$ rather than $\otimes(U, V)$; similarly, we will write $f \otimes g$ for $f, g \in \mathrm{Hom}(\mathcal{C})$ rather than $\otimes(f, g)$. Explicitly, the fact that $\otimes$ is a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ means that

- for each pair $(U, V) \in \mathcal{C} \times \mathcal{C}$ we have an object $U \otimes V \in \mathcal{C}$.
- for each pair $(f, g) \in \mathrm{Hom}(\mathcal{C}) \times \mathrm{Hom}(\mathcal{C})$ we have a morphism $f \otimes g \in \mathrm{Hom}(\mathcal{C})$ such that $s(f \otimes g) = s(f) \otimes s(g)$ and $b(f \otimes g) = b(f) \otimes b(g)$;
- if f, f', g, g' are morphisms in $\mathcal{C}$ with $s(f') = b(f)$ and $s(g') = b(g)$ then we have the *interchange law*

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g);$$

- If $U, V \in \mathcal{C}$ then $\mathrm{id}_U \otimes \mathrm{id}_V = \mathrm{id}_{U\otimes V}$.

[4]This clearly defines an equivalence relation.

Note that we did not require a tensor product to be associative, so if $U, V, W \in \mathcal{C}$ then $(U \otimes V) \otimes W$ need not be equal to $U \otimes (V \otimes W)$. The first piece of structure in a tensor category assures that such triple products are related to each other in a nice way in the sense of the following definition.

Definition 2.2.2 Let $\mathcal{C}$ be a category and let $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a tensor product. Then an associativity constraint for $\otimes$ is a natural isomorphism $a : \otimes \circ (\otimes \times \mathrm{id}_{\mathcal{C}}) \to \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \otimes)$ of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, i.e. a family $\{a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)\}_{U,V,W \in \mathcal{C}}$ of isomorphisms in $\mathcal{C}$ such that the square

$$\begin{array}{ccc} (U \otimes V) \otimes W & \xrightarrow{a_{U,V,W}} & U \otimes (V \otimes W) \\ \big\downarrow{\scriptstyle (f \otimes g) \otimes h} & & \big\downarrow{\scriptstyle f \otimes (g \otimes h)} \\ (U' \otimes V') \otimes W' & \xrightarrow{a_{U',V',W'}} & U' \otimes (V' \otimes W') \end{array}$$

commutes for all $U, U', V, V', W, W' \in \mathcal{C}$ and $f : U \to U'$, $g : V \to V'$ and $h : W \to W'$.

Definition 2.2.3 Let $\mathcal{C}$ be a category with tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and associativity constraint $a : \otimes \circ (\otimes \times \mathrm{id}_{\mathcal{C}}) \to \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \otimes)$. Then the associativity constraint is said to satisfy the *pentagon axiom* if the pentagonal diagram

$$\begin{array}{ccccc} & & ((U \otimes V) \otimes W) \otimes X & & \\ & \swarrow{\scriptstyle a_{U,V,W} \otimes \mathrm{id}_X} & & \searrow{\scriptstyle a_{U \otimes V, W, X}} & \\ (U \otimes (V \otimes W)) \otimes X & & & & (U \otimes V) \otimes (W \otimes X) \\ \big\downarrow{\scriptstyle a_{U, V \otimes W, X}} & & & & \big\downarrow{\scriptstyle a_{U,V,W \otimes X}} \\ U \otimes ((V \otimes W) \otimes X) & & \xrightarrow{\mathrm{id}_U \otimes a_{V,W,X}} & & U \otimes (V \otimes (W \otimes X)) \end{array}$$

commutes for all $U, V, W, X \in \mathcal{C}$.

Thus if $\otimes$ is a tensor product on a category $\mathcal{C}$, then an associativity constraint satisfying the pentagon axiom controls the non-associativity of the tensor product. The next ingredient that will be needed for the definition of a tensor category is a unit object, analogous to the unit element in a monoid. For this we first need some notation.

Definition 2.2.4 If $\mathcal{C}$ is a category and if $U \in \mathcal{C}$, then we define a functor $U \times \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ by $(U \times \mathrm{id}_{\mathcal{C}})(V) := (U, V)$ and $(U \times \mathrm{id}_{\mathcal{C}})(f) = (\mathrm{id}_U, f)$ for any $V \in \mathcal{C}$ and $f \in \mathrm{Hom}(\mathcal{C})$. Similarly, we also define the functor $\mathrm{id}_{\mathcal{C}} \times U : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$.

Definition 2.2.5 Let $\mathcal{C}$ be a category with tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and fix an object $I \in \mathcal{C}$.

(1) A *left unit constraint* with respect to I is a natural isomorphism $l : \otimes \circ (I \times \mathrm{id}_{\mathcal{C}}) \to \mathrm{id}_{\mathcal{C}}$, i.e. a family $\{l_V : I \otimes V \to V\}_{V \in \mathcal{C}}$ of isomorphisms in $\mathcal{C}$ such that the square

$$\begin{array}{ccc} I \otimes V & \xrightarrow{l_V} & V \\ \big\downarrow{\scriptstyle \mathrm{id}_I \otimes f} & & \big\downarrow{\scriptstyle f} \\ I \otimes W & \xrightarrow{l_W} & W \end{array}$$

commutes for all $V, W \in \mathcal{C}$ and $f : V \to W$.

(2) A *right unit constraint* with respect to I is a natural isomorphism $r : \otimes \circ (\mathrm{id}_{\mathcal{C}} \times I) \to \mathrm{id}_{\mathcal{C}}$, i.e. a family $\{r_V : V \otimes I \to V\}_{V \in \mathcal{C}}$ of isomorphisms in $\mathcal{C}$ such that the square

$$\begin{array}{ccc} V \otimes I & \xrightarrow{r_V} & V \\ {\scriptstyle f \otimes \mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle f} \\ W \otimes I & \xrightarrow{r_W} & W \end{array}$$

commutes for all $V, W \in \mathcal{C}$ and $f : V \to W$.

We have now defined the two basic structures that are needed for defining tensor categories: the associativity constraint (satisfying the pentagon axiom) and unit constraints. The only thing that is still left to define is a compatibility condition between these two structures.

Definition 2.2.6 Let $\mathcal{C}$ be a category with tensor product $\otimes$, associativity constraint a and left and right unit constraints l and r with respect to an object $I \in \mathcal{C}$. Then we say that $\otimes$, a, I, l and r satisfy the *triangle axiom* if the triangle

$$\begin{array}{ccc} (V \otimes I) \otimes W & \xrightarrow{a_{V,I,W}} & V \otimes (I \otimes W) \\ & {\scriptstyle r_V \otimes \mathrm{id}_W} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_V \otimes l_W} & \\ & V \otimes W & \end{array}$$

commutes for all $V, W \in \mathcal{C}$.

Definition 2.2.7 A *tensor category* $(\mathcal{C}, \otimes, I, a, l, r)$ is a category $\mathcal{C}$ which is equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, with an object $I \in \mathcal{C}$ (called the *unit object* of the tensor category), with an associativity constraint a and with left and right unit constraints l and r with respect to I such that the pentagon axiom and the triangle axiom are satisfied.

Tensor categories are often called *monoidal categories* in the literature, although some authors use these two terms for different things. For instance, in [28] monoidal categories are defined as in Definition 2.2.7 above, but tensor categories are defined to be monoidal categories with some extra structure.

A subcategory of a tensor category $(\mathcal{C}, \otimes, I, a, l, r)$ is called a *tensor subcategory* if it is also a tensor category with respect to the tensor product and unit object inherited from $\mathcal{C}$.

Example 2.2.8 We mention some examples of tensor categories.

(1) An easy example is given by the category $\mathrm{Vect}(\mathbb{F})$ of vector spaces over a field $\mathbb{F}$. The objects of this category are the vector spaces over $\mathbb{F}$ and the morphisms between them are the $\mathbb{F}$-linear maps. It becomes a (non-strict) tensor category if we define the tensor product to be the usual tensor product of vectors spaces and linear maps. We will write $\mathrm{Vect}_f(\mathbb{F})$ to denote the full tensor subcategory of $\mathrm{Vect}(\mathbb{F})$ determined by objects of $\mathrm{Vect}(\mathbb{F})$ that are finite-dimensional.

(2) If G is a group and $\mathbb{F}$ is a field, then we define the representation category $\mathrm{Rep}(G; \mathbb{F})$ of G as follows. The objects of $\mathrm{Rep}(G; \mathbb{F})$ are representations (V, π_V) of G, where V is a vector space over $\mathbb{F}$ and $\pi_V : G \to \mathrm{Aut}(V)$ is a group homomorphism. The morphisms from (V, π_V) to (W, π_W) are the $\mathbb{F}$-linear maps $T : V \to W$ that intertwine the two representations, i.e. for any $q \in G$ we have $\pi_W(q)T = T\pi_V(q)$. The tensor product in this category is given by the tensor product of representations and of intertwiners which are well-known from representation theory.

A tensor category $(\mathcal{C}, \otimes, I, a, l, r)$ is called *strict* if for all $U, V, W \in \mathcal{C}$ and $f, g, h \in \mathrm{Hom}(\mathcal{C})$ it satisfies $(U \otimes V) \otimes W = U \otimes (V \otimes W)$, $(f \otimes g) \otimes h = f \otimes (g \otimes h)$, $I \otimes V = V = V \otimes I$, $\mathrm{id}_I \otimes f = f = f \otimes \mathrm{id}_I$, $a_{U,V,W} = \mathrm{id}_{U\otimes V\otimes W}$ and $l_V = \mathrm{id}_V = r_V$. Note that the pentagon and triangle axioms are trivially satisfied in this case. Because the notion of a strict tensor category will be very important in what follows, we will define it again explicitly, without reference to the more general definition of a (non-strict) tensor category.

Definition 2.2.9 A *strict tensor category* $(\mathcal{C}, \otimes, I)$ consists of the following data:

- a category $\mathcal{C}$;
- an associative tensor product $\otimes$, i.e. a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying the equality $\otimes \circ [\otimes \times \mathrm{id}_{\mathcal{C}}] = \otimes \circ [\mathrm{id}_{\mathcal{C}} \times \otimes]$ of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;
- an object $I \in \mathcal{C}$ (called the unit object) satisfying $I \otimes V = V = V \otimes I$ for all $V \in \mathcal{C}$ and $\mathrm{id}_I \otimes f = f = f \otimes \mathrm{id}_I$ for all $f \in \mathrm{Hom}(\mathcal{C})$.

We will now discuss an example of a strict tensor category that will be very important to us. Let $\mathcal{C}$ be a category and consider the category $\mathrm{End}(\mathcal{C})$. It is clear that for any $F, G \in \mathrm{End}(\mathcal{C})$ their composition $F \circ G$ is again in $\mathrm{End}(\mathcal{C})$, which allows us to define the operation

$$F \otimes G := F \circ G$$

on the objects of $\mathrm{End}(\mathcal{C})$. Now suppose that $\varphi \in \mathrm{Hom}_{\mathrm{End}(\mathcal{C})}(F, F')$ and $\psi \in \mathrm{Hom}_{\mathrm{End}(\mathcal{C})}(G, G')$. Then for any $V \in \mathcal{C}$ we define a morphism $(\varphi \otimes \psi)_V \in \mathrm{Hom}_{\mathcal{C}}(F(G(V)), F'(G'(V)))$ by

$$(\varphi \otimes \psi)_V := \varphi_{G'(V)} \circ F(\psi_V) = F'(\psi_V) \circ \varphi_{G(V)}.$$

It can be shown that $\varphi \otimes \psi$ is a natural transformation from $F \circ G$ to $F' \circ G'$, i.e. that $\varphi \otimes \psi \in \mathrm{Hom}_{\mathrm{End}(\mathcal{C})}(F \otimes G, F' \otimes G')$. In fact, it is straightforward to check that $\otimes$ defines an associative tensor product on $\mathrm{End}(\mathcal{C})$ and that $\mathrm{End}(\mathcal{C})$ becomes a strict tensor category with unit object given by $\mathrm{id}_{\mathcal{C}}$. Furthermore, the subcategory $\mathrm{Aut}(\mathcal{C})$ of $\mathrm{End}(\mathcal{C})$ is a full tensor subcategory.

2.2.1 Tensor functors and natural tensor transformations

When we want to consider functors between tensor categories, it is important that these functors behave nicely with respect to the tensor products and unit objects in both tensor categories. Even in case both tensor categories are strict, there is no reason to demand that such a functor $F : \mathcal{C} \to \mathcal{D}$ is strict in the sense that it satisfies $F(V \otimes_{\mathcal{C}} W) = F(V) \otimes_{\mathcal{D}} F(W)$ or $F(I_{\mathcal{C}}) = I_{\mathcal{D}}$. On the other hand, it is also possible that a functor between two non-strict tensor categories does satisfy these strictness conditions[5].

Definition 2.2.10 Let $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, a_{\mathcal{C}}, l_{\mathcal{C}}, r_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, I_{\mathcal{D}}, a_{\mathcal{D}}, l_{\mathcal{D}}, r_{\mathcal{D}})$ be tensor categories. A *tensor functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varepsilon^F, \delta^F)$ consisting of

- a functor $F : \mathcal{C} \to \mathcal{D}$;
- a natural isomorphism $\delta^F : \otimes_{\mathcal{D}} \circ (F \times F) \to F \circ \otimes_{\mathcal{C}}$ of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{D}$, i.e. a family

$$\left\{\delta^F_{U,V} : F(U) \otimes_{\mathcal{D}} F(V) \to F(U \otimes_{\mathcal{C}} V)\right\}_{U,V\in\mathcal{C}}$$

  of isomorphisms in $\mathcal{D}$ such that for any objects $U, V, U', V' \in \mathcal{C}$ and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(U, U')$ and $g \in \mathrm{Hom}_{\mathcal{C}}(V, V')$ the square

[5] This is possible if and only if $F(a_{U,V,W}) = a_{F(U),F(V),F(W)}$, $F(l_U) = l_{F(U)}$ and $F(r_U) = r_{F(U)}$, i.e. if the associativity constraint and unit constraints of $\mathcal{C}$ are mapped to those of $\mathcal{D}$. This can easily be seen from the definition of a tensor functor below.

$$\begin{array}{ccc} F(U)\otimes_{\mathcal{D}} F(V) & \xrightarrow{\delta^F_{U,V}} & F(U\otimes_{\mathcal{C}} V) \\ {\scriptstyle F(f)\otimes_{\mathcal{D}}F(g)}\Big\downarrow & & \Big\downarrow{\scriptstyle F(f\otimes_{\mathcal{C}} g)} \\ F(U')\otimes_{\mathcal{D}} F(V') & \xrightarrow{\delta^F_{U',V'}} & F(U'\otimes_{\mathcal{C}} V') \end{array}$$

commutes, satisfying the additional property that the diagram

$$\begin{array}{ccc} (F(U)\otimes_{\mathcal{D}} F(V))\otimes_{\mathcal{D}} F(W) & \xrightarrow{(a_{\mathcal{D}})_{F(U),F(V),F(W)}} & F(U)\otimes_{\mathcal{D}}(F(V)\otimes_{\mathcal{D}} F(W)) \\ {\scriptstyle \delta^F_{U,V}\otimes_{\mathcal{D}}\mathrm{id}_{F(W)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}_{F(U)}\otimes_{\mathcal{D}}\delta^F_{V,W}} \\ F(U\otimes_{\mathcal{C}} V)\otimes_{\mathcal{D}} F(W) & & F(U)\otimes_{\mathcal{D}} F(V\otimes_{\mathcal{C}} W) \\ {\scriptstyle \delta^F_{U\otimes_{\mathcal{C}}V,W}}\Big\downarrow & & \Big\downarrow{\scriptstyle \delta^F_{U,V\otimes_{\mathcal{C}}W}} \\ F((U\otimes_{\mathcal{C}} V)\otimes_{\mathcal{C}} W) & \xrightarrow{F((a_{\mathcal{C}})_{U,V,W})} & F(U\otimes_{\mathcal{C}}(V\otimes_{\mathcal{C}} W)) \end{array}$$

commutes for all $U,V,W\in\mathcal{C}$;

- an isomorphism $\varepsilon^F : I_{\mathcal{D}} \to F(I_{\mathcal{C}})$ such that the diagrams

$$\begin{array}{ccc} I_{\mathcal{D}}\otimes_{\mathcal{D}} F(U) & \xrightarrow{(l_{\mathcal{D}})_{F(U)}} & F(U) \\ {\scriptstyle \varepsilon^F\otimes_{\mathcal{D}}\mathrm{id}_{F(U)}}\Big\downarrow & & \Big\uparrow{\scriptstyle F((l_{\mathcal{C}})_U)} \\ F(I_{\mathcal{C}})\otimes_{\mathcal{D}} F(U) & \xrightarrow{\delta_{I_{\mathcal{C}},U}} & F(I_{\mathcal{C}}\otimes_{\mathcal{C}} U) \end{array} \quad\text{and}\quad \begin{array}{ccc} F(U)\otimes_{\mathcal{D}} I_{\mathcal{D}} & \xrightarrow{(r_{\mathcal{D}})_{F(U)}} & F(U) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{F(U)}\otimes_{\mathcal{D}}\varepsilon^F} & & \Big\uparrow{\scriptstyle F((r_{\mathcal{C}})_U)} \\ F(U)\otimes_{\mathcal{D}} F(I_{\mathcal{C}}) & \xrightarrow{\delta_{U,I_{\mathcal{C}}}} & F(U\otimes_{\mathcal{C}} I_{\mathcal{C}}) \end{array}$$

  commute for all $U\in\mathcal{C}$.

The tensor functor $(F,\varepsilon^F,\delta^F)$ is said to be a *strict tensor functor* if $F(I_{\mathcal{C}}) = I_{\mathcal{D}}$ and $F(U\otimes_{\mathcal{C}} V) = F(U)\otimes_{\mathcal{D}} F(V)$ for all $U,V\in\mathcal{C}$ and if $\varepsilon^F = \mathrm{id}_{I_{\mathcal{D}}} = \mathrm{id}_{F(I_{\mathcal{C}})}$ and $\delta^F_{U,V} = \mathrm{id}_{F(U)\otimes_{\mathcal{D}}F(V)} = \mathrm{id}_{F(U\otimes_{\mathcal{C}}V)}$ for all $U,V\in\mathcal{C}$.

Remark 2.2.11 If both tensor categories are strict, then the hexagonal diagram reduces to the square

$$\begin{array}{ccc} F(U)\otimes_{\mathcal{D}} F(V)\otimes_{\mathcal{D}} F(W) & \xrightarrow{\mathrm{id}_{F(U)}\otimes_{\mathcal{D}}\delta^F_{V,W}} & F(U)\otimes_{\mathcal{D}} F(V\otimes_{\mathcal{C}} W) \\ {\scriptstyle \delta^F_{U,V}\otimes_{\mathcal{D}}\mathrm{id}_{F(W)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \delta^F_{U,V\otimes_{\mathcal{C}}W}} \\ F(U\otimes_{\mathcal{C}} V)\otimes_{\mathcal{D}} F(W) & \xrightarrow{\delta^F_{U\otimes_{\mathcal{C}}V,W}} & F(U\otimes_{\mathcal{C}} V\otimes_{\mathcal{C}} W) \end{array}$$

and the two square diagrams involving ε^F can then be reduced to the statement that the compositions

$$F(U) = I_{\mathcal{D}}\otimes_{\mathcal{D}} F(U) \xrightarrow{\varepsilon^F\otimes_{\mathcal{D}}\mathrm{id}_{F(U)}} F(I_{\mathcal{C}})\otimes_{\mathcal{D}} F(U) \xrightarrow{\delta^F_{I_{\mathcal{C}},U}} F(I_{\mathcal{C}}\otimes_{\mathcal{C}} U) = F(U)$$

and

$$F(U) = F(U)\otimes_{\mathcal{D}} I_{\mathcal{D}} \xrightarrow{\mathrm{id}_{F(U)}\otimes_{\mathcal{D}}\varepsilon^F} F(U)\otimes_{\mathcal{D}} F(I_{\mathcal{C}}) \xrightarrow{\delta^F_{U,I_{\mathcal{C}}}} F(U\otimes_{\mathcal{C}} I_{\mathcal{C}}) = F(U)$$

are both equal to $\mathrm{id}_{F(U)}$.

Let $(\mathcal{C},\otimes_{\mathcal{C}},I_{\mathcal{C}},a_{\mathcal{C}},l_{\mathcal{C}},r_{\mathcal{C}})$, $(\mathcal{D},\otimes_{\mathcal{D}},I_{\mathcal{D}},a_{\mathcal{D}},l_{\mathcal{D}},r_{\mathcal{D}})$ and $(\mathcal{E},\otimes_{\mathcal{E}},I_{\mathcal{E}},a_{\mathcal{E}},l_{\mathcal{E}},r_{\mathcal{E}})$ be tensor categories and let $(G,\varepsilon^G,\delta^G) : \mathcal{C}\to\mathcal{D}$ and $(F,\varepsilon^F,\delta^F) : \mathcal{D}\to\mathcal{E}$ be tensor functors. Then the composition $F\circ G$ can be given the structure of a tensor functor $(F\circ G,\varepsilon^{F\circ G},\delta^{F\circ G})$ by defining

$$\varepsilon^{F\circ G} := F(\varepsilon^G)\circ\varepsilon^F \qquad\text{and}\qquad \delta^{F\circ G}_{U,V} := F(\delta^G_{U,V})\circ\delta^F_{G(U),G(V)} \tag{2.2.1}$$

for any $U, V \in \mathcal{C}$.

When we have a natural transformation from one tensor functor to another tensor functor, we demand that this natural transformation behaves nicely with respect to the tensor structure of these functors.

Definition 2.2.12 Let $(\mathcal{C}, \otimes_\mathcal{C}, I_\mathcal{C}, a_\mathcal{C}, l_\mathcal{C}, r_\mathcal{C})$ and $(\mathcal{D}, \otimes_\mathcal{D}, I_\mathcal{D}, a_\mathcal{D}, l_\mathcal{D}, r_\mathcal{D})$ be tensor categories and let

$$(F, \varepsilon^F, \delta^F), (G, \varepsilon^G, \delta^G) : \mathcal{C} \to \mathcal{D}$$

be tensor functors. Then a *natural tensor transformation*

$$\varphi : (F, \varepsilon^F, \delta^F) \to (G, \varepsilon^G, \delta^G)$$

is a natural transformation $\varphi : F \to G$ such that the following diagrams commute for each pair (U, V) of objects in $\mathcal{C}$:

$$\begin{array}{ccc} & & F(I_\mathcal{C}) \\ & \overset{\varepsilon^F}{\nearrow} & \\ I_\mathcal{D} & & \downarrow \varphi_{I_\mathcal{C}} \\ & \underset{\varepsilon^G}{\searrow} & \\ & & G(I_\mathcal{C}) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} F(U) \otimes_\mathcal{D} F(V) & \xrightarrow{\delta^F_{U,V}} & F(U \otimes_\mathcal{C} V) \\ \Big\downarrow \varphi_U \otimes_\mathcal{D} \varphi_V & & \Big\downarrow \varphi_{U \otimes_\mathcal{C} V} \\ G(U) \otimes_\mathcal{D} G(V) & \xrightarrow{\delta^G_{U,V}} & G(U \otimes_\mathcal{C} V). \end{array}$$

A *natural tensor isomorphism* is a natural tensor transformation that is also a natural isomorphism. In this case we call $(F, \varepsilon^F, \delta^F)$ and $(G, \varepsilon^G, \delta^G)$ equivalent tensor functors.

Let $(\mathcal{C}, \otimes_\mathcal{C}, I_\mathcal{C}, a_\mathcal{C}, l_\mathcal{C}, r_\mathcal{C})$ and $(\mathcal{D}, \otimes_\mathcal{D}, I_\mathcal{D}, a_\mathcal{D}, l_\mathcal{D}, r_\mathcal{D})$ be tensor categories, let

$$(F, \varepsilon^F, \delta^F), (G, \varepsilon^G, \delta^G), (H, \varepsilon^H, \delta^H) : \mathcal{C} \to \mathcal{D}$$

be tensor functors and let $\varphi : (F, \varepsilon^F, \delta^F) \to (G, \varepsilon^G, \delta^G)$ and $\psi : (G, \varepsilon^G, \delta^G) \to (H, \varepsilon^H, \delta^H)$ be natural tensor transformations. Then it is straightforward to check that the natural transformation $\psi \circ \varphi : F \to H$ is in fact a natural tensor transformation $\psi \circ \varphi : (F, \varepsilon^F, \delta^F) \to (H, \varepsilon^H, \delta^H)$. This implies that the tensor functors from $\mathcal{C}$ to $\mathcal{D}$ form a subcategory $\mathrm{Fun}^\otimes(\mathcal{C}, \mathcal{D})$ of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, where the morphism between two tensor functors are defined to be those morphisms in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ that are natural tensor transformations.

Definition 2.2.13 Let $(\mathcal{C}, \otimes_\mathcal{C}, I_\mathcal{C}, a_\mathcal{C}, l_\mathcal{C}, r_\mathcal{C})$ and $(\mathcal{D}, \otimes_\mathcal{D}, I_\mathcal{D}, a_\mathcal{D}, l_\mathcal{D}, r_\mathcal{D})$ be tensor categories and let $(F, \varepsilon^F, \delta^F) : \mathcal{C} \to \mathcal{D}$ be a tensor functor. Then $(F, \varepsilon^F, \delta^F)$ is called a *tensor equivalence* if there exists a tensor functor $(G, \varepsilon^G, \delta^G) : \mathcal{D} \to \mathcal{C}$ together with natural tensor isomorphisms $\mathrm{id}_\mathcal{D} \to F \circ G$ and $G \circ F \to \mathrm{id}_\mathcal{C}$. If there exists a tensor equivalence between $\mathcal{C}$ and $\mathcal{D}$, we will say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent tensor categories.*

The full subcategory of $\mathrm{End}^\otimes(\mathcal{C})$ determined by the tensor functors from $\mathcal{C}$ to itself that are also a tensor equivalence is denoted by $\mathrm{Aut}^\otimes(\mathcal{C})$.

To check that a given tensor functor is a tensor equivalence, the following lemma is often convenient.

Lemma 2.2.14 *A tensor functor is a tensor equivalence if and only if it is fully faithful and essentially surjective.*

Remark 2.2.15 The fundamental result in the theory of tensor categories is that any tensor category is tensor equivalent to a strict tensor category. This is one of the several equivalent ways to formulate Mac Lane's famous coherence theorem. For an explicit construction of the 'strictification' of a tensor category we refer to [48].

2.2.2 The tensor category of tensor automorphisms

Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category. Using equation (2.2.1) we can define an operation $\otimes$ on the objects of $\mathrm{End}^{\otimes}(\mathcal{C})$ by

$$(G, \varepsilon^G, \delta^G) \otimes (F, \varepsilon^F, \delta^F) := (G \circ F, \varepsilon^G \diamond \varepsilon^F, \delta^G \diamond \delta^F),$$

where we define

$$\begin{aligned} \varepsilon^G \diamond \varepsilon^F &:= G(\varepsilon^F) \circ \varepsilon^G, \\ (\delta^G \diamond \delta^F)_{U,V} &:= G(\delta^F_{U,V}) \circ \delta^G_{F(U),F(V)} \end{aligned}$$

for $U, V \in \mathcal{C}$. On the morphisms of $\mathrm{End}^{\otimes}(\mathcal{C})$ we define $\otimes$ in the same manner as in $\mathrm{End}(\mathcal{C})$. It turns out that if $\varphi, \psi \in \mathrm{Hom}(\mathrm{End}^{\otimes}(\mathcal{C}))$, then $\varphi \otimes \psi \in \mathrm{Hom}(\mathrm{End}^{\otimes}(\mathcal{C}))$, from which it follows that $\mathrm{End}^{\otimes}(\mathcal{C})$ becomes a strict tensor category with $\otimes$ as defined above and with unit object given by $(\mathrm{id}_{\mathcal{C}}, \varepsilon^0, \delta^0)$, where $\varepsilon^0 = \mathrm{id}_I$ and $\delta^0_{U,V} = \mathrm{id}_{U \otimes V}$. The category $\mathrm{Aut}^{\otimes}(\mathcal{C})$ is a full tensor subcategory of $\mathrm{End}^{\otimes}(\mathcal{C})$.

2.3 Duality in strict tensor categories

In this section we will introduce the notion of duality for tensor categories, the definition of which is based on the observation that to each vector space we can assign a dual vector space.

2.3.1 Left and right duality; two-sided duality

Definition 2.3.1 Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category and let $V \in \mathcal{C}$. If $W \in \mathcal{C}$ and there exist morphisms $b : I \to V \otimes W$ and $d : W \otimes V \to I$ satisfying

$$\begin{aligned} (\mathrm{id}_V \otimes d) \circ (b \otimes \mathrm{id}_V) &= \mathrm{id}_V, \\ (d \otimes \mathrm{id}_W) \circ (\mathrm{id}_W \otimes b) &= \mathrm{id}_W, \end{aligned}$$

then (W, b, d) is called a *left dual of* V. Similarly, if $W' \in \mathcal{C}$ and there exist morphisms $b' : I \to W' \otimes V$ and $d' : V \otimes W' \to I$ satisfying

$$\begin{aligned} (d' \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes b') &= \mathrm{id}_V, \\ (\mathrm{id}_{W'} \otimes d') \circ (b' \otimes \mathrm{id}_{W'}) &= \mathrm{id}_{W'}, \end{aligned}$$

then (W', b', d') is called a *right dual of* V. If (W, b, d, b', d') is such that (W, b, d) is a left dual of V and (W, b', d') is a right dual of V, then it is called a *two-sided dual* of V.

The following lemma shows that duals are unique up to isomorphism and that they behave nicely with respect to tensor products. The proof is a straightforward computation.

Lemma 2.3.2 *Let $\mathcal{C}$ be a tensor category.*

(1) Let $(W_1, b_1, d_1, b_1', d_1')$ be a two-sided dual for $V \in \mathcal{C}$. If $W_2 \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(W_1, W_2)$ is an isomorphism, then $(W_2, b_2, d_2, b_2', d_2')$ is also a two-sided dual of V, where

$$\begin{aligned} b_2 &= [\mathrm{id}_V \otimes f] \circ b_1 & b_2' &= [f \otimes \mathrm{id}_V] \circ b_2 \\ d_2 &= d_1 \circ [f \otimes \mathrm{id}_V] & d_2' &= d_2 \circ [\mathrm{id}_V \otimes f]. \end{aligned}$$

Conversely, if $(W_2, b_2, d_2, b_2', d_2')$ is a two-sided dual of V, then there exists an isomorphism $f \in \mathrm{Hom}_{\mathcal{C}}(W_1, W_2)$ such that $(W_2, b_2, d_2, b_2', d_2')$ is of the form above.

(2) If $(W_1, b_1, d_1, b'_1, d'_1)$ and $(W_2, b_2, d_2, b'_2, d'_2)$ are two-sided duals for $V_1, V_2 \in \mathcal{C}$, respectively, then $(W_2 \otimes W_1, b, d, b', d')$ is a two-sided dual for $V_1 \otimes V_2$, where

$$b = [\mathrm{id}_{V_1} \otimes b_2 \otimes \mathrm{id}_{W_1}] \circ b_1 \qquad b' = [\mathrm{id}_{W_2} \otimes b'_1 \otimes \mathrm{id}_{V_2}] \circ b'_2$$
$$d = d_2 \circ [\mathrm{id}_{W_2} \otimes d_1 \otimes \mathrm{id}_{V_2}] \qquad d' = d'_1 \circ [\mathrm{id}_{V_1} \otimes d'_2 \otimes \mathrm{id}_{W_1}].$$

Thus the full subcategory of $\mathcal{C}$ determined by the objects that have a two-sided dual is a tensor subcategory of $\mathcal{C}$.

Now that we have introduced duals of objects, we can define the notion of duality. This basically means that for each object in $\mathcal{C}$ we have chosen some particular dual.

Definition 2.3.3 Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category.
(1) A *left duality* $((.)^\vee, b, d)$ for $\mathcal{C}$ is an assignment $V \mapsto (V^\vee, b_V, d_V)$ on the objects of $\mathcal{C}$ such that $(V^\vee, b_V, d_V)$ is a left dual for V.
(2) A *right duality* $({}^\vee(.), b', d')$ for $\mathcal{C}$ is an assignment $V \mapsto ({}^\vee V, b'_V, d'_V)$ such that $({}^\vee V, b'_V, d'_V)$ is a right dual for V.
(2) A *two-sided duality* for $\mathcal{C}$ is an assignment $V \mapsto (\overline{V}, b_V, d_V, b'_V, d'_V)$ such that for each $V \in \mathcal{C}$ we have that $(\overline{V}, b_V, d_V)$ is a left dual for V and $(\overline{V}, b'_V, d'_V)$ is a right dual for V.

Remark 2.3.4 Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category and let $(I^\vee, b_I, d_I)$ and $({}^\vee I, b'_I, d'_I)$ be a left and right dual for I, respectively. Then we have

$$\mathrm{id}_I = [\mathrm{id}_I \otimes d_I] \circ [b_I \otimes \mathrm{id}_I] = d_I \circ b_I$$
$$\mathrm{id}_I = [d'_I \otimes \mathrm{id}_I] \circ [\mathrm{id}_I \otimes b'_I] = d'_I \circ b'_I.$$

In case we choose $I^\vee = I = {}^\vee I$ (which is always possible, for instance by choosing $b_I = d_I = b'_I = d'_I = \mathrm{id}_I$; in particular, the unit object always has a two-sided dual), the other two equations become

$$\mathrm{id}_I = \mathrm{id}_{I^\vee} = [d_I \otimes \mathrm{id}_{I^\vee}] \circ [\mathrm{id}_{I^\vee} \otimes b_I] = [d_I \otimes \mathrm{id}_I] \circ [\mathrm{id}_I \otimes b_I] = d_I \circ b_I$$
$$\mathrm{id}_I = \mathrm{id}_{{}^\vee I} = [\mathrm{id}_{{}^\vee I} \otimes d'_I] \circ [b'_I \otimes \mathrm{id}_{{}^\vee I}] = [\mathrm{id}_I \otimes d'_I] \circ [b'_I \otimes \mathrm{id}_I] = d'_I \circ b'_I,$$

which are precisely the first two equations. As a consequence, saying that (I, b_I, d_I) is a left dual for I (or that (I, b'_I, d'_I) is a right dual for I) is the same as saying that it is a retract[6] of the idempotent $p_I := b_I \circ d_I$ (respectively of the idempotent $p'_I := b'_I \circ d'_I$).

Definition 2.3.5 Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category, let $V, W \in \mathcal{C}$ and let $f \in \mathrm{Hom}_{\mathcal{C}}(V, W)$.
(1) If $\mathcal{C}$ has a left duality $((.)^\vee, b, d)$, then we define the *left transpose* $f^\vee \in \mathrm{Hom}_{\mathcal{C}}(W, V)$ of f by

$$f^\vee := [d_W \otimes \mathrm{id}_{V^\vee}] \circ [\mathrm{id}_{W^\vee} \otimes f \otimes \mathrm{id}_{V^\vee}] \circ [\mathrm{id}_{W^\vee} \otimes b_V].$$

(1) If $\mathcal{C}$ has a right duality $({}^\vee(.), b', d')$, then we define the *right transpose* ${}^\vee f \in \mathrm{Hom}_{\mathcal{C}}(W, V)$ of f by

$${}^\vee f := [\mathrm{id}_{{}^\vee V} \otimes d'_W] \circ [\mathrm{id}_{{}^\vee V} \otimes f \otimes \mathrm{id}_{{}^\vee W}] \circ [b'_V \otimes \mathrm{id}_{{}^\vee W}].$$

Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category and let $(\overline{(.)}, b, d, b', d')$ be a two-sided duality for $\mathcal{C}$. If $V \in \mathcal{C}$ and $f \in \mathrm{End}_{\mathcal{C}}(V)$, then we define the *left trace* $\mathrm{Tr}_L(f) \in \mathrm{End}_{\mathcal{C}}(I)$ and *right trace* $\mathrm{Tr}_R(f) \in \mathrm{End}_{\mathcal{C}}(I)$ of f by

$$\mathrm{Tr}_L(f) = d'_V \circ [f \otimes \mathrm{id}_{\overline{V}}] \circ b_V$$
$$\mathrm{Tr}_R(f) = d_V \circ [\mathrm{id}_{\overline{V}} \otimes f] \circ b'_V.$$

If $V \in \mathcal{C}$, we define the *left dimension* of V by $d_L(V) := \mathrm{Tr}_L(\mathrm{id}_V)$ and we define the *right dimension* of V by $d_R(V) := \mathrm{Tr}_R(V)$. We emphasize that these left/right traces and dimensions depend on the particular choice of the two-sided duality.

[6] Let $\mathcal{C}$ be a category and let $V \in \mathcal{C}$. If there exists a $U \in \mathcal{C}$ together with morphisms $i : U \to V$ and $r : V \to U$ satisfying $r \circ i = \mathrm{id}_U$, then the triple (U, i, r) is called a *retract* of V. We will come back to retracts at the beginning of Subsection 2.7.4.

2.3.2 Pivotal and spherical categories

For some of our purposes it will not be enough that a tensor category simply has a two-sided duality and we have to require some additional properties of the two-sided duality. The following definition can be found in [74].

Definition 2.3.6 Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category. Then this category is called *pivotal* if there exists an assignment $V \mapsto \overline{V}$ on the objects of $\mathcal{C}$ satisfying

$$\overline{\overline{V}} = V, \qquad \overline{V \otimes W} = \overline{W} \otimes \overline{V}, \qquad \overline{I} = I$$

together with morphisms $\varepsilon_V : I \to V \otimes \overline{V}$ and $\overline{\varepsilon}_V : V \otimes \overline{V} \to I$ for each $V \in \mathcal{C}$ such that the following three conditions are satisfied:

(1) For each $V \in \mathcal{C}$ the equations

$$[\overline{\varepsilon}_V \otimes \mathrm{id}_V] \circ [\mathrm{id}_V \otimes \varepsilon_{\overline{V}}] = \mathrm{id}_V \qquad \text{and} \qquad [\mathrm{id}_V \otimes \overline{\varepsilon}_{\overline{V}}] \circ [\varepsilon_V \otimes \mathrm{id}_V] = \mathrm{id}_V$$

hold.

(2) For all $V, W \in \mathcal{C}$ we have

$$\varepsilon_{V \otimes W} = [\mathrm{id}_V \otimes \varepsilon_W \otimes \mathrm{id}_{\overline{V}}] \circ \varepsilon_V \qquad \text{and} \qquad \overline{\varepsilon}_{V \otimes W} = \overline{\varepsilon}_V \circ [\mathrm{id}_V \otimes \overline{\varepsilon}_W \otimes \mathrm{id}_{\overline{V}}].$$

(3) If $V, W \in \mathcal{C}$, then for every $f : V \to W$ we have the equality

$$[\overline{\varepsilon}_{\overline{W}} \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_{\overline{W}} \otimes f \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_{\overline{W}} \otimes \varepsilon_V] = [\mathrm{id}_{\overline{V}} \otimes \overline{\varepsilon}_W] \circ [\mathrm{id}_{\overline{V}} \otimes f \otimes \mathrm{id}_{\overline{W}}] \circ [\varepsilon_{\overline{V}} \otimes \mathrm{id}_{\overline{W}}]$$

of morphisms $\overline{W} \to \overline{V}$.

Remark 2.3.7 For future reference, we will mention the relation with two-sided duality explicitly.

(1) If we replace V by $\overline{V}$ in the two equations in part (1) of the definition, we get a total of four equations, which (together with the fact that $\overline{\overline{V}} = V$) state that $(\overline{V}, b_V, d_V, b'_V, d'_V)$ is a two-sided dual for V, where

$$b_V := \varepsilon_V, \qquad d_V := \overline{\varepsilon}_{\overline{V}}, \qquad b'_V := \varepsilon_{\overline{V}} \qquad d'_V := \overline{\varepsilon}_V.$$

Note that $b'_V = \varepsilon_{\overline{V}} = b_{\overline{V}}$ and $d'_V = \overline{\varepsilon}_V = \overline{\varepsilon}_{\overline{\overline{V}}} = d_{\overline{V}}$. Similarly, we also get $b_V = b'_{\overline{V}}$ and $d_V = d'_{\overline{V}}$. So the right duality can be expressed in terms of the left duality, and vice versa.

(2) The equations in part (2) of the definition (together with the fact that $\overline{V \otimes W} = \overline{W} \otimes \overline{V}$) state that this two-sided duality is very well-behaved under tensor products: the morphisms $\varepsilon_{V \otimes W}$ and $\overline{\varepsilon}_{V \otimes W}$ are completely determined by the morphisms ε_V and ε_W, respectively, $\overline{\varepsilon}_V$ and $\overline{\varepsilon}_W$.

(3) The equations in part (3) of the definition state that the left transpose $f^\vee$ and right transpose ${}^\vee f$ of any morphism f coincide. In what follows, we will denote it by $\overline{f}$.

Lemma 2.3.8 *If $\mathcal{C}$ is a pivotal category, then $\varepsilon_I = \overline{\varepsilon}_I = \mathrm{id}_I$.*

Proof. For any $V \in \mathcal{C}$ we have (using the b and d notation, rather than the ε and $\overline{\varepsilon}$ notation)

$$\begin{aligned}
\mathrm{id}_V &= [\mathrm{id}_V \otimes d_V] \circ [b_V \otimes \mathrm{id}_V] = [\mathrm{id}_V \otimes d_V] \circ [b_{V \otimes I} \otimes \mathrm{id}_V] \\
&= [\mathrm{id}_{V \otimes I \otimes I} \otimes d_V] \circ [\mathrm{id}_V \otimes b_I \otimes \mathrm{id}_{\overline{V} \otimes V}] \circ [b_V \otimes \mathrm{id}_V] \\
&= [\mathrm{id}_V \otimes b_I] \circ [\mathrm{id}_V \otimes d_V] \circ [b_V \otimes \mathrm{id}_V] = \mathrm{id}_V \otimes b_I.
\end{aligned}$$

Taking $V = I$, we get $\mathrm{id}_I = b_I = \varepsilon_I$. Using Remark 2.3.4, this in turn gives us $\mathrm{id}_I = d_I \circ b_I = d_I = \overline{\varepsilon}_{\overline{I}} = \overline{\varepsilon}_I$. □

Since every pivotal category has a two-sided duality, we always have left and right traces on a pivotal category. An interesting class of tensor categories is obtained by demanding that these left and right traces coincide.

Definition 2.3.9 A pivotal category for which $\mathrm{Tr}_L = \mathrm{Tr}_R$ is called a *spherical category.*

In a spherical category we can thus simply write Tr to denote the trace, and if $V \in \mathcal{C}$ we define the *dimension* of V by $d(V) := \mathrm{Tr}(V)$. Note that $d(V) = d_L(V) = d_R(V)$ in this case.

2.4 Braided tensor categories

If $\mathcal{C}$ and $\mathcal{D}$ are categories, we will write $\tau_{\mathcal{C},\mathcal{D}} : \mathcal{C}\times\mathcal{D} \to \mathcal{D}\times\mathcal{C}$ to denote the *flip functor*, i.e. the functor defined by $\tau_{\mathcal{C},\mathcal{D}}((V,W)) := (W,V)$ and $\tau_{\mathcal{C},\mathcal{D}}((f,g)) := (g,f)$ for $V \in \mathcal{C}$, $W \in \mathcal{D}$, $f \in \mathrm{Hom}(\mathcal{C})$ and $g \in \mathrm{Hom}(\mathcal{D})$.

Definition 2.4.1 Let $\mathcal{C}$ be a category with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

(1) A *commutativity constraint* c is a natural isomorphism $c : \otimes \to \otimes \circ \tau_{\mathcal{C},\mathcal{C}}$, i.e. a family $\{c_{V,W} : V \otimes W \to W \otimes V\}_{V,W\in\mathcal{C}}$ of isomorphisms in $\mathcal{C}$ such that the square

$$\begin{array}{ccc} V \otimes W & \xrightarrow{c_{V,W}} & W \otimes V \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ V' \otimes W' & \xrightarrow{c_{V',W'}} & W' \otimes V' \end{array}$$

commutes for all $f : V \to V'$ and $g : W \to W'$.

(1) If $\mathcal{C}$ has an associativity constraint a and if c is a commutativity constraint, then c is said to satisfy the *hexagon axiom* if the two hexagonal diagrams

$$\begin{array}{ccccc} & & (U \otimes V) \otimes W & & \\ & {\scriptstyle c_{U,V}\otimes \mathrm{id}_W}\swarrow & & \searrow{\scriptstyle a_{U,V,W}} & \\ (V \otimes U) \otimes W & & & & U \otimes (V \otimes W) \\ {\scriptstyle a_{V,U,W}}\downarrow & & & & \downarrow{\scriptstyle c_{U,V\otimes W}} \\ V \otimes (U \otimes W) & & & & (V \otimes W) \otimes U \\ & {\scriptstyle \mathrm{id}_V\otimes c_{U,W}}\searrow & & \swarrow{\scriptstyle a_{V,W,U}} & \\ & & V \otimes (W \otimes U) & & \end{array}$$

and

$$\begin{array}{ccccc} & & U \otimes (V \otimes W) & & \\ & {\scriptstyle \mathrm{id}_U\otimes c_{V,W}}\swarrow & & \searrow{\scriptstyle a^{-1}_{U,V,W}} & \\ U \otimes (W \otimes V) & & & & (U \otimes V) \otimes W \\ {\scriptstyle a^{-1}_{U,W,V}}\downarrow & & & & \downarrow{\scriptstyle c_{U\otimes V,W}} \\ (U \otimes W) \otimes V & & & & W \otimes (U \otimes V) \\ & {\scriptstyle c_{U,W}\otimes \mathrm{id}_V}\searrow & & \swarrow{\scriptstyle a^{-1}_{W,U,V}} & \\ & & (W \otimes U) \otimes V & & \end{array}$$

commute for all $U, V, W \in \mathcal{C}$.

Remark 2.4.2 Note that these hexagons can be reformulated as

$$c_{U,V\otimes W} = a^{-1}_{V,W,U} \circ [\mathrm{id}_V \otimes c_{U,W}] \circ a_{V,U,W} \circ [c_{U,V} \otimes \mathrm{id}_W] \circ a^{-1}_{U,V,W},$$

$$c_{U\otimes V,W} = a_{W,U,V} \circ [c_{U,W} \otimes \mathrm{id}_V] \circ a^{-1}_{U,W,V} \circ [\mathrm{id}_U \otimes c_{V,W}] \circ a_{U,V,W}.$$

These equations can be convenient in some situations.

Definition 2.4.3 Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category.
(1) A *braiding* in $\mathcal{C}$ is a commutativity constraint satisfying the hexagon axiom.
(2) A *braided tensor category* $(\mathcal{C}, \otimes, I, a, l, r, c)$ is a tensor category with a choice of braiding.

Remark 2.4.4 If $\mathcal{C}$ is a strict tensor category, the hexagonal diagrams become the triangles

$$\begin{array}{ccccc} & & U\otimes V\otimes W & & \\ & \swarrow^{c_{U,V}\otimes \mathrm{id}_W} & & \searrow^{c_{U,V\otimes W}} & \\ V\otimes U\otimes W & & \xrightarrow{\mathrm{id}_V\otimes c_{U,W}} & & V\otimes W\otimes U \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & U\otimes V\otimes W & & \\ & \swarrow^{\mathrm{id}_U\otimes c_{V,W}} & & \searrow^{c_{U\otimes V,W}} & \\ U\otimes W\otimes V & & \xrightarrow{c_{U,W}\otimes \mathrm{id}_V} & & W\otimes U\otimes V \end{array}$$

i.e.

$$\begin{aligned} c_{U,V\otimes W} &= [\mathrm{id}_V \otimes c_{U,W}] \circ [c_{U,V} \otimes \mathrm{id}_W], \\ c_{U\otimes V,W} &= [c_{U,W} \otimes \mathrm{id}_V] \circ [\mathrm{id}_U \otimes c_{V,W}]. \end{aligned}$$

These equations will be used very often in what follows.

If c is a braiding in $\mathcal{C}$, then so is $\widetilde{c}$, where we define

$$\widetilde{c}_{V,W} := c^{-1}_{W,V} \tag{2.4.1}$$

for $V, W \in \mathcal{C}$. If $\mathcal{C}$ is a braided tensor category with braiding c, then we will write $\widetilde{\mathcal{C}}$ to denote the tensor category $\mathcal{C}$ with braiding $\widetilde{c}$. The notations $\widetilde{c}$ and $\widetilde{\mathcal{C}}$ will be used very often in later chapters without explanation, so it is important to remember them.

If $\mathcal{C}$ is a braided tensor category with braiding c and if $V, W \in \mathcal{C}$, then we write

$$c^M_{V,W} := c_{W,V} \circ c_{V,W} \tag{2.4.2}$$

and call this the *monodromy* of V and W. An object $V \in \mathcal{C}$ is called *degenerate* if $c^M_{V,W} = \mathrm{id}_{V\otimes W}$ for all $W \in \mathcal{C}$. A braided tensor category is called a *symmetric tensor category* if all its objects are degenerate and in this case the braiding is also called a *symmetry*.

Definition 2.4.5 Let $(\mathcal{C}, \otimes, I, a, l, r, c)$ and $(\mathcal{C}', \otimes', I', a', l', r', c')$ be braided tensor categories and let $(F, \varepsilon^F, \delta^F) : \mathcal{C} \to \mathcal{C}'$ be a tensor functor. Then $(F, \varepsilon^F, \delta^F)$ is called a *braided tensor functor* if the square

$$\begin{array}{ccc} F(V)\otimes' F(W) & \xrightarrow{c'_{F(V),F(W)}} & F(W)\otimes' F(V) \\ \downarrow{\scriptstyle \delta^F_{V,W}} & & \downarrow{\scriptstyle \delta^F_{W,V}} \\ F(V\otimes W) & \xrightarrow{F(c_{V,W})} & F(W\otimes V) \end{array}$$

commutes for all $V, W \in \mathcal{C}$.

2.4.1 Ribbon categories

Suppose that $(\mathcal{C}, \otimes, I, c)$ is a braided strict tensor category. If $\mathcal{C}$ has a left duality $((.)^\vee, b, d)$, then we can easily obtain a right duality $({}^\vee(.), b', d')$ on $\mathcal{C}$ by defining ${}^\vee V := V^\vee$ and

$$b'_V := c_{V,V^\vee} \circ b_V \tag{2.4.3}$$
$$d'_V := d_V \circ c^{-1}_{V^\vee,V}. \tag{2.4.4}$$

Similarly, if $\mathcal{C}$ has a right duality $({}^\vee(.), b', d')$, then we obtain a left duality $((.)^\vee, b, d)$ by defining $V^\vee := {}^\vee V$ and

$$b_V := c_{{}^\vee V,V} \circ b'_V \tag{2.4.5}$$
$$d_V := d'_V \circ c^{-1}_{V,{}^\vee V}. \tag{2.4.6}$$

Thus a braided tensor category with a one-sided duality can always be equipped with a two-sided duality. However, it is not true that this two-sided duality automatically leads to a spherical structure. In order to assure that we obtain a spherical structure, we need to introduce the notion of a twist in a braided tensor category with a one-sided duality. We will choose this one-sided duality to be a left duality, since this is the convention that is found most often in the literature.

Definition 2.4.6 Let $(\mathcal{C}, \otimes, I, c)$ be a braided strict tensor category with left duality $((.)^\vee, b, d)$. A *twist* is a natural isomorphism $\theta : \mathrm{id}_\mathcal{C} \to \mathrm{id}_\mathcal{C}$, i.e. a family $\{\theta_V : V \to V\}$ of isomorphisms in $\mathcal{C}$ such that for any $V, W \in \mathcal{C}$ and $f \in \mathrm{Hom}_\mathcal{C}(V, W)$ we have $\theta_W \circ f = f \circ \theta_V$, satisfying the additional properties that

$$\theta_{V\otimes W} = [\theta_V \otimes \theta_W] \circ c^M_{V,W}$$
$$\theta_{V^\vee} = (\theta_V)^\vee$$

for all $V, W \in \mathcal{C}$. A braided strict tensor category equipped with a left duality and a twist is called a *ribbon category*.

If $\mathcal{C}$ is a ribbon category, then we can obtain a right duality $({}^\vee(.), b', d')$ by defining ${}^\vee V := V^\vee$, $b'_V := [\mathrm{id}_{V^\vee} \otimes \theta_V] \circ c_{V,V^\vee} \circ b_V$ and $d'_V := d_V \circ c_{V,V^\vee} \circ [\theta_V \otimes \mathrm{id}_{V^\vee}]$. Because ${}^\vee V = V^\vee$ we thus obtain a two-sided duality $(\overline{(.)}, b, d, b', d')$ on $\mathcal{C}$ and it can be shown that $\mathcal{C}$ becomes spherical with respect to this two-sided duality.[7] Furthermore, the twist can now be expressed in terms of the two-sided duality as

$$\theta_V = [d_V \otimes \mathrm{id}_V] \circ [\mathrm{id}_{\overline{V}} \otimes c_{V,V}] \circ [b'_V \otimes \mathrm{id}_V]. \tag{2.4.7}$$

Later we will encounter a particular situation where we are given a two-sided duality on a braided tensor category $\mathcal{C}$ and where equation (2.4.7) can be used to define a twist. This will then equip $\mathcal{C}$ with the structure of a ribbon category (and using the left duality, braiding and twist to define a right duality as above, we obtain the original right duality again).

2.4.2 The (relative) Drinfeld center

In this subsection we will introduce the (relative) Drinfeld center of a tensor category. This will be very important to us, because we will later extend the construction of the Drinfeld center to tensor categories with a group action and we will study this construction in great detail. The definition of the (relative) Drinfeld center makes use of the notion of a half braiding (relative to a tensor subcategory).

[7] However, the underlying pivotal structure is not as strict as defined in Subsection 2.3.2, but this will be of no importance to us. We only introduce the notion of ribbon categories as a motivation for the formula (2.4.7), which will be used later in another setting.

Definition 2.4.7 Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category, let $\mathcal{D} \subset \mathcal{C}$ be a tensor subcategory with inclusion functor $\mathscr{I} : \mathcal{D} \to \mathcal{C}$ and let $V \in \mathcal{C}$. A *half braiding for V relative to $\mathcal{D}$* is a natural isomorphism $\Phi_V : \otimes \circ [V \times \mathscr{I}] \to \otimes \circ [\mathscr{I} \times V]$ of functors $\mathcal{D} \to \mathcal{C}$, i.e. a family $\{\Phi_V(X) : V \otimes \mathscr{I}(X) \to \mathscr{I}(X) \otimes V\}_{X \in \mathcal{D}}$ such that the square

$$\begin{array}{ccc} V \otimes \mathscr{I}(X) & \xrightarrow{\Phi_V(X)} & \mathscr{I}(X) \otimes V \\ {\scriptstyle \mathrm{id}_V \otimes \mathscr{I}(f)} \downarrow & & \downarrow {\scriptstyle \mathscr{I}(f) \otimes \mathrm{id}_V} \\ V \otimes \mathscr{I}(Y) & \xrightarrow{\Phi_V(Y)} & \mathscr{I}(Y) \otimes V \end{array}$$

commutes for all $X, Y \in \mathcal{D}$ and $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$, satisfying the additional property that for any $X, Y \in \mathcal{D}$ we have

$$\Phi_V(X \otimes Y) = [\mathrm{id}_{\mathscr{I}(X)} \otimes \Phi_V(Y)] \circ [\Phi_V(X) \otimes \mathrm{id}_{\mathscr{I}(Y)}].$$

In case $\mathcal{D} = \mathcal{C}$, Φ_V is called a *half braiding for V*.

Let $\mathcal{C}$ be a braided tensor category with braiding c, let $\mathcal{D} \subset \mathcal{C}$ be a tensor subcategory and let $V \in \mathcal{C}$. If for each $X \in \mathcal{D}$ we define $\Phi_V(X) := c_{V,X}$, then Φ_V is a half braiding for V relative to $\mathcal{D}$. Thus, half braidings can be obtained by fixing the first argument of a braiding in a braided tensor category. Fixing the second argument does not give a half braiding according to the definition above. This observation motivates the following alternative definition of a half braiding, which will be needed in Section 4.4.

Definition 2.4.8 Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category, let $\mathcal{D} \subset \mathcal{C}$ be a tensor subcategory with inclusion functor $\mathscr{I} : \mathcal{D} \to \mathcal{C}$ and let $V \in \mathcal{C}$. A *half braiding of the second kind for V relative to $\mathcal{D}$* is a natural isomorphism $\Psi_V : \otimes \circ [\mathscr{I} \times V] \to \otimes \circ [V \times \mathscr{I}]$ of functors $\mathcal{D} \to \mathcal{C}$, i.e. a family $\{\Psi_V(X) : \mathscr{I}(X) \otimes V \to V \otimes \mathscr{I}(X)\}_{X \in \mathcal{D}}$ such that the square

$$\begin{array}{ccc} \mathscr{I}(X) \otimes V & \xrightarrow{\Psi_V(X)} & V \otimes \mathscr{I}(X) \\ {\scriptstyle \mathscr{I}(f) \otimes \mathrm{id}_V} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_V \otimes \mathscr{I}(f)} \\ \mathscr{I}(Y) \otimes V & \xrightarrow{\Psi_V(Y)} & V \otimes \mathscr{I}(Y) \end{array}$$

commutes for all $X, Y \in \mathcal{D}$ and $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$, satisfying the additional property that for any $X, Y \in \mathcal{D}$ we have

$$\Psi_V(X \otimes Y) = [\Psi_V(X) \otimes \mathrm{id}_{\mathscr{I}(Y)}] \circ [\mathrm{id}_{\mathscr{I}(X)} \otimes \Psi_V(Y)].$$

In case $\mathcal{D} = \mathcal{C}$, Ψ_V is called a *half braiding of the second kind for V*.

With the terminology in this definition we already anticipate for the terminology that will be introduced in a more general setting in Chapter 4. Note that fixing the second argument of the braiding in a braided tensor category will indeed give rise to a half braiding of the second kind.

If $\mathcal{C}$ is a tensor category and $\mathcal{D} \subset \mathcal{C}$ is a tensor subcategory, then we define a category $Z(\mathcal{C}; \mathcal{D})$ as follows. Its objects are given by

$$\mathrm{Obj}(Z(\mathcal{C}; \mathcal{D})) := \{(V, \Phi_V) : V \in \mathcal{C} \text{ and } \Phi_V \text{ is a half braiding for } V \text{ relative to } \mathcal{D}\}$$

and for $(V, \Phi_V), (W, \Phi_W) \in \mathrm{Obj}(Z(\mathcal{C}; \mathcal{D}))$ we define $\mathrm{Hom}_{Z(\mathcal{C};\mathcal{D})}((V, \Phi_V), (W, \Phi_W))$ to be

$$\{f \in \mathrm{Hom}_{\mathcal{C}}(V, W) : [\mathrm{id}_{\mathscr{I}(X)} \otimes f] \circ \Phi_V(X) = \Phi_W(X) \circ [f \otimes \mathrm{id}_{\mathscr{I}(X)}] \ \forall X \in \mathcal{D}\}.$$

The composition of morphisms in $Z(\mathcal{C};\mathcal{D})$ is defined to be the same as in $\mathcal{C}$. The category $Z(\mathcal{C};\mathcal{D})$ can be equipped with the structure of a strict tensor category by defining the tensor product on objects by

$$(V,\Phi_V)\otimes(W,\Phi_W) := (V\otimes W, \Phi_V\otimes\Phi_W),$$

where

$$(\Phi_V\otimes\Phi_W)(X) := [\Phi_V(X)\otimes \mathrm{id}_W]\circ[\mathrm{id}_V\otimes\Phi_W(X)]$$

and by letting the tensor product on morphisms be the one inherited from $\mathcal{C}$. The unit object of $Z(\mathcal{C};\mathcal{D})$ is (I,Φ_I^0), where $\Phi_I^0(X) = \mathrm{id}_{\mathscr{I}(X)}$ for all $X\in\mathcal{D}$. The tensor category $Z(\mathcal{C};\mathcal{D})$ is called the *Drinfeld center of* $\mathcal{C}$ *relative to* $\mathcal{D}$. If we write[8] $Z(\mathcal{D})$ to denote the full tensor subcategory of $Z(\mathcal{C};\mathcal{D})$ determined by the objects of the form (U,Φ_U) with $U\in\mathcal{D}$, then for any $(V,\Phi_V)\in Z(\mathcal{C};\mathcal{D})$ and $(W,\Phi_W)\in Z(\mathcal{D})$ we can define

$$C_{(V,\Phi_V),(W,\Phi_W)} := \Phi_V(W). \tag{2.4.8}$$

The restriction of this C to $Z(\mathcal{D})$ defines a braiding on $Z(\mathcal{D})$. In particular, if we choose $\mathcal{D}$ to be equal to $\mathcal{C}$ then we obtain a braided tensor category $Z(\mathcal{C}) := Z(\mathcal{C};\mathcal{C})$ which is called the *Drinfeld center of* $\mathcal{C}$.

Although the restriction of C in (2.4.8) to objects in $Z(\mathcal{D})$ gives us a braiding on $Z(\mathcal{D})$, it is clear that C does not give us a braiding on $Z(\mathcal{C};\mathcal{D})$ because we cannot take the second argument of C to be an object in $Z(\mathcal{C};\mathcal{D})$. To describe what C is on $Z(\mathcal{C};\mathcal{D})$ we introduce the notion of a partial braiding relative to a tensor subcategory.

Definition 2.4.9 Let $(\mathcal{C},\otimes,I)$ be a strict tensor category, let $\mathcal{D}$ be a tensor subcategory of $\mathcal{C}$ and denote the inclusion functor by $\mathscr{I}:\mathcal{D}\to\mathcal{C}$. A *partial braiding on* $\mathcal{C}$ *relative to* $\mathcal{D}$ is a natural isomorphism $c:\otimes\circ[\mathrm{id}_\mathcal{C}\times\mathscr{I}]\to\otimes\circ[\mathscr{I}\times\mathrm{id}_\mathcal{C}]\circ\tau_{\mathcal{C},\mathcal{D}}$ of functors $\mathcal{C}\times\mathcal{D}\to\mathcal{C}$, i.e. a family $\{c_{V,X}: V\otimes\mathscr{I}(X)\to\mathscr{I}(X)\otimes V\}_{V\in\mathcal{C},X\in\mathcal{D}}$ of isomorphisms in $\mathcal{C}$ such that for any $f\in\mathrm{Hom}_\mathcal{C}(V,V')$ and $g\in\mathrm{Hom}_\mathcal{D}(X,X')$ we have

$$c_{V',X'}\circ[f\otimes\mathscr{I}(g)] = [\mathscr{I}(g)\otimes f]\circ c_{V,X},$$

satisfying the additional property that

$$\begin{aligned} c_{V,X\otimes Y} &= [\mathrm{id}_{\mathscr{I}(X)}\otimes c_{V,Y}]\circ[c_{V,X}\otimes\mathrm{id}_{\mathscr{I}(Y)}]\\ c_{V\otimes W,X} &= [c_{V,X}\otimes\mathrm{id}_W]\circ[\mathrm{id}_V\otimes c_{W,X}]\end{aligned}$$

for all $V,W\in\mathcal{C}$ and $X,Y\in\mathcal{D}$. If $\mathcal{C}$ has a partial braiding relative to $\mathcal{D}$, then $\mathcal{C}$ will be called *partially braided relative to* $\mathcal{D}$.

Remark 2.4.10 Similarly, by using half braidings of the second kind relative to $\mathcal{D}$ in the construction of the relative Drinfeld center above, we obtain an alternative relative Drinfeld center that we will denote by $Z^{(2)}(\mathcal{C};\mathcal{D})$. The tensor product on $Z^{(2)}(\mathcal{C};\mathcal{D})$ is given by

$$(V,\Psi_V)\otimes(W,\Psi_W) = (V\otimes W,\Psi_V\otimes\Psi_W),$$

where

$$(\Psi_V\otimes\Psi_W)(X) = [\mathrm{id}_V\otimes\Psi_W(X)]\circ[\Psi_V(X)\otimes\mathrm{id}_W].$$

It has a partial braiding of the second kind relative to $\mathcal{D}$, the definition of which should be obvious now. If $\mathcal{D}=\mathcal{C}$, then we will write $Z^{(2)}(\mathcal{C})$.

2.5 Algebra in a strict tensor category

In this section we will give a categorical definition of algebras and coalgebras in the context of strict tensor categories. When one generalizes these definitions to non-strict tensor categories and applies them to the category of vector spaces, the usual definition of algebras and coalgebras is obtained. We will also consider modules and comodules in this categorical setting.

[8]The reason for this choice of notation will become clear in a moment, when we define the Drinfeld center $Z(\mathcal{C})$.

2.5.1 Algebras and coalgebras

The categorical version of the definitions of an algebra and a coalgebra are as follows.

Definition 2.5.1 Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category.

(1) An *algebra* A in $\mathcal{C}$ is a triple $\mathsf{A} = (A, \mu, \eta)$ consisting of an object $A \in \mathcal{C}$ and morphisms $\mu : A \otimes A \to A$ and $\eta : I \to A$ such that

$$\mu \circ [\mathrm{id}_A \otimes \mu] = \mu \circ [\mu \otimes \mathrm{id}_A] \qquad \text{and} \qquad \mu \circ [\eta \otimes \mathrm{id}_A] = \mathrm{id}_A = \mu \circ [\mathrm{id}_A \otimes \eta].$$

If $\mathsf{A} = (A, \mu, \eta)$ and $\mathsf{A}' = (A', \mu', \eta')$ are algebras in $\mathcal{C}$, then a morphism $f : A \to A'$ is called a *morphism of algebras* if $\mu' \circ [f \otimes f] = f \circ \mu$ and $f \circ \eta = \eta'$.

(2) A *coalgebra* C in $\mathcal{C}$ is a triple $\mathsf{C} = (C, \Delta, \varepsilon)$ consisting of an object $C \in \mathcal{C}$ and morphisms $\Delta : C \to C \otimes C$ and $\varepsilon : C \to I$ such that

$$[\mathrm{id}_C \otimes \Delta] \circ \Delta = [\Delta \otimes \mathrm{id}_C] \circ \Delta \qquad \text{and} \qquad [\varepsilon \otimes \mathrm{id}_C] \circ \Delta = \mathrm{id}_C = [\mathrm{id}_C \otimes \varepsilon] \circ \Delta.$$

If $\mathsf{C} = (C, \Delta, \varepsilon)$ and $\mathsf{C}' = (C', \Delta', \varepsilon')$ are coalgebras in $\mathcal{C}$, then a morphism $f : C \to C'$ is called a *morphism of coalgebras* if $[f \otimes f] \circ \Delta = \Delta' \circ f$ and $\varepsilon' \circ f = \varepsilon$.

Remark 2.5.2 More generally, one can define algebras and coalgebras in non-strict tensor categories by inserting the associativity constraint and the unit constraints in the appropriate places in the equations of the definition above.

As a trivial example, the unit object obtains the structure of an algebra (I, μ, η) if we define $\mu = \eta = \mathrm{id}_I$ and it obtains the structure of a coalgebra (I, Δ, ε) if we define $\Delta = \varepsilon = \mathrm{id}_I$.

If $\mathsf{A} = (A, \mu, \eta)$ is an algebra in a braided strict tensor category $\mathcal{C}$, then its *opposite algebra* is the algebra $\mathsf{A}^{\mathrm{op}} = (A, \mu^{\mathrm{op}}, \eta)$, where $\mu^{\mathrm{op}} = \mu \circ c_{A,A}$. We say that A is *commutative* if $\mathsf{A}^{\mathrm{op}} = \mathsf{A}$. Similarly, one also defines C^{op} for a coalgebra C in a braided tensor category by setting $\Delta^{\mathrm{op}} = c_{C,C} \circ \Delta$. Then C will be called *cocommutative* if $\mathsf{C}^{\mathrm{op}} = \mathsf{C}$. Note that a (co)algebra is (co)commutative with respect to the braiding c if and only if it is with respect to the braiding $\widetilde{c}$, where $\widetilde{c}$ is as in (2.4.1).

2.5.2 Modules over an algebra

Now that we have introduced a categorical version of the definition of (co)algebras, we will do the same with the definition of a module over an algebra.

Definition 2.5.3 Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category, let $\mathsf{A} = (A, \mu, \eta)$ be an algebra in $\mathcal{C}$ and let $\mathsf{C} = (C, \Delta, \varepsilon)$ be a coalgebra in $\mathcal{C}$. A left A-*module* in $\mathcal{C}$ is a pair (V, π_V), where $V \in \mathcal{C}$ and $\pi_V : A \otimes V \to V$ is a morphism satisfying

$$\pi_V \circ [\mu \otimes \mathrm{id}_V] = \pi_V \circ [\mathrm{id}_A \otimes \pi_V] \qquad \text{and} \qquad \pi_V \circ [\eta \otimes \mathrm{id}_V] = \mathrm{id}_V.$$

If (V, π_V) and $(V', \pi_{V'})$ are left A-modules, then a morphism $f : V \to V'$ is called a *morphism of left* A-*modules* if it satisfies $\pi_{V'} \circ [\mathrm{id}_A \otimes f] = f \circ \pi_V$.

The notion of a right module is obtained by making the obvious adjustments to the previous definition.

Example 2.5.4 An easy example of a module over an algebra is given by the algebra itself. Namely, if $\mathsf{A} = (A, \mu, \eta)$ is an algebra in a tensor category $\mathcal{C}$, then (A, μ) is a both a left and right A-module in $\mathcal{C}$.

The following proposition states that the modules over an algebra in a strict tensor category form a category. The proof is easy.

Proposition 2.5.5 *Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category and let $\mathsf{A} = (A, \mu, \eta)$ be an algebra in $\mathcal{C}$. Then the left A-modules form a category $\mathrm{Mod}_{\mathcal{C}}(\mathsf{A})$, where the objects are left A-modules and the morphisms are morphisms of left A-modules in the sense of the definition above, with the composition in $\mathrm{Mod}_{\mathcal{C}}(\mathsf{A})$ given by the composition in $\mathcal{C}$. Similarly, the right A-modules form a category $\mathrm{Mod}^R_{\mathcal{C}}(A)$.*

2.6 Module categories

Analogously to monoids acting on sets, a tensor category can act on a category. Categories that are acted upon by a tensor category are called module categories and will be important to us in Section 4.5.

2.6.1 Left and right module categories; bimodule categories

Let X be a set and let G be a group. Recall from elementary algebra that a (left) G-action on X is a homomorphism $\psi : G \to S(X)$, where $S(X)$ denotes the group of bijections of X. In the following definition, the action of a tensor category on a category is defined analogously.

Definition 2.6.1 Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \otimes, I)$ be two strict tensor categories.

(1) A *left $\mathcal{C}$-module category* is a pair $(\mathcal{M}, (F, \varepsilon^F, \delta^F))$, where $\mathcal{M}$ is a category and $(F, \varepsilon^F, \delta^F) : \mathcal{C} \to \mathrm{End}(\mathcal{M})$ is a tensor functor.

(2) A *right $\mathcal{D}$-module category* is a pair $(\mathcal{M}, (G, \varepsilon^G, \delta^G))$, where $\mathcal{M}$ is a category and $(G, \varepsilon^G, \delta^G) : \mathcal{D}^{\mathrm{rev}} \to \mathrm{End}(\mathcal{M})$ is a tensor functor.

(3) A *$(\mathcal{C}, \mathcal{D})$-bimodule category* is a pair $(\mathcal{M}, (H, \varepsilon^H, \delta^H))$, where $\mathcal{M}$ is a category and $(H, \varepsilon^H, \delta^H) : \mathcal{C} \times \mathcal{D}^{\mathrm{rev}} \to \mathrm{End}(\mathcal{M})$ is a tensor functor.

In part (3) the category $\mathcal{C} \times \mathcal{D}^{\mathrm{rev}}$ is equipped with the structure of a strict tensor category by defining the tensor product componentwise and by choosing $(I_{\mathcal{C}}, I_{\mathcal{D}})$ as unit object.

Consider again the case of a group G acting (from the left) on a set X. Such an action can also be defined as a map $G \times X \to X$, denoted by $(q, x) \mapsto q.x$, that satisfies $e.x = x$ and $(qr).x = q.(r.x)$ for all $q, r \in G$ and $x \in X$. The equivalence of the two definitions can be seen immediately by considering the correspondence $q.x = \psi(q)(x)$ for all $q \in G$ and $x \in X$. Analogously to this alternative definition of a group acting on a set, there is also an alternative definition of module categories. The alternative definition below can be found in [28], although some of our conditions concerning the unit object of the tensor category seem to be missing there.

Definition 2.6.2 Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \otimes, I)$ be two strict tensor categories.

(1) A *left $\mathcal{C}$-module category* $(\mathcal{M}, \rhd, \alpha, \lambda)$ consists of

- a category $\mathcal{M}$;
- a functor $\rhd : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$;
- a natural isomorphism $\alpha : \rhd \circ (\otimes \times \mathrm{id}_{\mathcal{M}}) \to \rhd \circ (\mathrm{id}_{\mathcal{C}} \times \rhd)$ of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, i.e. a family $\{\alpha_M(X, Y) : (X \otimes Y) \rhd M \to X \rhd (Y \rhd M)\}_{X,Y \in \mathcal{C}, M \in \mathcal{M}}$ of isomorphisms in $\mathcal{M}$ such that the square

$$\begin{array}{ccc} (X \otimes Y) \rhd M & \xrightarrow{\alpha_M(X,Y)} & X \rhd (Y \rhd M) \\ \Big\downarrow{\scriptstyle (f \otimes g) \rhd m} & & \Big\downarrow{\scriptstyle f \rhd (g \rhd m)} \\ (X' \otimes Y') \rhd M' & \xrightarrow{\alpha_{M'}(X',Y')} & X' \rhd (Y' \rhd M') \end{array} \tag{2.6.1}$$

commutes for all $X, X', Y, Y' \in \mathcal{C}$, $M, M' \in \mathcal{M}$, $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Y')$ and $m \in \mathrm{Hom}_{\mathcal{M}}(M, M')$, satisfying the additional property that the square

$$\begin{array}{ccc} (X \otimes Y \otimes Z) \rhd M & \xrightarrow{\alpha_M(X \otimes Y, Z)} & (X \otimes Y) \rhd (Z \rhd M) \\ \Big\downarrow{\scriptstyle \alpha_M(X, Y \otimes Z)} & & \Big\downarrow{\scriptstyle \alpha_{Z \rhd M}(X,Y)} \\ X \rhd ((Y \otimes Z) \rhd M) & \xrightarrow{\mathrm{id}_X \rhd \alpha_M(Y,Z)} & X \rhd (Y \rhd (Z \rhd M)) \end{array} \tag{2.6.2}$$

commutes for all $X, Y, Z \in \mathcal{C}$ and $M \in \mathcal{M}$;

- a natural isomorphism $\lambda : \triangleright \circ (I \times \mathrm{id}_{\mathcal{M}}) \to \mathrm{id}_{\mathcal{M}}$ of functors $\mathcal{M} \to \mathcal{M}$, i.e. a family $\{\lambda_M : I \triangleright M \to M\}_{M\in\mathcal{M}}$ of isomorphisms in $\mathcal{M}$ such that the square

$$\begin{array}{ccc} I \triangleright M & \xrightarrow{\lambda_M} & M \\ {\scriptstyle \mathrm{id}_I \triangleright m}\downarrow & & \downarrow{\scriptstyle m} \\ I \triangleright M' & \xrightarrow{\lambda_{M'}} & M' \end{array} \tag{2.6.3}$$

commutes for all $M, M' \in \mathcal{M}$ and $m \in \mathrm{Hom}_{\mathcal{M}}(M, M')$, satisfying the additional property that

$$\alpha_M(X, I)^{-1} = \mathrm{id}_X \triangleright \lambda_M \qquad \text{and} \qquad \alpha_M(I, X)^{-1} = \lambda_{X \triangleright M}$$

for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$.

(2) A *right $\mathcal{D}$-module category* $(\mathcal{M}, \triangleleft, \beta, \rho)$ consists of
- a category $\mathcal{M}$;
- a functor $\triangleleft : \mathcal{M} \times \mathcal{D} \to \mathcal{M}$;
- a natural isomorphism $\beta : \triangleleft \circ (\mathrm{id}_{\mathcal{M}} \times \otimes) \to \triangleleft \circ (\triangleleft \times \mathrm{id}_{\mathcal{D}})$ of functors $\mathcal{M} \times \mathcal{D} \times \mathcal{D} \to \mathcal{M}$, i.e. a family $\{\beta_M(X, Y) : M \triangleleft (U \otimes V) \to (M \triangleleft U) \triangleleft V\}_{U,V\in\mathcal{D},M\in\mathcal{M}}$ of isomorphisms in $\mathcal{M}$ such that the square

$$\begin{array}{ccc} M \triangleleft (U \otimes V) & \xrightarrow{\beta_M(U,V)} & (M \triangleleft U) \triangleleft V \\ {\scriptstyle m\triangleleft(f\otimes g)}\downarrow & & \downarrow{\scriptstyle (m\triangleleft f)\triangleleft g} \\ M' \triangleleft (U' \otimes V') & \xrightarrow{\beta_{M'}(U',V')} & (M' \triangleleft U') \triangleleft V' \end{array} \tag{2.6.4}$$

commutes for all $U, U', V, V' \in \mathcal{D}$, $M, M' \in \mathcal{M}$, $f \in \mathrm{Hom}_{\mathcal{D}}(U, U')$, $g \in \mathrm{Hom}_{\mathcal{D}}(V, V')$ and $m \in \mathrm{Hom}_{\mathcal{M}}(M, M')$, satisfying the additional property that the square

$$\begin{array}{ccc} M \triangleleft (U \otimes V \otimes W) & \xrightarrow{\beta_M(U,V\otimes W)} & (M \triangleleft U) \triangleleft (V \otimes W) \\ {\scriptstyle \beta_M(U\otimes V,W)}\downarrow & & \downarrow{\scriptstyle \beta_{M\triangleleft U}(V,W)} \\ (M \triangleleft (U \otimes V)) \triangleleft W & \xrightarrow{\beta_M(U,V)\triangleleft \mathrm{id}_W} & ((M \triangleleft U) \triangleleft V) \triangleleft W \end{array} \tag{2.6.5}$$

commutes for all $U, V, W \in \mathcal{D}$ and $M \in \mathcal{M}$;
- a natural isomorphism $\rho : \triangleleft \circ (\mathrm{id}_{\mathcal{M}} \times I) \to \mathrm{id}_{\mathcal{M}}$ of functors $\mathcal{M} \to \mathcal{M}$, i.e. a family $\{\rho_M : I \triangleleft M \to M\}_{M\in\mathcal{M}}$ of isomorphisms in $\mathcal{M}$ such that the square

$$\begin{array}{ccc} M \triangleleft I & \xrightarrow{\rho_M} & M \\ {\scriptstyle m\triangleleft \mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle m} \\ M' \triangleleft I & \xrightarrow{\rho_{M'}} & M' \end{array} \tag{2.6.6}$$

commutes for all $M, M' \in \mathcal{M}$ and $m \in \mathrm{Hom}_{\mathcal{M}}(M, M')$, satisfying the additional property that

$$\beta_M(I, U)^{-1} = \rho_M \triangleleft \mathrm{id}_U \qquad \text{and} \qquad \beta_M(U, I)^{-1} = \rho_{M \triangleleft U}$$

for all $U \in \mathcal{D}$ and $M \in \mathcal{M}$.

(3) A $(\mathcal{C}, \mathcal{D})$-*bimodule category* $(\mathcal{M}, \triangleright, \alpha, \lambda, \triangleleft, \beta, \rho, \gamma)$ consists of
- a left $\mathcal{C}$-module category $(\mathcal{M}, \triangleright, \alpha, \lambda)$;

- a right $\mathcal{D}$-module category category $(\mathcal{M}, \triangleleft, \beta, \rho)$;
- a natural isomorphism $\gamma : \triangleleft \circ (\triangleright \times \mathrm{id}_{\mathcal{D}}) \to \triangleright \circ (\mathrm{id}_{\mathcal{C}} \times \triangleleft)$ of functors $\mathcal{C} \times \mathcal{M} \times \mathcal{D} \to \mathcal{M}$, i.e. a family $\{\gamma_M(X,U) : (X \triangleright M) \triangleleft U \to X \triangleright (M \triangleleft U)\}_{X \in \mathcal{C}, U \in \mathcal{D}, M \in \mathcal{M}}$ of isomorphisms in $\mathcal{M}$ such that the square

$$\begin{array}{ccc} (X \triangleright M) \triangleleft U & \xrightarrow{\gamma_M(X,U)} & X \triangleright (M \triangleleft U) \\ {\scriptstyle (f \triangleright m) \triangleleft g}\downarrow & & \downarrow{\scriptstyle f \triangleright (m \triangleleft g)} \\ (X' \triangleright M') \triangleleft U' & \xrightarrow{\gamma_{M'}(X',U')} & X' \triangleright (M' \triangleleft U') \end{array} \tag{2.6.7}$$

commutes for all $X, X' \in \mathcal{C}$, $U, U' \in \mathcal{D}$, $M, M' \in \mathcal{M}$, $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$, $g \in \mathrm{Hom}_{\mathcal{D}}(U, U')$ and $m \in \mathrm{Hom}_{\mathcal{M}}(M, M')$, with the additional property that the diagrams

$$\begin{array}{ccccc} & & ((X \otimes Y) \triangleright M) \triangleleft U & & \\ & \swarrow{\scriptstyle \alpha_M(X,Y) \triangleleft \mathrm{id}_U} & & {\scriptstyle \gamma_M(X \otimes Y, U)}\searrow & \\ (X \triangleright (Y \triangleright M)) \triangleleft U & & & & (X \otimes Y) \triangleright (M \triangleleft U) \\ {\scriptstyle \gamma_{Y \triangleright M}(X,U)}\downarrow & & & & \downarrow{\scriptstyle \alpha_{M \triangleleft U}(X,Y)} \\ X \triangleright ((Y \triangleright M) \triangleleft U) & & \xrightarrow{\mathrm{id}_X \triangleright \gamma_M(Y,U)} & & X \triangleright (Y \triangleright (M \triangleleft U)) \end{array} \tag{2.6.8}$$

and

$$\begin{array}{ccccc} & & X \triangleright (M \triangleleft (V \otimes W)) & & \\ & \swarrow{\scriptstyle \mathrm{id}_X \triangleright \beta_M(V,W)} & & \nwarrow{\scriptstyle \gamma_M(X, V \otimes W)} & \\ X \triangleright ((M \triangleleft V) \triangleleft W) & & & & (X \triangleright M) \triangleleft (V \otimes W) \\ {\scriptstyle \gamma_{M \triangleleft V}(X,W)}\uparrow & & & & \downarrow{\scriptstyle \beta_{X \triangleright M}(V,W)} \\ (X \triangleright (M \triangleleft V)) \triangleleft W & & \xleftarrow{\gamma_M(X,V) \triangleleft \mathrm{id}_W} & & ((X \triangleright M) \triangleleft V) \triangleleft W \end{array} \tag{2.6.9}$$

commute for all $X, Y \in \mathcal{C}$, $V, W \in \mathcal{D}$ and $M \in \mathcal{M}$, as well as the triangles

$$\begin{array}{ccc} (I \triangleright M) \triangleleft Y & \xrightarrow{\gamma_M(I,Y)} & I \triangleright (M \triangleleft Y) \\ & {\scriptstyle \lambda_M \triangleleft \mathrm{id}_Y}\searrow \quad \swarrow{\scriptstyle \lambda_{M \triangleleft Y}} & \\ & M \triangleleft Y & \end{array}$$

and

$$\begin{array}{ccc} (X \triangleright M) \triangleleft I & \xrightarrow{\gamma_M(X,I)} & X \triangleright (M \triangleleft I) \\ & {\scriptstyle \rho_{X \triangleright M}}\searrow \quad \swarrow{\scriptstyle \mathrm{id}_X \triangleright \rho_M} & \\ & X \triangleright M & \end{array}$$

for all $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and $M \in \mathcal{M}$. If $\mathcal{C} = \mathcal{D}$, then we will call $(\mathcal{M}, \triangleright, \alpha, \lambda, \triangleleft, \beta, \rho, \gamma)$ a $\mathcal{C}$-*bimodule category*, rather than a $(\mathcal{C}, \mathcal{C})$-bimodule category.

If $\mathcal{M}$ is a left $\mathcal{C}$-module category for which $\triangleright\circ[I\times\mathrm{id}_{\mathcal{M}}] = \mathrm{id}_{\mathcal{M}}$ and $\lambda_M = \mathrm{id}_M$ for all $M \in \mathcal{M}$, then we will say that $\mathcal{M}$ is a *semi-strict left $\mathcal{C}$-module category.* If, in addition, we also have that $\triangleright\circ[\otimes\times\mathrm{id}_{\mathcal{M}}] = \triangleright\circ[\mathrm{id}_{\mathcal{C}}\times\triangleright]$ and $\alpha_M(X,Y) = \mathrm{id}_{(X\otimes Y)\triangleright M} = \mathrm{id}_{X\triangleright(Y\triangleright M)}$ for all $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$, we will say that $\mathcal{M}$ is a *strict left $\mathcal{C}$-module category.* Similarly, ones defines (semi-)strict right $\mathcal{D}$-module categories and (semi-)strict $(\mathcal{C},\mathcal{D})$-bimodule categories.

Example 2.6.3 The following examples of module categories will be important to us.

(1) If $\mathcal{C}$ is a strict tensor category, then $\mathcal{C}$ obviously is a $\mathcal{C}$-bimodule category if we define $\triangleright = \otimes$ and $\triangleleft = \otimes$. In fact, a similar statement can be made for non-strict tensor categories, but then one first has to define module categories over non-strict tensor categories.

(2) Let $\mathcal{C}$ be a strict tensor category and let $\mathsf{A} = (A,\mu,\eta)$ be an algebra in $\mathcal{C}$. Then the category $\mathrm{Mod}_{\mathcal{C}}(\mathsf{A})$ of left A-modules can be equipped with the structure of a strict right $\mathcal{C}$-module category as follows. The functor $\triangleleft : \mathrm{Mod}_{\mathcal{C}}(\mathsf{A}) \times \mathcal{C} \to \mathrm{Mod}_{\mathcal{C}}(\mathsf{A})$ is defined by $(V,\pi_V)\triangleleft X := (V\otimes X, \pi_V\otimes\mathrm{id}_X)$ and $f\triangleleft g = f\otimes g$. It is easy to check that $(V,\pi_V)\triangleleft X \in \mathrm{Mod}_{\mathcal{C}}(\mathsf{A})$ and that $f\triangleleft g \in \mathrm{Hom}_{\mathrm{Mod}_{\mathcal{C}}(\mathsf{A})}((V\otimes X,\pi_V\otimes\mathrm{id}_X),(W\otimes Y,\pi_W\otimes\mathrm{id}_Y))$ if $f\in\mathrm{Hom}_{\mathrm{Mod}_{\mathcal{C}}(\mathsf{A})}((V,\pi_V),(W,\pi_W))$ and $g\in\mathrm{Hom}_{\mathcal{C}}(X,Y)$, and that $\mathrm{Mod}_{\mathcal{C}}(\mathsf{A})$ is indeed a strict right $\mathcal{C}$-module category.

We will now show that if $\mathcal{C}$ is braided, then left $\mathcal{C}$-module categories give rise to right $\mathcal{C}$-module categories and vice versa. This result will be needed in Subsection 2.8.4.

Lemma 2.6.4 *Let $(\mathcal{C},\otimes,I,c)$ be a braided strict tensor category.*

(1) If $(\mathcal{M},\triangleright,\alpha,\lambda)$ is a left $\mathcal{C}$-module category, then we can equip $\mathcal{M}$ with the structure $(\mathcal{M},\triangleleft,\beta,\rho)$ of a right $\mathcal{C}$-module category by defining $M\triangleleft X := X\triangleright M$, $m\triangleleft f := f\triangleright m$, $\beta_M(X,Y) := \alpha_M(Y,X)\circ[c_{X,Y}\triangleright\mathrm{id}_M]$ and $\rho_M := \lambda_M$ for all $X,Y\in\mathcal{C}$, $M\in\mathcal{M}$, $f\in\mathrm{Hom}(\mathcal{C})$ and $m\in\mathrm{Hom}(\mathcal{M})$. In fact, if we define $\gamma_M(X,U) = \alpha_M(X,U)\circ[c_{U,X}\triangleright\mathrm{id}_M]\circ\alpha_M(U,X)^{-1}$, then $\mathcal{M}$ becomes a $\mathcal{C}$-bimodule category.

(2) If $(\mathcal{M},\triangleleft,\beta,\rho)$ is a right $\mathcal{C}$-module category, then we can equip $\mathcal{M}$ with the structure $(\mathcal{M},\triangleright,\alpha,\lambda)$ of a left $\mathcal{C}$-module category by defining $X\triangleright M := M\triangleleft X$, $f\triangleright m := m\triangleleft f$, $\alpha_M(X,Y) := \beta_M(Y,X)\circ[\mathrm{id}_M\triangleleft c_{X,Y}]$ and $\lambda_M := \rho_M$ for all $X,Y\in\mathcal{C}$, $M\in\mathcal{M}$, $f\in\mathrm{Hom}(\mathcal{C})$ and $m\in\mathrm{Hom}(\mathcal{M})$. In fact, if we define $\gamma_M(X,U) = \beta_M(U,X)\circ[\mathrm{id}_M\triangleleft c_{X,U}]\circ\beta_M(X,U)^{-1}$, then $\mathcal{M}$ becomes a $\mathcal{C}$-bimodule category.

Proof. We will only prove (2); the proof of (1) proceeds similarly. It is clear that $\triangleright$ is a functor. To check naturality of α, let $X,X',Y,Y'\in\mathcal{C}$, $M,M'\in\mathcal{M}$, $f\in\mathrm{Hom}_{\mathcal{C}}(X,X')$, $g\in\mathrm{Hom}_{\mathcal{C}}(Y,Y')$ and $m\in\mathrm{Hom}_{\mathcal{M}}(M,M')$, and consider the diagram

$$\begin{array}{ccccc}
M\triangleleft(X\otimes Y) & \xrightarrow{\mathrm{id}_M\triangleleft c_{X,Y}} & M\triangleleft(Y\otimes X) & \xrightarrow{\beta_M(Y,X)} & (M\triangleleft Y)\triangleleft X \\
\downarrow{\scriptstyle m\triangleleft(f\otimes g)} & & \downarrow{\scriptstyle m\triangleleft(g\otimes f)} & & \downarrow{\scriptstyle (m\triangleleft g)\triangleleft f} \\
M'\triangleleft(X'\otimes Y') & \xrightarrow{\mathrm{id}_{M'}\triangleleft c_{X',Y'}} & M'\triangleleft(Y'\otimes X') & \xrightarrow{\beta_{M'}(Y',X')} & (M'\triangleleft Y')\triangleleft X'.
\end{array}$$

The left square commutes by naturality of c and the right square commutes by naturality of β. Hence the big outer rectangle commutes as well. But this means that the square (2.6.1) commutes for our choice of α. Now let $X,Y,Z\in\mathcal{C}$, let $M\in\mathcal{M}$ and consider the diagram

$$\begin{array}{ccccc}
M\triangleleft(X\otimes Y\otimes Z) & \xrightarrow{\mathrm{id}_M\triangleleft c_{X\otimes Y,Z}} & M\triangleleft(Z\otimes X\otimes Y) & \xrightarrow{\beta_M(Z,X\otimes Y)} & (M\triangleleft Z)\triangleleft(X\otimes Y) \\
\downarrow{\scriptstyle \mathrm{id}_M\triangleleft c_{X,Y\otimes Z}} & & \downarrow{\scriptstyle \mathrm{id}_M\triangleleft(\mathrm{id}_Z\otimes c_{X,Y})} & & \downarrow{\scriptstyle \mathrm{id}_{M\triangleleft Z}\triangleleft c_{X,Y}} \\
M\triangleleft(Y\otimes Z\otimes X) & \xrightarrow{\mathrm{id}_M\triangleleft(c_{Y,Z}\otimes\mathrm{id}_X)} & M\triangleleft(Z\otimes Y\otimes X) & \xrightarrow{\beta_M(Z,Y\otimes X)} & (M\triangleleft Z)\triangleleft(Y\otimes X) \\
\downarrow{\scriptstyle \beta_M(Y\otimes Z,X)} & & \downarrow{\scriptstyle \beta_M(Z\otimes Y,X)} & & \downarrow{\scriptstyle \beta_{M\triangleleft Z}(Y,X)} \\
(M\triangleleft(Y\otimes Z))\triangleleft X & \xrightarrow{(\mathrm{id}_M\triangleleft c_{Y,Z})\triangleleft\mathrm{id}_X} & (M\triangleleft(Z\otimes Y))\triangleleft X & \xrightarrow{\beta_M(Z,Y)\triangleleft\mathrm{id}_X} & ((M\triangleleft Z)\triangleleft Y)\triangleleft X.
\end{array}$$

The upper left square commutes by compatibility of the braiding with the tensor product, the upper right and lower left squares commute by naturality of β and the lower right square commutes by the additional property (2.6.5) of β. Hence the big outer square commutes as well, which means that the square (2.6.2) commutes for our choice of α.

Naturality of λ follows immediately from naturality of ρ, since we have defined $\lambda_M = \rho_M$ for all $M \in \mathcal{M}$. For any $X \in \mathcal{C}$ and $M \in \mathcal{M}$ we have the equations

$$\alpha_M(X,I)^{-1} = [\beta_M(I,X) \circ (\mathrm{id}_M \lhd c_{X,I})]^{-1} = \beta_M(I,X)^{-1} = \rho_M \lhd \mathrm{id}_X = \mathrm{id}_X \rhd \lambda_M,$$
$$\alpha_M(I,X)^{-1} = [\beta_M(X,I) \circ (\mathrm{id}_M \lhd c_{I,X})]^{-1} = \beta_M(X,I)^{-1} = \rho_{M\lhd X} = \lambda_{X \rhd M},$$

showing that we have indeed a left $\mathcal{C}$-module category.

We will now show that we actually get a $\mathcal{C}$-bimodule category. Naturality of γ follows from naturality of β and c (the proof proceeds in a similar manner as the proof of naturality of α above, i.e. by splitting the diagram into smaller subdiagrams). To check commutativity of (2.6.8), we compute

$$\begin{aligned}
&[\mathrm{id}_X \rhd \gamma_M(Y,U)] \circ \gamma_{Y\rhd M}(X,U) \circ [\alpha_M(X,Y) \lhd \mathrm{id}_U]\\
&= [\gamma_M(Y,U) \lhd \mathrm{id}_X] \circ \gamma_{M\lhd Y}(X,U) \circ [\alpha_M(X,Y) \lhd \mathrm{id}_U]\\
&= [\beta_M(U,Y) \lhd \mathrm{id}_X] \circ [(\mathrm{id}_M \lhd c_{Y,U}) \lhd \mathrm{id}_X] \circ [\beta_M(Y,U)^{-1} \lhd \mathrm{id}_X] \circ \beta_{M\lhd Y}(U,X)\\
&\quad \circ [\mathrm{id}_{M\lhd Y} \lhd c_{X,U}] \circ \beta_{M\lhd Y}(X,U)^{-1} \circ [\beta_M(Y,X) \lhd \mathrm{id}_U] \circ [(\mathrm{id}_M \lhd c_{X,Y}) \lhd \mathrm{id}_U]\\
&= [\beta_M(U,Y) \lhd \mathrm{id}_X] \circ [(\mathrm{id}_M \lhd c_{Y,U}) \lhd \mathrm{id}_X] \circ \beta_M(Y\otimes U,X) \circ \beta_M(Y,U\otimes X)^{-1}\\
&\quad \circ [\mathrm{id}_{M\lhd Y} \lhd c_{X,U}] \circ \beta_M(Y,X\otimes U) \circ \beta_M(Y\otimes X,U)^{-1} \circ [(\mathrm{id}_M \lhd c_{X,Y}) \lhd \mathrm{id}_U]\\
&= [\beta_M(U,Y) \lhd \mathrm{id}_X] \circ \beta_M(U\otimes Y,X) \circ [\mathrm{id}_M \lhd (c_{Y,U} \otimes \mathrm{id}_X)] \circ [\mathrm{id}_M \lhd (\mathrm{id}_Y \otimes c_{X,U})]\\
&\quad \circ [\mathrm{id}_M \lhd (c_{X,Y} \otimes \mathrm{id}_U)] \circ \beta_M(X\otimes Y,U)^{-1}\\
&= \beta_{M\lhd U}(Y,X) \circ \beta_M(U,Y\otimes X) \circ [\mathrm{id}_M \lhd (\mathrm{id}_U \otimes c_{X,Y})] \circ [\mathrm{id}_M \lhd c_{X\otimes Y,U}] \circ \beta_M(X\otimes Y,U)^{-1}\\
&= \beta_{M\lhd U}(Y,X) \circ [\mathrm{id}_{M\lhd U} \lhd c_{X,Y}] \circ \beta_M(U,X\otimes Y) \circ [\mathrm{id}_M \lhd c_{X\otimes Y,U}] \circ \beta_M(X\otimes Y,U)^{-1}\\
&= \alpha_{M\lhd U}(X,Y) \circ \gamma_M(X\otimes Y,U),
\end{aligned}$$

and commutativity of (2.6.9) follows from

$$\begin{aligned}
&\gamma_{M\lhd V}(X,W) \circ [\gamma_M(X,V) \lhd \mathrm{id}_W] \circ \beta_{X\rhd M}(V,W)\\
&= \beta_{M\lhd V}(W,X) \circ [\mathrm{id}_{M\lhd V} \lhd c_{X,W}] \circ \beta_{M\lhd V}(X,W)^{-1} \circ [\beta_M(V,X) \lhd \mathrm{id}_W]\\
&\quad \circ [(\mathrm{id}_M \lhd c_{X,V}) \lhd \mathrm{id}_W] \circ [\beta_M(X,V)^{-1} \lhd \mathrm{id}_W] \circ \beta_{M\lhd X}(V,W)\\
&= \beta_{M\lhd V}(W,X) \circ [\mathrm{id}_{M\lhd V} \lhd c_{X,W}] \circ \beta_M(V,X\otimes W) \circ \beta_M(V\otimes X,W)^{-1}\\
&\quad \circ [(\mathrm{id}_M \lhd c_{X,V}) \lhd \mathrm{id}_W] \circ \beta_M(X\otimes V,W) \circ \beta_M(X,V\otimes W)^{-1}\\
&= \beta_{M\lhd V}(W,X) \circ \beta_M(V,W\otimes X) \circ [\mathrm{id}_M \lhd (\mathrm{id}_V \otimes c_{X,W})] \circ [\mathrm{id}_M \lhd (c_{X,V} \otimes \mathrm{id}_W)]\\
&\quad \circ \beta_M(X\otimes V,W)^{-1} \circ \beta_M(X\otimes V,W) \circ \beta_M(X,V\otimes W)^{-1}\\
&= [\beta_M(V,W) \lhd \mathrm{id}_X] \circ \beta_M(V\otimes W,X) \circ [\mathrm{id}_M \lhd c_{X,V\otimes W}] \circ \beta_M(X,V\otimes W)^{-1}\\
&= [\mathrm{id}_X \rhd \beta_M(V,W)] \circ \gamma_M(X,V\otimes W).
\end{aligned}$$

□

2.6.2 Functors of module categories; module natural transformations

In the previous subsection we defined (bi)module categories over a tensor category and we proved a lemma concerning the case where the tensor category is braided. The present subsection only contains two definitions, but these definitions are quite involved. We begin by defining the correct notion of functors between module categories.

Definition 2.6.5 Let $\mathcal{C}$ and $\mathcal{D}$ be strict tensor categories.

(1) If $(\mathcal{M}, \triangleright, \alpha, \lambda)$ and $(\mathcal{M}', \blacktriangleright, \alpha', \lambda')$ are left $\mathcal{C}$-module categories, then a *functor of left $\mathcal{C}$-module categories* (H, s) from $\mathcal{M}$ to $\mathcal{M}'$ consists of a functor $H : \mathcal{M} \to \mathcal{M}'$ and a natural isomorphism $s : H \circ \triangleright \to \blacktriangleright \circ [\mathrm{id}_{\mathcal{C}} \times H]$ of functors $\mathcal{C} \times \mathcal{M} \to \mathcal{M}'$, i.e. a family

$$\{s_M(X) : H(X \triangleright M) \to X \blacktriangleright H(M)\}_{X \in \mathcal{C},\ M \in \mathcal{M}}$$

of isomorphisms in $\mathcal{M}'$ such that the square

$$\begin{array}{ccc} H(X \triangleright M) & \xrightarrow{s_M(X)} & X \blacktriangleright H(M) \\ {\scriptstyle H(f \triangleright m)} \downarrow & & \downarrow {\scriptstyle f \blacktriangleright H(m)} \\ H(Y \triangleright N) & \xrightarrow{s_N(Y)} & Y \blacktriangleright H(N) \end{array}$$

commutes for all $X, Y \in \mathcal{C}$, $M, N \in \mathcal{M}$, $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $m \in \mathrm{Hom}_{\mathcal{M}}(M, N)$, satisfying the additional property that the diagrams

$$\begin{array}{ccccc} & & H((X \otimes Y) \triangleright M) & & \\ & \swarrow {\scriptstyle H(\alpha_M(X,Y))} & & {\scriptstyle s_M(X \otimes Y)} \searrow & \\ H(X \triangleright (Y \triangleright M)) & & & & (X \otimes Y) \blacktriangleright H(M) \\ {\scriptstyle s_{Y \triangleright M}(X)} \downarrow & & & & \downarrow {\scriptstyle \alpha'_{H(M)}(X,Y)} \\ X \blacktriangleright H(Y \triangleright M) & & \xrightarrow{\mathrm{id}_X \blacktriangleright s_M(Y)} & & X \blacktriangleright (Y \blacktriangleright H(M)) \end{array}$$

and

$$\begin{array}{ccc} H(I \triangleright M) & \xrightarrow{s_M(I)} & I \blacktriangleright H(M) \\ & {\scriptstyle H(\lambda_M)} \searrow \quad \swarrow {\scriptstyle \lambda'_{H(M)}} & \\ & H(M) & \end{array}$$

commute for all $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$.

(2) If $(\mathcal{M}, \triangleleft, \beta, \rho)$ and $(\mathcal{M}', \blacktriangleleft, \beta', \rho')$ are right $\mathcal{D}$-module categories, then a *functor of right $\mathcal{D}$-module categories* (H, t) from $\mathcal{M}$ to $\mathcal{M}'$ consists of a functor $H : \mathcal{M} \to \mathcal{M}'$ and a natural isomorphism $t : H \circ \triangleleft \to \blacktriangleleft \circ [H \times \mathrm{id}_{\mathcal{C}}]$ of functors $\mathcal{M} \times \mathcal{D} \to \mathcal{M}'$, i.e. a family

$$\{t_M(X) : H(M \triangleleft X) \to H(M) \blacktriangleleft X\}_{X \in \mathcal{D},\ M \in \mathcal{M}}$$

of isomorphisms in $\mathcal{M}'$ such that the square

$$\begin{array}{ccc} H(M \triangleleft X) & \xrightarrow{t_M(X)} & H(M) \blacktriangleleft X \\ {\scriptstyle H(m \triangleleft f)} \downarrow & & \downarrow {\scriptstyle H(M) \blacktriangleleft f} \\ H(N \triangleleft Y) & \xrightarrow{t_N(Y)} & H(N) \blacktriangleleft Y \end{array}$$

commutes for all $X, Y \in \mathcal{D}$, $M, N \in \mathcal{M}$, $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ and $m \in \mathrm{Hom}_{\mathcal{M}}(M, N)$, satisfying the additional property that the diagrams

$$\begin{array}{ccccc} & & H(M \triangleleft (X \otimes Y)) & & \\ & \swarrow^{H(\beta_M(X,Y))} & & \searrow^{t_M(X \otimes Y)} & \\ H((M \triangleleft X) \triangleleft Y) & & & & H(M) \blacktriangleleft (X \otimes Y) \\ \downarrow{\scriptstyle t_{M \triangleleft X}(Y)} & & & & \downarrow{\scriptstyle \beta'_{H(M)}(X,Y)} \\ H(M \triangleleft X) \blacktriangleleft Y & & \xrightarrow{t_M(X) \blacktriangleleft \mathrm{id}_Y} & & (H(M) \blacktriangleleft X) \blacktriangleleft Y \end{array}$$

and

$$\begin{array}{ccc} H(M \triangleleft I) & \xrightarrow{t_M(I)} & H(M) \blacktriangleleft I \\ & {\scriptstyle H(\rho_M)}\searrow \quad \swarrow{\scriptstyle \rho'_{H(M)}} & \\ & H(M) & \end{array}$$

commute for all $X, Y \in \mathcal{D}$ and $M \in \mathcal{M}$.

(3) If $(\mathcal{M}, \triangleright, \alpha, \lambda, \triangleleft, \beta, \rho, \gamma)$ and $(\mathcal{M}', \blacktriangleright, \alpha', \lambda', \blacktriangleleft, \beta', \rho', \gamma')$ are $(\mathcal{C}, \mathcal{D})$-bimodule categories, then a *functor of $(\mathcal{C}, \mathcal{D})$-bimodule categories* (H, s, t) from $\mathcal{M}$ to $\mathcal{M}'$ consists of a functor (H, s) of left $\mathcal{C}$-module categories and a functor (H, t) of right $\mathcal{D}$-module categories such that the diagram

$$\begin{array}{ccc} H((X \triangleright M) \triangleleft Y) & \xrightarrow{t_{X \triangleright M}(Y)} & H(X \triangleright M) \blacktriangleleft Y \\ \downarrow{\scriptstyle H(\gamma_M(X,Y))} & & \downarrow{\scriptstyle s_M(X) \blacktriangleleft \mathrm{id}_Y} \\ H(X \triangleright (M \triangleleft Y)) & & (X \blacktriangleright H(M)) \blacktriangleleft Y \\ \downarrow{\scriptstyle s_{M \triangleleft Y}(X)} & & \downarrow{\scriptstyle \gamma'_{H(M)}(X,Y)} \\ X \blacktriangleright H(M \triangleleft Y) & \xrightarrow{\mathrm{id}_X \blacktriangleright t_M(Y)} & X \blacktriangleright (H(M) \blacktriangleleft Y) \end{array}$$

commutes for all $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and $M \in \mathcal{M}$.

Now that we have introduced functors between module categories, we will define natural transformations between such functors.

Definition 2.6.6 Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \otimes, I)$ be strict tensor categories.

(1) If $(H_1, s^1), (H_2, s^2) : (\mathcal{M}, \triangleright, \alpha, \lambda) \to (\mathcal{N}, \blacktriangleright, \alpha', \lambda')$ are functors of left $\mathcal{C}$-module categories, then a *left module natural transformation* $\varphi : (H_1, s^1) \to (H_2, s^2)$ is a natural transformation $\varphi : H_1 \to H_2$ such that the diagram

$$\begin{array}{ccc} H_1(X \triangleright M) & \xrightarrow{\varphi_{X \triangleright M}} & H_2(X \triangleright M) \\ \downarrow{\scriptstyle s^1_M(X)} & & \downarrow{\scriptstyle s^2_M(X)} \\ X \blacktriangleright H_1(M) & \xrightarrow{\mathrm{id}_X \blacktriangleright \varphi_M} & X \blacktriangleright H_2(M) \end{array}$$

commutes for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$.

(2) If $(H_1, t^1), (H_2, t^2) : (\mathcal{M}, \triangleleft, \beta, \rho) \to (\mathcal{N}, \blacktriangleleft, \beta', \rho')$ are functors of right $\mathcal{D}$-module categories, then a *right module natural transformation* $\varphi : (H_1, t^1) \to (H_2, t^2)$ is a natural transformation $\varphi : H_1 \to H_2$ such that the diagram

$$\begin{array}{ccc}
H_1(M \lhd X) & \xrightarrow{\varphi_{M\lhd X}} & H_2(M \lhd X) \\
{\scriptstyle t^1_M(X)}\big\downarrow & & \big\downarrow{\scriptstyle t^2_M(X)} \\
H_1(M) \blacktriangleleft X & \xrightarrow{\varphi_M \blacktriangleleft \mathrm{id}_X} & H_2(M) \blacktriangleleft X
\end{array}$$

commutes for all $X \in \mathcal{D}$ and $M \in \mathcal{M}$.

(3) If $(H_1, s^1, t^1), (H_2, s^2, t^2) : (\mathcal{M}, \rhd, \alpha, \lambda, \lhd, \beta, \rho, \gamma) \to (\mathcal{N}, \blacktriangleright, \alpha', \lambda', \blacktriangleleft, \beta', \rho', \gamma')$ are functors of $(\mathcal{C}, \mathcal{D})$-bimodule categories, then a *bimodule natural transformation* $\varphi : (H_1, s^1, t^1) \to (H_2, s^2, t^2)$ is a natural transformation $\varphi : H_1 \to H_2$ that is both a left module natural transformation $(H_1, s^1) \to (H_2, s^2)$ and a right module natural transformation $(H_1, t^1) \to (H_2, t^2)$.

If $(H_1, s^1), (H_2, s^2), (H_3, s^3) : (\mathcal{M}, \rhd, \alpha, \lambda) \to (\mathcal{N}, \blacktriangleright, \alpha', \lambda')$ are functors of left $\mathcal{C}$-module categories and if $\varphi : (H_1, s^1) \to (H_2, s^2)$ and $\psi : (H_2, s^2) \to (H_3, s^3)$ are left module natural transformations, then the natural transformation $\psi \circ \varphi : H_1 \to H_3$ is a left module natural transformation. To see this, consider the diagram

$$\begin{array}{ccccc}
H_1(X \rhd M) & \xrightarrow{\varphi_{X\rhd M}} & H_2(X \rhd M) & \xrightarrow{\psi_{X\rhd M}} & H_3(X \rhd M) \\
{\scriptstyle s^1_M(X)}\big\downarrow & & \big\downarrow{\scriptstyle s^2_M(X)} & & \big\downarrow{\scriptstyle s^3_M(X)} \\
X \blacktriangleright H_1(M) & \xrightarrow{\mathrm{id}_X \blacktriangleright \varphi_M} & X \blacktriangleright H_2(M) & \xrightarrow{\mathrm{id}_X \blacktriangleright \psi_M} & X \blacktriangleright H_3(M)
\end{array}$$

for $X \in \mathcal{C}$ and $M \in \mathcal{M}$. The left and right square commute because φ and ψ are left module natural transformations, so the big outer rectangle commutes as well, showing that the natural transformation $\psi \circ \varphi$ is indeed a left module natural transformation. Similarly, the composition of two right module natural transformations is again a right module natural transformation, and consequently the composition of two bimodule natural transformations is also again a bimodule natural transformation. This implies that if we have two tensor categories $\mathcal{C}$ and $\mathcal{D}$, as well as two $(\mathcal{C}, \mathcal{D})$-bimodule categories $\mathcal{M}$ and $\mathcal{N}$, then the $(\mathcal{C}, \mathcal{D})$-bimodule functors from $\mathcal{M}$ to $\mathcal{N}$ form a subcategory $\mathrm{Fun}_{(\mathcal{C},\mathcal{D})}(\mathcal{M}, \mathcal{N})$ of $\mathrm{Fun}(\mathcal{M}, \mathcal{N})$.

2.6.3 Structures of a bimodule category on a tensor category

Our results of this subsection will be essential for our discussion in Section 4.5. Let $\mathcal{C}$ be a strict tensor category. It is obvious that $\mathcal{C}$ can be considered as a strict $\mathcal{C}$-bimodule category $(\mathcal{C}, \otimes, \otimes)$; we will simply refer to it as $\mathcal{C}$. Now suppose that we are also given another $\mathcal{C}$-bimodule structure $(\mathcal{C}, \rhd, \alpha, \lambda, \lhd, \beta, \rho, \gamma)$ on $\mathcal{C}$. In this subsection we will refer to this $\mathcal{C}$-bimodule category as $\mathcal{C}'$, for simplicity. We will investigate the $\mathcal{C}$-bimodule functors from $\mathcal{C}$ to $\mathcal{C}'$ in some detail.

In what follows, for any $V \in \mathcal{C}$ we will write $\mathcal{L}_V$ to denote the functor $\lhd \circ [V \times \mathrm{id}_{\mathcal{C}}]$, i.e. $\mathcal{L}_V(M) = V \lhd M$ and $\mathcal{L}_V(m) = \mathrm{id}_V \lhd m$ for $M \in \mathcal{C}$ and $m \in \mathrm{Hom}(\mathcal{C})$. Note that if $\lhd$ would be equal to $\otimes$, then this functor $\mathcal{L}_V$ would correspond to tensoring from the left with V.

Lemma 2.6.7 *Let $(H, s, t) : \mathcal{C} \to \mathcal{C}'$ be a functor of $\mathcal{C}$-bimodule categories. Then we obtain another functor $(\mathcal{L}_H, s', t') : \mathcal{C} \to \mathcal{C}'$ of $\mathcal{C}$-bimodule categories by defining the functor $\mathcal{L}_H$ by $\mathcal{L}_H := \mathcal{L}_{H(I)}$ and by defining*

$$s'_M(X) : \underbrace{\mathcal{L}_H(X \otimes M)}_{=H(I)\lhd(X\otimes M)} \to \underbrace{X \rhd \mathcal{L}_H(M)}_{=X\rhd(H(I)\lhd M)}$$

$$t'_M(X) : \underbrace{\mathcal{L}_H(M \otimes X)}_{=H(I)\lhd(M\otimes X)} \to \underbrace{\mathcal{L}_H(M) \lhd X}_{=(H(I)\lhd M)\lhd X}$$

by

$$s'_M(X) := \gamma_{H(I)}(X,M) \circ [s_I(X) \triangleleft \mathrm{id}_M] \circ [t_I(X)^{-1} \triangleleft \mathrm{id}_M] \circ \beta_{H(I)}(X,M)$$
$$t'_M(X) := \beta_{H(I)}(M,X).$$

Furthermore, if for each $M \in \mathcal{C}$ we define an isomorphism

$$\varphi_M^{(H,s,t)} : \underbrace{H(M)}_{=H(I\otimes M)} \to \underbrace{\mathcal{L}_H(M)}_{=H(I)\triangleleft M}$$

by $\varphi_M^{(H,s,t)} := t_I(M)$, then this defines a natural bimodule isomorphism

$$\varphi^{(H,s,t)} : (H,s,t) \to (\mathcal{L}_H, s', t').$$

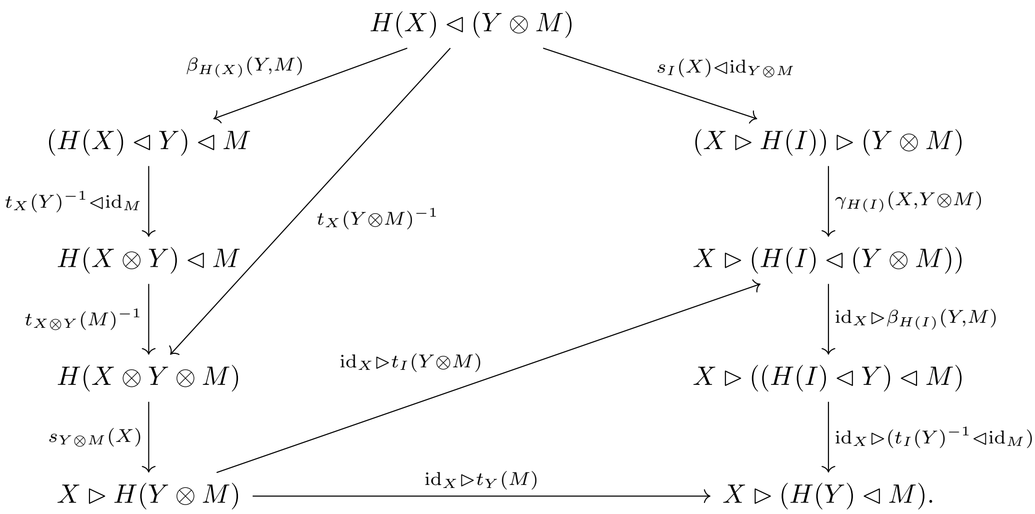
Proof. Naturality of s' follows from naturality of β, t, s and γ. We now have to show that the square

$$\begin{array}{ccc} H(I) \triangleleft (X \otimes Y \otimes M) & \xrightarrow{s'_M(X\otimes Y)} & (X\otimes Y) \triangleright (H(I) \triangleleft M) \\ \Big\downarrow{\scriptstyle s'_{Y\otimes M}(X)} & & \Big\downarrow{\scriptstyle \alpha_{H(I)\triangleleft M}(X,Y)} \\ X \triangleright (H(I) \triangleleft (Y \otimes M)) & \xrightarrow{\mathrm{id}_X \triangleright s'_M(Y)} & X \triangleright (Y \triangleright (H(I) \triangleleft M)) \end{array}$$

commutes for all $X, Y, M \in \mathcal{C}$. This follows from the computation

$$\begin{aligned}
&\alpha_{H(I)\triangleleft M}(X,Y) \circ s'_M(X\otimes Y) \\
&= \alpha_{H(I)\triangleleft M}(X,Y) \circ \gamma_{H(I)}(X\otimes Y, M) \circ [s_I(X\otimes Y) \triangleleft \mathrm{id}_M] \\
&\quad \circ [t_I(X\otimes Y)^{-1} \triangleleft \mathrm{id}_M] \circ \beta_{H(I)}(X \otimes Y, M) \\
&= [\mathrm{id}_X \triangleright \gamma_{H(I)}(Y,M)] \circ \gamma_{Y\triangleright H(I)}(X,M) \circ [\alpha_{H(I)}(X,Y) \triangleleft \mathrm{id}_M] \circ [s_I(X\otimes Y) \triangleleft \mathrm{id}_M] \\
&\quad \circ [t_I(X\otimes Y)^{-1} \triangleleft \mathrm{id}_M] \circ [\beta_{H(I)}(X,Y)^{-1} \triangleleft \mathrm{id}_M] \circ \beta_{H(I)\triangleleft X}(Y,M) \circ \beta_{H(I)}(X, Y\otimes M) \\
&= [\mathrm{id}_X \triangleright \gamma_{H(I)}(Y,M)] \circ \gamma_{Y\triangleright H(I)}(X,M) \circ [(\mathrm{id}_X \triangleright s_I(Y)) \triangleleft \mathrm{id}_M] \circ [s_{Y\otimes I}(X) \triangleleft \mathrm{id}_M] \\
&\quad \circ [t_{I\otimes X}(Y)^{-1} \triangleleft \mathrm{id}_M] \circ [(t_I(X)^{-1} \triangleleft \mathrm{id}_Y) \triangleleft \mathrm{id}_M] \circ \beta_{H(I)\triangleleft X}(Y,M) \circ \beta_{H(I)}(X, Y\otimes M) \\
&= [\mathrm{id}_X \triangleright \gamma_{H(I)}(Y,M)] \circ [\mathrm{id}_X \triangleright (s_I(Y) \triangleleft \mathrm{id}_M)] \circ \gamma_{H(Y)}(X,M) \circ [s_Y(X) \triangleleft \mathrm{id}_M] \\
&\quad \circ [t_X(Y)^{-1} \triangleleft \mathrm{id}_M] \circ \beta_{H(X)}(Y,M) \circ [t_I(X)^{-1} \triangleleft \mathrm{id}_{Y\otimes M}] \circ \beta_{H(I)}(X, Y\otimes M) \\
&= [\mathrm{id}_X \triangleright \gamma_{H(I)}(Y,M)] \circ [\mathrm{id}_X \triangleright (s_I(Y) \triangleleft \mathrm{id}_M)] \circ [\mathrm{id}_X \triangleright t_Y(M)] \circ s_{Y\otimes M}(X) \circ t_{X\otimes Y}(M)^{-1} \\
&\quad \circ [t_X(Y)^{-1} \triangleleft \mathrm{id}_M] \circ \beta_{H(X)}(Y,M) \circ [t_I(X)^{-1} \triangleleft \mathrm{id}_{Y\otimes M}] \circ \beta_{H(I)}(X, Y\otimes M) \\
&\overset{*}{=} [\mathrm{id}_X \triangleright \gamma_{H(I)}(Y,M)] \circ [\mathrm{id}_X \triangleright (s_I(Y) \triangleleft \mathrm{id}_M)] \circ [\mathrm{id}_X \triangleright (t_I(Y)^{-1} \triangleleft \mathrm{id}_M)] \circ [\mathrm{id}_X \triangleright \beta_{H(I)}(Y,M)] \\
&\quad \circ \gamma_{H(I)}(X, Y\otimes M) \circ [s_I(X) \triangleleft \mathrm{id}_{Y\otimes M}] \circ [t_I(X)^{-1} \triangleleft \mathrm{id}_{Y\otimes M}] \circ \beta_{H(I)}(X, Y\otimes M) \\
&= [\mathrm{id}_X \triangleright s'_M(Y)] \circ s'_{Y\otimes M}(X).
\end{aligned}$$

The equality $\overset{*}{=}$ can be understood by considering the diagram



The upper left and lower right subdiagrams commute because (H, t) is a functor of right $\mathcal{C}$-module categories and the middle subdiagram commutes by the compatibility condition between (H, s) and (H, t). Thus the big outer diagram commutes as well, proving the equality $\overset{*}{=}$. We also have

$$\begin{aligned} s'_M(I) &= \gamma_{H(I)}(I, M) \circ [s_I(I) \triangleleft \mathrm{id}_M] \circ [t_I(I)^{-1} \triangleleft \mathrm{id}_M] \circ \beta_{H(I)}(I, M) \\ &= \gamma_{H(I)}(I, M) \circ [\lambda^{-1}_{H(I)} \triangleleft \mathrm{id}_M] \circ [\rho_{H(I)} \triangleleft \mathrm{id}_M] \circ \beta_{H(I)}(I, M) \\ &= \gamma_{H(I)}(I, M) \circ [\lambda^{-1}_{H(I)} \triangleleft \mathrm{id}_M] \\ &= \lambda_{H(I)\triangleleft M}. \end{aligned}$$

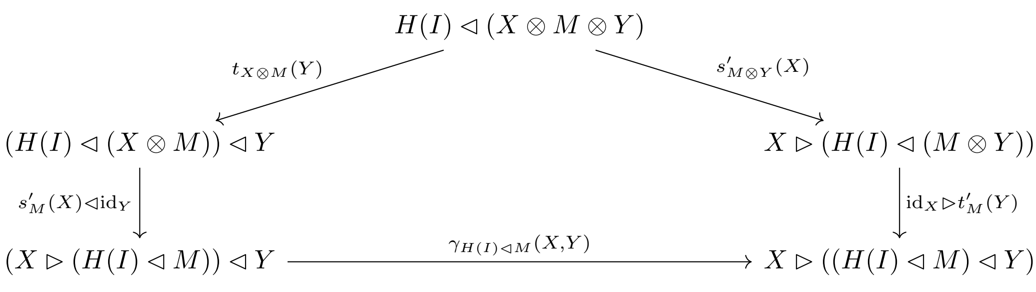
This proves that $(\mathcal{L}_H, s')$ is a functor of left $\mathcal{C}$-module categories. Naturality of t' follows directly from naturality of β. Also, we have

$$\begin{aligned} \beta_{\mathcal{L}_H(M)}(X, Y) \circ t'_M(X \otimes Y) &= \beta_{H(I)\triangleleft M}(X, Y) \circ \beta_{H(I)}(M, X \otimes Y) \\ &= [\beta_{H(I)}(M, X) \triangleleft \mathrm{id}_Y] \circ \beta_{H(I)}(M \otimes X, Y) \\ &= [t'_M(X) \triangleleft \mathrm{id}_Y] \circ t'_{M\otimes X} \end{aligned}$$

for all $X, Y, M \in \mathcal{C}$, and

$$t'_M(I) = \beta_{H(I)}(M, I) = \rho^{-1}_{H(I)\triangleleft M} = \rho^{-1}_{\mathcal{L}_H(M)},$$

showing that $(\mathcal{L}_H, t')$ is a functor of right $\mathcal{C}$-module categories. Finally, we have to show the diagram



commutes for all $X, Y, M \in \mathcal{C}$. This follows from

$$[\mathrm{id}_X \triangleright t'_M(Y)] \circ s'_{M\otimes Y}(X) = [\mathrm{id}_X \triangleright \beta_{H(I)}(M, Y)] \circ \gamma_{H(I)}(X, M \otimes Y) \circ [s_I(X) \triangleleft \mathrm{id}_{M\otimes Y}]$$

$$\begin{aligned}
&\circ [t_I(X)^{-1} \lhd \mathrm{id}_{M\otimes Y}] \circ \beta_{H(I)}(X, M \otimes Y)\\
&= \gamma_{H(I)\lhd M}(X,Y) \circ [\gamma_{H(I)}(X,M) \lhd \mathrm{id}_Y] \circ \beta_{X \rhd H(I)}(M,Y)\\
&\quad \circ [s_I(X) \lhd \mathrm{id}_{M\otimes Y}] \circ [t_I(X)^{-1} \lhd \mathrm{id}_{M\otimes Y}] \circ \beta_{H(I)}(X, M\otimes Y)\\
&= \gamma_{H(I)\lhd M}(X,Y) \circ [\gamma_{H(I)}(X,M) \lhd \mathrm{id}_Y] \circ [(s_I(X) \lhd \mathrm{id}_M) \lhd \mathrm{id}_Y]\\
&\quad \circ [(t_I(X)^{-1} \lhd \mathrm{id}_M) \lhd \mathrm{id}_Y] \circ \beta_{H(I)\lhd X}(M,Y) \circ \beta_{H(I)}(X, M\otimes Y)\\
&= \gamma_{H(I)\lhd M}(X,Y) \circ [\gamma_{H(I)}(X,M) \lhd \mathrm{id}_Y] \circ [(s_I(X) \lhd \mathrm{id}_M) \lhd \mathrm{id}_Y]\\
&\quad \circ [(t_I(X)^{-1} \lhd \mathrm{id}_M) \lhd \mathrm{id}_Y] \circ [\beta_{H(I)}(X,M) \lhd \mathrm{id}_Y] \circ \beta_{H(I)}(X\otimes M, Y)\\
&= \gamma_{H(I)\lhd M}(X,Y) \circ [s'_M(X) \lhd \mathrm{id}_Y] \circ t'_{X\otimes M}(Y).
\end{aligned}$$

We will now prove the second statement in the lemma. Naturality of $\varphi^{(H,s,t)}$ follows directly from naturality of $t_I(-)$, and we also know that $\varphi_M^{(H,s,t)}$ is an isomorphism for each $M \in \mathcal{C}$, so $\varphi^{(H,s,t)} : H \to \mathcal{L}_H$ is a natural isomorphism. To see that the diagram

$$\begin{array}{ccc}
H(X\otimes M) & \xrightarrow{\varphi^{(H,s,t)}_{X\otimes M} = t_I(X\otimes M)} & H(I) \lhd (X \otimes M)\\
\downarrow{\scriptstyle s_M(X)} & & \downarrow{\scriptstyle s'_M(X)}\\
X \rhd H(M) & \xrightarrow{\mathrm{id}_X \rhd \varphi^{(H,s,t)}_M = \mathrm{id}_X \rhd t_I(M)} & X \rhd (H(I) \lhd M)
\end{array} \tag{2.6.10}$$

commutes for all $X, M \in \mathcal{C}$, we compute

$$\begin{aligned}
&s'_M(X) \circ \varphi^{(H,s,t)}_{X\otimes M}\\
&= \gamma_{H(I)}(X,M) \circ [s_I(X) \lhd \mathrm{id}_M] \circ [t_I(X)^{-1} \lhd \mathrm{id}_M] \circ \beta_{H(I)}(X,M) \circ t_I(X \otimes M)\\
&= \gamma_{H(I)}(X,M) \circ [s_I(X) \lhd \mathrm{id}_M] \circ t_X(M) = [\mathrm{id}_X \rhd t_I(M)] \circ s_{I\otimes M}(X)\\
&= [\mathrm{id}_X \rhd \varphi^{(H,s,t)}_M] \circ s_M(X).
\end{aligned}$$

We will now show that the diagram

$$\begin{array}{ccc}
H(M\otimes X) & \xrightarrow{\varphi^{(H,s,t)}_{M\otimes X} = t_I(M\otimes X)} & H(I) \lhd (M \otimes X)\\
\downarrow{\scriptstyle t_M(X)} & & \downarrow{\scriptstyle t'_M(X) = \beta_{H(I)}(M,X)}\\
H(M) \lhd X & \xrightarrow{\varphi^{(H,s,t)}_M \lhd \mathrm{id}_X = t_I(M) \lhd \mathrm{id}_X} & (H(I) \lhd M) \lhd X
\end{array} \tag{2.6.11}$$

commutes for all $X, M \in \mathcal{C}$. This follows from

$$t'_M(X) \circ \varphi^{(H,s,t)}_{M\otimes X} = \beta_{H(I)}(M,X) \circ t_I(M \otimes X) = [t_I(M) \lhd \mathrm{id}_X] \circ t_{I\otimes M}(X) = [\varphi^{(H,s,t)}_M \lhd \mathrm{id}_X] \circ t_M(X).$$

This finishes the proof of the lemma.
□

The essence of the lemma is that we can replace any bimodule functor $(H, s, t) : \mathcal{C} \to \mathcal{C}'$ by the bimodule functor $(\mathcal{L}_H, s', t')$, up to a natural bimodule isomorphism. The next lemma shows that the assignment $(H, s, t) \mapsto (\mathcal{L}_H, s', t')$ can be made into a functor from the category of bimodule functors $\mathcal{C} \to \mathcal{C}'$ to itself.

Lemma 2.6.8 *We obtain a functor $T : \mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C}, \mathcal{C}') \to \mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C}, \mathcal{C}')$ by defining*

$$T[(H, s, t)] := (\mathcal{L}_H, s', t')$$

on the objects and by defining for $\sigma : (H_1, s^1, t^1) \to (H_2, s^2, t^2)$ *the morphism* $T(\sigma) : (\mathcal{L}_{H_1}, (s^1)', (t^1)') \to (\mathcal{L}_{H_2}, (s^2)', (t^2)')$ *by*

$$T(\sigma)_M := (t^2)_I(M) \circ \sigma_M \circ (t^1)_I(M)^{-1}.$$

Proof. Let $\sigma : (H_1, s^1, t^1) \to (H_2, s^2, t^2)$. We will first show that

$$T(\sigma) \in \mathrm{Hom}_{\mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C},\mathcal{C}')}((\mathcal{L}_{H_1}, (s^1)', (t^1)'), (\mathcal{L}_{H_2}, (s^2)', (t^2)')). \tag{2.6.12}$$

To prove naturality of $T(\sigma) : \mathcal{L}_{H_1} \to \mathcal{L}_{H_2}$, let $M, N \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ and consider the diagram

$$\begin{array}{ccccccc}
H_1(I) \lhd M & \xrightarrow{t_I^1(M)^{-1}} & H_1(M) & \xrightarrow{\sigma_M} & H_2(M) & \xrightarrow{t_I^2(M)} & H_2(I) \lhd M \\
\big\downarrow{\scriptstyle \mathrm{id}_{H_1(I)} \lhd f} & & \big\downarrow{\scriptstyle H_1(f)} & & \big\downarrow{\scriptstyle H_2(f)} & & \big\downarrow{\scriptstyle \mathrm{id}_{H_2(I)} \lhd f} \\
H_1(I) \lhd N & \xrightarrow{t_I^1(N)^{-1}} & H_1(N) & \xrightarrow{\sigma_N} & H_2(N) & \xrightarrow{t_I^2(N)} & H_2(I) \lhd N.
\end{array}$$

The left and right squares commute by naturality of t, and the middle square commutes by naturality of σ. Hence the outer rectangle commutes as well, showing naturality of $T(\sigma)$. Now consider the diagrams

$$\begin{array}{ccccccc}
H_1(I) \lhd (X \otimes M) & \xrightarrow{t_I^1(X \otimes M)^{-1}} & H_1(X \otimes M) & \xrightarrow{\sigma_{X \otimes M}} & H_2(X \otimes M) & \xrightarrow{t_I^2(X \otimes M)} & H_2(I) \lhd (X \otimes M) \\
\big\downarrow{\scriptstyle (s^1)'_M(X)} & & \big\downarrow{\scriptstyle s_M^1(X)} & & \big\downarrow{\scriptstyle s_M^2(X)} & & \big\downarrow{\scriptstyle (s^2)'_M(X)} \\
X \rhd (H_1(I) \lhd M) & \xrightarrow{\mathrm{id}_X \rhd t_I^1(M)^{-1}} & X \rhd H_1(M) & \xrightarrow{\mathrm{id}_X \rhd \sigma_M} & X \rhd H_2(M) & \xrightarrow{\mathrm{id}_X \rhd t_I^2(M)} & X \rhd (H_2(I) \lhd M)
\end{array}$$

and

$$\begin{array}{ccccccc}
H_1(I) \lhd (M \otimes X) & \xrightarrow{t_I^1(M \otimes X)^{-1}} & H_1(M \otimes X) & \xrightarrow{\sigma_{M \otimes X}} & H_2(M \otimes X) & \xrightarrow{t_I^2(M \otimes X)} & H_2(I) \lhd (M \otimes X) \\
\big\downarrow{\scriptstyle (t^1)'_M(X)} & & \big\downarrow{\scriptstyle t_M^1(X)} & & \big\downarrow{\scriptstyle t_M^2(X)} & & \big\downarrow{\scriptstyle (t^2)'_M(X)} \\
(H(I) \lhd M) \lhd X & \xrightarrow{t_I^1(M)^{-1} \lhd \mathrm{id}_X} & H_1(M) \lhd X & \xrightarrow{\sigma_M \lhd \mathrm{id}_X} & H_2(M) \lhd X & \xrightarrow{t_I^2(M) \lhd \mathrm{id}_X} & (H_2(I) \lhd M).
\end{array}$$

In both diagrams, the middle square commutes because σ is a bimodule natural transformation and the other small squares commute by (2.6.10) and (2.6.11). Thus in both diagrams the big outer rectangles commute as well, showing that $T(\sigma)$ is a bimodule natural transformation, i.e. that we have (2.6.12).

Now suppose that $\sigma : (H_1, s^1, t^1) \to (H_2, s^2, t^2)$ and $\tau : (H_2, s^2, t^2) \to (H_3, s^3, t^3)$. Then $\tau \circ \sigma : (H_1, s^1, t^1) \to (H_3, s^3, t^3)$ and hence

$$\begin{aligned}
T(\tau \circ \sigma)_M &= (t^3)_I(M) \circ (\tau \circ \sigma)_M \circ (t^1)_I(M)^{-1} = (t^3)_I(M) \circ \tau_M \circ \sigma_M \circ (t^1)_I(M)^{-1} \\
&= (t^3)_I(M) \circ \tau_M \circ (t^2)_I(M)^{-1} \circ (t^2)_I(M) \circ \sigma_M \circ (t^1)_I(M)^{-1} \\
&= T(\tau)_M \circ T(\sigma)_M = [T(\tau) \circ T(\sigma)]_M,
\end{aligned}$$

so T is indeed a functor.
□

Now that we have seen that we have a functor $T : \mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C},\mathcal{C}') \to \mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C},\mathcal{C}')$, we will show that this functor is naturally isomorphic to the identity functor on $\mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C},\mathcal{C}')$. Fortunately, the proof is easy and short.

Lemma 2.6.9 *We have a natural isomorphism* $\varphi : \mathrm{id}_{\mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C},\mathcal{C}')} \to T$ *of functors* $\mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C},\mathcal{C}') \to \mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C},\mathcal{C}')$*, where*

$$\varphi^{(H,s,t)} : (H,s,t) \to T[(H,s,t)] = (\mathcal{L}_H, s', t')$$

is as defined in Lemma 2.6.7 above[9].

Proof. Let $(H_1, s^1, t^1), (H_2, s^2, t^2) \in \mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C},\mathcal{C}')$ and let $\sigma \in \mathrm{Hom}_{\mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C},\mathcal{C}')}((H_1, s^1, t^1), (H_2, s^2, t^2))$. We must show that the square

$$\begin{array}{ccc} (H_1, s^1, t^1) & \xrightarrow{\varphi^{(H_1,s^1,t^1)}} & T[(H_1, s^1, t^1)] \\ \sigma \big\downarrow & & \big\downarrow T(\sigma) \\ (H_2, s^2, t^2) & \xrightarrow{\varphi^{(H_2,s^2,t^2)}} & T[(H_2, s^2, t^2)] \end{array}$$

commutes. But this follows from the fact that for each $M \in \mathcal{C}$ we have

$$T(\sigma)_M \circ \varphi_M^{(H_1,s^1,t^1)} = t_I^2(M) \circ \sigma_M \circ t_I^1(M)^{-1} \circ t_I^1(M) = t_I^2(M) \circ \sigma_M = \varphi_M^{(H_2,s^2,t^2)} \circ \sigma_M.$$

□

For future reference, we will now summarize our results in the following theorem, specialized to the case where the $\mathcal{C}$-bimodule structure is strict.

Theorem 2.6.10 *Let* $(\mathcal{C}, \otimes, I)$ *be a strict tensor category, also considered as a strict* $\mathcal{C}$*-bimodule category* $(\mathcal{C}, \otimes, \otimes)$ *which will be denoted by* ${}^{\otimes}\mathcal{C}^{\otimes}$ *for simplicity, and suppose that we have some structure* $(\mathcal{C}, \triangleright, \triangleleft)$ *of a strict* $\mathcal{C}$*-bimodule category on* $\mathcal{C}$*; we will refer to this* $\mathcal{C}$*-bimodule category as* ${}^{\triangleright}\mathcal{C}^{\triangleleft}$.

(1) We obtain a functor $T : \mathrm{Fun}_{(\mathcal{C},\mathcal{C})}({}^{\otimes}\mathcal{C}^{\otimes}, {}^{\triangleright}\mathcal{C}^{\triangleleft}) \to \mathrm{Fun}_{(\mathcal{C},\mathcal{C})}({}^{\otimes}\mathcal{C}^{\otimes}, {}^{\triangleright}\mathcal{C}^{\triangleleft})$ *by defining*

$$T[(H,s,t)] := (\mathcal{L}_H, s', t')$$

for $(H,s,t) \in \mathrm{Fun}_{(\mathcal{C},\mathcal{C})}({}^{\otimes}\mathcal{C}^{\otimes}, {}^{\triangleright}\mathcal{C}^{\triangleleft})$*, where*

$$s'_M(X) := [s_I(X) \triangleleft \mathrm{id}_M] \circ [t_I(X)^{-1} \triangleleft \mathrm{id}_M]$$

$$t'_M(X) := \mathrm{id}$$

and by defining $T(\sigma)$ *by*

$$T(\sigma)_M := t_I^2(M) \circ \sigma_M \circ t_I^1(M)^{-1}$$

for $\sigma : (H_1, s^1, t^1) \to (H_2, s^2, t^2)$.

(2) We have a natural isomorphism

$$\varphi : \mathrm{id}_{\mathrm{Fun}_{(\mathcal{C},\mathcal{C})}({}^{\otimes}\mathcal{C}^{\otimes}, {}^{\triangleright}\mathcal{C}^{\triangleleft})} \to T,$$

where $\varphi^{(H,s,t)} : (H,s,t) \to T[(H,s,t)] = (\mathcal{L}_H, s', t')$ *is given by*

$$\varphi_M^{(H,s,t)} := t_I(M) : H(M) \to H(I) \triangleleft M.$$

[9] Usually our convention is to write lower indices for natural transformations. However, in our computations above it was very convenient to use an upper index for φ rather than a lower index, because otherwise we would have to write expressions like $\left(\varphi_{(H,s,t)}\right)_M$ instead of $\varphi_M^{(H,s,t)}$.

2.7 Linear categories

A category $\mathcal{C}$ is called an *Ab-category* (or *pre-additive category*) if $\mathrm{Hom}_{\mathcal{C}}(V, W)$ is an abelian group for all $V, W \in \mathcal{C}$ and if the composition is bi-additive. We use the term 'bi-additive' here because we will always write the abelian groups $\mathrm{Hom}_{\mathcal{C}}(V, W)$ additively and we will denote the neutral elements in these groups by 0.

Remark 2.7.1 Recall the definition of the disjoint union $\bigsqcup_{\alpha \in A} \mathcal{C}_\alpha$ of categories $\{\mathcal{C}_\alpha\}_{\alpha \in A}$ as introduced in (2.1.1). In case all $\mathcal{C}_\alpha$ are Ab-categories, we replace the empty set $\emptyset$ in the definition of the disjoint union with $\{0\}$. This assures that the disjoint union is also an Ab-category in this case. The reader should keep this convention in mind, especially in Chapter 4 where we will define the category $Z_G(\mathcal{C})$ as a disjoint union.

If $V, V_1, V_2 \in \mathcal{C}$ are objects in an Ab-category, then V is called a *direct sum* of V_1 and V_2, written $V \cong V_1 \oplus V_2$, if there exist morphisms $f_j : V_j \to V$ and $f'_j : V \to V_j$ such that $f'_j \circ f_j = \mathrm{id}_{V_j}$ and $f_1 \circ f'_1 + f_2 \circ f'_2 = \mathrm{id}_V$, where $j \in \{1, 2\}$. Consequently, $f'_j \circ f_k = 0$ if $j \neq k$. An Ab-category $\mathcal{C}$ is said to *have direct sums* if for any $V_1, V_2 \in \mathcal{C}$ there exists a direct sum $V \cong V_1 \oplus V_2$.

The following lemma shows that direct sums in a tensor category behave nicely with respect to duals. The proof is a straightforward computation.

Lemma 2.7.2 *Let $\mathcal{C}$ be a tensor category that has direct sums. Suppose that $V_1, V_2 \in \mathcal{C}$ have two-sided duals $(\overline{V_1}, b_1, d_1, b'_1, d'_1)$ and $(\overline{V_2}, b_2, d_2, b'_2, d'_2)$, respectively, and let $V \cong V_1 \oplus V_2$ and $\overline{V} \cong \overline{V_1} \oplus \overline{V_2}$ be direct sums implemented by morphisms $f_j \in \mathrm{Hom}_{\mathcal{C}}(V_j, V)$, $f'_j \in \mathrm{Hom}_{\mathcal{C}}(V, V_j)$ and $g_j \in \mathrm{Hom}_{\mathcal{C}}(\overline{V_j}, \overline{V})$, $g'_j \in \mathrm{Hom}_{\mathcal{C}}(\overline{V}, \overline{V_j})$, respectively. Then $(\overline{V}, b, d, b', d')$ is a two-sided dual for V with*

$$\begin{aligned} b &= [f_1 \otimes g_1] \circ b_1 + [f_2 \otimes g_2] \circ b_2 \\ d &= d_1 \circ [g_1 \otimes f_1] + d_2 \circ [g_2 \otimes f_2] \\ b' &= [g_1 \otimes f_1] \circ b'_1 + [g_2 \otimes f_2] \circ b'_2 \\ d' &= d'_1 \circ [f_1 \otimes g_1] + d'_2 \circ [f_2 \otimes g_2]. \end{aligned}$$

If $\mathbb{F}$ is a field, then a category $\mathcal{C}$ is called $\mathbb{F}$-*linear* if $\mathrm{Hom}_{\mathcal{C}}(V, W)$ is an $\mathbb{F}$-vector space for all $V, W \in \mathcal{C}$ and if the composition is $\mathbb{F}$-bilinear. If V is an object in an $\mathbb{F}$-linear category $\mathcal{C}$ then V is called *irreducible* if $\mathrm{End}_{\mathcal{C}}(V) = \mathbb{F} \cdot \mathrm{id}_V$. Note that if I is irreducible and $\mathcal{C}$ is spherical, then the dimension $d(V) \in \mathrm{End}_{\mathcal{C}}(I) = \mathbb{F} \cdot \mathrm{id}_I$ of any object can be considered as an element in $\mathbb{F}$. An $\mathbb{F}$-linear category is called *rational* if it has finitely many isomorphism classes of irreducible objects.

Definition 2.7.3 An $\mathbb{F}$-linear category $\mathcal{C}$ is called *semisimple* if
- it has direct sums;
- idempotents split[10];
- any object is a finite direct sum of irreducible objects.

Definition 2.7.4 An $\mathbb{F}$-linear tensor category[11] $\mathcal{C}$ is called a *fusion category* (over the field $\mathbb{F}$) if
- it is semisimple;
- it is rational;
- every object has a two-sided dual;
- $\mathrm{Hom}_{\mathcal{C}}(V, W)$ is finite-dimensional for every $V, W \in \mathcal{C}$;
- its unit object is irreducible.

[10] A category is said to have split idempotents if for any idempotent $p \in \mathrm{End}_{\mathcal{C}}(V)$ in the category $\mathcal{C}$ there exists an object $U \in \mathcal{C}$ together with morphisms $i : U \to V$ and $r : V \to U$ with $r \circ i = \mathrm{id}_U$ and $p = i \circ r$. In Subsection 2.7.4 we will consider this concept in detail.

[11] This means that $\mathcal{C}$ is a $\mathbb{F}$-linear category that is also a tensor category, satisfying the additional property that the tensor product is $\mathbb{F}$-bilinear on the morphisms.

If $\mathcal{C}$ is a fusion category over the field $\mathbb{F}$ and if $X \in \mathcal{C}$ is irreducible, then we can define the *square dimension* $d^2(X)$ of X by

$$d^2(X) := d_L(X) \cdot d_R(X) = (d'_X \circ b_X) \cdot (d_X \circ b'_X).$$

It is easily checked that the square dimension of an irreducible object is independent of the choice of the two-sided dual and is therefore well-defined. If two irreducible objects are isomorphic, then their square dimensions are equal. Thus for each fusion category $\mathcal{C}$ we can define its *dimension* by

$$\dim(\mathcal{C}) := \sum_i d^2(X_i),$$

where $\{X_i\}_i$ is a complete set of representatives of isomorphism classes of irreducible objects of $\mathcal{C}$. In case $\mathcal{C}$ is a spherical fusion category and $X \in \mathcal{C}$ is irreducible, we have $d^2(X) = d(X)^2$ and hence $\dim(\mathcal{C}) = \sum_i d(X_i)^2$.

A braided fusion category (over a field $\mathbb{F}$) that is a ribbon category is called a *pre-modular tensor category.* If $\mathcal{C}$ is a pre-modular category and if $V, W \in \mathcal{C}$, then we define

$$S(V,W) := [d_V \otimes d'_W] \circ [\mathrm{id}_{\overline{V}} \otimes c^M_{V,W} \otimes \mathrm{id}_{\overline{W}}] \circ [b'_V \otimes b_W],$$

which is an element in $\mathrm{End}_{\mathcal{C}}(I) = \mathbb{F} \cdot \mathrm{id}_I$ that only depends on the equivalence classes of V and W. If $\{V_i\}_i$ is a complete set of representatives of equivalence classes of irreducible objects in $\mathcal{C}$, then we define the matrix $S_{ij} := S(V_i, V_j)$. If this matrix is invertible, then $\mathcal{C}$ is called a *modular tensor category.* It can be shown that a pre-modular tensor category is modular if and only if its only degenerate objects are isomorphic to $I^{\oplus n}$ for some n.

2.7.1 $*$-categories

Definition 2.7.5 Let $\mathcal{C}$ be a $\mathbb{C}$-linear category. Then a *positive $*$-operation* on $\mathcal{C}$ is a family of conjugate-linear maps $\{* : \mathrm{Hom}_{\mathcal{C}}(V,W) \to \mathrm{Hom}_{\mathcal{C}}(W,V)\}_{V,W \in \mathcal{C}}$, denoted by $f \mapsto f^*$, satisfying

- $f^{**} := (f^*)^* = f$ for any $f \in \mathrm{Hom}_{\mathcal{C}}(V,W)$ and $V, W \in \mathcal{C}$;
- $(g \circ f)^* = f^* \circ g^*$ for any composable $f, g \in \mathrm{Hom}(\mathcal{C})$;
- $f^* \circ f = 0$ implies $f = 0$ if $f \in \mathrm{Hom}(\mathcal{C})$.

A category equipped with a positive $*$-operation is called a *$*$-category.*

Let $\mathcal{C}$ be a $*$-category and let $f \in \mathrm{Hom}_{\mathcal{C}}(V,W)$ with $V, W \in \mathcal{C}$. Then f is called an *isometry* if $f^* \circ f = \mathrm{id}_V$, f is called a *co-isometry* if $f \circ f^* = \mathrm{id}_W$ and f is called *unitary* if it is both an isometry and a co-isometry. In case $V = W$, then f is called *self-adjoint* if $f = f^*$ and f is called a *projection* if $f = f^* = f^2$. A $*$-category $\mathcal{C}$ is said to *have subobjects* if for any $V \in \mathcal{C}$ and any projection $p \in \mathrm{End}_{\mathcal{C}}(V)$ there exists a $U \in \mathcal{C}$ together with an isometry $u \in \mathrm{Hom}_{\mathcal{C}}(U,V)$ such that $u \circ u^* = p$. Direct sums in $*$-categories are defined in the same way as in the more general $\mathbb{F}$-linear categories, except that we now demand that the f_j are isometries u_j and that f'_j is replaced by u_j^*. We say that a $*$-category *has direct sums* if for every two objects V_1 and V_2 there exists a direct sum $V \cong V_1 \oplus V_2$.

If V is an irreducible object in a $*$-category $\mathcal{C}$, then for any $W \in \mathcal{C}$ the vector space $\mathrm{Hom}_{\mathcal{C}}(V,W)$ is an inner product space with inner product defined by $\langle f, g \rangle \cdot \mathrm{id}_V = g^* \circ f$.

A tensor category $\mathcal{C}$ that is also a $*$-category is called a *tensor $*$-category* if $(f \otimes g)^* = f^* \otimes g^*$ for all $f, g \in \mathrm{Hom}(\mathcal{C})$. If V is an object in a tensor $*$-category and if (W, b, d) is a left dual of V, then (W, d^*, b^*) is a right dual of V; similarly, if (W, b', d') is a right dual of V, then (W, d'^*, b'^*) is a left dual of V. Thus, any one-sided duality in a tensor $*$-category is automatically a two-sided duality. For this reason, the notion of duality is defined somewhat more symmetrically in a tensor $*$-category:

Definition 2.7.6 Let $\mathcal{C}$ be a tensor $*$-category and let $V \in \mathcal{C}$. Then a *conjugate* of V is a triple $(\overline{V}, r, \overline{r})$, where $\overline{V} \in \mathcal{C}$ and $r : I \to \overline{V} \otimes V$ and $\overline{r} : I \to V \otimes \overline{V}$ are morphisms satisfying

$$[\mathrm{id}_V \otimes r^*] \circ [\overline{r} \otimes \mathrm{id}_V] = \mathrm{id}_V$$

$$[\mathrm{id}_{\overline{V}} \otimes \overline{r}^*] \circ [r \otimes \mathrm{id}_{\overline{V}}] = \mathrm{id}_{\overline{V}}.$$

We say that $\mathcal{C}$ *has conjugates* if every object has a conjugate.

The following lemma demonstrates that conjugates are uniquely determined up to isomorphism. This result will be important to us later.

Lemma 2.7.7 *Let $\mathcal{C}$ be a tensor $*$-category and let $(W_1, r_1, \overline{r}_1)$ be a conjugate for $V \in \mathcal{C}$.*

(1) If $W_2 \in \mathcal{C}$ and $f : W_1 \to W_2$ is an isomorphism, then $(W_2, r_2, \overline{r}_2)$ is also a conjugate for V, where $r_2 := [f \otimes \mathrm{id}_V] \circ r_1$ and $\overline{r}_2 := [\mathrm{id}_V \otimes (f^{-1})^*] \circ \overline{r}_1$.

(2) If $(W_2, r_2, \overline{r}_2)$ is a conjugate for V, then there exists an isomorphism $f : W_1 \to W_2$ such that $r_2 := [f \otimes \mathrm{id}_V] \circ r_1$ and $\overline{r}_2 := [\mathrm{id}_V \otimes (f^{-1})^*] \circ \overline{r}_1$.

In fact, $(W_2, f) \mapsto (W_2, [f \otimes \mathrm{id}_V] \circ r_1, [\mathrm{id}_V \otimes (f^{-1})^*] \circ \overline{r}_1)$ is a bijection with inverse given by $(W_2, r_2, \overline{r}_2) \mapsto (W_2, [r_1^* \otimes \mathrm{id}_V] \circ [\mathrm{id}_{W_1} \otimes \overline{r}_2])$.

Proof. The proof of part (1) is a straightforward computation. For part (2) we note that $f : W_1 \to W_2$ is defined by $f := [r_1^* \otimes \mathrm{id}_V] \circ [\mathrm{id}_{W_1} \otimes \overline{r}_2]$, which is indeed an isomorphism with inverse given by $f^{-1} = [\mathrm{id}_{W_1} \otimes \overline{r}_2^*] \circ [r_1 \otimes \mathrm{id}_{W_2}]$. Indeed,

$$\begin{aligned} r_2 &= [\mathrm{id}_{W_2} \otimes \overline{r}_1^* \otimes \mathrm{id}_V] \circ [\mathrm{id}_{W_2 \otimes V} \otimes r_1] \circ r_2 = [\mathrm{id}_{W_2} \otimes \overline{r}_1^* \otimes \mathrm{id}_V] \circ [r_2 \otimes \mathrm{id}_{W_1 \otimes V}] \circ r_1 \\ &= [f \otimes \mathrm{id}_V] \circ r_1 \\ \overline{r}_2 &= [\mathrm{id}_V \otimes r_1^* \otimes \mathrm{id}_{W_2}] \circ [\overline{r}_1 \otimes \mathrm{id}_{V \otimes W_2}] \circ \overline{r}_2 = [\mathrm{id}_V \otimes r_1^* \otimes \mathrm{id}_{W_2}] \circ [\mathrm{id}_{V \otimes W_1} \otimes \overline{r}_2] \circ \overline{r}_1 \\ &= [\mathrm{id}_V \otimes (f^{-1})^*]. \end{aligned}$$

□

We now introduce a class of categories that is extremely important in algebraic quantum field theory, because the category of the so-called DHR endomorphisms of a quantum field theory belongs automatically to this class.

Definition 2.7.8 A tensor $*$-category is called a TC^* if

- it has direct sums, subobjects and conjugates;
- $\mathrm{Hom}_{\mathcal{C}}(V, W)$ is finite-dimensional for all $V, W \in \mathcal{C}$;
- its unit object is irreducible.

A TC^* will be called a BTC^* if it has a unitary braiding and a BTC^* is called an STC^* if the braiding is a symmetry.

We mention that a TC^* is automatically semisimple. Consequently, a rational $(B)TC^*$ is a (braided) fusion category.

2.7.2 Completion with respect to direct sums

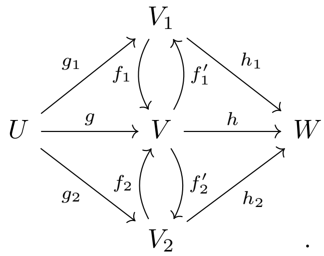
We now wish to extend an Ab-category in such a way that it has direct sums. To motivate this extension, we first make the following observation. Suppose that $\mathcal{C}$ is an Ab-category and that we are given a direct sum $V \cong V_1 \oplus V_2$ with corresponding morphisms f_j and f'_j. If $U \in \mathcal{C}$ then we will write $\mathrm{Hom}_{\mathcal{C}}(U, V_1) \oplus \mathrm{Hom}_{\mathcal{C}}(U, V_2)$ to denote the direct product of the two groups $\mathrm{Hom}_{\mathcal{C}}(U, V_j)$, i.e. the set of elements in $\mathrm{Hom}_{\mathcal{C}}(U, V_1) \times \mathrm{Hom}_{\mathcal{C}}(U, V_2)$ with componentwise addition. Similarly, if $W \in \mathcal{C}$ then we also define $\mathrm{Hom}_{\mathcal{C}}(V_1, W) \oplus \mathrm{Hom}_{\mathcal{C}}(V_2, W)$. Now fix some $U, W \in \mathcal{C}$. Then it is easy to check that the assignment

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(U, V) &\to \mathrm{Hom}_{\mathcal{C}}(U, V_1) \oplus \mathrm{Hom}_{\mathcal{C}}(U, V_2) \\ g &\mapsto (f'_1 \circ g, f'_2 \circ g) \end{aligned}$$

is an additive bijection with inverse given by $(g_1, g_2) \mapsto f_1 \circ g_1 + f_2 \circ g_2$. Similarly, the assignment

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(V, W) &\to \mathrm{Hom}_{\mathcal{C}}(V_1, W) \oplus \mathrm{Hom}_{\mathcal{C}}(V_2, W) \\ h &\mapsto (h \circ f_1, h \circ f_2) \end{aligned}$$

is also an additive bijection with inverse given by $(h_1, h_2) \mapsto h_1 \circ f_1' + h_2 \circ f_2'$.



These two bijections show how the morphisms to and from a direct sum $V \cong V_1 \oplus V_2$ are related to the morphisms to and from V_1 and V_2. This will now be used to extend an Ab-category in such a way that it has direct sums.

Let $\mathcal{C}$ be an Ab-category. We then define a new category $\mathcal{C}^{\oplus}$ as follows. The objects of $\mathcal{C}^{\oplus}$ are finite sequences of objects in $\mathcal{C}$, i.e. $\mathrm{Obj}(\mathcal{C}^{\oplus}) = \{(U_1, \ldots, U_n) \in \mathcal{C}^{\times n}, n \in \mathbb{Z}_{\geq 1}\}$. The morphisms in $\mathcal{C}^{\oplus}$ are matrices with entries that are morphisms in $\mathcal{C}$. More precisely, if $(U_1, \ldots, U_l), (V_1, \ldots, V_m) \in \mathcal{C}^{\oplus}$ then we define

$$\mathrm{Hom}_{\mathcal{C}^{\oplus}}((U_1, \ldots, U_l), (V_1, \ldots, V_m)) = \{F = (F_{ij}) : F_{ij} \in \mathrm{Hom}_{\mathcal{C}}(U_j, V_i)\},$$

where of course $1 \leq i \leq m$ and $1 \leq j \leq l$, so F is an $m \times l$-matrix. If $F \in \mathrm{Hom}_{\mathcal{C}^{\oplus}}((U_1, \ldots, U_l), (V_1, \ldots, V_m))$ and $G \in \mathrm{Hom}_{\mathcal{C}^{\oplus}}((V_1, \ldots, V_m), (W_1, \ldots, W_n))$, then we define the composition $G \circ F$ by

$$(G \circ F)_{ij} := \sum_{k=1}^{m} G_{ik} \circ F_{kj}.$$

Because both $\circ$ and matrix multiplication are associative, the composition in $\mathcal{C}^{\oplus}$ is associative. If $(U_1, \ldots, U_l) \in \mathcal{C}^{\oplus}$ then we define $\mathrm{id}_{(U_1, \ldots, U_l)}$ to be the diagonal matrix

$$\mathrm{id}_{(U_1, \ldots, U_l)} = \mathrm{diag}(\mathrm{id}_{U_1}, \ldots, \mathrm{id}_{U_l}),$$

which obviously serves as the identity morphism of $(U_1, \ldots, U_l)$. Thus $\mathcal{C}^{\oplus}$ is a category and we can identify $\mathcal{C}$ with the full subcategory of $\mathcal{C}^{\oplus}$ determined by the objects that are 1-tuples. It is clear that $\mathcal{C}^{\oplus}$ is again an Ab-category with addition of morphisms defined by matrix addition. We will now show that $\mathcal{C}^{\oplus}$ has direct sums. Let $U = (U_1, \ldots, U_l), V = (V_1, \ldots, V_m) \in \mathcal{C}^{\oplus}$ and consider $W := (U_1, \ldots, U_l, V_1, \ldots, V_m) \in \mathcal{C}^{\oplus}$. We then define morphisms $F \in \mathrm{Hom}_{\mathcal{C}^{\oplus}}(U, W)$, $F' \in \mathrm{Hom}_{\mathcal{C}^{\oplus}}(W, U)$, $G \in \mathrm{Hom}_{\mathcal{C}^{\oplus}}(V, W)$ and $G' \in \mathrm{Hom}_{\mathcal{C}^{\oplus}}(W, V)$ by

$$F := \begin{pmatrix} \mathrm{id}_{(U_1, \ldots, U_l)} \\ 0_{m \times l} \end{pmatrix}, \quad F' := \begin{pmatrix} \mathrm{id}_{(U_1, \ldots, U_l)} & 0_{l \times m} \end{pmatrix}, \quad G := \begin{pmatrix} 0_{l \times m} \\ \mathrm{id}_{(V_1, \ldots, V_m)} \end{pmatrix}, \quad G' := \begin{pmatrix} 0_{m \times l} & \mathrm{id}_{(V_1, \ldots, V_m)} \end{pmatrix}.$$

Then $F' \circ F = \mathrm{id}_{(U_1, \ldots, U_l)}$, $G' \circ G = \mathrm{id}_{(V_1, \ldots, V_m)}$ and

$$F \circ F' + G \circ G' = \begin{pmatrix} \mathrm{id}_{(U_1, \ldots, U_l)} & 0_{l \times m} \\ 0_{m \times l} & 0_{m \times m} \end{pmatrix} + \begin{pmatrix} 0_{l \times l} & 0_{l \times m} \\ 0_{m \times l} & \mathrm{id}_{(V_1, \ldots, V_m)} \end{pmatrix} = \mathrm{id}_W,$$

showing that these morphisms make W into a direct sum of U and V. Thus $\mathcal{C}^{\oplus}$ indeed has direct sums. Note that we can now interpret any object $U = (U_1, \ldots, U_l) \in \mathcal{C}^{\oplus}$ as a direct sum $U \cong U_1 \oplus \ldots \oplus U_l$ of the

$U_j \in \mathcal{C}$. It is clear that if $\mathcal{C}$ is $\mathbb{F}$-linear, then so is $\mathcal{C}^{\oplus}$. In case $\mathcal{C}$ is a strict tensor category, $\mathcal{C}^{\oplus}$ can be made into a strict tensor category as follows. On the objects we define

$$(U_1, \ldots, U_l) \otimes (V_1, \ldots, V_m) = (U_1 \otimes V_1, \ldots, U_1 \otimes V_m, U_2 \otimes V_1, \ldots, U_l \otimes V_m),$$

i.e. we use the lexicographic ordering of the indices i and j in order to determine the order of the $U_i \otimes V_j$. We can rewrite this equation as

$$((U_1, \ldots, U_l) \otimes (V_1, \ldots, V_m))_{(i,j)} = U_i \otimes V_j,$$

where the set of double indices (i, j) is ordered (so the left-hand side describes a well-defined object of $\mathcal{C}^{\oplus}$, i.e. it describes an ordered $l \times m$-tuple of objects in $\mathcal{C}$). Now let $F \in \mathrm{Hom}_{\mathcal{C}^{\oplus}}((U_1, \ldots, U_l), (U'_1, \ldots, U'_{l'}))$ and $G \in \mathrm{Hom}_{\mathcal{C}^{\oplus}}((V_1, \ldots, V_m), (V'_1, \ldots, V'_{m'}))$. Then we define their tensor product by

$$(F \otimes G)_{(i',j'),(i,j)} = F_{i'i} \otimes G_{j'j}.$$

The unit object is simply the 1-tuple (I). If $\mathcal{C}$ also has a $*$-operation making it a tensor $*$-category, we can define a $*$-operation on $\mathcal{C}^{\oplus}$ by $(F^*)_{ij} := F^*_{ji}$. In this way $\mathcal{C}^{\oplus}$ also becomes a tensor $*$-category.

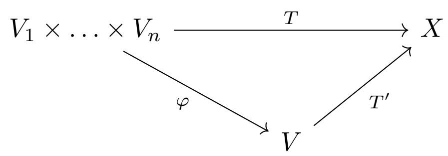
2.7.3 The enriched product of linear categories

Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathbb{F}$-linear categories. We then define an $\mathbb{F}$-linear category $\mathcal{C} \boxtimes_0 \mathcal{D}$ as follows. Its objects are pairs $(V, W) \in \mathcal{C} \times \mathcal{D}$, denoted by $V \boxtimes W$, and the morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}\boxtimes_0\mathcal{D}}(V \boxtimes W, V' \boxtimes W') := \mathrm{Hom}_{\mathcal{C}}(V, V') \boxtimes \mathrm{Hom}_{\mathcal{D}}(W, W'), \tag{2.7.1}$$

where $\boxtimes$ on the right-hand side is the tensor product of vector spaces.

Before we define the composition (and later also the tensor product in case $\mathcal{C}$ and $\mathcal{D}$ are tensor categories) in $\mathcal{C} \boxtimes_0 \mathcal{D}$, it is good to briefly recall the definition of a tensor product of vector spaces. If $(V_1, \ldots, V_n)$ is an ordered n-tuple of $\mathbb{F}$-linear vector spaces, then a *tensor product* of $(V_1, \ldots, V_n)$ is a pair (V, φ) consisting of an $\mathbb{F}$-linear vector space V together with a multilinear map $\varphi : V_1 \times \ldots \times V_n \to V$ such that for any $\mathbb{F}$-linear vector space X and any multilinear map $T : V_1 \times \ldots \times V_n \to X$ there exists a unique linear map $T' : V \to X$ such that $T = T' \circ \varphi$, i.e. any multilinear map from $V_1 \times \ldots \times V_n$ factors uniquely through V.



It can be shown that tensor products exist and are unique up to isomorphism. Once some tensor product (V, φ) has been chosen, it is convenient to denote the vector space V by $V_1 \otimes \ldots \otimes V_n$ and to write $v_1 \otimes \ldots \otimes v_n := \varphi(v_1, \ldots, v_n)$ when $v_j \in V_j$. The vector space $V_1 \otimes \ldots \otimes V_n$ is spanned by the vectors of the form $v_1 \otimes \ldots \otimes v_n$ (these vectors are often called homogeneous). If all V_j are finite-dimensional and if for each j we choose a basis $(e^{(j)}_\alpha)_{\alpha=1}^{\dim(V_j)}$ for V_j, then the vectors $e^{(1)}_{\alpha_1} \otimes \ldots \otimes e^{(n)}_{\alpha_n}$ form a basis for $V_1 \otimes \ldots \otimes V_n$. In particular, $\dim(V_1 \otimes \ldots \otimes V_n) = \dim(V_1)\dim(V_2)\ldots\dim(V_n)$.

The composition in the category $\mathcal{C} \boxtimes_0 \mathcal{D}$ is now defined by

$$(f_2 \boxtimes g_2) \circ (f_1 \boxtimes g_1) := (f_2 \circ f_1) \boxtimes (g_2 \circ g_1)$$

on homogeneous elements and this definition is then extended by linearity to arbitrary elements. This composition is associative and for any $V \boxtimes W \in \mathcal{C} \boxtimes_0 \mathcal{D}$ the morphism $\mathrm{id}_V \boxtimes \mathrm{id}_W$ is the identity morphism

$\mathrm{id}_{V\boxtimes W}$, so $\mathcal{C}\boxtimes_0\mathcal{C}$ is a category that is obviously $\mathbb{F}$-linear. Note that an object in this category is irreducible if and only if it is of the form $V\boxtimes W$ with both V and W irreducible.

If $\mathcal{C}$ and $\mathcal{D}$ are $\mathbb{F}$-linear strict tensor categories, then we can define a tensor product

$$(V_1\boxtimes W_1)\otimes(V_2\boxtimes W_2):=(V_1\otimes V_2)\boxtimes(W_1\otimes W_2)$$

on the objects of $\mathcal{C}\boxtimes_0\mathcal{D}$ and similarly on the morphisms. This makes $\mathcal{C}\boxtimes_0\mathcal{D}$ into a strict tensor category with unit object $I=I_{\mathcal{C}}\boxtimes I_{\mathcal{D}}$. Completing this category with respect to direct sums, we obtain a new category

$$\mathcal{C}\boxtimes\mathcal{D}:=(\mathcal{C}\boxtimes_0\mathcal{D})^{\oplus},$$

which will be called[12] the *enriched product* of $\mathcal{C}$ and $\mathcal{D}$ from now on. Many authors call $\mathcal{C}\boxtimes\mathcal{D}$ the Deligne product of $\mathcal{C}$ and $\mathcal{D}$, but we will not use this terminology.

2.7.4 Splitting idempotents and the Karoubi envelope

Let $\mathcal{C}$ be a category and let $V\in\mathcal{C}$. If there exists a $U\in\mathcal{C}$ together with morphisms $i:U\to V$ and $r:V\to U$ satisfying $r\circ i=\mathrm{id}_U$, then the triple (U,i,r) is called a *retract* of V. Note that $i\circ r\in\mathrm{End}_{\mathcal{C}}(V)$ is automatically an idempotent, since $(i\circ r)\circ(i\circ r)=i\circ(r\circ i)\circ r=i\circ r$. Not every idempotent in a category comes necessarily from a retract in this way, but when it does we will give it a special name:

Definition 2.7.9 If $\mathcal{C}$ is a category, if $V\in\mathcal{C}$ and if $f\in\mathrm{End}_{\mathcal{C}}(V)$, then f is called a *split idempotent* if there exists a retract (U,i,r) of V with $f=i\circ r$. A category in which each idempotent is a split idempotent is said to *have splitting idempotents* or is called a *Karoubian category.*

Now suppose that $\mathcal{C}$ is a Karoubian tensor category, let $p\in\mathrm{End}_{\mathcal{C}}(V)$ be an idempotent and let (U,i,r) be a retract corresponding to p. The following lemma demonstrates that if V has a left or right dual, then so has U. The proof is a straighforward computation.

Lemma 2.7.10 *Let $\mathcal{C}$ be a Karoubian tensor category, let (W,b,d,b',d') be a two-sided dual for $V\in\mathcal{C}$ and let (U,i,r) be a retract of V corresponding to some idempotent $p\in\mathrm{End}_{\mathcal{C}}(V)$. If (W_1,i_1,r_1) and (W_2,i_2,r_2) are retracts of W corresponding to the idempotents*

$$p_1=[d\otimes\mathrm{id}_W]\circ[\mathrm{id}_W\otimes p\otimes\mathrm{id}_W]\circ[\mathrm{id}_W\otimes b]\in\mathrm{End}_{\mathcal{C}}(W)$$
$$p_2=[\mathrm{id}_W\otimes d']\circ[\mathrm{id}_W\otimes p\otimes\mathrm{id}_W]\circ[b'\otimes\mathrm{id}_W]\in\mathrm{End}_{\mathcal{C}}(W),$$

respectively, then (W_1,b,d) is a left dual for U and (W_2,b',d') is a right dual for U with

$$b=[r\otimes r_1]\circ b$$
$$d=d\circ[i_1\otimes i]$$
$$b'=[r_2\otimes r]\circ b'$$
$$d'=d'\circ[i\otimes i_2].$$

If $p_1=p_2$ then we can choose $(W_1,i_1,r_1)=(W_2,i_2,r_2)\equiv(W,i,r)$ and in that case (W,b,d,b',d') is a two-sided dual for U.

There is a canonical way to make a category Karoubian. If $\mathcal{C}$ is a category, then we define its *Karoubi envelope* $\overline{\mathcal{C}}$ to be the category with objects given by

$$\mathrm{Obj}(\overline{\mathcal{C}})=\{(V,e):V\in\mathcal{C},e\in\mathrm{End}_{\mathcal{C}}(V),e\circ e=e\},$$

[12] More precisely, one should call this the product of $\mathcal{C}$ and $\mathcal{D}$ enriched over the category of $\mathbb{F}$-vector spaces. However, since we will not be considering any other types of enriched products, there will be no confusion. See also [53] for more details.

with morphisms given by

$$\begin{aligned}\mathrm{Hom}_{\overline{\mathcal{C}}}((V_1,e_1),(V_2,e_2)) &= \{f \in \mathrm{Hom}_{\mathcal{C}}(V_1,V_2) : e_2 \circ f = f = f \circ e_1\} \\ &= \{f \in \mathrm{Hom}_{\mathcal{C}}(V_1,V_2) : f = e_2 \circ f \circ e_1\} \\ &= e_2 \circ \mathrm{Hom}_{\mathcal{C}}(V_1,V_2) \circ e_1.\end{aligned}$$

and with composition given by the composition in $\mathcal{C}$. To see that $g \circ f \in \mathrm{Hom}_{\overline{\mathcal{C}}}((V_1,e_1),(V_3,e_3))$ whenever $f \in \mathrm{Hom}_{\overline{\mathcal{C}}}((V_1,e_1),(V_2,e_2))$ and $g \in \mathrm{Hom}_{\overline{\mathcal{C}}}((V_2,e_2),(V_3,e_3))$, we notice that

$$e_3 \circ (g \circ f) \circ e_1 = e_3 \circ (e_3 \circ g \circ e_2) \circ (e_2 \circ f \circ e_1) \circ e_1 = (e_3 \circ g \circ e_2) \circ (e_2 \circ f \circ e_1) = g \circ f.$$

The identity morphism on (V,e) is $\mathrm{id}_{(V,e)} = e$. It is straightforward to check that $\overline{\mathcal{C}}$ has splitting idempotents. In fact, if $(V,e) \in \overline{\mathcal{C}}$ and if $p \in \mathrm{End}_{\overline{\mathcal{C}}}((V,e))$ is an idempotent, then $((V,p),p,p)$ is a retract of (V,e) that splits the idempotent p. The functor $\mathscr{I}_K : \mathcal{C} \to \overline{\mathcal{C}}$ defined by $\mathscr{I}_K(V) = (V,\mathrm{id}_V)$ and $\mathscr{I}_K(f) = f$ is fully faithful and is also injective on the objects.

In case $\mathcal{C}$ is a strict tensor category, the Karoubi envelope can also be equipped with the structure of a strict tensor category. Namely, if $(V_1,e_1),(V_2,e_2) \in \overline{\mathcal{C}}$, then we define

$$(V_1,e_1)\overline{\otimes}(V_2,e_2) := (V_1 \otimes V_2, e_1 \otimes e_2),$$

where we have used that $(e_1 \otimes e_2)^2 = {e_1}^2 \otimes {e_2}^2 = e_1 \otimes e_2$ by the interchange law. On the morphisms we simply define $f_1 \overline{\otimes} f_2 := f_1 \otimes f_2$ for $f_1, f_2 \in \mathrm{Hom}(\overline{\mathcal{C}})$. To see that $f_1 \otimes f_2 \in \mathrm{Hom}_{\overline{\mathcal{C}}}((V_1 \otimes V_2, e_1 \otimes e_2),(V_1' \otimes V_2', e_1' \otimes e_2'))$ whenever $f_j \in \mathrm{Hom}_{\overline{\mathcal{C}}}((V_j,e_j),(V_j',e_j'))$ for $j \in \{1,2\}$, we note that

$$\begin{aligned}[f_1 \otimes f_2] \circ [e_1 \otimes e_2] &= [f_1 \circ e_1] \otimes [f_2 \circ e_2] = f_1 \otimes f_2 \\ [e_1' \otimes e_2'] \circ [f_1 \otimes f_2] &= [e_1' \circ f_1] \otimes [e_2' \circ f_2] = f_1 \otimes f_2.\end{aligned}$$

Finally, if we define $\overline{I} := (I,\mathrm{id}_I)$, then for any $(V,e) \in \overline{\mathcal{C}}$ we have $\overline{I}\overline{\otimes}(V,e) = (V,e) = (V,e)\overline{\otimes}\overline{I}$, and for any $f \in \mathrm{Hom}(\overline{\mathcal{C}})$ we obviously have $\mathrm{id}_{\overline{I}}\overline{\otimes} f = f = f \overline{\otimes} \mathrm{id}_{\overline{I}}$. Thus $\overline{\mathcal{C}}$ is indeed a strict tensor category. Also, the functor $\mathscr{I}_K : \mathcal{C} \to \overline{\mathcal{C}}$ defined above is a strict tensor functor.

Now let $(V,e) \in \overline{\mathcal{C}}$ and suppose that $(V^\vee, b_V, d_V)$ is a left dual for $V \in \mathcal{C}$. We can then make $(V^\vee, e^\vee) \in \overline{\mathcal{C}}$ into a left dual for (V,e) as follows. We define $b_{(V,e)} \in \mathrm{Hom}_{\mathcal{C}}(I, V \otimes V^\vee)$ and $d_{(V,e)} \in \mathrm{Hom}_{\mathcal{C}}(V^\vee \otimes V, I)$ by

$$\begin{aligned}b_{(V,e)} &= [e \otimes \mathrm{id}_{V^\vee}] \circ b_V \\ d_{(V,e)} &= d_V \circ [\mathrm{id}_{V^\vee} \otimes e].\end{aligned}$$

These morphisms satisfy

$$\begin{aligned}&[e \otimes e^\vee] \circ b_{(V,e)} = [e \otimes e^\vee] \circ [e \otimes \mathrm{id}_{V^\vee}] \circ b_V = [e \otimes e^\vee] \circ b_V = [e^2 \otimes \mathrm{id}_{V^\vee}] \circ b_V = b_{(V,e)} \\ &d_{(V,e)} \circ [e^\vee \otimes e] = d_V \circ [\mathrm{id}_{V^\vee} \otimes e] \circ [e^\vee \otimes e] = d_V \circ [e^\vee \otimes e] = d_V \circ [\mathrm{id}_{V^\vee} \otimes e^2] = d_{(V,e)}\end{aligned}$$

and of course $b_{(V,e)} \circ \mathrm{id}_I = b_{(V,e)}$ and $\mathrm{id}_I \circ d_{(V,e)} = d_{(V,e)}$, which means that $b_{(V,e)} \in \mathrm{Hom}_{\overline{\mathcal{C}}}((I,\mathrm{id}_I),(V \otimes V^\vee, e \otimes e^\vee))$ and $d_{(V,e)} \in \mathrm{Hom}_{\overline{\mathcal{C}}}((V^\vee \otimes V, e^\vee \otimes e),(I,\mathrm{id}_I))$. Also,

$$\begin{aligned}[d_{(V,e)}\overline{\otimes}\mathrm{id}_{(V^\vee,e^\vee)}]\overline{\circ}[\mathrm{id}_{(V^\vee,e^\vee)}\overline{\otimes} b_{(V,e)}] &= [d_V \otimes e^\vee] \circ [\mathrm{id}_{V^\vee} \otimes e^2 \otimes \mathrm{id}_{V^\vee}] \circ [e^\vee \otimes b_V] \\ &= e^\vee \circ [d_V \otimes \mathrm{id}_{V^\vee}] \circ [\mathrm{id}_{V^\vee} \otimes e \otimes \mathrm{id}_{V^\vee}] \circ [\mathrm{id}_{V^\vee} \otimes b_V] \circ e^\vee \\ &= (e^\vee)^3 = e^\vee = \mathrm{id}_{(V^\vee,e^\vee)} \\ [\mathrm{id}_{(V,e)}\overline{\otimes} d_{(V,e)}]\overline{\circ}[b_{(V,e)}\overline{\otimes}\mathrm{id}_{(V,e)}] &= [e \otimes d_V] \circ [\mathrm{id}_{V \otimes V^\vee} \otimes e] \circ [e \otimes \mathrm{id}_{V^\vee \otimes V}] \circ [b_V \otimes e] \\ &= e^2 \circ [\mathrm{id}_V \otimes d_V] \circ [b_V \otimes \mathrm{id}_V] \circ e^2 = e^4 = e = \mathrm{id}_{(V,e)},\end{aligned}$$

so we conclude that $((V^\vee, e^\vee), b_{(V,e)}, d_{(V,e)})$ is a left dual for (V,e). Thus, if $\mathcal{C}$ has left duals, then $\overline{\mathcal{C}}$ also has left duals. The same statement is also true for right duals. Namely, if $({}^\vee V, b'_V, d'_V)$ is a right dual for $V \in \mathcal{C}$, then $(({}^\vee V, {}^\vee e), b'_{(V,e)}, d'_{(V,e)})$ is a right dual for $(V,e) \in \overline{\mathcal{C}}$, where $b'_{(V,e)} := [\mathrm{id}_{{}^\vee V} \otimes e] \circ b'_V$ and $d'_{(V,e)} := d'_V \circ [e \otimes \mathrm{id}_{{}^\vee V}]$. The proof goes almost the same as for left duals.

Let $\mathcal{C}$ be an Ab-category that has direct sums and let $(V_1,p_1),(V_2,p_2) \in \overline{\mathcal{C}}$. Because $\mathcal{C}$ has direct sums, there exists an object $V \in \mathcal{C}$ together with morphisms $f_j \in \mathrm{Hom}_{\mathcal{C}}(V_j, V)$ and $f'_j \in \mathrm{Hom}_{\mathcal{C}}(V, V_j)$ such that $f'_j \circ f_j = \mathrm{id}_{U_j}$ and $f_1 \circ f'_1 + f_2 \circ f'_2 = \mathrm{id}_V$ for $j \in \{1,2\}$. Define $\overline{f}_j := f_j \circ p_j$, $\overline{f}'_j := p_j \circ f'_j$ and $p := f_1 \circ p_1 \circ f'_1 + f_2 \circ p_2 \circ f'_2$. Then $p^2 = p$, so $(V,p) \in \overline{\mathcal{C}}$, and we also have $p \circ \overline{f}_j \circ p_j = \overline{f}_j$ and $p_j \circ \overline{f}'_j \circ p = \overline{f}'_j$, i.e. $\overline{f}_j \in \mathrm{Hom}_{\overline{\mathcal{C}}}((V_j,p_j),(V,p))$ and $\overline{f}'_j \in \mathrm{Hom}_{\overline{\mathcal{C}}}((V,p),(V_j,p_j))$. It is a straightforward computation to show that (V,p) is a direct sum $(V,p) \cong (V_1,p_1) \oplus (V_2,p_2)$ implemented by the morphisms $\overline{f}_j$ and $\overline{f}'_j$. Thus we conclude that if $\mathcal{C}$ has direct sums, then so has $\overline{\mathcal{C}}$.

Now suppose that $\mathcal{C}$ is a $*$-category. Then the objects of $\overline{\mathcal{C}}$ are also defined to be pairs (V,p) as above, but we also require that $p^* = p$ (i.e. that p is a projection rather than just an idempotent). Then $\overline{\mathcal{C}}$ also becomes a $*$-category, where we define the $*$-operation to be the same as in $\mathcal{C}$.

2.8 Group actions on tensor categories

If G is a group, let $\mathcal{G}$ be the category with $\mathrm{Obj}(\mathcal{G}) = G$, with $\mathrm{End}_{\mathcal{G}}(q) = \{\mathrm{id}_q\}$ for $q \in G$ and with $\mathrm{Hom}_{\mathcal{G}}(q,r) = \emptyset$ if $q,r \in G$ with $q \neq r$. It becomes a strict tensor category with tensor product $q \otimes r = qr$ and $\mathrm{id}_q \otimes \mathrm{id}_r = \mathrm{id}_{qr}$ and with unit object e.

Definition 2.8.1 If G is a group, then a *G-action* on a tensor category $(\mathcal{C}, \otimes, I, a, l, r)$ is a tensor functor $(F, \varepsilon^F, \delta^F) : \mathcal{G} \to \mathrm{Aut}^{\otimes}(\mathcal{C})$. A tensor category that is equipped with a G-action is called a *G-category.*

Since the objects of $\mathrm{Aut}^{\otimes}(\mathcal{C})$ are again tensor functors, the functor F assigns to each object $q \in \mathcal{G}$ a triple $(F_q, \varepsilon^{F_q}, \delta^{F_q})$, which we will simply denote by $(F_q, \varepsilon^q, \delta^q)$. In the following two remarks we will unfold the definition of a G-action in great detail, mainly in order to introduce the notation.

Remark 2.8.2 Let $q \in G$ be fixed. The fact that $(F_q, \varepsilon^q, \delta^q)$ is a tensor functor means that

- $F_q : \mathcal{C} \to \mathcal{C}$ is a functor.
- $\delta^q : \otimes \circ (F_q \times F_q) \to F_q \circ \otimes$ is a natural isomorphism of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, i.e. a family

$$\{\delta^q_{U,V} : F_q(U) \otimes F_q(V) \to F_q(U \otimes V)\}_{U,V \in \mathcal{C}}$$

of isomorphisms in $\mathcal{C}$ such that the following square commutes

$$\begin{array}{ccc} F_q(U) \otimes F_q(V) & \xrightarrow{\delta^q_{U,V}} & F_q(U \otimes V) \\ {\scriptstyle F_q(f) \otimes F_q(g)} \downarrow & & \downarrow {\scriptstyle F_q(f \otimes g)} \\ F_q(U') \otimes F_q(V') & \xrightarrow{\delta^q_{U',V'}} & F_q(U' \otimes V') \end{array}$$

for any objects $U, V, U', V' \in \mathcal{C}$ and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(U,U')$ and $g \in \mathrm{Hom}_{\mathcal{C}}(V,V')$, satisfying the additional property that the diagram

$$\begin{array}{ccc} (F_q(U) \otimes F_q(V)) \otimes F_q(W) & \xrightarrow{a_{F_q(U),F_q(V),F_q(W)}} & F_q(U) \otimes (F_q(V) \otimes F_q(W)) \\ {\scriptstyle \delta^q_{U,V} \otimes \mathrm{id}_{F_q(W)}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{F_q(U)} \otimes \delta^q_{V,W}} \\ F_q(U \otimes V) \otimes F_q(W) & & F_q(U) \otimes F_q(V \otimes W) \\ {\scriptstyle \delta^q_{U \otimes V, W}} \downarrow & & \downarrow {\scriptstyle \delta^q_{U, V \otimes W}} \\ F_q((U \otimes V) \otimes W) & \xrightarrow{F_q(a_{U,V,W})} & F_q(U \otimes (V \otimes W)) \end{array} \tag{2.8.1}$$

commutes any $U, V, W \in \mathcal{C}$. In case $\mathcal{C}$ is a strict tensor category, this diagram reduces to

$$\begin{array}{ccc} F_q(U)\otimes F_q(V)\otimes F_q(W) & \xrightarrow{\mathrm{id}_{F_q(U)}\otimes\delta^q_{V,W}} & F_q(U)\otimes F_q(V\otimes W) \\ \Big\downarrow{\scriptstyle \delta^q_{U,V}\otimes\mathrm{id}_{F_q(W)}} & & \Big\downarrow{\scriptstyle \delta^q_{U,V\otimes W}} \\ F_q(U\otimes V)\otimes F_q(W) & \xrightarrow{\delta^q_{U\otimes V,W}} & F_q(U\otimes V\otimes W). \end{array}$$

- $\varepsilon^q : I \to F_q(I)$ is an isomorphism such that the diagrams

$$\begin{array}{ccc} I\otimes F_q(U) & \xrightarrow{l_{F_q(U)}} & F_q(U) \\ \Big\downarrow{\scriptstyle \varepsilon^q\otimes\mathrm{id}_{F_q(U)}} & & \Big\uparrow{\scriptstyle F_q(l_U)} \\ F_q(I)\otimes F_q(U) & \xrightarrow{\delta^q_{I,U}} & F_q(I\otimes U) \end{array} \quad \text{and} \quad \begin{array}{ccc} F_q(U)\otimes I & \xrightarrow{r_{F_q(U)}} & F_q(U) \\ \Big\downarrow{\scriptstyle \mathrm{id}_{F_q(U)}\otimes\varepsilon^q} & & \Big\uparrow{\scriptstyle F_q(r_U)} \\ F_q(U)\otimes F_q(I) & \xrightarrow{\delta^q_{U,I}} & F_q(U\otimes I) \end{array}$$

commute for all $U \in \mathcal{C}$. In case $\mathcal{C}$ is a strict tensor category, these diagrams become the identities

$$\begin{aligned} \delta^q_{I,U}\circ(\varepsilon^q\otimes\mathrm{id}_{F_q(U)}) &= \mathrm{id}_{F_q(U)} \\ \delta^q_{U,I}\circ(\mathrm{id}_{F_q(U)}\otimes\varepsilon^q) &= \mathrm{id}_{F_q(U)} \end{aligned}$$

which hold for all $U \in \mathcal{C}$.

Remark 2.8.3 The fact that $(F, \varepsilon^F, \delta^F)$ is a tensor functor means that

- $F : \mathcal{G} \to \mathrm{Aut}^{\otimes}(\mathcal{C})$ is a functor. Note that the functor F is quite trivial on the morphisms since the only morphisms in the category $\mathcal{G}$ are the identity morphisms.
- $\delta^F : \otimes_{\mathrm{Aut}^{\otimes}(\mathcal{C})} \circ (F \times F) \to F \circ \otimes_{\mathcal{G}}$ is a natural[13] isomorphism of functors $\mathcal{G} \times \mathcal{G} \to \mathrm{Aut}^{\otimes}(\mathcal{C})$, i.e. a family

$$\{\delta^F_{q,r} : (F_q \circ F_r, \varepsilon^q \diamond \varepsilon^r, \delta^q \diamond \delta^r) \to (F_{qr}, \varepsilon^{qr}, \delta^{qr})\}_{q,r\in G}$$

of isomorphisms in the category $\mathrm{Aut}^{\otimes}(\mathcal{C})$. Thus each $\delta^F_{q,r}$ is in fact a family

$$\{(\delta^F_{q,r})_V : (F_q \circ F_r)(V) \to F_{qr}(V)\}_{V\in\mathcal{C}}$$

of isomorphisms in $\mathcal{C}$ such that the following square commutes

$$\begin{array}{ccc} (F_q\circ F_r)(V) & \xrightarrow{(\delta^F_{q,r})_V} & F_{qr}(V) \\ \Big\downarrow{\scriptstyle (F_q\circ F_r)(f)} & & \Big\downarrow{\scriptstyle F_{qr}(f)} \\ (F_q\circ F_r)(V') & \xrightarrow{(\delta^F_{q,r})_{V'}} & F_{qr}(V') \end{array}$$

for all $V, V' \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(V, V')$, and such that the diagrams

$$\begin{array}{ccc} & & F_q(F_r(I)) \\ & \nearrow^{F_q(\varepsilon^r)\circ\varepsilon^q} & \Big\downarrow{\scriptstyle (\delta^F_{q,r})_I} \\ I & \searrow_{\varepsilon^{qr}} & \\ & & F_{qr}(I) \end{array}$$

[13]The naturality gives no extra conditions here because $\mathcal{G}$ is discrete.

and

$$\begin{array}{ccc} F_q(F_r(U)) \otimes F_q(F_r(V)) & \xrightarrow{F_q(\delta^r_{U,V})\circ\delta^q_{F_r(U),F_r(V)}} & F_q(F_r(U\otimes V)) \\ \Big\downarrow{\scriptstyle (\delta^F_{q,r})_U\otimes(\delta^F_{q,r})_V} & & \Big\downarrow{\scriptstyle (\delta^F_{q,r})_{U\otimes V}} \\ F_{qr}(U)\otimes F_{qr}(V) & \xrightarrow{\delta^{qr}_{U,V}} & F_{qr}(U\otimes V) \end{array}$$

commute. The $\delta^F_{q,r}$ are required to satisfy the additional property that the square

$$\begin{array}{ccc} (F_q\circ F_r\circ F_s)(V) & \xrightarrow{F_q\left((\delta^F_{r,s})_V\right)} & (F_q\circ F_{rs})(V) \\ \Big\downarrow{\scriptstyle (\delta^F_{q,r})_{F_s(V)}} & & \Big\downarrow{\scriptstyle (\delta^F_{q,rs})_V} \\ (F_{qr}\circ F_s)(V) & \xrightarrow{(\delta^F_{qr,s})_V} & (F_{qrs})(V) \end{array}$$

commutes for all $V\in\mathcal{C}$ and $q,r,s\in G$.

- $\varepsilon^F : \mathrm{id}_{\mathcal{C}} \to F_e$ is an isomorphism in the category $\mathrm{Aut}^{\otimes}(\mathcal{C})$, i.e. a family

$$\{\varepsilon^F_V : V \to F_e(V)\}_{V\in\mathcal{C}}$$

of isomorphisms such that the following square commutes

$$\begin{array}{ccc} V & \xrightarrow{\varepsilon^F_V} & F_e(V) \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle F_e(f)} \\ V' & \xrightarrow{\varepsilon^F_{V'}} & F_e(V') \end{array}$$

for all $V,V'\in\mathcal{C}$ and $f\in\mathrm{Hom}_{\mathcal{C}}(V,V')$ and because it is also a tensor natural isomorphism, it must also satisfy the equalities

$$\varepsilon^F_I = \varepsilon^e$$
$$\varepsilon^F_{U\otimes V} = \delta^e_{U,V}\circ[\varepsilon^F_U\otimes\varepsilon^F_V].$$

Also, ε^F satisfies the additional property that

$$\begin{aligned} \mathrm{id}_{F_q(V)} &= (\mathrm{id}_{F_q})_V = (\delta^F_{e,q})_V\circ\varepsilon^F_{F_q(V)} \\ \mathrm{id}_{F_q(V)} &= (\mathrm{id}_{F_q})_V = (\delta^F_{q,e})_V\circ F_q(\varepsilon^F_V) \end{aligned}$$

for all $V\in\mathcal{C}$, where id_{F_q} denotes the identity natural tensor isomorphism from F_q to itself.

Definition 2.8.4 A G-action F on a strict tensor category $(\mathcal{C},\otimes,I)$ is called *strict* if

- $F_q(V\otimes W) = F_q(V)\otimes F_q(W)$ and $F_q(I)=I$ for all $q\in G$ and $V,W\in\mathcal{C}$;
- $F_{qr} = F_q\circ F_r$ for all $q,r\in G$ and $F_e = \mathrm{id}_{\mathcal{C}}$;
- for all $q,r\in G$ and $V,W\in\mathcal{C}$ we have $\varepsilon^F_V = \mathrm{id}_V = \mathrm{id}_{F_e(V)}$, $(\delta^F_{q,r})_V = \mathrm{id}_{F_q(F_r(V))} = \mathrm{id}_{F_{qr}(V)}$, $\varepsilon^q = \mathrm{id}_I = \mathrm{id}_{F_q(I)}$ and $\delta^q_{V,W} = \mathrm{id}_{F_q(V)\otimes F_q(W)} = \mathrm{id}_{F_q(V\otimes W)}$.

A strict tensor category with a strict G-action is called a *strict G-category.*

If $\mathcal{C}$ is a strict tensor category with a strict G-action F, then we define the fixpoint subcategory $\mathcal{C}^G$ of $\mathcal{C}$ as follows. The objects of $\mathcal{C}^G$ are the objects $V \in \mathcal{C}$ for which $F_q(V) = V$ for all $q \in G$ and the morphisms from $V \in \mathcal{C}^G$ to $W \in \mathcal{C}^G$ are the morphisms in f in $\mathrm{Hom}_{\mathcal{C}}(V, W)$ for which $F_q(f) = f$ for all $q \in G$. Thus

$$\begin{aligned} \mathrm{Obj}(\mathcal{C}^G) &:= \{V \in \mathcal{C} : F_q(V) = V \text{ for all } q \in G\} \\ \mathrm{Hom}_{\mathcal{C}^G}(V, W) &:= \{f \in \mathrm{Hom}_{\mathcal{C}}(V, W) : F_q(f) = f \text{ for all } q \in G\} \end{aligned}$$

It follows from functoriality of F_q for each $q \in G$ that $\mathcal{C}^G$ is a subcategory of $\mathcal{C}$. Because each F_q is a tensor functor, it is also easy to see that $\mathcal{C}^G$ is a tensor subcategory of $\mathcal{C}$.

2.8.1 G-functors and natural G-transformations

We will now consider the extra conditions that a tensor functor between G-categories has to satisfy in order to respect the G-actions of both categories.

Definition 2.8.5 Let G be a group and let $(\mathcal{C}, \otimes, I, a, l, r)$ and $(\mathcal{C}', \otimes', I', a', l', r')$ be tensor categories with G-actions F and F', respectively. Then a *G-functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a tensor functor $(K, \varepsilon^K, \delta^K)$ from $\mathcal{C}$ to $\mathcal{C}'$ together with a natural isomorphism $\xi^K(q) : K \circ F_q \to F'_q \circ K$ for each $q \in G$, i.e. a family

$$\{\xi^K(q)_X : K(F_q(X)) \to F'_q(K(X))\}_{X \in \mathcal{C}}$$

of isomorphism in $\mathcal{C}'$ such that for any $X, Y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the diagram

$$\begin{array}{ccc} K(F_q(X)) & \xrightarrow{\xi^K(q)_X} & F'_q(K(X)) \\ \downarrow{\scriptstyle K(F_q(f))} & & \downarrow{\scriptstyle F'_q(K(f))} \\ K(F_q(Y)) & \xrightarrow{\xi^K(q)_Y} & F'_q(K(Y)) \end{array}$$

commutes, satisfying the following additional properties:

- for each $X \in \mathcal{C}$ and $q, r \in G$ the diagram

$$\begin{array}{ccccc} K(F_q(F_r(X))) & \xrightarrow{\xi^K(q)_{F_r(X)}} & F'_q(K(F_r(X))) & \xrightarrow{F'_q(\xi^K(r)_X)} & F'_q(F'_r(K(X))) \\ \downarrow{\scriptstyle K((\delta^F_{q,r})_X)} & & & & \downarrow{\scriptstyle (\delta^{F'}_{q,r})_{K(X)}} \\ K(F_{qr}(X)) & & \xrightarrow{\xi^K(qr)_X} & & F'_{qr}(K(X)) \end{array}$$

  commutes;
- for each $X \in \mathcal{C}$ the diagram

$$\begin{array}{ccc} K(X) & \searrow{\scriptstyle \varepsilon^{F'}_{K(X)}} & \\ \downarrow{\scriptstyle K(\varepsilon^F_X)} & & \\ K(F_e(X)) & \xrightarrow{\xi^K(e)_X} & F'_e(K(X)) \end{array}$$

  commutes;
- for any $X, Y \in \mathcal{C}$ and $q \in G$ the diagram

$$\begin{array}{ccccc}
K(F_q(X\otimes Y)) & \xleftarrow{K(\delta^{F_q}_{X,Y})} & K(F_q(X)\otimes F_q(Y)) & \xleftarrow{\delta^K_{F_q(X),F_q(Y)}} & K(F_q(X))\otimes' K(F_q(Y)) \\
\downarrow{\scriptstyle \xi^K(q)_{X\otimes Y}} & & & & \downarrow{\scriptstyle \xi^K(q)_X\otimes'\xi^K(q)_Y} \\
F'_q(K(X\otimes Y)) & \xleftarrow{F'_q(\delta^K_{X,Y})} & F'_q(K(X)\otimes' K(Y)) & \xleftarrow{\delta^{F'_q}_{K(X),K(Y)}} & F'_q(K(X))\otimes' F'_q(K(Y))
\end{array}$$

commutes.

Let $(K,\varepsilon^K,\delta^K,\xi^K):\mathcal{C}\to\mathcal{C}'$ and $(L,\varepsilon^L,\delta^L,\xi^L):\mathcal{C}'\to\mathcal{C}''$ be two G-functors. We have already seen that the functor $L\circ K$ can be equipped with the structure of a tensor functor. In addition, it also becomes a G-functor by using the composition

$$L(K(F_q(X)))\xrightarrow{L(\xi^K(q)_X)} L(F'_q(K(X)))\xrightarrow{\xi^L(q)_{K(X)}} F''_q(L(K(X)))$$

for $q\in G$ and $X\in\mathcal{C}$. Thus we conclude that the composition of two G-functors can again be equipped with the structure of a G-functor.

Definition 2.8.6 Let G be a group and let $(\mathcal{C},\otimes,I,a,l,r)$ and $(\mathcal{C}',\otimes',I',a',l',r')$ be tensor categories with G-actions F and F', respectively. If $(K,\varepsilon^K,\delta^K,\xi^K)$ and $(L,\varepsilon^L,\delta^L,\xi^L)$ are G-functors from $\mathcal{C}$ to $\mathcal{C}'$ then a *natural G-transformation* $\varphi:(K,\varepsilon^K,\delta^K,\xi^K)\to(L,\varepsilon^L,\delta^L,\xi^L)$ is a natural tensor transformation $\varphi:(K,\varepsilon^K,\delta^K)\to(L,\varepsilon^L,\delta^L)$ such that for each $X\in\mathcal{C}$ and $q\in G$ the diagram

$$\begin{array}{ccc}
K(F_q(X)) & \xrightarrow{\varphi_{F_q(X)}} & L(F_q(X)) \\
\downarrow{\scriptstyle \xi^K(q)_X} & & \downarrow{\scriptstyle \xi^L(q)_X} \\
F'_q(K(X)) & \xrightarrow{F'_q(\varphi_X)} & F'_q(L(X))
\end{array}$$

commutes.

Now that we have introduced G-functors and natural G-transformations, we can use these notions to define an equivalence between G-categories.

Definition 2.8.7 Let G be a group and let $(\mathcal{C},\otimes,I,a,l,r)$ and $(\mathcal{C}',\otimes',I',a',l',r')$ be tensor categories with G-actions F and F', respectively. A *G-equivalence* (or *equivalence of G-categories*) from $\mathcal{C}$ to $\mathcal{C}'$ is a G-functor $(K,\varepsilon^K,\delta^K,\xi^K):\mathcal{C}\to\mathcal{C}'$ such that there exists a G-functor $(L,\varepsilon^L,\delta^L,\xi^L):\mathcal{C}'\to\mathcal{C}$ together with natural G-isomorphisms $\varphi:\mathrm{id}_{\mathcal{C}'}\to K\circ L$ and $\psi:L\circ K\to\mathrm{id}_{\mathcal{C}}$. If there exists such a G-equivalence then $\mathcal{C}$ and $\mathcal{C}'$ will be called equivalent G-categories.

2.8.2 Braiding and duality in G-categories; modular tensor G-categories

We have already defined braidings and duality for tensor categories, and these definitions remain unchanged when a tensor category is also equipped with a group action. However, in the presence of a group action it would be nice if these notions are compatible with the group action. We will now define the corresponding compatibility conditions, beginning with the braiding.

Definition 2.8.8 If G is a group and if $\mathcal{C}$ is a G-category, then a braiding c on $\mathcal{C}$ is called *compatible with the G-action* (or *G-compatible*) if for any $V,W\in\mathcal{C}$ and $q\in G$ the square

$$\begin{array}{ccc} F_q(V) \otimes F_q(W) & \xrightarrow{c_{F_q(V),F_q(W)}} & F_q(W) \otimes F_q(V) \\ \Big\downarrow \delta^q_{V,W} & & \Big\downarrow \delta^q_{W,V} \\ F_q(V \otimes W) & \xrightarrow{F_q(c_{V,W})} & F_q(W \otimes V) \end{array}$$

commutes. If $\mathcal{C}$ is a strict G-category, then this diagram reduces to the equation $F_q(c_{V,W}) = c_{F_q(V),F_q(W)}$.

When we compare the square diagram in this definition with the square diagram in Definition 2.4.5, we see that c is G-compatible if and only if $(F_q, \varepsilon^q, \delta^q)$ is a braided tensor functor for each $q \in G$.

Definition 2.8.9 Let $\mathcal{C}$ be a strict tensor category with strict G-action F.

(1) If $\mathcal{C}$ has a left duality $((.)^\vee, b, d)$, then we call this a *left G-duality* if it has the property that for all $q \in G$ and $V \in \mathcal{C}$ we have $F_q(V^\vee) = F_q(V)^\vee$, $F_q(b_V) = b_{F_q(V)}$ and $F_q(d_V) = d_{F_q(V)}$.

(2) If $\mathcal{C}$ has a right duality $({}^\vee(.), b', d')$, then we call this a *right G-duality* if it has the property that for all $q \in G$ and $V \in \mathcal{C}$ we have $F_q({}^\vee V) = {}^\vee F_q(V)$, $F_q(b'_V) = b'_{F_q(V)}$ and $F_q(d'_V) = d'_{F_q(V)}$.

(3) If $\mathcal{C}$ has a two-sided duality $(\overline{(.)}, b, d, b', d')$, then we will call this a *two-sided G-duality* if $(\overline{(.)}, b, d)$ is a left G-duality and $(\overline{(.)}, b', d')$ is a right G-duality.

(4) If $\mathcal{C}$ admits a two-sided G-duality and is pivotal or spherical with respect to some given two-sided G-duality, then $\mathcal{C}$ will be called *G-pivotal* or *G-spherical*, respectively.

Remark 2.8.10 Note that for the definition in part (4) it is actually enough to assume that $\mathcal{C}$ admits a left G-duality, since this implies automatically that $\mathcal{C}$ admits a two-sided G-duality. Namely, because $\mathcal{C}$ is pivotal we have $F_q(b'_V) = F_q(b_{\overline{V}}) = b_{F_q(\overline{V})} = b_{\overline{F_q(V)}} = b'_{F_q(V)}$ and $F_q(d'_V) = F_q(d_{\overline{V}}) = d_{F_q(\overline{V})} = d_{\overline{F_q(V)}} = d'_{F_q(V)}$.

The following definition will be very important for us when we discuss algebraic quantum field theory in Chapter 3.

Definition 2.8.11 If $\mathcal{C}$ is a modular tensor category with a strict G-action F, then we will call $\mathcal{C}$ a *modular tensor G-category* if its braiding is G-compatible and if its duality is a G-duality.

2.8.3 Braided G-crossed categories

In this subsection we will introduce the class of categories that will be most important to us, namely the class of braided G-crossed categories. Besides having a G-action, these categories also have a G-grading.

Definition 2.8.12 If G is a group and $(\mathcal{C}, \otimes, I, a, l, r)$ is a tensor category, then a *G-grading* on $\mathcal{C}$ consists of a full tensor subcategory $\mathcal{C}_{\mathrm{hom}} \subset \mathcal{C}$ (the objects of which are called the *homogeneous objects* of $\mathcal{C}$) together with a map $\partial : \mathcal{C}_{\mathrm{hom}} \to G$ on the objects of $\mathcal{C}_{\mathrm{hom}}$ that is constant on isomorphism classes and satisfies $\partial(V \otimes W) = \partial(V)\partial(W)$ for all $V, W \in \mathcal{C}_{\mathrm{hom}}$. If $V \in \mathcal{C}_{\mathrm{hom}}$, we will call $\partial(V)$ the *degree* of V. A tensor category with a G-grading is called a *G-graded category*. The set $\partial(\mathcal{C}_{\mathrm{hom}}) \subset G$ is called the *G-spectrum* of $\mathcal{C}$ and for each $q \in G$ we will write $\mathcal{C}_q$ to denote the full subcategory of $\mathcal{C}$ determined by the objects of degree q.

Note that $\mathrm{Obj}(\mathcal{C}_{\mathrm{hom}}) = \bigsqcup_{q \in \partial(\mathcal{C})} \mathrm{Obj}(\mathcal{C}_q)$. If $V, W \in \mathcal{C}_q$ for some $q \in G$ and if $f \in \mathrm{Hom}_{\mathcal{C}}(V, W)$, then it will sometimes be convenient to write that $\partial(f) = q$.

Remark 2.8.13 In later chapters we will mainly encounter situations where $\mathcal{C}_{\mathrm{hom}} = \mathcal{C}$. The only two exceptions to this are the crossed product defined in Subsection 3.1.3 and Theorem 3.2.20. In these exceptional cases we are always in the situation where any object in the G-crossed category is a finite direct sum of homogeneous objects. See also [78] and [80].

Now suppose that we have two groups G_1 and G_2 together with a tensor category $\mathcal{C}$ which has a G_1-action F and a G_2-grading $\partial : \mathcal{C}_{\text{hom}} \to G$. If there is also a group action α (written $q \mapsto \alpha_q$) of G_1 on G_2, then we can use α to define a compatibility condition between F and ∂. Namely, we can say that F and ∂ are compatible if

$$\partial(F_q(V)) = \alpha_q(\partial(V)) \tag{2.8.2}$$

for all $V \in \mathcal{C}_{\text{hom}}$. As a special case, we can consider the case when $G_1 = G_2 = G$ and where α is the action of G on itself given by conjugation, i.e. $\alpha_q(r) = qrq^{-1}$. Then equation (2.8.2) becomes $\partial(F_q(V)) = q\partial(V)q^{-1}$. As indicated in the following definition, such categories have a special name.

Definition 2.8.14 Let G be a group and let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category with G-action F. Suppose that $\mathcal{C}$ has a full tensor subcategory $\mathcal{C}_{\text{hom}} \subset \mathcal{C}$ that has a G-grading $\partial : \mathcal{C}_{\text{hom}} \to G$. Then $\mathcal{C}$ is called a *G-crossed category* if the G-grading is compatible with the G-action in the sense that $\partial(F_q(V)) = q\partial(V)q^{-1}$ for all $q \in G$ and $V \in \mathcal{C}_{\text{hom}}$.

The slightly more general setting sketched above the definition will be encountered by us in one place. Namely, in Chapter 3 we will define a G-category that is $G \times G$-graded. The action of G on $G \times G$ is then given by $\alpha_q((r,s)) = (qrq^{-1}, qsq^{-1})$ and the compatibility condition (2.8.2) is satisfied in this case.

Definition 2.8.15 A G-crossed category that is also a TC^* will be called a G-crossed TC^*.

The functors and natural transformations between G-crossed categories are defined as follows.

Definition 2.8.16 Let G be a group and let $(\mathcal{C}, \otimes, I, a, l, r, F, \partial)$ and $(\mathcal{C}', \otimes', I', a', l', r', F', \partial')$ be G-crossed categories. Then a *G-crossed functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a G-functor $(K, \varepsilon^K, \delta^K, \xi^K) : \mathcal{C} \to \mathcal{C}'$ that satisfies $\partial'(K(V)) = \partial(V)$ for all $V \in \mathcal{C}_{\text{hom}}$. A *natural G-crossed transformation* from one G-crossed functor to another is just a natural G-transformation of the underlying G-functors.

Definition 2.8.17 A *braiding (of the first kind* [14]*)* on a G-crossed category $(\mathcal{C}, \otimes, I, a, l, r, F, \partial)$ is a family of isomorphisms $\{c_{V,W} : V \otimes W \to F_{\partial(V)}(W) \otimes V\}_{V \in \mathcal{C}_{\text{hom}}, W \in \mathcal{C}}$ satisfying naturality in the sense that

$$\begin{array}{ccc} V \otimes W & \xrightarrow{c_{V,W}} & F_{\partial(V)}(W) \otimes V \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle F_{\partial(V)}(g)\otimes f} \\ V' \otimes W' & \xrightarrow{c_{V',W'}} & F_{\partial(V')}(W') \otimes V' \end{array}$$

commutes for all $f \in \text{Hom}_{\mathcal{C}}(V, V')$ and $g \in \text{Hom}_{\mathcal{C}}(W, W')$ with $\partial(V) = \partial(V')$, as well as commutativity of the diagrams

$$\begin{array}{ccc} & (U \otimes V) \otimes W & \\ {\scriptstyle c_{U,V}\otimes \text{id}_W}\swarrow & & \searrow{\scriptstyle a_{U,V,W}} \\ (F_{\partial(U)}(V) \otimes U) \otimes W & & U \otimes (V \otimes W) \\ \downarrow{\scriptstyle a_{F_{\partial(U)}(V),U,W}} & & \downarrow{\scriptstyle c_{U,V\otimes W}} \\ F_{\partial(U)}(V) \otimes (U \otimes W) & & F_{\partial(U)}(V \otimes W) \otimes U \\ \downarrow{\scriptstyle \text{id}_{F_{\partial(U)}(V)}\otimes c_{U,W}} & & \downarrow{\scriptstyle (\delta^{\partial(U)}_{V,W})^{-1}\otimes \text{id}_U} \\ F_{\partial(U)}(V) \otimes (F_{\partial(U)}(W) \otimes U) & \xrightarrow{a^{-1}_{F_{\partial(U)}(V),F_{\partial(U)}(W),U}} & (F_{\partial(U)}(V) \otimes F_{\partial(U)}(W)) \otimes U \end{array}$$

[14]We will only write 'of the first kind' if we want to distinguish it explicitly from a braiding of the second kind. Braidings of the second kind will be defined later.

for all $U \in \mathcal{C}_{\text{hom}}$ and $V, W \in \mathcal{C}$;

$$
\begin{array}{ccc}
 & U \otimes (V \otimes W) & \\
\swarrow^{\mathrm{id}_U \otimes c_{V,W}} & & \searrow^{a^{-1}_{U,V,W}} \\
U \otimes (F_{\partial(V)}(W) \otimes V) & & (U \otimes V) \otimes W \\
\downarrow a^{-1}_{U,F_{\partial(V)}(W),V} & & \downarrow c_{U \otimes V, W} \\
(U \otimes F_{\partial(V)}(W)) \otimes V & & F_{\partial(U)\partial(V)}(W) \otimes (U \otimes V) \\
\downarrow c_{U,F_{\partial(V)}(W)} \otimes \mathrm{id}_V & & \downarrow (\delta^F_{\partial(U),\partial(V)})^{-1}_W \otimes \mathrm{id}_{U \otimes V} \\
(F_{\partial(U)}(F_{\partial(V)}(W)) \otimes U) \otimes V & \xrightarrow{a_{F_{\partial(U)}(F_{\partial(V)}(W)),U,V}} & F_{\partial(U)}(F_{\partial(V)}(W)) \otimes (U \otimes V)
\end{array}
$$

for all $U, V \in \mathcal{C}_{\text{hom}}$ and $W \in \mathcal{C}$;

$$
\begin{array}{ccc}
F_q(V) \otimes F_q(W) & \xrightarrow{c_{F_q(V),F_q(W)}} & F_{q\partial(V)q^{-1}}(F_q(W)) \otimes F_q(V) \\
\downarrow \delta^q_{V,W} & & \downarrow (\delta^F_{q\partial(V)q^{-1},q})_W \otimes \mathrm{id}_{F_q(V)} \\
F_q(V \otimes W) & & F_{q\partial(V)}(W) \otimes F_q(V) \\
\downarrow F_q(c_{V,W}) & & \downarrow (\delta^F_{q,\partial(V)})^{-1}_W \otimes \mathrm{id}_{F_q(V)} \\
F_q(F_{\partial(V)}(W) \otimes V) & \xrightarrow{(\delta^q_{F_{\partial(V)}(W),V})^{-1}} & F_q(F_{\partial(V)}(W)) \otimes F_q(V)
\end{array}
$$

for all $V \in \mathcal{C}_{\text{hom}}$, $W \in \mathcal{C}$ and $q \in G$.

Remark 2.8.18 Note that the heptagonal diagrams are equivalent to the equations

$$
\begin{aligned}
c_{U,V\otimes W} &= [(\delta^{\partial(U)}_{V,W}) \otimes \mathrm{id}_U] \circ a^{-1}_{F_{\partial(U)}(V),F_{\partial(U)}(W),U} \circ [\mathrm{id}_{F_{\partial(U)}(V)} \otimes c_{U,W}] \\
&\quad \circ a_{F_{\partial(U)}(V),U,W} \circ [c_{U,V} \otimes \mathrm{id}_W] \circ a^{-1}_{U,V,W} \\
c_{U\otimes V,W} &= [(\delta^F_{\partial(U),\partial(V)})_W \otimes \mathrm{id}_{U\otimes V}] \circ a_{F_{\partial(U)}(F_{\partial(V)}(W)),U,V} \circ [c_{U,F_{\partial(V)}(W)} \otimes \mathrm{id}_V] \\
&\quad \circ a^{-1}_{U,F_{\partial(V)}(W),V} \circ [\mathrm{id}_U \otimes c_{V,W}] \circ a_{U,V,W}.
\end{aligned}
$$

In case the G-crossed category is strict, these equations reduce to

$$
\begin{aligned}
c_{U,V\otimes W} &= [\mathrm{id}_{F_{\partial(U)}(V)} \otimes c_{U,W}] \circ [c_{U,V} \otimes \mathrm{id}_W] \\
c_{U\otimes V,W} &= [c_{U,F_{\partial(V)}(W)} \otimes \mathrm{id}_V] \circ [\mathrm{id}_U \otimes c_{V,W}].
\end{aligned}
$$

The hexagonal diagram is equivalent to the equation

$$
\begin{aligned}
&c_{F_q(V),F_q(W)} \\
&= [(\delta^F_{q\partial(V)q^{-1},q})^{-1}_W \otimes \mathrm{id}_{F_q(V)}] \circ [(\delta^F_{q,\partial(V)})_W \otimes \mathrm{id}_{F_q(V)}] \circ (\delta^q_{F_{\partial(V)}(W),V})^{-1} \circ F_q(c_{V,W}) \circ \delta^q_{V,W},
\end{aligned}
$$

which in the strict case reduces to $c_{F_q(V),F_q(W)} = F_q(c_{V,W})$.

Now suppose that $\mathcal{C}$ is a (non-strict) G-category. Recall that in Subsection 2.8.2 a braiding on $\mathcal{C}$ was defined in the same way as for tensor categories without a group action, and in Definition 2.8.8 we introduced the notion of G-compatibility for a braiding on a G-category. However, we can also consider $\mathcal{C}$ as a G-crossed category with trivial G-spectrum $\partial(\mathcal{C}) = \{e\}$ by defining $\partial(V) := e$ for all $V \in \mathcal{C}$. Thus we can also use Definition 2.8.17 to define the notion of a braiding on $\mathcal{C}$, which we will now temporarily call a G-crossed braiding in order to distinguish it from ordinary braidings. These two definitions are not the same. Namely, a (G-compatible) braiding on $\mathcal{C}$ is a family $\{c_{V,W} : V \otimes W \to W \otimes V\}_{V,W \in \mathcal{C}}$, whereas a G-crossed braiding on $\mathcal{C}$ is a family $\{c_{V,W} : V \otimes W \to F_e(W) \otimes V\}_{V,W \in \mathcal{C}}$. Fortunately, there is a nice correspondence between G-compatible braidings and G-crossed braidings on a G-category, as we will now explain. If c is a G-compatible braiding, then for each $V, W \in \mathcal{C}$ we define an isomorphism $\widehat{c}_{V,W} : V \otimes W \to F_e(W) \to V$ by

$$\widehat{c}_{V,W} := [\varepsilon^F_W \otimes \mathrm{id}_V] \circ c_{V,W}.$$

A straightforward computation shows that this defines a G-crossed braiding $\widehat{c}$ on $\mathcal{C}$. Conversely, if $\widehat{c}$ is a G-crossed braiding on $\mathcal{C}$ then for each $V, W \in \mathcal{C}$ we define an isomorphism $c_{V,W} : V \otimes W \to W \otimes V$ by

$$c_{V,W} := [(\varepsilon^F_W)^{-1} \otimes \mathrm{id}_V] \circ \widehat{c}_{V,W}.$$

This defines a G-compatible braiding c on $\mathcal{C}$. The two assignments $c \mapsto \widehat{c}$ and $\widehat{c} \mapsto c$ are clearly inverse to each other and establish a one-to-one correspondence between G-compatible braidings and G-crossed braidings on $\mathcal{C}$. This correspondence allows one to be a bit sloppy about braidings on a G-category. For instance, given a braided G-crossed category $\mathcal{D}$ with braiding c, one often says that its full G-subcategory $\mathcal{D}_e$ determined by the objects of degree e has a G-compatible braiding, although the restriction of c to $\mathcal{D}_e$ is actually a G-crossed braiding on $\mathcal{D}_e$. Note that these subtleties do not arise at all in strict G-categories, because in the strict case G-compatible braidings and G-crossed braidings are the same.

Definition 2.8.19 Let G be a group and let $(\mathcal{C}, \otimes, I, a, l, r, F, \partial, c)$ and $(\mathcal{C}', \otimes', I', a', l', r', F', \partial', c')$ be braided G-crossed categories. A *braided G-crossed functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a G-crossed functor $(K, \varepsilon^K, \delta^K, \xi^K) : \mathcal{C} \to \mathcal{C}'$ such that the diagram

$$\begin{array}{ccccc}
K(V) \otimes' K(W) & \xrightarrow{\delta^K_{V,W}} & K(V \otimes W) & \xrightarrow{K(c_{V,W})} & K(F_{\partial(V)}(W) \otimes V) \\
\big\downarrow{\scriptstyle c'_{K(V),K(W)}} & & & & \big\downarrow{\scriptstyle (\delta^K_{F_{\partial(V)}(W),V})^{-1}} \\
F'_{\partial(V)}(K(W)) \otimes' K(V) & & \xleftarrow{\xi^K(\partial(V))_W \otimes' \mathrm{id}_{K(V)}} & & K(F_{\partial(V)}(W)) \otimes' K(V)
\end{array}$$

commutes for all $V \in \mathcal{C}_{\mathrm{hom}}$ and $W \in \mathcal{C}$. A *natural braided G-crossed transformation* from one braided G-crossed functor to another is just a natural G-crossed transformation of the underlying G-crossed functors.

Definition 2.8.20 Let G be a group and let $(\mathcal{C}, \otimes, I, a, l, r, F, \partial, c)$ and $(\mathcal{C}', \otimes', I', a', l', r', F', \partial', c')$ be braided G-crossed categories. A *braided G-crossed equivalence* (or *equivalence of braided G-crossed categories*) from $\mathcal{C}$ to $\mathcal{C}'$ is a braided G-crossed functor $(K, \varepsilon^K, \delta^K, \xi^K) : \mathcal{C} \to \mathcal{C}'$ such that there exists a braided G-crossed functor $(L, \varepsilon^L, \delta^L, \xi^L) : \mathcal{C}' \to \mathcal{C}$ together with natural braided G-crossed isomorphisms $\varphi : \mathrm{id}_{\mathcal{C}'} \to K \circ L$ and $\psi : L \circ K \to \mathrm{id}_{\mathcal{C}}$. If there exists such a braided G-crossed equivalence, then $\mathcal{C}$ and $\mathcal{C}'$ will be called equivalent braided G-crossed categories.

Note that the definition of a braiding for a crossed G-category is quite asymmetrical in the two objects. It should come as no surprise that there is also another possible definition for a braiding in a G-crossed category:

Definition 2.8.21 A *braiding of the second kind* on a G-crossed category $(\mathcal{C}, \otimes, I, a, l, r, F, \partial)$ is a family of isomorphisms $\{c_{V,W} : V \otimes W \to W \otimes F_{\partial(W)^{-1}}(V)\}_{V \in \mathcal{C}, W \in \mathcal{C}_{\mathrm{hom}}}$ satisfying naturality in the sense that

$$
\begin{CD}
V \otimes W @>{c_{V,W}}>> W \otimes F_{\partial(W)^{-1}}(V) \\
@V{f\otimes g}VV @VV{g \otimes F_{\partial(W)^{-1}}(f)}V \\
V' \otimes W' @>{c_{V',W'}}>> W' \otimes F_{\partial(W')^{-1}}(V')
\end{CD}
$$

commutes for all $V \in \mathcal{C}$, $W \in \mathcal{C}_{\mathrm{hom}}$, $f \in \mathrm{Hom}_{\mathcal{C}}(V,V')$ and $g \in \mathrm{Hom}_{\mathcal{C}}(W,W')$ with $\partial(W) = \partial(W')$, as well as commutativity of the diagrams

$$
\begin{CD}
(V \otimes F_{\partial(V)^{-1}}(U)) \otimes W @<{c_{U,V}\otimes \mathrm{id}_W}<< (U \otimes V) \otimes W @>{a_{U,V,W}}>> U \otimes (V \otimes W) \\
@V{a_{V,F_{\partial(V)^{-1}}(U),W}}VV @. @VV{c_{U,V\otimes W}}V \\
V \otimes (F_{\partial(V)^{-1}}(U) \otimes W) @. @. (V \otimes W) \otimes F_{\partial(W)^{-1}\partial(V)^{-1}}(U) \\
@V{\mathrm{id}_V \otimes c_{F_{\partial(V)^{-1}}(U),W}}VV @. @VV{\mathrm{id}_{V\otimes W} \otimes (\delta^F_{\partial(W)^{-1},\partial(V)^{-1}})^{-1}_U}V \\
V \otimes (W \otimes F_{\partial(W)^{-1}}(F_{\partial(V)^{-1}}(U))) @>{a^{-1}_{V,W,F_{\partial(W)^{-1}}(F_{\partial(V)^{-1}}(U))}}>> {} @>>> (V \otimes W) \otimes F_{\partial(W)^{-1}}(F_{\partial(V)^{-1}}(U))
\end{CD}
$$

for all $U \in \mathcal{C}$, $V, W \in \mathcal{C}_{\mathrm{hom}}$;

$$
\begin{CD}
U \otimes (W \otimes F_{\partial(W)^{-1}}(V)) @<{\mathrm{id}_U \otimes c_{V,W}}<< U \otimes (V \otimes W) @>{a^{-1}_{U,V,W}}>> (U \otimes V) \otimes W \\
@V{a^{-1}_{U,W,F_{\partial(W)^{-1}}(V)}}VV @. @VV{c_{U\otimes V,W}}V \\
(U \otimes W) \otimes F_{\partial(W)^{-1}}(V) @. @. W \otimes F_{\partial(W)^{-1}}(U \otimes V) \\
@V{c_{U,W} \otimes \mathrm{id}_{F_{\partial(W)^{-1}}(V)}}VV @. @VV{\mathrm{id}_W \otimes (\delta^{\partial(W)^{-1}}_{U,V})^{-1}}V \\
(W \otimes F_{\partial(W)^{-1}}(U)) \otimes F_{\partial(W)^{-1}}(V) @>{a_{W,F_{\partial(W)^{-1}}(U),F_{\partial(W)^{-1}}(V)}}>> {} @>>> W \otimes (F_{\partial(W)^{-1}}(U) \otimes F_{\partial(W)^{-1}}(V))
\end{CD}
$$

for all $U, V \in \mathcal{C}$ and $W \in \mathcal{C}_{\mathrm{hom}}$;

$$
\begin{CD}
F_q(V) \otimes F_q(W) @>{c_{F_q(V),F_q(W)}}>> F_q(W) \otimes F_{q\partial(W)^{-1}q^{-1}}(F_q(V)) \\
@V{\delta^q_{V,W}}VV @VV{\mathrm{id}_{F_q(W)} \otimes (\delta^F_{q\partial(W)^{-1}q^{-1},q})_V}V \\
F_q(V \otimes W) @. F_q(W) \otimes F_{q\partial(W)^{-1}}(V) \\
@V{F_q(c_{V,W})}VV @VV{\mathrm{id}_{F_q(W)} \otimes (\delta^F_{q,\partial(W)^{-1}})^{-1}_V}V \\
F_q(W \otimes F_{\partial(W)^{-1}}(V)) @>{(\delta^q_{W,F_{\partial(W)^{-1}}(V)})^{-1}}>> F_q(W) \otimes F_q(F_{\partial(W)^{-1}}(V))
\end{CD}
$$

for all $V \in \mathcal{C}$, $W \in \mathcal{C}_{\mathrm{hom}}$ and $q \in G$.

Remark 2.8.22 We have two remarks concerning braidings of the second kind, which will be important to us later.

(1) In case the G-crossed category is strict, the two heptagonal diagrams reduce to the two equations

$$c_{U,V\otimes W} = [\mathrm{id}_V \otimes c_{F_{\partial(V)^{-1}}(U),W}] \circ [c_{U,V} \otimes \mathrm{id}_W]$$
$$c_{U\otimes V,W} = [c_{U,W} \otimes \mathrm{id}_{F_{\partial(W)^{-1}}(V)}] \circ [\mathrm{id}_U \otimes c_{V,W}].$$

(2) Suppose that $\mathcal{C}$ is a strict G-crossed category. If c is a braiding of the first kind, then $\widetilde{c}_{U,V} = c^{-1}_{V,F_{\partial(V)^{-1}}(U)}$ defines a braiding of the second kind. Conversely, if c is a braiding of the second kind, then $\widetilde{c}_{U,V} = c^{-1}_{F_{\partial(U)}(V),U}$ defines a braiding of the first kind. These two operations on braidings are inverse to each other and define a bijective correspondence between braidings of the first and second kind.

2.8.4 Braided G-crossed categories as a collection of module categories

This subsection is somewhat independent of the content of the rest of this thesis. We will only come back to the content of this subsection in Section 5.2 of our final chapter.

Let $(\mathcal{C}, \otimes, I, F, \partial, c)$ be a braided strict G-crossed category. Then for each $q \in G$ the full subcategory $\mathcal{C}_q$ determined by the objects of degree q is a strict $\mathcal{C}_e$-bimodule category, where both the left and right $\mathcal{C}_e$ actions $\triangleright$ and $\triangleleft$ on $\mathcal{C}_q$ are defined to be the tensor product $\otimes$. Because $\mathcal{C}$ is braided, we expect that it should be possible to somehow describe this $\mathcal{C}_e$-bimodule structure of $\mathcal{C}_q$ in terms of either $\triangleright$ alone or $\triangleleft$ alone. We will now make this more precise. For this we will for the moment consider $\mathcal{C}_q$ as a strict right $\mathcal{C}_e$-module category $(\mathcal{C}_q, \otimes)$. Because $\mathcal{C}_e$ is braided, we can apply Lemma 2.6.4 to equip $\mathcal{C}_q$ with the structure of a (non-strict) $\mathcal{C}_e$-bimodule category $(\mathcal{C}_q, \otimes^{\mathrm{op}}, \alpha, \otimes, \gamma)$, where the isomorphisms

$$\alpha_M(X,Y) : \underbrace{(X \otimes Y) \otimes^{\mathrm{op}} M}_{=M\otimes X\otimes Y} \to \underbrace{X \otimes^{\mathrm{op}} (Y \otimes^{\mathrm{op}} M)}_{=M\otimes Y\otimes X}$$
$$\gamma_M(X,U) : \underbrace{(X \otimes^{\mathrm{op}} M) \otimes U}_{=M\otimes X\otimes U} \to \underbrace{X \otimes^{\mathrm{op}} (M \otimes U)}_{=M\otimes U\otimes X}$$

are defined by $\alpha_M(X,Y) := \mathrm{id}_M \otimes c_{X,Y}$ and $\gamma_M(X,U) = \mathrm{id}_M \otimes c_{X,U}$ for $M \in \mathcal{C}_q$ and $X, Y, U \in \mathcal{C}_e$.

Proposition 2.8.23 *Let $(\mathcal{C}, \otimes, I, F, \partial, c)$ be a braided G-crossed category and let $(\mathcal{C}_q, \otimes^{\mathrm{op}}, \alpha, \otimes, \gamma)$ be the $\mathcal{C}_e$-bimodule category defined above. Then we have an equivalence*

$$(H, s, t) : (\mathcal{C}_q, \otimes, \otimes) \to (\mathcal{C}_q, \otimes^{\mathrm{op}}, \alpha, \otimes, \gamma)$$

of $\mathcal{C}_e$-bimodule categories, where $H = \mathrm{id}_{\mathcal{C}_q}$, $s_M(X) = c_{X,M}$ and $t_M(X) = \mathrm{id}_{M\otimes X}$ for all $X \in \mathcal{C}_e$ and $M \in \mathcal{C}_q$.

Proof. It is obvious that H is an equivalence of categories, so we only need to show that (H, s, t) is a functor of $\mathcal{C}_e$-bimodule categories. That (H, t) is a functor of right $\mathcal{C}_e$-module catories is trivial. To see that (H, s) is a functor of left $\mathcal{C}_e$-module categories, we note that naturality of s follows from naturality of c and that

$$[\mathrm{id}_X \otimes^{\mathrm{op}} s_M(Y)] \circ s_{Y\otimes M}(X) = [s_M(Y) \otimes \mathrm{id}_X] \circ s_{Y\otimes M}(X) = [c_{Y,M} \otimes \mathrm{id}_X] \circ c_{X,Y\otimes M}$$
$$= [\mathrm{id}_M \otimes c_{X,Y}] \circ c_{X\otimes Y,M} = \alpha_{H(M)}(X,Y) \circ s_M(X \otimes Y).$$

Finally, (H, s, t) is also a functor of $\mathcal{C}_e$-bimodule categories, since

$$s_{M\otimes Y}(X) = c_{X,M\otimes Y} = [\mathrm{id}_M \otimes c_{X,Y}] \circ [c_{X,M} \otimes \mathrm{id}_Y] = \gamma_M(X,Y) \circ [s_M(X) \otimes \mathrm{id}_Y].$$

□

2.8.5 The mirror image of a braided G-crossed category

Our next goal is to show that if we are given a braided G-crossed category $\mathcal{C}$, then we can alter the tensor product, G-grading and braiding in such a way that we again obtain a braided G-crossed category $\mathcal{C}^\bullet$ which we will call the mirror image of $\mathcal{C}$. The reason for this terminology is that when we will study the relationship between left and right G-localized endomorphisms of a quantum field theory in Chapter 3, we will find that the categories of these left and right G-localized endomorphisms are related to each other in exactly the same way as $\mathcal{C}$ and $\mathcal{C}^\bullet$ are related to each other.

Theorem 2.8.24 *Let $(\mathcal{D}, \otimes, I, G, F, \partial, C)$ be a braided*[15] *G-crossed category. Then we obtain a braided G-crossed category $(\mathcal{D}, \bullet, I, G, F, \partial_\bullet, C^\bullet)$, where the tensor product $\bullet$ is defined by*

$$U \bullet V := U \otimes F_{q^{-1}}(V)$$
$$f \bullet g := f \otimes F_{r^{-1}}(g)$$

for $U, V \in \mathcal{D}$, $q = \partial(U)$, $f, g \in \mathrm{Hom}(\mathcal{D})$ and $r = \partial(f)$, the degree map $\partial_\bullet$ is defined by $\partial_\bullet(U) := \partial(U)^{-1}$ for $U \in \mathcal{D}$ and the braiding is defined by

$$C^\bullet_{U,V} = C^{-1}_{F_{\partial(U)^{-1}}(V), F_{\partial(U)^{-1}\partial(V)^{-1}\partial(U)}(U)}$$

for any $U, V \in \mathcal{D}$.

Proof. To see that $\bullet : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ is a functor, let $f \in \mathrm{Hom}_\mathcal{D}(U, U')$ and $g \in \mathrm{Hom}_\mathcal{D}(V, V')$ with $\partial(U) = \partial(U') =: q$. Then

$$f \bullet g = f \otimes F_{q^{-1}}(g) \in \mathrm{Hom}_\mathcal{D}(U \otimes F_{q^{-1}}(V), U' \otimes F_{q^{-1}}(V')) = \mathrm{Hom}_\mathcal{D}(U \bullet V, U' \bullet V').$$

The interchange law for $\bullet$ is a consequence of the interchange law for $\otimes$ and of the fact that F_r is a functor for each $r \in G$ as we will show now[16]. If $f_1, f_2, g_1, g_2 \in \mathrm{Hom}(\mathcal{D})$ are such that $f_2 \circ f_1$ and $g_2 \circ g_1$ are well-defined, then

$$\begin{aligned}(f_2 \circ f_1) \bullet (g_2 \circ g_1) &= (f_2 \circ f_1) \otimes F_{q^{-1}}(g_2 \circ g_1) = (f_2 \circ f_1) \otimes (F_{q^{-1}}(g_2) \circ F_{q^{-1}}(g_1)) \\ &= (f_2 \otimes F_{q^{-1}}(g_2)) \circ (f_1 \otimes F_{q^{-1}}(g_1)) = (f_2 \bullet g_2) \circ (f_1 \bullet g_1),\end{aligned}$$

where $q = \partial(f)$, proving the interchange law for $\bullet$. If $U, V \in \mathcal{D}$ and $q = \partial(U)$, then

$$\mathrm{id}_U \bullet \mathrm{id}_V = \mathrm{id}_U \otimes F_{q^{-1}}(\mathrm{id}_V) = \mathrm{id}_{U \otimes F_{q^{-1}}(V)} = \mathrm{id}_{U \bullet V},$$

showing that $\bullet$ is indeed a functor. If $U, V, W \in \mathcal{D}$ with $q = \partial(U)$ and $r = \partial(V)$, then

$$\begin{aligned}(U \bullet V) \bullet W &= (U \otimes F_{q^{-1}}(V)) \bullet W = (U \otimes F_{q^{-1}}(V)) \otimes F_{(qq^{-1}rq)^{-1}}(W) \\ &= U \otimes F_{q^{-1}}(V) \otimes F_{q^{-1}r^{-1}}(W) \\ &= U \otimes F_{q^{-1}}(V \otimes F_{r^{-1}}(W)) = U \bullet (V \otimes F_{r^{-1}}(W)) \\ &= U \bullet (V \bullet W)\end{aligned}$$

and if $f, g, h \in \mathrm{Hom}(\mathcal{D})$ with $q = \partial(f)$ and $r = \partial(g)$ then a similar computation gives us

$$(f \bullet g) \bullet h = (f \otimes F_{q^{-1}}(g)) \bullet h = (f \otimes F_{q^{-1}}(g)) \otimes F_{(qq^{-1}rq)^{-1}}(h)$$

[15] In this proposition we will use a capital letter C to denote the braiding, because this looks better than a small letter c when large subindices are involved.

[16] In fact, the fact that $\bullet$ is a functor actually follows from the fact that $\bullet$ can be written as a composition $\bullet = \otimes \circ (\mathrm{id}_\mathcal{D} \times F_r)$ of functors, for a certain r depending on the particular objects or morphisms involved.

$$\begin{aligned}&= f \otimes F_{q^{-1}}(g) \otimes F_{q^{-1}r^{-1}}(h)\\ &= f \otimes F_{q^{-1}}(g \otimes F_{r^{-1}}(h)) = f \bullet (g \otimes F_{r^{-1}}(h))\\ &= f \bullet (g \bullet h),\end{aligned}$$

showing that $\bullet$ is associative. For the unit object we find that for any $U \in \mathcal{D}$ with $q = \partial(U)$

$$U \bullet I = U \otimes F_{q^{-1}}(I) = U \otimes I = U$$
$$I \bullet U = I \otimes F_{e^{-1}}(U) = I \otimes U = U$$

and for any $f \in \mathrm{Hom}(\mathcal{D})$ with $q = \partial(f)$ we have

$$f \bullet \mathrm{id}_I = f \otimes F_{q^{-1}}(\mathrm{id}_I) = f \otimes \mathrm{id}_I = f$$
$$\mathrm{id}_I \bullet f = \mathrm{id}_I \otimes F_{e^{-1}}(f) = \mathrm{id}_I \otimes f = f,$$

showing that $(\mathcal{D}, \bullet, I)$ is a strict tensor category. If $U, V \in \mathcal{D}$ with $q = \partial(U)$ and if $t \in G$, then

$$\begin{aligned}F_t(U \bullet V) &= F_t(U \otimes F_{q^{-1}}(V)) = F_t(U) \otimes F_{tq^{-1}}(V) = F_t(U) \otimes F_{(tqt^{-1})^{-1}}(F_t(V))\\ &= F_t(U) \bullet F_t(V).\end{aligned}$$

If $f, g \in \mathrm{Hom}(\mathcal{D})$ with $q = \partial(f)$, then a similar computation gives

$$\begin{aligned}F_t(f \bullet g) &= F_t(f \otimes F_{q^{-1}}(g)) = F_t(f) \otimes F_{tq^{-1}}(g) = F_t(f) \otimes F_{(tqt^{-1})^{-1}}(F_t(g))\\ &= F_t(f) \bullet F_t(g).\end{aligned}$$

Together with the fact that $F_{ut} = F_u \circ F_t$, this shows that F defines a G-action on $(\mathcal{D}, \bullet, I)$. The map $\partial_\bullet$ satisfies

$$\begin{aligned}\partial_\bullet(U \bullet V) &= \partial(U \otimes F_{\partial(U)^{-1}}(V))^{-1} = [\partial(U)\partial(F_{\partial(U)^{-1}}(V))]^{-1} = [\partial(U)\partial(U)^{-1}\partial(V)\partial(U)]^{-1}\\ &= \partial(U)^{-1}\partial(V)^{-1} = \partial_\bullet(U)\partial_\bullet(V),\end{aligned}$$

so it is indeed a degree map. Also, for any $U \in \mathcal{D}$ and $t \in G$ we have

$$\partial_\bullet(F_t(U)) = \partial(F_t(U))^{-1} = [t\partial(U)t^{-1}]^{-1} = t\partial(U)^{-1}t^{-1} = t\partial_\bullet(U)t^{-1},$$

which shows that $(\mathcal{D}, \bullet, I, G, F, \partial_\bullet)$ is G-crossed.

Finally, we will show that $C^\bullet$ is indeed a braiding. Let $U, V \in \mathcal{D}$ and consider the braiding isomorphism $C_{F_{\partial(U)^{-1}}(V), F_{\partial(U)^{-1}\partial(V)^{-1}\partial(U)}(U)}$. This is a an isomorphism

$$F_{\partial(U)^{-1}}(V) \otimes F_{\partial(U)^{-1}\partial(V)^{-1}\partial(U)}(U) \to F_{\partial(U)^{-1}\partial(V)\partial(U)}[F_{\partial(U)^{-1}\partial(V)^{-1}\partial(U)}(U)] \otimes F_{\partial(U)^{-1}}(V).$$

The object on the left equals $F_{\partial(U)^{-1}}(V) \bullet U = F_{\partial_\bullet(U)}(V) \bullet U$ and the object on the right equals $U \otimes F_{\partial(U)^{-1}}(V) = U \bullet V$, so $C^\bullet_{U,V}$ is indeed an isomorphism from $U \bullet V$ to $F_{\partial_\bullet(U)}(V) \bullet U$, as it should. To show naturality, let $U, U', V, V' \in \mathcal{C}$ with $\partial_\bullet(U) = \partial_\bullet(U')$ and $\partial_\bullet(V) = \partial_\bullet(V')$, and let $f : U \to U'$ and $g : V \to V'$. In what follows, we will write $q := \partial(U) = \partial_\bullet(U)^{-1}$ and $r := \partial(V) = \partial_\bullet(V)^{-1}$. Because C is a braiding, we have

$$[f \otimes F_{q^{-1}}(g)] \circ C_{F_{q^{-1}}(V), F_{q^{-1}r^{-1}q}(U)} = C_{F_{q^{-1}}(V'), F_{q^{-1}r^{-1}q}(U')} \circ [F_{q^{-1}}(g) \otimes F_{q^{-1}r^{-1}q}(f)]$$

by naturality. Composing this equation from the left with $C^{-1}_{F_{q^{-1}}(V'), F_{q^{-1}r^{-1}q}(U')}$ and from the right with $C^{-1}_{F_{q^{-1}}(V), F_{q^{-1}r^{-1}q}(U)}$, we get

$$C^{-1}_{F_{q^{-1}}(V'), F_{q^{-1}r^{-1}q}(U')} \circ [f \otimes F_{q^{-1}}(g)] = [F_{q^{-1}}(g) \otimes F_{q^{-1}r^{-1}q}(f)] \circ C^{-1}_{F_{q^{-1}}(V), F_{q^{-1}r^{-1}q}(U)},$$

which is precisely

$$C^{\bullet}_{U',V'} \circ [f \bullet g] = [F_{\partial_{\bullet}(U)}(g) \bullet f] \circ C^{\bullet}_{U,V},$$

showing naturality of $C^{\bullet}$. Now let $U, V, W \in \mathcal{D}$. Then

$$\begin{aligned}
&{C^{\bullet}_{U\bullet V,W}}^{-1} \\
&= C_{F_{\partial(U\bullet V)^{-1}}(W), F_{(U\bullet V)^{-1}\partial(W)^{-1}\partial(U\bullet V)}(U\bullet V)} \\
&= C_{F_{\partial(U)^{-1}\partial(V)^{-1}}(W), F_{\partial(U)^{-1}\partial(V)^{-1}\partial(W)^{-1}\partial(V)\partial(U)}(U\otimes F_{\partial(U)^{-1}}(V))} \\
&= C_{F_{\partial(U)^{-1}\partial(V)^{-1}}(W), F_{\partial(U)^{-1}\partial(V)^{-1}\partial(W)^{-1}\partial(V)\partial(U)}(U)\otimes F_{\partial(U)^{-1}\partial(V)^{-1}\partial(W)^{-1}\partial(V)}(V)} \\
&= F_{\partial(U)^{-1}\partial(V)^{-1}}\left(C_{W,F_{\partial(W)^{-1}\partial(V)\partial(U)}(U)\otimes F_{\partial(W)^{-1}\partial(V)}(V)}\right) \\
&= F_{\partial(U)^{-1}\partial(V)^{-1}}\left\{[\mathrm{id}_{F_{\partial(V)\partial(U)}(U)} \otimes C_{W,F_{\partial(W)^{-1}\partial(V)}(V)}] \circ [C_{W,F_{\partial(W)^{-1}\partial(V)\partial(U)}(U)} \otimes \mathrm{id}_{F_{\partial(W)^{-1}\partial(V)}(V)}]\right\} \\
&= [\mathrm{id}_U \otimes C_{F_{\partial(U)^{-1}\partial(V)^{-1}}(W), F_{\partial(U)^{-1}\partial(V)^{-1}\partial(W)^{-1}\partial(V)}(V)}] \\
&\quad \circ [C_{F_{\partial(U)^{-1}\partial(V)^{-1}}(W), F_{\partial(U)^{-1}\partial(V)^{-1}\partial(W)^{-1}\partial(V)\partial(U)}(U)} \otimes \mathrm{id}_{F_{\partial(U)^{-1}\partial(V)^{-1}\partial(W)^{-1}\partial(V)}(V)}] \\
&= [\mathrm{id}_U \otimes F_{\partial(U)^{-1}}(C_{F_{\partial(V)^{-1}}(W), F_{\partial(V)^{-1}\partial(W)^{-1}\partial(V)}(V)})] \\
&\quad \circ [C_{F_{\partial(U)^{-1}\partial(V)^{-1}}(W), F_{\partial(U)^{-1}\partial(V)^{-1}\partial(W)^{-1}\partial(V)\partial(U)}(U)} \otimes F_{\partial(U)^{-1}\partial(V)^{-1}\partial(W)^{-1}\partial(V)}(\mathrm{id}_V)] \\
&= [\mathrm{id}_U \bullet {C^{\bullet}_{V,W}}^{-1}] \circ [{C^{\bullet}_{U,F_{\partial_{\bullet}(V)}(W)}}^{-1} \bullet \mathrm{id}_V]
\end{aligned}$$

and similarly

$$\begin{aligned}
{C^{\bullet}_{U,V\bullet W}}^{-1} &= C_{F_{\partial(U)^{-1}}(V\bullet W), F_{\partial(U)^{-1}\partial(V\bullet W)^{-1}\partial(U)}(U)} \\
&= C_{F_{\partial(U)^{-1}}(V\otimes F_{\partial(V)^{-1}}(W)), F_{\partial(U)^{-1}\partial(V)^{-1}\partial(W)^{-1}\partial(U)}(U)} \\
&= C_{F_{\partial(U)^{-1}}(V)\otimes F_{\partial(U)^{-1}\partial(V)^{-1}}(W), F_{\partial(U)^{-1}\partial(V)^{-1}\partial(W)^{-1}\partial(U)}(U)} \\
&= F_{\partial(U)^{-1}}\left(C_{V\otimes F_{\partial(V)^{-1}}(W), F_{\partial(V)^{-1}\partial(W)^{-1}\partial(U)}(U)}\right) \\
&= F_{\partial(U)^{-1}}\left\{[C_{V,F_{\partial(V)^{-1}\partial(U)}(U)} \otimes \mathrm{id}_{F_{\partial(V)^{-1}}(W)}] \circ [\mathrm{id}_V \otimes C_{F_{\partial(V)^{-1}}(W), F_{\partial(V)^{-1}\partial(W)^{-1}\partial(U)}(U)}]\right\} \\
&= [C_{F_{\partial(U)^{-1}}(V), F_{\partial(U)^{-1}\partial(V)^{-1}\partial(U)}(U)} \otimes \mathrm{id}_{F_{\partial(U)^{-1}\partial(V)^{-1}}(W)}] \\
&\quad \circ [\mathrm{id}_{F_{\partial(U)^{-1}}(V)} \otimes C_{F_{\partial(U)^{-1}\partial(V)^{-1}}(W), F_{\partial(U)^{-1}\partial(V)^{-1}\partial(W)^{-1}\partial(U)}(U)}] \\
&= [C_{F_{\partial(U)^{-1}}(V), F_{\partial(U)^{-1}\partial(V)^{-1}\partial(U)}(U)} \otimes F_{\partial(U)^{-1}\partial(V)^{-1}}(\mathrm{id}_W)] \\
&\quad \circ [\mathrm{id}_{F_{\partial(U)^{-1}}(V)} \otimes F_{\partial(U)^{-1}\partial(V)^{-1}\partial(U)}\left(C_{F_{\partial(U)^{-1}}(W), F_{\partial(U)^{-1}\partial(W)^{-1}\partial(U)}(U)}\right)] \\
&= [{C^{\bullet}_{U,V}}^{-1} \bullet \mathrm{id}_W] \circ [\mathrm{id}_{F_{\partial_{\bullet}(U)}(V)} \bullet {C^{\bullet}_{U,W}}^{-1}].
\end{aligned}$$

Taking the inverse on both sides in these two equations gives

$$\begin{aligned}
C^{\bullet}_{U\bullet V,W} &= [C^{\bullet}_{U,F_{\partial_{\bullet}(V)}(W)} \bullet \mathrm{id}_V] \circ [\mathrm{id}_U \bullet C^{\bullet}_{V,W}] \\
C^{\bullet}_{U,V\bullet W} &= [\mathrm{id}_{F_{\partial_{\bullet}(U)}(V)} \bullet C^{\bullet}_{U,W}] \circ [C^{\bullet}_{U,V} \bullet \mathrm{id}_W].
\end{aligned}$$

□

In what follows, if $\mathcal{D}$ is a G-crossed category then we will simply write $\mathcal{D}^{\bullet}$ to denote the G-crossed category that is obtained as in the previous proposition.

2.9 Constructions from Frobenius algebras

We will now consider several important constructions that arise from Frobenius algebras in a tensor category. Of course, we have to begin by providing the definition of a Frobenius algebra in tensor category.

Definition 2.9.1 Let $(\mathcal{D}, \otimes, I)$ be a strict tensor category. A *Frobenius algebra* in $\mathcal{D}$ is a quintuple $\mathsf{Q} = (Q, \mu, \eta, \Delta, \varepsilon)$, where $Q \in \mathcal{D}$ and (Q, μ, η) is an algebra in $\mathcal{C}$ and (Q, Δ, ε) is a coalgebra in $\mathcal{C}$, satisfying the additional property that

$$[\mu \otimes \mathrm{id}_Q] \circ [\mathrm{id}_Q \otimes \Delta] = \Delta \circ \mu = [\mathrm{id}_Q \otimes \mu] \circ [\Delta \otimes \mathrm{id}_Q]. \tag{2.9.1}$$

The property in (2.9.1) is also called the *Frobenius property*. In case $\mathcal{D}$ is $\mathbb{F}$-linear for some field $\mathbb{F}$, the Frobenius algebra is called *strongly separable* (or *special*) if

$$\mu \circ \Delta = (\kappa_{\mathsf{Q}})_1 \mathrm{id}_Q \tag{2.9.2}$$

$$\varepsilon \circ \eta = (\kappa_{\mathsf{Q}})_2 \mathrm{id}_I \tag{2.9.3}$$

with $(\kappa_Q)_1, (\kappa_Q)_2 \in \mathbb{F}^*$. A strongly separable Frobenius algebra is called *normalized* if $(\kappa_{\mathsf{Q}})_1 = (\kappa_{\mathsf{Q}})_2$, in which case we simply write κ_{Q} to denote both $(\kappa_{\mathsf{Q}})_1$ and $(\kappa_{\mathsf{Q}})_2$ and we will call κ_{Q} the normalization constant of Q.

Example 2.9.2 If $(\mathcal{D}, \otimes, I)$ is a strict tensor category, then $(I, \mu^0, \eta^0, \Delta^0, \varepsilon^0)$ is a Frobenius algebra if we take $\mu^0 = \eta^0 = \Delta^0 = \varepsilon^0 = \mathrm{id}_I$. We will call this the *trivial Frobenius algebra* from now on and we will denote it by Q_0. If $\mathcal{D}$ is $\mathbb{F}$-linear it is obvious that Q_0 is normalized[17] with $\kappa_{\mathsf{Q}_0} = 1$.

Remark 2.9.3 As a direct consequence of the axioms for a Frobenius algebra, any Frobenius algebra $\mathsf{Q} = (Q, \mu, \eta, \Delta, \varepsilon)$ is a two-sided dual of itself, where we can take $b_Q = b'_Q = \Delta \circ \eta$ and $d_Q = d'_Q = \varepsilon \circ \mu$.

Example 2.9.4 If $\mathcal{D}$ is a strict tensor category and $(\overline{V}, b, d, b', d')$ is a two-sided dual for $V \in \mathcal{D}$, then we can make $Q := \overline{V} \otimes V$ into a Frobenius algebra by defining

$$\mu := \mathrm{id}_{\overline{V}} \otimes d' \otimes \mathrm{id}_V, \qquad \eta := b', \qquad \Delta := \mathrm{id}_{\overline{V}} \otimes b \otimes \mathrm{id}_V, \qquad \varepsilon := d.$$

Similarly, we can also make $Q' = V \otimes \overline{V}$ into a Frobenius algebra by defining

$$\mu' := \mathrm{id}_V \otimes d \otimes \mathrm{id}_{\overline{V}}, \qquad \eta' := b, \qquad \Delta' := \mathrm{id}_V \otimes b' \otimes \mathrm{id}_{\overline{V}}, \qquad \varepsilon' := d'.$$

In Subsection 2.9.3 we will slightly generalize this.

If $\mathsf{Q} = (Q, \mu, \eta, \Delta, \varepsilon)$ is a Frobenius algebra, then we will often write μ_2 and Δ_2 to denote the morphisms

$$\mu_2 := \mu \circ [\mu \otimes \mathrm{id}_Q] = \mu \circ [\mathrm{id}_Q \otimes \mu]$$
$$\Delta_2 := [\Delta \otimes \mathrm{id}_Q] \circ \Delta = [\mathrm{id}_Q \otimes \Delta] \circ \Delta,$$

since this can save space significantly. The following easy lemma can be convenient sometimes.

Lemma 2.9.5 *Let $(\mathcal{D}, \otimes, I)$ be a strict tensor category and let $\mathsf{Q} = (Q, \mu, \eta, \Delta, \varepsilon)$ be a Frobenius algebra in $\mathcal{D}$. Then*

$$[\mathrm{id}_{Q\otimes Q} \otimes \mu_2] \circ [\Delta_2 \otimes \mathrm{id}_{Q\otimes Q}] = [\mathrm{id}_{Q\otimes Q} \otimes \mu] \circ [\mathrm{id}_Q \otimes (\Delta \circ \mu) \otimes \mathrm{id}_Q] \circ [\Delta \otimes \mathrm{id}_{Q\otimes Q}].$$

[17] This can also be established by choosing the morphisms to be scalar multiples of id_I satisfying some restriction, but we will not be needing this.

Proof. We have

$$\begin{aligned}
&[\mathrm{id}_{Q\otimes Q} \otimes \mu_2] \circ [\Delta_2 \otimes \mathrm{id}_{Q\otimes Q}] \\
&= [\mathrm{id}_{Q\otimes Q} \otimes \mu] \circ [\mathrm{id}_{Q\otimes Q} \otimes \mu \otimes \mathrm{id}_Q] \circ [\mathrm{id}_Q \otimes \Delta \otimes \mathrm{id}_{Q\otimes Q}] \circ [\Delta \otimes \mathrm{id}_{Q\otimes Q}] \\
&= [\mathrm{id}_{Q\otimes Q} \otimes \mu] \circ [\mathrm{id}_Q \otimes ((\mathrm{id}_Q \otimes \mu) \circ (\Delta \otimes \mathrm{id}_Q)) \otimes \mathrm{id}_Q] \circ [\Delta \otimes \mathrm{id}_{Q\otimes Q}] \\
&= [\mathrm{id}_{Q\otimes Q} \otimes \mu] \circ [\mathrm{id}_Q \otimes (\Delta \circ \mu) \otimes \mathrm{id}_Q] \circ [\Delta \otimes \mathrm{id}_{Q\otimes Q}],
\end{aligned}$$

where in the second step we used the Frobenius property (2.9.1).
□

2.9.1 Example: Frobenius algebras in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}$

We will now consider an important example of a Frobenius algebra, which is a slight generalization of the Frobenius algebra that appears in Proposition 4.1 of [75]. For us it is important because we will need it to prove our statements in Section 4.9, but we would like to mention that it also has some implications for subfactor theory as indicated in Subsection 8.3 of [75].

In this subsection $\mathbb{F}$ is a quadratically closed field and $\mathcal{C}$ will be a spherical fusion category over $\mathbb{F}$. In particular, to each irreducible object $X \in \mathcal{C}$ we can assign its dimension $d(X) \in \mathbb{F}$ and from now on we will assume that for each irreducible X we have chosen a square root $d(X)^{1/2}$ of its dimension. We will always assume that $\dim(\mathcal{C}) \neq 0$ and that we have chosen a square root $\kappa := \dim(\mathcal{C})^{1/2}$, together with a square root $\kappa^{1/2}$. Also, we will assume that we have chosen some fixed complete set $\{X_i : i \in \Gamma\}$ of representatives of equivalence classes of irreducible objects, and we will also assume that for each triple $(i, j, k) \in \Gamma^{\times 3}$ we have chosen a basis

$$\{(t^k_{ij})_\alpha : \alpha = 1, \ldots, N^k_{ij}\}$$

for $\mathrm{Hom}_{\mathcal{C}}(X_k, X_i \otimes X_j) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X_i^{\mathrm{op}} \otimes X_j^{\mathrm{op}}, X_k^{\mathrm{op}})$, together with a dual basis

$$\{(t^{ij}_k)_\alpha : \alpha = 1, \ldots, N^k_{ij}\}$$

for $\mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_k) = \mathrm{Hom}_{\mathcal{C}}(X_k^{\mathrm{op}}, X_i^{\mathrm{op}} \otimes X_j^{\mathrm{op}})$. For Γ we will take a finite subset of the form $\{0, 1, \ldots, n\}$, where $X_0 = I$. Furthermore, we will write $\mathcal{D} := \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}$ and we will denote the tensor product in $\mathcal{D}$ by $\otimes_2$ to distinguish it from the tensor product $\otimes$ in $\mathcal{C}$.

Proposition 2.9.6 *Let $F : \mathcal{C} \to \mathcal{C}$ be a strict tensor equivalence. Then there exists a normalized strongly separable Frobenius algebra $Q_F = (Q_F, \mu^F, \eta^F, \Delta^F, \varepsilon^F)$ in $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}$ such that*

$$Q_F \cong \bigoplus_{i\in\Gamma} F(X_i) \boxtimes X_i^{\mathrm{op}}$$

and with normalization constant $\kappa_{Q_F} = \kappa$.

We will only sketch how the Frobenius structure is defined. The verification of the axioms is done in the same way as in [75]. Let $Q_F \in \mathcal{D}$ be a direct sum $\bigoplus_{i\in\Gamma} F(X_i) \boxtimes X_i^{\mathrm{op}}$, i.e. we choose morphisms

$$\begin{aligned}
u_i &\in \mathrm{Hom}_{\mathcal{D}}(F(X_i) \boxtimes X_i^{\mathrm{op}}, Q_F) \\
v_i &\in \mathrm{Hom}_{\mathcal{D}}(Q_F, F(X_i) \boxtimes X_i^{\mathrm{op}})
\end{aligned}$$

such that $v_i \circ u_i = \mathrm{id}_{F(X_i)\boxtimes X_i^{\mathrm{op}}}$ and $\sum_{i\in\Gamma} u_i \circ v_i = \mathrm{id}_{Q_F}$. We then define

$$\begin{aligned}
\eta^F &:= \kappa^{1/2} u_0 \\
\varepsilon^F &:= \kappa^{1/2} v_0.
\end{aligned}$$

Next for each triple $(i,j,k) \in \Gamma^{\times 3}$ we define

$$T(F)_{ij}^k := \sum_\alpha F((t_{ij}^k)_\alpha) \boxtimes (t_k^{ij})_\alpha \in \mathrm{Hom}_{\mathcal{D}}(F(X_k) \boxtimes X_k^{\mathrm{op}}, (F(X_i) \boxtimes X_i^{\mathrm{op}}) \otimes_2 (F(X_j) \boxtimes X_j^{\mathrm{op}}))$$
$$T(F)_k^{ij} := \sum_\alpha F((t_k^{ij})_\alpha) \boxtimes (t_{ij}^k)_\alpha \in \mathrm{Hom}_{\mathcal{D}}((F(X_i) \boxtimes X_i^{\mathrm{op}}) \otimes_2 (F(X_j) \boxtimes X_j^{\mathrm{op}}), F(X_k) \boxtimes X_k^{\mathrm{op}}).$$

These are both independent of the choice of bases, as can be checked easily. The multiplication and comultiplication are defined by

$$\mu^F := \sum_{i,j,k} \beta_k^{ij} u_k \circ T(F)_k^{ij} \circ [v_i \otimes_2 v_j]$$
$$\Delta^F := \sum_{i,j,k} \beta_{ij}^k [u_i \otimes_2 u_j] \circ T(F)_{ij}^k \circ v_k$$

where $\beta_{ij}^k, \beta_k^{ij} \in \mathbb{F}$ are given by

$$\beta_{ij}^k = \beta_k^{ij} = \kappa^{-1/2} \sqrt{\frac{d(X_i)d(X_j)}{d(X_k)}}.$$

Here $\kappa^{-1/2} := (\kappa^{1/2})^{-1}$ and $\sqrt{\frac{d(X_i)d(X_j)}{d(X_k)}} := (d(X_k)^{1/2})^{-1} d(X_i)^{1/2} d(X_j)^{1/2}$.

2.9.2 A category constructed from a pair of Frobenius algebras

In this subsection we will show how to construct a category $\mathcal{D}(\mathsf{Q}_1, \mathsf{Q}_2)$ from a pair $(\mathsf{Q}_1, \mathsf{Q}_2)$ of Frobenius algebras in a tensor category $\mathcal{D}$. This construction is a small generalization of the construction in [74] where one Frobenius algebra Q was given and three categories were constructed. In our setting, these three categories would be called $\mathcal{D}(\mathsf{Q}_0, \mathsf{Q})$, $\mathcal{D}(\mathsf{Q}, \mathsf{Q}_0)$ and $\mathcal{D}(\mathsf{Q}, \mathsf{Q})$. The category $\mathcal{D}(\mathsf{Q}_1, \mathsf{Q}_2)$ will be used later for two different purposes. The first is the same as in [74], i.e. to construct a 2-category. This construction of a 2-category will take place immediately in the next subsection. Secondly, $\mathcal{D}(\mathsf{Q}_1, \mathsf{Q}_2)$ will be used to construct a crossed product of $\mathcal{D}$ with a symmetric subcategory, which will be done in Subsection 3.1.3. For this second application we have to assume that $\mathcal{D}$ is braided, as we will do in the second part of the present subsection.

Theorem 2.9.7 *Let $(\mathcal{D}, \otimes, I)$ be a strict tensor category and let $\mathsf{Q}_1 = (Q_1, \mu^1, \eta^1, \Delta^1, \varepsilon^1)$ and $\mathsf{Q}_2 = (Q_2, \mu^2, \eta^2, \Delta^2, \varepsilon^2)$ be two Frobenius algebras in $\mathcal{D}$. We will write*[18]

$$\mathrm{Obj}(\mathcal{D}(\mathsf{Q}_1, \mathsf{Q}_2)) := \{\overline{J}_2 V J_1 : V \in \mathcal{D}\}$$
$$\mathrm{Hom}_{\mathcal{D}(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 V J_1, \overline{J}_2 W J_1) := \mathrm{Hom}_{\mathcal{D}}(V \otimes Q_1, Q_2 \otimes W).$$

Then $\mathcal{D}(\mathsf{Q}_1, \mathsf{Q}_2)$ becomes a category if we define the composition of $f \in \mathrm{Hom}_{\mathcal{D}(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 U J_1, \overline{J}_2 V J_1)$ and $g \in \mathrm{Hom}_{\mathcal{D}(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 V J_1, \overline{J}_2 W J_1)$ by

$$g \bullet f := [\mu^2 \otimes \mathrm{id}_W] \circ [\mathrm{id}_{Q_2} \otimes g] \circ [f \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_U \otimes \Delta^1]$$

and if we define the identity morphisms $\mathrm{id}_{\overline{J}_2 V J_1} \in \mathrm{End}_{\mathcal{D}(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 V J_1) = \mathrm{Hom}_{\mathcal{D}}(V \otimes Q_1, Q_2 \otimes V)$ by

$$\mathrm{id}_{\overline{J}_2 V J_1} := \eta^2 \otimes \mathrm{id}_V \otimes \varepsilon^1.$$

[18] At this stage $\overline{J}_2$ and J_1 have no other purpose (yet) but to distinguish manifestly the objects of $\mathcal{D}(\mathsf{Q}_1, \mathsf{Q}_2)$ from those of $\mathcal{D}$. Later we will see that $\overline{J}_2$ and J_1 will obtain a particular meaning.

Furthermore, we obtain a functor $\mathscr{I}_0 : \mathcal{D} \to \mathcal{D}(\mathsf{Q}_1, \mathsf{Q}_2)$ *by defining*

$$\mathscr{I}_0(V) := \overline{J}_2 V J_1$$
$$\mathscr{I}_0(f) := \eta^2 \otimes f \otimes \varepsilon^1$$

and this functor is bijective on the objects. If $\mathcal{D}$ *is* $\mathbb{F}$*-linear and if* $\varepsilon^1 \circ \eta^1 = \alpha_1 \mathrm{id}_I$ *and* $\varepsilon^2 \circ \eta^2 = \alpha_2 \mathrm{id}_I$ *for some* $\alpha_1, \alpha_2 \in \mathbb{F}^*$*, then* $\mathscr{I}_0$ *is faithful.*

Proof. For the proof that $\mathcal{D}(\mathsf{Q}_1, \mathsf{Q}_2)$ is a category we refer to [74]. If $f \in \mathrm{Hom}_{\mathcal{D}}(U, V)$ then

$$\begin{aligned}\mathscr{I}_0(f) = \eta^2 \otimes f \otimes \varepsilon^1 \in &\mathrm{Hom}_{\mathcal{D}}(U \otimes Q_1, Q_2 \otimes V) = \mathrm{Hom}_{\mathcal{D}(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 U J_1, \overline{J}_2 V J_1)\\ &= \mathrm{Hom}_{\mathcal{D}(\mathsf{Q}_1,\mathsf{Q}_2)}(\mathscr{I}_0(U), \mathscr{I}_0(V))\end{aligned}$$

and if we also have $g \in \mathrm{Hom}_{\mathcal{D}}(V, W)$ then

$$\begin{aligned}\mathscr{I}_0(g \circ f) &= \eta^2 \otimes (g \circ f) \otimes \varepsilon^1 = [\mu^2 \otimes \mathrm{id}_W] \circ [\mathrm{id}_{Q_2} \otimes \eta^2 \otimes g \otimes \varepsilon^1] \circ [\eta^2 \otimes f \otimes \varepsilon^2 \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_U \otimes \Delta^1]\\ &= [\mu^2 \otimes \mathrm{id}_W] \circ [\mathrm{id}_{Q_2} \otimes \mathscr{I}(g)] \circ [\mathscr{I}(f) \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_U \otimes \Delta^1] = \mathscr{I}_0(g) \bullet \mathscr{I}_0(f).\end{aligned}$$

Also, for any $V \in \mathcal{D}$ we have $\mathscr{I}_0(\mathrm{id}_V) = \eta^2 \otimes \mathrm{id}_V \otimes \varepsilon^1 = \mathrm{id}_{\overline{J}_2 V J_1}$. This finishes the proof that $\mathscr{I}_0$ is a functor. Bijectivity of $\mathscr{I}_0$ on the objects is trivial. Now assume the additional assumptions at the end of the theorem and suppose that $f, g \in \mathrm{Hom}_{\mathcal{D}}(V, W)$ with $\mathscr{I}_0(f) = \mathscr{I}_0(g)$. Then

$$\begin{aligned}f &= \alpha_1^{-1}\alpha_2^{-1}[\varepsilon^2 \otimes \mathrm{id}_W] \circ [\eta^2 \otimes f \otimes \varepsilon^1] \circ [\mathrm{id}_V \otimes \eta^1] = \alpha_1^{-1}\alpha_2^{-1}[\varepsilon^2 \otimes \mathrm{id}_W] \circ \mathscr{I}_0(f) \circ [\mathrm{id}_V \otimes \eta^1]\\ &= \alpha_1^{-1}\alpha_2^{-1}[\varepsilon^2 \otimes \mathrm{id}_W] \circ \mathscr{I}_0(g) \circ [\mathrm{id}_V \otimes \eta^1] = \alpha_1^{-1}\alpha_2^{-1}[\varepsilon^2 \otimes \mathrm{id}_W] \circ [\eta^2 \otimes g \otimes \varepsilon^1] \circ [\mathrm{id}_V \otimes \eta^1]\\ &= g\end{aligned}$$

showing that $\mathscr{I}_0$ is faithful.
□

In the next subsection we will use $\mathcal{D}(\mathsf{Q}_1, \mathsf{Q}_2)$ to construct a 2-category from a collection of Frobenius algebras. As already mentioned at the beginning of this subsection, we will also need $\mathcal{D}(\mathsf{Q}_1, \mathsf{Q}_2)$ later to construct a certain crossed product of a braided tensor category with a symmetric tensor subcategory. As a preparation for this second purpose of $\mathcal{D}(\mathsf{Q}_1, \mathsf{Q}_2)$ we will prove that $\mathcal{D}(\mathsf{Q}_1, \mathsf{Q}_2)$ can be made into a tensor category if $\mathcal{D}$ is braided.

Theorem 2.9.8 *Let* $(\mathcal{D}, \otimes, I, c)$ *be a braided strict tensor category and let* Q_1 *and* Q_2 *be two Frobenius algebras in* $\mathcal{D}$*, both of which are commutative and cocommutative.*

(1) The category $\mathcal{D}(\mathsf{Q}_1, \mathsf{Q}_2)$ *can be equipped with the structure of a strict tensor category by defining the tensor product on the objects by*

$$\overline{J}_2 V J_1 \circledast \overline{J}_2 W J_1 := \overline{J}_2 (V \otimes W) J_1$$

and on the morphisms by[19]

$$\begin{aligned}f_1 \circledast f_2 := &[\mu^2 \otimes \mathrm{id}_{W_1 \otimes W_2}] \circ [\mathrm{id}_{Q_2} \otimes c_{W_1, Q_2} \otimes \mathrm{id}_{W_2}] \circ [f_1 \otimes f_2]\\ &\circ [\mathrm{id}_{V_1} \otimes \widetilde{c}_{V_2, Q_1} \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{V_1 \otimes V_2} \otimes \Delta^1],\end{aligned}$$

where $f_j \in \mathrm{Hom}_{\mathcal{D}(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 V_j J_1, \overline{J}_2 W_j J_1) = \mathrm{Hom}_{\mathcal{D}}(V_j \otimes Q_1, Q_2 \otimes W_j)$*, and by defining the unit object to be* $\overline{J}_2 I J_1$*. To emphasize the dependence on c, we will write* $\mathcal{D}_c(\mathsf{Q}_1, \mathsf{Q}_2)$ *to denote this tensor category.*

[19]The braiding $\widetilde{c}$ was defined in equation (2.4.1) in Section 2.4.

(2) *The functor* $\mathscr{I}_0 : \mathcal{D} \to \mathcal{D}_c(Q_1, Q_2)$ *is a tensor functor.*
(3) *If* $\mathcal{D}$ *has left or right duals, then so has* $\mathcal{D}_c(Q_1, Q_2)$.

Proof. (1) If $f_j \in \mathrm{Hom}_{\mathcal{D}(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 V_j J_1, \overline{J}_2 W_j J_1) = \mathrm{Hom}_{\mathcal{D}}(V_j \otimes Q_1, Q_2 \otimes W_j)$ for $j \in \{1, 2\}$, then

$$\begin{aligned} f_1 \circledast f_2 \in & \mathrm{Hom}_{\mathcal{D}}(V_1 \otimes V_2 \otimes Q_1, Q_2 \otimes W_1 \otimes W_2) = \mathrm{Hom}_{\mathcal{D}(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2(V_1 \otimes V_2)J_1, \overline{J}_2(W_1 \otimes W_2)J_1) \\ &= \mathrm{Hom}_{\mathcal{D}(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 V_1 J_1 \circledast \overline{J}_2 V_2 J_1, \overline{J}_2 W_1 J_1 \circledast \overline{J}_2 W_2 J_1) \end{aligned}$$

and

$$\begin{aligned} \mathrm{id}_{\overline{J}_2 V J_1} \circledast \mathrm{id}_{\overline{J}_2 W J_1} &= [\mu^2 \otimes \mathrm{id}_{V \otimes W}] \circ [\mathrm{id}_{Q_2} \otimes c_{V,Q_2} \otimes \mathrm{id}_W] \circ [\eta^2 \otimes \mathrm{id}_V \otimes \varepsilon^1 \otimes \eta^2 \otimes \mathrm{id}_W \otimes \varepsilon^1] \\ &\quad \circ [\mathrm{id}_V \otimes \widetilde{c}_{W,Q_1} \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{V \otimes W} \otimes \Delta^1] \\ &= \eta^2 \otimes \mathrm{id}_{V \otimes W} \otimes \varepsilon^1 = \mathrm{id}_{\overline{J}_2 V J_1 \circledast \overline{J}_2 W J_1}. \end{aligned}$$

To prove the interchange law, we consider morphisms f_1, f_2, g_1 and g_2 with

$$\begin{aligned} f_j &\in \mathrm{Hom}_{\mathcal{D}(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 U_j J_1, \overline{J}_2 V_j J_1) \\ g_j &\in \mathrm{Hom}_{\mathcal{D}(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 V_j J_1, \overline{J}_2 W_j J_1) \end{aligned}$$

with $j \in \{1, 2\}$. We first note that

$$\begin{aligned} &[\mu^2 \otimes \mathrm{id}_{W_1 \otimes V_2}] \circ [\mathrm{id}_{Q_2} \otimes g_1 \otimes \mathrm{id}_{V_2}] \circ [c_{V_1,Q_2} \otimes \widetilde{c}_{V_2,Q_1}] \circ [\mathrm{id}_{V_1} \otimes f_2 \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{V_1 \otimes U_2} \otimes \Delta^1] \\ &= [\mu^2 \otimes \mathrm{id}_{W_1 \otimes V_2}] \circ [\mathrm{id}_{Q_2} \otimes g_1 \otimes \mathrm{id}_{V_2}] \circ [c_{V_1,Q_2} \otimes \mathrm{id}_{Q_1 \otimes V_2}] \circ [\mathrm{id}_{V_1} \otimes (c_{Q_1,Q_2} \circ \widetilde{c}_{Q_2,Q_1}) \otimes \mathrm{id}_{V_2}] \\ &\quad \circ [\mathrm{id}_{V_1 \otimes Q_2} \otimes \widetilde{c}_{V_2,Q_1}] \circ [\mathrm{id}_{V_1} \otimes f_2 \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{V_1 \otimes U_2} \otimes \Delta^1] \\ &= [\mu^2 \otimes \mathrm{id}_{W_1 \otimes V_2}] \circ [c_{Q_2,Q_2} \otimes \mathrm{id}_{W_1 \otimes V_2}] \circ [\mathrm{id}_{Q_2} \otimes c_{W_1,Q_2} \otimes \mathrm{id}_{V_2}] \circ [g_1 \otimes f_2] \\ &\quad \circ [\mathrm{id}_{V_1} \otimes \widetilde{c}_{U_2,Q_1} \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{V_1 \otimes U_2} \otimes \widetilde{c}_{Q_1,Q_1}] \circ [\mathrm{id}_{V_1 \otimes U_2} \otimes \Delta^1] \\ &= [(\mu^2)^{\mathrm{op}} \otimes \mathrm{id}_{W_1 \otimes V_2}] \circ [\mathrm{id}_{Q_2} \otimes c_{W_1,Q_2} \otimes \mathrm{id}_{V_2}] \circ [g_1 \otimes f_2] \circ [\mathrm{id}_{V_1} \otimes \widetilde{c}_{U_2,Q_1} \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{V_1 \otimes U_2} \otimes (\Delta^1)^{\mathrm{op}}] \\ &= [\mu^2 \otimes \mathrm{id}_{W_1 \otimes V_2}] \circ [\mathrm{id}_{Q_2} \otimes c_{W_1,Q_2} \otimes \mathrm{id}_{V_2}] \circ [g_1 \otimes f_2] \circ [\mathrm{id}_{V_1} \otimes \widetilde{c}_{U_2,Q_1} \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{V_1 \otimes U_2} \otimes \Delta^1]. \end{aligned}$$

Here $(\Delta^1)^{\mathrm{op}}$ refers to $\widetilde{c}$ in this case, but as we have mentioned before, cocommutativity with respect to c is equivalent to cocommutativity with respect to $\widetilde{c}$. Using this in the equality $\overset{*}{=}$ below, we get that

$$\begin{aligned} &(g_1 \circledast g_2) \bullet (f_1 \circledast f_2) \\ &= [\mu_2^2 \otimes \mathrm{id}_{W_1 \otimes W_2}] \circ [\mathrm{id}_{Q_2 \otimes Q_2} \otimes c_{W_1,Q_2} \otimes \mathrm{id}_{W_2}] \circ [\mathrm{id}_{Q_2} \otimes g_1 \otimes g_2] \circ [\mu^2 \otimes \mathrm{id}_{V_1} \otimes \widetilde{c}_{V_2,Q_1} \otimes \mathrm{id}_{Q_1}] \\ &\quad \circ [\mathrm{id}_{Q_2} \otimes c_{V_1,Q_2} \otimes \mathrm{id}_{V_2} \otimes \Delta^1] \circ [f_1 \otimes f_2 \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{U_1} \otimes \widetilde{c}_{U_2,Q_1} \otimes \mathrm{id}_{Q_1 \otimes Q_1}] \circ [\mathrm{id}_{U_1 \otimes U_2} \otimes \Delta_2^1] \\ &= [\mu_2^2 \otimes \mathrm{id}_{W_1 \otimes W_2}] \circ [\mathrm{id}_{Q_2 \otimes Q_2} \otimes c_{W_1,Q_2} \otimes \mathrm{id}_{W_2}] \circ [\mathrm{id}_{Q_2 \otimes Q_2 \otimes W_1} \otimes g_2] \\ &\quad \circ [\mathrm{id}_{Q_2} \otimes \mu^2 \otimes \mathrm{id}_{W_1 \otimes V_2 \otimes Q_1}] \circ [\mathrm{id}_{Q_2 \otimes Q_2} \otimes g_1 \otimes \mathrm{id}_{V_2 \otimes Q_1}] \circ [\mathrm{id}_{Q_2} \otimes c_{V_1,Q_2} \otimes \widetilde{c}_{V_2,Q_1} \otimes \mathrm{id}_{Q_1}] \\ &\quad \circ [\mathrm{id}_{Q_2 \otimes V_1} \otimes f_2 \otimes \mathrm{id}_{Q_1 \otimes Q_1}] \circ [\mathrm{id}_{Q_2 \otimes V_1 \otimes U_2} \otimes \Delta^1 \otimes \mathrm{id}_{Q_1}] \\ &\quad \circ [f_1 \otimes \mathrm{id}_{U_2 \otimes Q_1 \otimes Q_1}] \circ [\mathrm{id}_{U_1} \otimes \widetilde{c}_{U_2,Q_1} \otimes \mathrm{id}_{Q_1 \otimes Q_1}] \circ [\mathrm{id}_{U_1 \otimes U_2} \otimes \Delta_2^1] \\ &\overset{*}{=} [\mu_2^2 \otimes \mathrm{id}_{W_1 \otimes W_2}] \circ [\mathrm{id}_{Q_2 \otimes Q_2} \otimes c_{W_1,Q_2} \otimes \mathrm{id}_{W_2}] \circ [\mathrm{id}_{Q_2 \otimes Q_2 \otimes W_1} \otimes g_2] \\ &\quad \circ [\mathrm{id}_{Q_2} \otimes \mu^2 \otimes \mathrm{id}_{W_1 \otimes V_2 \otimes Q_1}] \circ [\mathrm{id}_{Q_2 \otimes Q_2} \otimes c_{W_1,Q_2} \otimes \mathrm{id}_{V_2 \otimes Q_1}] \circ [\mathrm{id}_{Q_2} \otimes g_1 \otimes f_2 \otimes \mathrm{id}_{Q_1}] \\ &\quad \circ [\mathrm{id}_{Q_2 \otimes V_1} \otimes \widetilde{c}_{U_2,Q_1} \otimes \mathrm{id}_{Q_1 \otimes Q_1}] \circ [\mathrm{id}_{Q_2 \otimes V_1 \otimes U_2} \otimes \Delta^1 \otimes \mathrm{id}_{Q_1}] \\ &\quad \circ [f_1 \otimes \mathrm{id}_{U_2 \otimes Q_1 \otimes Q_1}] \circ [\mathrm{id}_{U_1} \otimes \widetilde{c}_{U_2,Q_1} \otimes \mathrm{id}_{Q_1 \otimes Q_1}] \circ [\mathrm{id}_{U_1 \otimes U_2} \otimes \Delta_2^1] \\ &= [\mu^2 \otimes \mathrm{id}_{W_1 \otimes W_2}] \circ [\mathrm{id}_{Q_2} \otimes c_{W_1,Q_2} \otimes \mathrm{id}_{W_2}] \circ [\mu^2 \otimes \mathrm{id}_{W_1} \otimes \mu^2 \otimes \mathrm{id}_{W_2}] \circ [\mathrm{id}_{Q_2} \otimes g_1 \otimes \mathrm{id}_{Q_2} \otimes g_2] \end{aligned}$$

$$\circ [f_1 \otimes \mathrm{id}_{Q_1} \otimes f_2 \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{U_1} \otimes \Delta^1 \otimes \mathrm{id}_{U_2} \otimes \Delta^1] \circ [\mathrm{id}_{U_1} \otimes \widetilde{c}_{U_2,Q_1} \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{U_1 \otimes U_2} \otimes \Delta^1]$$
$$= (g_1 \bullet f_1) \circledast (g_2 \bullet f_2).$$

Associativity of $\circledast$ on the objects is obvious. If $f_j \in \mathrm{Hom}_{\mathcal{D}(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 V_j J_1, \overline{J}_2 W_j J_1) = \mathrm{Hom}_{\mathcal{D}}(V_j \otimes Q_1, Q_2 \otimes W_j)$ for $j \in \{1,2,3\}$ then

$$\begin{aligned}
&(f_1 \circledast f_2) \circledast f_3 \\
&= [\mu^2 \otimes \mathrm{id}_{W_1 \otimes W_2 \otimes W_3}] \circ [\mathrm{id}_{Q_2} \otimes c_{W_1 \otimes W_2, Q_2} \otimes \mathrm{id}_{W_3}] \circ [(f_1 \circledast f_2) \otimes f_3] \\
&\quad \circ [\mathrm{id}_{V_1 \otimes V_2} \otimes \widetilde{c}_{V_3,Q_1} \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{V_1 \otimes V_2 \otimes V_3} \otimes \Delta^1] \\
&= [\mu^2_2 \otimes \mathrm{id}_{W_1 \otimes W_2 \otimes W_3}] \circ [\mathrm{id}_{Q_2 \otimes Q_2} \otimes c_{W_1 \otimes W_2, Q_2} \otimes \mathrm{id}_{W_3}] \circ [\mathrm{id}_{Q_2} \otimes c_{W_1,Q_2} \otimes \mathrm{id}_{W_2 \otimes Q_2 \otimes W_3}] \circ [f_1 \otimes f_2 \otimes f_3] \\
&\quad \circ [\mathrm{id}_{V_1} \otimes \widetilde{c}_{V_2,Q_1} \otimes \mathrm{id}_{Q_1 \otimes V_3 \otimes Q_1}] \circ [\mathrm{id}_{V_1 \otimes V_2} \otimes \widetilde{c}_{V_3,Q_1 \otimes Q_1} \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{V_1 \otimes V_2 \otimes V_3} \otimes \Delta^1_2] \\
&= [\mu^2_2 \otimes \mathrm{id}_{W_1 \otimes W_2 \otimes W_3}] \circ [\mathrm{id}_{Q_2} \otimes c_{W_1,Q_2 \otimes Q_2} \otimes \mathrm{id}_{W_2 \otimes W_3}] \circ [\mathrm{id}_{Q_2 \otimes W_1 \otimes Q_2} \otimes c_{W_2,Q_2} \otimes \mathrm{id}_{W_3}] \circ [f_1 \otimes f_2 \otimes f_3] \\
&\quad \circ [\mathrm{id}_{V_1 \otimes Q_1 \otimes V_2} \otimes \widetilde{c}_{V_3,Q_1} \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{V_1} \otimes \widetilde{c}_{V_2 \otimes V_3,Q_1} \otimes \mathrm{id}_{Q_1 \otimes Q_1}] \circ [\mathrm{id}_{V_1 \otimes V_2 \otimes V_3} \otimes \Delta^1_2] \\
&= [\mu^2 \otimes \mathrm{id}_{W_1 \otimes W_2 \otimes W_3}] \circ [\mathrm{id}_{Q_2} \otimes c_{W_1,Q_2} \otimes \mathrm{id}_{W_2 \otimes W_3}] \circ [f_1 \otimes (f_2 \circledast f_3)] \\
&\quad \circ [\mathrm{id}_{V_1} \otimes \widetilde{c}_{V_2 \otimes V_3,Q_1} \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{V_1 \otimes V_2 \otimes V_3} \otimes \Delta^1] \\
&= f_1 \circledast (f_2 \circledast f_3).
\end{aligned}$$

For any $V \in \mathcal{D}$ we have $\overline{J}_2 I J_1 \circledast \overline{J}_2 V J_1 = \overline{J}_2 V J_1 = \overline{J}_2 V J_1 \circledast \overline{J}_2 I J_1$ and for any $f : \overline{J}_2 V J_1 \to \overline{J}_2 W J_1$ we have

$$\begin{aligned}
&\mathrm{id}_{\overline{J}_2 I J_1} \circledast f \\
&= [\mu^2 \otimes \mathrm{id}_{I \otimes W}] \circ [\mathrm{id}_{Q_2} \otimes c_{I,Q_2} \otimes \mathrm{id}_W] \circ [\eta^2 \otimes \varepsilon^1 \otimes f] \circ [\mathrm{id}_I \otimes \widetilde{c}_{V,Q_1} \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{I \otimes V} \otimes \Delta^1] = f
\end{aligned}$$

and

$$\begin{aligned}
&f \circledast \mathrm{id}_{\overline{J}_2 I J_1} \\
&= [\mu^2 \otimes \mathrm{id}_{W \otimes I}] \circ [\mathrm{id}_{Q_2} \otimes c_{W,Q_2} \otimes \mathrm{id}_I] \circ [f \otimes \eta^2 \otimes \varepsilon^1] \circ [\mathrm{id}_V \otimes \widetilde{c}_{I,Q_1} \otimes \mathrm{id}_{Q_1}] \circ [\mathrm{id}_{V \otimes I} \otimes \Delta^1] = f.
\end{aligned}$$

(2) We will now show that $\mathscr{I}_0$ is a strict tensor functor. On the objects we have

$$\mathscr{I}_0(V) \circledast \mathscr{I}_0(W) = \overline{J}_2 V J_1 \circledast \overline{J}_2 W J_1 = \overline{J}_2 (V \otimes W) J_1 = \mathscr{I}_0(V \otimes W).$$

If $f_j \in \mathrm{Hom}_{\mathcal{D}}(V_j, W_j)$ for $j \in \{1,2\}$, then

$$\begin{aligned}
&\mathscr{I}_0(f_1) \circledast \mathscr{I}_0(f_2) \\
&= [\mu^2 \otimes \mathrm{id}_{W_1 \otimes W_2}] \circ [\mathrm{id}_{Q_2} \otimes c_{W_1,Q_2} \otimes \mathrm{id}_{W_2}] \circ [\mathscr{I}_0(f_1) \otimes \mathscr{I}_0(f_2)] \circ [\mathrm{id}_{V_1} \otimes \widetilde{c}_{V_2,Q_1} \otimes \mathrm{id}_{Q_2}] \\
&\quad \circ [\mathrm{id}_{V_1 \otimes V_2} \otimes \Delta^1] \\
&= [\mu^2 \otimes \mathrm{id}_{W_1 \otimes W_2}] \circ [\mathrm{id}_{Q_2} \otimes c_{W_1,Q_2} \otimes \mathrm{id}_{W_2}] \circ [\eta^2 \otimes f_1 \otimes \varepsilon^1 \otimes \eta^2 \otimes f_2 \otimes \varepsilon^1] \\
&\quad \circ [\mathrm{id}_{V_1} \otimes \widetilde{c}_{V_2,Q_1} \otimes \mathrm{id}_{Q_2}] \circ [\mathrm{id}_{V_1 \otimes V_2} \otimes \Delta^1] \\
&= \eta^2 \otimes f_1 \otimes f_2 \otimes \varepsilon^1 = \mathscr{I}_0(f_1 \otimes f_2).
\end{aligned}$$

Also, $\mathscr{I}_0(I) = \overline{J}_2 I J_1$, which is the unit object in $\mathcal{D}_c(\mathsf{Q}_1, \mathsf{Q}_2)$.
(3) If $(V^\vee, b_V, d_V)$ is a left dual for V in the category $\mathcal{D}$, then $(\overline{J}_2 V^\vee J_1, b_{\overline{J}_2 V J_1}, d_{\overline{J}_2 V J_1})$ is a left dual for $\overline{J}_2 V J_1$ in the category $\mathcal{D}_c(\mathsf{Q}_1, \mathsf{Q}_2)$, where $b_{\overline{J}_2 V J_1} := \mathscr{I}_0(b_V)$ and $d_{\overline{J}_2 V J_1} := \mathscr{I}_0(d_V)$. This follows directly from the fact that $\mathscr{I}_0$ is a tensor functor. Because $\mathscr{I}_0$ is bijective on the objects, it follows that if $\mathcal{D}$ has left duals, then so has $\mathcal{D}_c(\mathsf{Q}_1, \mathsf{Q}_2)$. The same reasoning can also be applied to right duals.

□

The last thing we will check about the tensor category $\mathcal{D}_c(\mathsf{Q}_1, \mathsf{Q}_2)$ in this subsection is whether the image of the braiding of $\mathcal{D}$ under the functor $\mathscr{I}_0$ is natural in case either Q_1 or Q_2 is equal to Q_0. This result will be essential later.

Lemma 2.9.9 *Let $\mathcal{D}$, Q_1 and Q_2 be as in the theorem above and let $f \in \mathrm{Hom}_{\mathcal{D}_c(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 X J_1, \overline{J}_2 Y J_1)$. Then for any $V \in \mathcal{D}$ we have the following four equations:*

$$\mathscr{I}_0(c_{Y,V}) \bullet [f \circledast \mathrm{id}_{\overline{J}_2 V J_1}] = [\mathrm{id}_{Q_2} \otimes c_{Y,V}] \circ [f \otimes \mathrm{id}_V] \circ [\mathrm{id}_X \otimes \widetilde{c}_{V,Q_1}] \tag{2.9.4}$$
$$\mathscr{I}_0(c_{V,Y}) \bullet [\mathrm{id}_{\overline{J}_2 V J_1} \circledast f] = [\mathrm{id}_{Q_2} \otimes c_{V,Y}] \circ [c_{V,Q_2} \otimes \mathrm{id}_Y] \circ [\mathrm{id}_V \otimes f] \tag{2.9.5}$$
$$[\mathrm{id}_{\overline{J}_2 V J_1} \circledast f] \bullet \mathscr{I}_0(c_{X,V}) = [c_{V,Q_2} \otimes \mathrm{id}_Y] \circ [\mathrm{id}_V \otimes f] \circ [c_{X,V} \otimes \mathrm{id}_{Q_1}] \tag{2.9.6}$$
$$[f \circledast \mathrm{id}_{\overline{J}_2 V J_1}] \bullet \mathscr{I}_0(c_{V,X}) = [f \otimes \mathrm{id}_V] \circ [\mathrm{id}_X \otimes \widetilde{c}_{V,Q_1}] \circ [c_{V,X} \otimes \mathrm{id}_{Q_1}]. \tag{2.9.7}$$

As a consequence of these equations, we can make the following statements.

(1) In case $\mathsf{Q}_2 = \mathsf{Q}_0$, we have that

$$\mathscr{I}_0(c_{Y,V}) \bullet [f \circledast \mathrm{id}_{\overline{J}_0 V J_1}] = [\mathrm{id}_{\overline{J}_0 V J_1} \circledast f] \bullet \mathscr{I}_0(c_{X,V})$$

for all $f \in \mathrm{Hom}_{\mathcal{D}_c(\mathsf{Q}_1,\mathsf{Q}_0)}(\overline{J}_0 X J_1, \overline{J}_0 Y J_1)$ and $V \in \mathcal{D}$. Furthermore, if $c_{V,Q_1} \circ c_{Q_1,V} = \mathrm{id}_{Q_1 \otimes V}$ then we also have

$$\mathscr{I}_0(c_{V,Y}) \bullet [\mathrm{id}_{\overline{J}_0 V J_1} \circledast f] = [f \circledast \mathrm{id}_{\overline{J}_0 V J_1}] \bullet \mathscr{I}_0(c_{V,X})$$

for all $f \in \mathrm{Hom}_{\mathcal{D}_c(\mathsf{Q}_1,\mathsf{Q}_0)}(\overline{J}_0 X J_1, \overline{J}_0 Y J_1)$.

(2) In case $\mathsf{Q}_1 = \mathsf{Q}_0$, we have that

$$\mathscr{I}_0(c_{V,Y}) \bullet [\mathrm{id}_{\overline{J}_2 V J_0} \circledast f] = [f \circledast \mathrm{id}_{\overline{J}_2 V J_0}] \bullet \mathscr{I}_0(c_{V,X})$$

for all $f \in \mathrm{Hom}_{\mathcal{D}_c(\mathsf{Q}_0,\mathsf{Q}_2)}(\overline{J}_2 X J_0, \overline{J}_2 Y J_0)$ and $V \in \mathcal{D}$. Furthermore, if $c_{V,Q_2} \circ c_{Q_2,V} = \mathrm{id}_{Q_2 \otimes V}$ then we also have

$$\mathscr{I}_0(c_{Y,V}) \bullet [f \circledast \mathrm{id}_{\overline{J}_2 V J_0}] = [\mathrm{id}_{\overline{J}_2 V J_0} \circledast f] \bullet \mathscr{I}_0(c_{X,V})$$

for all $f \in \mathrm{Hom}_{\mathcal{D}_c(\mathsf{Q}_0,\mathsf{Q}_2)}(\overline{J}_2 X J_0, \overline{J}_2 Y J_0)$.

Proof. Let X, Y and f be as given and fix some $V \in \mathcal{D}$. We first observe that

$$f \circledast \mathrm{id}_{\overline{J}_2 V J_1} = [f \otimes \mathrm{id}_V] \circ [\mathrm{id}_X \otimes \widetilde{c}_{V,Q_1}]$$
$$\mathrm{id}_{\overline{J}_2 V J_1} \circledast f = [c_{V,Q_2} \otimes \mathrm{id}_Y] \circ [\mathrm{id}_V \otimes f].$$

A simple computation shows that for any $g \in \mathrm{Hom}_{\mathcal{D}}(X \otimes V \otimes Q_1, Q_2 \otimes Y \otimes V)$ and $h \in \mathrm{Hom}_{\mathcal{D}}(V \otimes X \otimes Q_1, Q_2 \otimes V \otimes Y)$ we have

$$\mathscr{I}_0(c_{Y,V}) \bullet g = [\mathrm{id}_{Q_2} \otimes c_{Y,V}] \circ g$$
$$\mathscr{I}_0(c_{V,Y}) \bullet h = [\mathrm{id}_{Q_2} \otimes c_{V,Y}] \circ h$$
$$h \bullet \mathscr{I}_0(c_{X,V}) = h \circ [c_{X,V} \otimes \mathrm{id}_{Q_1}]$$
$$g \bullet \mathscr{I}_0(c_{V,X}) = g \circ [c_{V,X} \otimes \mathrm{id}_{Q_1}].$$

If we substitute $g = f \circledast \mathrm{id}_{\overline{J}_2 V J_1}$ and $h = \mathrm{id}_{\overline{J}_2 V J_1} \circledast f$, the four equations follow immediately.
(1) Now suppose that $\mathsf{Q}_2 = \mathsf{Q}_0$ and note that $f \in \mathrm{Hom}_{\mathcal{D}}(X \otimes Q_1, Y)$ in this case. Then the right-hand sides of (2.9.4) and (2.9.6) are equal to $c_{Y,V} \circ [f \otimes \mathrm{id}_V] \circ [\mathrm{id}_X \otimes \widetilde{c}_{V,Q_1}]$ and $[\mathrm{id}_V \otimes f] \circ [c_{X,V} \otimes \mathrm{id}_{Q_1}]$, respectively. By naturality of c, these two coincide.

The right-hand sides of (2.9.5) and (2.9.7) are equal to $c_{V,Y} \circ [\mathrm{id}_V \otimes f] = [f \otimes \mathrm{id}_V] \circ c_{V,X\otimes Q_1}$ and $[f\otimes \mathrm{id}_V] \circ [\mathrm{id}_X \otimes \widetilde{c}_{V,Q_1}] \circ [c_{V,X} \otimes \mathrm{id}_{Q_1}]$, respectively. When we compose both expressions from the right with the isomorphism $[c^{-1}_{V,X} \otimes \mathrm{id}_{Q_1}]$, they become equal to $[f \otimes \mathrm{id}_V] \circ [\mathrm{id}_X \otimes c_{V,Q_1}]$ and $[f \otimes \mathrm{id}_V] \circ [\mathrm{id}_X \otimes \widetilde{c}_{V,Q_1}]$. Hence, if $c_{V,Q_1} \circ c_{Q_1,V} = \mathrm{id}_{Q_1\otimes V}$, the two are equal.
(2) Now suppose that $\mathsf{Q}_1 = \mathsf{Q}_0$ and note that $f \in \mathrm{Hom}_{\mathcal{D}}(X, Q_2 \otimes Y)$ in this case. Then the right-hand sides of (2.9.5) and (2.9.7) are equal to $c_{V,Q_2\otimes Y} \circ [\mathrm{id}_V \otimes f]$ and $[f \otimes \mathrm{id}_V] \circ c_{V,X}$, respectively. Hence by naturality of c these two are equal to each other.

The right-hand sides of (2.9.4) and (2.9.6) are equal to $[\mathrm{id}_{Q_2} \otimes c_{Y,V}] \circ [f \otimes \mathrm{id}_V]$ and $[c_{V,Q_2} \otimes \mathrm{id}_Y] \circ [\mathrm{id}_V \otimes f] \circ c_{X,V} = [(c_{V,Q_2} \circ c_{Q_2,V}) \otimes \mathrm{id}_Y] \circ [\mathrm{id}_{Q_2} \otimes c_{Y,V}] \circ [f \otimes \mathrm{id}_V]$, respectively. Hence, if $c_{V,Q_2} \circ c_{Q_2,V} = \mathrm{id}_{Q_2\otimes V}$, these are two are equal.
□

Thus in both cases (1) and (2) of the lemma, the image of the braiding is only natural in one of its arguments. Naturality in the other argument holds if the monodromy of the Frobenius algebra with any other object is trivial. We will come back to this in Subsection 3.1.3.

2.9.3 2-categories from a collection of Frobenius algebras

We will now introduce 2-categories. Very similar to the case of tensor categories, there are both strict and non-strict versions of 2-categories. For our purposes, we will need some kind of intermediate version (similar to the case of a tensor category where the associativity constraint is trivial, but the unit constraints are not). For this reason we will use the intermediate version as our definition of a 2-category.

Definition 2.9.10 A 2-*category* $\mathscr{E}$ consists of the following data:

- a class of 0-*cells*;
- a category $\mathcal{C}(\mathcal{A}_1, \mathcal{A}_2)$ for any two 0-cells $\mathcal{A}_1$ and $\mathcal{A}_2$, the objects of which are called 1-*cells*, the morphisms of which are called 2-*cells* and the composition of which is called *vertical composition*;
- a functor $\diamond : \mathcal{C}(\mathcal{A}_2, \mathcal{A}_3) \times \mathcal{C}(\mathcal{A}_1, \mathcal{A}_2) \to \mathcal{C}(\mathcal{A}_1, \mathcal{A}_3)$ for any three 0-cells $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ called the *horizontal composition*;
- a distinguished 1-cell $I_{\mathcal{A}} \in \mathcal{C}(\mathcal{A}, \mathcal{A})$ for each 0-cell $\mathcal{A}$, together with natural isomorphisms $l : \diamond \circ [I_{\mathcal{A}} \times \mathrm{id}_{\mathcal{C}(\mathcal{A}',\mathcal{A})}] \to \mathrm{id}_{\mathcal{C}(\mathcal{A}',\mathcal{A})}$ and $r : \diamond \circ [\mathrm{id}_{\mathcal{C}(\mathcal{A},\mathcal{A}')} \times I_{\mathcal{A}}] \to \mathrm{id}_{\mathcal{C}(\mathcal{A},\mathcal{A}')}$ for any 0-cell $\mathcal{A}'$.

These data are required to satisfy the following conditions:

(1) the horizontal composition is associative;
(2) if $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are 0-cells, then for any two 1-cells $V \in \mathcal{C}(\mathcal{A}_2, \mathcal{A}_3)$ and $W \in \mathcal{C}(\mathcal{A}_1, \mathcal{A}_2)$ we have $\mathrm{id}_V \diamond l_W = r_V \diamond \mathrm{id}_W$.

Remark 2.9.11 As mentioned before the definition, this is not the most general definition of a 2-category since we have chosen the horizontal composition to be associative. It is straightforward to obtain the more general version of a 2-category by introducing the analogue of an associativity constraint. When this is done, the definition of a tensor category coincides with the definition of a 2-category that has precisely one 0-cell. Consequently, for each 0-cell $\mathcal{A}$ in a 2-category we see that $\mathcal{C}(\mathcal{A}, \mathcal{A})$ is a tensor category. With our somewhat restricted definition of a 2-category above, the tensor categories $\mathcal{C}(\mathcal{A}, \mathcal{A})$ have an associative tensor product, but they might have non-trivial unit constraints.

Definition 2.9.12 Let $\mathscr{E}$ be a 2-category, let $\mathcal{A}_1$ and $\mathcal{A}_2$ be 0-cells and let $V \in \mathcal{C}(\mathcal{A}_1, \mathcal{A}_2)$ be a 1-cell.

(1) A *left dual* of V is a 1-cell $W \in \mathcal{C}(\mathcal{A}_2, \mathcal{A}_1)$ together with 2-cells $b_V \in \mathrm{Hom}_{\mathcal{C}(\mathcal{A}_2,\mathcal{A}_2)}(I_{\mathcal{A}_2}, V \diamond W)$ and $d_V \in \mathrm{Hom}_{\mathcal{C}(\mathcal{A}_1,\mathcal{A}_1)}(W \diamond V, I_{\mathcal{A}_1})$ satisfying

$$[\mathrm{id}_V \diamond d_V] \circ [b_V \diamond \mathrm{id}_V] = \mathrm{id}_V$$
$$[d_V \diamond \mathrm{id}_W] \circ [\mathrm{id}_W \diamond b_V] = \mathrm{id}_W,$$

where in the first equation the composition is in $\mathcal{C}(\mathcal{A}_1, \mathcal{A}_2)$ and in the second equation it is in $\mathcal{C}(\mathcal{A}_2, \mathcal{A}_1)$.

(2) A *right dual* of V is a 1-cell $W \in \mathcal{C}(\mathcal{A}_2, \mathcal{A}_1)$ together with 2-cells $b'_V \in \mathrm{Hom}_{\mathcal{C}(\mathcal{A}_1,\mathcal{A}_1)}(I_{\mathcal{A}_1}, W \diamond V)$ and $d'_V \in \mathrm{Hom}_{\mathcal{C}(\mathcal{A}_2,\mathcal{A}_2)}(V \diamond W, I_{\mathcal{A}_2})$ satisfying

$$[d'_V \diamond \mathrm{id}_V] \circ [\mathrm{id}_V \diamond b'_V] = \mathrm{id}_V$$
$$[\mathrm{id}_W \diamond d'_V] \circ [b'_V \diamond \mathrm{id}_W] = \mathrm{id}_W,$$

where in the first equation the composition is in $\mathcal{C}(\mathcal{A}_1, \mathcal{A}_2)$ and in the second equation it is in $\mathcal{C}(\mathcal{A}_2, \mathcal{A}_1)$.

Let $\mathscr{E}$ be a 2-category and let $V \in \mathcal{C}(\mathcal{A}_1, \mathcal{A}_2)$ and $\overline{V} \in \mathcal{C}(\mathcal{A}_2, \mathcal{A}_1)$ be 1-cells such that $(\overline{V}, b, d, b', d')$ is a two-sided dual for V. We can then construct a Frobenius algebra $\mathsf{Q} = (Q, \mu, \eta, \Delta, \varepsilon)$ in the tensor category $\mathcal{C}(\mathcal{A}_1, \mathcal{A}_1)$ by defining

$$Q = \overline{V} \diamond V, \qquad \mu = \mathrm{id}_{\overline{V}} \diamond d'_V \diamond \mathrm{id}_V, \qquad \eta = b'_V, \qquad \Delta = \mathrm{id}_{\overline{V}} \diamond b_V \diamond \mathrm{id}_V, \qquad \varepsilon = d_V.$$

Similarly, we can also construct a Frobenius algebra $\mathsf{Q}' = (Q', \mu', \eta', \Delta', \varepsilon')$ in the tensor category $\mathcal{C}(\mathcal{A}_2, \mathcal{A}_2)$ by defining

$$Q' = V \diamond \overline{V}, \qquad \mu' = \mathrm{id}_V \diamond d_V \diamond \mathrm{id}_{\overline{V}}, \qquad \eta' = b_V, \qquad \Delta' = \mathrm{id}_V \diamond b'_V \diamond \mathrm{id}_{\overline{V}}, \qquad \varepsilon' = d'_V.$$

Note that this is a generalization of the situation in Example 2.9.4.

We will now consider the situation where we are given several Frobenius algebras $\{\mathsf{Q}_q\}_q$ in a tensor category $\mathcal{D}$. For any such pair $(\mathsf{Q}_q, \mathsf{Q}_r)$ we define $\mathcal{D}(\mathsf{Q}_q, \mathsf{Q}_r)$ as in the previous subsection. However, we will now simply write $\mathcal{D}^q_r$ rather than $\mathcal{D}(\mathsf{Q}_q, \mathsf{Q}_r)$. In order to construct a 2 category with 0-cells $\{\mathsf{Q}_q\}$, we will first introduce a horizontal composition in the following lemma.

Lemma 2.9.13 *Let $(\mathcal{D}, \otimes, I)$ be a strict tensor category, let Q_q, Q_r and Q_s be three Frobenius algebras in $\mathcal{D}$.*

(1) We obtain a functor $\diamond : \mathcal{D}^r_s \times \mathcal{D}^q_r \to \mathcal{D}^q_s$ by

$$\overline{J}_s V J_r \diamond \overline{J}_r W J_q := \overline{J}_s (V \otimes Q_r \otimes W) J_q$$
$$f_2 \diamond f_1 := [\mathrm{id}_{Q_s \otimes W_2} \otimes \mu^r \otimes \mathrm{id}_{W_1}] \circ [f_2 \otimes \mathrm{id}_{Q_r} \otimes f_1] \circ [\mathrm{id}_{V_2} \otimes \Delta^r \otimes \mathrm{id}_{V_1 \otimes Q_q}],$$

where $f_1 \in \mathrm{Hom}_{\mathcal{D}^q_r}(\overline{J}_r V_1 J_q, \overline{J}_r W_1 J_q)$ and $f_2 \in \mathrm{Hom}_{\mathcal{D}^r_s}(\overline{J}_s V_2 J_r, \overline{J}_s W_2 J_r)$.

(2) If Q_t is a fourth Frobenius algebra in $\mathcal{D}$, then the functors

$$\diamond \circ (\diamond \times \mathrm{id}_{\mathcal{D}^q_r}) : \mathcal{D}^s_t \times \mathcal{D}^r_s \times \mathcal{D}^q_r \to \mathcal{D}^q_t$$
$$\diamond \circ (\mathrm{id}_{\mathcal{D}^s_t} \times \diamond) : \mathcal{D}^s_t \times \mathcal{D}^r_s \times \mathcal{D}^q_r \to \mathcal{D}^q_t$$

are equal.

Proof. The only part of the proof that is not a very short computation is the interchange law, so this is the only part we will prove. If $f_j \in \mathrm{Hom}_{\mathcal{D}^q_r}(\overline{J}_r U_j J_q, \overline{J}_r V_j J_q)$ and $g_j \in \mathrm{Hom}_{\mathcal{D}^q_r}(\overline{J}_r V_j J_q, \overline{J}_r W_j J_q)$ for $j \in \{1, 2\}$ then we have

$$\begin{aligned}
&(g_2 \bullet f_2) \diamond (g_1 \bullet f_1) \\
&= [\mathrm{id}_{Q_s \otimes W_2} \otimes \mu^r \otimes \mathrm{id}_{W_1}] \circ [\mu^s \otimes \mathrm{id}_{W_2 \otimes Q_r} \otimes \mu^r \otimes \mathrm{id}_{W_1}] \circ [\mathrm{id}_{Q_s} \otimes g_2 \otimes \mathrm{id}_{Q_r \otimes Q_r} \otimes g_1] \\
&\quad \circ [f_2 \otimes \mathrm{id}_{Q_r \otimes Q_r} \otimes f_1 \otimes \mathrm{id}_{Q_q}] \circ [\mathrm{id}_{U_2} \otimes \Delta^r \otimes \mathrm{id}_{Q_r \otimes U_1} \otimes \Delta^q] \circ [\mathrm{id}_{U_2} \otimes \Delta^r \otimes \mathrm{id}_{U_1 \otimes Q_q}] \\
&= [\mu^s \otimes \mathrm{id}_{W_2 \otimes Q_r \otimes W_1}] \circ [\mathrm{id}_{Q_s} \otimes g_2 \otimes \mathrm{id}_{Q_r \otimes W_1}] \circ [f_2 \otimes \mathrm{id}_{Q_r \otimes Q_r \otimes W_1}] \\
&\quad \circ [\mathrm{id}_{U_2 \otimes Q_r \otimes Q_r} \otimes \mu^r_2 \otimes \mathrm{id}_{W_1}] \circ [\mathrm{id}_{U_2} \otimes \Delta^r_2 \otimes \mathrm{id}_{Q_r \otimes Q_r \otimes W_1}] \\
&\quad \circ [\mathrm{id}_{U_2 \otimes Q_r \otimes Q_r} \otimes g_1] \circ [\mathrm{id}_{U_2 \otimes Q_r} \otimes f_1 \otimes \mathrm{id}_{Q_q}] \circ [\mathrm{id}_{U_2 \otimes Q_r \otimes U_1} \otimes \Delta^q]
\end{aligned}$$

$$
\begin{aligned}
&= [\mu^s \otimes \mathrm{id}_{W_2\otimes Q_r\otimes W_1}] \circ [\mathrm{id}_{Q_s} \otimes g_2 \otimes \mathrm{id}_{Q_r\otimes W_1}] \circ [f_2 \otimes \mathrm{id}_{Q_r\otimes Q_r\otimes W_1}] \\
&\quad \circ [\mathrm{id}_{U_2\otimes Q_r\otimes Q_r} \otimes \mu^r \otimes \mathrm{id}_{W_1}] \circ [\mathrm{id}_{U_2\otimes Q_r} \otimes (\Delta^r \circ \mu^r) \otimes \mathrm{id}_{Q_r\otimes W_1}] \circ [\mathrm{id}_{U_2} \otimes \Delta^r \otimes \mathrm{id}_{Q_r\otimes Q_r\otimes W_1}] \\
&\quad \circ [\mathrm{id}_{U_2\otimes Q_r\otimes Q_r} \otimes g_1] \circ [\mathrm{id}_{U_2\otimes Q_r} \otimes f_1 \otimes \mathrm{id}_{Q_q}] \circ [\mathrm{id}_{U_2\otimes Q_r\otimes U_1} \otimes \Delta^q] \\
&= [\mu^s \otimes \mathrm{id}_{W_2\otimes Q_r\otimes W_1}] \circ [\mathrm{id}_{Q_s\otimes Q_s\otimes W_2} \otimes \mu^r \otimes \mathrm{id}_{W_1}] \circ [\mathrm{id}_{Q_s} \otimes g_2 \otimes \mathrm{id}_{Q_r} \otimes g_1] \\
&\quad \circ [\mathrm{id}_{Q_s\otimes V_2} \otimes \Delta^r \otimes \mathrm{id}_{V_1\otimes Q_q}] \circ [\mathrm{id}_{Q_s\otimes V_2} \otimes \mu^r \otimes \mathrm{id}_{V_1\otimes Q_q}] \circ [f_2 \otimes \mathrm{id}_{Q_r} \otimes f_1 \otimes \mathrm{id}_{Q_q}] \\
&\quad \circ [\mathrm{id}_{U_2} \otimes \Delta^r \otimes \mathrm{id}_{U_1\otimes Q_q\otimes Q_q}] \circ [\mathrm{id}_{U_2\otimes Q_r\otimes U_1} \otimes \Delta^q] \\
&= (g_2 \diamond g_1) \bullet (f_2 \diamond f_1)
\end{aligned}
$$

where in the third equality we used Lemma 2.9.5.
□

Recall that if $\mathcal{D}$ is a category, then we write $\overline{\mathcal{D}}$ to denote its Karoubian envelope. If $\mathcal{D}$ is a strict tensor category and if Q_q, Q_r and Q_s are Frobenius algebras in $\mathcal{D}$, then we can extend the functor $\diamond : \mathcal{D}^r_s \times \mathcal{D}^q_r \to \mathcal{D}^q_s$ to a functor $\bar{\diamond}$: $\overline{\mathcal{D}^r_s} \times \overline{\mathcal{D}^q_r} \to \overline{\mathcal{D}^q_s}$ by defining

$$(\overline{J}_s V' J_r, p') \mathbin{\bar{\diamond}} (\overline{J}_r V J_q, p) := (\overline{J}_s V' J_r \diamond \overline{J}_r V J_q, p' \diamond p) \tag{2.9.8}$$

and

$$f' \mathbin{\bar{\diamond}} f := f' \diamond f. \tag{2.9.9}$$

To see that $\bar{\diamond}$ is indeed a functor, let

$$
\begin{aligned}
f &\in \mathrm{Hom}_{\overline{\mathcal{D}^q_r}}((\overline{J}_r V J_q, p_1), (\overline{J}_r W J_q, p_2)) \\
f' &\in \mathrm{Hom}_{\overline{\mathcal{D}^r_s}}((\overline{J}_s V' J_r, p_1'), (\overline{J}_s W' J_r, p_2'))).
\end{aligned}
$$

Then $f' \mathbin{\bar{\diamond}} f = f' \diamond f \in \mathrm{Hom}_{\mathcal{D}^q_s}(\overline{J}_s V' J_r \diamond \overline{J}_r V J_q, \overline{J}_s W' J_r \diamond \overline{J}_r W J_q)$, since $\diamond$ is a functor. Also,

$$
\begin{aligned}
[p_2' \diamond p_2] \bullet [f' \mathbin{\bar{\diamond}} f] \bullet [p_1' \diamond p_1] &= [p_2' \diamond p_2] \bullet [f' \diamond f] \bullet [p_1' \diamond p_1] = (p_2' \bullet f' \bullet p_1') \diamond (p_2 \bullet f \bullet p_1) \\
&= f' \diamond f = f' \mathbin{\bar{\diamond}} f,
\end{aligned}
$$

so indeed we have that

$$
\begin{aligned}
f' \mathbin{\bar{\diamond}} f &\in \mathrm{Hom}_{\overline{\mathcal{D}^q_s}} \left((\overline{J}_s V' J_r \diamond \overline{J}_r V J_q, p_1' \diamond p_1), (\overline{J}_s W' J_r \diamond \overline{J}_r W J_q, p_2' \diamond p_2)\right) \\
&= \mathrm{Hom}_{\overline{\mathcal{D}^q_s}} \Big((\overline{J}_s V' J_r, p_1') \mathbin{\bar{\diamond}} (\overline{J}_r V J_q, p_1), (\overline{J}_s W' J_r, p_2') \mathbin{\bar{\diamond}} (\overline{J}_r W J_q, p_2)\Big).
\end{aligned}
$$

Furthermore,

$$(g_2 \bullet f_2) \mathbin{\bar{\diamond}} (g_1 \bullet f_1) = (g_2 \bullet f_2) \diamond (g_1 \bullet f_1) = (g_2 \diamond g_1) \bullet (f_2 \diamond f_1) = (g_2 \mathbin{\bar{\diamond}} g_1) \bullet (f_2 \mathbin{\bar{\diamond}} f_1),$$

so $\bar{\diamond}$ is indeed a functor.

Lemma 2.9.14 *Let $(\mathcal{D}, \otimes, I)$ be an $\mathbb{F}$-linear strict tensor category and let Q_q, Q_r and Q_s be three Frobenius algebras in $\mathcal{D}$ of which Q_r is normalized with $\kappa_r := \kappa_{\mathsf{Q}_r}$. Then there exists an object $I_r \in \overline{\mathcal{D}^r_r}$ such that*

(1) there is a natural isomorphism $l : \bar{\diamond} \circ [I_r \times \mathrm{id}_{\overline{\mathcal{D}^q_r}}] \to \mathrm{id}_{\overline{\mathcal{D}^q_r}}$, i.e. a family

$$\{l_{(\overline{J}_r V J_q, p)} : I_r \mathbin{\bar{\diamond}} (\overline{J}_r V J_q, p) \to (\overline{J}_r V J_q, p)\}_{(\overline{J}_r V J_q, p) \in \overline{\mathcal{D}^q_r}}$$

of isomorphisms satisfying

$$l_{(\overline{J}_r V' J_q, p')} \bullet [\mathrm{id}_{I_r} \mathbin{\bar{\diamond}} f] = f \bullet l_{(\overline{J}_r V J_q, p)}$$

for all $f \in \mathrm{Hom}_{\overline{\mathcal{D}^q_r}}((\overline{J}_r V J_q, p), (\overline{J}_r V' J_q, p'))$;

(2) *there is a natural isomorphism* $r : \bar{\diamond} \circ [\mathrm{id}_{\overline{\mathcal{D}_s^r}} \times I_r] \to \mathrm{id}_{\overline{\mathcal{D}_s^r}}$, *i.e, a family*

$$\{r_{(\overline{J}_s W J_r, u)} : (\overline{J}_s W J_r, u) \,\bar{\diamond}\, I_r \to (\overline{J}_s W J_r, u)\}_{(\overline{J}_s W J_r, u) \in \overline{\mathcal{D}_s^r}}$$

satisfying

$$r_{(\overline{J}_s W' J_r, u')} \bullet [g \,\bar{\diamond}\, \mathrm{id}_{I_r}] = g \bullet r_{(\overline{J}_s W J_r, u)}$$

for all $g \in \mathrm{Hom}_{\overline{\mathcal{D}_r^q}}((\overline{J}_s W J_r, u), (\overline{J}_s W' J_r, u'))$*;*

(3) *for all* $(\overline{J}_r V J_q, p) \in \overline{\mathcal{D}_r^q}$ *and* $(\overline{J}_s W J_r, u) \in \overline{\mathcal{D}_s^r}$ *we have*

$$\mathrm{id}_{(\overline{J}_s W J_r, u)} \,\bar{\diamond}\, l_{(\overline{J}_r V J_q, p)} = r_{(\overline{J}_s W J_r, u)} \,\bar{\diamond}\, \mathrm{id}_{(\overline{J}_r V J_q, p)}.$$

Proof. For the details of the proof we refer to [74]. Here we will only mention how the I_r, l and r are defined. We first define $p_r \in \mathrm{End}_{\mathcal{D}_r^r}(\overline{J}_r I J_r) = \mathrm{End}_{\mathcal{D}}(Q_r)$ by $p_r := \kappa_r^{-1} \mathrm{id}_{Q_r}$, which can easily be seen to be an idempotent. The object I_r is then defined by

$$I_r := (\overline{J}_r I J_r, p_r).$$

We then define $l_{(\overline{J}_r V J_q, p)} : I_r \,\bar{\diamond}\, (\overline{J}_r V J_q, p) \to (\overline{J}_r V J_q, p)$ by

$$l_{(\overline{J}_r V J_q, p)} := [\mu^r \otimes \mathrm{id}_V] \circ [\mathrm{id}_{Q_r} \otimes p],$$

which is an isomorphism with inverse $l^{-1}_{(\overline{J}_r V J_q, p)} := [\Delta^r \otimes \mathrm{id}_V] \circ p$ and $r_{(\overline{J}_s W J_r, p)} : (\overline{J}_s W J_r, p) \,\bar{\diamond}\, I_r \to (\overline{J}_s W J_r, p)$ is defined by

$$r_{(\overline{J}_s W J_r, p)} := p \circ [\mathrm{id}_W \otimes \mu^r]$$

which is an isomorphism with inverse $r^{-1}_{(\overline{J}_s W J_r, p)} = [p \otimes \mathrm{id}_{Q_r}] \circ [\mathrm{id}_W \otimes \Delta^r]$.
□

We will now combine all our results in the following theorem, which states that a collection of normalized Frobenius algebras gives rise to a 2-category.

Theorem 2.9.15 *Let* $(\mathcal{D}, \otimes, I)$ *be an* $\mathbb{F}$*-linear strict tensor category and let* $\{Q_q : q \in S\}$ *be a collection of normalized Frobenius algebras in* $\mathcal{D}$*. Then we obtain a 2-category* $\mathcal{E}$ *as follows:*

- *the* 0*-cells are the Frobenius algebras* Q_q*;*
- *the category* $\mathcal{C}(\mathsf{Q}_q, \mathsf{Q}_r)$ *is defined to be* $\overline{\mathcal{D}_r^q}$*, i.e the Karoubi envelope of the category* $\mathcal{D}(\mathsf{Q}_q, \mathsf{Q}_r)$ *as defined in Theorem 2.9.7;*
- *the horizontal composition is given by* $\bar{\diamond}$ *as defined in Lemma 2.9.13 and equations* (2.9.8) *and* (2.9.9)*;*
- *for each* $q \in S$ *the unit object is defined by* $I_q = (\overline{J}_q I J_q, p_q)$*, where* $p_q = \kappa_q^{-1} \mathrm{id}_{Q_q}$ *and* l *and* r *are defined as in Lemma 2.9.14.*

Furthermore, for any $q \in S$ *we can give* $\overline{J}_0 I J_q \in \overline{\mathcal{D}_0^q}$ *the structure of a two-sided dual for* $\overline{J}_q I J_0 \in \overline{\mathcal{D}_q^0}$ *such that the Frobenius algebra* Q_q *can be reconstructed from this structure by the procedure that is described after Definition 2.9.12.*

For the final statement in this theorem we refer to parts 4 and 5 of Theorem 3.11 in [74]. In fact, if we take the index set S to be $\{0, 1\}$ in our theorem above (so that $\{\mathsf{Q}_q : q \in S\}$ consists of the trivial Frobenius algebra Q_0 together with one other Frobenius algebra $\mathsf{Q}_1 =: \mathsf{Q}$), we are precisely in the setting of Theorem 3.11 of [74]. Note that our notation here is slightly more efficient than the one in [74]. Namely, in [74] the three categories $\mathcal{D}(\mathsf{Q}_0, \mathsf{Q})$, $\mathcal{D}(\mathsf{Q}, \mathsf{Q}_0)$ and $\mathcal{D}(\mathsf{Q}, \mathsf{Q})$ were constructed independently of each other[20], whereas in our notation these three categories are just examples of one single category $\mathcal{D}(\mathsf{Q}_q, \mathsf{Q}_r)$.

[20] In [74] they were denoted by $HOM_{\mathcal{E}}(\mathfrak{A}, \mathfrak{B})$, $HOM_{\mathcal{E}}(\mathfrak{B}, \mathfrak{A})$ and $HOM_{\mathcal{E}}(\mathfrak{B}, \mathfrak{B})$, respectively.

Chapter 3

Algebraic quantum field theory

In this chapter we will provide the reader with a short introduction to certain concepts in algebraic quantum field theory (AQFT). Our exposition here should not be considered as a general introduction to AQFT, but rather as an efficient way to quickly understand those aspects of AQFT that will be relevant to us. The relevance of AQFT to our research is twofold. Firstly, our original research problem concerned a categorical construction in the representation theory of AQFTs and our new research problem arose from generalizing the original problem to a more abstract categorical setting, see also Subsection 3.2.5 below. In this sense, AQFT formed the main motivation for our research. Secondly (and more importantly), certain observations about the braided G-crossed categories that arise in AQFT inspired us to introduce new categorical definitions and to formulate certain statements about the braided G-crossed category $Z_G(\mathcal{C})$ that we will define in Chapter 4. As we will explain in Subsection 3.2.5, certain properties of the categories that arise in AQFT gave us the idea to construct a group-crossed generalization of the Drinfeld center. Also, our results concerning left and right G-localized endomorphisms in Subsection 3.2.2 of the present chapter inspired us to define the mirror image of a braided G-crossed category, as we did in Subsection 2.8.5, and it also inspired us to carry out the constructions in Subsection 4.10.1 and to formulate Theorem 4.10.7.

The selection of topics in AQFT that occur in this chapter is very similar to that in Müger's paper [78], but the particular manner in which we display the material here is quite different from [78]. Namely, we have chosen to rewrite the content of [78] in the language of [65], i.e. in the language of subfactor theory. As a consequence, some of the proofs of statements in [78] had to be adjusted considerably. For example, our proof of part (1) of Theorem 3.2.20 in Subsection 3.2.3 is different from the proof in [78]. Also, in [78] there is no analogue of our Lemma 3.2.14, which forms an important part of our proof of Theorem 3.2.20, but which can also by itself be considered as an interesting result in AQFT. Also, our more symmetrical approach to left/right G-localized endomorphisms in Subsection 3.2.1 is also quite different from the literature. Besides Lemma 3.2.14 and the alternative proof of Theorem 3.2.20, the complete content of Subsection 3.2.2 is also new.

3.1 Operator algebras

Since AQFT is formulated in the language of operator algebras, we will begin this chapter with a section on operator algebras. In the first subsection we will start our discussion of operator algebras from the very beginning by giving the definitions of C^*-algebras and von Neumann algebras. In the subsequent subsections we will discuss some more advanced topics such as C^*-tensor categories, crossed products of a BTC^* with a symmetric subcategory, C^*-2-categories and type III subfactors.

3.1.1 C^*-algebras, W^*-algebras and von Neumann algebras

We now give a brief overview of the basic facts from the theory of operator algebras that will be relevant to us when we will discuss AQFT, without going into any details or proofs here. We assume that the reader is familiar with the basic notions from functional analysis, although we will recall some of these notions again here because of their importance for AQFT. For a detailed discussion on operator algebras we refer to the standard textbooks [46]-[47], [83], [93] and [101]-[103].

An *associative algebra* $\mathcal{A}$ over $\mathbb{C}$ is a complex vector space together with a bilinear map $(a, b) \mapsto ab$ from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$ (called the multiplication in $\mathcal{A}$) that satisfies $(ab)c = a(bc)$ for all $a, b, c \in \mathcal{A}$. In this chapter, whenever we speak of an algebra we will always mean an associative algebra over $\mathbb{C}$. An algebra $\mathcal{A}$ will be called *unital* if there is an element $1 \in \mathcal{A}$ (called the unit element of $\mathcal{A}$) such that $1a = a = a1$ for all $a \in \mathcal{A}$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are algebras, then an *algebra homomorphism* φ from $\mathcal{A}_1$ to $\mathcal{A}_2$ is a linear map $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}_1$. If both algebras are unital, then φ is called a *unital* algebra homomorphism if $\varphi(1) = 1$.

If $\mathcal{A}$ is an algebra, then an *involution* on $\mathcal{A}$ is a conjugate-linear map $* : \mathcal{A} \to \mathcal{A}$, denoted by $a \mapsto a^*$, that satisfies $(ab)^* = b^*a^*$ and $a^{**} := (a^*)^* = a$ for all $a, b \in \mathcal{A}$. An algebra with an involution will be called a $*$-*algebra*. A $*$-algebra is called unital if it has a unit element. In that case we automatically have $1^* = 1$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are (unital) $*$-algebras, then a *(unital)* $*$-*homomorphism* is a (unital) algebra homomorphism $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\varphi(a^*) = \varphi(a)^*$ for all $a \in \mathcal{A}_1$.

If $\mathcal{A}$ is an algebra and has a norm $\|\cdot\|$ with respect to which $\mathcal{A}$ is complete, i.e. if $(\mathcal{A}, \|\cdot\|)$ becomes a Banach space, then $\mathcal{A}$ is called a *Banach algebra* if the norm is submultiplicative: $\|ab\| \leq \|a\| \cdot \|b\|$ for all $a, b \in \mathcal{A}$. If $\mathcal{A}$ also has a unit, then it is called a unital Banach algebra if $\|1\| = 1$.

A triple $(\mathcal{A}, *, \|\cdot\|)$ consisting of an (unital) algebra $\mathcal{A}$, an involution $*$ on $\mathcal{A}$ and a norm $\|\cdot\|$ on $\mathcal{A}$ such that $(\mathcal{A}, *)$ is a (unital) $*$-algebra and such that $(\mathcal{A}, \|\cdot\|)$ is a (unital) Banach algebra is called a *(unital) Banach* $*$-*algebra* if $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$. A (unital) Banach $*$-algebra $\mathcal{A}$ is called a C^*-*algebra* if it satisfies the extra condition that[1] $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$. In a C^*-algebra $\mathcal{A}$ an element $n \in \mathcal{A}$ is called *normal* if $n^*n = nn^*$, an element $a \in \mathcal{A}$ is called *self-adjoint* if $a^* = a$ and an element $e \in \mathcal{A}$ is called a *projection* if $e^* = e = e^2$ (i.e. a projection is a self-adjoint idempotent). If $\mathcal{A}$ is unital, an element $u \in \mathcal{A}$ is called an *isometry* if $u^*u = 1$ (in which case we automatically have that uu^* is a projection) and it is called *unitary* if $u^*u = 1 = uu^*$ (i.e. a unitary element is a normal isometry). In any unital C^*-algebra we define the *spectrum* $\sigma(a) \subset \mathbb{C}$ of an element $a \in \mathcal{A}$ by

$$\sigma(a) := \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible}\},$$

which is automatically a compact subset of $\mathbb{C}$. For a self-adjoint element $a \in \mathcal{A}$ we have $\sigma(a) \subset \mathbb{R}$ and we say that a is *positive* if $\sigma(a) \subset \mathbb{R}_{\geq 0}$. Equivalently, positive elements can be characterized by the property that they can be written in the form x^*x for some $x \in \mathcal{A}$. The set of positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}_+$ and is closed under sums and scalar multiplication with non-negative real numbers. If $a \in \mathcal{A}_+$, then we will also write $a \geq 0$. If we write $\mathcal{A}_{sa}$ to denote the set of self-adjoint elements of $\mathcal{A}$, then for any $a, b \in \mathcal{A}_{sa}$ we will write $a \leq b$ if $b - a \geq 0$. A linear map $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ between C^*-algebras is called *positive* if $\varphi((\mathcal{A}_1)_+) \subset (\mathcal{A}_2)_+$, and in that case φ will be called *faithful* if $\varphi(a) \neq 0$ for all non-zero $a \in (\mathcal{A}_1)_+$. A stronger condition than positivity is complete positivity. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are C^*-algebras and $n \in \mathbb{Z}_{\geq 1}$, then a linear map $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ is called n-*positive* if $\sum_{i=1,j=1}^{n} y_i^* \varphi(x_i^* x_j) y_j \geq 0$ for all $x_1, \ldots, x_n \in \mathcal{A}_1$ and $y_1, \ldots, y_n \in \mathcal{A}_2$. If $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ is n-positive for every $n \in \mathbb{Z}_{\geq 1}$ then it is called *completely positive*. We will now define two important kinds of positive linear maps in the theory of C^*-algebras.

Definition 3.1.1 Let $\mathcal{A}$ be a unital C^*-algebra. Then a *state* on $\mathcal{A}$ is a positive linear map $\omega : \mathcal{A} \to \mathbb{C}$ with $\omega(1) = 1$. If $\mathcal{B} \subset \mathcal{A}$ is a unital C^*-subalgebra, then a positive linear map $\varepsilon : \mathcal{A} \to \mathcal{B}$ with $\varepsilon(1) = 1$ is called a *conditional expectation* from $\mathcal{A}$ onto $\mathcal{B}$ if it satisfies the property that

$$\varepsilon(b^*ab) = b^*\varepsilon(a)b \tag{3.1.1}$$

[1] This extra condition automatically ensures that the earlier conditions $\|a^*\| = \|a\|$ and $\|1\| = 1$ are satisfied.

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. The condition in (3.1.1) is called the *bimodule property.*

Note that the bimodule property automatically ensures that a conditional expectation is completely positive.

Because every positive element in a C^*-algebra is of the form x^*x, a $*$-homomorphism $\rho : \mathcal{A}_1 \to \mathcal{A}_2$ of C^*-algebras is automatically positive, since $\rho(x^*x) = \rho(x^*)\rho(x) = \rho(x)^*\rho(x) \in (\mathcal{A}_2)_+$ and it can also be shown that it is automatically continuous with respect to the topologies induced by the norms on $\mathcal{A}_1$ and $\mathcal{A}_2$. If $\rho : \mathcal{A} \to \mathcal{A}$ is a $*$-endomorphism of a C^*-algebra $\mathcal{A}$, then a *left inverse* for ρ is a positive linear map $\varphi : \mathcal{A} \to \mathcal{A}$ such that $\varphi(1) = 1$ and $\varphi(a\rho(b)) = \varphi(a)b$ for all $a, b \in \mathcal{A}$. The composition $\varepsilon := \rho \circ \varphi$ is then a conditional expectation from $\mathcal{A}$ onto $\rho(\mathcal{A})$.

Example 3.1.2 Let H be a complex Hilbert space. Then a linear map $T : H \to H$ is called *bounded* if there exists a $C \geq 0$ such that[2] $\|Th\| \leq C\|h\|$ for all $h \in H$. The infimum of all C for which this inequality holds (for all $h \in H$) is called the *operator norm* of T and is denoted by $\|T\|$. The set of all bounded operators on H is denoted by $B(H)$. If $T \in B(H)$, then there exists a unique operator $T^* \in B(H)$, called the *adjoint* of T, such that $\langle Th_1, h_2 \rangle = \langle h_1, T^*h_2 \rangle$ for all h_1, h_2. It is straightforward to check that $\|.\|$ is indeed a norm on $B(H)$, that $T \mapsto T^*$ defines an involution on $B(H)$ and that $B(H)$ becomes a unital C^*-algebra in this way; the topology on $B(H)$ induced by the operator norm is called the *norm topology* on $B(H)$. Any $*$-subalgebra $\mathcal{A} \subset B(H)$ that is closed with respect to the norm topology on $B(H)$ is a C^*-algebra. If $\mathcal{A} \subset B(H)$ is such a C^*-algebra acting on the Hilbert space H, then a vector $\Omega \in H$ is called *cyclic* for $\mathcal{A}$ if $\mathcal{A}\Omega$ is dense in H and it is called *separating* for $\mathcal{A}$ if $A\Omega \neq 0$ for all non-zero $A \in \mathcal{A}$.

The example above shows that closed $*$-subalgebras of $B(H)$ for some Hilbert space H provide us with concrete examples of C^*-algebras. In analogy with group theory, this observation leads us to the definition of a representation of a C^*-algebra.

Definition 3.1.3 Let $\mathcal{A}$ be a C^*-algebra. A *representation* (H, π) of $\mathcal{A}$ consists of a Hilbert space H together with a $*$-homomorphism $\pi : \mathcal{A} \to B(H)$. If π is injective, then (H, π) is called a *faithful* representation of $\mathcal{A}$. A vector $\Omega \in H$ is called *cyclic* for (H, π) if $\pi(\mathcal{A})\Omega$ is dense in H. If such a cyclic vector exists, then (H, π) is called a *cyclic representation.*

If (H, π) is a cyclic representation of $\mathcal{A}$ and if $\Omega \in H$ is a unit vector that is cyclic for (H, π), then the map $a \mapsto \langle \pi(a)\Omega, \Omega \rangle$ is a state on $\mathcal{A}$.

An important example of a representation of a C^*-algebra is the so-called GNS representation corresponding to a state. We will now briefly mention how the construction of this representation is carried out, without proving any details.

Example 3.1.4 Let $\mathcal{A}$ be a C^*-algebra and let $\omega : \mathcal{A} \to \mathbb{C}$ be a state. Then $N_\omega := \{a \in \mathcal{A} : \omega(a^*a) = 0\}$ is a closed left ideal in $\mathcal{A}$ and we can define an inner product on the vector space $\mathcal{A}/N_\omega$ by $\langle a + N_\omega, b + N_\omega \rangle := \omega(b^*a)$. Thus $\mathcal{A}/N_\omega$ becomes an inner product space and we will denote its Hilbert space completion by H_ω. Next we define a $*$-homomorphism $\pi_\omega : \mathcal{A} \to B(H_\omega)$ by first defining $\pi_\omega(a)$ on $\mathcal{A}/N_\omega$ by $\pi_\omega(a)(b + N_\omega) := ab + N_\omega$ and then extending this bounded operator on A/N_ω uniquely to a bounded operator on H_ω. We thus obtain a representation (H_ω, π_ω), which is known as the *Gelfand-Naimark-Segal representation* (or *GNS representation* for short) corresponding to the state ω. It has the property that faithfulness of ω (in the sense of positive linear maps) implies faithfulness of π_ω (in the sense of representations). Note that if $\mathcal{A}$ is unital, then $\Omega_\omega := 1 + N_\omega \in H_\omega$ is a cyclic vector for (H_ω, π_ω) that has the property that $\langle \pi_\omega(a)\Omega_\omega, \Omega_\omega \rangle = \omega(a)$ for all $a \in \mathcal{A}$. In the non-unital case there also exists a vector Ω_ω with these properties, but this is not directly relevant to our discussion of AQFT, where we will always be given a unital C^*-algebra. The details of the GNS construction can be found in any textbook on operator algebras, such as the ones mentioned at the beginning of this subsection.

[2] It is very convenient to write Th rather than $T(h)$.

Let H be a Hilbert space. An operator $T \in B(H)$ is called *trace-class* if for every orthonormal basis $\mathcal{E}$ of H the series $\sum_{e\in\mathcal{E}} |\langle Te, e\rangle|$ converges. The set of trace-class operators on H is denoted by $B_1(H)$. If $T \in B_1(H)$, then its *trace* is defined by

$$\mathrm{Tr}(T) := \sum_{e\in\mathcal{E}} \langle Te, e\rangle$$

and is independent of the choice of the orthonormal basis $\mathcal{E}$. On $B_1(H)$ we define a norm $\|.\|_1$ by[3]

$$\|T\|_1 := \mathrm{Tr}(|T|),$$

which is called the *trace norm*. It can be shown that $(B_1(H), \|.\|_1)$ is a Banach space. Furthermore, $B_1(H)$ is a two-sided ideal of $B(H)$, so for any $T \in B_1(H)$ and $A \in B(H)$ we can define $\mathrm{Tr}(TA) \in \mathbb{C}$. Thus for each $T \in B_1(H)$ we can define a seminorm $p_T : B(H) \to \mathbb{R}_{\geq 0}$ by $p_T(A) := |\mathrm{Tr}(TA)|$. The locally convex topology on $B(H)$ generated by the collection[4] $\{p_T : T \in B_1(H)\}$ of seminorms is called the *σ-weak topology* or *ultraweak topology* on $B(H)$. This topology is weaker than the norm topology on $B(H)$. As a consequence, any $*$-subalgebra of $B(H)$ that is ultraweakly closed, is automatically closed in the norm topology and is therefore a C^*-subalgebra of $B(H)$. If $M \subset B(H)$ is such an ultraweakly closed $*$-subalgebra of $B(H)$, then we define $M^\perp := \{T \in B_1(H) : \mathrm{Tr}(TA) = 0 \text{ for all } A \in M\}$, which is a closed linear subspace of the Banach space $(B_1(H), \|.\|_1)$. Hence the quotient space $M_* := B_1(H)/M^\perp$ becomes a Banach space when equipped with the quotient norm. For any $A \in M$ we now define a bounded linear functional $\psi_A : M_* \to \mathbb{C}$ on the Banach space M_* by $\psi_A(T) := \mathrm{Tr}(TA)$. In this way we obtain a map

$$\begin{aligned} \psi : M &\to (M_*)^* \\ T &\mapsto \psi_T \end{aligned}$$

which turns out to be an isometric linear isomorphism of Banach spaces. We thus conclude that each ultraweakly closed $*$-subalgebra M of $B(H)$ is isomorphic as a Banach space to the dual of a Banach space M_*. The weak*-topology $\sigma(M, M_*)$ on M is precisely the ultraweak topology inherited from $B(H)$ by the inclusion $M \subset B(H)$. This last fact indicates that the ultraweak topology on an ultraweakly closed $*$-subalgebra of $B(H)$ is particularly important.

Motivated by the observations above, there is also the notion of a W^*-algebra, which is defined to be a C^*-algebra M that is isomorphic as a Banach space to the dual of a Banach space M_*, called the *predual* of M, which can be shown to be unique up to isometric isomorphism. It follows directly from our discussion above that if H is a Hilbert space, then any ultraweakly closed $*$-subalgebra of $B(H)$ is a W^*-algebra. Conversely, any W^*-algebra is $*$-isomorphic to an ultraweakly closed $*$-subalgebra of $B(H)$ for some Hilbert space H. See Section 1.16 of [93] for details on this representation theorem for W^*-algebras.

If H is a Hilbert space and if $S \subset B(H)$ is a subset, then we define its *commutant* S' to be the set $S' := \{T \in B(H) : TA = AT \text{ for all } A \in S\}$. A $*$-subalgebra M of $B(H)$ is called a *von Neumann algebra* on H if $M'' := (M')' = M$. Any von Neumann algebra on H is in particular a C^*-subalgebra of $B(H)$ containing the unit operator 1_H. A deep theorem in the theory of operator algebras states that the von Neumann algebras on H are precisely the W^*-subalgebras of $B(H)$ that contain the unit operator 1_H. We have already mentioned that $*$-homomorphisms between C^*-algebras are automatically continuous. Because the ultraweak topology is an important part of the structure of a von Neumann algebra, we also introduce the notion of a *normal* $*$-homomorphism between von Neumann algebras, which is a $*$-homomorphism that is continuous with respect to the ultraweak topologies of the von Neumann algebras.

If $\{M_\alpha\}_\alpha$ is a family of von Neumann algebras on H, then we will write $\bigvee_\alpha M_\alpha$ to denote the smallest von Neumann algebra on H that contains all M_α, and it will be called the von Neumann algebra generated by $\{M_\alpha\}_\alpha$.

[3] Here $|T| := \sqrt{T^*T}$, where the square root of the positive operator T^*T is defined using the so-called functional calculus.

[4] We refer to chapter IV of [15] for the definition of the locally convex topology generated by a collection of seminorms.

If M is a von Neumann algebra on H, then its center $Z(M)$ consists of those elements in M that commute with every element of M, so we can write $Z(M) = M' \cap M$. A von Neumann algebra M is called a *factor* if $Z(M) = \mathbb{C}1$. A factor M on a separable Hilbert space H is said to be of *type III* if for every $P = P^* = P^2 \in M$ there exists a $U \in M$ with $U^*U = 1_H$ and $UU^* = P$. Endomorphisms of such type III factors are automatically injective. There is a more general definition of type III factors that also holds on non-separable Hilbert spaces, but we will only be concerned with separable Hilbert spaces when we come to discuss AQFT.

We will end this subsection with the definition of a group action on an operator algebra. If $\mathcal{A}$ is a C^*-algebra then we will write $\mathrm{Aut}(\mathcal{A})$ to denote the group of $*$-automorphisms of $\mathcal{A}$.

Definition 3.1.5 Let $\mathcal{A}$ be a C^*-algebra and let G be a group. Then a *G-action* on $\mathcal{A}$ is a homomorphism

$$\begin{aligned} \beta : G &\to \mathrm{Aut}(\mathcal{A}) \\ q &\mapsto \beta_q. \end{aligned}$$

We write $\mathcal{A}^G$ to denote the algebra of G-invariant elements in $\mathcal{A}$, i.e.

$$\mathcal{A}^G = \{A \in \mathcal{A} : \beta_q(A) = A \text{ for all } q \in G\}.$$

In case $\mathcal{A}$ is a von Neumann algebra on a Hilbert space H, we will say that the G-action β is *unitarily implemented* if there exists a unitary representation[5] $V : G \to B(H)$ of G such that for any $q \in G$ we have $\beta_q(A) = V(q)AV(q)^*$ for all $A \in \mathcal{A}$.

Let G be a group and let β be a G-action on the unital[6] C^*-algebra $\mathcal{A}$. Suppose that $\omega : \mathcal{A} \to \mathbb{C}$ is a state that is invariant under the G-action β on $\mathcal{A}$, i.e. that $\omega \circ \beta_q = \omega$ for all $q \in G$. Then for each $q \in G$ we define a bounded operator $V(q) : \pi_\omega(\mathcal{A})\Omega \to H_\omega$ by

$$V(q)\pi_\omega(a)\Omega_\omega := \pi_\omega(\beta_q(a))\Omega_\omega,$$

where we write $\Omega_\omega = 1 + N_\omega \in H_\omega$ as in Example 3.1.4. Then the unique bounded extension of $V(q)$ to H_ω is a unitary operator, which we also denote by $V(q)$, and this gives rise to a unitary representation $V : G \to B(H_\omega)$ of G that satisfies

$$V(q)\Omega_\omega = \Omega_\omega \qquad \text{and} \qquad V(q)\pi_\omega(a)V(q)^* = \pi_\omega(\beta_q(a))$$

for all $q \in G$ and $a \in \mathcal{A}$. For the proof of these statements we refer to Theorem 2.33 of [1].

3.1.2 C^*-tensor categories

The categories that we will encounter in AQFT all have the property that the set of morphisms between any two objects is a Banach space. The set of endomorphisms of one particular object is even a C^*-algebra in these categories. For this reason the following definition will be of great importance to us.

Definition 3.1.6 A $*$-category $\mathcal{C}$ is called a *C^*-category* if:
- for all $V, W \in \mathcal{C}$ the $\mathbb{C}$-vector space $\mathrm{Hom}_\mathcal{C}(V, W)$ is a Banach space with norm $\|.\|_{V,W}$;
- $\|g \circ f\|_{U,W} \leq \|g\|_{V,W} \cdot \|f\|_{U,V}$ for all $f \in \mathrm{Hom}_\mathcal{C}(U, V)$ and $g \in \mathrm{Hom}_\mathcal{C}(V, W)$;
- for any $V \in \mathcal{C}$ and $f \in \mathrm{End}_\mathcal{C}(V)$ we have $\|f^* \circ f\|_{V,V} = \|f\|^2_{V,V}$.

[5] In the situation of a unitarily implemented group action we will also refer to the representation V as being the G-action, because β is completely determined by V.

[6] This also works for a non-unital $\mathcal{A}$, but for our purposes it is enough to consider the unital case.

If $\mathcal{C}$ is a C^*-category and if $V \in \mathcal{C}$ is irreducible, then $\mathrm{Hom}_{\mathcal{C}}(V, W)$ is a Hilbert space with respect to the inner product defined in Subsection 2.7.1. This follows from the fact that the norm $\|.\|$ induced by the inner product coincides with the norm $\|.\|_{V,W}$, namely $\|f\|^2 = \langle f, f\rangle = \langle f, f\rangle \|\mathrm{id}_V\|_{\mathrm{End}_{\mathcal{C}}(V)} = \|f^* \circ f\|_{V,V} = \|f\|^2_{V,W}$. An orthonormal basis for $\mathrm{Hom}_{\mathcal{C}}(V, W)$ is precisely a maximal collection of isometries $\{u_\alpha\}_\alpha$ satisfying $u^*_\alpha \circ u_\beta = \delta_{\alpha,\beta}\mathrm{id}_V$.

A C^**-tensor category* is a C^*-category that is also a tensor category and satisfies

$$\|f_1 \otimes f_2\|_{V_1\otimes V_2, W_1\otimes W_2} \leq \|f_1\|_{V_1,W_1}\|f_2\|_{V_2,W_2}$$

for $f_j \in \mathrm{Hom}_{\mathcal{C}}(V_j, W_j)$. In the rest of this subsection, $\mathcal{C}$ will always denote a C^*-tensor-category with irreducible unit object (i.e. $\mathrm{End}_{\mathcal{C}}(I) = \mathbb{C}\mathrm{id}_I$) that has subobjects and direct sums.

Lemma 3.1.7 *If $V, W \in \mathcal{C}$ have conjugates, then $\mathrm{Hom}_{\mathcal{C}}(V, W)$ is finite-dimensional. As a consequence, any object that has a conjugate is a finite direct sum of irreducible objects.*

Proof. It is shown in Lemma 3.2 of [66] that $\mathrm{End}_{\mathcal{C}}(V)$ and $\mathrm{End}_{\mathcal{C}}(W)$ are finite-dimensional. We will now use the argument in Proposition A.47 of [79]. Let $X \cong V \oplus W$ be a direct sum implemented by isometries $u \in \mathrm{Hom}_{\mathcal{C}}(V, X)$ and $v \in \mathrm{Hom}_{\mathcal{C}}(W, X)$. Then X has a conjugate, so $\mathrm{End}_{\mathcal{C}}(X)$ is finite-dimensional. But the linear map $\mathrm{Hom}_{\mathcal{C}}(V, W) \to \mathrm{End}_{\mathcal{C}}(X)$ given by $f \mapsto v \circ f \circ u$ is injective, so $\mathrm{Hom}_{\mathcal{C}}(V, W)$ must be finite-dimensional.
□

If $(V, r, \overline{r})$ is a conjugate of $V \in \mathcal{C}$, then $(V, r, \overline{r})$ is called *normalized* if $\|r\| = \|\overline{r}\|$. Note that this condition is equivalent to the condition $r^* \circ r = \overline{r}^* \circ \overline{r}$, because $r^*r = \|r^* \circ r\|\mathrm{id}_I = \|r\|^2\mathrm{id}_I$ and $\overline{r}^* \circ \overline{r} = \|\overline{r}^* \circ \overline{r}\|\mathrm{id}_I = \|\overline{r}\|^2\mathrm{id}_I$. If $(\overline{V}, r, \overline{r})$ is any conjugate of V, then $(\overline{V}, \alpha \cdot r, \overline{\alpha}^{-1}\overline{r})$ is also a conjugate of V for any non-zero value of $\alpha \in \mathbb{C}$ (note that α and $\overline{\alpha}^{-1}$ have the same phase but inverse modulus). In particular, we can make any conjugate into a normalized conjugate by rescaling it with the appropriate value of α. In case V is irreducible, $\mathrm{Hom}_{\mathcal{C}}(I, \overline{V} \otimes V)$ and $\mathrm{Hom}_{\mathcal{C}}(I, V \otimes \overline{V})$ are one-dimensional, so up to a phase factor there are unique r and $\overline{r}$ such that $(\overline{V}, r, \overline{r})$ is normalized. In the following two lemmas we will show that direct sums and tensor products of normalized conjugates are normalized conjugates again.

Lemma 3.1.8 *Let $(\overline{V_j}, r_j, \overline{r}_j)$ be normalized conjugates of $V_j \in \mathcal{C}$ for $j \in \{1, \ldots, n\}$ and let $V \cong \bigoplus_{j=1}^n V_j$ and $\overline{V} \cong \bigoplus_{j=1}^n V_j$ be direct sums implemented by isometries $u_j \in \mathrm{Hom}_{\mathcal{C}}(V_j, V)$ and $\overline{u}_j \in \mathrm{Hom}_{\mathcal{C}}(\overline{V_j}, \overline{V})$. Then $(\overline{V}, r, \overline{r})$ is a normalized conjugate of V, where $r = \sum_{j=1}^n [\overline{u}_j \otimes u_j] \circ r_j$ and $\overline{r} = \sum_{j=1}^n [u_j \otimes \overline{u}_j] \circ \overline{r}_j$.*

Proof. First of all we notice that $r^* \circ r$ is equal to

$$\{\sum_i [\overline{u}_i \otimes u_i] \circ r_i\}^* \circ \{\sum_j [\overline{u}_j \otimes u_j] \circ r_j\} = \sum_{i,j} r_i^* \circ [(\overline{u}_i^* \circ \overline{u}_j) \otimes (u_i^* \circ u_j)] \circ r_j = \sum_i r_i^* \circ r_i$$

and similarly $\overline{r}^* \circ \overline{r} = \sum_i \overline{r}_i^* \circ \overline{r}_i$. So normalization of the $(V_j, r_j, \overline{r}_j)$ gives $r^* \circ r = \sum_i r_i^* \circ r_i = \sum_i \overline{r}_i^* \circ \overline{r}_i = \overline{r}^* \circ \overline{r}$, showing that $(\overline{V}, r, \overline{r})$ is normalized.
□

The proof of the following lemma is an easy computation that we omit here.

Lemma 3.1.9 *Let $(\overline{V_1}, r_1, \overline{r}_1)$ and $(\overline{V_2}, r_2, \overline{r}_2)$ be normalized conjugates of V_1 and V_2, respectively. Then $(\overline{V_2} \otimes \overline{V_1}, r, \overline{r})$ is a normalized conjugate of $V_1 \otimes V_2$, where $r = [\mathrm{id}_{\overline{V_2}} \otimes r_1 \otimes \mathrm{id}_{V_2}] \circ r_2$ and $\overline{r} = [\mathrm{id}_{V_1} \otimes \overline{r}_2 \otimes \mathrm{id}_{\overline{V_1}}] \circ r_1$.*

In Subsection 3.1.1 we gave the definition of a left inverse for an endomorphism of a C^*-algebra. We will now define the notion of a left inverse for an object in a C^*-tensor category, which will be used very often in what follows. At the end of Subsection 3.1.5 we will see that the operator algebraic definition of a left inverse is related to the categorical definition of a left inverse.

Definition 3.1.10 If $V \in \mathcal{C}$, then a *left inverse* for V is a family $\{\mathsf{L}_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(V \otimes X, V \otimes Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)\}_{X,Y\in\mathcal{C}}$ of linear maps such that

- for any $f \in \mathrm{Hom}_{\mathcal{C}}(X,X')$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Y')$ we have

$$\mathsf{L}_{X',Y'}\{[\mathrm{id}_V \otimes g] \circ h \circ [\mathrm{id}_V \otimes f^*]\} = g \circ \mathsf{L}_{X,Y}(h) \circ f^*$$

whenever $h \in \mathrm{Hom}_{\mathcal{C}}(V \otimes X, V \otimes Y)$;

- for any $h \in \mathrm{Hom}_{\mathcal{C}}(V \otimes X, V \otimes Y)$ we have $\mathsf{L}_{X\otimes Z,Y\otimes Z}(h \otimes \mathrm{id}_Z) = \mathsf{L}_{X,Y}(h) \otimes \mathrm{id}_Z$ whenever $Z \in \mathcal{C}$.

If L is a left inverse for $V \in \mathcal{C}$, then it is called *normalized* if $\mathsf{L}_{I,I}(\mathrm{id}_V) = \mathrm{id}_I$, it is called *positive* if $\mathsf{L}_{X,X}$ is positive for all $X \in \mathcal{C}$ and it is called *faithful* if $\mathsf{L}_{X,X}$ is faithful for all $X \in \mathcal{C}$.

In a similar way, one also defines the notion of a right inverse of an object in a C^*-tensor category.

If $V \in \mathcal{C}$ and if $(\overline{V}, r, \overline{r})$ is a conjugate for V, then we obtain a left inverse $\mathsf{L}^{(\overline{V},r,\overline{r})}$ for V by defining

$$\mathsf{L}^{(\overline{V},r,\overline{r})}_{X,Y}(h) := [r^* \otimes \mathrm{id}_Y] \circ [\mathrm{id}_{\overline{V}} \otimes h] \circ [r \otimes \mathrm{id}_X]$$

for $h \in \mathrm{Hom}_{\mathcal{C}}(V \otimes X, V \otimes Y)$. We will call this the *left inverse determined by* $(\overline{V}, r, \overline{r})$. The following lemma can be found as Lemma 2.7 in [66].

Lemma 3.1.11 *Let $(\overline{V}, r, \overline{r})$ be a conjugate for $V \in \mathcal{C}$ and let L be the left inverse determined by $(\overline{V}, r, \overline{r})$. Then for any $X, Y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(V \otimes X, V \otimes Y)$ we have*

$$f^* \circ f \leq [(\overline{r}^* \circ \overline{r}) \otimes \mathrm{id}_{V\otimes X}] \circ [\mathrm{id}_V \otimes \mathsf{L}_{X,X}(f^* \circ f)] \tag{3.1.2}$$

and equality holds for $f^* \circ f = \overline{r} \circ \overline{r}^*$. Hence $\overline{r}^* \circ \overline{r}$ is the smallest element $g \in \mathrm{End}_{\mathcal{C}}(I)$ such that the inequality $f^* \circ f \leq [g \otimes \mathrm{id}_{V\otimes X}] \circ [\mathrm{id}_V \otimes \mathsf{L}_{X,X}(f^* \circ f)]$ holds.

If $(\overline{V}, r_1, \overline{r}_1)$ and $(\overline{V}, r_2, \overline{r}_2)$ are two conjugates for V, then they determine the same left inverse if and only if there exists a unitary $u \in \mathrm{End}_{\mathcal{C}}(\overline{V})$ such that $r_2 = [u \otimes \mathrm{id}_V] \circ r_1$ and $\overline{r}_2 = [\mathrm{id}_V \otimes u] \circ \overline{r}_1$, see Lemma 3.3 of [65]. As mentioned above, for irreducible V and a given conjugate object $\overline{V}$, there are r and $\overline{r}$ that are unique up to a phase such that $(\overline{V}, r, \overline{r})$ is a normalized conjugate (i.e. $(\overline{V}, r, \overline{r})$ is unique up to unitary equivalence). Hence there is a unique normalized left inverse for irreducible objects.

Now let L be a left inverse for $V \in \mathcal{C}$ and suppose that V has a conjugate. If $h \in \mathrm{Hom}_{\mathcal{C}}(V \otimes X, V \otimes Y)$ and if we choose a conjugate $(V, r, \overline{r})$ for V, then we can rewrite h as

$$h = [\mathrm{id}_V \otimes r^* \otimes \mathrm{id}_Y] \circ [(\overline{r} \circ \overline{r}^*) \otimes \mathrm{id}_{V\otimes Y}] \circ [\mathrm{id}_{V\otimes\overline{V}} \otimes h] \circ [\mathrm{id}_V \otimes r \otimes \mathrm{id}_X].$$

Using this, the action of $\mathsf{L}_{X,Y}$ on h is given by

$$\begin{aligned}
\mathsf{L}_{X,Y}(h) &= \mathsf{L}_{X,Y}\{[\mathrm{id}_V \otimes r^* \otimes \mathrm{id}_Y] \circ [(\overline{r} \circ \overline{r}^*) \otimes \mathrm{id}_{V\otimes Y}] \circ [\mathrm{id}_{V\otimes\overline{V}} \otimes h] \circ [\mathrm{id}_V \otimes r \otimes \mathrm{id}_X]\} \\
&= [r^* \otimes \mathrm{id}_Y] \circ \{\mathsf{L}_{\overline{V}\otimes V\otimes Y,\overline{V}\otimes V\otimes Y}[(\overline{r} \circ \overline{r}^*) \otimes \mathrm{id}_{V\otimes Y}]\} \circ [\mathrm{id}_{\overline{V}} \otimes h] \circ [r \otimes \mathrm{id}_X] \\
&= [r^* \otimes \mathrm{id}_Y] \circ [\mathsf{L}_{\overline{V},\overline{V}}(\overline{r} \circ \overline{r}^*) \otimes \mathrm{id}_{V\otimes Y}] \circ [\mathrm{id}_{\overline{V}} \otimes h] \circ [r \otimes \mathrm{id}_X] \\
&= [r^* \otimes \mathrm{id}_Y] \circ [\mathsf{L}_{\overline{V},\overline{V}}(\overline{r} \circ \overline{r}^*) \otimes h] \circ [r \otimes \mathrm{id}_X],
\end{aligned}$$

which shows that (in case V has a conjugate) L is completely determined by $\mathsf{L}_{\overline{V},\overline{V}}(\overline{r} \circ \overline{r}^*)$, where $(\overline{V}, r, \overline{r})$ is an arbitrary conjugate for V. In particular, considering different conjugate objects of V would not add any new left inverses for V, so we can just as well choose one particular conjugate object $\overline{V}$ and express all left inverses for V in terms of this conjugate, where different left inverses are obtained by taking different values of $\mathsf{L}_{\overline{V},\overline{V}}(\overline{r} \circ \overline{r}^*)$. The left inverse L is positive if and only if $\mathsf{L}_{\overline{V},\overline{V}}(\overline{r} \circ \overline{r}^*)$ is positive, which is equivalent to the statement that $\mathsf{L}_{\overline{V},\overline{V}}(\overline{r} \circ \overline{r}^*) = x^* x$ for some $x \in \mathrm{End}_{\mathcal{C}}(\overline{V})$. Thus if L is a positive left inverse of V, then we can express it as

$$\mathsf{L}_{X,Y}(h) = \{[r^* \circ (x^* \otimes \mathrm{id}_V)] \otimes \mathrm{id}_Y\} \circ [\mathrm{id}_{\overline{V}} \otimes h] \circ \{[(x \otimes \mathrm{id}_V) \circ r] \otimes \mathrm{id}_X\}$$

for any $h \in \mathrm{Hom}_{\mathcal{C}}(V \otimes X, V \otimes Y)$. Furthermore, the left inverse L is faithful if and only if x is an isomorphism.

Once we have chosen some particular conjugate object $\overline{V}$ of V, the faithful left inverses for V are in bijective correspondence with unitary equivalent classes of conjugates for V. Because for any two conjugates within such a unitary equivalence class we have that the quantity $\|r\| \cdot \|\overline{r}\|$ is the same, the following notion of a dimension of a left inverse is well-defined.

Definition 3.1.12 Let L be a faithful left inverse for V and suppose that V has a conjugate. Then we define the *dimension* of L by $d(\mathsf{L}) := \|r\| \cdot \|\overline{r}\|$, where $(\overline{V}, r, \overline{r})$ is any conjugate implementing L.

The dimension of a left inverse is always ≥ 1, since

$$1 = \|\mathrm{id}_V\| = \|(\overline{r}^* \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes r)\| \leq \|\overline{r}\| \cdot \|r\| = d(\mathsf{L}).$$

In case L comes from a normalized conjugate $(\overline{V}, r, \overline{r})$, the dimension can be written as $d(\mathsf{L}) = \|r\|^2 = \|r^* \circ r\| = \|\overline{r}^* \circ \overline{r}\|$ and $d(\mathsf{L}) = 1$ if and only if r is unitary, see Lemma 3.5 of [66].

It follows directly from the computation in the proof of Lemma 3.1.8 that if V, V_j, $\overline{V}_j$ and $\overline{V}$ are as in the lemma and if we write L_j to denote the left inverse determined by $(\overline{V}_j, r_j, \overline{r}_j)$, we have that

$$d(\mathsf{L}) = r^* \circ r = \sum_{j=1}^{n} r_j^* \circ r_j = \sum_{j=1}^{n} d(\mathsf{L}_j),$$

where in the first equality we used that $(\overline{V}, r, \overline{r})$ is normalized. Also, in the situation of Lemma 3.1.9 we have that

$$d(\mathsf{L}) = d(\mathsf{L}_1) d(\mathsf{L}_2).$$

As we have already seen, for irreducible objects that have a conjugate there exists a unique normalized left inverse. We now wish to define a special class of left inverses for objects that have conjugates but are not irreducible. If $V \in \mathcal{C}$ is an arbitrary object that has a conjugate, then it is a direct sum of finitely many irreducible objects $V_1, \ldots, V_n$. For each of these objects we can take a normalized conjugate $(\overline{V}_j, r_j, \overline{r}_j)$ (uniquely determined up to unitary equivalence) and we can form a direct sum $\overline{V} \cong \bigoplus_{j=1}^n \overline{V}_j$ implemented by isometries $u_j \in \mathrm{Hom}_{\mathcal{C}}(\overline{V}_j, \overline{V})$. As in Lemma 3.1.8 above we can then take the direct sum $(\overline{V}, r, \overline{r})$ of the $(\overline{V}_j, r_j, \overline{r}_j)$ to obtain a normalized conjugate for V. The unitary equivalence class of $(\overline{V}, r, \overline{r})$ does not depend on the choice of isometries, nor on the choice of the $(\overline{V}_j, r_j, \overline{r}_j)$ within their unitary equivalence classes. Hence in this way we obtain a well-defined priviledged unitary equivalence class of conjugates for V, called *standard conjugates.*

Definition 3.1.13 If $V \in \mathcal{C}$ has a conjugate then we define its *standard left inverse* to be the left inverse that is determined by a standard conjugate for V. The *dimension* $d(V)$ of V is defined to be the dimension of its standard left inverse.

In other words, if $(\overline{V}, r, \overline{r})$ is a standard conjugate for V, then $d(V) = r^* \circ r = \overline{r}^* \circ \overline{r}$. Note that for irreducible objects, standard conjugates coincide with normalized conjugates. If $V \in \mathcal{C}$ has a conjugate, we will denote its standard left inverse by $\mathsf{L}^{(V)}$. The following theorem is now immediate.

Theorem 3.1.14 *Let $\mathcal{C}$ be a C^*-tensor category with direct sums and conjugates. Then its full subcategory $\mathcal{C}_f$ determined by the objects that have conjugates is a TC^* and each object in $\mathcal{C}_f$ has a well-defined dimension that is additive with respect to direct sums and multiplicative with respect to tensor products.*

In particular, it is possible to define a notion of dimension in any TC^* (just take $\mathcal{C}$ to be a TC^* in the theorem, so that $\mathcal{C}_f = \mathcal{C}$) in the abscence of a spherical structure. As shown in [112] any TC^* can be equipped with a spherical structure.

As explained in [72] (proposition 2.1), any TC^* can be equipped with a unique norm on the sets of morphisms that makes it into a C^*-tensor category. In particular, we can define standard conjugates for the objects in a TC^*. Now if $\mathcal{C}$ is a BTC^*, then for any $V \in \mathcal{C}$ we define $\theta_V \in \mathrm{End}_{\mathcal{C}}(V)$ by

$$\theta_V := \mathsf{L}^{(V)}_{V,V}(c_{V,V}) = [r_V^* \otimes \mathrm{id}_V] \circ [\mathrm{id}_{\overline{V}} \otimes c_{V,V}] \circ [r_V \otimes \mathrm{id}_V],$$

where $(\overline{V}, r, \overline{r})$ is a standard conjugate for V and we call θ_V the twist of V because it can be shown to be a twist in the sense of Definition 2.4.6. Thus if $\mathcal{C}$ is a rational BTC^* then it can be considered as a pre-modular category.

The next theorem, which is also known as Doplicher-Roberts duality, characterizes so-called even STC^*s and will be very important to us in the next subsection. An STC^* $\mathcal{C}$ is called *even* if $\theta_V = \mathrm{id}_V$ for all $V \in \mathcal{C}$. If G is a compact group, then the category $\mathrm{Rep}_f(G)$ of finite-dimensional unitary representations gives us an example of an even STC^*. Doplicher-Roberts duality states that all STC^*s arise in this way.

Theorem 3.1.15 ***(Doplicher-Roberts duality)*** *If $\mathcal{C}$ is an even STC^*, then there exists a compact group G, unique up to isomorphism, such that there exists an equivalence $\mathcal{C} \simeq \mathrm{Rep}_f(G)$ of STC^*s.*

For a proof of this theorem we refer to the paper [25] by Doplicher and Roberts. An alternative proof was given by Müger in [79].

3.1.3 The crossed product of a BTC^* with a symmetric subcategory

When we come to discuss group actions on a quantum field theory in Subsection 3.2.3, we would like to consider a theorem by Müger which states that the braided G-crossed category of left or right G-localized endomorphisms of the quantum field theory is equivalent to a crossed product of the category of localized endomorphisms of the corresponding orbifold quantum field theory with a certain symmetric subcategory thereof. As a preparation for this, we will now discuss in some detail the general construction of such crossed products. This construction uses as a starting point our construction of the category $\overline{\mathcal{D}_c(\mathsf{Q}_1, \mathsf{Q}_2)}$ in Subsection 2.9.2 where $\mathcal{D}$ was a braided tensor category with braiding c and Q_1 and Q_2 were Frobenius algebras in $\mathcal{D}$. However, this time we will further restrict ourselves to a much smaller class of situations than in Subsection 2.9.2 because we will not need the full generality for our discussion of group actions on a quantum field theory. The first restriction that we will make is the assumption that our categories have a $*$-operation.

If $\mathcal{D}$ is a tensor $*$-category, then any algebra (Q, μ, η) gives rise to a coalgebra (Q, μ^*, η^*) and similarly every coalgebra gives rise to an algebra. Because Frobenius algebras are both algebras and coalgebras, we can thus consider the special class of Frobenius algebras in a tensor $*$-category for which the algebra and coalgebra structures are related by the $*$-operation in this way. Because we also want to consider dimensions of Frobenius algebras, we restrict ourselves to C^*-tensor categories.

Definition 3.1.16 Let $\mathcal{D}$ be a C^*-tensor category. Then a *$*$-Frobenius algebra* is an algebra $\mathsf{Q} = (Q, \mu, \eta)$ that satisfies

$$[\mu \otimes \mathrm{id}_Q] \circ [\mathrm{id}_Q \otimes \mu^*] = \mu^* \circ \mu = [\mathrm{id}_Q \otimes \mu] \circ [\mu^* \otimes \mathrm{id}_Q]. \tag{3.1.3}$$

It is called *strongly separable* (or *special*) if $\mu \circ \mu^* = \kappa_1 \cdot \mathrm{id}_Q$ and $\eta^* \circ \eta = \kappa_2 \cdot \mathrm{id}_I$ for some $\kappa_1, \kappa_2 \in \mathbb{C}$. In this case, if $\kappa_1 = \kappa_2$ then it is called *normalized.* If in addition $\mathcal{D}$ has an irreducible unit and if Q has a conjugate, then Q will be called *standard* if $\kappa_1 = \kappa_2 = \sqrt{d(Q)}$.

Equivalently, a $*$-Frobenius algebra is a coalgebra $\mathsf{Q} = (Q, \Delta, \varepsilon)$ satisfying

$$[\Delta^* \otimes \mathrm{id}_Q] \circ [\mathrm{id}_Q \otimes \Delta] = \Delta \circ \Delta^* = [\mathrm{id}_Q \otimes \Delta^*] \circ [\Delta \otimes \mathrm{id}_Q].$$

When we are working with $*$-Frobenius algebras, we will often use both of the symbols μ and Δ (and also both η and ε), although we could of course choose to replace every Δ by a μ^*, etcetera.

Example 3.1.17 If $\mathcal{D}$ is a C^*-tensor category and $(\overline{V}, r, \overline{r})$ is a conjugate for $V \in \mathcal{D}$, then $Q := \overline{V} \otimes V$ becomes a $*$-Frobenius algebra $\mathsf{Q} = (Q, \mu, \eta)$ if we define $\mu := \mathrm{id}_{\overline{V}} \otimes \overline{r}^* \otimes \mathrm{id}_V$ and $\eta := r$. Now suppose that $\mathcal{D}$ has an irreducible unit and that $(\overline{V}, r, \overline{r})$ is normalized, i.e. $r^* \circ r = \overline{r}^* \circ \overline{r} \equiv \kappa \cdot \mathrm{id}_I$. Then $\mu \circ \mu^* = \mathrm{id}_{\overline{V}} \otimes (\overline{r}^* \circ \overline{r}) \otimes \mathrm{id}_V = \kappa \cdot \mathrm{id}_{\overline{V} \otimes V} = \kappa \cdot \mathrm{id}_Q$ and $\eta^* \circ \eta = r^* \circ r = \kappa \cdot \mathrm{id}_I$, so Q is normalized. If $(\overline{V}, r, \overline{r})$ is also standard, then $\kappa = d(V) = d(\overline{V})$ and hence $\kappa^2 = d(V)d(\overline{V}) = d(Q)$, so Q is standard.

If $\mathsf{Q} = (Q, \mu, \eta)$ is a $*$-Frobenius algebra in a tensor $*$-category $\mathcal{D}$, then we will write $\mathrm{Aut}^*(Q, \mu, \eta)$ (respectively $\mathrm{Aut}^*(Q, \Delta, \varepsilon)$) to denote the set of all unitary algebra (respectively coalgebra) automorphisms of Q. Both $\mathrm{Aut}^*(Q, \mu, \eta)$ and $\mathrm{Aut}^*(Q, \Delta, \varepsilon)$ are obviously groups with respect to the composition in $\mathrm{End}_{\mathcal{D}}(Q)$. Because the algebra structure and coalgebra structure on Q are related by the $*$-operation, the sets $\mathrm{Aut}^*(Q, \mu, \eta)$ and $\mathrm{Aut}^*(Q, \Delta, \varepsilon)$ coincide. Namely, if $u \in \mathrm{Aut}^*(Q, \mu, \eta)$ then it is easy to see that $u^* \in \mathrm{Aut}^*(Q, \Delta, \varepsilon)$ (and the other way around), so unitarity $u^* = u^{-1}$ implies that the two groups are indeed the same as sets, and hence also as groups because the multiplication is the same. We will therefore simply write $\mathrm{Aut}^*(\mathsf{Q})$ to denote both.

Before we will start our discussion of the crossed product, we will make one important observation. Suppose that we are given two $*$-Frobenius algebras Q_1 and Q_2 in a braided tensor $*$-category $\mathcal{D}$ with braiding c. Then we can equip the tensor category $\mathcal{D}_c(\mathsf{Q}_1, \mathsf{Q}_2)$ with a $*$-operation as follows. If $f \in \mathrm{Hom}_{\mathcal{D}_c(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 U J_1, \overline{J}_2 V J_1) = \mathrm{Hom}_{\mathcal{D}}(U \otimes Q_1, Q_2 \otimes V)$, then we define $f^\times \in \mathrm{Hom}_{\mathcal{D}_c(\mathsf{Q}_1,\mathsf{Q}_2)}(\overline{J}_2 V J_1, \overline{J}_2 U J_1) = \mathrm{Hom}_{\mathcal{D}}(V \otimes Q_1, Q_2 \otimes U)$ by

$$f^\times := [\mathrm{id}_{Q_2 \otimes U} \otimes (\eta^{1^*} \circ \mu^1)] \circ [\mathrm{id}_{Q_2} \otimes f^* \otimes \mathrm{id}_{Q_1}] \circ [(\mu^{2^*} \circ \eta^2) \otimes \mathrm{id}_{V \otimes Q_1}].$$

It is an easy computation to check that this defines a $*$-operation on $\mathcal{D}_c(\mathsf{Q}_1, \mathsf{Q}_2)$.

We are now ready to consider the crossed product of a BTC^* with an even symmetric subcategory. Recall that, by Doplicher-Roberts duality, every even STC^* is equivalent to $\mathrm{Rep}_f(G)$ for some compact group G (which is uniquely determined up to isomorphism). In case it is also rational, more can be said. The following proposition can be found in [77].

Proposition 3.1.18 *If $\mathcal{S}$ is an even STC^* that is rational, then there exists a commutative $*$-Frobenius algebra $\mathsf{Q}_{\mathcal{S}} = (Q, \mu, \eta)$ in $\mathcal{S}$ with the following properties:*

(1) there is an equivalence $\mathcal{S} \simeq \mathrm{Rep}_f(G_{\mathcal{S}})$ of STC^*s, where $G_{\mathcal{S}}$ is the finite group $G_{\mathcal{S}} = \mathrm{Aut}^*(\mathsf{Q}_{\mathcal{S}})$;

(2) the image of $\mathsf{Q}_{\mathcal{S}}$ under the equivalence $\mathcal{S} \simeq \mathrm{Rep}_f(G_{\mathcal{S}})$ is isomorphic to the left regular representation of $G_{\mathcal{S}}$;

(3) $\mathsf{Q}_{\mathcal{S}}$ is strongly separable and $\kappa_1 \cdot \kappa_2 = |G_{\mathcal{S}}|$;

(4) $\mathsf{Q}_{\mathcal{S}}$ is absorbing, i.e. for any $X \in \mathcal{S}$ we have $Q \otimes X \cong Q^{\oplus d(X)}$;

(5) $\dim(\mathrm{Hom}_{\mathcal{S}}(I, Q)) = 1$.

The setting for the rest of this subsection will be as follows. We will always assume that we are given a category $\mathcal{D}$ which is a BTC^* (with braiding denoted by c). Whenever we are working with conjugates in $\mathcal{D}$, we will always implicitly assume that they are standard. In particular, these conjugates give rise to standard left inverses L and standard right inverses R and to a well-defined dimension of the objects in $\mathcal{D}$. We also assume there is a full subcategory $\mathcal{S} \subset \mathcal{D}$ which is a rational even STC^* and we will write $\mathsf{Q}_{\mathcal{S}} = (Q, \mu, \eta)$ to denote the corresponding $*$-Frobenius algebra in $\mathcal{S}$ given by Proposition 3.1.18 above and we will write $G_{\mathcal{S}}$ to denote the corresponding group $\mathrm{Aut}^*(\mathsf{Q}_{\mathcal{S}})$. We scale the morphisms μ and η such that

$\kappa_1 = |G|$ and $\kappa_2 = 1$. In the rest of this subsection we will use the notation[7] $(\mathcal{D} \rtimes_0 \mathcal{S}, c)_1 := \mathcal{D}_c(\mathsf{Q}_{\mathcal{S}}, \mathsf{Q}_0)$ and $(\mathcal{D} \rtimes_0 \mathcal{S}, c)_2 := \mathcal{D}_c(\mathsf{Q}_0, \mathsf{Q}_{\mathcal{S}})$ and

$$(\mathcal{D} \rtimes \mathcal{S}, c)_1 := \overline{\mathcal{D}_c(\mathsf{Q}_{\mathcal{S}}, \mathsf{Q}_0)}$$
$$(\mathcal{D} \rtimes \mathcal{S}, c)_2 := \overline{\mathcal{D}_c(\mathsf{Q}_0, \mathsf{Q}_{\mathcal{S}})}.$$

The next lemma, the content of which can be found in Proposition 3.3 of [77], shows that these tensor categories can be equipped with a $G_{\mathcal{S}}$-action.

Lemma 3.1.19 *Both $(\mathcal{D} \rtimes \mathcal{S}, c)_1$ and $(\mathcal{D} \rtimes \mathcal{S}, c)_2$ can be equipped with the structure of a strict $G_{\mathcal{S}}$-category as follows.*

(1) For any $f \in \mathrm{Hom}_{(\mathcal{D} \rtimes_0 \mathcal{S}, c)_1}(\overline{J}_0 V J_{\mathcal{S}}, \overline{J}_0 W J_{\mathcal{S}}) = \mathrm{Hom}_{\mathcal{D}}(V \otimes Q, W)$ and $q \in G$ we first define

$$F_q^1(f) := f \circ [\mathrm{id}_V \otimes q^{-1}] \in \mathrm{Hom}_{\mathcal{D}}(V \otimes Q, W) = \mathrm{Hom}_{(\mathcal{D} \rtimes_0 \mathcal{S}, c)_1}(\overline{J}_0 V J_{\mathcal{S}}, \overline{J}_0 W J_{\mathcal{S}}). \quad (3.1.4)$$

If $(\overline{J}_0 X J_{\mathcal{S}}, e) \in (\mathcal{D} \rtimes \mathcal{S}, c)_1$, then we define the action of $q \in G_{\mathcal{S}}$ on $(\overline{J}_0 X J_{\mathcal{S}}, e)$ by

$$F_q^1[(\overline{J}_0 X J_{\mathcal{S}}, e)] := (\overline{J}_0 X J_{\mathcal{S}}, F_q^1(e)).$$

If $f \in \mathrm{Hom}_{(\mathcal{D} \rtimes \mathcal{S}, c)_1}((\overline{J}_0 V J_{\mathcal{S}}, e), (\overline{J}_0 W J_{\mathcal{S}}, p))$, then we define $F_q^1(f)$ by the same formula as in (3.1.4).

(2) For any $f \in \mathrm{Hom}_{(\mathcal{D} \rtimes_0 \mathcal{S}, c)_2}(\overline{J}_{\mathcal{S}} V J_0, \overline{J}_{\mathcal{S}} W J_0) = \mathrm{Hom}_{\mathcal{D}}(V, Q \otimes W)$ and $q \in G_{\mathcal{S}}$ we first define

$$F_q^2(f) := [q \otimes \mathrm{id}_W] \circ f \in \mathrm{Hom}_{\mathcal{D}}(V, Q \otimes W) = \mathrm{Hom}_{(\mathcal{D} \rtimes_0 \mathcal{S}, c)_2}(\overline{J}_{\mathcal{S}} V J_0, \overline{J}_{\mathcal{S}} W J_0). \quad (3.1.5)$$

If $(\overline{J}_{\mathcal{S}} X J_0, e) \in (\mathcal{D} \rtimes \mathcal{S}, c)_2$, then we define the action of $q \in G_{\mathcal{S}}$ on $(\overline{J}_{\mathcal{S}} X J_0, e)$ by

$$F_q^2[(\overline{J}_{\mathcal{S}} X J_0, e)] := (\overline{J}_{\mathcal{S}} X J_0, F_q^2(e)).$$

If $f \in \mathrm{Hom}_{(\mathcal{D} \rtimes \mathcal{S}, c)_2}((\overline{J}_{\mathcal{S}} V J_0, e), (\overline{J}_{\mathcal{S}} W J_0, p))$, then we define $F_q^2(f)$ by the same formula as in (3.1.5). *Furthermore, we have $(\mathcal{D} \rtimes \mathcal{S}, c)_1^{G_{\mathcal{S}}} \simeq \mathcal{D}$ and $(\mathcal{D} \rtimes \mathcal{S}, c)_2^{G_{\mathcal{S}}} \simeq \mathcal{D}$ as tensor categories.*

Now that we know that both categories are $G_{\mathcal{S}}$-categories, we will focus on defining a $G_{\mathcal{S}}$-grading on them. This is a difficult procedure, the details of which can be found in Section 3.2 of [77]. However, because we have chosen some different conventions than the ones in [77] which will cause our formulas for the $G_{\mathcal{S}}$-degree of objects to be different, we will give a sketch of the proof of the next lemma. In this way the reader can check that our formulas are correct.

Lemma 3.1.20 *Let $\widetilde{X} = (\overline{J}_0 X J_{\mathcal{S}}, e) \in (\mathcal{D} \rtimes \mathcal{S}, c)_1$ and $\widetilde{Y} = (\overline{J}_{\mathcal{S}} Y J_0, p) \in (\mathcal{D} \rtimes \mathcal{S}, c)_2$ be irreducible.*

(1) There exists a unique morphism $\partial_1(\widetilde{X}) \in \mathrm{End}_{\mathcal{D}}(Q)$ such that

$$\widetilde{c}_{Q,X} \circ \widetilde{c}_{X,Q} \circ [e \otimes \mathrm{id}_Q] \circ [\mathrm{id}_X \otimes \Delta] = [e \otimes \partial_1(\widetilde{X})] \circ [\mathrm{id}_X \otimes \Delta]. \quad (3.1.6)$$

Explicitly, this morphism $\partial_1(\widetilde{X})$ is given by

$$\partial_1(\widetilde{X}) = \left(\mathsf{L}_{Q,I}^{(X)}(e) \circ \eta\right)^{-1} \cdot \left\{\mathsf{L}_{Q\otimes Q,Q}^{(X)}\left[\widetilde{c}_{Q,X} \circ \widetilde{c}_{X,Q} \circ (e \otimes \mathrm{id}_Q)\right]\right\} \circ \Delta$$

and we have $\partial_1(\widetilde{X}) \in G_{\mathcal{S}}$.

[7] Perhaps we have to say some words about this choice of notation. The notation $\mathcal{D} \rtimes_0 \mathcal{S}$ and $\mathcal{D} \rtimes \mathcal{S}$ is chosen simply because it also occurs in [77], although a slightly different convention was chosen there (concerning the place of the Frobenius algebra in the definition of the sets of morphisms). The explicit appearance of the braiding c in our notation was already explained when we introduced the notation $\mathcal{D}_c(\mathsf{Q}_1, \mathsf{Q}_2)$. Finally, the subindices 1 and 2 refer to the fact that, when these categories will be equipped with the structure of a $G_{\mathcal{S}}$-crossed category, the one with the subindex 1 (respectively 2) will have a braiding of the first (respectively second) kind extending c.

(2) There exists a unique morphism $\partial_2(\widetilde{Y}) \in \mathrm{End}_{\mathcal{D}}(Q)$ *such that*

$$[\mu \otimes \mathrm{id}_Y] \circ [\mathrm{id}_Q \otimes p] \circ c_{Y,Q} \circ c_{Q,Y} = [\mu \otimes \mathrm{id}_Y] \circ [\partial_2(\widetilde{Y}) \otimes p]. \tag{3.1.7}$$

Explicitly, this morphism $\partial_2(\widetilde{Y})$ *is given by*

$$\partial_2(\widetilde{Y}) = \left(\varepsilon \circ \mathsf{R}^{(Y)}_{I,Q}(p)\right)^{-1} \cdot \mu \circ \left\{\mathsf{R}^{(Y)}_{Q,Q\otimes Q}\left[(\mathrm{id}_Q \otimes p) \circ c_{Y,Q} \circ c_{Q,Y}\right]\right\}$$

and we have $\partial_2(\widetilde{Y}) \in G_{\mathcal{S}}$.

Proof. The proofs of (1) and (2) are very similar; we will only give the proof of (2) here. We will proceed in the same way as in the discussion after Lemma 3.10 of [77]. Let $v \in \mathrm{Hom}_{(\mathcal{D}\rtimes\mathcal{S},c)_2}(\widetilde{Y}, \mathscr{I}(Y))$ be such that $v^\times \bullet v = \mathrm{id}_{\widetilde{Y}}$ and $v \bullet v^\times = p$. We define $\partial_2''\widetilde{Y} \in \mathrm{End}_{(\mathcal{D}\rtimes\mathcal{S},c)_2}(\mathscr{I}(Q) \circledast \widetilde{Y})$ by

$$\partial_2''(\widetilde{Y}) := [\mathrm{id}_{\mathscr{I}(Q)} \circledast v] \bullet \mathscr{I}(c_{Y,Q} \circ c_{Q,Y}) \bullet [\mathrm{id}_{\mathscr{I}(Q)} \circledast v^\times].$$

This morphism is invertible, with inverse given by the same expression but with c replaced by $\widetilde{c}$. Thus $\partial_2''(\widetilde{Y}) \in \mathrm{Aut}_{(\mathcal{D}\rtimes\mathcal{S},c)_2}(\mathscr{I}(Q) \circledast \widetilde{Y})$. Because $\widetilde{Y}$ is irreducible and $\mathscr{I}(Q) \cong \mathscr{I}(I)^{\oplus|G|}$ (which follows from the absorbing property), there exists $\partial_2'(\widetilde{Y}) \in \mathrm{Aut}_{(\mathcal{D}\rtimes\mathcal{S},c)_2}(\mathscr{I}(Q))$ such that

$$\partial_2''(\widetilde{Y}) = \partial_2'(\widetilde{Y}) \circledast \mathrm{id}_{\widetilde{Y}}. \tag{3.1.8}$$

In the category $(\mathcal{D} \rtimes_0 \mathcal{S}, c)_2$ this equation corresponds to[8]

$$[\mathrm{id}_{\mathscr{I}_0(Q)} \circledast p] \bullet \mathscr{I}_0(c_{Y,Q} \circ c_{Q,Y}) = \partial_2'(\widetilde{Y}) \circledast p,$$

which corresponds in the category $\mathcal{D}$ to

$$[c_{Q,Q} \otimes \mathrm{id}_Y] \circ [\mathrm{id}_Q \otimes p] \circ c_{Y,Q} \circ c_{Q,Y} = [\mu \otimes \mathrm{id}_{Q\otimes Y}] \circ [\mathrm{id}_Q \otimes c_{Q,Q} \otimes \mathrm{id}_Y] \circ [\partial_2'(\widetilde{Y}) \otimes p].$$

If we compose both sides from the left with $\mu \otimes \mathrm{id}_Y$ and use commutativity of $\mathsf{Q}_{\mathcal{S}}$, we obtain

$$[\mu \otimes \mathrm{id}_Y] \circ [\mathrm{id}_Q \otimes p] \circ c_{Y,Q} \circ c_{Q,Y} = [\mu_2 \otimes \mathrm{id}_Y] \circ [\partial_2'(\widetilde{Y}) \otimes p] =: [\mu \otimes \mathrm{id}_Y] \circ [\partial_2(\widetilde{Y}) \otimes p],$$

where we have defined $\partial_2(\widetilde{Y}) := \mu \circ \partial_2'(\widetilde{Y}) \in \mathrm{End}_{\mathcal{D}}(Q)$. Applying the standard right inverse of $\widetilde{Y}$ on both sides of (3.1.8) gives

$$\partial_2'(\widetilde{Y}) = d(\widetilde{Y})^{-1} \cdot \mathsf{R}^{(\widetilde{Y})}_{\mathscr{I}(Q),\mathscr{I}(Q)}(\partial_2''(\widetilde{Y})),$$

which in $\mathcal{D}$ corresponds to

$$\partial_2'(\widetilde{Y}) = \left(\varepsilon \circ \mathsf{R}^{(Y)}_{I,Q}(p)\right)^{-1} \cdot \mathsf{R}^{(Y)}_{Q,Q\otimes Q}\left[(\mathrm{id}_Q \otimes p) \circ c_{Y,Q} \circ c_{Q,Y}\right]$$

and hence we have

$$\partial_2(\widetilde{Y}) = \left(\varepsilon \circ \mathsf{R}^{(Y)}_{I,Q}(p)\right)^{-1} \mu \circ \left\{\mathsf{R}^{(Y)}_{Q,Q\otimes Q}\left[(\mathrm{id}_Q \otimes p) \circ c_{Y,Q} \circ c_{Q,Y}\right]\right\}$$

Now that we have an explicit expression for $\partial_2(\widetilde{Y}) \in \mathrm{End}_{\mathcal{D}}(Q)$ we can prove that it is in $G_{\mathcal{S}}$. Because $\dim(\mathrm{Hom}_{\mathcal{D}}(I, Q)) = 1$, we must have that $\partial_2(\widetilde{Y}) \circ \eta$ is a multiple of η. Using naturality of the braiding in

[8] Here we use Lemma 2.9.9.

$\mathcal{D}$ and the fact that $c_{I,Q} = c_{Q,I} = \mathrm{id}_Q$, we easily see that this multiple is 1, i.e. $\partial_2(\widetilde{Y}) \circ \eta = \eta$. Analogous to the computation in [77], one can now show that

$$[\mu \otimes \mathrm{id}_Y] \circ [\mathrm{id}_Q \otimes p] \circ [\partial_2(\widetilde{Y}) \otimes \mathrm{id}_Y] \circ [\mu \otimes \mathrm{id}_Y] = [\mu_2 \otimes \mathrm{id}_Y] \circ [\mathrm{id}_{Q\otimes Q} \otimes p] \circ [\partial_2(\widetilde{Y}) \otimes \partial_2(\widetilde{Y}) \otimes \mathrm{id}_Y].$$

Letting the right inverse $\mathsf{R}^{(Y)}_{Q\otimes Q,Q}$ act on both sides and noting that $\mathsf{R}^{(Y)}_{I,Q}(p) \in \mathrm{Hom}_{\mathcal{D}}(I,Q)$ must be a scalar multiple of η, we find that

$$\partial_2(\widetilde{Y}) \circ \mu = \mu \circ [\partial_2(\widetilde{Y}) \otimes \partial_2(\widetilde{Y})],$$

so $\partial_2(\widetilde{Y})$ is indeed an algebra morphism.
□

Thus to each irreducible object in these two categories we have assigned an element in the group $G_{\mathcal{S}}$. We can now use this to define a $G_{\mathcal{S}}$-grading on these categories. If $q \in G$ then an object $\widetilde{X}$ in either of these categories will be called *homogeneous of degree* q if it is a finite direct sum of irreducible objects $\widetilde{X}_j$ with $\partial_{1/2}(\widetilde{X}_j) = q$ for all j. It is easy to see that if some (not necessarily irreducible) object in these categories is homogeneous of degree q, then the formulas for $\partial_{1/2}$ in the lemma also correctly give q. So for any homogeneous object in these categories its degree is given by these formulas. Also, if two homogeneous objects are isomorphic to each other, then their degrees must be the same. These facts can be found in Lemma 3.13 of [77]. Furthermore, if $\widetilde{X}$ and $\widetilde{Y}$ are two homogeneous objects in either (but the same) of these two categories, then $\partial_{1/2}(\widetilde{X} \circledast \widetilde{Y}) = \partial_{1/2}(\widetilde{X})\partial_{1/2}(\widetilde{Y})$. For the proof we refer to Proposition 3.17 of [77].

In the same way as in the proof of Proposition 3.14 of [77], one can show that both categories are $G_{\mathcal{S}}$-crossed. In fact, they are both braided $G_{\mathcal{S}}$-crossed TC^*s. We will now define a braiding on both categories.

Lemma 3.1.21 *We will write $\mathscr{I}_0$ to denote both inclusion functors $\mathcal{D} \to (\mathcal{D} \rtimes_0 \mathcal{S}, c)_1$ and $\mathcal{D} \to (\mathcal{D} \rtimes_0 \mathcal{S}, c)_2$.*

(1) If $\widetilde{X} = (\overline{J}_0 X J_{\mathcal{S}}, e) \in (\mathcal{D} \rtimes \mathcal{S}, c)_1$ is homogeneous and if $f \in \mathrm{Hom}_{(\mathcal{D}\rtimes_0\mathcal{S},c)_1}(\overline{J}_0 Y J_{\mathcal{S}}, \overline{J}_0 Z J_{\mathcal{S}})$, then

$$\mathscr{I}_0(\widetilde{c}_{Z,X}) \bullet [f \circledast e] = [\mathrm{id}_{\overline{J}_0 X J_{\mathcal{S}}} \circledast F_{\partial(\widetilde{X})^{-1}}(f)] \bullet \mathscr{I}_0(\widetilde{c}_{Y,X}) \bullet [\mathrm{id}_{\overline{J}_0 Y J_{\mathcal{S}}} \circledast e].$$

As a consequence, if $\widetilde{Y} = (\overline{J}_0 Y J_{\mathcal{S}}, p)$ is homogeneous, then

$$\mathscr{I}_0(\widetilde{c}_{Y,X}) \bullet [p \circledast e] = [e \circledast F_{\partial(\widetilde{X})^{-1}}(p)] \bullet \mathscr{I}_0(\widetilde{c}_{Y,X})$$

and this morphism is in $\mathrm{Hom}_{(\mathcal{D}\rtimes\mathcal{S},c)_1}(\widetilde{Y} \circledast \widetilde{X}, \widetilde{X} \circledast F_{\partial_1(\widetilde{X})^{-1}}(\widetilde{Y}))$.

(2) If $\widetilde{X} = (\overline{J}_{\mathcal{S}} X J_0, e) \in (\mathcal{D} \rtimes \mathcal{S}, c)_2$ is homogeneous and if $f \in \mathrm{Hom}_{(\mathcal{D}\rtimes_0\mathcal{S},c)_2}(\overline{J}_{\mathcal{S}} Y J_0, \overline{J}_{\mathcal{S}} Z J_0)$, then

$$[e \circledast F_{\partial_2(\widetilde{X})^{-1}}(f)] \bullet \mathscr{I}_0(c_{Y,X}) = [e \circledast \mathrm{id}_{\overline{J}_{\mathcal{S}} Z J_0}] \bullet \mathscr{I}_0(c_{Z,X}) \bullet [f \circledast \mathrm{id}_{\overline{J}_{\mathcal{S}} X J_0}].$$

As a consequence, if $\widetilde{Y} = (\overline{J}_{\mathcal{S}} Y J_0, p)$ is homogeneous, then

$$\mathscr{I}_0(c_{Y,X}) \bullet [p \circledast e] = [e \circledast F_{\partial_2(\widetilde{X})^{-1}}(p)] \bullet \mathscr{I}_0(c_{Y,X})$$

and this morphism is in $\mathrm{Hom}_{(\mathcal{D}\rtimes\mathcal{S},c)_2}(F_{\partial_2(\widetilde{X})}(\widetilde{Y}) \circledast \widetilde{X}, \widetilde{X} \circledast (\widetilde{Y}))$.

The proof of this lemma makes use of Lemma 2.9.9 and proceeds in the same way as in [77], so we will not provide it here. An immediate consequence of the lemma is the following corollary.

Corollary 3.1.22 *The $G_{\mathcal{S}}$-crossed categories can be equipped with a braiding as follows.*

(1) If $\widetilde{X} = (\overline{J}_0 X J_{\mathcal{S}}, e), \widetilde{Y} = (\overline{J}_0 Y J_{\mathcal{S}}, p) \in (\mathcal{D} \rtimes \mathcal{S}, c)_1$ *are homogeneous, then*

$$\tilde{c}^1_{\widetilde{X},\widetilde{Y}} := \mathscr{I}_0(\widetilde{c}_{X,Y}) \bullet [e \circledast p]$$

defines a braiding of the second kind on $(\mathcal{D} \rtimes \mathcal{S}, c)_1$ *which extends the braiding* $\widetilde{c}$ *on* $\mathcal{D}$.

(2) If $\widetilde{X} = (\overline{J}_{\mathcal{S}} X J_0, e), \widetilde{Y} = (\overline{J}_{\mathcal{S}} Y J_0, p) \in (\mathcal{D} \rtimes \mathcal{S}, c)_2$ *are homogeneous, then*

$$c^2_{\widetilde{X},\widetilde{Y}} := \mathscr{I}_0(c_{X,Y}) \bullet [e \circledast p]$$

defines a braiding of the second kind on $(\mathcal{D} \rtimes \mathcal{S}, c)_2$ *which extends the braiding* c *on* $\mathcal{D}$.

If we define $c^1_{\widetilde{X},\widetilde{Y}} := (\tilde{c}^1_{F^1_{\partial_1(\widetilde{X})}(\widetilde{Y}),\widetilde{X}})^{-1}$, we obtain a braiding of the first kind on $(\mathcal{D} \rtimes \mathcal{S}, c)_1$ that extends the braiding c. Thus we have constructed two braided $G_{\mathcal{S}}$-crossed TC^*s $(\mathcal{D} \rtimes \mathcal{S}, c)_1$ and $(\mathcal{D} \rtimes \mathcal{S}, c)_2$, the first of which has a braiding of the first kind extending c and the second of which has a braiding of the second kind extending c.

3.1.4 C^*-2-categories from pairs of operator algebras

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two unital C^*-algebras. We will write $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$ to denote the set of all unital $*$-homomorphisms from $\mathcal{A}_1$ to $\mathcal{A}_2$. If $\rho, \sigma \in \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$, then we define the set

$$(\rho, \sigma) := \{R \in \mathcal{A}_2 : R\rho(A) = \sigma(A)R \text{ for all } A \in \mathcal{A}_1\}.$$

If $R \in (\rho, \sigma)$ and $S \in (\sigma, \tau)$ for $\rho, \sigma, \tau \in \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$, then for any $A \in \mathcal{A}_1$ we have

$$SR\rho(A) = S\sigma(A)R = \tau(A)SR,$$

so $SR \in (\rho, \tau)$. For any $\rho \in \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$ we obviously have that $1 \in (\rho, \rho)$; whenever we want to emphasize that we are considering $1 \in \mathcal{A}_2$ as an element in (ρ, ρ), we will write it as 1_ρ. If $\rho, \sigma \in \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$ and if $R \in (\rho, \sigma)$, then we obviously have $R1_\rho = R = 1_\sigma R$. This shows that we obtain a category, where the objects are defined to be elements in $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$, the set of morphisms from ρ to σ is defined to be (ρ, σ), the composition of morphisms is given by the multiplication in $\mathcal{A}_2$ and the identity morphism 1_ρ in (ρ, ρ) is the unit element $1 \in \mathcal{A}_2$. We will denote this category by $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$. In case $\mathcal{A}_1 = \mathcal{A}_2 \equiv \mathcal{A}$, we will write $\mathrm{End}(\mathcal{A})$ rather than $\mathrm{Hom}(\mathcal{A}, \mathcal{A})$. It is straightforward to check that for any $\rho, \sigma \in \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$ the set (ρ, σ) is a Banach subspace of $\mathcal{A}_2$, and if $R \in (\rho, \sigma)$ then for any $A \in \mathcal{A}_1$ we have

$$R^*\sigma(A) = R^*\sigma(A^*)^* = (\sigma(A^*)R)^* = (R\rho(A^*))^* = \rho(A^*)^*R^* = \rho(A)R^*,$$

so $R^* \in (\sigma, \rho)$. One easily shows that $\mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2)$ is in fact a C^*-category.

If $\sigma \in \mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j)$ and $\rho \in \mathrm{Hom}(\mathcal{A}_j, \mathcal{A}_k)$ for $i, j, k \in \{1, 2\}$, then of course $\rho \otimes \sigma := \rho \circ \sigma \in \mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_k)$. Now suppose that $\sigma, \sigma' \in \mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j)$, $\rho, \rho' \in \mathrm{Hom}(\mathcal{A}_j, \mathcal{A}_k)$ for $i, j, k \in \{1, 2\}$ and that $R \in (\rho, \rho')$ and $S \in (\sigma, \sigma')$. Define

$$R \times S := R\rho(S) = \rho'(S)R. \tag{3.1.9}$$

Then for any $A \in \mathcal{A}_i$ we have

$$\begin{aligned}(R \times S)[(\rho \otimes \sigma)(A)] &= R\rho(S)\rho(\sigma(A)) = R\rho(S\sigma(A)) = \rho'(S\sigma(A))R \\ &= \rho'(\sigma'(A)S)R = \rho'(\sigma'(A))\rho'(S)R \\ &= [(\rho' \otimes \sigma')(A)](R \times s),\end{aligned}$$

showing that $R \times S \in (\rho \otimes \sigma, \rho' \otimes \sigma')$. Now let $\iota_j \in \mathrm{End}(\mathcal{A}_j)$ denote the identity endomorphism of $\mathcal{A}_j$ and let $\rho \in \mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j)$ and $\sigma \in \mathrm{Hom}(\mathcal{A}_j, \mathcal{A}_k)$ for $i, j, k \in \{1, 2\}$. Then $\iota_j \otimes \rho = \rho$ and $\sigma \otimes \iota_j = \sigma$. Furthermore, if $R \in \mathrm{Hom}(\mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j))$ and $S \in \mathrm{Hom}(\mathrm{Hom}(\mathcal{A}_j, \mathcal{A}_k))$, then $1_{\iota_j} \times R = R$ and $S \times 1_{\iota_j} = S$. Hence we conclude that we obtain a 2-category with two 0-cells $\mathcal{A}_1$ and $\mathcal{A}_2$, where (in the notation of Definition 2.9.10) $\mathcal{C}(\mathcal{A}_i, \mathcal{A}_j) = \mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j)$ is a C^*-category, the horizontal composition $\diamond$ is given by $\otimes$ on the 1-cells and by $\times$ on the 2-cells, and the unit objects are given by $\iota_1 \in \mathrm{End}(\mathcal{A}_1)$ and $\iota_2 \in \mathrm{End}(\mathcal{A}_2)$ with trivial unit constraints. As a consequence, $\mathrm{End}(\mathcal{A})$ is a C^*-tensor category for any C^*-algebra $\mathcal{A}$.

Now suppose that $\mathcal{A}$ is a type III factor on a separable Hilbert space H. If $\rho \in \mathrm{End}(\mathcal{A})$ and if $E \in (\rho, \rho)$ is a projection, then there exists an isometry $U \in \mathcal{A}$ such that $UU^* = E$. If we define the linear map $\rho_E : \mathcal{A} \to \mathcal{A}$ by

$$\rho_E(A) := U^* \rho(A) U$$

for each $A \in \mathcal{A}$, then $\rho_E(1) = U^*\rho(1)U = U^*U = 1$ and for any $A, B \in \mathcal{A}$ we have

$$\rho_E(AB) = U^*\rho(AB)U = U^*\rho(A)\rho(B)U = U^*E\rho(A)\rho(B)U = U^*\rho(A)E\rho(B)U = \rho_E(A)\rho_E(B)$$

and $\rho_E(A^*) = U^*\rho(A)^*U = (U^*\rho(A)U)^* = \rho_E(A)^*$. We thus have $\rho_E \in \mathrm{End}(\mathcal{A})$. Also, $U\rho_E(A) = E\rho(A)U = \rho(A)EU = \rho(A)U$, so $U \in (\rho_E, \rho)$. This shows that $\mathrm{End}(\mathcal{A})$ has subobjects. Now let $\rho_1, \rho_2 \in \mathrm{End}(\mathcal{A})$. Because von Neumann algebras are the norm-closures of the linear spans of their projections and because type III factors cannot be finite-dimensional as vector spaces, there exists a non-zero projection $E \in \mathcal{A}$ with $E \neq 1$. Then $1 - E \in \mathcal{A}$ is also a non-zero projection with $1 - E \neq 1$ and hence there exist isometries $V_1, V_2 \in \mathcal{A}$ such that $V_1V_1^* = E$ and $V_2V_2^* = 1 - E$. For each $A \in \mathcal{A}$ we then define the linear map $\rho : \mathcal{A} \to \mathcal{A}$ by

$$\rho(A) := V_1\rho_1(A)V_1^* + V_2\rho_2(A)V_2^*.$$

It is easy to see that $\rho(A^*) = \rho(A)^*$ for all $A \in \mathcal{A}$ and that $\rho(1) = 1$. For $j \neq k$ the computation $V_j^*V_k = V_j^*(V_jV_j^* + V_kV_k^*)V_k = 2V_j^*V_k$ shows that $V_1^*V_2 = 0 = V_2^*V_1$. Using this it is easy to check that $\rho(AB) = \rho(A)\rho(B)$ for all $A, B \in \mathcal{A}$. Thus $\mathrm{End}(\mathcal{A})$ has direct sums. We thus conclude that if $\mathcal{A}$ is a type III factor, then $\mathrm{End}(\mathcal{A})$ has subobjects and direct sums.

If G is a group and $\beta : G \to \mathrm{Aut}(\mathcal{A})$ is a G-action on a C^*-algebra $\mathcal{A}$ and $q \in G$, then for any $\rho \in \mathrm{End}(\mathcal{A})$ we define $\beta_q(\rho) \in \mathrm{End}(\mathcal{A})$ by $\beta_q(\rho) := \beta_q \circ \rho \circ \beta_{q^{-1}}$. It is straightforward to check that in this way we obtain a G-action β on the category $\mathrm{End}(\mathcal{A})$.

3.1.5 Subfactors of type III

In this subsection we will briefly mention the main facts about type III subfactors, as can be found in Section 2 of [65]. We will not provide any proofs here.

Definition 3.1.23 Two von Neumann algebras M and N on some common Hilbert space H with $N \subset M$ are called *an inclusion of von Neumann algebras*. We will write $i : N \to M$ to denote the inclusion map. If both M and N are factors, then the inclusion $N \subset M$ is called a *subfactor*. A subfactor is called *irreducible* if $N' \cap M = \mathbb{C}1$.

Note that if $N \subset M$ is an inclusion of von Neumann algebras, then so is $M' \subset N'$.

In the rest of this subsection, we will always assume that we are given a subfactor $N \subset M$, where both N and M are factors of type III acting on a separable Hilbert space H. In this situation there exists a vector $\Phi \in H$ that is cyclic and separating for both N and M. We will write φ to denote the state on N determined by Φ, i.e. $\varphi(n) = \langle n\Phi, \Phi \rangle$ for all $n \in N$, which is a faithful normal state on N. Let J_N and J_M be the modular conjugation operators[9] with respect to Φ for N and M, respectively, and write $j_N := \mathrm{Ad}(J_N)$ and $j_M := \mathrm{Ad}(J_M)$. It is a standard result that $j_N(N) = N'$ and $j_N(N') = N$ and the same equations also hold when we replace N with M, of course. Then the *canonical endomorphism* $\gamma \in \mathrm{End}(M)$ (with respect to Φ) is defined by

$$\gamma := j_N \circ j_M.$$

[9]For the definition of these modular conjugation operators we refer to theorem 9.2.9 of [47] (Tomita's theorem).

A different choice of Φ only changes γ by conjugation with a unitary in N. If we write

$$N_1 := \gamma(M) = j_N(j_M(M))$$
$$M_1 := j_M(j_N(N))$$

then $N_1 = j_N(j_M(M)) = j_N(M') \subset j_N(N') = N$ and $M_1 = j_M(j_N(N)) = j_M(N') \supset j_M(M') = M$, so we obtain two new subfactors $N_1 \subset N$ and $M \subset M_1$. The subfactor $N_1 \subset N$ has canonical endomorphism $\lambda = j_{N_1} \circ j_N$, which is also called the *dual canonical endomorphism.* Because $J_N J_M = J_{N_1} J_N$, the dual canonical endomorphism is the restriction of the canonical endomorphism to N, i.e. $\lambda = \gamma|_N$. Because $\gamma(M) \subset N$, we can also interpret $\gamma \in \mathrm{End}(M)$ as a map $M \to N$, which we will denote by $\bar{i}$. Thus $\gamma = i \circ \bar{i}$ and we can write $\lambda \in \mathrm{End}(N)$ as $\lambda = \gamma|_N = \bar{i} \circ i$. The canonical endomorphism for the subfactor $M \subset M_1$ will be denoted by γ_1, so $\gamma_1 = j_M \circ j_{M_1}$. Because $J_N J_M = J_M J_{M_1}$, we have $\gamma_1|_M = \gamma$.

Definition 3.1.24 The set of all faithful normal conditional expectations from M onto N is denoted by $C(M, N)$. If $\varepsilon \in C(M, N)$, then its *index* $\mathrm{Ind}(\varepsilon) \in [1, \infty]$ is defined by

$$\mathrm{Ind}(\varepsilon)^{-1} := \sup\{\lambda \geq 0 : \varepsilon(m) \geq \lambda m \text{ for all } m \in M_+\}, \tag{3.1.10}$$

where we set $\mathrm{Ind}(\varepsilon) = \infty$ if the right-hand side is 0.

Suppose from now on that there exists an $\varepsilon \in C(M, N)$ and that we have chosen such a fixed ε. Then $\omega := \varphi \circ \varepsilon$ is a faithful normal state on M, so we can choose a unit vector $\Omega \in H$ that represents it in the sense that $\omega(m) = \langle m\Omega, \Omega\rangle$ for all $m \in M$. The projection e_N onto the closure of $N\Omega$ lies in N' and is called the *Jones projection* with respect to $\omega = \varphi \circ \varepsilon$ and we have $\Omega \in e_N H$. The von Neumann algebra generated by M and the projection e_N coincides with M_1 defined above. Because for any two $n_1, n_2 \in N$ we have

$$\langle n_1\Omega, n_2\Omega\rangle = \omega(n_2^* n_1) = \varphi(\varepsilon(n_2^* n_1)) = \varphi(n_2^* n_1) = \langle n_1\Phi, n_2\Phi\rangle$$

and because Φ is separating for N, the assignment $n\Phi \mapsto n\Omega$ defines an isometry $v' \in N'$ with $v'H = e_N H$. This isometry satisfies $J_M v' = v' J_N$ and $v'v'^* = e_N$. Using v', we introduce a second isometry $v_1 := j_M(v')$ which lies in M_1 (by definition of M_1) and satisfies

$$v_1 v_1^* = J_M v' J_M J_M v'^* J_M = J_M v' v'^* J_M = j_M(e_N) = e_N$$
$$v'^* v_1 = v'^* J_M v' J_M = v'^* v' J_N J_M = J_N J_M$$
$$v_1 J_{M_1} = J_M v' J_M J_{M_1} = J_M v' J_N J_M = J_M v' v'^* v_1 = J_M v_1 v_1^* v_1 = J_M v_1.$$

As a consequence of the second equation we get $\gamma = \mathrm{Ad}(v'^* v_1)$ and as a consequence of the third equality we have for all $m_1 \in M_1$ and $n \in N$ that

$$v_1 m_1 = \gamma_1(m_1) v_1$$
$$wn = \lambda(n) w,$$

where we have defined $w := \gamma_1(v_1)$. Note that these equations mean that $v_1 \in (\iota_{M_1}, \gamma_1)$ in the category $\mathrm{End}(M_1)$ and that $w \in (\iota_N, \lambda)$ in the category $\mathrm{End}(N)$. A useful fact is that the isometry $w \in N$ induces the conditional expectation according to $\varepsilon(m) = w^* \gamma(m) w$ for any $m \in M$. The content of the following lemma can be found in Subsection 2.7 of [65].

Lemma 3.1.25 *The following two statements are equivalent:*

(1) the index $\mathrm{Ind}(\varepsilon)$ *is finite;*

(2) there exists $\nu \in \mathbb{R}_{>0}$ *together with an isometry* $v \in M$ *with* $vm = \gamma(m)v$ *for all* $m \in M$ *that satisfies* $w^* v = \nu^{-1/2} 1 = w^* \gamma(v)$.

In case one (and hence both) of these statements holds, we have $\nu = \mathrm{Ind}(\varepsilon)$.

The isometry $v \in M$ in part (2) has the property that for any $m \in M$ we have

$$m = \mathrm{Ind}(\varepsilon) \cdot \varepsilon(mv^*)v = \mathrm{Ind}(\varepsilon) \cdot v^*\varepsilon(vm). \tag{3.1.11}$$

This equality means that we can write each element $m \in M$ of the form $m = nv$ and in fact this n is unique. This can in turn be used to obtain the formula

$$\gamma(m) = \mathrm{Ind}(\varepsilon) \cdot \varepsilon(vmv^*) \tag{3.1.12}$$

for any $m \in M$. These facts can be restated in more categorical terms as follows. If $\varepsilon \in C(M,N)$, then there always exists a $w \in (\iota_N, \bar{i} \circ i)$ in the category $\mathrm{End}(N)$. The index of ε is finite if and only if there also exists a $v \in (\iota_M, i \circ \bar{i})$ in the category $\mathrm{End}(M)$ such that

$$[1_i \otimes w^*] \circ [v \otimes 1_i] = \nu^{-1/2} 1_i$$
$$[w^* \otimes 1_{\bar{i}}] \circ [1_{\bar{i}} \otimes v] = \nu^{-1/2} 1_{\bar{i}}$$

in which case we have $\nu = \mathrm{Ind}(\varepsilon)$. These two equations imply that the 1-cell $\bar{i}$ can be given the structure of a conjugate $(\bar{i}, r, \bar{r})$ for the 1-cell i in the C^*-2-category determined by N and M if we choose $r = \sqrt{\kappa_2}w$ and $\bar{r} = \sqrt{\kappa_1}v$ with $\kappa_1, \kappa_2 \in \mathbb{R}_{>0}$ satisfying $\kappa_1\kappa_2 = \nu$. By using the procedure described after Definition 2.9.12, the canonical endomorphism γ becomes a special $*$-Frobenius algebra in $\mathrm{End}(M)$ with unit and comultiplication

$$\eta_\gamma = \sqrt{\kappa_1}v : \iota_M \to \gamma$$
$$\Delta_\gamma = \sqrt{\kappa_2}w : \gamma \to \gamma \otimes \gamma$$

and the dual canonical endomorphism λ is a special $*$-Frobenius algebra in $\mathrm{End}(N)$ with unit and comultiplication

$$\eta_\lambda = \sqrt{\kappa_2}w : \iota_N \to \lambda$$
$$\Delta_\lambda = \sqrt{\kappa_1}\gamma(v) : \lambda \to \lambda \otimes \lambda.$$

The subfactor $N \subset M$ can be obtained from the Frobenius algebra γ by using that $N = \varepsilon(M) = w^*\gamma(M)w$. It is also possible to reconstruct the subfactor $N \subset M$ from this Frobenius algebra λ as shown in [65] (after Corollary 4.8) and [8] (Theorem 3.11).

We now turn to the important notion of the index of a subfactor.

Definition 3.1.26 If there exists an $\varepsilon \in C(M,N)$ with $\mathrm{Ind}(\varepsilon) < \infty$, then the *index* $[M : N]$ of N in M is defined as

$$[M : N] = \inf\{\mathrm{Ind}(\varepsilon) : \varepsilon \in C(M,N)\}.$$

If such ε does not exist, we set $[M : N] = \infty$.

If $[M : N] < \infty$ then there exists a unique $\varepsilon_0 \in C(M,N)$ with $\mathrm{Ind}(\varepsilon_0) = [M : N]$, called the *minimal conditional expectation.*

We end this subsection with an important application. Let M be a type III factor and let $(\bar{\rho}, r, \bar{r})$ be a conjugate for $\rho \in \mathrm{End}(M)$. If L denotes the corresponding categorical left inverse, then for any $a \in (\rho \otimes \sigma, \rho \otimes \tau)$ we have $\mathsf{L}_{\sigma,\tau}(a) = r^*\bar{\rho}(a)r$. If for each $m \in M$ we now define

$$\varphi(m) := (r^*r)^{-1}r^*\bar{\rho}(m)r,$$

it is easy to see that φ defines a(n operator algebraic) left inverse for ρ and hence $\varepsilon := \rho \circ \varphi$ is a conditional expectation from M onto $\rho(M)$. The corresponding categorical left inverse L' that is obtained by letting $\mathsf{L}'_{\sigma,\tau}$

be the restriction of φ to $(\rho\otimes\sigma, \rho\otimes\tau)$ is related to L by $\mathsf{L}'_{\sigma,\tau} = (r^*r)^{-1}\mathsf{L}_{\sigma,\tau}$. Note that $d(\mathsf{L}') = d(\mathsf{L}) = \|r\|\cdot\|\overline{r}\|$ because the dimension is invariant under scaling of categorical left inverses. Now if $\tau, \sigma \in \mathrm{End}(M)$ and $a \in (\rho\otimes\sigma, \rho\otimes\tau)$, then according to Lemma 3.1.11 we have the inequality

$$a^*a \leq \overline{r}^*\overline{r}\rho(\mathsf{L}_{\sigma,\sigma}(a^*a)) = \underbrace{\overline{r}^*\overline{r}r^*r}_{=\|\overline{r}\|^2\|r\|^2} \rho(\mathsf{L}'_{\sigma,\sigma}(a^*a)) = d(\mathsf{L})^2\rho(\mathsf{L}'_{\sigma,\sigma}(a^*a)) = d(\mathsf{L})^2\varepsilon(a^*a)$$

or

$$\varepsilon(a^*a) \geq d(\mathsf{L})^{-2}a^*a$$

and this is the best bound possible. In fact this inequality holds if we replace a^*a by any $m \in M_+$, so $\mathrm{Ind}(\varepsilon) = d(\mathsf{L})^2$. By Theorem 3.11 of [65] the value of $d(\mathsf{L})$ is minimal if L is determined by a standard conjugate of ρ, i.e. $\mathrm{Ind}(\varepsilon) \geq d(\rho)^2$ for any ε that comes from a left inverse for ρ as above[10], and equality holds when ε comes from the standard left inverse. We now want to conclude that $[M : \rho(M)] = d(\rho)^2$, but for this we need to show that the minimal conditional expectation comes from a left inverse. If $\varepsilon' \in C(M, \rho(M))$ is arbitrary, then $\phi := \rho^{-1}\circ\varepsilon'$ is a left inverse for ρ, where ρ^{-1} denotes the inverse of the $*$-homomorphism $\rho : M \to \rho(M)$ (note that ρ is injective). But then $\varepsilon' = \rho\circ\phi$, so ε' comes from a left inverse. In particular this is true for the minimal conditional expectation ε_0, so we conclude that

$$[M : \rho(M)] = \mathrm{Ind}(\varepsilon_0) = d(\rho)^2.$$

As a corollary, we find that if $(\overline{\rho}, r_0, \overline{r}_0)$ is a standard conjugate for ρ, then the minimal conditional expectation ε_0 is given by $\varepsilon_0(m) = (r_0^*r_0)^{-1}\rho(r_0^*\overline{\rho}(m)r_0)$.

3.2 Nets of von Neumann algebras

Now that we have discussed all relevant results from category theory and operator algebras, we can consider AQFT. We begin by stating some elementary definitions, which can be found in [78].

We will write $\mathcal{K}$ to denote the set of all bounded open intervals in $\mathbb{R}$, so $\mathcal{K}$ is the set of all subsets $I \subset \mathbb{R}$ of the form $I = (a, b)$ with $a, b \in \mathbb{R}$. If $I = (a, b), J = (c, d) \in \mathcal{K}$ then the notation $I < J$ (also written $J > I$) will mean that $b \leq c$, i.e. that I and J are disjoint and that I lies to the left of J. We will say that two intervals $I, J \in \mathcal{K}$ are *adjacent*, if $\overline{I}\cap\overline{J} = \{p\}$ for some $p \in \mathbb{R}$. Finally, for any open subset $O \subset \mathbb{R}$ we will use the notation $O^\perp$ to denote the interior of the complement of O, so $O^\perp = \mathbb{R}\backslash\overline{O}$. Note that if $I, J \in \mathcal{K}$ are adjacent with $I < J$, then we can write $I = (a, b)$ and $J = (b, c)$ for certain $a, b, c \in \mathbb{R}$ and we have $(I \cup J)^{\perp\perp} = (a, c) = I \cup \{b\} \cup J$. Unbounded intervals of the form $(-\infty, a)$ or (b, ∞) with $a, b \in \mathbb{R}$ are called left half-lines and right half-lines, respectively. If I is a left/right half-line, then $I^\perp$ is clearly a right/left half line.

Definition 3.2.1 A *net of von Neumann algebras* on $\mathbb{R}$ is a pair $(H, \mathcal{A})$ consisting of a separable Hilbert space H and an assignment

$$\mathcal{K} \ni I \mapsto \mathcal{A}(I) \subset B(H)$$

where each $\mathcal{A}(I)$ is a von Neumann algebra, satisfying the property that $\mathcal{A}(I) \subset \mathcal{A}(J)$ whenever $I \subset J$; this property of $\mathcal{A}$ is called *isotony*.

If $(H, \mathcal{A})$ is a net of von Neumann algebras, then for each open subset $O \subset \mathbb{R}$ we define

$$\mathcal{A}(O) := \bigvee_{I\in\mathcal{K}, I\subset O} \mathcal{A}(I).$$

[10]Note that it is not necessary to demand that this left inverse is determined by a conjugate, since this is always the case.

If O happens to be in $\mathcal{K}$, then $\mathcal{A}(O)$ was already defined by the assignment $I \mapsto \mathcal{A}(I)$ itself, but by isotony the two different definitions of $\mathcal{A}(O)$ will then coincide. This shows that the notation is consistent.

We will also introduce the $*$-algebra $\mathcal{A}_\infty := \bigcup_{I\in\mathcal{K}} \mathcal{A}(I)$. With some abuse of notation, we will denote the norm closure of $\mathcal{A}_\infty$ by $\mathcal{A}$, so

$$\mathcal{A} := \overline{\mathcal{A}_\infty}^{\|\cdot\|} = \overline{\bigcup_{I\in\mathcal{K}} \mathcal{A}(I)}^{\|\cdot\|}.$$

The C^*-algebra $\mathcal{A}$ is called the *quasi-local algebra* corresponding to the net $(H, \mathcal{A})$.

Definition 3.2.2 A net $(H, \mathcal{A})$ of von Neumann algebras on $\mathbb{R}$ is said to satisfy:

- *irreducibility* if $\bigvee_{I\in\mathcal{K}} \mathcal{A}(I) = B(H)$;
- *locality* if $\mathcal{A}(I) \subset \mathcal{A}(J)'$ whenever $I, J \in \mathcal{K}$ and $I \subset J^\perp$;
- *half-line duality* if $\mathcal{A}(I^\perp)' = \mathcal{A}(I)$ for all half-lines $I \subset \mathbb{R}$;
- *Haag duality* if $\mathcal{A}(I^\perp)' = \mathcal{A}(I)$ for all $I \in \mathcal{K}$;
- *additivity* if for any open set $O \subset \mathbb{R}$ and for any collection $\{I_\alpha\}_\alpha$ with $I_\alpha \in \mathcal{K}$ for all α and $O = \bigcup_\alpha I_\alpha$ we have that $\mathcal{A}(O) = \bigvee_\alpha \mathcal{A}(I_\alpha)$;
- *strong additivity* if it satisfies additivity and if for any two adjacent $I, J \in \mathcal{K}$ we have $\mathcal{A}((I \cup J)^{\perp\perp}) = \mathcal{A}(I) \vee \mathcal{A}(J)$;
- the *split property* if for any $I, J \in \mathcal{K}$ with $\overline{I} \cap \overline{J} = \emptyset$ the map $\mathcal{A}(I) \otimes_{\mathrm{alg}} \mathcal{A}(J) \to \mathcal{A}(I) \vee \mathcal{A}(J)$, $x \otimes y \mapsto xy$ extends to an isomorphism of von Neumann algebras.

Definition 3.2.3 Let $(H, \mathcal{A})$ be a net of von Neumann algebras and let G be a group. Then a *G-action* on $(H, \mathcal{A})$ is a G-action $\beta : G \to \mathrm{Aut}(\mathcal{A})$ on the quasi-local algebra $\mathcal{A}$ satisfying the following two properties:

- $\beta_q(\mathcal{A}(I)) = \mathcal{A}(I)$ for all $q \in G$ and $I \in \mathcal{K}$;
- if for some $I \in \mathcal{K}$ the restriction of β_q to $\mathcal{A}(I)$ is the identity map on $\mathcal{A}(I)$, then $q = e$.

A *G-net of von Neumann algebras* is a triple $(H, \mathcal{A}, \beta)$, where $(H, \mathcal{A})$ is a net of von Neumann algebras and β is a G-action on it.

As mentioned before, a G-action β on a C^*-algebra induces a G-action on its tensor category of endomorphisms. This means that we have a G-action on the tensor category $\mathrm{End}(\mathcal{A})$ of endomorphisms of the quasi-local algebra of a G-net of von Neumann algebras.

3.2.1 G-localized endomorphisms

In [78] an endomorphism $\rho \in \mathrm{End}(\mathcal{A})$ of the quasi-local algebra of a G-net $(H, \mathcal{A}, \beta)$ of von Neumann algebras was called q-localized (where $q \in G$) in some interval $I \in \mathcal{K}$ if $\rho(A) = A$ for all $A \in \mathcal{A}(J)$ with $J < I$ and if $\rho(A) = \beta_q(A)$ for all $A \in \mathcal{A}(J)$ with $J > I$. Here we will introduce a more symmetrical definition, because this will be needed later.

Definition 3.2.4 Let $(H, \mathcal{A}, \beta)$ be a G-net of von Neumann algebras on $\mathbb{R}$.

(1) If $q, r \in G$ and $I \in \mathcal{K}$, then $\rho \in \mathrm{End}(\mathcal{A})$ is called *(q, r)-localized in I* if
  - for any $J < I$ and $A \in \mathcal{A}(J)$ we have $\rho(A) = \beta_q(A)$;
  - for any $J > I$ and $A \in \mathcal{A}(J)$ we have $\rho(A) = \beta_r(A)$.

  An endomorphism $\rho \in \mathrm{End}(\mathcal{A})$ is called *(q, r)-localized* if it is (q, r)-localized in some $I \in \mathcal{K}$.

(2) If ρ is (q, r)-localized, then ρ is called *transportable* if for any $J \in \mathcal{K}$ there exists an endomorphism ρ' that is (q, r)-localized in J together with a unitary $W \in (\rho, \rho')$. We will denote the set of all transportable (q, r)-localized endomorphisms by $G - \mathrm{Loc}(\mathcal{A})_{q,r}$.

The unitary operator W in the definition of transportability of ρ is often called a *charge transporter* for ρ in the literature on AQFT; we will also use this term occasionally. If $\rho \in G - \mathrm{Loc}(\mathcal{A})_{q,r}$ is (q, r)-localized in $I \in \mathcal{K}$, then we will write this simply as $\rho \in G - \mathrm{Loc}(\mathcal{A})_{q,r}(I)$.

For any $q, r \in G$, the full subcategory of $\mathrm{End}(\mathcal{A})$ determined by the endomorphisms in $\rho \in G-\mathrm{Loc}(\mathcal{A})_{q,r}$ will be denoted by $G-\mathrm{Loc}(\mathcal{A})_{q,r}$. We then define the category[11]

$$G-\mathrm{Loc}(\mathcal{A}) := \bigsqcup_{q,r\in G} G-\mathrm{Loc}(\mathcal{A})_{q,r}.$$

If $\rho \in G-\mathrm{Loc}(\mathcal{A})_{q,r}$ and $\rho' \in G-\mathrm{Loc}(\mathcal{A})_{q',r'}$, then their tensor product $\rho \otimes \rho'$ in $\mathrm{End}(\mathcal{A})$ is easily seen to be in $G-\mathrm{Loc}(\mathcal{A})_{qq',rr'}$, which implies that $G-\mathrm{Loc}(\mathcal{A})$ is a $G \times G$-graded tensor subcategory of $\mathrm{End}(\mathcal{A})$. Furthermore, the G-action on $\mathrm{End}(\mathcal{A})$ restricts to a G-action on $G-\mathrm{Loc}(\mathcal{A})$ and it satisfies $\beta_q(G-\mathrm{Loc}(\mathcal{A})_{r,s}) \subset G-\mathrm{Loc}(\mathcal{A})_{qrq^{-1},qsq^{-1}}$. As already announced after Definition 2.8.14, this $G-\mathrm{Loc}(\mathcal{A})$ gives a slightly more general notion of a G-crossed category, where there is a G-action, a $G\times G$-grading and an action of G on $G \times G$ given by $\alpha_q(r,s) = (qrq^{-1}, qsq^{-1})$. The following lemma is a slight generalization of (a part of) Lemmas 2.12 and 2.13 in [78].

Lemma 3.2.5 *Let $(H, \mathcal{A}, \beta)$ be a G-net of von Neumann algebras satisfying Haag duality.*
(1) If ρ is (q,r)-localized in $I \in \mathcal{K}$, then $\rho(\mathcal{A}(I)) \subset \mathcal{A}(I)$.
(2) If ρ and σ are both (q,r)-localized in $I \in \mathcal{K}$, then $(\rho,\sigma) \subset \mathcal{A}(I)$.

Proof. (1) If $J \in \mathcal{K}$ with $J \subset I^\perp$, then the restriction of ρ to $\mathcal{A}(J)$ is either β_q or β_r, so we have $\rho(\mathcal{A}(J)) = \mathcal{A}(J)$. This implies that $\rho(\mathcal{A}(I^\perp)) = \mathcal{A}(I^\perp)$. Thus we have

$$\rho(\mathcal{A}(I)) \subset \mathcal{A}(I^\perp)' = \mathcal{A}(I),$$

where the inclusion follows from the fact that $\rho(\mathcal{A}(I))$ commutes with $\rho(\mathcal{A}(I^\perp)) = \mathcal{A}(I^\perp)$ by locality.
(2) Suppose that $R \in (\rho,\sigma)$. Let $J \in \mathcal{K}$ with $J < I$. If $A \in \mathcal{A}(J)$, then

$$R\beta_q(A) = R\rho(A) = \sigma(A)R = \beta_q(A)R,$$

so $R \in \mathcal{A}(J)'$. With the same reasoning, the same holds when $J > I$. Hence we conclude that $R \in \mathcal{A}(I^\perp)' = \mathcal{A}(I)$.
□

Later we will consider two subcategories of $G-\mathrm{Loc}(\mathcal{A})$ that are G-crossed, see also Definition 3.2.8 below, and in order to define a braiding on these two G-crossed categories we will need the following lemma, which is a generalization of Lemma 2.14 in [78].

Lemma 3.2.6 *Assume that $(H, \mathcal{A}, \beta)$ is a G-net of von Neumann algebras satisfying Haag duality and strong additivity. Let $I, J \in \mathcal{K}$ with $I < J$, let ρ be (q,r)-localized in I and let ρ' be (q',r')-localized in J. If $rq' = q'r$, then*

$$\rho \otimes \rho' = \beta_r(\rho') \otimes \beta_{q'^{-1}}(\rho) \tag{3.2.1}$$
$$\rho' \otimes \rho = \beta_{q'}(\rho) \otimes \beta_{r^{-1}}(\rho'). \tag{3.2.2}$$

Proof. Write $I = (a,b)$ and $J = (c,d)$. Choose $m < a$ and $M > d$ and write $K := (m,c)$ and $L := (b,M)$. It is enough to show that the equations hold on both $\mathcal{A}(K)$ and $\mathcal{A}(L)$, which is what we will do now.
(1) If $A \in \mathcal{A}(K)$, then

$$\begin{aligned}(\rho \otimes \rho')(A) = \rho(\rho'(A)) = \rho(\beta_{q'}(A)) = (\beta_{q'} \circ \beta_{q'^{-1}} \circ \rho \circ \beta_{\rho'})(A) \\ = \beta_{q'}[\beta_{q'^{-1}}(\rho)(A)] = (\beta_r \circ \beta_{q'} \circ \beta_{r^{-1}})[\beta_{q'^{-1}}(\rho)(A)]\end{aligned}$$

[11]Note that this is different from the category $G-\mathrm{Loc}(\mathcal{A})$ as defined in [78]. The $G-\mathrm{Loc}(\mathcal{A})$ in [78] is equal to $(\bigsqcup_{q\in G} G-\mathrm{Loc}(\mathcal{A})_{e,q})^\oplus$.

$$= (\beta_r \circ \rho' \circ \beta_{r^{-1}})(\beta_{q'^{-1}}(\rho)(A)) = (\beta_r(\rho') \otimes \beta_{q'^{-1}}(\rho))(A)$$

and if $A \in \mathcal{A}(L)$, then

$$\begin{aligned}(\rho \otimes \rho')(A) &= \rho(\rho'(A)) = \beta_r(\rho'(A)) = (\beta_r \circ \rho' \circ \beta_{r^{-1}} \circ \beta_r)(A) \\ &= \beta_r(\rho')[\beta_r(A)] = \beta_r(\rho')[(\beta_{q'^{-1}} \circ \beta_r \circ \beta_{q'})(A)] \\ &= \beta_r(\rho')[(\beta_{q'^{-1}} \circ \rho \circ \beta_{q'})(A)] = \beta_r(\rho')[\beta_{q'^{-1}}(\rho)(A)] \\ &= (\beta_r(\rho') \otimes \beta_{q'^{-1}}(\rho))(A).\end{aligned}$$

(2) If $A \in \mathcal{A}(K)$, then

$$\begin{aligned}(\rho' \otimes \rho)(A) &= \rho'(\rho(A)) = \beta_{q'}(\rho(A)) = (\beta_{q'} \circ \rho \circ \beta_{q'^{-1}} \circ \beta_{q'})(A) \\ &= \beta_{q'}(\rho)[\beta_{q'}(A)] = \beta_{q'}(\rho)[(\beta_{r^{-1}} \circ \beta_{q'} \circ \beta_r)(A)] \\ &= \beta_{q'}(\rho)[(\beta_{r^{-1}} \circ \rho' \circ \beta_r)(A)] = \beta_{q'}(\rho)[\beta_{r^{-1}}(\rho')(A)] \\ &= (\beta_{q'}(\rho) \otimes \beta_{r^{-1}}(\rho'))(A)\end{aligned}$$

and if $A \in \mathcal{A}(L)$, then

$$\begin{aligned}(\rho' \otimes \rho)(A) &= \rho'(\rho(A)) = \rho'(\beta_r(A)) = (\beta_r \circ \beta_{r^{-1}} \circ \rho' \circ \beta_r)(A) \\ &= \beta_r[\beta_{r^{-1}}(\rho')(A)] = (\beta_{q'} \circ \beta_r \circ \beta_{q'^{-1}})[\beta_{r^{-1}}(\rho)(A)] \\ &= (\beta_{q'} \circ \rho \circ \beta_{q'^{-1}})[\beta_{r^{-1}}(\rho')(A)] = (\beta_{q'}(\rho) \otimes \beta_{r^{-1}}(\rho'))(A).\end{aligned}$$

□

Definition 3.2.7 Let $(H, \mathcal{A}, \beta)$ be a G-net of von Neumann algebras satisfying Haag duality and strong additivity. Suppose that $\rho_1 \in G - \mathrm{Loc}(\mathcal{A})_{q_1,r_1}$, $\rho_2 \in G - \mathrm{Loc}(\mathcal{A})_{q_2,r_2}$ and let $I, J \in \mathcal{K}$ with $I < J$.

(1) Let $\widetilde{\rho}_1 \in G - \mathrm{Loc}(\mathcal{A})_{q_1,r_1}(I)$ and $\widetilde{\rho}_2 \in G - \mathrm{Loc}(\mathcal{A})_{q_2,r_2}(J)$, together with unitaries $U_j \in (\rho_j, \widetilde{\rho}_j)$ for $j \in \{1, 2\}$. If $r_1 q_2 = q_2 r_1$, then we define the isomorphism $C^l_{\rho_1,\rho_2} : \rho_1 \otimes \rho_2 \to \beta_{r_1}(\rho_2) \otimes \beta_{q_2^{-1}}(\rho_1)$ to be the composition

$$\rho_1 \otimes \rho_2 \xrightarrow{U_1 \times U_2} \widetilde{\rho}_1 \otimes \widetilde{\rho}_2 = \beta_{r_1}(\widetilde{\rho}_2) \otimes \beta_{q_2^{-1}}(\widetilde{\rho}_1) \xrightarrow{\beta_{r_1}(U_2)^* \times \beta_{q_2^{-1}}(U_1)^*} \beta_{r_1}(\rho_2) \otimes \beta_{q_2^{-1}}(\rho_1),$$

i.e. $C^l_{\rho_1,\rho_2} = [\beta_{r_1}(U_2)^* \times \beta_{q_2^{-1}}(U_1)^*] \circ [U_1 \times U_2]$.

(2) Let $\widehat{\rho}_1 \in G - \mathrm{Loc}(\mathcal{A})_{q_1,r_1}(J)$ and $\widehat{\rho}_2 \in G - \mathrm{Loc}(\mathcal{A})_{q_2,r_2}(I)$, together with unitaries $V_j \in (\rho_j, \widehat{\rho}_j)$ for $j \in \{1, 2\}$. If $q_1 r_2 = r_2 q_1$, then we define the isomorphism $C^r_{\rho_1,\rho_2} : \rho_1 \otimes \rho_2 \to \beta_{q_1}(\rho_2) \otimes \beta_{r_2^{-1}}(\rho_1)$ to be the composition

$$\rho_1 \otimes \rho_2 \xrightarrow{V_1 \times V_2} \widehat{\rho}_1 \otimes \widehat{\rho}_2 = \beta_{q_1}(\widehat{\rho}_2) \otimes \beta_{r_2^{-1}}(\widehat{\rho}_1) \xrightarrow{\beta_{q_1}(V_2)^* \times \beta_{r_2^{-1}}(V_1)^*} \beta_{q_1}(\rho_2) \otimes \beta_{r_2^{-1}}(\rho_1),$$

i.e. $C^r_{\rho_1,\rho_2} = [\beta_{q_1}(V_2)^* \times \beta_{r_2^{-1}}(V_1)^*] \circ [V_1 \times V_2]$.

Actually, we should have written these isomorphisms as $C^l_{\rho_1,\rho_2}(I, J, \widetilde{\rho}_j, U_j)$ and $C^r_{\rho_1,\rho_2}(I, J, \widehat{\rho}_j, V_j)$, showing that there were some choices involved. However, by using some standard arguments it can be shown that these isomorphisms do not depend on these particular choices. This justifies our notation.

Definition 3.2.8 If $(H, \mathcal{A}, \beta)$ is a G-net of von Neumann algebras, then for each $q \in G$ we define the categories $G - \mathrm{Loc}^L(\mathcal{A})_q := G - \mathrm{Loc}(\mathcal{A})_{q,e}$ and $G - \mathrm{Loc}^R(\mathcal{A})_q := G - \mathrm{Loc}(\mathcal{A})_{e,q}$ and

$$G - \mathrm{Loc}^L(\mathcal{A}) := \coprod_{q \in G} G - \mathrm{Loc}^L(\mathcal{A})_q$$

$$G - \mathrm{Loc}^R(\mathcal{A}) := \bigsqcup_{q \in G} G - \mathrm{Loc}^R(\mathcal{A})_q.$$

We also write $\mathrm{Loc}(\mathcal{A})$ to denote $G - \mathrm{Loc}(\mathcal{A})_{e,e} = G - \mathrm{Loc}^L(\mathcal{A})_e = G - \mathrm{Loc}^R(\mathcal{A})_e$. The objects in $\mathrm{Loc}(\mathcal{A})$ are also called *DHR endomorphisms*[12].

The categories $G - \mathrm{Loc}^L(\mathcal{A})$ and $G - \mathrm{Loc}^R(\mathcal{A})$ are obviously G-crossed. We will now use the expressions C^l and C^r to define braidings on these two G-crossed categories.

Theorem 3.2.9 *Let $(H, \mathcal{A})$ be a G-net of von Neumann algebras satisfying Haag duality and strong additivity. Then $G - \mathrm{Loc}^L(\mathcal{A})$ and $G - \mathrm{Loc}^R(\mathcal{A})$ are G-crossed categories. Furthermore, both categories can be equipped with a braiding in the following way.*
(1) On $G - \mathrm{Loc}^L(\mathcal{A})$ we can define a braiding $c^{L,r}$ and a braiding of the second kind $c^{L,l}$ by

$$c^{L,r}_{\rho,\sigma} := C^r_{\rho,\sigma}$$
$$c^{L,l}_{\rho,\sigma} := C^l_{\rho,\sigma}$$

(2) On $G - \mathrm{Loc}^R(\mathcal{A})$ we can define a braiding $c^{R,l}$ and a braiding of the second kind $c^{R,r}$ by

$$c^{R,l}_{\rho,\sigma} = C^l_{\rho,\sigma}$$
$$c^{R,r}_{\rho,\sigma} = C^r_{\rho,\sigma}.$$

We will write $G-\mathrm{Loc}^{(L,r)}(\mathcal{A})$ to denote the category $G-\mathrm{Loc}^L(\mathcal{A})$ equipped with the braiding $c^{L,r}$. Likewise we also introduce the notation $G - \mathrm{Loc}^{(L,l)}(\mathcal{A})$, $G - \mathrm{Loc}^{(R,l)}(\mathcal{A})$ and $G - \mathrm{Loc}^{(R,r)}(\mathcal{A})$. Note that $G - \mathrm{Loc}^{(L,r)}(\mathcal{A})_e$ and $G - \mathrm{Loc}^{(R,r)}(\mathcal{A})_e$ are both equal to $\mathrm{Loc}(\mathcal{A})$ equipped with the same braiding, which we will denote by c^r. Similarly, $G - \mathrm{Loc}^{(L,l)}(\mathcal{A})_e$ and $G - \mathrm{Loc}^{(R,l)}(\mathcal{A})_e$ are both equal to $\mathrm{Loc}(\mathcal{A})$ equipped with the same braiding, which will be denoted by c^l. These two braidings on $\mathrm{Loc}(\mathcal{A})$ are related by $c^l_{\rho,\sigma} = (c^r_{\sigma,\rho})^{-1}$. We will often write $\mathrm{Loc}^{l/r}(\mathcal{A})$ if we want to emphasize which of these two braidings on $\mathrm{Loc}(\mathcal{A})$ we are considering. It thus follows that $G-\mathrm{Loc}^{(L,r)}(\mathcal{A})$ and $G-\mathrm{Loc}^{(R,r)}(\mathcal{A})$ are braided G-crossed extensions of $\mathrm{Loc}^r(\mathcal{A})$ and that $G - \mathrm{Loc}^{(L,l)}(\mathcal{A})$ and $G - \mathrm{Loc}^{(R,l)}(\mathcal{A})$ are braided G-crossed extensions of $\mathrm{Loc}^l(\mathcal{A})$.

3.2.2 The categorical relation between $G - \mathrm{Loc}^{(L,r)}(\mathcal{A})$ and $G - \mathrm{Loc}^{(R,l)}(\mathcal{A})$

In this subsection we will show that the braided G-crossed structures of the categories $G - \mathrm{Loc}^{(L,r)}(\mathcal{A})$ and $G - \mathrm{Loc}^{(R,l)}(\mathcal{A})$ can be transported to one another. When comparing the two structures on the same category, we find that they are the mirror image of one another. In fact, the results in this subsection motivated us to define the notion of a mirror image of a braided G-crossed category, as we did in Subsection 2.8.5.

Let $(H, \mathcal{A}, \beta)$ be a G-net of von Neumann algebras satisfying Haag duality and strong additivity. If $q \in G$ and $\rho \in G - \mathrm{Loc}^L(\mathcal{A})_q$, then it is clear that $\mathscr{S}_q(\rho) := \rho \circ \beta_{q^{-1}} \in G - \mathrm{Loc}^R(\mathcal{A})_{q^{-1}}$, and the same is true if we interchange L and R. Similarly, if $\rho, \sigma \in G - \mathrm{Loc}^L(\mathcal{A})_q$ and $S \in (\rho, \sigma)$, then $\mathscr{S}_q(S) := S \in (\mathscr{S}_q(\rho), \mathscr{S}_q(\sigma))$, and again the same is true if we interchange L and R. This obviously defines functors $\mathscr{S}_q$ between $G - \mathrm{Loc}^L(\mathcal{A})_q$ and $G - \mathrm{Loc}^R(\mathcal{A})_{q^{-1}}$ for each $q \in G$, and together all $\{\mathscr{S}_q\}_{q \in G}$ constitute a functor $\mathscr{S}$ between $G - \mathrm{Loc}^L(\mathcal{A})$ and $G - \mathrm{Loc}^R(\mathcal{A})$. We can use these functors $\mathscr{S}$ to transport the G-crossed structure from one of these two categories to the other. We will only show how to transport the G-crossed structure of $G - \mathrm{Loc}^L(\mathcal{A})$ to $G - \mathrm{Loc}^R(\mathcal{A})$; the opposite direction goes similarly.

[12]Here DHR stands for Doplicher-Haag-Roberts. These endomorphisms were first studied in the series of papers [18], [19], [20] and [21].

We start with the G-grading. It is clear that if $\rho \in G - \mathrm{Loc}^R(\mathcal{A})_q$, then the degree of $\mathscr{S}_q(\rho)$ is q^{-1}. Hence if we transport the degree ∂ of $G - \mathrm{Loc}^L(\mathcal{A})$ to $G - \mathrm{Loc}^R(\mathcal{A})$ we obtain a new degree $\partial_\bullet$ on $G - \mathrm{Loc}^R(\mathcal{A})$ by defining

$$\partial_\bullet(\rho) := \partial(\rho)^{-1}. \tag{3.2.3}$$

We will now consider the tensor product. Note that both the tensor product and group action on the categories $G - \mathrm{Loc}^L(\mathcal{A})$ and $G - \mathrm{Loc}^R(\mathcal{A})$ are restrictions of the tensor product and group action on $\mathrm{End}(\mathcal{A})$, so there is no need to label them by $\otimes_{L/R}$ or $\beta^{L/R}$. If $\rho \in G - \mathrm{Loc}^R(\mathcal{A})_q$ and $\sigma \in G - \mathrm{Loc}^R(\mathcal{A})_r$, then their images in $G - \mathrm{Loc}^L(\mathcal{A})$ under the functor $\mathscr{S}$ are $\rho \circ \beta_{q^{-1}}$ and $\sigma \circ \beta_{r^{-1}}$. We then take their tensor product in $G - \mathrm{Loc}^L(\mathcal{A})$ to obtain

$$\begin{aligned}\rho \circ \beta_{q^{-1}} \circ \sigma \circ \beta_{r^{-1}} &= \rho \circ \beta_{q^{-1}} \circ \sigma \circ \beta_{r^{-1}} \circ \beta_{rq} \circ \beta_{(rq)^{-1}} = \rho \circ \beta_{q^{-1}} \circ \sigma \circ \beta_{q^{-1}} \circ \beta_{(rq)^{-1}} \\ &= \mathscr{S}_{rq}(\rho \otimes \beta_q(\sigma))\end{aligned}$$

and then we transport this result back to $G - \mathrm{Loc}^R(\mathcal{A})$ to obtain a new tensor product

$$\rho \bullet \sigma := \rho \otimes \beta_q(\sigma) \tag{3.2.4}$$

on the objects of $G - \mathrm{Loc}^R(\mathcal{A})$. If $S \in (\rho_1, \rho_2)$ and $T \in (\sigma_1, \sigma_2)$ with $\rho_j \in G - \mathrm{Loc}^R(\mathcal{A})_q$ and $\sigma_j \in G - \mathrm{Loc}^R(\mathcal{A})$, then

$$\mathscr{S}(S) \times \mathscr{S}(T) = S(\rho_1 \circ \beta_{q^{-1}})(T) = S\rho_1(\beta_{q^{-1}}(T)) = \mathscr{S}(S \times \beta_{q^{-1}}(T)),$$

so when we transport back to $G - \mathrm{Loc}^R(\mathcal{A})$ we obtain the new tensor product

$$S \bullet T := S \times \beta_{q^{-1}}(T) \tag{3.2.5}$$

on the morphisms of $G - \mathrm{Loc}^R(\mathcal{A})$. This is the tensor product of $G - \mathrm{Loc}^L(\mathcal{A})$ transported to $G - \mathrm{Loc}^R(\mathcal{A})$. If $q \in G$ and $\rho \in G - \mathrm{Loc}^R(\mathcal{A})_r$, then for the group action on the objects we find

$$\beta_q(\mathscr{S}_r(\rho)) = \beta_q \circ \rho \circ \beta_{r^{-1}} \circ \beta_{q^{-1}} = \beta_q \circ \rho \circ \beta_{q^{-1}} \circ \beta_{(qrq^{-1})^{-1}} = \mathscr{S}_{qrq^{-1}}(\beta_q(\rho)),$$

and if $q \in G$ and S is a morphism in $G - \mathrm{Loc}^R(\mathcal{A})_r$, then

$$\beta_q(\mathscr{S}_r(S)) = \beta_q(S) = \mathscr{S}_{qrq^{-1}}(\beta_q(S)),$$

so the group action of $G - \mathrm{Loc}^L(\mathcal{A})$ transported to $G - \mathrm{Loc}^R(\mathcal{A})$ is just the original group action on $G - \mathrm{Loc}^R(\mathcal{A})$. Finally, we will consider the braiding. Let $\rho \in G - \mathrm{Loc}^{(R,l)}(\mathcal{A})_q$ and $\sigma \in G - \mathrm{Loc}^{(R,l)}(\mathcal{A})_r$. Then the braiding of $\mathscr{S}(\rho)$ and $\mathscr{S}(\sigma)$ in $G - \mathrm{Loc}^{(L,r)}(\mathcal{A})$ is given by

$$\begin{aligned}c^{L,r}_{\mathscr{S}(\rho),\mathscr{S}(\sigma)} &= C^r_{\mathscr{S}(\rho),\mathscr{S}(\sigma)} = C^r_{\rho\circ\beta_{q^{-1}},\sigma\circ\beta_{r^{-1}}} = C^r_{\rho,\beta_{q^{-1}}(\sigma)} = c^{R,r}_{\rho,\beta_{q^{-1}}(\sigma)} \\ &= (c^{R,l}_{\beta_{q^{-1}}(\sigma),\beta_{q^{-1}r^{-1}q}(\rho)})^{-1}.\end{aligned}$$

We will explain the third step in some detail, because it is non-trivial. Choose $\widetilde{\rho} \in G - \mathrm{Loc}^R(\mathcal{A})_q$ localized in J and $\widetilde{\sigma} \in G - \mathrm{Loc}^R(\mathcal{A})_r$ localized in I with $I < J$, together with unitaries $V_1 \in (\rho, \widetilde{\rho})$ and $V_2 \in (\sigma, \widetilde{\sigma})$. Then $\widetilde{\rho} \circ \beta_{q^{-1}} \in G - \mathrm{Loc}^L(\mathcal{A})_{q^{-1}}$ and $\widetilde{\sigma} \circ \beta_{r^{-1}} \in G - \mathrm{Loc}^L(\mathcal{A})_{r^{-1}}$ are localized in J and I, respectively. Also, for any $A \in \mathcal{A}$ we then have $\widetilde{\rho}(\beta_{q^{-1}}(A)) = V_1 \rho(\beta_{q^{-1}}(A)) V_1^*$, so $V_1 \in (\rho \circ \beta_{q^{-1}}, \widetilde{\rho} \circ \beta_{q^{-1}})$. Similarly, we also have $V_2 \in (\sigma \circ \beta_{r^{-1}}, \widetilde{\sigma} \circ \beta_{r^{-1}})$. Thus we can use V_1 and V_2 to compute $C^r_{\rho\circ\beta_{q^{-1}},\sigma\circ\beta_{r^{-1}}}$. Namely,

$$C^r_{\rho\circ\beta_{q^{-1}},\sigma\circ\beta_{r^{-1}}} = [\beta_{q^{-1}}(V_2) \times \beta_{e^{-1}}(V_1)]^* \circ [V_1 \times V_2]$$

$$= \{\beta_{q^{-1}}(V_2)[\beta_{q^{-1}}(\sigma \circ \beta_{r^{-1}})](V_1)\}^* V_1(\rho \circ \beta_{q^{-1}})(V_2)$$
$$= [(\beta_{q^{-1}} \circ \sigma \circ \beta_{r^{-1}} \circ \beta_q)(V_1)]^* \beta_{q^{-1}}(V_2)^* V_1 \rho(\beta_{q^{-1}}(V_2)).$$

On the other hand, since $\beta_{q^{-1}}(V_2) \in (\beta_{q^{-1}}(\sigma), \beta_{q^{-1}}(\widetilde{\sigma}))$, we can use V_1 and $W_2 := \beta_{q^{-1}}(V_2)$ to compute

$$\begin{aligned} C^r_{\rho,\beta_{q^{-1}}(\sigma)} &= [\beta_e(W_2) \times \beta_{(qrq^{-1})^{-1}}(V_1)]^* \circ [V_1 \times W_2] \\ &= [\beta_{q^{-1}}(V_2) \times \beta_{qr^{-1}q^{-1}}(V_1)]^* \circ [V_1 \times \beta_{q^{-1}}(V_2)] \\ &= \{\beta_{q^{-1}}(V_2)\beta_{q^{-1}}(\sigma)(\beta_{qr^{-1}q^{-1}}(V_1))\}^* V_1 \rho(\beta_{q^{-1}}(V_2)) \\ &= [(\beta_{q^{-1}} \circ \sigma \circ \beta_{r^{-1}} \circ \beta_q)(V_1)]^* \beta_{q^{-1}}(V_2)^* V_1 \rho(\beta_{q^{-1}}(V_2)). \end{aligned}$$

So the two expressions are indeed equal. Thus we conclude that the braiding of $G - \mathrm{Loc}^L(\mathcal{A})$ transported to $G - \mathrm{Loc}^R(\mathcal{A})$ is

$$(c^{R,l}_{\rho,\sigma})^\bullet = (c^{R,l}_{\beta_{q^{-1}}(\sigma),\beta_{q^{-1}r^{-1}q}(\rho)})^{-1}. \tag{3.2.6}$$

We have thus defined a new braided G-crossed structure on $G - \mathrm{Loc}^R(\mathcal{A})$ by transporting the braided G-crossed category from $G - \mathrm{Loc}^L(\mathcal{A})$ to $G - \mathrm{Loc}^R(\mathcal{A})$. The tensor product is given by (3.2.4) and (3.2.5), the G-grading is given by (3.2.3), the braiding is given by (3.2.6) and the G-action remains unchanged. Comparing this with our discussion in Subsection 2.8.5 we see that this new braided G-crossed structure on $G - \mathrm{Loc}^R(\mathcal{A})$ gives us precisely the mirror image of $G - \mathrm{Loc}^R(\mathcal{A})$.

3.2.3 Group actions on quantum field theories

Originally, the purpose of this subsection was to only state Theorem 3.2.20 below (together with the needed definitions), which was proven in Müger's paper [78]. One of the statements in this theorem is that the braided G-crossed category of G-localized endomorphisms of an AQFT with a G-action is equivalent to the crossed product of the category of DHR endomorphisms of the orbifold theory with a certain symmetric subcategory thereof. This equivalence could be useful in relating the categories in AQFT to our abstract categorical results in Chapter 4. Unfortunately, we were not able to accomplish such a relation based on this equivalence (although we did find some other relations that were not related to this equivalence), so in this sense the current subsection is somewhat independent of the rest of the thesis. However, we have decided to include this subsection because an important part of the content can be considered as being new. Namely, while studying [78] we decided to reformulate it in the language of [65], which led us to new proofs of some of the results in [78]. In fact, our proof of Theorem 3.2.20 strongly relies on our Lemma 3.2.14, which is not found in [78] (or elsewhere, as far as we know).

So far we have only constructed the categories $G - \mathrm{Loc}^L(\mathcal{A})$ and $G - \mathrm{Loc}^R(\mathcal{A})$ as braided G-crossed categories. For this we did not need much more than the assumption that $(H, \mathcal{A}, \beta)$ is a G-net of von Neumann algebras satisfying Haag duality and strong additivity. To obtain further results on these categories, we need to make some additional assumptions.

Definition 3.2.10 A *quantum field theory* (*QFT* for short) on $\mathbb{R}$ is a triple $(H, \mathcal{A}, \Omega)$, where H is a separable Hilbert space, $(H, \mathcal{A})$ is a net of von Neumann algebras and $\Omega \in H$ is a unit vector, satisfying the following additional properties:

- for each $I \in \mathcal{K}$ the von Neumann algebra $\mathcal{A}(I)$ is a type III factor;
- for each $I \in \mathcal{K}$ the vector $\Omega \in H$ is both cyclic and separating for $\mathcal{A}(I)$;
- the net $(H, \mathcal{A})$ satisfies locality, irreducibility, strong additivity and Haag duality.

If G is a topological group, then a *G-action* on a QFT $(H, \mathcal{A}, \Omega)$ on $\mathbb{R}$ is a strongly continuous unitary representation $V : G \to B(H)$ that induces a G-action β on the net $(H, \mathcal{A})$ as in Definition 3.1.5, satisfying $V(q)\Omega = \Omega$ for all $q \in G$.

If $(H, \mathcal{A}, \Omega, V)$ is a QFT with G-action and if $\rho \in G - \text{Loc}(\mathcal{A})_{q,r}$ is (q,r)-localized in $I \in \mathcal{K}$, then we have already seen that $\rho(\mathcal{A}(I)) \subset \mathcal{A}(I)$. Because we now also have that $\mathcal{A}(I)$ is a type III factor on a separable Hilbert space and because unital $*$-endomorphisms thereof are automatically normal, the restriction of ρ to $\mathcal{A}(I)$ is a normal $*$-endomorphism of $\mathcal{A}(I)$, see Lemma 2.12 of [78]. Hence we can consider the index $[\mathcal{A}(I) : \rho(\mathcal{A}(I))] \in [1, \infty]$ of the subfactor $\rho(\mathcal{A}(I)) \subset \mathcal{A}(I)$. This index is independent of the choice of $I \in \mathcal{K}$ as long as I is chosen such that ρ is (q,r)-localized in $\mathcal{A}(I)$. Hence we can define

$$\text{Ind}(\rho) := [\mathcal{A}(I) : \rho(\mathcal{A}(I))]$$

where $I \in \mathcal{K}$ is chosen so that ρ is localized in I. The following useful lemma can be found as Lemma 2.4 in [9].

Lemma 3.2.11 *Let $(H, \mathcal{A}, \Omega)$ be a QFT on $\mathbb{R}$ and let $\rho, \sigma \in \text{Loc}(\mathcal{A})(I)$. Then*[13]

$$\text{Hom}_{\mathcal{A}}(\rho, \sigma) = \text{Hom}_{\mathcal{A}(I)}(\rho|_{\mathcal{A}(I)}, \sigma|_{\mathcal{A}(I)}).$$

As can be seen in [9], the proof of this lemma is based on strong additivity and on the fact that the restrictions of ρ and σ to $\mathcal{A}(I)$ are normal. The content of the lemma is often formulated informally as the statement that local intertwiners coincide with global intertwiners.

The full subcategory of $G-\text{Loc}(\mathcal{A})$ determined by the objects with finite index is denoted by $G-\text{Loc}_f(\mathcal{A})$. Similarly, we also define $G - \text{Loc}_f^L(\mathcal{A})$ and $G - \text{Loc}_f^R(\mathcal{A})$. The following important result can be found as Proposition 2.19 in [78], which also includes a detailed proof.

Proposition 3.2.12 *Let $(H, \mathcal{A}, \Omega, V)$ be a QFT with G-action. Then $G - \text{Loc}_f^{(L,l)/(L,r)}(\mathcal{A})$ and $G - \text{Loc}_f^{(R,l)/(R,r)}(\mathcal{A})$ are braided G-crossed TC^*s. Furthermore, for any $\rho \in G - \text{Loc}^{L/R}(\mathcal{A})$ we have $d(\rho) = \text{Ind}(\rho)^2$.*

In order to assure that this rather large subsection is actually readable, we have subdivided it into smaller subsubsections. In this way the reader will have a better overview of the content.

Application of subfactor theory to QFT

We will now begin our discussion on how the results on type III subfactors can be applied to group actions on a QFT. If $(H, \mathcal{A}, \Omega, V)$ is a QFT with G-action, then we define

$$H^G := \{\Psi \in H : V(q)\Psi = \Psi \text{ for all } q \in G\}$$

and for each $I \in \mathcal{K}$ we define

$$\mathcal{A}(I)^G := \{A \in \mathcal{A}(I) : \beta_q(A) = A \text{ for all } q \in G\}.$$

We then introduce two new nets $(H, \mathcal{A}^G)$ and $(H^G, \mathcal{B})$ of von Neumann algebras by defining

$$\mathcal{A}^G(I) := \mathcal{A}(I)^G$$
$$\mathcal{B}(I) := \mathcal{A}(I)^G|_{H^G}$$

for all $I \in \mathcal{K}$. Observe that the operators in $\mathcal{B}(I)$ indeed map H^G into itself and that $\Omega \in H^G$. It is not necessarily true that $(H^G, \mathcal{B}, \Omega)$ is automatically again a QFT on $\mathbb{R}$. Therefore, in the sequel we will simply assume that this is the case. Furthermore, we will always assume that G is compact. Thus:

[13] To prevent any possible confusion in this subsection, we will often replace our earlier simplistic notation (ρ, σ) with notation such as $\text{Hom}_{\mathcal{A}}(\rho, \sigma)$ to indicate that we are considering the set of intertwiners from ρ to σ in the category $\text{End}(\mathcal{A})$. An even more precise notation would be $\text{Hom}_{\text{End}(\mathcal{A})}(\rho, \sigma)$, but we think that this is a bit exaggerative.

From now on we assume that we are given a QFT $(H, \mathcal{A}, \Omega, V)$ on $\mathbb{R}$ with action V of a compact group G such that $(H^G, \mathcal{B}, \Omega)$ is again a QFT on $\mathbb{R}$.

Although our assumption concerns the net $\mathcal{B}$, we will be mainly interested in the net $\mathcal{A}^G$ because we want to apply the results of Subsection 3.1.5 on type III subfactors to the present setting (and only $\mathcal{A}^G$ gives rise to a type III subfactor). We thus want to translate our assumption on $\mathcal{B}$ to certain properties of $\mathcal{A}^G$, which will be done by introducing the $*$-homomorphism

$$\begin{aligned} \pi_0 : \mathcal{A}^G &\to \mathcal{B} \\ A &\mapsto A|_{H^G}, \end{aligned}$$

Considered as a representation of $\mathcal{A}^G$ on H^G, π_0 is unitarily equivalent to the GNS representation of $\mathcal{A}^G$ corresponding to the state ω determined by the unit vector Ω. Because Ω is separating for $\mathcal{A}^G$, it follows that π_0 is faithful and can therefore be interpreted as a $*$-isomorphism from $\mathcal{A}^G$ to $\mathcal{B}$. In particular, we have $\mathcal{B}(I) = \pi_0(\mathcal{A}^G(I))$ and $\mathcal{A}^G(I) = \pi_0^{-1}(\mathcal{B}(I))$ for all $I \in \mathcal{K}$ and each $\mathcal{A}^G(I)$ is a type III factor. The strong additivity of $\mathcal{B}$ implies that the net $\mathcal{A}^G$ also satisfies strong additivity and Haag duality of $\mathcal{B}$ implies that $\mathcal{A}^G(I) = \mathcal{A}^G(I^\perp)' \cap \mathcal{A}^G$ for all $I \in \mathcal{K}$. To each endomorphism $\rho \in \mathrm{End}(\mathcal{A}^G)$ we can assign the endomorphism $\pi_0 \circ \rho \circ \pi_0^{-1} \in \mathrm{End}(\mathcal{B})$. In fact, the assignments $A \mapsto \pi_0(A)$ and $\rho \mapsto \pi_0 \circ \rho \circ \pi_0^{-1}$ establish a strict tensor equivalence

$$\Pi_0 : \mathrm{End}(\mathcal{A}^G) \to \mathrm{End}(\mathcal{B})$$

that is bijective on the objects (and thus invertible). Now suppose that $\rho \in \mathrm{Loc}(\mathcal{A}^G)(I)$ for some $I \in \mathcal{K}$. Then it is clear that $\Pi_0(\rho) \in \mathrm{End}(\mathcal{B})$ is also localized in I, i.e. that it acts trivially on $\mathcal{B}(I^\perp)$. If $J \in \mathcal{K}$, then by transportability of ρ we can find a unitary $u \in \mathcal{A}^G$ such that $\widetilde{\rho} := \mathrm{Ad}(u) \circ \rho$ is localized in J and if we define $U := \Pi_0(u) \in \mathcal{B}$ then $\mathrm{Ad}(U) \circ \Pi_0(\rho)$ is localized in J. This shows that Π_0 restricts to a (strict tensor) functor $\mathrm{Loc}(\mathcal{A}^G) \to \mathrm{Loc}(\mathcal{B})$. We can thus use the inverse of the functor Π_0 to transport the braiding of $\mathrm{Loc}(\mathcal{B})$ to $\mathrm{Loc}(\mathcal{A}^G)$ and it can be shown (see the discussion in [9], after corollary 2.7) that this braiding on $\mathrm{Loc}(\mathcal{A}^G)$ can be obtained in the same way from unitary charge transporters as the braiding in $\mathrm{End}(\mathcal{B})$, despite the fact that $\mathcal{A}^G$ does not satisfy Haag duality. These facts will be used implicitly in Proposition 3.2.16 below.

Now that the precise setting has been made clear, we will apply the theory of type III subfactors as discussed in Subsection 3.1.5. For each $I \in \mathcal{K}$ we have an irreducible type III subfactor $\mathcal{A}^G(I) \subset \mathcal{A}(I)$ and the vector $\Omega \in H^G$ is cyclic and separating for both $\mathcal{A}(I)$ and $\mathcal{A}^G(I)$, where in the first case cyclicity refers to H and in the second case it refers to H^G. In particular, the Jones projection is the same for every $I \in \mathcal{K}$, namely the projection E_G onto H^G. If $\varepsilon : I \mapsto \varepsilon_I$ is a mapping that assigns to each $I \in \mathcal{K}$ an element $\varepsilon_I \in C(\mathcal{A}(I), \mathcal{A}^G(I))$, then ε is called *consistent* if $\varepsilon_J|_{\mathcal{A}(I)} = \varepsilon_I$ whenever $I \subset J$, in which case we will write $\varepsilon \in C(\mathcal{A}, \mathcal{A}^G)$. If $\varepsilon \in C(\mathcal{A}, \mathcal{A}^G)$, then it is called *standard* if $\omega \circ \varepsilon_I = \omega$ for every $I \in \mathcal{K}$, where ω denotes the state determined by Ω. An example of a standard $\varepsilon \in C(\mathcal{A}, \mathcal{A}^G)$ is given by

$$\varepsilon_I(A) := \int_G \beta_q(A) d\mu(q)$$

for $A \in \mathcal{A}(I)$, where μ denotes the normalized Haar measure on G. Note that this ε_I must be the unique conditional expectation in $C(\mathcal{A}(I), \mathcal{A}^G(I))$ by irreducibility of $\mathcal{A}^G(I) \subset \mathcal{A}(I)$ and is therefore minimal. In case G is finite, this formula for ε simplifies to

$$\varepsilon_I(A) = \frac{1}{|G|} \sum_{q \in G} \beta_q(A) \tag{3.2.7}$$

for $A \in \mathcal{A}(I)$. For any $I \in \mathcal{K}$ we also have $\mathrm{Ind}(\varepsilon_I) = |G|$, which in particular implies that the index $[\mathcal{A}(I) : \mathcal{A}^G(I)]$ equals $|G|$ for all $I \in \mathcal{K}$ and is thus independent of the choice of I.

From now on we will always assume that the group G is finite.

Thus, ε will always refer to (3.2.7). The next theorem, which can be found in [65], shows that we can extend canonical endomorphisms. The intertwiners w and v in this theorem (and in the rest of this subsection) refer to the ones in Lemma 3.1.25.

Theorem 3.2.13 *Fix some interval $I \in \mathcal{K}$.*

(1) If γ is a canonical endomorphism for $\mathcal{A}^G(I) \subset \mathcal{A}(I)$, then for any $J \in \mathcal{K}$ with $J \supset I$ there exists an extension $\widetilde{\gamma}$ of γ to $\mathcal{A}(J)$ that is a canonical endomorphism for $\mathcal{A}^G(J) \subset \mathcal{A}(J)$ and it satisfies

$$\widetilde{\gamma}|_{\mathcal{A}(I)' \cap \mathcal{A}^G(J)} = \mathrm{id}.$$

We will write $\widetilde{\lambda} = \widetilde{\gamma}|_{\mathcal{A}^G(J)}$ and $\lambda = \gamma|_{\mathcal{A}^G(I)} = \widetilde{\lambda}|_{\mathcal{A}^G(I)}$.

(2) The isometry $w : \iota_{\mathcal{A}^G(I)} \to \lambda$ in $\mathcal{A}^G(I)$ which induces ε_I is also an isometry $w : \iota_{\mathcal{A}^G(J)} \to \widetilde{\lambda}$ that induces ε_J.

(3) The isometry $v : \iota_{\mathcal{A}(I)} \to \gamma$ in $\mathcal{A}(I)$ is also an intertwiner $v : \iota_{\mathcal{A}(J)} \to \widetilde{\gamma}$.[14]

Consequently, for each $I \in \mathcal{K}$ there exists an endomorphism $\gamma^{(I)} \in \mathrm{End}(\mathcal{A})$ of the quasi-local algebra such that $\gamma^{(I)}|_{\mathcal{A}(I)' \cap \mathcal{A}^G} = \mathrm{id}$ and such that for any $J \in \mathcal{K}$ with $J \supset I$ the restriction of $\gamma^{(I)}$ to $\mathcal{A}(J)$ is a canonical endomorphism for the subfactor $\mathcal{A}^G(J) \subset \mathcal{A}(J)$. In particular we have that

$$\gamma^{(I)}|_{\mathcal{A}^G(I^\perp)} = \mathrm{id}.$$

If $I_1, I_2 \in \mathcal{K}$, then $\gamma^{(I_1)}$ and $\gamma^{(I_2)}$ are related by conjugation by a unitary that can be chosen in $\mathcal{A}^G(K)$ for any $K \in \mathcal{K}$ with $K \supset I_1 \cup I_2$. If for $I \in \mathcal{K}$ we define $\lambda^{(I)} \in \mathrm{End}(\mathcal{A}^G)$ by

$$\lambda^{(I)} := \gamma^{(I)}|_{\mathcal{A}^G},$$

then $\lambda^{(I)} \in \mathrm{Loc}(\mathcal{A}^G)$ is localized in I and there is an isometry $w : \iota_{\mathcal{A}^G(I)} \to \lambda^{(I)}$ in $\mathcal{A}^G(I)$ which induces ε according to $\varepsilon_J(A) = w^* \gamma^{(I)}(A) w$ for any $J \supset I$. We also have an isometry $v : \iota_{\mathcal{A}(I)} \to \gamma^{(I)}$ in $\mathcal{A}(I)$ which satisfies $w^* v = |G|^{-1/2} 1 = w^* \gamma^{(I)}(v)$. Hence $\varepsilon(vv^*) = w^* \gamma^{(I)}(vv^*) w = |G|^{-1} 1$. According to equation (3.1.11), if $J \in \mathcal{K}$ with $J \supset I$, then any $A \in \mathcal{A}(J)$ can be written uniquely as a product of an element in $\mathcal{A}^G(J)$ and v, namely

$$A = |G| \varepsilon(A v^*) v.$$

In other words, if $J \in \mathcal{K}$ with $J \supset I$ then the local algebra $\mathcal{A}(J)$ is generated by $\mathcal{A}^G(J)$ and the element $v \in \mathcal{A}(I)$, where $I \in \mathcal{K}$ was chosen to define $\gamma^{(I)} \in \mathrm{End}(\mathcal{A})$ and $v : \iota_{\mathcal{A}(I)} \to \gamma^{(I)}|_{\mathcal{A}(I)}$ is the intertwiner in $\mathcal{A}(I)$ that comes with the finiteness of $\mathrm{Ind}(\varepsilon)$ as discussed before.

Charged intertwiners and the categories $\mathcal{S}$ and $\mathcal{H}$

Given any $\rho \in \mathrm{Loc}(\mathcal{A}^G)$ we say that $\psi \in \mathcal{A}$ is a *charged intertwiner* for ρ if

$$\psi A = \rho(A) \psi \tag{3.2.8}$$

for all $A \in \mathcal{A}^G$. In case ρ is localized in I, we have $\psi \in \mathcal{A}(I)$. The set of all charged intertwiners for ρ is denoted by $\mathscr{H}_\rho$, which is obviously a $\mathbb{C}$-vector space. By irreducibility of $\mathcal{A}^G(I) \subset \mathcal{A}(I)$, for any $\psi_1, \psi_2 \in \mathscr{H}_\rho$ we have $\psi_2^* \psi_1 \in \mathbb{C}1$ and we can therefore define a sesquilinear map on $\mathscr{H}_\rho$ by $\langle \psi', \psi \rangle 1 = \psi^* \psi'$, which is an inner product that makes $\mathscr{H}_\rho$ into a Hilbert space. We mention that not every $\rho \in \mathrm{Loc}(\mathcal{A}^G)$ has non-zero charged intertwiners, but those $\rho \in \mathrm{Loc}_f(\mathcal{A}^G)$ for which $\mathscr{H}_\rho$ is non-trivial will be very important to us.

[14] In the more general setting of [65], where ε is any standard element in $C(M, N)$ for a 'quantum field theoretical net of subfactors' $N \subset M$, this part of the theorem implies that $\mathrm{Ind}(\varepsilon_I) = \mathrm{Ind}(\varepsilon_J)$ by Lemma 3.1.25 (in case $\mathrm{Ind}(\varepsilon_I) < \infty$), showing that $[\mathcal{A}(I) : \mathcal{A}^G(I)] = [\mathcal{A}(J) : \mathcal{A}^G(J)]$.

From now on we will write $\mathcal{S}$ to denote the full subcategory of $\mathrm{Loc}_f(\mathcal{A}^G)$ determined by all objects $\sigma \in \mathrm{Loc}_f(\mathcal{A}^G)$ for which $\mathscr{H}_\sigma$ is non-trivial.

It can be shown that $\mathcal{S}$ is a symmetric tensor $*$-subcategory of the BTC^* $\mathrm{Loc}_f(\mathcal{A}^G)$, so $\mathcal{S}$ is an STC^*.

We now define a category $\mathcal{H}$ as follows. The set of objects of $\mathcal{H}$ is defined to be $\{\mathscr{H}_\sigma : \sigma \in \mathcal{S}\}$ and if $\mathscr{H}_\sigma, \mathscr{H}_{\sigma'} \in \mathcal{H}$, then we define

$$\mathrm{Hom}_{\mathcal{H}}(\mathscr{H}_\sigma, \mathscr{H}_{\sigma'}) := \{T \in \mathcal{A}^G : T\mathscr{H}_\sigma \subset \mathscr{H}_{\sigma'}\}$$

and the composition of morphisms is simply defined as multiplication in $\mathcal{A}^G$. The Hilbert spaces $\mathscr{H}_\sigma$ are all finite-dimensional with dimension $\dim(\mathscr{H}_\sigma) = d(\sigma)$. Thus, if $\sigma \in \mathcal{S}$ and if $\{\psi_\sigma^i\}_i$ is an orthonormal basis for $\mathscr{H}_\sigma$ then $\sum_i \psi_\sigma^i {\psi_\sigma^i}^* = 1$ and it follows from (3.2.8) that σ can be written as

$$\sigma(A) = \sum_i \sigma(A)\psi_\sigma^i {\psi_\sigma^i}^* = \sum_i \psi_\sigma^i A {\psi_\sigma^i}^*$$

for $A \in \mathcal{A}^G$. We now claim that $\mathrm{Hom}_{\mathcal{H}}(\mathscr{H}_\sigma, \mathscr{H}_{\sigma'}) = \mathrm{Hom}_{\mathcal{A}^G}(\sigma, \sigma')$. Namely, if $T \in \mathrm{Hom}_{\mathcal{H}}(\mathscr{H}_\sigma, \mathscr{H}_{\sigma'})$, then for any $B \in \mathcal{A}^G$ and $\psi \in \mathscr{H}_\sigma$ we have $\sigma'(B)T\psi = T\psi B = T\sigma(B)\psi$, where in the first step we used that $T\psi \in \mathscr{H}_{\sigma'}$. So if we take an orthonormal basis $\{\psi_\sigma^i\}_i$ of $\mathscr{H}_\sigma$ we get

$$\sigma'(B)T = \sum_i \sigma'(B)T\psi_\sigma^i {\psi_\sigma^i}^* = \sum_i T\sigma(B)\psi_\sigma^i {\psi_\sigma^i}^* = T\sigma(B).$$

Conversely, if $S \in \mathrm{Hom}_{\mathcal{A}^G}(\sigma, \sigma')$, then for for any $B \in \mathcal{A}^G$ and $\psi \in \mathscr{H}_\sigma$ we have $S\psi B = S\sigma(B)\psi = \sigma'(B)S\psi$. This proves our claim and implies that there exists a fully faithful functor[15]

$$\mathcal{S} \to \mathcal{H}$$

given on the objects by $\sigma \mapsto \mathscr{H}_\sigma$ and on the morphisms by the identity map $T \mapsto T$. The category $\mathcal{H}$ becomes a strict tensor category with unit object $\mathscr{H}_\iota = \mathbb{C}1$ and with tensor product on the objects given by $\mathscr{H}_{\sigma_1} \otimes \mathscr{H}_{\sigma_2} := \mathscr{H}_{\sigma_1 \otimes \sigma_2}$ and on the morphisms given by the tensor product of morphisms in $\mathrm{End}(\mathcal{A}^G)$. It is trivial to see that the functor $\mathcal{S} \to \mathcal{H}$ is a tensor functor. An easy computation shows that $\mathscr{H}_{\sigma_1}\mathscr{H}_{\sigma_2} \subset \mathscr{H}_{\sigma_1 \otimes \sigma_2}$ and that if $\{\psi_{\sigma_1}^i\}_i$ and $\{\psi_{\sigma_2}^i\}_i$ are orthonormal bases then the elements $\{\psi_{\sigma_1}^i \psi_{\sigma_2}^j\}_{i,j}$ are mutually orthonormal and hence form an orthonormal basis for $\mathscr{H}_{\sigma_1 \otimes \sigma_2}$ because $d(\sigma_1 \otimes \sigma_2) = d(\sigma_1)d(\sigma_2)$. Thus $\mathscr{H}_{\sigma_1 \otimes \sigma_2}$ can really be interpreted as the tensor product of $\mathscr{H}_{\sigma_1}$ and $\mathscr{H}_{\sigma_2}$ in the sense of Hilbert spaces. The crucial fact about these Hilbert spaces $\mathscr{H}_\sigma$ is that they carry a representations of G. If $q \in G$ and $\psi \in \mathscr{H}_\sigma$ with $\sigma \in \mathcal{S}$, then for any $A \in \mathcal{A}^G$ we have $\beta_q(\psi)A = \beta_q(\psi A) = \beta_q(\sigma(A)\psi) = \sigma(A)\beta_q(\psi)$, so $\beta_q(\psi) \in \mathscr{H}_\rho$. By linearity of β_q, we thus have a representation V_σ of G on $\mathscr{H}_\sigma$ given by

$$V_\sigma(q)\psi := \beta_q(\psi).$$

For any $q \in G$ and $\psi_1, \psi_2 \in \mathscr{H}_\sigma$ we have $\langle V_\sigma(q)\psi_1, V_\sigma(q)\psi_2\rangle = \beta_q(\psi_2^*\psi_1) = \langle \psi_1, \psi_2\rangle$, so $V_\sigma(q)$ is an invertible isometry on the Hilbert space $\mathscr{H}_\sigma$ for any $q \in G$, and hence $(\mathscr{H}_\sigma, V_\sigma)$ is a unitary representation of G. Also, if $T \in \mathrm{Hom}_{\mathcal{S}}(\mathscr{H}_\sigma, \mathscr{H}_{\sigma'})$, then for any $q \in G$ and $\psi \in \mathscr{H}_\sigma$ we have

$$TV_\sigma(q)\psi = T\beta_q(\psi) = \beta_q(T\psi) = V_{\sigma'}(q)T\psi,$$

which implies that $TV_\sigma(q) = V_{\sigma'}(q)T$. Thus if we interpret T as a map $\mathscr{H}_\sigma \to \mathscr{H}_{\sigma'}$ given by $\psi \mapsto T\psi$, then $T \in \mathrm{Hom}_{\mathrm{Rep}_f(G)}((\mathscr{H}_\sigma, V_\sigma), (\mathscr{H}_{\sigma'}, V_{\sigma'}))$. In this way we obtain a fully faithful tensor functor $\mathcal{S} \to \mathrm{Rep}_f(G)$, which can be shown to be essentially surjective and even an equivalence of STC^*s.

[15] We would like to emphasize that $\mathcal{H}$ is not a full subcategory of the category of finite-dimensional Hilbert spaces, because it has less morphisms. This will become clear later on.

The Frobenius algebra λ

According to equation (3.1.12) the canonical endomorphism $\gamma^{(I)} \in \mathrm{End}(\mathcal{A})$ obtained from an interval $I \in \mathcal{K}$ can be expressed as

$$\gamma^{(I)}(A) = \sum_{q\in G} \beta_q(vAv^*) = \sum_{q\in G} v_q\beta_q(A)v_q^*,$$

where we have introduced the notation $v_q := \beta_q(v) \in \mathcal{A}(I)$. The equation holds for all $A \in \mathcal{A}(J)$ with $J \in \mathcal{K}$ and $J \supset I$, and hence for all $A \in \mathcal{A}$. Because $v_q^* v_q = 1_{\beta_q} \in \mathrm{End}_{\mathcal{A}}(\beta_q)$ and because $1 = \gamma^{(I)}(1) = \sum_{q\in G} v_q v_q^*$, this formula for $\gamma^{(I)}$ shows that

$$\gamma^{(I)} \cong \bigoplus_{q\in G} \beta_q$$

in the category $\mathrm{End}(\mathcal{A})$. Restricting to $\mathcal{A}^G$ we obtain the formula

$$\lambda^{(I)}(B) = \sum_{q\in G} v_q B v_q^*$$

for the dual canonical endomorphism, which immediately implies that $\{v_q\}_{q\in G}$ forms an orthonormal basis for $\mathscr{H}_\lambda$. To simplify the expressions, we will now write λ rather than $\lambda^{(I)}$. The representation V_λ of G on $\mathscr{H}_\lambda$ is given by $V_\lambda(q)v_r = \beta_q(v_r) = v_{qr}$, which is precisely the regular representation of G. We can give the regular representation $(\mathscr{H}_\lambda, V_\lambda)$ the structure of a commutative $*$-Frobenius algebra by defining its unit $\eta : (\mathscr{H}_\iota, V_\iota) \to (\mathscr{H}_\lambda, V_\lambda)$ by

$$1 \mapsto \frac{\sqrt{\kappa_2}}{\sqrt{|G|}} \sum_{q\in G} v_q$$

and its multiplication $\mu : (\mathscr{H}_\lambda, V_\lambda) \otimes (\mathscr{H}_\lambda, V_\lambda) \to (\mathscr{H}_\lambda, V_\lambda)$ by

$$v_q v_r \mapsto \sqrt{\kappa_1}\delta_{q,r} v_q$$

for some $\kappa_1, \kappa_2 \in \mathbb{R}_{>0}$ with $\kappa_1\kappa_2 = |G|$. The group $\mathrm{Aut}^*((\mathscr{H}_\lambda, V_\lambda), \mu, \eta)$ of unitary Frobenius automorphisms can be identified with G. Namely, these unitaries are of the form $\phi_q : v_r \mapsto v_{qr}$. Under the functor $\mathcal{S} \to \mathrm{Rep}_f(G)$ the Frobenius structure on λ, with unit $\sqrt{\kappa_2} w$ and multiplication $\sqrt{\kappa_1}\gamma(v)^*$, corresponds precisely to the Frobenius structure of $(\mathscr{H}_\lambda, V_\lambda)$ because we have

$$\sqrt{\kappa_2} w 1 = \sqrt{\kappa_2}\mathrm{Ind}(\varepsilon)^{1/2}\varepsilon(v) = \sqrt{\kappa_2}|G|^{1/2}\frac{1}{|G|}\sum_{q\in G}\beta_q(v) = \frac{\sqrt{\kappa_2}}{\sqrt{|G|}}\sum_{q\in G} v_q$$

and

$$\sqrt{\kappa_1}\gamma(v)^* v_q v_r = \sqrt{\kappa_1}\beta_q(\gamma(v^*)v)v_r = \sqrt{\kappa_1}\beta_q(vv^*)v_r = \sqrt{\kappa_1}v_q v_q^* v_r = \sqrt{\kappa_1} v_q \delta_{q,r}.$$

This implies in particular that we can identify $\mathrm{Aut}^*(\lambda, \sqrt{\kappa_1}\gamma(v)^*, \sqrt{\kappa_2}w)$ with G, but we will be a bit more explicit about this. Of course we can identify any unitary Frobenius automorphism of the regular representation of G (as characterized above) with a unitary Frobenius automorphism of λ, but as we will see in the following lemma there is also a more direct way to obtain them, which will be necessary later. In what follows we will always choose the normalization $\kappa_1 = |G|$ and $\kappa_2 = 1$, so the $*$-Frobenius structure $(\lambda, \mu_\lambda, \eta_\lambda)$ will always be given by $\mu_\lambda = \sqrt{|G|}\gamma(v)^*$ and $\eta_\lambda = w$.

Lemma 3.2.14 *Let $I \in \mathcal{K}$ and let $\gamma^{(I)}$ and $\lambda^{(I)}$ be as before. There exists a group isomorphism*

$$G \to \mathrm{Aut}^*(\lambda^{(I)})^{\mathrm{op}}$$
$$q \mapsto D_q$$

such that the G-action on $A \in \mathcal{A}(J)$ with $J \supset I$ can be expressed as $\beta_q(A) = BD_q v$, where $B = |G|\varepsilon(Av^*)v \in \mathcal{A}^G(J)$ is the unique element such that $A = Bv$.

Proof. Recall that each $A \in \mathcal{A}(I)$ can be written as Bv for some uniquely determined $B \in \mathcal{A}^G(I)$. In particular this means that for each $q \in G$ there exists a unique $D_q \in \mathcal{A}^G(I)$ such that

$$\beta_q(v) = D_q v.$$

Obviously, we have $D_e = 1$. For $q, r \in G$ we have $D_{qr}v = \beta_{qr}(v) = \beta_q(D_r v) = D_r\beta_q(v) = D_r D_q v$, so $D_{qr} = D_r D_q$. Thus $q \mapsto D_q$ is a homomorphism $G \to \mathcal{A}^G(I)^{\mathrm{op}}$. In particular we have $D_{q^{-1}} = D_q^{-1}$. Before we can prove that $D_q \in \mathrm{Aut}^*(\lambda^{(I)})$ for all $q \in G$, we first have to calculate some expressions involving the D_q and v. The intertwining property of v gives $vD_q = \gamma(D_q)v = \lambda(D_q)v$ and thus

$$v_r D_q = \beta_r(vD_q) = \lambda(D_q)v_r.$$

To see how the D_q act on v^* we note that $D_q v^* v = v^* v D_q = v^*\gamma(D_q)v = v^*\lambda(D_q)v$, which implies that $D_q v^* = v^*\lambda(D_q)$ and hence

$$D_q v_r^* = v_r^*\lambda(D_q).$$

Also, $\beta_q(v^*)D_q v = \beta_q(v^*v) = 1 = v^*v$, which implies that $\beta_q(v^*)D_q = v^*$. Letting $\beta_{q^{-1}}$ act on both sides, we get $v^*D_q = \beta_{q^{-1}}(v^*)$ and hence

$$v_r^* D_q = \beta_{rq^{-1}}(v^*).$$

Using this, we find that $D_q^* v = (v^*D_q)^* = \beta_{q^{-1}}(v^*)^* = D_{q^{-1}}v = D_q^{-1}v$, showing that $D_q^* = D_q^{-1}$, i.e. that the D_q are unitary.

We are now ready to show that these unitaries D_q are automorphisms of the Frobenius algebra λ. For any $B \in \mathcal{A}^G(I)$ we have

$$\begin{aligned}\lambda(B)D_q &= \sum_{r\in G} v_r B v_r^* D_q = \sum_{r\in G} v_r B\beta_{rq^{-1}}(v^*) = \sum_{r\in G}\beta_r(v)B\beta_{rq^{-1}}(v^*)\\ &= \sum_{r\in G}\beta_{rq}(v)B\beta_r(v^*) = \sum_{r\in G}\beta_{rq}(v)Bv_r^* = \sum_{r\in G} D_q v_r B v_r^* = D_q\lambda(B),\end{aligned}$$

so the D_q are indeed self-intertwiners of λ. Next, we have

$$v^*D_q w = \beta_{q^{-1}}(v^*)w = \beta_{q^{-1}}(v^*w) = \mathrm{Ind}(\varepsilon)^{-1/2}1 = \mathrm{Ind}(\varepsilon)^{-1/2}v^*\varepsilon(v) = v^*w,$$

which shows that $D_q w = w$, i.e. the D_q preserve the unit. Finally, we have

$$D_q\lambda(D_q)\gamma(v)v = D_q\lambda(D_q)vv = D_q v D_q v = \beta_q(vv) = \beta_q(\gamma(v)v) = \gamma(v)\beta_q(v) = \gamma(v)D_q v,$$

so we also have $D_q\lambda(D_q)\gamma(v) = \gamma(v)D_q$, which means precisely that the D_q are comultiplicative. We have thus proved that the D_q are unitary Frobenius automorphisms of λ. In fact, these are all of them, because they are all different and we have seen before that the group of unitary Frobenius automorphisms is isomorphic to G. The relation between these $D_q \in \mathrm{Aut}^*(\lambda^{(I)})$ and the G-action on $\mathcal{A}(I)$ is now evident. Namely, the G-action on an element $A = Bv \in \mathcal{A}(I)$ can be expressed in terms of these D_q as

$$\beta_q(A) = B\beta_q(v) = BD_q v.$$

□

Observe that we have just shown that unitary Frobenius automorphisms of λ completely characterize the G-action on $\mathcal{A}$, which will be used later. Of course, any $A \in \mathcal{A}(J)$ can also be written uniquely as v^*B for some $B \in \mathcal{A}^G(J)$. The G-action can thus also be expressed as

$$\beta_q(A) = \beta_q(v)^*B = v^*D_q^*B = v^*D_q^{-1}B,$$

but we will not need this. The lemma above will be used later to define a G-crossed structure on the category $(\mathrm{Loc}_f(\mathcal{A}^G) \rtimes \mathcal{S}, c^{l/r})_2$, rather than a $G_{\mathcal{S}}$-crossed structure with $G_{\mathcal{S}} = \mathrm{Aut}^*(\lambda)$.

α-induction

We now want to extend objects in $\mathrm{Loc}(\mathcal{A}^G)$ to endomorphisms of $\mathcal{A}$ by a procedure that is known by the name of α-induction. Before we do this, we first state the following useful lemma:

Lemma 3.2.15 *Let $I \in \mathcal{K}$ and write $\gamma = \gamma^{(I)}$ and $\lambda = \lambda^{(I)}$. We have the identities*

$$c^{l/r}_{\lambda,\lambda} v^2 = c^{l/r\,-1}_{\lambda,\lambda} v^2 = v^2, \tag{3.2.9}$$

$$c^{l/r}_{\lambda,\lambda} \gamma(v) = c^{l/r\,-1}_{\lambda,\lambda} \gamma(v) = \gamma(v). \tag{3.2.10}$$

Also, if $\rho \in \mathrm{Loc}(\mathcal{A}^G)$ then

$$[\mathrm{Ad}(c^{l/r}_{\rho,\lambda}) \circ \rho \circ \gamma](v) = \lambda(c^{l/r\,-1}_{\rho,\lambda})\gamma(v). \tag{3.2.11}$$

For the easy proof of (3.2.9) and (3.2.10) we refer to either [65] (Corollary 4.4) or [9] (Lemma 3.4). The equation (3.2.11) is proven in Lemma 3.1 of [9], but can also be obtained by noticing that it can be restated categorically as $[c^{l/r}_{\rho,\lambda} \times \mathrm{id}_\lambda] \circ [\mathrm{id}_\rho \times \gamma(v)] \circ c^{l/r\,-1}_{\rho,\lambda} = [\mathrm{id}_\lambda \times c^{l/r\,-1}_{\rho,\lambda}] \circ [\gamma(v) \times \mathrm{id}_\rho]$ and by using the fact that $\gamma(v) \in \mathrm{Hom}_{\mathrm{End}(\mathcal{A}^G)}(\lambda, \lambda \otimes \lambda)$.

Proposition 3.2.16 ***(α-induction)*** *Let $\rho \in \mathrm{Loc}(\mathcal{A}^G)$ be localized in $J = (a, b) \in \mathcal{K}$.*

(1) The endomorphisms $E^l(\rho), E^r(\rho) \in \mathrm{End}(\mathcal{A})$ given by[16]

$$E^{l/r}(\rho) := \gamma^{(J)^{-1}} \circ \mathrm{Ad}(c^{l/r}_{\rho,\lambda^{(J)}}) \circ \rho \circ \gamma^{(J)}$$

are extensions of ρ and we have $E^{l/r}(\rho) \in \mathrm{End}(\mathcal{A})^G$.

(2) $E^l(\rho)$ acts trivially on $\mathcal{A}((b, \infty))$ and $E^r(\rho)$ acts trivially on $\mathcal{A}((-\infty, a))$.

(3) If $\rho_1, \rho_2 \in \mathrm{Loc}(\mathcal{A}^G)$ and $S \in \mathrm{Hom}_{\mathrm{Loc}(\mathcal{A}^G)}(\rho_1, \rho_2)$ then also $S \in \mathrm{Hom}_{\mathrm{End}(\mathcal{A})}(E^{l/r}(\rho_1), E^{l/r}(\rho_2))$ and hence $S \in \mathrm{Hom}_{\mathrm{End}(\mathcal{A})^G}(E^{l/r}(\rho_1), E^{l/r}(\rho_2))$.

Proof. (1) Let $J_1 \in \mathcal{K}$ and $A \in \mathcal{A}(J_1)$. We will show that $E^{l/r}(\rho)(A)$ is well-defined. Choose $J_2 \in \mathcal{K}$ such that $J_2 \supset J \cup J_1$; in particular, $A \in \mathcal{A}(J_2)$. By transportability of ρ we can choose $K_l, K_r \in \mathcal{K}$ with $K_l < J_2 < K_r$ together with $\rho_l \in \mathrm{End}(\mathcal{A}^G)(K_l)$, $\rho_r \in \mathrm{End}(\mathcal{A}^G)(K_r)$ and unitaries $U_l \in (\rho, \rho_l)$ and $U_r \in (\rho, \rho_r)$ in $\mathcal{A}^G$. With these unitaries we can write the braiding as $c^{l/r}_{\rho,\lambda^{(J)}} = \lambda^{(J)}(U_{l/r})^* U_{l/r}$. But then

$$\begin{aligned}
&\left(\mathrm{Ad}(c^{l/r}_{\rho,\lambda^{(J)}}) \circ \rho \circ \gamma^{(J)}\right)(A) \\
&= \left(\mathrm{Ad}(\lambda^{(J)}(U^*_{l/r})U_{l/r}) \circ \rho \circ \gamma^{(J)}\right)(A) = \left(\mathrm{Ad}(\lambda^{(J)}(U^*_{l/r})) \circ \rho_{l/r} \circ \gamma^{(J)}\right)(A) \\
&= \lambda^{(J)}(U^*_{l/r})\rho_{l/r}(\gamma^{(J)}(A))\lambda^{(J)}(U_{l/r}) = \lambda^{(J)}(U^*_{l/r})\gamma^{(J)}(A)\lambda^{(J)}(U_{l/r}) \\
&= \gamma^{(J)}(U^*_{l/r} A U_{l/r}) = \left(\gamma^{(J)} \circ \mathrm{Ad}(U^*_{l/r})\right)(A) \\
&\in \gamma^{(J)}(\mathcal{A}),
\end{aligned}$$

which shows that $E^{l/r}(\rho)(A)$ is well-defined[17] and that $E^{l/r}(\rho)(A) = U^*_{l/r} A U_{l/r}$. Because $A \in \mathcal{A}(J_1)$ and J_1 were arbitrary, we conclude that $E^{l/r}(\rho)$ is well-defined on $\bigcup_{K \in \mathcal{K}} \mathcal{A}(K)$ and it is clearly a $*$-homomorphism. By continuity, $E^{l/r}(\rho) \in \mathrm{End}(\mathcal{A})$. Because $\gamma^{(J)}|_{\mathcal{A}^G} = \lambda^{(J)}$, we get

$$E^{l/r}(\rho)|_{\mathcal{A}^G} = \gamma^{(J)^{-1}} \circ \mathrm{Ad}(c^{l/r}_{\rho,\lambda^{(J)}}) \circ \rho \circ \lambda^{(J)} = \gamma^{(J)^{-1}} \circ \lambda^{(J)} \circ \rho = \rho,$$

[16] Because $\gamma^{(J)}$ is injective, we can consider it as a bijection onto its image. The inverse of this bijection is denoted by $\gamma^{(J)^{-1}}$.

[17] This can also be derived from equation (3.2.11) by using the fact that any $A \in \mathcal{A}$ can be written as the product Bv of an element $B \in \mathcal{A}^G$ with v.

so $E^{l/r}(\rho)$ is indeed an extension of ρ. Note that the formula $E^{l/r}(\rho)(A) = U_{l/r}^* A U_{l/r}$ implies that for each $q \in G$ we have

$$[\beta_q \circ E^{l/r}(\rho) \circ \beta_{q^{-1}}](A) = \beta_q(U_{l/r}^* \beta_{q^{-1}}(A) U_{l/r}) = U_{l/r}^* A U_{l/r} = E^{l/r}(\rho)(A),$$

where we have used that $U_{l/r} \in \mathcal{A}^G$. Thus $E^{l/r}(\rho) \in \mathrm{End}(\mathcal{A})^G$.

(2) Now suppose that $J_1 \in \mathcal{K}$ with $J < J_1$ and let $A \in \mathcal{A}(J_1)$. We thus have $J = (a, b)$ and $J_1 = (c, d)$ for some $c, d \in \mathbb{R}$ with $b < c < d$, and we define $J_2 = (a, d)$; in particular, $J_2 \supset J \cup J_1$. This is a special case of the situation in part (1) of this proof, so we still have the formula $E^l(\rho)(A) = U_l^* A U_l$, where $U_l \in (\rho, \rho_l)$ is a unitary in $\mathcal{A}^G(K)$, where $K \in \mathcal{K}$ can now be chosen such that $K \supset K_l \cup J$ and $K < J_1$. But then U_l and A commute with each other by locality, so $E^l(\rho)(A) = U_l^* A U_l = A$. This shows that $E^l(\rho)$ acts trivially on $\mathcal{A}((b, \infty))$. Similarly, one can show that $E^r(\rho)$ acts trivially on $\mathcal{A}((-\infty, a))$.

(3) Now suppose that $\rho_1, \rho_2 \in \mathrm{Loc}(\mathcal{A}^G)$ and $S \in \mathrm{Hom}_{\mathrm{Loc}(\mathcal{A}^G)}(\rho_1, \rho_2)$. We then choose $J \in \mathcal{K}$ such that ρ_1 and ρ_2 are both localized in J, so that we can use one and the same $\gamma^{(J)}$ in the expressions for $E^{l/r}(\rho_j)$ for $j \in \{1, 2\}$, which we will simply denote by γ in the rest of this proof (and we denote its restriction by λ). We thus have $\gamma(S) \in \mathrm{Hom}_{\mathrm{End}(\mathcal{A}^G)}(\lambda \otimes \rho_1, \lambda \otimes \rho_2)$ and hence

$$[\gamma(S) \times \mathrm{id}_\lambda] \circ [\mathrm{id}_\lambda \times {c^{l/r}_{\rho_1,\lambda}}^{-1}] = {c^{l/r}_{\lambda\otimes\rho_2,\lambda}}^{-1} \circ [\mathrm{id}_\lambda \times \gamma(S)] \circ [c^{l/r}_{\lambda,\lambda} \times \mathrm{id}_{\rho_1}],$$

which can be rewritten as

$$\gamma(S)\lambda({c^{l/r}_{\rho_1,\lambda}}^{-1}) = \lambda({c^{l/r}_{\rho_2,\lambda}}^{-1}){c^{l/r}_{\lambda,\lambda}}^{-1}\lambda(\gamma(S))c^{l/r}_{\lambda,\lambda}. \tag{3.2.12}$$

We now get

$$\begin{aligned}\gamma(S)[\mathrm{Ad}(c^{l/r}_{\rho_1,\lambda}) \circ \rho_1 \circ \gamma](v) &= \gamma(S)\lambda({c^{l/r}_{\rho_1,\lambda}}^{-1})\gamma(v) = \lambda({c^{l/r}_{\rho_2,\lambda}}^{-1}){c^{l/r}_{\lambda,\lambda}}^{-1}\lambda(\gamma(S))c^{l/r}_{\lambda,\lambda}\gamma(v)\\ &= \lambda({c^{l/r}_{\rho_2,\lambda}}^{-1}){c^{l/r}_{\lambda,\lambda}}^{-1}\lambda(\gamma(S))\gamma(v) = \lambda({c^{l/r}_{\rho_2,\lambda}}^{-1}){c^{l/r}_{\lambda,\lambda}}^{-1}\gamma(v)\gamma(S)\\ &= \lambda({c^{l/r}_{\rho_2,\lambda}}^{-1})\gamma(v)\gamma(S) = [\mathrm{Ad}(c^{l/r}_{\rho_2,\lambda}) \circ \rho_2 \circ \gamma](v)\gamma(S),\end{aligned}$$

where in the first and last step we used (3.2.11), in the second step we used (3.2.12), in the third and fifth step we used (3.2.10) and in the fourth step we used that $\lambda(\gamma(S))\gamma(v) = \gamma^2(S)\gamma(v) = \gamma(\gamma(S)v) = \gamma(vS) = \gamma(v)\gamma(S)$. If we now apply γ^{-1} to both sides, we get

$$S[E^{l/r}(\rho_1)](v) = [E^{l/r}(\rho_2)](v)S.$$

Together with the fact that any $A \in \mathcal{A}$ can be written as Bv for some $B \in \mathcal{A}^G$ and the fact that $S \in \mathrm{Hom}_{\mathrm{End}(\mathcal{A}^G)}(\rho_1, \rho_2)$, we obtain $S \in \mathrm{Hom}_{\mathrm{End}(\mathcal{A})^G}(E^{l/r}(\rho_1), E^{l/r}(\rho_2))$.
□

Remark 3.2.17 In Theorem 3.8 of [78], for each $\rho \in \mathrm{Loc}(\mathcal{A}^G)$ an endomorphism $E(\rho)$ was defined on (elements like) v as $E(\rho)(v) = c_{\lambda,\rho}v$, where c refers to the left braiding c^l. To understand this expression, we observe that it follows from (3.2.11) that

$$E^{l/r}(\rho)(v) = {c^{l/r}_{\rho,\lambda}}^{-1} v = c^{r/l}_{\lambda,\rho} v, \tag{3.2.13}$$

so we conclude that we must read the equation in [78] as $E^r(\rho)(v) = c^l_{\lambda,\rho}v$ (where we have substituted v for the x in the equation in [78]). Indeed, the extension of ρ that is considered in [78] is supposed to act trivially on left half-lines, which is a second indication that the extension $E(\rho)$ in [78] is our $E^r(\rho)$ and not $E^l(\rho)$, despite the fact that c^l is used in [78].

It follows from the proposition that we obtain functors

$$\mathscr{E}^{l/r} : \mathrm{Loc}(\mathcal{A}^G) \to \mathrm{End}(\mathcal{A})^G, \tag{3.2.14}$$

given on the objects by $E^{l/r}$ and on the morphisms by the inclusion $\mathcal{A}^G \to \mathcal{A}$. It is shown in [9] that this is in fact a strict tensor functor. It is also clear that it is faithful and that it is injective on the objects, since $E^{l/r}(\rho)$ coincides with ρ on $\mathcal{A}^G$. Furthermore, if $\overline{\rho} \in \mathrm{Loc}(\mathcal{A}^G)$ is a conjugate for $\rho \in \mathrm{Loc}(\mathcal{A}^G)$, then $E^{l/r}(\overline{\rho}) \in \mathrm{End}(\mathcal{A})^G$ is a conjugate for $E^{l/r}(\rho) \in \mathrm{End}(\mathcal{A})^G$.

The next lemma and the remark following it will be essential when we want to construct the functor $\mathscr{K}^{l/r}$ in the proof of Theorem 3.2.20. The content of the lemma was inspired by the proof of Theorem 3.9 of [9].

Lemma 3.2.18 *If $\rho_1, \rho_2 \in \mathrm{Loc}(\mathcal{A}^G)(I)$, then we have linear maps*

$$\begin{aligned} K : \mathrm{Hom}_{\mathcal{A}(I)}(E^{l/r}(\rho_1), E^{l/r}(\rho_2)) &\to \mathrm{Hom}_{\mathcal{A}^G(I)}(\lambda \otimes \rho_1, \rho_2) \\ S &\mapsto w^*\gamma(S) \end{aligned}$$

and

$$\begin{aligned} L : \mathrm{Hom}_{\mathcal{A}^G(I)}(\lambda \otimes \rho_1, \rho_2) &\to \mathrm{Hom}_{\mathcal{A}(I)}(E^{l/r}(\rho_1), E^{l/r}(\rho_2)) \\ R &\mapsto Rv \end{aligned}$$

that satisfy $K(L(R)) = |G|^{-1/2}R$ and $L(K(S)) = |G|^{-1/2}S$.

Proof. Let $S \in \mathrm{Hom}_{\mathcal{A}(I)}(E^{l/r}(\rho_1), E^{l/r}(\rho_2))$. Then $w^*\gamma(S) \in \mathcal{A}^G(I)$ and by assumption we have $S[E^{l/r}(\rho_1)](A) = [E^{l/r}(\rho_2)](A)S$ for all $A \in \mathcal{A}(I)$. Restricting to $\mathcal{A}^G(I)$ gives us $S\rho_1(B) = \rho_2(B)S$ for all $B \in \mathcal{A}^G(I)$. When we apply γ to both sides of this equation, we get that $\gamma(S)\gamma(\rho_1(B)) = \gamma(\rho_2(B))\gamma(S)$ for all $B \in \mathcal{A}^G(I)$, which can be rewritten as

$$\gamma(S)\lambda(\rho_1(B)) = \lambda(\rho_2(B))\gamma(S)$$

for all $B \in \mathcal{A}^G(I)$. This means that $\gamma(S) \in \mathrm{Hom}_{\mathcal{A}^G(I)}(\lambda \otimes \rho_1, \lambda \otimes \rho_2)$. Using this, we now find that for each $B \in \mathcal{A}^G(I)$ we have

$$w^*\gamma(S)\lambda(\rho_1(B)) = w^*\lambda(\rho_2(B))\gamma(S) = \iota_{\mathcal{A}^G(I)}(\rho_2(B))w^*\gamma(S) = \rho_2(B)w^*\gamma(S),$$

i.e. $w^*\gamma(S) \in \mathrm{Hom}_{\mathcal{A}^G(I)}(\lambda \otimes \rho_1, \rho_2)$, showing that the first map is well-defined; its linearity is obvious. We will now consider the map L. If $R \in \mathrm{Hom}_{\mathcal{A}^G(I)}(\lambda \otimes \rho_1, \rho_2)$ then $Rv \in \mathcal{A}(I)$ and for all $B \in \mathcal{A}^G(I)$ we have

$$Rv\rho_1(B) = Rv\iota_{\mathcal{A}(I)}(\rho_1(B)) = R\gamma(\rho_1(B))v = R\lambda(\rho_1(B))v = \rho_2(B)Rv,$$

so $Rv \in \mathrm{Hom}_{\mathcal{A}^G(I)}(\rho_1, \rho_2)$ and hence $Rv \in \mathrm{Hom}_{\mathcal{A}(I)}(E^{l/r}(\rho_1), E^{l/r}(\rho_2))$ by Proposition 3.2.16; linearity is again obvious. Finally, if $S \in \mathrm{Hom}_{\mathcal{A}(I)}(E^{l/r}(\rho_1), E^{l/r}(\rho_2))$ and $R \in \mathrm{Hom}_{\mathcal{A}^G(I)}(\lambda \otimes \rho_1, \rho_2)$ then

$$\begin{aligned} L(K(S)) &= L(w^*\gamma(S)) = w^*\gamma(S)v = w^*vS = |G|^{-1/2}S \\ K(L(R)) &= K(Rv) = w^*\gamma(Rv) = w^*\gamma(R)\gamma(v) = w^*\lambda(R)\gamma(v) = Rw^*\gamma(v) = |G|^{-1/2}R. \end{aligned}$$

□

Remark 3.2.19 (1) If we scale L with a factor of $|G|^{1/2}$ then it becomes the inverse of K. Under this bijective correspondence, the elements of $\mathrm{Hom}_{\mathcal{A}(I)}(E^{l/r}(\rho_1), E^{l/r}(\rho_2)) \cap \mathcal{A}^G(I)$ correspond to the elements

in $\mathrm{Hom}_{\mathcal{A}(I)}(E^{l/r}(\rho_1), E^{l/r}(\rho_2))$ of the form $w^*\lambda(S) = Sw^* = w^* \times S$.
(2) Analogous to K and L, we can also define linear maps

$$K' : \mathrm{Hom}_{\mathcal{A}(I)}(E^{l/r}(\rho_1), E^{l/r}(\rho_2)) \to \mathrm{Hom}_{\mathcal{A}^G(I)}(\rho_1, \lambda \otimes \rho_2)$$
$$S \mapsto \gamma(S)w$$

and

$$L' : \mathrm{Hom}_{\mathcal{A}^G(I)}(\rho_1, \lambda \otimes \rho_2) \to \mathrm{Hom}_{\mathcal{A}(I)}(E^{l/r}(\rho_1), E^{l/r}(\rho_2))$$
$$R \mapsto v^* R.$$

that satisfy $K'(L'(S)) = |G|^{-1/2}S$ and $L'(K'(R)) = |G|^{-1/2}R$. The details are almost the same as in the proof of the lemma above.

Müger's theorem

We will now consider the category

$$(\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2 := (\mathrm{Loc}_f(\mathcal{A}^G) \rtimes \mathcal{S}, c^{l/r})_2$$

as constructed in Subsection 3.1.3. This category was shown to be a $G_\mathcal{S}$-crossed category with a braiding of the second kind, where $G_\mathcal{S} = \mathrm{Aut}^*(\lambda) \cong G$ in the present case. However, this $G_\mathcal{S}$-crossed structure with $G_\mathcal{S} \cong G$ will not be good enough for our present purposes. Namely, we want to show that this category is equivalent as a group-crossed category to $G - \mathrm{Loc}_f^{(L,l)/(R,r)}(\mathcal{A})$ and for this we really need a concrete correspondence between $G_\mathcal{S}$ and G, so that we can make $(\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2$ into a G-crossed category and we can construct a functor between these two categories that is G-crossed. This can be done by using Lemma 3.2.14, which provided us with a group isomorphism $\phi : G \to G_\mathcal{S}^{\mathrm{op}}$, where ϕ was given by $\phi(q) = D_q$. We can use this ϕ to define a group isomorphism $\Gamma : G \to G_\mathcal{S}$ by

$$\Gamma(q) = \phi(q)^{-1} = D_q^{-1}.$$

The $G_\mathcal{S}$-action F^2 and the $G_\mathcal{S}$-grading ∂_2 on $(\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2$ as defined in Subsection 3.1.3 can now be turned into a G-action F and a G-grading ∂ by defining

$$F_q := F^2_{\Gamma(q)}$$
$$\partial := \Gamma^{-1} \circ \partial_2.$$

Because $(\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2$ is $G_\mathcal{S}$-crossed, we have for any $(\rho, p) \in (\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2$ and $q \in G$ that

$$\partial(F_q(\rho, p)) = \Gamma^{-1}(\partial_2(F^2_{\Gamma(q)}(\rho, p))) = \Gamma^{-1}(\Gamma(q)\partial_2(\rho, p)\Gamma(q)^{-1}) = q\partial(\rho, p)q^{-1},$$

showing that $(\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2$ is G-crossed. Note also that for any $(\rho, p) \in (\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2$ we have $F_{\partial(\rho,p)} = F^2_{\Gamma(\Gamma^{-1}(\partial_2(\rho,p)))} = F^2_{\partial_2(\rho,p)}$, which implies that the braiding on the $G_\mathcal{S}$-crossed category $(\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2$ remains a braiding on the corresponding G-crossed category. In what follows, we will always consider $(\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2$ as a G-crossed category.

The following theorem summarizes the main results of [78]. Since our approach to α-induction has been different from [78], we have included a proof of this theorem which agrees more with our perspective than the one in [78]. However, some parts of the proof can still be copied from [78] and are therefore not included here. In contrast to our conventions before, in this theorem we assume that the categories $G - \mathrm{Loc}_f^{L/R}(\mathcal{A})$ also contain all finite direct sums of homogeneous objects, because this was also the case for the crossed product category. Finally, it should be obvious that similar statements as in this theorem can also be made when the categories are equipped with a braiding of the first kind.

Theorem 3.2.20 *Let $(H, \mathcal{A}, \Omega, V)$ be a QFT on $\mathbb{R}$ with G-action, where G is a finite group, and suppose that $(H^G, \mathcal{B}, \Omega)$ is a QFT on $\mathbb{R}$.*

(1) There exist equivalences

$$\mathscr{K}^{l/r} : (\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2 \to G - \mathrm{Loc}_f^{(L,l)/(R,r)}(\mathcal{A})$$

of braided G-crossed categories.

(2) There exist strict braided tensor functors

$$\begin{aligned}\mathscr{E}^{l/r} &: \mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \to (G - \mathrm{Loc}_f^{(L,l)/(R,r)}(\mathcal{A}))^G \\ \mathscr{R}^{l/r} &: (G - \mathrm{Loc}_f^{(L,l)/(R,r)}(\mathcal{A}))^G \to \mathrm{Loc}_f^{l/r}(\mathcal{A}^G)\end{aligned}$$

that are inverse to each other.

(3) If $\mathscr{I} : \mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \to \mathrm{Loc}_f^{(L,l)/(R,r)}(\mathcal{A}^G) \rtimes \mathcal{S}$ denotes the inclusion functor, then $\mathscr{E}^{l/r} = \mathscr{K}^{l/r} \circ \mathscr{I}$, i.e. the diagram

$$\begin{array}{ccc} \mathrm{Loc}_f^{l/r}(\mathcal{A}^G) & \xrightarrow{\ \mathscr{I}\ } & (\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2 \\ & \searrow^{\mathscr{E}^{l/r}} & \downarrow{\scriptstyle \mathscr{K}^{l/r}} \\ & & G - \mathrm{Loc}_f^{(L,l)/(R,r)}(\mathcal{A}) \end{array}$$

commutes. In other words, $\mathscr{E}^{l/r}$ factors through $(\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2$.

Proof. By restriction of the functors in (3.2.14) to $\mathrm{Loc}_f^{l/r}(\mathcal{A}^G)$, we obtain functors $\mathscr{E}^{l/r} : \mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \to \mathrm{End}(\mathcal{A})^G$. We will now define a functor $\mathscr{K}_0^{l/r} : (\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes_0 \mathcal{S})_2 \to \mathrm{End}(\mathcal{A})$ that will later be extended to the functor $\mathscr{K}^{l/r}$ that appears in part (1) of the theorem. It follows from the diagram in part (3) that we have to define $\mathscr{K}_0^{l/r}$ on the objects of $(\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes_0 \mathcal{S})_2$ as

$$\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}} \rho J_0) := \mathscr{E}^{l/r}(\rho).$$

Concerning the morphisms of $(\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes_0 \mathcal{S})_2$, part (2) of Remark 3.2.19 indicates that we have to choose

$$\mathscr{K}_0^{l/r}(R) := \sqrt{|G|} v^* R,$$

where the normalization constant $\sqrt{|G|}$ ensures that $\mathscr{K}_0^{l/r}(\mathrm{id}_{\overline{J}_{\mathcal{S}} \rho J_0}) = \sqrt{|G|} v^* w = 1_{\mathscr{K}_0^{l/r}(\rho)}$. This indeed defines a functor, since

$$\begin{aligned}\mathscr{K}_0^{l/r}(R \bullet S) &= \mathscr{K}_0^{l/r}(\sqrt{|G|}\gamma(v)^* \lambda(R) S) = |G| v^* \gamma(v)^* \lambda(R) S = |G| v^* v^* \lambda(R) S \\ &= \sqrt{|G|}\sqrt{|G|} v^* R v^* S = \mathscr{K}_0^{l/r}(R) \mathscr{K}_0^{l/r}(S),\end{aligned}$$

where we have used that $vv = \gamma(v)v$ by the intertwining property of v. It is easy to see that $\mathscr{K}_0^{l/r}$ is $\mathbb{C}$-linear. Note that $\mathscr{K}_0^{l/r}$ is fully faithful by part (2) of Remark 3.2.19. It is clear that we have $\mathscr{E}^{l/r} = \mathscr{K}_0^{l/r} \circ \mathscr{I}$ on the objects. On the morphisms we have $\mathscr{K}_0^{l/r}(\mathscr{I}(R)) = \mathscr{K}_0^{l/r}(wR) = \sqrt{|G|} v^* w R = R = \mathscr{E}^{l/r}(R)$.

Because $\mathscr{E}^{l/r}$ is a strict tensor functor, we have for any objects $\overline{J}_{\mathcal{S}} \rho_1 J_0$ and $\overline{J}_{\mathcal{S}} \rho_2 J_0$

$$\begin{aligned}\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}} \rho_1 J_0 \circledast \overline{J}_{\mathcal{S}} \rho_2 J_0) &= \mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}} (\rho_1 \otimes \rho_2) J_0) = \mathscr{E}^{l/r}(\rho_1 \otimes \rho_2) = \mathscr{E}^{l/r}(\rho_1) \otimes \mathscr{E}^{l/r}(\rho_2) \\ &= \mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}} \rho_1 J_0) \otimes \mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}} \rho_2 J_0).\end{aligned}$$

To see that $\mathscr{K}_0^{l/r}$ respects the tensor product on the morphisms, we calculate for morphisms R and S with $R \in \mathrm{Hom}_{\mathcal{A}^G}(\rho_1, \lambda \otimes \rho_2)$

$$\begin{aligned}\mathscr{K}_0^{l/r}(R \circledast S) &= \sqrt{|G|}\sqrt{|G|}v^*\gamma(v)^*\lambda(c^{l/r}_{\rho_2,\lambda})R\rho_1(S) = |G|v^*v^*\lambda(c^{l/r}_{\rho_2,\lambda})R\rho_1(S)\\ &= |G|v^*v^*c^{l/r}_{\lambda,\lambda}\lambda(c^{l/r}_{\rho_2,\lambda})R\rho_1(S) = |G|v^*v^*\lambda(R)c^{l/r}_{\rho_1,\lambda}\rho_1(S)\\ &= |G|v^*Rv^*c^{l/r}_{\rho_1,\lambda}\rho_1(S) = |G|v^*R\left({c^{l/r}_{\rho_1,\lambda}}^{-1}v\right)^* E^{l/r}(\rho_1)(S)\\ &= \sqrt{|G|}\sqrt{|G|}v^*RE^{l/r}(\rho_1)(v^*S) = \mathscr{K}_0^{l/r}(R) \times \mathscr{K}_0^{l/r}(S),\end{aligned}$$

where we used (3.2.9) in the third step, naturality of the braiding in the fourth step and (3.2.13) in the seventh step. The functor is also a $*$-functor, since

$$\mathscr{K}_0^{l/r}(S^\times) = \sqrt{|G|}\sqrt{|G|}v^*\lambda(S^*)\gamma(v)w = |G|v^*\gamma(S^*v)w = |G|S^*vv^*w = \sqrt{|G|}S^*v = \mathscr{K}_0^{l/r}(S)^*.$$

We thus conclude that we have a fully faithful strict tensor $*$-functor $\mathscr{K}_0^{l/r} : (\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes_0 \mathcal{S})_2 \to \mathrm{End}(\mathcal{A})$ that satisfies $\mathscr{E}^{l/r} = \mathscr{K}_0^{l/r} \circ \mathscr{I}$.

This functor $\mathscr{K}_0^{l/r}$ will now be extended to a functor $\mathscr{K}^{l/r} : (\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2 \to \mathrm{End}(\mathcal{A})$ as follows. Let $(\overline{J}_{\mathcal{S}}\rho J_0, p) \in (\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes \mathcal{S})_2$. Then functoriality of $\mathscr{K}_0^{l/r}$ implies that $\mathscr{K}_0^{l/r}(p) = \sqrt{|G|}v^*p$ is a projection in $\mathcal{A}(I)$. We can thus choose an isometry[18] $u_{(\rho,p)} \in \mathcal{A}(I)$ such that $u_{(\rho,p)}u^*_{(\rho,p)} = \mathscr{K}_0^{l/r}(p) = \sqrt{|G|}v^*p$. In case p is the identity morphism of $\overline{J}_{\mathcal{S}}\rho J_0$, i.e. $p = w$ (in which case $\mathscr{K}_0^{l/r}(p) = 1$), we will always choose $u_{(\rho,p)} = 1$. In this way, we have an isometry $u_{(\rho,p)}$ for each object $(\overline{J}_{\mathcal{S}}\rho J_0, p)$. We then define the functor $\mathscr{K}^{l/r}$ on the objects by

$$\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0, p)(A) := u^*_{(\rho,p)}E^{l/r}(\rho)(A)u_{(\rho,p)} = u^*_{(\rho,p)}\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0)(A)u_{(\rho,p)}$$

and we notice that $u_{(\rho,p)} \in \mathrm{Hom}_{\mathcal{A}}(\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0, p), \mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0))$, since

$$\begin{aligned}u_{(\rho,p)}\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0, p)(A) &= \mathscr{K}_0^{l/r}(p)\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0)(A)u_{(\rho,p)} = \mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0)(A)\mathscr{K}_0^{l/r}(p)u_{(\rho,p)}\\ &= \mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0)(A)u_{(\rho,p)}.\end{aligned}$$

On the morphisms it is defined as

$$\mathscr{K}^{l/r}(R) := u^*_{(\rho_2,p_2)}\mathscr{K}_0^{l/r}(R)u_{(\rho_1,p_1)}$$

if $R : (\overline{J}_{\mathcal{S}}\rho_1 J_0, p_1) \to (\overline{J}_{\mathcal{S}}\rho_2 J_0, p_2)$. To see that $\mathscr{K}^{l/r}(R)$ is indeed a morphism from $\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho_1 J_0, p_1)$ to $\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho_2 J_0, p_2)$ in the category $\mathrm{End}(\mathcal{A})$, we note that for $A \in \mathcal{A}$ we have

$$\begin{aligned}&\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho_2 J_0, p_2)(A)\mathscr{K}^{l/r}(R)\\ &= u^*_{(\rho_2,p_2)}\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho_2 J_0)(A)u_{(\rho_2,p_2)}u^*_{(\rho_2,p_2)}\mathscr{K}_0^{l/r}(R)u_{(\rho_1,p_1)} = u^*_{(\rho_2,p_2)}\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho_2 J_0)(A)\mathscr{K}_0^{l/r}(p_2 \bullet R)u_{(\rho_1,p_1)}\\ &= u^*_{(\rho_2,p_2)}\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho_2 J_0)(A)\mathscr{K}_0^{l/r}(R \bullet p_1)u_{(\rho_1,p_1)} = u^*_{(\rho_2,p_2)}\mathscr{K}_0^{l/r}(R \bullet p_1)\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho_1 J_0)(A)u_{(\rho_1,p_1)}\\ &= u^*_{(\rho_2,p_2)}\mathscr{K}_0^{l/r}(R)u_{(\rho_1,p_1)}u^*_{(\rho_1,p_1)}\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho_1 J_0)(A)u_{(\rho_1,p_1)}\\ &= \mathscr{K}^{l/r}(R)\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho_1 J_0, p_1)(A)\end{aligned}$$

[18]We will simply write $u_{(\rho,p)}$, rather than the more cumbersome $u_{(\overline{J}_{\mathcal{S}}\rho J_0,p)}$.

where we have used that $p_2 \bullet R = R \bullet p_1$, as well as functoriality of $\mathscr{K}_0^{l/r}$. That $\mathscr{K}^{l/r}$ is a functor follows from

$$\begin{aligned}\mathscr{K}^{l/r}(S \bullet R) &= u^*_{(\rho_3,p_3)}\mathscr{K}_0^{l/r}(S \bullet R)u_{(\rho_1,p_1)} = u^*_{(\rho_3,p_3)}\mathscr{K}_0^{l/r}(S \bullet R \bullet p_1)u_{(\rho_1,p_1)}\\ &= u^*_{(\rho_3,p_3)}\mathscr{K}_0^{l/r}(S \bullet p_2 \bullet R)u_{(\rho_1,p_1)} = \mathscr{K}^{l/r}(S)\mathscr{K}^{l/r}(R).\end{aligned}$$

if $S : (\overline{J}_{\mathcal{S}}\rho_2 J_0, p_2) \to (\overline{J}_{\mathcal{S}}\rho_3 J_0, p_3)$. It is useful to observe that

$$u_{(\rho_2,p_2)}\mathscr{K}^{l/r}(R)u^*_{(\rho_1,p_1)} = \mathscr{K}_0^{l/r}(p)\mathscr{K}_0^{l/r}(R)\mathscr{K}_0^{l/r}(p) = \mathscr{K}_0^{l/r}(p \bullet R \bullet p),$$

which implies that $\mathscr{K}^{l/r}$ is fully faithful because $\mathscr{K}_0^{l/r}$ is fully faithful. Our choice $u_{(\rho,p)} = 1$ for the case where p is the identity morphism of $\overline{J}_{\mathcal{S}}\rho J_0$ guarantees that $\mathscr{K}^{l/r}$ is indeed an extension of $\mathscr{K}_0^{l/r}$. We thus conclude that $\mathscr{K}^{l/r}$ is a fully faithful $*$-functor.

To see that it can be made into a tensor functor, we first observe that $\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\iota_{\mathcal{A}^G}J_0, 1) = \iota_{\mathcal{A}}$ and that for any $A \in \mathcal{A}$ we have on the one hand

$$\begin{aligned}&[\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho_1 J_0, p_1) \otimes \mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho_2 J_0, p_2)](A)\\ &\qquad = u_1^*\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho_1 J_0)(u_2)^*\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}(\rho_1 \otimes \rho_2)J_0)\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho_1 J_0)(u_2)u_1\end{aligned}$$

and on the other

$$\mathscr{K}^{l/r}[(\overline{J}_{\mathcal{S}}\rho_1 J_0, p_1) \circledast (\overline{J}_{\mathcal{S}}\rho_2 J_0, p_2)](A) = u^*_{12}\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}(\rho_1 \otimes \rho_2)J_0)(A)u_{12},$$

where we write $u_j := u_{(\rho_j,p_j)}$ and $u_{12} := u_{(\rho_1\otimes\rho_2,p_1\otimes p_2)}$. Because we have both $u_{12}u_{12}^* = \mathscr{K}_0^{l/r}(p_1 \circledast p_2) = \mathscr{K}^{l/r}(p_1) \times \mathscr{K}^{l/r}(p_2)$ and

$$\begin{aligned}\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho_1 J_0)(u_2)u_1u_1^*\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho_1 J_0)(u_2)^* &= \mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho_1 J_0)(u_2)\mathscr{K}_0^{l/r}(p_1)\mathscr{K}_0^{l/r}(\overline{J}_{\mathcal{S}}\rho_1 J_0)(u_2)^*\\ &= \mathscr{K}^{l/r}(p_1) \times \mathscr{K}^{l/r}(p_2)\end{aligned}$$

(where we used the interchange law in the tensor category $\mathrm{End}(\mathcal{A})$), we see that the isometries u_{12} and $\mathscr{K}_0^{l/r}(\rho_1)(u_2)u_1$ have the same range projections. Hence we obtain a unitary

$$u^*_{12}\mathscr{K}_0^{l/r}(u_2)u_1 \in \mathrm{Hom}_{\mathcal{A}}\left(\mathscr{K}^{l/r}[(\overline{J}_{\mathcal{S}}\rho_1 J_0, p_1) \circledast (\overline{J}_{\mathcal{S}}\rho_2 J_0, p_2)], \mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho_1 J_0, p_1) \otimes \mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho_2 J_0, p_2)\right).$$

A straightforward computation shows that this gives $\mathscr{K}^{l/r}$ the structure of a tensor functor.

Next we will show that $\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0, p) \in G-\mathrm{Loc}_f^{(L,l)/(R,r)}(\mathcal{A})$ for any object $(\overline{J}_{\mathcal{S}}\rho J_0, p)$. It is enough to prove this for irreducible objects, so suppose that $(\overline{J}_{\mathcal{S}}\rho J_0, p) \in (\mathrm{Loc}_f^{l/r}(\mathcal{A}^G) \rtimes_0 \mathcal{S})_2$ is irreducible with degree $\partial_2(\overline{J}_{\mathcal{S}}\rho J_0, p)$. Assume that ρ is localized in $I \in \mathcal{K}$; we may also assume that $p \in \mathcal{A}^G(I)$ and hence also that $u_{(\rho,p)} \in \mathcal{A}(I)$. Consequently, $\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0, p)$ acts trivially on intervals that are to the right/left of I. Now suppose that $I_l, I_r \in \mathcal{K}$ with $I_l < I < I_r$ and let $\gamma^{l/r} \in \mathrm{End}(\mathcal{A})$ be constructed from an interval in $I_{l/r}$ with corresponding restriction $\lambda^{l/r}$ and isometries $w^{l/r}$ and $v^{l/r}$. Our task is to calculate $\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0, p)(v^{l/r})$ because any $A \in \mathcal{A}(I_{l/r})$ can be written as $Bv^{l/r}$ for some $B \in \mathcal{A}^G(I_{l/r})$. We proceed in the same way as in [78]. To keep the expressions readable, we will suppress the indices l/r and we will simply write $u := u_{(\rho,p)}$ all the time. We have

$$\begin{aligned}\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0, p)(v) &= u^*\mathscr{K}_0(\overline{J}_{\mathcal{S}}\rho J_0)(v)u = u^*c^{-1}_{\rho,\lambda}vu = u^*c^{-1}_{\lambda,\rho}c^{-1}_{\rho,\lambda}vu = E(\lambda)(u^*)c^{-1}_{\lambda,\rho}c^{-1}_{\rho,\lambda}E(\lambda)(u)v\\ &= \left[\mathscr{K}_0(\overline{J}_{\mathcal{S}}\lambda J_0)(u^*)c^{-1}_{\lambda,\rho}c^{-1}_{\rho,\lambda}\mathscr{K}_0(\overline{J}_{\mathcal{S}}\lambda J_0)(u)\right]v,\end{aligned}$$

where we have used that u and v commute by locality of the net $\mathcal{A}$, that $c_{\lambda,\rho} = 1$ and that $E(\lambda)$ acts trivially on $\mathcal{A}(I)$ and hence on u. Since $c^{-1}_{\lambda,\rho}c^{-1}_{\rho,\lambda} \in \mathrm{End}_{\mathcal{A}^G}(\lambda \otimes \rho)$, it follows from the tensor functoriality of $\mathscr{E}$ that we also have

$$c^{-1}_{\lambda,\rho}c^{-1}_{\rho,\lambda} \in \mathrm{End}_{\mathcal{A}}(\mathscr{K}_0(\overline{J}_{\mathcal{S}}(\lambda \otimes \rho)J_0)) = \mathrm{End}_{\mathcal{A}}(\mathscr{K}_0(\overline{J}_{\mathcal{S}}\lambda J_0) \otimes \mathscr{K}_0(\overline{J}_{\mathcal{S}}\rho J_0)).$$

Thus the expression in the square brackets above can be written as a morphism in the category $\mathrm{End}(\mathcal{A})$, namely

$$[1_{\mathscr{K}_0(\overline{J}_{\mathcal{S}}\lambda J_0)} \times u^*]c^{-1}_{\lambda,\rho}c^{-1}_{\rho,\lambda}[1_{\mathscr{K}_0(\overline{J}_{\mathcal{S}}\lambda J_0)} \times u] \in \mathrm{End}_{\mathcal{A}}(\mathscr{K}_0(\overline{J}_{\mathcal{S}}\lambda J_0) \otimes \mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0, p))$$

and noticing that $\mathscr{K}_0(\overline{J}_{\mathcal{S}}\lambda J_0)(A) = E(\lambda)(A) = \lambda(A) = \sum_{q\in G} v_q A v_q^*$ and hence that v_q is in $\mathrm{Hom}_{\mathcal{A}}(\iota_{\mathcal{A}}, \mathscr{K}_0(\overline{J}_{\mathcal{S}}\lambda J_0))$, we find that $v_q^*\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0, p)(v) \in \mathrm{End}_{\mathcal{A}}(\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0, p)) = \mathbb{C}1_{\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0,p)}$ by irreducibility of $(\overline{J}_{\mathcal{S}}\rho J_0, p)$. Thus it can be calculated as[19]

$$\begin{aligned}
v_q^*\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0, p)(v) &= d(\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0, p))^{-1}\mathrm{Tr}_{\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0,p)}\left[v_q^*\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0, p)(v)\right] \\
&= d(\overline{J}_{\mathcal{S}}\rho J_0, p)^{-1}\mathrm{Tr}_{\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0,p)}\left[v_q^* u^* c^{-1}_{\lambda,\rho}c^{-1}_{\rho,\lambda} uv\right] \\
&= d(\overline{J}_{\mathcal{S}}\rho J_0, p)^{-1}\mathrm{Tr}_{\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0,p)}\left[u^* v_q^* c^{-1}_{\lambda,\rho}c^{-1}_{\rho,\lambda} vu\right] \\
&= d(\overline{J}_{\mathcal{S}}\rho J_0, p)^{-1}\mathrm{Tr}_{\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0,p)}[u^* \underbrace{(v_q^* \times 1_{\mathscr{K}(\rho,1_\rho)})c^{-1}_{\lambda,\rho}c^{-1}_{\rho,\lambda}(v \times 1_{\mathscr{K}(\rho,1_\rho)})}_{=:Z_q} u] \\
&= d(\overline{J}_{\mathcal{S}}\rho J_0, p)^{-1}\mathrm{Tr}_{\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0,p)}(u^* Z_q u) = d(\overline{J}_{\mathcal{S}}\rho J_0, p)^{-1}\mathrm{Tr}_{\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0,1_{\overline{J}_{\mathcal{S}}\rho J_0})}(u^* u Z_q) \\
&= d(\overline{J}_{\mathcal{S}}\rho J_0, p)^{-1}\mathrm{Tr}_{\mathscr{K}_0(\overline{J}_{\mathcal{S}}\rho J_0)}(\mathscr{K}_0(p) Z_q),
\end{aligned}$$

where at the end we used the cyclic property of the trace. This expression is useful because it allows us to calculate

$$\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0, p)(v) = \sum_{q\in G} v_q v_q^* \mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0, p)(v) = d(\overline{J}_{\mathcal{S}}\rho J_0, p)^{-1}\sum_{q\in G} v_q \mathrm{Tr}_{\mathscr{K}_0(\overline{J}_{\mathcal{S}}\rho J_0)}(\mathscr{K}_0(p)Z_q).$$

Note that this already shows that $\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0, p)(v) \in \mathrm{span}\{v_q : q \in G\} = \mathrm{span}\{\beta_q(v) : q \in G\}$, but we want to be more precise by calculating the trace. We have

$$\begin{aligned}
\mathrm{Tr}_{\mathscr{K}_0(\overline{J}_{\mathcal{S}}\rho J_0)}(\mathscr{K}_0(p)Z_q) &= \mathsf{R}^{(\mathscr{K}_0(\overline{J}_{\mathcal{S}}\rho J_0))}_{\iota,\iota}(\mathscr{K}_0(p)Z_q) = v_q^*\left\{\mathsf{R}^{(\mathscr{K}_0(\overline{J}_{\mathcal{S}}\rho J_0))}_{\iota,\iota}\left([1_{\mathscr{K}_0(\overline{J}_{\mathcal{S}}\lambda J_0)} \times \mathscr{K}_0(p)]c^{-1}_{\lambda,\rho}c^{-1}_{\rho,\lambda}\right)\right\} v \\
&=: v_q^* Y v = v_q^*\mathscr{K}_0(\mathscr{K}_0^{-1}(Y))v = v_q^* v^* \mathscr{K}_0^{-1}(Y)v,
\end{aligned}$$

where we have used that $\mathscr{K}_0$ is fully faithful. Substitution of this into the expression for $\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0, p)(v)$ above, we obtain

$$\begin{aligned}
\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0, p)(v) &= d(\overline{J}_{\mathcal{S}}\rho J_0, p)^{-1}\sum_{q\in G} v_q v_q^* v^* \mathscr{K}_0^{-1}(Y)v \\
&= d(\overline{J}_{\mathcal{S}}\rho J_0, p)^{-1}\sqrt{|G|}\sum_{q\in G} v_q v_q^* \gamma(v)^* \mathscr{K}_0^{-1}(Y)v \\
&= d(\overline{J}_{\mathcal{S}}\rho J_0, p)^{-1}\sqrt{|G|}\gamma(v)^*\mathscr{K}_0^{-1}(Y)v.
\end{aligned}$$

[19]Actually we should multiply all expressions on the right with $1_{\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0,p)}$ to be precise, but we will not do this in order to save space.

A straightforward calculation shows that the morphism ${\mathscr{K}_0}^{-1}(Y)$ corresponds to the element

$$c_{\lambda,\lambda}\left\{\mathsf{R}^{(\rho)}_{\lambda,\lambda\otimes\lambda}\left([1_\lambda\times p]c^{-1}_{\lambda,\rho}c^{-1}_{\rho,\lambda}\right)\right\}$$

of $\mathrm{Hom}_{\mathcal{A}^G}(\lambda,\lambda\otimes\lambda)$. When we compose this expression from the left with $\sqrt{|G|}\gamma(v)^*$ and use the fact that λ is a commutative $*$-Frobenius algebra, we obtain precisely $d(\overline{J}_{\mathcal{S}}\rho J_0,p)\cdot\partial_2(\overline{J}_{\mathcal{S}}\rho J_0,p)^{-1}$. Thus we conclude that

$$\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0,p)(v)=\partial_2(\overline{J}_{\mathcal{S}}\rho J_0,p)^{-1}v=\Gamma(\partial(\overline{J}_{\mathcal{S}}\rho J_0,p))^{-1}v=D_{\partial(\overline{J}_{\mathcal{S}}\rho J_0,p)}v=\beta_{\partial(\overline{J}_{\mathcal{S}}\rho J_0,p)}(v),$$

which proves that $\mathscr{K}(\overline{J}_{\mathcal{S}}\rho J_0,p)$ is indeed $\partial(\overline{J}_{\mathcal{S}}\rho J_0,p)$-localized. So we have shown that $\mathscr{K}^{l/r}$ is a fully faithful tensor $*$-functor $(\mathrm{Loc}^{l/r}_f(\mathcal{A}^G)\rtimes\mathcal{S})_2\to G-\mathrm{Loc}^{(L,l)/(R,r)}_f(\mathcal{A})$. Consequently, it now follows from part (3) that $\mathscr{E}^{l/r}$ is a functor $\mathrm{Loc}^{l/r}_f(\mathcal{A}^G)\to(G-\mathrm{Loc}^{(L,l)/(R,r)}_f(\mathcal{A}))^G$.

We next show that it is a G-functor. If $q\in G$ then for any $A\in\mathcal{A}$ we have

$$\mathscr{K}^{l/r}(F_q(\overline{J}_{\mathcal{S}}\rho J_0,p))(A)=\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0,F_q(p))(A)=u^*_{(\rho,F_q(p))}E^{l/r}(\rho)(A)u_{(\rho,F_q(p))}$$

and

$$[\beta_q(\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0,p))(A)]=\beta_q(u_{(\rho,p)})^*E^{l/r}(\rho)(A)\beta_q(u_{(\rho,p)}),$$

where we have used that $E^{l/r}(\rho)$ is G-invariant. In view of $\beta_q(u_{(\rho,p)})\beta_q(u_{(\rho,p)})^*=\beta_q(u_{(\rho,p)}u^*_{(\rho,p)})=\beta_q(\mathscr{K}_0^{l/r}(p))$ and

$$\begin{aligned}u_{(\rho,F_q(p))}u^*_{(\rho,F_q(p))}&=\mathscr{K}_0^{l/r}(F_q(p))=\mathscr{K}_0^{l/r}(D_q^{-1}p)=\sqrt{|G|}v^*D_q^*p=\sqrt{|G|}\beta_q(v)^*p\\&=\sqrt{|G|}\beta_q(v^*p)=\beta_q(\mathscr{K}_0^{l/r}(p)),\end{aligned}$$

the isometries $\beta_q(u_{(\rho,p)})$ and $u_{((\rho,F_q(p)))}$ have the same range projection. Hence we obtain a unitary

$$\beta_q(u_{(\rho,p)})^*u_{(\rho,F_q(p))}\in\mathrm{Hom}_{\mathcal{A}}\left(\mathscr{K}^{l/r}(F_q(\overline{J}_{\mathcal{S}}\rho J_0,p)),\beta_q(\mathscr{K}^{l/r}(\overline{J}_{\mathcal{S}}\rho J_0,p))\right)$$

and a straightforward calculation shows that this gives $\mathscr{K}^{l/r}$ the structure of a G-functor. We also need to prove that $\mathscr{K}^{l/r}$ is essentially surjective, but this is precisely Proposition 3.14 of [78], which includes a very clear proof. Checking that $\mathscr{K}^{l/r}$ is braided is just a computation and does not involve any subtleties; we leave this to the reader. The functor $\mathscr{R}^{l/r}$ is defined on the objects by $\mathscr{R}^{l/r}(\rho)=\rho|_{\mathcal{A}^G}$ and on the morphisms simply as $\mathscr{R}^{l/r}(R)=R$. It is clear that $\mathscr{R}^{l/r}(\rho)$ is localized in $I\in\mathcal{K}$ if ρ was G-localized in I, but transportability requires a little work, see Proposition 3.5 of [78]. The statement in part (2) can be found as Theorem 3.12 in [78].
□

Thus the category $G-\mathrm{Loc}^R_f(\mathcal{A})$ can be constructed up to equivalence from $\mathrm{Loc}_f(\mathcal{A}^G)$ and its subcategory $\mathcal{S}$ by taking the crossed product, and conversely $\mathrm{Loc}_f(\mathcal{A}^G)$ can be constructed from $G-\mathrm{Loc}^R_f(\mathcal{A})$ by considering the fixed-point subcategory.

3.2.4 The relation with conformal field theory

In this subsection we will briefly introduce the notion of a chiral conformal field theory. We will explain that such theories automatically give rise to the QFTs on $\mathbb{R}$ that we discussed in the preceding subsection. We will use this fact to make our discussion in Subsection 3.2.5 somewhat more natural.

If S^1 denotes the unit circle, we will write $\mathcal{I}$ to denote the collection of all non-empty and non-dense connected open subsets of S^1. A set $I\in\mathcal{I}$ can thus be interpreted as an open interval in S^1 and can be written as $I = (a, b)$ for $a, b \in S^1$ with $a \neq b$, where we use the convention that moving from a to b along I is in the counterclockwise direction on S^1. If $I \subset S^1$, then we write I' to denote the interior of $S^1 \backslash I$. Note that if $I \in \mathcal{I}$ then $I' \in \mathcal{I}$ and that if $I = (a, b)$, then $I' = (b, a)$. A subset $E \subset S^1$ is called an n-interval if it can be written as $E = I_1 \cup \ldots \cup I_n$ for some $I_j \in \mathcal{I}$ with $\overline{I_j} \cap \overline{I_k} = \emptyset$ for all $j \neq k$. The set of all n-intervals is denoted by $\mathcal{I}^n$; in particular $\mathcal{I}^1 = \mathcal{I}$.

Analogous to Definition 3.2.1 we can also define nets of von Neumann algebras on S^1, where the role of $\mathcal{K}$ is now taken by $\mathcal{I}$. The notions of locality, Haag duality, irreducibility, strong additivity and the split property are also defined in an anologous way for these nets, simply by replacing $\mathcal{K}$ by $\mathcal{I}$ and by replacing $I^{\perp}$ by I'. If $\mathcal{A}$ is a net of von Neumann algebras on S^1, then $\mathcal{A}$ is called n-regular if for any n points $\zeta_1, \ldots, \zeta_n \in S^1$ we have $\mathcal{A}(S^1 \backslash \{\zeta_1, \ldots, \zeta_n\}) = B(H)$. If G is a group, then a G-action on net $(H, \mathcal{A})$ of von Neumann algebras is defined in a similar way as for nets of von Neumann algebras on $\mathbb{R}$.

Definition 3.2.21 A *chiral conformal field theory* (or *chiral CFT* for short) is a quadruple $(H, \mathcal{A}, U, \Omega)$ consisting of a net $(H, \mathcal{A})$ of von Neumann algebras on S^1, a strongly continuous unitary representation $U : PSU(1, 1) = SU(1, 1)/\{1, -1\} \to B(H)$ of the Möbius group on H and a unit vector $\Omega \in H$ satisfying the following conditions:

- the net $(H, \mathcal{A})$ satisfies locality and irreducibility;
- for any $a \in PSU(1, 1)$ and $I \in \mathcal{I}$ we have $U(a)\mathcal{A}(I)U(a)^* = \mathcal{A}(aI)$;
- if L_0 denotes the generator of the rotation subgroup of $PSU(1, 1)$, then $L_0 \geq 0$;
- the subspace of $PSU(1, 1)$-invariant vectors in H equals $\mathbb{C}\Omega$.

If G is a topological group, then a G-action on a chiral CFT $(H, \mathcal{A}, U, \Omega)$ is a strongly continuous unitary representation $V : G \to B(H)$ that induces a G-action β on the net $(H, \mathcal{A})$ satisfying $V\Omega = \Omega$.

We briefly mention some consequences of these axioms. Let $(H, \mathcal{A}, U, \Omega)$ be a chiral CFT. Then it automatically satisfies additivity and Haag duality and for each $I \in \mathcal{I}$ the von Neumann algebra $\mathcal{A}(I)$ is a type III factor for which the vector Ω is both cyclic and separating. If $I \in \mathcal{I}$, then Haag duality gives

$$\mathcal{A}(I) \vee \mathcal{A}(I') = (\mathcal{A}(I)' \cap \mathcal{A}(I')')' = (\mathcal{A}(I)' \cap \mathcal{A}(I))' = Z(\mathcal{A}(I))' = B(H),$$

so $\mathcal{A}$ is 2-regular. In fact, this calculation shows that 2-regularity is equivalent to the statement that $\mathcal{A}(I)$ is a factor for all $I \in \mathcal{I}$. If $\mathcal{A}$ satisfies strong additivity, then it is n-regular for every $n \in \mathbb{Z}_{\geq 1}$. It then follows from proposition 1 in [52] that the inclusion $\mathcal{A}(E) \subset \mathcal{A}(E')'$ is irreducible for any n-interval $E \in \mathcal{I}^n$.

A representation $(H_\pi, \{\pi_I\}_{I \in \mathcal{I}})$ of a chiral CFT $(H, \mathcal{A}, U, \Omega)$ consists of a Hilbert space H_π together with a collection of unital representations $\pi_I : \mathcal{A}(I) \to B(H_\pi)$ satisfying the condition that $\pi_J|_{\mathcal{A}(I)} = \pi_I$ for all $I, J \in \mathcal{I}$ with $I \subset J$. It is easy to see that the category $\mathrm{Rep}(\mathcal{A})$ of representations of $\mathcal{A}$ (with morphisms given by the intertwiners between representations) is a C^*-category. As stated in [78], if $\mathcal{A}$ satisfies strong additivity then the index $[\pi_I(\mathcal{A}(I'))' : \pi_I(\mathcal{A}(I))]$ is independent of $I \in \mathcal{I}$ and we can therefore define the dimension

$$d(\pi) := [\pi_I(\mathcal{A}(I'))' : \pi_I(\mathcal{A}(I))]^{1/2} \in [1, \infty].$$

The full subcategory of $\mathrm{Rep}(\mathcal{A})$ determined by the representations with finite dimension is denoted by $\mathrm{Rep}_f(\mathcal{A})$.

If $(H, \mathcal{A}, U, \Omega)$ is a chiral CFT satisfying strong additivity and the split property, then for any $E \in \mathcal{I}^2$ the index $[\mathcal{A}(E')' : \mathcal{A}(E)]$ is independent of the choice of E, as shown in proposition 5 of [52]. So if $(H, \mathcal{A}, U, \Omega)$ is a chiral CFT satisfying strong additivity and the split property, then we can define the quantity

$$\mu(\mathcal{A}) := [\mathcal{A}(E')' : \mathcal{A}(E)] \in [1, \infty],$$

where $E \in \mathcal{I}^2$ is an arbitrarily chosen 2-interval. This quantity is called the μ-index of $\mathcal{A}$. A chiral CFT satisfying strong additivity and the split property is called *completely rational* if its μ-index is finite. If G is a finite group acting on a completely rational chiral CFT $(H, \mathcal{A}, U, \Omega)$, then the orbifold theory $(H^G, \mathcal{B}, U|_{H^G}, \Omega)$ is again a completely rational chiral CFT, as shown in proposition 4.2 of [109].

We will now discuss the relation between chiral CFTs on S^1 and QFTs on $\mathbb{R}$. Let $(H, \mathcal{A}, U, \Omega)$ be a chiral CFT. If $\zeta \in S^1$, then we can identify $S^1 \backslash \{\zeta\}$ with $\mathbb{R}$ by the use of a stereographic projection. Recall that on $\mathbb{R}$ we denoted the collection of bounded open intervals by $\mathcal{K}$; the corresponding collection of subsets of $S^1 \backslash \{\zeta\}$ will be denoted by $\mathcal{K}_\zeta$. Note that $\mathcal{K}_\zeta \subset \mathcal{I}$, so we obtain a net $(H, \mathcal{A}_\zeta)$ of von Neumann algebras on $\mathbb{R}$ (or rather $S^1 \backslash \{\zeta\}$) by defining $\mathcal{A}_\zeta(I) := \mathcal{A}(I)$ for all $I \in \mathcal{K}_\zeta$. It follows directly that for each $I \in \mathcal{K}_\zeta$ the von Neumann algebra $\mathcal{A}_\zeta(I)$ is a type III factor for which the vector Ω is cyclic and separating and it is also clear that the net $\mathcal{A}_\zeta$ satisfies half-line duality. Now suppose that $\mathcal{A}$ is also strongly additive. Then $\mathcal{A}$ is in particular 1-regular, so it follows from irreducibility of $\mathcal{A}$ that $\mathcal{A}_\zeta$ is also irreducible. It is also obvious that $\mathcal{A}_\zeta$ satisfies strong additivity. Now suppose that $I \in \mathcal{K}_\zeta$. Then

$$\mathcal{A}_\zeta(I^\perp)' = \mathcal{A}(I^\perp)' = \mathcal{A}(I^\perp \cup \{\zeta\})' = \mathcal{A}(I')' = \mathcal{A}(I) = \mathcal{A}_\zeta(I),$$

where in the second step we used strong additivity of the net $\mathcal{A}$ and in the fourth step we used Haag duality of the net $\mathcal{A}$. This shows that the net $\mathcal{A}_\zeta$ satisfies Haag duality as well, and hence that $(H, \mathcal{A}_\zeta, \Omega)$ is a QFT on $\mathbb{R}$. As shown in theorem 2.31 of [78], we have equivalences

$$\begin{aligned} \mathrm{Rep}(\mathcal{A}) &\simeq \mathrm{Loc}(\mathcal{A}_\zeta) \\ \mathrm{Rep}_f(\mathcal{A}) &\simeq \mathrm{Loc}_f(\mathcal{A}_\zeta) \end{aligned}$$

of $*$-categories, which allow us to transport the braided tensor structure from $\mathrm{Loc}_{(f)}(\mathcal{A}_\zeta)$ to $\mathrm{Rep}_{(f)}(\mathcal{A})$.

If $(H, \mathcal{A}, U, \Omega, V)$ is completely rational with action V of a finite group G, then $(H^G, \mathcal{B}, U|_{H^G}, \Omega)$ is again completely rational, as mentioned above. So in this case it follows that $(H, \mathcal{A}_\zeta, \Omega, V)$ is a QFT on $\mathbb{R}$ with G-action V and that $(H^G, \mathcal{B}_\zeta, \Omega)$ is automatically a QFT on $\mathbb{R}$, where $\mathcal{B}_\zeta(I) := \mathcal{A}_\zeta^G(I)|_{H^G}$. This is precisely the setting that we assumed in the previous subsection, showing that it was not an artificial situation. Furthermore, the categories $\mathrm{Loc}_f(\mathcal{A}_\zeta)$ and $\mathrm{Loc}_f(\mathcal{B}_\zeta)$ are modular and $\mu(\mathcal{A}) = \dim(\mathrm{Loc}_f(\mathcal{A}_\zeta))$. If for any $q \in G$ we choose a complete set of representatives $\rho_{q,1}, \ldots, \rho_{q,N_q}$ of equivalence classes of irreducible objects in $G - \mathrm{Loc}_f^{L/R}(\mathcal{A}_\zeta)_q$, then we have

$$\mu(\mathcal{A}) = \sum_{i=1}^{N_q} d(\rho_{q,i})^2, \tag{3.2.15}$$

see also theorem 4.2 of [78]. Since $d(\rho) \geq 1$, a special case occurs when $\mu(\mathcal{A}) = 1$. Namely, if $\mu(\mathcal{A}) = 1$, then for every $q \in G$ there is precisely one equivalence class of irreducible objects of degree q and any irreducible object is invertible. These special kind of completely rational chiral CFTs $\mathcal{A}$ are called *holomorphic*.

3.2.5 Our main problem and the idea of a G-crossed Drinfeld center

Let G be a finite group and let $(H, \mathcal{A}, U, \Omega, V)$ be a completely rational chiral CFT with a G-action. If $\zeta \in S^1$, then $(H, \mathcal{A}_\zeta, \Omega, V)$ is a QFT on $\mathbb{R}$ with a G-action and $\mathrm{Loc}_f^{l/r}(\mathcal{A}_\zeta)$ is a modular tensor category. This modular tensor category has G-crossed extensions $G - \mathrm{Loc}_f^{L/R}(\mathcal{A}_\zeta)$ of left/right G-localized endomorphisms and these have full G-spectrum. More precisely, $\mathrm{Loc}_f^l(\mathcal{A}_\zeta)$ has braided G-crossed extensions $G - \mathrm{Loc}_f^{(L,l)}(\mathcal{A}_\zeta)$ and $G - \mathrm{Loc}_f^{(R,l)}(\mathcal{A}_\zeta)$, the first of which has a braiding of the second kind and the second of which has a braiding of the first kind,

$$
\begin{array}{lll}
 & G-\mathrm{Loc}_f^{(L,l)}(\mathcal{A}_\zeta) & \text{(left } G\text{-localization; braiding of the second kind)} \\
 & \nearrow & \\
\mathrm{Loc}_f^l(\mathcal{A}_\zeta) & & \\
 & \searrow & \\
 & G-\mathrm{Loc}_f^{(R,l)}(\mathcal{A}_\zeta) & \text{(right } G\text{-localization; braiding of the first kind)}
\end{array}
$$

and $\mathrm{Loc}_f^r(\mathcal{A}_\zeta)$ has braided G-crossed extensions $G-\mathrm{Loc}_f^{(L,r)}(\mathcal{A}_\zeta)$ and $G-\mathrm{Loc}_f^{(R,r)}(\mathcal{A}_\zeta)$, the first of which has a braiding of the first kind and the second of which has a braiding of the second kind,

$$
\begin{array}{lll}
 & G-\mathrm{Loc}_f^{(L,r)}(\mathcal{A}_\zeta) & \text{(left } G\text{-localization; braiding of the first kind)} \\
 & \nearrow & \\
\mathrm{Loc}_f^r(\mathcal{A}_\zeta) & & \\
 & \searrow & \\
 & G-\mathrm{Loc}_f^{(R,r)}(\mathcal{A}_\zeta) & \text{(right } G\text{-localization; braiding of the second kind).}
\end{array}
$$

The orbifold net $(H^G, \mathcal{B}_\zeta, \Omega)$ is automatically a QFT on $\mathbb{R}$ and $\mathrm{Loc}_f^{l/r}(\mathcal{B}_\zeta) \simeq \mathrm{Loc}_f^{l/r}(\mathcal{A}_\zeta^G)$ is also a modular tensor category, where $\mathcal{B}_\zeta(I) = \mathcal{A}_\zeta(I)^G|_{H^G}$. In particular, we are in the situation of Subsection 3.2.3. In Theorem 3.2.20 we have seen that $G-\mathrm{Loc}_f^{(L/R,\, l/r)}(\mathcal{A}_\zeta)$ can be constructed up to equivalence as the crossed product of $\mathrm{Loc}_f^{l/r}(\mathcal{A}_\zeta^G)$ with a certain symmetric tensor subcategory $\mathcal{S}$ which satisfied $\mathcal{S} \simeq \mathrm{Rep}_f(G)$.

As already mentioned in Section 1.2 of the introduction, at the beginning of this project we wondered whether it was possible to construct $G-\mathrm{Loc}_f^{(L/R,\, l/r)}(\mathcal{A}_\zeta)$ already from $\mathrm{Loc}_f^{l/r}(\mathcal{A}_\zeta)$ in the special case of the permutation action of S_N on $\mathcal{A}^{\otimes N}$. We explained in Section 1.2 that more recent results imply that such a construction does not exist, or at least not without any further assumptions on $\mathcal{A}$. Since we were not aware of this at the time, in the rest of this subsection we will briefly display our reasoning at the time of our ignorance, mainly because it will motivate our more abstract construction in the following chapter. As this construction goes beyond permutation groups, we will continue our discussion of AQFT here in a more general setting than the setting of permutation actions on the N-fold tensor product of an AQFT, although this more general setting was never motivated by our expectations concerning AQFT.

Suppose that the chiral CFT that we started with has the property that it is possible to construct $G-\mathrm{Loc}_f^{(L/R,\, l/r)}(\mathcal{A}_\zeta)$ categorically from $\mathrm{Loc}_f^{l/r}(\mathcal{A}_\zeta)$ together with its G-action[20]. For simplicity, we will write $\mathcal{C} = \mathrm{Loc}_f^l(\mathcal{A}_\zeta)$ and $\widetilde{\mathcal{C}} = \mathrm{Loc}_f^r(\mathcal{A}_\zeta)$. Then $\mathcal{C}$ has braided G-crossed extensions $\mathcal{D}_1 := G-\mathrm{Loc}_f^{(R,l)}(\mathcal{A}_\zeta)$ (with a braiding of the first kind) and $\mathcal{D}_2 := G-\mathrm{Loc}_f^{(L,l)}(\mathcal{A}_\zeta)$ (with a braiding of the second kind). Similarly, $\widetilde{\mathcal{C}}$ has braided G-crossed extensions $\widetilde{\mathcal{D}}_1 := G-\mathrm{Loc}_f^{(L,r)}(\mathcal{A}_\zeta)$ (with a braiding of the first kind) and $\widetilde{\mathcal{D}}_2 := G-\mathrm{Loc}_f^{(R,r)}(\mathcal{A}_\zeta)$ (with a braiding of the second kind). Note that the notation $\widetilde{\mathcal{C}}$ is consistent with the one introduced after equation (2.4.1) in Section 2.4. Furthermore, $\widetilde{\mathcal{D}}_1$ and $\widetilde{\mathcal{D}}_2$ can be obtained from $\mathcal{D}_2$ and $\mathcal{D}_1$, respectively, by using the procedure in part (2) of Remark 2.8.22. We thus see that there is some redundancy here and we can continue by considering only $\mathcal{C}$, $\mathcal{D}_1$ and $\mathcal{D}_2$. Note that the equivalence $\mathrm{Loc}_f(\mathcal{A}_\zeta) \simeq \mathrm{Rep}_f(\mathcal{A})$ mentioned in the preceding subsection means that we are looking for a G-crossed

[20] Note that this G-action on $\mathrm{Loc}_f^{l/r}(\mathcal{A}_\zeta)$ is completely determined by N in the case of the S_N action on the N-fold product of an AQFT. In the general setting the information analogous to the number N seems to be the G-action on $\mathrm{Loc}_f^{l/r}(\mathcal{A}_\zeta)$.

extension of a category of representations in AQFT. Hence also the title of this thesis. Forgetting about AQFT for the moment, we are thus in the situation where we are given a modular tensor category $\mathcal{C}$ with a G-action and we want to find two categorical constructions that give us the categories $\mathcal{D}_1$ and $\mathcal{D}_2$ out of $\mathcal{C}$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are G-crossed extensions of $\mathcal{C}$ with braidings of the first and second kind, respectively, that both extend the braiding of $\mathcal{C}$.[21] Finding such constructions has become our main problem.

The first question one might ask is how these two constructions differ from one another. In AQFT the main the difference between $\mathcal{D}_1$ and $\mathcal{D}_2$ is that $\mathcal{D}_1$ consists of right G-localized endomorphisms and that $\mathcal{D}_2$ consists of left G-localized endomorphisms. However, categorically we do not have a notion of left and right G-localization and the only structural difference between $\mathcal{D}_1$ and $\mathcal{D}_2$ is the type of braiding. This leads us to believe that the braiding might play a leading role in both constructions of G-extensions. Namely, the definition of the extra objects that are added to $\mathcal{C}$ in the construction of $\mathcal{D}_1$ should be such that it is somehow obvious from the beginning that we will end up with a braiding of the first kind.[22] A typical example of a known construction of a braided tensor category where the definition of the objects is such that the existence of the braiding is almost obvious, is the construction of the Drinfeld center. Indeed, the objects of the Drinfeld center of a tensor category are pairs (V, Φ_V), where V is an object in the given tensor category and Φ_V is a half braiding for V, see also Subsection 2.4.2. In case a tensor category is already braided, it is contained in its Drinfeld center as a braided tensor subcategory. Now recall that a half braiding behaves in the same way as a braiding for which one of the arguments is kept fixed. This gave us the idea to consider the construction of the Drinfeld center for a G-category, but with half braidings of the type that arise when one of the arguments is held fixed in the braiding of a braided G-crossed category. This construction of a more general Drinfeld center will be the subject of the next chapter.

[21]We cannot expect that such extensions will always have full G-spectra, because it is known that not every G-modular tensor category has a braided G-crossed extension with full G-spectrum.

[22]Note that in AQFT the categories $\mathcal{D}_1$ and $\mathcal{D}_2$ have different objects. In fact, in AQFT it is impossible to define a braiding of the second kind on $\mathcal{D}_1$ that extends c^l (rather than c^r).

Chapter 4

A G-crossed generalization of the Drinfeld center

As we explained at the very end of the preceding chapter, the situation of a completely rational chiral CFT with a group action motivated us to look for a G-crossed version of the Drinfeld center. The starting point in this chapter will be the construction of such a G-crossed Drinfeld center $Z_G(\mathcal{C})$ of a strict G-category $\mathcal{C}$. We will then repeat this construction also for the non-strict case and we will show that if $\mathcal{C}$ and $\mathcal{C}'$ are (non-strict) G-categories with $\mathcal{C} \simeq \mathcal{C}'$, then $Z_G(\mathcal{C}) \simeq Z_G(\mathcal{C}')$. The detailed proofs in the non-strict case are extremely involved and are therefore not included here; the interested reader is referred to Appendices A and B. Next, we will prove an equivalence between $Z_G(\mathcal{C})$ and a certain relative Drinfeld center and we will show that $Z_G(\mathcal{C})$ is equivalent to a category of functors of bimodule categories. After that we will show that some properties of $\mathcal{C}$ carry over to $Z_G(\mathcal{C})$, such as having direct sums and subobjects, being spherical and being semisimple. To make things a little less abstract we will also consider a concrete example of the construction of $Z_G(\mathcal{C})$ for some particular class of G-categories. This example turns out to have some relevance for completely rational chiral CFTs that are holomorphic. Finally, we will consider some remarkable properties of $Z_G(\mathcal{C})$ in case $\mathcal{C}$ has a braiding.

Nearly everything in this chapter is new[1], but some results are not that impressive. For instance, the statements in Sections 4.1, 4.6, 4.7 and 4.9 are just straightforward generalizations of the ones in [75]. Our deeper results in this chapter are Theorem 4.3.4, Theorem 4.4.7, Theorem 4.5.4, Theorem 4.5.5, Theorem 4.8.2, Proposition 4.10.3, Theorem 4.10.4, Proposition 4.10.6, Theorem 4.10.7 and Theorem 4.10.12.

4.1 Half braidings in G-categories

We have seen that for a strict tensor category $\mathcal{C}$ the Drinfeld center $Z(\mathcal{C})$ is constructed by considering pairs (V, Φ_V), where $V \in \mathcal{C}$ and Φ_V is a half braiding for V. In case $\mathcal{C}$ has a strict G-action, it is easy to see that if Φ_V is a half-braiding for $V \in \mathcal{C}$, then so is $\mathcal{F}_q\Phi_V$, where we define

$$\mathcal{F}_q\Phi_V(X) := F_q(\Phi_V(F_{q^{-1}}(X)))$$

for $X \in \mathcal{C}$. This implies that we can define a G-action $\mathcal{F}$ on $Z(\mathcal{C})$ by

$$\mathcal{F}_q[(V, \Phi_V)] := (F_q(V), \mathcal{F}_q\Phi_V)$$

[1] At the final stage of this research project we learned that our construction of $Z_G(\mathcal{C})$ was already carried out by Barvels in [6]. However, the only overlap between [6] and the present chapter is the construction of $Z_G(\mathcal{C})$ (in the strict case). What surprised the author was that the exact same notation $Z_G(\mathcal{C})$ was used in [6]. Apparently, this notation is very natural to choose, although one could of course argue that the notation $Z_{G,F}(\mathcal{C})$ would be more precise, where F denotes the group action $F : G \to \mathrm{Aut}^{\otimes}(\mathcal{C})$.

and

$$\mathcal{F}_q(f) = F_q(f)$$

for any $(V, \Phi_V) \in Z(\mathcal{C})$ and $f \in \mathrm{Hom}(Z(\mathcal{C}))$. Hence $Z(\mathcal{C})$ becomes a G-category in case $\mathcal{C}$ was a G-category. Now recall that in a braided G-crossed category $\mathcal{D}$ the degree $q = \partial(V)$ of an object $V \in \mathcal{D}$ occurs in the target object of the braiding $c_{V,X} : V \otimes X \to F_q(X) \otimes V$ for any $X \in \mathcal{D}$. This suggests that we can perhaps define a G-grading on a more rich version of $Z(\mathcal{C})$ by allowing more exotic kinds of half braidings in the G-category $\mathcal{C}$, namely the half braidings of the type that arise when one of the arguments of a braiding in a braided G-crossed category is held fixed. Indeed, such exotic half braidings can indeed be defined in G-categories, as we will see in the next definition. The fact that there are two possible kinds of braidings in G-crossed categories, as we have seen in Subsection 2.8.3, is reflected by the fact that we can also define two kinds of half braidings[2], which for obvious reasons will be called half braidings of the first and second kind.

Definition 4.1.1 Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category with strict G-action F.

(1) If $V \in \mathcal{C}$ and $q \in G$, then a *half q-braiding (of the first kind [3]) for V* is a natural isomorphism

$$\Phi_V : \otimes \circ (V \times \mathrm{id}_{\mathcal{C}}) \quad \to \quad \otimes \circ (F_q \times V)$$

of functors $\mathcal{C} \to \mathcal{C}$, i.e. a family $\{\Phi_V(X) : V \otimes X \to F_q(X) \otimes V\}_{X \in \mathcal{C}}$ of isomorphisms in $\mathcal{C}$ such that for all $X, Y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the square

$$\begin{array}{ccc} V \otimes X & \xrightarrow{\Phi_V(X)} & F_q(X) \otimes V \\ \downarrow{\scriptstyle \mathrm{id}_V \otimes f} & & \downarrow{\scriptstyle F_q(f) \otimes \mathrm{id}_V} \\ V \otimes Y & \xrightarrow{\Phi_V(Y)} & F_q(Y) \otimes V \end{array}$$

commutes, satisfying the additional property that for all $X, Y \in \mathcal{C}$ we have

$$\Phi_V(X \otimes Y) = [\mathrm{id}_{F_q(X)} \otimes \Phi_V(Y)] \circ [\Phi_V(X) \otimes \mathrm{id}_Y]. \tag{4.1.1}$$

(2) If $V \in \mathcal{C}$ and $q \in G$, then a *half q-braiding[4] of the second kind for V* is a natural isomorphism

$$\Psi_V : \otimes \circ (\mathrm{id}_{\mathcal{C}} \times V) \quad \to \quad \otimes \circ (V \times F_{q^{-1}})$$

of functors $\mathcal{C} \to \mathcal{C}$, i.e. a family $\{\Psi_V(X) : X \otimes V \to V \otimes F_{q^{-1}}(X)\}_{X \in \mathcal{C}}$ of isomorphisms in $\mathcal{C}$ such that for all $X, Y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the square

$$\begin{array}{ccc} X \otimes V & \xrightarrow{\Psi_V(X)} & V \otimes F_{q^{-1}}(X) \\ \downarrow{\scriptstyle f \otimes \mathrm{id}_V} & & \downarrow{\scriptstyle \mathrm{id}_V \otimes F_{q^{-1}}(f)} \\ Y \otimes V & \xrightarrow{\Psi_V(Y)} & V \otimes F_{q^{-1}}(Y) \end{array}$$

[2] Of course, in the construction of the Drinfeld center $Z(\mathcal{C})$ from a tensor category $\mathcal{C}$ we could also define two kinds of half braidings on $\mathcal{C}$, depending on whether we fix the left or right argument in the definition of a braiding. However, although there were already two possible definitions of a half braiding here, there is only one definition of a braiding, so both kinds of half braidings lead to the same kind of braiding for $Z(\mathcal{C})$. In the case where there is a G-action, choosing half braidings of the first (or second) kind, will lead to a G-crossed category $Z_G(\mathcal{C})$ with a braiding of the first (respectively second) kind.

[3] As for braidings in a G-crossed category, we will only write "of the first kind" when we want to distinguish it explicitly from a half q-braiding of the second kind; otherwise we will just call it a half q-braiding for V.

[4] Despite the fact that q^{-1} occurs here (rather than q), we will not call this a half q^{-1}-braiding of the second kind for V. This is because in a G-crossed category that has a braiding of the second kind, the object V would have degree q and not q^{-1}.

commutes, satisfying the additional property that for all $X, Y \in \mathcal{C}$ we have

$$\Psi_V(X \otimes Y) = [\Psi_V(X) \otimes \mathrm{id}_{F_{q^{-1}}(Y)}] \circ [\mathrm{id}_X \otimes \Psi_V(Y)]. \quad (4.1.2)$$

Remark 4.1.2 (1) Note that if Φ_V is a half q-braiding for V, then $\Phi_V(I) = \Phi_V(I \otimes I) = [\mathrm{id}_{F_q(I)} \otimes \Phi_V(I)] \circ [\Phi_V(I) \otimes \mathrm{id}_I] = \Phi_V(I) \circ \Phi_V(I)$, which implies that

$$\Phi_V(I) = \mathrm{id}_V. \quad (4.1.3)$$

We will often use this simple result in our calculations. For half q-braidings of the second kind a similar reasoning leads to $\Psi_V(I) = \mathrm{id}_V$.
(2) The unit object always has half e-braidings Φ_I^0 and Ψ_I^0 of the first and second kind, respectively, where $\Phi_I^0(X) = \Psi_I(X) = \mathrm{id}_X$ for all $X \in \mathcal{C}$. The notation Φ_I^0 and Ψ_I^0 will be used in the sequel.

As the following proposition shows, there is an operation that assigns half braidings of the second kind to half braidings of the first kind, and vice versa. The proof is a simple calculation that we omit here.

Proposition 4.1.3 *Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category with strict G-action F and let $V \in \mathcal{C}$ and $q \in G$.*

(1) If Φ_V is a half q-braiding for V, then $\Psi_V(-) := \Phi_V(F_{q^{-1}}(-))^{-1}$ is a half q-braiding of the second kind for V.

(2) If Ψ_V is a half q-braiding of the second kind for V, then $\Phi_V(-) := \Psi_V(F_q(-))^{-1}$ is a half q-braiding for V.

The two operations above are inverse to each other and thus establish a one-to-one correspondence between the two kinds of half braidings.

One might wonder what happens when $\mathcal{C}$ already is a braided G-crossed category and thus all its objects have a degree. Is it possible for an object of degree q to have a half r-braiding with $q \neq r$? The following proposition shows that this is certainly impossible if the center of the group G is trivial and if the G-spectrum $\partial(\mathcal{C}) \subset G$ is all of G.

Proposition 4.1.4 *Let $(\mathcal{C}, \otimes, I, G, F, \partial)$ be a G-crossed category and let $V \in \mathcal{C}$ with $\partial(V) = q$. If Φ_V is a half r-braiding for V, then $r^{-1}q$ commutes with $\partial(\mathcal{C})$. In particular, if $\partial(\mathcal{C}) = G$ and $Z(G) = \{e\}$, then $q = r$.*

Proof. For all $X \in \mathcal{C}$ we have an isomorphism $\Phi_V(X) : V \otimes X \to F_r(X) \otimes V$. Since ∂ is constant on isomorphism classes, we must have $\partial(V \otimes X) = \partial(F_r(X) \otimes V)$, or $q\partial(X) = r\partial(X)r^{-1}q$, or $r^{-1}q\partial(X) = \partial(X)r^{-1}q$.
□

4.1.1 Tensor products and group actions

The following lemma will be crucial in Section 4.2 when we want to define a tensor product and a G-action on our category $Z_G(\mathcal{C})$. In this lemma we will write down all statements for both kinds of half braidings. However, we will only prove these statements for half braidings of the first kind, since the proofs for half braidings of the second kind are completely analogous.

Lemma 4.1.5 *Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category with strict action F of the group G. Let $V \in \mathcal{C}$ and let Φ_V and Ψ_V be a half q-braiding and a half q-braiding of the second kind for V, respectively.*

(1) Let $W \in \mathcal{C}$ and $r \in G$. If Φ_W is a half r-braiding for W and if Ψ_W is a half r-braiding of the second kind for W, then we obtain a half qr-braiding $\Phi_V \otimes \Phi_W$ for $V \otimes W$ and a half qr-braiding of the second kind $\Psi_V \otimes \Psi_W$ for $V \otimes W$ by defining

$$\begin{aligned} (\Phi_V \otimes \Phi_W)(X) &:= [\Phi_V(F_r(X)) \otimes \mathrm{id}_W] \circ [\mathrm{id}_V \otimes \Phi_W(X)] \\ (\Psi_V \otimes \Psi_W)(X) &:= [\mathrm{id}_V \otimes \Psi_W(F_{q^{-1}}(X))] \circ [\Psi_V(X) \otimes \mathrm{id}_W]. \end{aligned}$$

(2) If $r \in G$, we obtain a half rqr^{-1}-braiding $\mathcal{F}_r\Phi_V$ for $F_r(V)$ and a half rqr^{-1}-braiding of the second kind $\mathcal{F}_r\Psi_V$ for $F_r(V)$ by defining

$$\begin{aligned}(\mathcal{F}_r\Phi_V)(X) &:= F_r(\Phi_V(F_{r^{-1}}(X)))\\ (\mathcal{F}_r\Psi_V)(X) &:= F_r(\Psi_V(F_{r^{-1}}(X)))\end{aligned}$$

Proof. (1) For each $X \in \mathcal{C}$ the morphism $(\Phi_V \otimes \Phi_W)(X) \in \mathrm{Hom}_{\mathcal{C}}(V \otimes W \otimes X, F_{qr}(X) \otimes V \otimes W)$ is an isomorphism, because $\Phi_V(-)$ and $\Phi_W(-)$ are isomorphisms. To check naturality of $\Phi_V \otimes \Phi_W$, let $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and consider the diagram

$$\begin{array}{ccccc} V\otimes W\otimes X & \xrightarrow{\mathrm{id}_V\otimes\Phi_W(X)} & V\otimes F_r(X)\otimes W & \xrightarrow{\Phi_V(F_r(X))\otimes\mathrm{id}_W} & F_{qr}(X)\otimes V\otimes W\\ \downarrow{\scriptstyle \mathrm{id}_{V\otimes W}\otimes f} & & \downarrow{\scriptstyle \mathrm{id}_V\otimes F_r(f)\otimes\mathrm{id}_W} & & \downarrow{\scriptstyle F_{qr}(f)\otimes\mathrm{id}_{V\otimes W}}\\ V\otimes W\otimes Y & \xrightarrow{\mathrm{id}_V\otimes\Phi_W(Y)} & V\otimes F_r(Y)\otimes W & \xrightarrow{\Phi_V(F_r(Y))\otimes\mathrm{id}_W} & F_{qr}(Y)\otimes V\otimes W.\end{array}$$

The left square commutes by naturality of Φ_W and the right square commutes by naturality of Φ_V, hence the big outer rectangle commutes as well, showing that $\Phi_V \otimes \Phi_W$ is natural. We will now show that $\Phi_V \otimes \Phi_W$ satisfies equation (4.1.1):

$$\begin{aligned}(\Phi_V \otimes \Phi_W)(X\otimes Y) &= [\Phi_V(F_r(X\otimes Y))\otimes \mathrm{id}_W]\circ[\mathrm{id}_V\otimes\Phi_W(X\otimes Y)]\\ &= [\mathrm{id}_{F_q(F_r(X))}\otimes\Phi_V(F_r(Y))\otimes\mathrm{id}_W]\circ[\Phi_V(F_r(X))\otimes\mathrm{id}_{F_r(Y)}\otimes\mathrm{id}_W]\\ &\quad\circ[\mathrm{id}_V\otimes\mathrm{id}_{F_r(X)}\otimes\Phi_W(Y)]\circ[\mathrm{id}_V\otimes\Phi_W(X)\otimes\mathrm{id}_Y]\\ &= [\mathrm{id}_{F_{qr}(X)}\otimes\Phi_V(F_r(Y))\otimes\mathrm{id}_W]\circ[\mathrm{id}_{F_{qr}(X)}\otimes\mathrm{id}_V\otimes\Phi_W(Y)]\\ &\quad\circ[\Phi_V(F_r(X))\otimes\mathrm{id}_W\otimes\mathrm{id}_Y]\circ[\mathrm{id}_V\otimes\Phi_W(X)\otimes\mathrm{id}_Y]\\ &= [\mathrm{id}_{F_{qr}(X)}\otimes(\Phi_V\otimes\Phi_W)(Y)]\circ[(\Phi_V\otimes\Phi_W)(X)\otimes\mathrm{id}_Y]\end{aligned}$$

where the third equality follows from the interchange law. We thus conclude that $\Phi_V \otimes \Phi_W$ is indeed a half qr-braiding for $V \otimes W$.

(2) By assumption, $\Phi_V(F_{r^{-1}}(X)) : V \otimes F_{r^{-1}}(X) \to F_{qr^{-1}}(X) \otimes V$ is an isomorphism, so $F_r(\Phi_V(F_{r^{-1}}(X)))$: $F_r(V) \otimes X \to F_{rqr^{-1}}(X) \otimes F_r(V)$ is also an isomorphism. To check naturality, let $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. Then

$$\begin{aligned}[F_{rqr^{-1}}(f)\otimes\mathrm{id}_{F_r(V)}]\circ\mathcal{F}_r\Phi_V(X) &= [F_{rqr^{-1}}(f)\otimes\mathrm{id}_{F_r(V)}]\circ F_r(\Phi_V(F_{r^{-1}}(X)))\\ &= F_r\{[F_q(F_{r^{-1}}(f))\otimes\mathrm{id}_V]\circ\Phi_V(F_{r^{-1}}(X))\}\\ &= F_r\{\Phi_V(F_{r^{-1}}(Y))\circ[\mathrm{id}_V\otimes F_{r^{-1}}(f)]\}\\ &= F_r(\Phi_V(F_{r^{-1}}(Y)))\circ[\mathrm{id}_{F_r(V)}\otimes f]\\ &= \mathcal{F}_r\Phi_V(Y)\circ[\mathrm{id}_{F_r(V)}\otimes f],\end{aligned}$$

so $\mathcal{F}_r\Phi_V$ is indeed natural. Furthermore, if $X, Y \in \mathcal{C}$, then

$$\begin{aligned}\mathcal{F}_r\Phi_V(X\otimes Y) &= F_r(\Phi_V(F_{r^{-1}}(X\otimes Y))) = F_r(\Phi_V(F_{r^{-1}}(X)\otimes F_{r^{-1}}(Y)))\\ &= F_r\{[\mathrm{id}_{F_{qr^{-1}}(X)}\otimes\Phi_V(F_{r^{-1}}(Y))]\circ[\Phi_V(F_{r^{-1}}(X))\otimes\mathrm{id}_{F_{r^{-1}}(Y)}]\}\\ &= [\mathrm{id}_{F_{rqr^{-1}}(X)}\otimes F_r(\Phi_V(F_{r^{-1}}(Y)))]\circ[F_r(\Phi_V(F_{r^{-1}}(X)))\otimes\mathrm{id}_Y]\\ &= [\mathrm{id}_{F_{rqr^{-1}}(X)}\otimes\mathcal{F}_r\Phi_V(Y)]\circ[\mathcal{F}_r\Phi_V(X)\otimes\mathrm{id}_Y],\end{aligned}$$

so $\mathcal{F}_r\Phi_V$ is indeed a half rqr^{-1}-braiding for $F_r(V)$.

□

4.1.2 Retracts and direct sums

The following lemma will not be of importance in defining $Z_G(\mathcal{C})$ as a braided G-crossed category, but it will be used in Section 4.6 to show that if $\mathcal{C}$ has retracts or direct sums, then so has $Z_G(\mathcal{C})$. The proof is a straightforward generalization of the corresponding one in [75].

Lemma 4.1.6 *Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category with strict action F of the group G. Let $V \in \mathcal{C}$ and let Φ_V be a half q-braiding.*

(1) If $U \in \mathcal{C}$ is a subobject of V with corresponding morphisms $i : U \to V$ and $r : V \to U$ satisfying $r \circ i = \mathrm{id}_U$, then for each $X \in \mathcal{C}$ we define $\Phi_U(X) : U \otimes X \to F_q(X) \otimes U$ by

$$\Phi_U(X) := [\mathrm{id}_{F_q(X)} \otimes r] \circ \Phi_V(X) \circ [i \otimes \mathrm{id}_X]$$

If the idempotent $p := i \circ r \in \mathrm{End}_{\mathcal{C}}(V)$ satisfies the equation

$$[\mathrm{id}_{F_q(X)} \otimes p] \circ \Phi_V(X) = \Phi_V(X) \circ [p \otimes \mathrm{id}_X]$$

then Φ_U is a half q-braiding for U.

(2) Suppose that $\mathcal{C}$ is an Ab-category and let Φ_W be a half q-braiding for $W \in \mathcal{C}$. If Z is a direct sum of V and W with corresponding morphisms $f : V \to Z$, $f' : Z \to V$, $g : W \to Z$ and $g' : Z \to W$ satisfying $f' \circ f = \mathrm{id}_V$, $g' \circ g = \mathrm{id}_W$ and $f \circ f' + g \circ g' = \mathrm{id}_Z$, then we obtain a half q-braidings Φ_Z for Z by

$$\Phi_Z(X) := [\mathrm{id}_{F_q(X)} \otimes f] \circ \Phi_V(X) \circ [f' \otimes \mathrm{id}_X] + [\mathrm{id}_{F_q(X)} \otimes g] \circ \Phi_W(X) \circ [g' \otimes \mathrm{id}_X]$$

Proof. (1) It is easy to check that each $\Phi_U(X)$ is invertible with inverse

$$\Phi_U(X)^{-1} := [r \otimes \mathrm{id}_X] \circ \Phi_V(X)^{-1} \circ [\mathrm{id}_{F_q(X)} \otimes i] \in \mathrm{Hom}_{\mathcal{C}}(F_q(X) \otimes U, U \otimes X).$$

Naturality of Φ_U follows directly from the naturality of Φ_V. It also satisfies

$$\begin{aligned}
&\Phi_U(X \otimes Y) \\
&= [\mathrm{id}_{F_q(X\otimes Y)} \otimes r] \circ [\mathrm{id}_{F_q(X)} \otimes \Phi_V(Y)] \circ [\Phi_V(X) \otimes \mathrm{id}_Y] \circ [(i \circ r \circ i) \otimes \mathrm{id}_{X\otimes Y}] \\
&= [\mathrm{id}_{F_q(X\otimes Y)} \otimes r] \circ [\mathrm{id}_{F_q(X)} \otimes \Phi_V(Y)] \circ [\mathrm{id}_{F_q(X)} \otimes (i \circ r) \otimes \mathrm{id}_Y] \circ [\Phi_V(X) \otimes \mathrm{id}_Y] \circ [i \otimes \mathrm{id}_{X\otimes Y}] \\
&= [\mathrm{id}_{F_q(X)} \otimes \Phi_U(Y)] \circ [\Phi_U(X) \otimes \mathrm{id}_Y],
\end{aligned}$$

where in the first step we used that $i = i \circ r \circ i$ and in the second step we used the given assumption on the idempotent $p = i \circ r$. So we conclude that Φ_U is a half q-braiding.
(2) For each $X \in \mathcal{C}$, $\Phi_Z(X)$ is invertible with inverse $\Phi_Z(X)^{-1} \in \mathrm{Hom}_{\mathcal{C}}(F_q(X) \otimes Z, Z \otimes X)$ given by

$$\Phi_Z(X)^{-1} := [f \otimes \mathrm{id}_X] \circ \Phi_V(X)^{-1} \circ [\mathrm{id}_{F_q(X)} \otimes f'] + [g \otimes \mathrm{id}_X] \circ \Phi_W(X)^{-1} \circ [\mathrm{id}_{F_q(X)} \otimes g'],$$

as the reader can easily check. Naturality of Φ_Z follows directly from the naturality of Φ_V and Φ_W. Checking that $\Phi_Z(X \otimes Y) = [\mathrm{id}_{F_q(X)} \otimes \Phi_Z(Y)] \circ [\Phi_Z(X) \otimes \mathrm{id}_Y]$ is a straightforward computation.
□

4.1.3 Duality

Since in Section 4.6 we will also be interested in the question whether $Z_G(\mathcal{C})$ inherits a left or right duality from $\mathcal{C}$, in this subsection we will focus on transporting the half braiding of a certain object to its dual. The proof of the following lemma is an easy exercise for the reader.

Lemma 4.1.7 *Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category with strict action F of the group G and let $V \in \mathcal{C}$. Suppose that Φ_V satisfies all axioms of a half q-braiding for V, except invertiblity, and suppose that $\Phi_V(I) = \mathrm{id}_V$. If $X \in \mathcal{C}$ has a right dual $({}^\vee X, b'_X, d'_X)$, then $\Phi_V(X)$ is invertible with inverse*

$$\Phi_V(X)^{-1} = [F_q(d'_X) \otimes \mathrm{id}_{V\otimes X}] \circ [\mathrm{id}_{F_q(X)} \otimes \Phi_V({}^\vee X) \otimes \mathrm{id}_X] \circ [\mathrm{id}_{F_q(X)\otimes V} \otimes b'_X].$$

The main application of Lemma 4.1.7 is the proof of the following important lemma, which shows that under certain circumstances we can transport a half braiding for an object V to its left dual.

Lemma 4.1.8 *Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category with strict action F of the group G, let $V \in \mathcal{C}$, let Φ_V be a half q-braiding for V. Suppose that each object in $\mathcal{C}$ has a right dual. If $(V^\vee, b_V, d_V)$ is a left dual of V, then we obtain a half q^{-1}-braiding $(\Phi_V)^\vee$ for $V^\vee$ by defining*

$$(\Phi_V)^\vee(X) := [d_V \otimes \mathrm{id}_{F_{q^{-1}}(X)\otimes V^\vee}] \circ [\mathrm{id}_{V^\vee} \otimes \Phi_V(F_{q^{-1}}(X))^{-1} \otimes \mathrm{id}_{V^\vee}] \circ [\mathrm{id}_{V^\vee \otimes X} \otimes b_V]$$

for all $X \in \mathcal{C}$.

Proof. Since $X \mapsto \Phi_V(F_{q^{-1}}(X))^{-1}$ is a half q-braiding of the second kind for V, it follows that $(\Phi_V)^\vee$ is natural and that for any $X, Y \in \mathcal{C}$ we have

$$(\Phi_V)^\vee(X \otimes Y) = [\mathrm{id}_{F_{q^{-1}}(X)} \otimes (\Phi_V)^\vee(Y)] \circ [(\Phi_V)^\vee(X) \otimes \mathrm{id}_Y].$$

In view of $\Phi_V(F_{q^{-1}}(I)) = \Phi_V(I) = \mathrm{id}_V$, we have $(\Phi_V)^\vee(I) = [d_V \otimes \mathrm{id}_{V^\vee}] \circ [\mathrm{id}_{V^\vee} \otimes b_V] = \mathrm{id}_{V^\vee}$, so it follows from the preceding lemma that $(\Phi_V)^\vee(X)$ is invertible for all $X \in \mathcal{C}$, the inverse being given by

$$(\Phi_V)^\vee(X)^{-1} = [F_{q^{-1}}(d'_X) \otimes \mathrm{id}_{V^\vee \otimes X}] \circ [\mathrm{id}_{F_{q^{-1}}(X)} \otimes (\Phi_V)^\vee({}^\vee X) \otimes \mathrm{id}_X] \circ [\mathrm{id}_{F_{q^{-1}}(X)\otimes V^\vee} \otimes b'_X].$$

□

Lemma 4.1.9 *Suppose that $\mathcal{C}$ is a G-pivotal category. Let $V \in \mathcal{C}$, let Φ_V be a half q-braiding for V. We obtain a half q^{-1}-braiding $\overline{\Phi_V}$ for $\overline{V}$ by defining*

$$\overline{\Phi_V}(X) = [d_V \otimes \mathrm{id}_{F_{q^{-1}}(X)\otimes\overline{V}}] \circ [\mathrm{id}_{\overline{V}} \otimes \Phi_V(F_{q^{-1}}(X))^{-1} \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_{\overline{V}\otimes X} \otimes b_V]$$

and it satisfies $\overline{\overline{\Phi_V}}(X) = \Phi_V(X)$ for all $X \in \mathcal{C}$. If Φ_W is a half r-braiding for $W \in \mathcal{C}$, then $\overline{\Phi_V \otimes \Phi_W} = \overline{\Phi_W} \otimes \overline{\Phi_V}$. Furthermore, we have $\overline{\Phi^0_I} = \Phi^0_I$.

Proof. That $\overline{\Phi_V}$ is a half q^{-1}-braiding for $\overline{V}$ was proven in the preceding lemma. By using the fact that the left and right transpose of a morphism in a pivotal category coincide, as well as the other special properties that b, b', d and d' have in a pivotal category, one can verify that

$$\begin{aligned}\overline{\Phi_V}(F_q(X))^{-1} = {} & [\mathrm{id}_{\overline{V}\otimes F_q(X)} \otimes d_{\overline{V}}] \circ [\mathrm{id}_{\overline{V}\otimes F_q(X)\otimes V} \otimes d_X \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_{\overline{V}\otimes F_q(X)} \otimes \Phi_V(\overline{X})^{-1} \otimes \mathrm{id}_{X\otimes\overline{V}}] \\ & \circ [\mathrm{id}_{\overline{V}} \otimes b_{F_q(X)} \otimes \mathrm{id}_{V\otimes X\otimes\overline{V}}] \circ [b_{\overline{V}} \otimes \mathrm{id}_{X\otimes\overline{V}}].\end{aligned}$$

Using this expression, we get

$$\begin{aligned}\overline{\overline{\Phi_V}}(X) &= [d_{\overline{V}} \otimes \mathrm{id}_{F_q(X)\otimes V}] \circ [\mathrm{id}_V \otimes \overline{\Phi_V}(F_q(X))^{-1} \otimes \mathrm{id}_V] \circ [\mathrm{id}_{V\otimes X} \otimes b_{\overline{V}}] \\ &= [\mathrm{id}_{F_q(X)\otimes V} \otimes d_X] \circ [\mathrm{id}_{F_q(X)} \otimes \Phi_V(\overline{X})^{-1} \otimes \mathrm{id}_X] \circ [b_{F_q(X)} \otimes \mathrm{id}_{V\otimes X}].\end{aligned}$$

To see that this equals $\Phi_V(X)$, we compute

$$\overline{\overline{\Phi_V}}(X) \circ \Phi_V(X)^{-1}$$

$$= [\mathrm{id}_{F_q(X)\otimes V} \otimes d_X] \circ [\mathrm{id}_{F_q(X)} \otimes \Phi_V(\overline{X})^{-1} \otimes \mathrm{id}_X] \circ [b_{F_q(X)} \otimes \mathrm{id}_{V\otimes X}] \circ \Phi_V(X)^{-1}$$
$$= [\mathrm{id}_{F_q(X)\otimes V} \otimes d_X] \circ [\mathrm{id}_{F_q(X)} \otimes \Phi_V(\overline{X} \otimes X)^{-1}] \circ [F_q(b_X) \otimes \mathrm{id}_{F_q(X)\otimes V}]$$
$$= [\mathrm{id}_{F_q(X)} \otimes \Phi_V(I)^{-1}] \circ [\mathrm{id}_{F_q(X)} \otimes F_q(d_X) \otimes \mathrm{id}_V] \circ [F_q(b_X) \otimes \mathrm{id}_{F_q(X)\otimes V}]$$
$$= \mathrm{id}_{F_q(X)\otimes V}$$

and similarly, one finds $\Phi_V(X)^{-1} \circ \overline{\overline{\Phi_V}}(X) = \mathrm{id}_{V\otimes X}$. Thus, we have indeed that $\overline{\overline{\Phi_V}}(X) = \Phi_V(X)$. Furthermore,

$$\begin{aligned}
&\overline{\Phi_V \otimes \Phi_W}(X) \\
&= [d_W \otimes \mathrm{id}_{F_{r^{-1}q^{-1}}(X)\otimes\overline{W}\otimes\overline{V}}] \circ [\mathrm{id}_{\overline{W}} \otimes d_V \otimes \mathrm{id}_{W\otimes F_{r^{-1}q^{-1}}(X)\otimes\overline{W}\otimes\overline{V}}] \\
&\quad \circ [\mathrm{id}_{\overline{W}\otimes\overline{V}\otimes V} \otimes \Phi_W(F_{r^{-1}q^{-1}}(X))^{-1} \otimes \mathrm{id}_{\overline{W}\otimes\overline{V}}] \circ [\mathrm{id}_{\overline{W}\otimes\overline{V}} \otimes \Phi_V(F_{q^{-1}}(X))^{-1} \otimes \mathrm{id}_{W\otimes\overline{W}\otimes\overline{V}}] \\
&\quad \circ [\mathrm{id}_{\overline{W}\otimes\overline{V}\otimes X\otimes V} \otimes b_W \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_{\overline{W}\otimes\overline{V}\otimes X} \otimes b_V] \\
&= [d_W \otimes \mathrm{id}_{F_{r^{-1}q^{-1}}(X)\otimes\overline{W}\otimes\overline{V}}] \circ [\mathrm{id}_{\overline{W}} \otimes \Phi_W(F_{r^{-1}q^{-1}}(X))^{-1} \otimes \mathrm{id}_{\overline{W}\otimes\overline{V}}] \circ [\mathrm{id}_{\overline{W}\otimes F_{q^{-1}}(X)} \otimes b_W \otimes \mathrm{id}_{\overline{V}}] \\
&\quad \circ [\mathrm{id}_{\overline{W}} \otimes d_V \otimes \mathrm{id}_{F_{q^{-1}}(X)\otimes\overline{V}}] \circ [\mathrm{id}_{\overline{W}\otimes\overline{V}} \otimes \Phi_V(F_{q^{-1}}(X))^{-1} \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_{\overline{W}\otimes\overline{V}\otimes X} \otimes b_V] \\
&= [\overline{\Phi_W}(F_{q^{-1}}(X)) \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_{\overline{W}} \otimes \overline{\Phi_V}(X)] \\
&= (\overline{\Phi_W} \otimes \overline{\Phi_V})(X).
\end{aligned}$$

Finally, for any $X \in \mathcal{C}$ we have

$$\overline{\Phi^0_I}(X) = [d_I \otimes \mathrm{id}_{X\otimes\overline{I}}] \circ [\mathrm{id}_{\overline{I}} \otimes \Phi^0_I(F_{e^{-1}}(X))^{-1} \otimes \mathrm{id}_{\overline{I}}] \circ [\mathrm{id}_{\overline{I}\otimes X} \otimes b_I] = \mathrm{id}_X = \Phi^0_I(X),$$

where we have used that $b_I = d_I = \mathrm{id}_I$ in a pivotal category, see Lemma 2.3.8.
□

4.2 Construction of $Z_G(\mathcal{C})$ in the strict case

After all our preparations concerning half braidings in G-categories, we are now ready to construct the G-crossed Drinfeld center $Z_G(\mathcal{C})$.

Theorem 4.2.1 *Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category with strict G-action F. We then define a braided G-crossed category $Z_G(\mathcal{C})$ as follows:*

- *For each $q \in G$ we first define a category $Z_G(\mathcal{C})_q$ as follows. The class of objects is given by*

$$\mathrm{Obj}(Z_G(\mathcal{C})_q) := \{(V, \Phi_V) : V \in \mathcal{C} \text{ and } \Phi_V \text{ is a half } q\text{-braiding for } V\}$$

and we define $\mathrm{Hom}_{Z_G(\mathcal{C})_q}((V, \Phi_V), (W, \Phi_W))$ to be

$$\{f \in \mathrm{Hom}_{\mathcal{C}}(V, W) : [\mathrm{id}_{F_q(X)} \otimes f] \circ \Phi_V(X) = \Phi_W(X) \circ [f \otimes \mathrm{id}_X]\ \forall X \in \mathcal{C}\}.$$

This defines a category $Z_G(\mathcal{C})_q$, where the composition of morphisms is the same as in $\mathcal{C}$ and the identity morphisms are given by $\mathrm{id}_{(V,\Phi_V)} = \mathrm{id}_V$. We then define the category $Z_G(\mathcal{C})$ as the disjoint union[5]

$$Z_G(\mathcal{C}) := \bigsqcup_{q\in G} Z_G(\mathcal{C})_q.$$

Thus, an object in $Z_G(\mathcal{C})$ is a triple (V, q, Φ_V) with $q \in G$ and $(V, \Phi_V) \in Z_G(\mathcal{C})_q$.

[5] As already announced in Remark 2.7.1, our definition of the disjoint union depends on whether we are dealing with Ab-categories or not.

- *The category $Z_G(\mathcal{C})$ can be equipped with the structure of a strict tensor category by defining the tensor product on the objects by*
$$(V, q, \Phi_V) \otimes (W, r, \Phi_W) := (V \otimes W, qr, \Phi_V \otimes \Phi_W),$$
where
$$(\Phi_V \otimes \Phi_W)(X) := [\Phi_V(F_r(X)) \otimes \mathrm{id}_W] \circ [\mathrm{id}_V \otimes \Phi_W(X)]$$
and by letting the tensor product on the morphisms be the same as in $\mathcal{C}$; the unit object is (I, e, Φ_I^0), where $\Phi_I^0(X) = \mathrm{id}_X$ for all $X \in \mathcal{C}$.
- *It becomes a G-graded tensor category if we define $\partial[(V, q, \Phi_V)] = q$.*
- *We can define an action of the group G on the objects of $Z_G(\mathcal{C})$ by*
$$\mathcal{F}_q[(V, r, \Phi_V)] = (F_q(V), qrq^{-1}, \mathcal{F}_q\Phi_V),$$
where
$$(\mathcal{F}_q\Phi_V)(X) := F_q(\Phi_V(F_{q^{-1}}(X)))$$
and on the morphisms we define $\mathcal{F}_q(f) := F_q(f)$. This gives $Z_G(\mathcal{C})$ the structure of a G-crossed category.
- *Furthermore, $Z_G(\mathcal{C})$ becomes a braided G-crossed category if we define a braiding*
$$C_{(V,q,\Phi_V),(W,r,\Phi_V)} : (V, q, \Phi_V) \otimes (W, r, \Phi_W) \to \mathcal{F}_q[(W, r, \Phi_W)] \otimes (V, q, \Phi_V),$$
by $C_{(V,q,\Phi_V),(W,r,\Phi_W)} := \Phi_V(W)$.

Proof. Let $f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((U, q, \Phi_U), (V, q, \Phi_V))$ and $g \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V, q, \Phi_V), (W, q, \Phi_W))$. Then for all $X \in \mathcal{C}$ the morphism $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(U, W)$ satisfies
$$[\mathrm{id}_{F_q(X)} \otimes (g \circ f)] \circ \Phi_U(X) = [\mathrm{id}_{F_q(X)} \otimes g] \circ \Phi_V(X) \circ [f \otimes \mathrm{id}_X] = \Phi_W(X) \circ [(g \circ f) \otimes \mathrm{id}_X].$$
So indeed $g \circ f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((U, q, \Phi_U), (W, q, \Phi_W))$. If $(V, q, \Phi_V) \in Z_G(\mathcal{C})$, then it is easy to see that $\mathrm{id}_V \in \mathrm{End}_{Z_G(\mathcal{C})}((V, q, \Phi_V))$. Since the composition in $Z_G(\mathcal{C})$ is the same as in $\mathcal{C}$, it is also clear that id_V acts as an identity morphism in $Z_G(\mathcal{C})$. Thus $Z_G(\mathcal{C})$ is a category.

We have already seen that $\Phi_V \otimes \Phi_W$ is a half qr-braiding for $V \otimes W$, so the tensor product is well-defined on the objects. Now suppose that we have two morphisms
$$f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V, q, \Phi_V), (W, q, \Phi_W)),$$
$$f' \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V', q', \Phi_{V'}), (W', q', \Phi_{W'})).$$
Then for each $X \in \mathcal{C}$ the morphism $f \otimes f' \in \mathrm{Hom}_{\mathcal{C}}(V \otimes V', W \otimes W')$ satisfies
$$\begin{aligned}
&[\mathrm{id}_{F_{qq'}(X)} \otimes f \otimes f'] \circ [(\Phi_V \otimes \Phi_{V'})(X)] \\
&= [\mathrm{id}_{F_{qq'}(X)} \otimes \mathrm{id}_W \otimes f'] \circ [\mathrm{id}_{F_{qq'}(X)} \otimes f \otimes \mathrm{id}_{V'}] \circ [\Phi_V(F_{q'}(X)) \otimes \mathrm{id}_{V'}] \circ [\mathrm{id}_V \otimes \Phi_{V'}(X)] \\
&= [\mathrm{id}_{F_{qq'}(X)} \otimes \mathrm{id}_W \otimes f'] \circ [\Phi_W(F_{q'}(X)) \otimes \mathrm{id}_{V'}] \circ [f \otimes \mathrm{id}_{F_{q'}(X)} \otimes \mathrm{id}_{V'}] \circ [\mathrm{id}_V \otimes \Phi_{V'}(X)] \\
&= [\Phi_W(F_{q'}(X)) \otimes \mathrm{id}_{W'}] \circ [\mathrm{id}_W \otimes \mathrm{id}_{F_{q'}(X)} \otimes f'] \circ [\mathrm{id}_W \otimes \Phi_{V'}(X)] \circ [f \otimes \mathrm{id}_{V'} \otimes \mathrm{id}_X] \\
&= [\Phi_W(F_{q'}(X)) \otimes \mathrm{id}_{W'}] \circ [\mathrm{id}_W \otimes \Phi_{W'}(X)] \circ [\mathrm{id}_W \otimes f' \otimes \mathrm{id}_X] \circ [f \otimes \mathrm{id}_{V'} \otimes \mathrm{id}_X] \\
&= [(\Phi_W \otimes \Phi_{W'})(X)] \circ [f \otimes f' \otimes \mathrm{id}_X].
\end{aligned}$$
This shows that $f \otimes f' \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V \otimes V', qq', \Phi_V \otimes \Phi_{V'}), (W \otimes W', qq', \Phi_W \otimes \Phi_{W'}))$ and hence that the tensor product on morphisms is well-defined. That the tensor product acts properly on identity morphisms and satisfies the interchange law follows directly from the fact that the tensor product on morphisms in

$Z_G(\mathcal{C})$ is defined to be the same as in $\mathcal{C}$. The proof that $(I, e, \Phi_I^0) \in Z_G(\mathcal{C})$ acts as a unit object is trivial. We thus conclude that $Z_G(\mathcal{C})$ is a strict tensor category.

The map ∂ has the property that

$$\partial[(V, q, \Phi_V) \otimes (W, r, \Phi_W)] \quad = \quad \partial[(V \otimes W, qr, \Phi_V \otimes \Phi_W)] = qr = \partial[(V, q, \Phi_V)]\partial[(W, r, \Phi_W)],$$

and it is clear that if $(V, q, \Phi_V), (W, r, \Phi_W) \in Z_G(\mathcal{C})$ are isomorphic, then $q = r$. Hence it follows that $Z_G(\mathcal{C})$ is a G-graded tensor category.

We will now show that each $\mathcal{F}_q$ is a functor $Z_G(\mathcal{C}) \to Z_G(\mathcal{C})$ for each $q \in G$. If f is a morphism in $\mathrm{Hom}_{Z_G(\mathcal{C})}((V, r, \Phi_V), (W, r, \Phi_W))$, then $\mathcal{F}_q(f) = F_q(f) \in \mathrm{Hom}_{\mathcal{C}}(F_q(V), F_q(W))$ and for $X \in \mathcal{C}$ we have

$$\begin{aligned}
&[\mathrm{id}_{F_{qrq^{-1}}(X)} \otimes \mathcal{F}_q(f)] \circ [\mathcal{F}_q\Phi_V(X)] \\
&= F_q[(\mathrm{id}_{F_{qr^{-1}}(X)} \otimes f) \circ \Phi_V(F_{q^{-1}}(X))] = F_q[\Phi_W(F_{q^{-1}}(X)) \circ (f \otimes \mathrm{id}_{F_{q^{-1}}(X)})] \\
&= [\mathcal{F}_q\Phi_W(X)] \circ [\mathcal{F}_q(f) \otimes \mathrm{id}_X],
\end{aligned}$$

showing that $\mathcal{F}_q(f) \in \mathrm{Hom}_{Z_G(\mathcal{C})}(\mathcal{F}_q[(V, r, \Phi_V)], \mathcal{F}_q[(W, r, \Phi_W)])$. It is also clear that we have $\mathcal{F}_q(g \circ f) = \mathcal{F}_q(g) \circ \mathcal{F}_q(f)$ and $\mathcal{F}_q(\mathrm{id}_{(V,r,\Phi_V)}) = \mathrm{id}_{\mathcal{F}_q[(V,r,\Phi_V)]}$ for any morphisms f and g and any object (V, r, Φ_V) in the category $Z_G(\mathcal{C})$, because the G-action on morphisms is the same in $Z_G(\mathcal{C})$ as in $\mathcal{C}$. Thus $\mathcal{F}_q$ is a functor. It also strictly preserves the tensor product on objects, since

$$\begin{aligned}
\mathcal{F}_q[(V, r, \Phi_V) \otimes (W, s, \Phi_W)] &= \mathcal{F}_q[(V \otimes W, rs, \Phi_V \otimes \Phi_W)] \\
&= (F_q(V \otimes W), qrsq^{-1}, \mathcal{F}_q(\Phi_V \otimes \Phi_W)) \\
&= (F_q(V) \otimes F_q(W), qrq^{-1}qsq^{-1}, \mathcal{F}_q\Phi_V \otimes \mathcal{F}_q\Phi_W) \\
&= (F_q(V), qrq^{-1}, \mathcal{F}_q\Phi_V) \otimes (F_q(W), qsq^{-1}, \mathcal{F}_q\Phi_W) \\
&= \mathcal{F}_q[(V, r, \Phi_V)] \otimes \mathcal{F}_q[(W, s, \Phi_W)],
\end{aligned}$$

where we have used that $\mathcal{F}_q(\Phi_V \otimes \Phi_W) = \mathcal{F}_q\Phi_V \otimes \mathcal{F}_q\Phi_W$, which follows from the computation

$$\begin{aligned}
[\mathcal{F}_q(\Phi_V \otimes \Phi_W)](X) &= F_q[(\Phi_V \otimes \Phi_W)(F_{q^{-1}}(X))] \\
&= F_q\left\{[\Phi_V(F_{sq^{-1}}(X)) \otimes \mathrm{id}_W] \circ [\mathrm{id}_V \otimes \Phi_W(F_{q^{-1}}(X))]\right\} \\
&= [F_q(\Phi_V(F_{sq^{-1}}(X))) \otimes \mathrm{id}_{F_q(W)}] \circ [\mathrm{id}_{F_q(V)} \otimes F_q(\Phi_W(F_{q^{-1}}(X)))] \\
&= [(\mathcal{F}_q\Phi_V)(F_{qsq^{-1}}(X)) \otimes \mathrm{id}_{F_q(W)}] \circ [\mathrm{id}_{F_q(V)} \otimes (\mathcal{F}_q\Phi_W)(X)] \\
&= (\mathcal{F}_q\Phi_V \otimes \mathcal{F}_q\Phi_W)(X).
\end{aligned}$$

Since $\mathcal{F}_q$ is defined on morphisms in the same way as in $\mathcal{C}$, it is clear that $\mathcal{F}_q$ strictly preserves the tensor structure on the morphisms as well. We also have

$$\mathcal{F}_q[(I, e, \Phi_I^0)] = (F_q(I), qeq^{-1}, F_q(\mathrm{id}_{F_{q^{-1}}(-)})) = (I, e, \mathrm{id}_-) = (I, e, \Phi_I^0),$$

so we conclude that $\mathcal{F}_q$ is a strict tensor functor. The map $q \mapsto F_q$ is a homomorphism, since

$$\begin{aligned}
\mathcal{F}_{qr}[(V, s, \Phi_V)] &= (F_{qr}(V), qrs(qr)^{-1}, \mathcal{F}_{qr}\Phi_V) = (F_q(F_r(V)), qrsr^{-1}q^{-1}, \mathcal{F}_q(\mathcal{F}_r\Phi_V)) \\
&= \mathcal{F}_q[(F_r(V), rsr^{-1}, \mathcal{F}_r\Phi_V)] = \mathcal{F}_q[\mathcal{F}_r[(V, s, \Phi_V)]],
\end{aligned}$$

where we have used that $\mathcal{F}_{qr}\Phi_V = \mathcal{F}_q(\mathcal{F}_r\Phi_V)$, which follows from the computation

$$\begin{aligned}
(\mathcal{F}_{qr}\Phi_V)(X) &= F_{qr}(\Phi_V(F_{(qr)^{-1}}(X))) = F_q(F_r(\Phi_V(F_{r^{-1}}(F_{q^{-1}}(X))))) \\
&= F_q[\mathcal{F}_r\Phi_V(F_{q^{-1}}(X))] = [\mathcal{F}_q(\mathcal{F}_r\Phi_V)](X).
\end{aligned}$$

On the morphisms we can simply use that $\mathcal{F}_{qr} = F_{qr} = F_q \circ F_r = \mathcal{F}_q \circ \mathcal{F}_r$. Thus $q \mapsto \mathcal{F}_q$ defines a strict G-action on $Z_G(\mathcal{C})$. Also,

$$\partial\{\mathcal{F}_q[(V, r, \Phi_V)]\} = \partial[(F_q(V), qrq^{-1}, \mathcal{F}_q\Phi_V)] = qrq^{-1},$$

so $Z_G(\mathcal{C})$ is a G-crossed category.

Let $(V, q, \Phi_V), (W, r, \Phi_W) \in Z_G(\mathcal{C})$. Invertibility of $C_{(V,q,\Phi_V),(W,r,\Phi_W)} = \Phi_V(W)$ is clear. For any $s \in G$ we have (in the category $\mathcal{C}$)

$$\begin{aligned} F_s[C_{(V,q,\Phi_V),(W,r,\Phi_W)}] &= F_s(\Phi_V(W)) = F_s[\Phi_V(F_{s^{-1}}(F_s(W)))] = (\mathcal{F}_s\Phi_V)(F_s(W)) \\ &= C_{\mathcal{F}_s[(V,q,\Phi_V)],\mathcal{F}_s[(W,r,\Phi_W)]}. \end{aligned}$$

Observe that $\mathrm{Hom}_{Z_G(\mathcal{C})}((V \otimes W, qr, \Phi_V \otimes \Phi_W), (F_q(W) \otimes V, qr, \mathcal{F}_q\Phi_W \otimes \Phi_V))$ consists of those $f \in \mathrm{Hom}_{\mathcal{C}}(V \otimes W, F_q(W) \otimes V)$ that satisfy

$$[\mathrm{id}_{F_{qr}(X)} \otimes f] \circ [(\Phi_V \otimes \Phi_W)(X)] = [(\mathcal{F}_q\Phi_W \otimes \Phi_V)(X)] \circ [f \otimes \mathrm{id}_X],$$

so we must check this equality for $f = \Phi_V(W)$. We have

$$\begin{aligned} &[\mathrm{id}_{F_{qr}(X)} \otimes \Phi_V(W)] \circ [(\Phi_V \otimes \Phi_W)(X)] \\ &= [\mathrm{id}_{F_{qr}(X)} \otimes \Phi_V(W)] \circ [\Phi_V(F_r(X)) \otimes \mathrm{id}_W] \circ [\mathrm{id}_V \otimes \Phi_W(X)] \\ &= \Phi_V(F_r(X) \otimes W) \circ [\mathrm{id}_V \otimes \Phi_W(X)] = [F_q(\Phi_W(X)) \otimes \mathrm{id}_V] \circ \Phi_V(W \otimes X) \\ &= [\mathcal{F}_q\Phi_W(F_q(X)) \otimes \mathrm{id}_V] \circ [\mathrm{id}_{F_q(W)} \otimes \Phi_V(X)] \circ [\Phi_V(W) \otimes \mathrm{id}_X] \\ &= [(\mathcal{F}_q\Phi_W \otimes \Phi_V)(X)] \circ [\Phi_V(W) \otimes \mathrm{id}_X], \end{aligned}$$

where in the third step we used naturality of Φ_V. It follows directly from the definition of a half q-braiding that C is natural in its second argument. But it is also natural in its first argument due to the definition of the set of morphisms between any two objects in $Z_G(\mathcal{C})$. By definition of a half q-braiding, the braiding in $Z_G(\mathcal{C})$ behaves properly with respect to tensor products in its second argument, and by definition of the tensor product on objects in $Z_G(\mathcal{C})$ (in particular the tensor product of half braidings) it also behaves properly with respect to tensor products in its first argument. This completes the proof that $Z_G(\mathcal{C})$ is a braided G-crossed category.
□

The following corollary follows directly from the construction of $Z_G(\mathcal{C})$ and from the fact that the definition of a half e-braiding coincides with the definition of a half braiding in the absence of a group action.

Corollary 4.2.2 *Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category with strict action F of the group G. Then $Z_G(\mathcal{C})_e = Z(\mathcal{C})$, where the expression on the right denotes the ordinary Drinfeld center (which is a braided G-category). Consequently, we have a full inclusion $Z(\mathcal{C}) \subset Z_G(\mathcal{C})$.*

In particular, it follows that $Z_G(\mathcal{C})$ is a braided G-crossed extension of $Z(\mathcal{C})$. If $\mathcal{C}$ is a strict tensor category (without any given group action), we can consider it as a G-category with $G = \{e\}$ and we get $Z_G(\mathcal{C}) = Z(\mathcal{C})$. This shows that $Z_G(\mathcal{C})$ is a generalization of $Z(\mathcal{C})$, as was already suggested in the title of this chapter.

Finally, we would like to mention that there is of course also a construction of $Z_G(\mathcal{C})$ that uses half braidings of the second kind instead. We will not consider this construction here, because it is completely analogous to the construction above.

4.3 The case when $\mathcal{C}$ is a non-strict G-category

In this section we will consider the construction of $Z_G(\mathcal{C})$ in case $\mathcal{C}$ is non-strict and we will show that if $\mathcal{C}$ and $\mathcal{C}'$ are two equivalent (non-strict) G-categories, then $Z_G(\mathcal{C}) \simeq Z_G(\mathcal{C}')$ as braided G-crossed categories. The detailed proofs of the statements in this section can be found in the appendices.

4.3.1 Construction of $Z_G(\mathcal{C})$ for a non-strict G-category $\mathcal{C}$

In order to carry out the construction of $Z_G(\mathcal{C})$ in the non-strict case, we first have to redefine half q-braidings in the non-strict setting. We limit ourselves to half q-braidings of the first kind.

Definition 4.3.1 Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category with a G-action $(F, \varepsilon^F, \delta^F)$.
If $V \in \mathcal{C}$ and $q \in G$, then a *half q-braiding for V* is a natural isomorphism

$$\Phi_V : \otimes \circ (V \times \mathrm{id}_{\mathcal{C}}) \quad \to \quad \otimes \circ (F_q \times V)$$

of functors $\mathcal{C} \to \mathcal{C}$, i.e. a family $\{\Phi_V(X) : V \otimes X \to F_q(X) \otimes V\}_{X \in \mathcal{C}}$ of isomorphisms in $\mathcal{C}$ such that for all $X, Y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the square

$$\begin{array}{ccc} V \otimes X & \xrightarrow{\Phi_V(X)} & F_q(X) \otimes V \\ \Big\downarrow{\scriptstyle \mathrm{id}_V \otimes f} & & \Big\downarrow{\scriptstyle F_q(f) \otimes \mathrm{id}_V} \\ V \otimes Y & \xrightarrow{\Phi_V(Y)} & F_q(Y) \otimes V \end{array}$$

commutes, satisfying the additional property that for all $X, Y \in \mathcal{C}$ we have

$$\begin{aligned} \Phi_V(X \otimes Y) = {} & [\delta^q_{X,Y} \otimes \mathrm{id}_V] \circ a^{-1}_{F_q(X), F_q(Y), V} \circ [\mathrm{id}_{F_q(X)} \otimes \Phi_V(Y)] \\ & \circ a_{F_q(X), V, Y} \circ [\Phi_V(X) \otimes \mathrm{id}_Y] \circ a^{-1}_{V,X,Y}. \end{aligned} \tag{4.3.1}$$

Now that we have a definition of half q-braidings in the non-strict setting, we have to generalize the lemmas of the strict case that enabled us to define the tensor product and G-action on $Z_G(\mathcal{C})$. The proof of the following lemma can be found in Section A.1 of Appendix A.

Lemma 4.3.2 *Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category with G-action (F, ε, δ).*
(1) If for each $X \in \mathcal{C}$ we define $\Phi^0_I(X) : I \otimes X \to F_e(X) \otimes I$ by $\Phi^0_I(X) := r^{-1}_{F_e(X)} \circ \varepsilon^F_X \circ l_X$, then Φ^0_I defines a half e-braiding for I.
(2) Let Φ_V be a half q-braiding for V and let Φ_W be a half r-braiding for W. Then we obtain a half qr-braiding $\Phi_V \otimes \Phi_W$ for $V \otimes W$ defined by

$$\begin{aligned} (\Phi_V \otimes \Phi_W)(X) := {} & [(\delta^F_{q,r})_X \otimes \mathrm{id}_{V \otimes W}] \circ a_{F_q(F_r(X)), V, W} \circ [\Phi_V(F_r(X)) \otimes \mathrm{id}_W] \\ & \circ a^{-1}_{V, F_r(X), W} \circ [\mathrm{id}_V \otimes \Phi_W(X)] \circ a_{V,W,X}. \end{aligned}$$

(3) If $r \in G$, we obtain a half rqr^{-1}-braiding $\mathcal{F}_r\Phi_V$ for $F_r(V)$ by defining

$$\begin{aligned} (\mathcal{F}_r\Phi_V)(X) := {} & [(\delta^F_{rq,r^{-1}})_X \otimes \mathrm{id}_{F_r(V)}] \circ [(\delta^F_{r,q})_{F_{r^{-1}}(X)} \otimes \mathrm{id}_{F_r(V)}] \circ (\delta^r_{F_q(F_{r^{-1}}(X)), V})^{-1} \\ & \circ F_r(\Phi_V(F_{r^{-1}}(X))) \circ \delta^r_{V, F_{r^{-1}}(X)} \circ [\mathrm{id}_{F_r(V)} \otimes (\delta^F_{r,r^{-1}})^{-1}_X] \circ [\mathrm{id}_{F_r(V)} \otimes \varepsilon^F_X]. \end{aligned}$$

Using this lemma, we can now construct $Z_G(\mathcal{C})$ in the non-strict case. The proof of the following theorem can be found in Sections A.2 through A.5 of Appendix A.

Theorem 4.3.3 *Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category with G-action $(F, \varepsilon^F, \delta^F)$. We then define a braided G-crossed category $Z_G(\mathcal{C})$ as follows.*

- *For each $q \in G$ we first define a category $Z_G(\mathcal{C})_q$ as follows. The class of objects is given by*

$$\mathrm{Obj}(Z_G(\mathcal{C})_q) := \{(V, \Phi_V) : V \in \mathcal{C} \text{ and } \Phi_V \text{ is a half } q\text{-braiding for } V\}$$

and we define $\mathrm{Hom}_{Z_G(\mathcal{C})_q}((V,\Phi_V),(W,\Phi_W))$ *to be*

$$\{f \in \mathrm{Hom}_{\mathcal{C}}(V,W) : [\mathrm{id}_{F_q(X)} \otimes f] \circ \Phi_V(X) = \Phi_W(X) \circ [f \otimes \mathrm{id}_X] \ \forall X \in \mathcal{C}\}.$$

This defines a category $Z_G(\mathcal{C})_q$*, where the composition of morphisms is the same as in* $\mathcal{C}$ *and the identity morphisms are given by* $\mathrm{id}_{(V,\Phi_V)} = \mathrm{id}_V$*. We then define the category* $Z_G(\mathcal{C})$ *as the disjoint union*[6]

$$Z_G(\mathcal{C}) := \bigsqcup_{q\in G} Z_G(\mathcal{C})_q.$$

Thus, an object in $Z_G(\mathcal{C})$ *is a triple* (V,q,Φ_V) *with* $q \in G$ *and* $(V,\Phi_V) \in Z_G(\mathcal{C})_q$*.*

- *The category* $Z_G(\mathcal{C})$ *can be equipped with the structure of a tensor category by defining the tensor product on the objects by*

$$(V,q,\Phi_V) \otimes (W,r,\Phi_W) := (V \otimes W, qr, \Phi_V \otimes \Phi_W),$$

where

$$\begin{aligned}(\Phi_V \otimes \Phi_W)(X) : = {} & [(\delta^F_{q,r})_X \otimes \mathrm{id}_{V\otimes W}] \circ a_{F_q(F_r(X)),V,W} \circ [\Phi_V(F_r(X)) \otimes \mathrm{id}_W] \\ & \circ a^{-1}_{V,F_r(X),W} \circ [\mathrm{id}_V \otimes \Phi_W(X)] \circ a_{V,W,X}\end{aligned}$$

and by letting the tensor product on the morphisms be the same as in $\mathcal{C}$*. The unit object is* (I,e,Φ^0_I)*, where* $\Phi^0_I(X) = r^{-1}_{F_e(X)} \circ \varepsilon^F_X \circ l_X$ *for all* $X \in \mathcal{C}$ *and the associativity constraint and the unit constraints are the ones of* $\mathcal{C}$*.*
- *It becomes a* G*-graded tensor category if we define* $\partial[(V,q,\Phi_V)] = q$*.*
- *We can define an action* $(\mathcal{F},e,\delta)$ *of the group* G *on the objects of* $Z_G(\mathcal{C})$ *by*

$$\mathcal{F}_q[(V,r,\Phi_V)] = (F_q(V), qrq^{-1}, \mathcal{F}_q\Phi_V),$$

where

$$\begin{aligned}(\mathcal{F}_r\Phi_V)(X) : = {} & [(\delta^F_{rq,r^{-1}})_X \otimes \mathrm{id}_{F_r(V)}] \circ [(\delta^F_{r,q})_{F_{r^{-1}}(X)} \otimes \mathrm{id}_{F_r(V)}] \circ (\delta^r_{F_q(F_{r^{-1}}(X)),V})^{-1} \\ & \circ F_r(\Phi_V(F_{r^{-1}}(X))) \circ \delta^r_{V,F_{r^{-1}}(X)} \circ [\mathrm{id}_{F_r(V)} \otimes (\delta^F_{r,r^{-1}})^{-1}_X] \circ [\mathrm{id}_{F_r(V)} \otimes \varepsilon^F_X],\end{aligned}$$

and on the morphisms we define $\mathcal{F}_q(f) := F_q(f)$*. The* ε *and* δ *are the same as for the* G*-action on* $\mathcal{C}$*. This gives* $Z_G(\mathcal{C})$ *the structure of a* G*-crossed category.*
- *Furthermore,* $Z_G(\mathcal{C})$ *becomes a braided* G*-crossed category if we define a braiding*

$$C_{(V,q,\Phi_V),(W,r,\Phi_V)} : (V,q,\Phi_V) \otimes (W,r,\Phi_W) \to \mathcal{F}_q[(W,r,\Phi_W)] \otimes (V,q,\Phi_V)$$

by $C_{(V,q,\Phi_V),(W,r,\Phi_W)} := \Phi_V(W)$*.*

Apparently the construction of $Z_G(\mathcal{C})$ is much more involved in the non-strict case than in the strict case. It would therefore be nice if we could somehow get rid of the non-strict case. In the next subsection we will show that this is indeed possible in a certain sense.

4.3.2 If $\mathcal{C} \simeq \mathcal{C}'$ then $Z_G(\mathcal{C}) \simeq Z_G(\mathcal{C}')$

The following theorem shows that if $\mathcal{C} \simeq \mathcal{C}'$ as G-categories, then $Z_G(\mathcal{C}) \simeq Z_G(\mathcal{C}')$ as braided G-crossed categories. We will only give a sketch of the proof here. The detailed proof can be found in Appendix B. Together with a coherence result of Galindo this theorem will then be used to argue that we can often restrict ourselves to the strict case.

[6] See also footnote 5.

Theorem 4.3.4 *If $\mathcal{C}$ and $\mathcal{C}'$ are equivalent G-categories, then $Z_G(\mathcal{C})$ and $Z_G(\mathcal{C}')$ are equivalent as braided G-crossed categories.*

Sketch of the proof. Suppose that we are given a group G and two tensor categories $(\mathcal{C}, \otimes, I, a, l, r)$ and $(\mathcal{C}', \otimes', I', a', l', r')$ with G-actions[7] F and F', respectively, such that $\mathcal{C}$ and $\mathcal{C}'$ are equivalent as G-categories. We may thus assume that we are given G-functors

$$(K, \varepsilon^K, \delta^K, \xi^K) : \mathcal{C} \to \mathcal{C}'$$
$$(L, \varepsilon^L, \delta^L, \xi^L) : \mathcal{C}' \to \mathcal{C}$$

together with natural G-isomorphisms

$$\varphi : \mathrm{id}_{\mathcal{C}'} \to K \circ L$$
$$\psi : L \circ K \to \mathrm{id}_{\mathcal{C}}.$$

If $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ then we define for each $X' \in \mathcal{C}'$ the isomorphism $\mathscr{K}\Phi_V(X') \in \mathrm{Hom}_{\mathcal{C}'}(K(V) \otimes' X', F'_q(X') \otimes' K(V))$ by

$$\begin{aligned}\mathscr{K}\Phi_V(X') := {} & [F'_q(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [\xi^K(q)_{L(X')} \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(L(X')),V})^{-1} \\ & \circ K(\Phi_V(L(X'))) \circ \delta^K_{V,L(X')} \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{X'}].\end{aligned}$$

Then $\mathscr{K}\Phi_V$ can be shown to be a half q-braiding for $K(V)$ and we obtain a functor $\mathscr{K} : Z_G(\mathcal{C}) \to Z_G(\mathcal{C}')$ by defining $\mathscr{K}[(V, q, \Phi_V)] := (K(V), q, \mathscr{K}\Phi_V)$ on the objects and $\mathscr{K}(f) := K(f)$ on the morphisms. The next step is to make $\mathscr{K}$ into a G-crossed functor. This is done by defining

$$\varepsilon^{\mathscr{K}} := \varepsilon^K, \qquad \delta^{\mathscr{K}}_{(V,q,\Phi_V),(W,r,\Phi_W)} := \delta^K_{V,W}, \qquad \xi^{\mathscr{K}}(q)_{(V,q,\Phi_V)} := \xi^K(q)_V$$

for any $(V, q, \Phi_V), (W, r, \Phi_W) \in Z_G(\mathcal{C})$ and $q \in G$. Of course it should first be proven that these morphisms are indeed morphisms in the category $Z_G(\mathcal{C}')$ and then it has to be shown that $(\mathscr{K}, \varepsilon^{\mathscr{K}}, \delta^{\mathscr{K}}, \xi^{\mathscr{K}})$ is a G-crossed functor. It can then be shown that this G-crossed functor is in fact braided, i.e. that

$$(\mathscr{K}, \varepsilon^{\mathscr{K}}, \delta^{\mathscr{K}}, \xi^{\mathscr{K}}) : Z_G(\mathcal{C}) \to Z_G(\mathcal{C}')$$

is a functor of braided G-crossed categories.

By interchanging the roles of K and L and interchanging the roles of φ and ψ^{-1}, it is trivial to see that we can also construct a braided G-crossed functor $\mathscr{L} : Z_G(\mathcal{C}') \to Z_G(\mathcal{C})$ by defining the functor $\mathscr{L}$ on the objects of $Z_G(\mathcal{C}')$ by $\mathscr{L}[(V', q, \Phi_{V'})] := (L(V'), q, \mathscr{L}\Phi_{V'})$, where the half q-braiding $\mathscr{L}\Phi_{V'}$ for $L(V')$ is given by

$$\begin{aligned}\mathscr{L}\Phi_{V'}(X) = {} & [F_q(\psi_X) \otimes \mathrm{id}_{L(V')}] \circ [\xi^L(q)_{K(X)} \otimes \mathrm{id}_{L(V')}] \circ (\delta^L_{F'_q(K(X)),V'})^{-1} \\ & \circ L(\Phi_{V'}(K(X))) \circ \delta^L_{V',K(X)} \circ [\mathrm{id}_{L(V')} \otimes \psi_X^{-1}]\end{aligned}$$

for $X \in \mathcal{C}$. On the morphisms $\mathscr{L}$ is defined as $\mathscr{L}(f') = L(f')$ and the $\varepsilon^{\mathscr{L}}$, $\delta^{\mathscr{L}}$ and $\xi^{\mathscr{L}}$ are given directly by the ε^L, δ^L and ξ^L, in the same manner as we did for $\mathscr{K}$.

The next step is to show that the functors $\mathscr{K}$ and $\mathscr{L}$ together constitute an equivalence of braided G-crossed categories. For each $(V', q, \Phi_{V'}) \in Z_G(\mathcal{C}')$ we define the isomorphism

$$\widetilde{\varphi}_{(V',q,\Phi_{V'})} \in \mathrm{Hom}_{Z_G(\mathcal{C}')}((V', q, \Phi_{V'}), \mathscr{K}\mathscr{L}[(V', q, \Phi_{V'})])$$

[7] In other places we were free to write e^q and δ^q instead of e^{F_q} and δ^{F_q}, but here we cannot do this because there are two group actions. These will be distinguished most easily by writing e^{F_q} or $e^{F'_q}$, etc.

by $\widetilde{\varphi}_{(V',q,\Phi_{V'})} := \varphi_{V'}$. It is then shown that this defines a natural braided G-crossed isomorphism from $\mathrm{id}_{Z_G(\mathcal{C}')}$ to $\mathscr{K} \circ \mathscr{L}$. In exactly the same manner one can construct a natural braided G-crossed isomorphism $\widetilde{\psi} : \mathscr{L} \circ \mathscr{K} \to \mathrm{id}_{Z_G(\mathcal{C})}$. Thus $Z_G(\mathcal{C}) \simeq Z_G(\mathcal{C}')$ as braided G-crossed categories.
□

The following coherence theorem for G-categories can be found in Galindo's paper [36].

Theorem 4.3.5 *If G is a discrete group, then every G-category is equivalent to a strict G-category.*

Together with our result above, this theorem has an interesting consequence. If G is a discrete group and $\mathcal{C}$ is a G-category, then the coherence theorem states that there is a strict G-category $\mathcal{C}'$ such that $\mathcal{C} \simeq \mathcal{C}'$ as G-categories. Our result above then implies that $Z_G(\mathcal{C}) \simeq Z_G(\mathcal{C}')$ as braided G-crossed categories. But $Z_G(\mathcal{C}')$ is a strict braided G-crossed category. Thus, if one is only interested in finding $Z_G(\mathcal{C})$ up to equivalence (of braided G-crossed categories) for some G-category $\mathcal{C}$ with G discrete, it may be assumed that $\mathcal{C}$ is a strict G-category.

4.4 $Z_G(\mathcal{C})$ as a relative Drinfeld center

In this section we will investigate the relationship between the G-crossed Drinfeld center $Z_G(\mathcal{C})$ and a certain instance of a relative Drinfeld center as defined in Subsection 2.4.2.

If $\mathcal{C}$ is G-crossed, then $Z^{(2)}(\mathcal{C};\mathcal{C}_e)$ is G-crossed

Let G be a group and let $(\mathcal{C}, \otimes, I, F, \partial)$ be a G-crossed category. It follows from our discussion in Subsection 2.4.2 that the relative Drinfeld center $Z(\mathcal{C};\mathcal{C}_e)$ is a tensor category. The same is true for $Z^{(2)}(\mathcal{C};\mathcal{C}_e)$, as defined in Remark 2.4.10. We can define a G-grading on $Z^{(2)}(\mathcal{C};\mathcal{C}_e)$ by letting $\partial[(V,\Psi_V)] := \partial(V)$, since

$$\begin{aligned}\partial[(V,\Psi_V)\otimes(W,\Psi_W)] &= \partial[(V\otimes W, \Psi_V\otimes\Psi_W)] = \partial(V\otimes W) = \partial(V)\partial(W)\\ &= \partial[(V,\Psi_V)]\partial[(W,\Psi_W)].\end{aligned}$$

If Ψ_V is a half braiding of the second kind for $V \in \mathcal{C}$ relative to $\mathcal{C}_e$ and if $q \in G$ then we obtain a half braiding of the second kind $\mathcal{F}_q\Psi_V$ for $F_q(V)$ relative to $\mathcal{C}_e$ by defining

$$\mathcal{F}_q\Psi_V(X) := F_q(\Psi_V(F_{q^{-1}}(X)))$$

for all $X \in \mathcal{C}_e$. This defines a group action $\mathcal{F}$ on $Z^{(2)}(\mathcal{C};\mathcal{C}_e)$ by letting

$$\begin{aligned}\mathcal{F}_q[(V,\Psi_V)] &= (F_q(V), \mathcal{F}_q\Psi_V),\\ \mathcal{F}_q(f) &= F_q(f)\end{aligned}$$

for any $(V,\Psi_V) \in Z^{(2)}(\mathcal{C};\mathcal{C}_e)$ and $f \in \mathrm{Hom}(Z^{(2)}(\mathcal{C};\mathcal{C}_e))$. In fact this equips $Z^{(2)}(\mathcal{C};\mathcal{C}_e)$ with the structure of a G-crossed category. Furthermore, $Z^{(2)}(\mathcal{C};\mathcal{C}_e)$ has a partial braiding of the second kind with respect to $Z^{(2)}(\mathcal{C}_e)$ given by

$$C_{(V,\Psi_V),(W,\Psi_W)} := \Psi_W(V),$$

where $(V,\Psi_V) \in Z^{(2)}(\mathcal{C}_e)$ and $(W,\Psi_W) \in Z^{(2)}(\mathcal{C};\mathcal{C}_e)$. Now that we have seen that $Z^{(2)}(\mathcal{C};\mathcal{C}_e)$ is a G-crossed category with a partial braiding of the second kind with respect to $Z^{(2)}(\mathcal{C}_e)$ whenever $\mathcal{C}$ is a G-crossed category, we want to apply this to a particular example. But first we need to introduce the following construction of a G-crossed category $\mathcal{C} \rtimes G$ from a G-category $\mathcal{C}$.

The G-crossed category $\mathcal{C} \rtimes G$

Let G be a group and let $(\mathcal{C}, \otimes, I)$ be a strict tensor category with a strict G-action F. Then we define a category $\mathcal{C} \rtimes G$ as follows. The objects are pairs[8] $\langle V, q\rangle$ where $V \in \mathcal{C}$ and $q \in G$. We define

$$\mathrm{Hom}_{\mathcal{C}\rtimes G}(\langle V,q\rangle, \langle W,r\rangle) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(V,W) & \text{if } q = r \\ \emptyset & \text{if } q \neq r, \end{cases}$$

where the composition in $\mathcal{C} \rtimes G$ is the same as in $\mathcal{C}$ and for any $\langle V, q\rangle \in \mathcal{C} \rtimes G$ we set $\mathrm{id}_{\langle V,q\rangle} := \mathrm{id}_V$. We will now define a tensor product $\otimes^{\rtimes}$ on the objects of $\mathcal{C} \rtimes G$ by

$$\langle V,q\rangle \otimes^{\rtimes} \langle W,r\rangle := \langle V \otimes F_q(W), qr\rangle.$$

It is easy to check that $\otimes^{\rtimes}$ is associative on the objects and that for any $\langle V,q\rangle \in \mathcal{C} \rtimes G$ we have $\langle I,e\rangle \otimes^{\rtimes} \langle V,q\rangle = \langle V,q\rangle = \langle V,q\rangle \otimes^{\rtimes} \langle I,e\rangle$. If $\langle V,q\rangle, \langle V',q\rangle, \langle W,r\rangle, \langle W',r\rangle \in \mathcal{C} \rtimes G$ and if we have $f \in \mathrm{Hom}_{\mathcal{C}\rtimes G}(\langle V,q\rangle, \langle V',q\rangle)$ and $g \in \mathrm{Hom}_{\mathcal{C}\rtimes G}(\langle W,r\rangle, \langle W',r\rangle)$, then we define

$$f \otimes^{\rtimes} g := f \otimes F_q(g).$$

This equips $\mathcal{C} \rtimes G$ with the structure of a strict tensor category. In fact, it is G-graded, with grading given by $\partial^{\rtimes}(\langle V,q\rangle) = q$. The group action F of $\mathcal{C}$ can be used to define a group action $F^{\rtimes}$ on $\mathcal{C} \rtimes G$ by

$$F_q^{\rtimes}(\langle V,r\rangle) := \langle F_q(V), qrq^{-1}\rangle$$

and on the morphisms we define $F_q^{\rtimes}(f) := F_q(f)$. In this way $\mathcal{C} \rtimes G$ becomes a G-crossed category with G-spectrum $\partial^{\rtimes}(\mathcal{C} \rtimes G) = G$. The G-category $\mathcal{C}$ can be identified with the G-subcategory $(\mathcal{C} \rtimes G)_e$ of $\mathcal{C} \rtimes G$ in the obvious way. Thus $\mathcal{C} \rtimes G$ can be considered as a G-crossed extension of $\mathcal{C}$.

The G-crossed category $\mathcal{C}^{\mathrm{rev}} \rtimes G$

Let G be a group and let $(\mathcal{C}, \otimes, I)$ be a strict tensor category with strict G-action F. The reversed tensor category $\mathcal{C}^{\mathrm{rev}}$, which is just $\mathcal{C}$ as a category but with $\otimes$ replaced by $\otimes^{\mathrm{rev}} = \otimes \circ \tau$ (where τ is the flip functor as defined at the beginning of Section 2.4), is again a G-category[9] with the same G-action as in $\mathcal{C}$. Hence we can construct its G-crossed extension $\mathcal{C}^{\mathrm{rev}} \rtimes G$. On the objects its tensor product is given by

$$\langle V,q\rangle \widehat{\otimes} \langle W,r\rangle = \langle V \otimes^{\mathrm{rev}} F_q(W), qr\rangle = \langle F_q(W) \otimes V, qr\rangle$$

and for the tensor product on the morphisms one finds a similar expression, namely

$$f \widehat{\otimes} g = F_q(g) \otimes f,$$

where q is the degree of both the domain object and target object of f. The group action is given by

$$\widehat{F}_q(\langle V,r\rangle) = \langle F_q(V), qrq^{-1}\rangle$$
$$\widehat{F}_q(f) = F_q(f)$$

and the G-grading is given by $\widehat{\partial}(\langle V,q\rangle) = q$.

Remark 4.4.1 Note that $\mathcal{C}^{\mathrm{rev}} \rtimes G$ is not the same as $(\mathcal{C} \rtimes G)^{\mathrm{rev}}$. The latter is not even a G-crossed category unless G is abelian, since the compatibility between the tensor product and the grading of $\mathcal{C} \rtimes G$ is lost after reversing the tensor product (one has to invert the degrees of the objects at the same time in order to retain this compatibility).

[8] We use the brackets $\langle\ ,\ \rangle$ rather than $(\ ,\ )$ because this will make the more complicated expressions at the end of this section more readable.

[9] This follows from $F_q(V \otimes^{\mathrm{rev}} W) = F_q(W \otimes V) = F_q(W) \otimes F_q(V) = F_q(V) \otimes^{\mathrm{rev}} F_q(W)$.

From now on we will write $\mathcal{D} := \mathcal{C}^{\text{rev}} \rtimes G$ for simplicity. In what follows, we will be interested in $Z^{(2)}(\mathcal{D}, \mathcal{D}_e)$, which was introduced in Remark 2.4.10 in Subsection 2.4.2. Note that the objects of $Z^{(2)}(\mathcal{D}, \mathcal{D}_e)$ are pairs $\big(\langle V, q\rangle, \Psi_{\langle V,q\rangle}\big)$, where $\Psi_{\langle V,q\rangle}$ is a half braiding of the second kind for $\langle V, q\rangle \in \mathcal{D}$ relative to $\mathcal{D}_e$. If $\big(\langle V, q\rangle, \Psi_{\langle V,q\rangle}\big)$ and $\big(\langle W, q\rangle, \Psi_{\langle W,q\rangle}\big)$ are objects in $Z^{(2)}(\mathcal{D}; \mathcal{D}_e)$, then $\text{Hom}_{Z^{(2)}(\mathcal{D};\mathcal{D}_e)}\left\{\big(\langle V, q\rangle, \Psi_{\langle V,q\rangle}\big), \big(\langle W, q\rangle, \Psi_{\langle W,q\rangle}\big)\right\}$ is equal to the set of all $f \in \text{Hom}_{\mathcal{D}}(\langle V, q\rangle, \langle W, q\rangle)$ that satisfy

$$\big[f \widehat{\otimes} \text{id}_{\langle X,e\rangle}\big] \circ \Psi_{\langle V,q\rangle}(\langle X, e\rangle) = \Psi_{\langle W,q\rangle}(\langle X, e\rangle) \circ \big[\text{id}_{\langle X,e\rangle} \widehat{\otimes} f\big]$$

for all $\langle X, e\rangle \in \mathcal{D}_e$, which can be slightly simplified by writing

$$\begin{aligned} &\text{Hom}_{Z^{(2)}(\mathcal{D};\mathcal{D}_e)}\left\{\big(\langle V, q\rangle, \Psi_{\langle V,q\rangle}\big), \big(\langle W, q\rangle, \Psi_{\langle W,q\rangle}\big)\right\} \\ &= \left\{f \in \text{Hom}_{\mathcal{C}}(V, W) : \big[f \widehat{\otimes} \text{id}_{\langle X,e\rangle}\big] \circ \Psi_{\langle V,q\rangle}(\langle X, e\rangle) = \Psi_{\langle W,q\rangle}(\langle X, e\rangle) \circ \big[\text{id}_{\langle X,e\rangle} \widehat{\otimes} f\big] \quad \forall X \in \mathcal{C}\right\}. \end{aligned}$$

On the objects the tensor product is given by

$$\begin{aligned} \big(\langle V, q\rangle, \Psi_{\langle V,q\rangle}\big) \widehat{\otimes} \big(\langle W, r\rangle, \Psi_{\langle W,r\rangle}\big) &= \big(\langle V, q\rangle \widehat{\otimes} \langle W, r\rangle, \Psi_{\langle V,q\rangle} \widehat{\otimes} \Psi_{\langle W,r\rangle}\big) \\ &= \big(\langle F_q(W) \otimes V, qr\rangle, \Psi_{\langle V,q\rangle} \widehat{\otimes} \Psi_{\langle W,r\rangle}\big), \end{aligned}$$

where $\Psi_{\langle V,q\rangle} \widehat{\otimes} \Psi_{\langle W,r\rangle}$ is the half braiding of the second kind for $\langle F_q(W) \otimes V, qr\rangle$ given by

$$\big(\Psi_{\langle V,q\rangle} \widehat{\otimes} \Psi_{\langle W,r\rangle}\big)(\langle X, e\rangle) = \big[\text{id}_{\langle V,q\rangle} \widehat{\otimes} \Psi_{\langle W,r\rangle}(\langle X, e\rangle)\big] \circ \big[\Psi_{\langle V,q\rangle}(\langle X, e\rangle) \widehat{\otimes} \text{id}_{\langle W,r\rangle}\big]$$

for $\langle X, e\rangle \in \mathcal{D}_e$, and on the morphisms the tensor product is the same as in $\mathcal{D}$. On the objects the group action is given by

$$\widehat{\mathcal{F}}_q\left[\big(\langle V, r\rangle, \Psi_{\langle V,r\rangle}\big)\right] = \left(\widehat{F}_q(\langle V, r\rangle), \widehat{\mathcal{F}}_q \Psi_{\langle V,r\rangle}\right) = \left(\langle F_q(V), qrq^{-1}\rangle, \widehat{\mathcal{F}}_q \Psi_{\langle V,r\rangle}\right),$$

where $\widehat{\mathcal{F}}_q \Psi_{\langle V,r\rangle}$ is the half braiding of the second kind for $\langle F_q(V), qrq^{-1}\rangle$ given by

$$\widehat{\mathcal{F}}_q \Psi_{\langle V,r\rangle}(\langle X, e\rangle) = \widehat{F}_q\left(\Psi_{\langle V,r\rangle}\left(\widehat{F}_{q^{-1}}(\langle X, e\rangle)\right)\right) = F_q\left(\Psi_{\langle V,r\rangle}\left(\langle F_{q^{-1}}(X), e\rangle\right)\right)$$

for $\langle X, e\rangle \in \mathcal{D}_e$, and on the morphisms the group action is the same as in $\mathcal{D}$. The G-grading is simply given by

$$\widehat{\partial}\left[\big(\langle V, q\rangle, \Psi_{\langle V,q\rangle}\big)\right] = \partial^{\rtimes}(\langle V, q\rangle) = q.$$

In this way, $Z^{(2)}(\mathcal{D}; \mathcal{D}_e)$ becomes a G-crossed category. It also has a partial braiding $\widehat{C}$ of the second kind relative to $Z^{(2)}(\mathcal{D}_e)$ given by

$$\widehat{C}_{(\langle V,e\rangle, \Psi_{\langle V,e\rangle}),(\langle W,r\rangle, \Psi_{\langle W,r\rangle})} = \Psi_{\langle W,r\rangle}(\langle V, e\rangle).$$

As we will see in Proposition 4.4.6 below, this partial braiding of the second kind relative to $Z^{(2)}(\mathcal{D}_e)$ can be extended to an ordinary braiding. But first we need to prove some lemmas that will also be useful in proving Theorem 4.4.7, which will be the main result of this section.

Some useful lemmas

Now let $\langle V, q\rangle \in \mathcal{D}$ and let $\Psi_{\langle V,q\rangle}$ be a half braiding of the second kind for $\langle V, q\rangle$ relative to $\mathcal{D}_e$. Then for any $\langle X, e\rangle \in \mathcal{D}_e$ we have that $\Psi_{\langle V,q\rangle}(\langle X, e\rangle)$ is an isomorphism

$$\Psi_{\langle V,q\rangle}(\langle X, e\rangle) : \underbrace{\langle X, e\rangle \widehat{\otimes} \langle V, q\rangle}_{=\langle V \otimes X, q\rangle} \to \underbrace{\langle V, q\rangle \widehat{\otimes} \langle X, e\rangle}_{=\langle F_q(X) \otimes V, q\rangle}.$$

In particular, we can write $\Psi_{\langle V,q\rangle}(\langle X, e\rangle) = \Phi_V(X)$ for some $\Phi_V(X) \in \text{Hom}_{\mathcal{C}}(V \otimes X, F_q(X) \otimes V)$. As the following lemma shows, this gives rise to a half q-braiding for V.

Lemma 4.4.2 *Let $\mathcal{C}$ be a G-category and let $\mathcal{D} = \mathcal{C}^{\text{rev}} \rtimes G$.*
(1) If $\Psi_{\langle V,q\rangle}$ is a half braiding of the second kind for $\langle V,q\rangle \in \mathcal{D}$ relative to $\mathcal{D}_e$, then

$$\Phi_V(X) := \Psi_{\langle V,q\rangle}(\langle X,e\rangle)$$

defines a half q-braiding for $V \in \mathcal{C}$.
(2) If Φ_V is a half q-braiding for $V \in \mathcal{C}$, then

$$\Psi_{\langle V,q\rangle}(\langle X,e\rangle) := \Phi_V(X)$$

defines a half braiding of the second kind for $\langle V,q\rangle \in \mathcal{D}$ relative to $\mathcal{D}_e$.

Proof. We can simply prove both statements at once, because in both statements Φ_V and $\Psi_{\langle V,q\rangle}$ are related in the same way. Naturality of $\Psi_{\langle V,q\rangle}$ means that for any $f \in \text{Hom}_{\mathcal{C}}(X,Y)$ we have

$$\Psi_{\langle V,q\rangle}(\langle Y,e\rangle) \circ \left[f \widehat{\otimes} \text{id}_{\langle V,q\rangle}\right] = \left[\text{id}_{\langle V,q\rangle} \widehat{\otimes} f\right] \circ \Psi_{\langle V,q\rangle}(\langle X,e\rangle),$$

which can be rewritten as

$$\Phi_V(Y) \circ [\text{id}_V \otimes f] = [F_q(f) \otimes \text{id}_V] \circ \Phi_V(X),$$

which is equivalent to naturality of Φ_V. Now note that the condition

$$\Psi_{\langle V,q\rangle}(\langle Y,e\rangle \widehat{\otimes} \langle X,e\rangle) = \left[\Psi_{\langle V,q\rangle}(\langle Y,e\rangle) \widehat{\otimes} \text{id}_{\langle X,e\rangle}\right] \circ \left[\text{id}_{\langle Y,e\rangle} \widehat{\otimes} \Psi_{\langle V,q\rangle}(\langle X,e\rangle)\right]$$

can be rewritten as

$$\Phi_V(X \otimes Y) = [\text{id}_{F_q(X)} \otimes \Phi_V(Y)] \circ [\Phi_V(X) \otimes \text{id}_Y].$$

Note that the different order of X and Y on the left-hand sides of the last two equations is not relevant, because these equations have to be satisfied for all $X, Y \in \mathcal{C}$. This completes the proof.
□

As the following lemma shows, the morphisms in $Z^{(2)}(\mathcal{D};\mathcal{D}_e)$ coincide in a certain sense with the morphisms in $Z_G(\mathcal{C})$.

Lemma 4.4.3 *Let $\mathcal{C}$ be a G-category and let $\mathcal{D} = \mathcal{C}^{\text{rev}} \rtimes G$ and $q \in G$. If $\Psi_{\langle V,q\rangle}$ and $\Psi_{\langle W,q\rangle}$ are half braidings of the second kind for $\langle V,q\rangle$ and $\langle W,q\rangle$ relative to $\mathcal{D}_e$, with corresponding half q-braidings Φ_V and Φ_W for V and W, then for any $f \in \text{Hom}_{\mathcal{C}}(V,W)$ the following two statements are equivalent:*
(1) In the category $Z^{(2)}(\mathcal{D};\mathcal{D}_e)$ we have $f \in \text{Hom}\left\{\left(\langle V,q\rangle, \Psi_{\langle V,q\rangle}\right), \left(\langle W,q\rangle, \Psi_{\langle W,q\rangle}\right)\right\}$.
(2) In the category $Z_G(\mathcal{C})$ we have $f \in \text{Hom}\left\{(V,q,\Phi_V), (W,q,\Phi_W)\right\}$.

Proof. Statement (1) is equivalent to the condition that

$$\Psi_{\langle W,q\rangle}(\langle X,e\rangle) \circ \left[\text{id}_{\langle X,e\rangle} \widehat{\otimes} f\right] = \left[f \widehat{\otimes} \text{id}_{\langle X,e\rangle}\right] \circ \Psi_{\langle V,q\rangle}(\langle X,e\rangle)$$

for all $\langle X,e\rangle \in \mathcal{D}_e$, which can be restated as the condition that

$$\Phi_W(X) \circ [f \otimes \text{id}_X] = [\text{id}_{F_q(X)} \otimes f] \circ \Phi_V(X)$$

for all $X \in \mathcal{C}$, which is equivalent to statement (2).
□

As shown in Lemma 4.4.2, there is a one-to-one correspondence between half braidings of the second kind $\Psi_{\langle V,q\rangle}$ relative to $\mathcal{D}_e$ for $\langle V,q\rangle$ on the one hand, and half q-braidings Φ_V for V on the other. In the following lemma we will show that the group action behaves very nicely with respect to this correspondence.

Lemma 4.4.4 *Let $\mathcal{C}$ be a G-category and let $\mathcal{D} = \mathcal{C}^{\mathrm{rev}} \rtimes G$ and $q \in G$. If $\Psi_{\langle V,r\rangle}$ is a half braiding of the second kind for $\langle V, r\rangle$ relative to $\mathcal{D}_e$ and if Φ_V is the corresponding half r-braiding for V, then for all $X \in \mathcal{C}$ we have the equality*

$$\widehat{\mathcal{F}}_q \Psi_{\langle V,r\rangle}(\langle X, e\rangle) = \mathcal{F}_q \Phi_V(X).$$

In other words, if $\Psi_{\langle V,r\rangle}$ corresponds to Φ_V, then $\widehat{\mathcal{F}}_q \Psi_{\langle V,r\rangle}$ corresponds to $\mathcal{F}_q \Phi_V$.

Proof. This follows from the computation

$$\begin{aligned}\widehat{\mathcal{F}}_q \Psi_{\langle V,r\rangle}(\langle X, e\rangle) &= \widehat{F}_q \left[\Psi_{\langle V,r\rangle}\left(\widehat{F}_{q^{-1}}(\langle X, e\rangle)\right)\right] = \widehat{F}_q \left[\Psi_{\langle V,r\rangle}(\langle F_{q^{-1}}(X), e\rangle)\right] \\ &= F_q \left(\Phi_V(F_{q^{-1}}(X))\right) = \mathcal{F}_q \Phi_V(X).\end{aligned}$$

□

Finally, we also have to investigate how the tensor product behaves with respect to the correspondence. This will be done in the following lemma.

Lemma 4.4.5 *Let $\mathcal{C}$ be a G-category and let $\mathcal{D} = \mathcal{C}^{\mathrm{rev}} \rtimes G$ and $q \in G$.*

(1) If $\Psi_{\langle V,q\rangle}$ and $\Psi_{\langle W,r\rangle}$ are half braidings of the second kind for $\langle V, q\rangle$ and $\langle W, r\rangle$ relative to $\mathcal{D}_e$ and if Φ_V and Φ_W are the corresponding half q-braiding for V and half r-braiding for W, then for any $X \in \mathcal{C}$ we have

$$\left(\Psi_{\langle V,q\rangle} \widehat{\otimes} \Psi_{\langle W,r\rangle}\right)(\langle X, e\rangle) = (\mathcal{F}_q \Phi_W \otimes \Phi_V)(X).$$

In other words, $\mathcal{F}_q \Phi_W \otimes \Phi_V$ is the half qr-braiding for $F_q(W) \otimes V$ corresponding to the half braiding of the second kind $\Psi_{\langle V,q\rangle} \widehat{\otimes} \Psi_{\langle W,r\rangle}$ relative to $\mathcal{D}_e$ for $\langle F_q(W) \otimes V, qr\rangle$.

(2) If Φ_V and Φ_W are a half q-braiding for V and a half r-braiding for W and if $\Psi_{\langle V,q\rangle}$ and $\Psi_{\langle W,r\rangle}$ are the corresponding half braidings of the second kind for $\langle V, q\rangle$ and $\langle W, r\rangle$ relative to $\mathcal{D}_e$, then for any $X \in \mathcal{C}$ we have

$$(\Phi_V \otimes \Phi_W)(X) = \left(\Psi_{\langle W,r\rangle} \widehat{\otimes} \widehat{\mathcal{F}}_{r^{-1}} \Psi_{\langle V,q\rangle}\right)(\langle X, e\rangle).$$

In other words $\Psi_{\langle W,r\rangle} \widehat{\otimes} \widehat{\mathcal{F}}_{r^{-1}} \Psi_{\langle V,q\rangle}$ is the half braiding of the second kind relative to $\mathcal{D}_e$ for $\langle V \otimes W, qr\rangle$ corresponding to the half qr-braiding $\Phi_V \otimes \Phi_W$ for $V \otimes W$.

Proof. (1) We have

$$\begin{aligned}\left(\Psi_{\langle V,q\rangle} \widehat{\otimes} \Psi_{\langle W,r\rangle}\right)(\langle X, e\rangle) &= \left[\mathrm{id}_{\langle V,q\rangle} \widehat{\otimes} \Psi_{\langle W,r\rangle}(\langle X, e\rangle)\right] \circ \left[\Psi_{\langle V,q\rangle}(\langle X, e\rangle) \widehat{\otimes} \mathrm{id}_{\langle W,r\rangle}\right] \\ &= [F_q(\Phi_W(X)) \otimes \mathrm{id}_V] \circ [\mathrm{id}_{F_q(W)} \otimes \Phi_V(X)] \\ &= [\mathcal{F}_q \Phi_W(F_q(X)) \otimes \mathrm{id}_V] \circ [\mathrm{id}_{F_q(W)} \otimes \Phi_V(X)] \\ &= (\mathcal{F}_q \Phi_W \otimes \Phi_V)(X).\end{aligned}$$

(2) A similar calculation as in part (1), but in reverse order, gives us

$$\begin{aligned}(\Phi_V \otimes \Phi_W)(X) &= [\Phi_V(F_r(X)) \otimes \mathrm{id}_W] \circ [\mathrm{id}_V \otimes \Phi_W(X)] \\ &= \left[\mathrm{id}_{\langle W,r\rangle} \widehat{\otimes} \widehat{\mathcal{F}}_{r^{-1}} \Psi_{\langle V,q\rangle}(\langle X, e\rangle)\right] \circ \left[\Psi_{\langle W,r\rangle}(\langle X, e\rangle) \widehat{\otimes} \mathrm{id}_{\widehat{F}_{r^{-1}}(\langle V,q\rangle)}\right] \\ &= \left(\Psi_{\langle W,r\rangle} \widehat{\otimes} \widehat{\mathcal{F}}_{r^{-1}} \Psi_{\langle V,q\rangle}\right)(\langle X, e\rangle).\end{aligned}$$

Note that we could also have used part (1) to conclude that

$$\left(\Psi_{\langle W,r\rangle} \widehat{\otimes} \widehat{\mathcal{F}}_{r^{-1}} \Psi_{\langle V,q\rangle}\right)(\langle X, e\rangle) = (\mathcal{F}_r \mathcal{F}_{r^{-1}} \Phi_V \otimes \Phi_W)(X) = (\Phi_V \otimes \Phi_W)(X).$$

□

The G-crossed category $Z^{(2)}(\mathcal{D};\mathcal{D}_e)$ has a braiding

From our discussion at the beginning of this section we know that $Z^{(2)}(\mathcal{D};\mathcal{D}_e)$ is a G-crossed category that has a partial braiding $\widehat{C}$ of the second kind. This statement is true for any G-crossed category $\mathcal{D}$ and is not based on the fact that $\mathcal{D}=\mathcal{C}\rtimes G$. As we will now show, in the case where $\mathcal{D}=\mathcal{C}\rtimes G$ we can equip $Z^{(2)}(\mathcal{D};\mathcal{D}_e)$ with a braiding that extends $\widehat{C}$, so that it becomes a braided G-crossed category.

Proposition 4.4.6 *If $\mathcal{C}$ is a G-category and $\mathcal{D}=\mathcal{C}^{\mathrm{rev}}\rtimes G$, then the G-crossed category $Z^{(2)}(\mathcal{D};\mathcal{D}_e)$ has a braiding (of the first kind).*

Proof. For any two objects $(\langle V,q\rangle,\Psi_{\langle V,q\rangle})$ and $(\langle W,r\rangle,\Psi_{\langle W,r\rangle})$ in $Z^{(2)}(\mathcal{D};\mathcal{D}_e)$ we define

$$\widehat{C}_{(\langle V,q\rangle,\Psi_{\langle V,q\rangle}),(\langle W,r\rangle,\Psi_{\langle W,r\rangle})} := \mathcal{F}_q\Phi_W(V),$$

where Φ_W is the half r-braiding for $W\in\mathcal{C}$ corresponding to $\Psi_{\langle W,r\rangle}$. For $\widehat{C}$ to be a braiding (of the first kind), the domain object of $\widehat{C}_{(\langle V,q\rangle,\Psi_{\langle V,q\rangle}),(\langle W,r\rangle,\Psi_{\langle W,r\rangle})}$ has to be

$$\left(\langle V,q\rangle,\Psi_{\langle V,q\rangle}\right)\widehat{\otimes}\left(\langle W,r\rangle,\Psi_{\langle W,r\rangle}\right)=\left(\langle F_q(W)\otimes V,qr\rangle,\Psi_{\langle V,q\rangle}\widehat{\otimes}\Psi_{\langle W,r\rangle}\right)$$

and its target object has to be

$$\begin{aligned}\widehat{\mathcal{F}}_q\left[(\langle W,r\rangle,\Psi_{\langle W,r\rangle})\right]\widehat{\otimes}\left(\langle V,q\rangle,\Psi_{\langle V,q\rangle}\right)&=\left(\langle F_q(W),qrq^{-1}\rangle,\widehat{\mathcal{F}}_q\Psi_{\langle W,r\rangle}\right)\widehat{\otimes}\left(\langle V,q\rangle,\Psi_{\langle V,q\rangle}\right)\\&=\left(\langle F_{qrq^{-1}}(V)\otimes F_q(W),qr\rangle,\widehat{\mathcal{F}}_q\Psi_{\langle W,r\rangle}\widehat{\otimes}\Psi_{\langle V,q,\rangle}\right).\end{aligned}$$

To check whether this is the case, we first note that $\mathcal{F}_q\Phi_W$ is a half qrq^{-1}-braiding for $F_q(W)$, so $\mathcal{F}_q\Phi_W(V)$ is an isomorphism from $F_q(W)\otimes V$ to $F_{qrq^{-1}}(V)\otimes F_q(W)$. Furthermore, for any $\langle X,e\rangle\in\mathcal{D}_e$ we have

$$\begin{aligned}&\left[\widehat{C}_{(\langle V,q\rangle,\Psi_{\langle V,q\rangle}),(\langle W,r\rangle,\Psi_{\langle W,r\rangle})}\widehat{\otimes}\mathrm{id}_{\langle X,e\rangle}\right]\circ\left(\Psi_{\langle V,q\rangle}\widehat{\otimes}\Psi_{\langle W,r\rangle}\right)(\langle X,e\rangle)\\&=\left[\widehat{C}_{(\langle V,q\rangle,\Psi_{\langle V,q\rangle}),(\langle W,r\rangle,\Psi_{\langle W,r\rangle})}\widehat{\otimes}\mathrm{id}_{\langle X,e\rangle}\right]\circ[\mathrm{id}_{\langle V,q\rangle}\widehat{\otimes}\Psi_{\langle W,r\rangle}(\langle X,e\rangle)]\circ[\Psi_{\langle V,q\rangle}(\langle X,e\rangle)\widehat{\otimes}\mathrm{id}_{\langle W,r\rangle}]\\&=[\mathrm{id}_{F_{qr}(X)}\otimes\mathcal{F}_q\Phi_W(V)]\circ[F_q(\Phi_W(X))\otimes\mathrm{id}_V]\circ[\mathrm{id}_{F_q(W)}\otimes\Phi_V(X)]\\&=F_q\left\{[\mathrm{id}_{F_r(X)}\otimes\Phi_W(F_{q^{-1}}(V))]\circ[\Phi_W(X)\otimes\mathrm{id}_{F_{q^{-1}}(V)}]\circ[\mathrm{id}_W\otimes F_{q^{-1}}(\Phi_V(X))]\right\}\\&=F_q\left\{\Phi_W(X\otimes F_{q^{-1}}(V))\circ[\mathrm{id}_W\otimes F_{q^{-1}}(\Phi_V(X))]\right\}\\&=F_q\left\{[F_{rq^{-1}}(\Phi_V(X))\otimes\mathrm{id}_W]\circ\Phi_W(F_{q^{-1}}(V\otimes X))\right\}\\&=[F_{qrq^{-1}}(\Phi_V(X))\otimes\mathrm{id}_{F_q(W)}]\circ\mathcal{F}_q\Phi_W(V\otimes X)\\&=[F_{qrq^{-1}}(\Phi_V(X))\otimes\mathrm{id}_{F_q(W)}]\circ[\mathrm{id}_{F_{rqr^{-1}}(V)}\otimes\mathcal{F}_q\Phi_W(X)]\circ[\mathcal{F}_q\Phi_W(V)\otimes\mathrm{id}_X]\\&=\left[\mathrm{id}_{\langle F_q(W),qrq^{-1}\rangle}\widehat{\otimes}\Psi_{\langle V,q\rangle}(\langle X,e\rangle)\right]\circ\left[\mathcal{F}_q\Psi_{\langle W,r\rangle}(\langle X,e\rangle)\widehat{\otimes}\mathrm{id}_{\langle V,q\rangle}\right]\circ\left[\mathrm{id}_{\langle X,e\rangle}\widehat{\otimes}\widehat{C}_{(\langle V,q\rangle,\Psi_{\langle V,q\rangle}),(\langle W,r\rangle,\Psi_{\langle W,r\rangle})}\right]\\&=\left(\widehat{\mathcal{F}}_q\Psi_{\langle W,r\rangle}\widehat{\otimes}\Psi_{\langle V,q\rangle}\right)(\langle X,e\rangle)\circ\left[\mathrm{id}_{\langle X,e\rangle}\widehat{\otimes}\widehat{C}_{(\langle V,q\rangle,\Psi_{\langle V,q\rangle}),(\langle W,r\rangle,\Psi_{\langle W,r\rangle})}\right].\end{aligned}$$

This proves that $\widehat{C}_{(\langle V,q\rangle,\Psi_{\langle V,q\rangle}),(\langle W,r\rangle,\Psi_{\langle W,r\rangle})}$ has indeed the correct domain and target objects. It is also clear that it is an isomorphism, since $\mathcal{F}_q\Phi_W(V)$ is an isomorphism in the category $\mathcal{C}$. To prove naturality, consider two morphisms

$$\begin{aligned}f&\in\mathrm{Hom}_{Z^{(2)}(\mathcal{D};\mathcal{D}_e)}\left\{\left(\langle V,q\rangle,\Psi_{\langle V,q\rangle}\right),\left(\langle V',q\rangle,\Psi_{\langle V',q\rangle}\right)\right\}\\g&\in\mathrm{Hom}_{Z^{(2)}(\mathcal{D};\mathcal{D}_e)}\left\{\left(\langle W,r\rangle,\Psi_{\langle W,r\rangle}\right),\left(\langle W',r\rangle,\Psi_{\langle W',r\rangle}\right)\right\}.\end{aligned}$$

For these morphisms we have

$$\begin{aligned}\widehat{C}_{(\langle V',q\rangle,\Psi_{\langle V',q\rangle}),(\langle W',r\rangle,\Psi_{\langle W',r\rangle})}\circ[f\widehat{\otimes}g] &= \mathcal{F}_q\Phi_{W'}(V')\circ[F_q(g)\otimes f]\\ &= [F_{qrq^{-1}}(f)\otimes \mathrm{id}_{F_q(W')}]\circ\mathcal{F}_q\Phi_{W'}(V)\circ[F_q(g)\otimes\mathrm{id}_V]\\ &= [F_{qrq^{-1}}(f)\otimes F_q(g)]\circ\mathcal{F}_q\Phi_W(V)\\ &= [F_q(g)\widehat{\otimes}f]\circ\widehat{C}_{(\langle V,q\rangle,\Psi_{\langle V,q\rangle}),(\langle W,r\rangle,\Psi_{\langle W,r\rangle})},\end{aligned}$$

where in the second step we used naturality of $\mathcal{F}_q\Phi_{W'}$ and in the third step we used that

$$f\in\mathrm{Hom}_{Z_G(\mathcal{C})}\{(V,q,\Phi_V),(W,r,\Phi_W)\}$$

by Lemma 4.4.3. Also,

$$\begin{aligned}&\widehat{C}_{(\langle U,q\rangle,\Psi_{\langle U,q\rangle}),(\langle V,r\rangle,\Psi_{\langle V,r\rangle})\widehat{\otimes}(\langle W,s\rangle,\Psi_{\langle W,s\rangle})} = \widehat{C}_{(\langle U,q\rangle,\Psi_{\langle U,q\rangle}),(\langle F_r(W)\otimes V,rs\rangle,\Psi_{\langle V,r\rangle}\widehat{\otimes}\Psi_{\langle W,s\rangle})}\\ &= [\mathcal{F}_q(\mathcal{F}_r\Phi_W\otimes\Phi_V)](U) = (\mathcal{F}_{qr}\Phi_W\otimes\mathcal{F}_q\Phi_V)(U)\\ &= [\mathcal{F}_{qr}\Phi_W(F_{qrq^{-1}}(U))\otimes\mathrm{id}_{F_q(V)}]\circ[\mathrm{id}_{F_{qr}(W)}\otimes\mathcal{F}_q\Phi_V(U)]\\ &= [F_{qr}(\Phi_W(F_{q^{-1}}(U)))\otimes\mathrm{id}_{F_q(V)}]\circ[\mathrm{id}_{F_{qr}(W)}\otimes\mathcal{F}_q\Phi_V(U)]\\ &= [F_{qrq^{-1}}(\mathcal{F}_q\Phi_W(U))\otimes\mathrm{id}_{F_q(V)}]\circ[\mathrm{id}_{F_{qr}(W)}\otimes\mathcal{F}_q\Phi_V(U)]\\ &= \left[\mathrm{id}_{\widehat{\mathcal{F}}_q[(\langle V,r\rangle,\Psi_{\langle V,r\rangle})]}\widehat{\otimes}\widehat{C}_{(\langle U,q\rangle,\Psi_{\langle U,q\rangle}),(\langle W,s\rangle,\Psi_{\langle W,s\rangle})}\right]\circ\left[\widehat{C}_{(\langle U,q\rangle,\Psi_{\langle U,q\rangle}),(\langle V,r\rangle,\Psi_{\langle V,r\rangle})}\widehat{\otimes}\mathrm{id}_{(\langle W,s\rangle,\Psi_{\langle W,s\rangle})}\right]\end{aligned}$$

and

$$\begin{aligned}&\widehat{C}_{(\langle U,q\rangle,\Psi_{\langle U,q\rangle})\widehat{\otimes}(\langle V,r\rangle,\Psi_{\langle V,r\rangle}),(\langle W,s\rangle,\Psi_{\langle W,s\rangle})} = \widehat{C}_{(\langle F_q(V)\otimes U,qr\rangle,\Psi_{\langle U,q\rangle}\widehat{\otimes}\Psi_{\langle V,r\rangle}),(\langle W,s\rangle,\Psi_{\langle W,s\rangle})}\\ &= \mathcal{F}_{qr}\Phi_W(F_q(V)\otimes U)\\ &= [\mathrm{id}_{F_{qrsr^{-1}q^{-1}}(F_q(V))}\otimes\mathcal{F}_{qr}\Phi_W(U)]\circ[\mathcal{F}_{qr}\Phi_W(F_q(V))\otimes\mathrm{id}_U]\\ &= [\mathrm{id}_{F_{qrsr^{-1}}(V)}\otimes\mathcal{F}_{qr}\Phi_W(U)]\circ[F_q(\mathcal{F}_r\Phi_W(V))\otimes\mathrm{id}_U]\\ &= \left[\widehat{C}_{(\langle U,q\rangle,\Psi_{\langle U,q\rangle}),\widehat{\mathcal{F}}_r[(\langle W,s\rangle,\Psi_{\langle W,s\rangle})]}\widehat{\otimes}\mathrm{id}_{(\langle V,r\rangle,\Psi_{\langle V,r\rangle})}\right]\circ\left[\mathrm{id}_{(\langle U,q\rangle,\Psi_{\langle U,q\rangle})}\widehat{\otimes}\widehat{C}_{(\langle V,r\rangle,\Psi_{\langle V,r\rangle}),(\langle W,s\rangle,\Psi_{\langle W,s\rangle})}\right].\end{aligned}$$

Finally, we also have

$$\begin{aligned}\widehat{\mathcal{F}}_q\left[\widehat{C}_{(\langle V,r\rangle,\Psi_{\langle V,r\rangle}),(\langle W,s\rangle,\Psi_{\langle W,s\rangle})}\right] &= F_q(\mathcal{F}_r\Phi_W(V)) = \mathcal{F}_{qr}\Phi_W(F_q(V)) = \mathcal{F}_{qrq^{-1}}\mathcal{F}_q\Phi_W(F_q(V))\\ &= \widehat{C}_{(\langle F_q(V),qrq^{-1}\rangle,\widehat{\mathcal{F}}_q\Psi_{\langle V,r\rangle}),(\langle F_q(W),qsq^{-1}\rangle,\widehat{\mathcal{F}}_q\Psi_{\langle W,s\rangle})}\\ &= \widehat{C}_{\widehat{\mathcal{F}}_q[(\langle V,r\rangle,\Psi_{\langle V,r\rangle})],\widehat{\mathcal{F}}_q[(\langle W,s\rangle,\Psi_{\langle W,s\rangle})]}.\end{aligned}$$

This proves that $\widehat{C}$ is a braiding (of the first kind) for $Z^{(2)}(\mathcal{D};\mathcal{D}_e)$.
□

The equivalence $Z_G(\mathcal{C})\simeq Z^{(2)}(\mathcal{D};\mathcal{D}_e)$

We are now ready to formulate the main result of this section. It states that $Z_G(\mathcal{C})$ is equivalent to $Z^{(2)}(\mathcal{D};\mathcal{D}_e)$ as a braided G-crossed category.

Theorem 4.4.7 *If $\mathcal{C}$ is a G-category, then there exists an equivalence*

$$K: Z^{(2)}(\mathcal{C}^{\mathrm{rev}}\rtimes G;(\mathcal{C}^{\mathrm{rev}}\rtimes G)_e)\to Z_G(\mathcal{C})$$

of braided G-crossed categories.

Proof. In what follows we will write $\mathcal{D} = \mathcal{C}^{\text{rev}} \rtimes G$ for simplicity. We define a functor $K : Z^{(2)}(\mathcal{D};\mathcal{D}_e) \to Z_G(\mathcal{C})$ by

$$K\left[\left(\langle V,q\rangle, \Psi_{\langle V,q\rangle}\right)\right] = (V, q, \Phi_V)$$
$$K(f) = f,$$

where Φ_V is the half q-braiding for V defined by $\Phi_V(X) = \Psi_{\langle V,q\rangle}(\langle X, e\rangle)$. Lemmas 4.4.2 and 4.4.3 show that K is well-defined. That K is a functor follows from the simple computation

$$K(g \circ f) = g \circ f = K(g) \circ K(f).$$

We will now equip K with the structure $(K, \varepsilon^K, \delta^K)$ of a tensor functor. To find δ^K, let $\left(\langle V,q\rangle, \Psi_{\langle V,q\rangle}\right)$ and $\left(\langle W,r\rangle, \Psi_{\langle W,r\rangle}\right)$ be objects in $Z^{(2)}(\mathcal{D};\mathcal{D}_e)$ and let Φ_V and Φ_W be the half q-braiding for V and half r-braiding for W corresponding to $\Psi_{\langle V,q\rangle}$ and $\Psi_{\langle W,r\rangle}$. Then on the one hand we have

$$K\left[\left(\langle V,q\rangle, \Psi_{\langle V,q\rangle}\right)\right] \otimes K\left[\left(\langle W,r\rangle, \Psi_{\langle W,r\rangle}\right)\right] = (V, q, \Phi_V) \otimes (W, r, \Phi_W)$$

and on the other hand we have

$$\begin{aligned} K\left[\left(\langle V,q\rangle, \Psi_{\langle V,q\rangle}\right) \widehat{\otimes} \left(\langle W,r\rangle, \Psi_{\langle W,r\rangle}\right)\right] &= K\left[\left(V \widehat{\otimes} W, \Psi_{\langle V,q\rangle} \widehat{\otimes} \Psi_{\langle W,r\rangle}\right)\right] \\ &= (F_q(W) \otimes V, qr, \mathcal{F}_q \Phi_W \otimes \Phi_V) \\ &= \mathcal{F}_q[(W, r, \Phi_W)] \otimes (V, q, \Phi_V), \end{aligned}$$

where in the second step we used Lemma 4.4.5. From this it follows that $\delta^K_{\left(\langle V,q\rangle, \Psi_{\langle V,q\rangle}\right), \left(\langle W,r\rangle, \Psi_{\langle W,r\rangle}\right)}$ has to be an isomorphism

$$\delta^K_{\left(\langle V,q\rangle, \Psi_{\langle V,q\rangle}\right), \left(\langle W,r\rangle, \Psi_{\langle W,r\rangle}\right)} : (V, q, \Phi_V) \otimes (W, r, \Phi_W) \to \mathcal{F}_q[(W, r, \Phi_W)] \otimes (V, q, \Phi_V),$$

which suggests that we should choose

$$\delta^K_{\left(\langle V,q\rangle, \Psi_{\langle V,q\rangle}\right), \left(\langle W,r\rangle, \Psi_{\langle W,r\rangle}\right)} := C_{(V,q,\Phi_V),(W,r,\Phi_W)},$$

where C is the braiding in $Z_G(\mathcal{C})$. Naturality of δ^K follows from naturality of C. Namely, if we have morphisms $f : \left(\langle V,q\rangle, \Psi_{\langle V,q\rangle}\right) \to \left(\langle V',q\rangle, \Psi_{\langle V',q\rangle}\right)$ and $g : \left(\langle W,r\rangle, \Psi_{\langle W,r\rangle}\right) \to \left(\langle W',r\rangle, \Psi_{\langle W',r\rangle}\right)$ then

$$\begin{aligned} \delta^K_{\left(\langle V',q\rangle, \Psi_{\langle V',q\rangle}\right), \left(\langle W',r\rangle, \Psi_{\langle W',r\rangle}\right)} \circ [K(f) \otimes K(g)] &= C_{(V',q,\Phi_{V'}),(W',r,\Phi_{W'})} \circ [f \otimes g] \\ &= [F_q(g) \otimes f] \circ C_{(V,q,\Phi_V),(W,r,\Phi_W)} \\ &= K(f \widehat{\otimes} g) \circ \delta^K_{\left(\langle V,q\rangle, \Psi_{\langle V,q\rangle}\right), \left(\langle W,r\rangle, \Psi_{\langle W,r\rangle}\right)}. \end{aligned}$$

We also have

$$\begin{aligned} &\delta^K_{\left(\langle U,q\rangle, \Psi_{\langle U,q\rangle}\right) \widehat{\otimes} \left(\langle V,r\rangle, \Psi_{\langle V,r\rangle}\right), \left(\langle W,s\rangle, \Psi_{\langle W,s\rangle}\right)} \circ \left[\delta^K_{\left(\langle U,q\rangle, \Psi_{\langle U,q\rangle}\right), \left(\langle V,r\rangle, \Psi_{\langle V,r\rangle}\right)} \otimes \mathrm{id}_{K\left[\left(\langle W,s\rangle, \Psi_{\langle W,s\rangle}\right)\right]}\right] \\ &= C_{\mathcal{F}_q[(V,r,\Phi_V)] \otimes (U,q,\Phi_U), (W,s,\Phi_W)} \circ \left[C_{(U,q,\Phi_U),(V,r,\Phi_V)} \otimes \mathrm{id}_{(W,s,\Phi_W)}\right] \\ &= C_{(U,q,\Phi_U), \mathcal{F}_r[(W,s,\Phi_W)] \otimes (V,r,\Phi_V)} \circ \left[\mathrm{id}_{(U,q,\Phi_U)} \otimes C_{(V,r,\Phi_V),(W,s,\Phi_W)}\right] \\ &= \delta^K_{\left(\langle U,q\rangle, \Psi_{\langle U,q\rangle}\right), \left(\langle V,r\rangle, \Psi_{\langle V,r\rangle}\right) \widehat{\otimes} \left(\langle W,s\rangle, \Psi_{\langle W,s\rangle}\right)} \circ \left[\mathrm{id}_{K\left[\left(\langle U,q\rangle, \Psi_{\langle U,q\rangle}\right)\right]} \otimes \delta^K_{\left(\langle V,r\rangle, \Psi_{\langle V,r\rangle}\right), \left(\langle W,s\rangle, \Psi_{\langle W,s\rangle}\right)}\right]. \end{aligned}$$

To find ε^K, we first observe that for any $X \in \mathcal{C}$ we have $\Psi^0_{\langle I,e\rangle}(\langle X, e\rangle) = \mathrm{id}_{\langle X,e\rangle} = \mathrm{id}_X = \Phi^0_I(X)$. This implies $K[(\langle I,e\rangle, \Psi^0_{\langle I,e\rangle})] = (I, e, \Phi^0_I)$, which suggests to choose $\varepsilon^K = \mathrm{id}_{(I,e,\Phi^0_I)}$. For this choice of ε^K we find

$$\delta^K_{\left(\langle I,e\rangle, \Psi_{\langle I,e\rangle}\right), \left(\langle V,q\rangle, \Psi_{\langle V,q\rangle}\right)} \circ \left[\varepsilon^K \otimes \mathrm{id}_{K\left[\left(\langle V,q\rangle, \Psi_{\langle V,q\rangle}\right)\right]}\right] = C_{(I,e,\Phi^0_I),(V,q,\Phi_V)} \circ \left[\mathrm{id}_{(I,e,\Phi^0_I)} \otimes \mathrm{id}_{(V,q,\Phi_V)}\right]$$

$$= \mathrm{id}_{(V,q,\Phi_V)}$$

and

$$\delta^K_{(\langle V,q\rangle,\Psi_{\langle V,q\rangle}),(\langle I,e\rangle,\Psi_{\langle I,e\rangle})} \circ \left[\mathrm{id}_{K[(\langle V,q\rangle,\Psi_{\langle V,q\rangle})]} \otimes \varepsilon^K\right] = C_{(V,q,\Phi_V),(I,e,\Phi_I^0)} \circ \left[\mathrm{id}_{(V,q,\Phi_V)} \otimes \mathrm{id}_{(I,e,\Phi_I^0)}\right]$$
$$= \mathrm{id}_{(V,q,\Phi_V)},$$

which proves that $(K,\varepsilon^K,\delta^K)$ is indeed a tensor functor. We will now show that K can be equipped with the structure of a G-functor. It follows from Lemma 4.4.4 that

$$\begin{aligned}(K \circ \widehat{\mathcal{F}}_q)\left[(\langle V,r\rangle,\Psi_{\langle V,r\rangle})\right] &= (F_q(V), qrq^{-1}, \mathcal{F}_q\Phi_V) = \mathcal{F}_q[(V,r,\Phi_V)] \\ &= (\mathcal{F}_q \circ K)\left[(\langle V,r\rangle,\Psi_{\langle V,r\rangle})\right].\end{aligned}$$

Also, on the morphisms,

$$(K \circ \widehat{\mathcal{F}}_q)(f) = K(F_q(f)) = F_q(f) = (\mathcal{F}_q \circ K)(f),$$

so we have the equality of functors $K \circ \widehat{\mathcal{F}}_q = \mathcal{F}_q \circ K$ for any $q \in G$. This suggests that we might take the natural isomorphisms $\{\xi^K(q) : K \circ \widehat{\mathcal{F}}_q \to \mathcal{F}_q \circ K\}_{q\in G}$ in the definition of a G-functor to be trivial. For this choice of ξ^K all diagrams in the definition of a G-functor trivialize, except the last one. But the commutativity of that diagram follows from the simple computation

$$\begin{aligned}\delta^K_{\widehat{\mathcal{F}}_q[(\langle V,r\rangle,\Psi_{\langle V,r\rangle})],\widehat{\mathcal{F}}_q[(\langle W,s\rangle,\Psi_{\langle W,s\rangle})]} &= \delta^K_{(\langle F_q(V),qrq^{-1}\rangle,\widehat{\mathcal{F}}_q\Psi_{\langle V,r\rangle}),(\langle F_q(W),qsq^{-1}\rangle,\widehat{\mathcal{F}}_q\Psi_{\langle W,s\rangle})} \\ &= C_{(F_q(V),qrq^{-1},\mathcal{F}_q\Phi_V),(F_q(W),qsq^{-1},\mathcal{F}_q\Phi_W)} \\ &= C_{\mathcal{F}_q[(V,r,\Phi_V)],\mathcal{F}_q[(W,s,\Phi_W)]} \\ &= \mathcal{F}_q\left(C_{(V,r,\Phi_V),(W,s,\Phi_W)}\right) \\ &= \mathcal{F}_q\left(\delta^K_{(\langle V,r\rangle,\Psi_{\langle V,r\rangle}),(\langle W,s\rangle,\Psi_{\langle W,s\rangle})}\right).\end{aligned}$$

This proves that $(K,\varepsilon^K,\delta^K,\xi^K)$ is a G-functor. It is also a G-crossed functor, since it preserves the G-grading. Finally, this functor is also braided, because

$$\begin{aligned}&\left[\delta^K_{\widehat{\mathcal{F}}_q[(\langle W,r\rangle,\Psi_{\langle W,r\rangle})],(\langle V,q\rangle,\Psi_{\langle V,q\rangle})}\right]^{-1} \circ K\left[\widehat{C}_{(\langle V,q\rangle,\Psi_{\langle V,q\rangle}),(\langle W,r\rangle,\Psi_{\langle W,r\rangle})}\right] \circ \delta^K_{(\langle V,q\rangle,\Psi_{\langle V,q\rangle}),(\langle W,r\rangle,\Psi_{\langle W,r\rangle})} \\ &= \left[C_{(F_q(W),qrq^{-1},\mathcal{F}_q\Phi_W),(V,q,\Phi_V)}\right]^{-1} \circ K[\mathcal{F}_q\Phi_W(V)] \circ C_{(V,q,\Phi_V),(W,r,\Phi_W)} \\ &= \mathcal{F}_q\Phi_W(V)^{-1} \circ \mathcal{F}_q\Phi_W(V) \circ C_{K[(\langle V,q\rangle,\Psi_{\langle V,q\rangle})],K[(\langle W,r\rangle,\Psi_{\langle W,r\rangle})]} \\ &= C_{K[(\langle V,q\rangle,\Psi_{\langle V,q\rangle})],K[(\langle W,r\rangle,\Psi_{\langle W,r\rangle})]}.\end{aligned}$$

This proves that K is a braided G-crossed functor. To prove that it is an equivalence, we define a functor $L : Z_G(\mathcal{C}) \to Z^{(2)}(\mathcal{D};\mathcal{D}_e)$ by

$$\begin{aligned}L[(V,q,\Phi_V)] &= (\langle V,q\rangle,\Psi_{\langle V,q\rangle}) \\ L(f) &= f,\end{aligned}$$

where $\Psi_{\langle V,q\rangle}$ is the half braiding of the second kind for $\langle V,q\rangle$ relative to $\mathcal{D}_e$ that is defined by $\Psi_{\langle V,q\rangle}(\langle X,e\rangle) = \Phi_V(X)$. The same reasoning as for K can be used to show that L is a well-defined functor. It is easy to see that K and L are inverse to each other. We can equip L with the structure $(L,\varepsilon^L,\delta^L)$ of a tensor functor by setting

$$\delta^L_{(V,q,\Phi_V),(W,r,\Phi_W)} = L\left(\delta^K_{L[(V,q,\Phi_V)],L[(W,r,\Phi_W)]}\right)^{-1} = L\left(C_{(V,q,\Phi_V),(W,r,\Phi_W)}\right)^{-1}$$

$$= L(\Phi_V(W))^{-1} = \Phi_V(W)^{-1} = (\mathcal{F}_r\mathcal{F}_{r^{-1}}\Phi_V(W))^{-1}$$
$$= \widehat{C}^{-1}_{(\langle W,r\rangle,\Psi_{\langle V,r\rangle}),\widehat{\mathcal{F}}_{r^{-1}}[(\langle V,q\rangle,\Psi_{\langle V,q\rangle})]}$$

and $\varepsilon^L = \mathrm{id}_{(\langle I,e\rangle,\Psi^0_{\langle I,e\rangle})}$. It becomes a G-functor if we choose the isomorphisms $\{\xi^L(q) : L\circ\mathcal{F}_q \to \widehat{\mathcal{F}}_q \circ L\}_{q\in G}$ to be trivial and this will in fact result in a braided G-crossed functor. The proof of these statements proceeds analogously to the case of K. The functors K and L are also inverse to each other in the sense of G-crossed functors, i.e. their compositions in the sense of G-crossed functors will result in the trivial G-crossed structure on the identity functors. The existence of such a functor L proves that the braided G-crossed functor K is an equivalence of braided G-crossed categories.
□

Remark 4.4.8 In the paper [38] by Gelaki, Naidu and Nikshych the construction of $Z(\mathcal{D};\mathcal{D}_e)$ is considered for the case where $\mathcal{D}$ is a G-graded fusion category with full G-spectrum. They show that $Z(\mathcal{D};\mathcal{D}_e)$ can be equipped with both a G-action and a braiding in a canonical way, so that it becomes a braided G-crossed category. In Section 3D of [38] this is applied to the case $\mathcal{D} = \mathcal{C}\rtimes G$ where $\mathcal{C}$ is a fusion category with G-action, which results in a similar[10] braided G-crossed structure on $Z(\mathcal{D};\mathcal{D}_e)$ as ours.

4.5 $Z_G(\mathcal{C})$ as a category of bimodule functors

The goal of this section is to show that $Z_G(\mathcal{C})$ is equivalent, as a braided G-crossed category, to a certain category of bimodule functors. In the first subsection we will demonstrate how a group action on a tensor category $\mathcal{C}$ gives rise to structures of a $\mathcal{C}$-bimodule category on $\mathcal{C}$. In the second subsection we will introduce a braided G-crossed structure on a certain category $\mathcal{D}$ of bimodule functors and in the third subsection we will prove that $\mathcal{D}$ and $Z_G(\mathcal{C})$ are equivalent as braided G-crossed categories. The proofs of the statements in the second and third subsection involve many computations. In Section 4.3 we omitted all the proofs because they involved computations that were all very long, see [96]. In contrast, the computations that we will encounter now are all quite short and are therefore included.

4.5.1 Bimodule categories from group actions

If we are given a G-action on a tensor category $\mathcal{C}$, then we can equip $\mathcal{C}$ with different structures of a $\mathcal{C}$-bimodule category, indexed by pairs of group elements in G.

Proposition 4.5.1 *Let $\mathcal{C}$ be a strict tensor category with strict action F of the group G. For each $q\in G$ we define bifunctors $\overset{q}{\triangleright}: \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ and $\overset{q}{\triangleleft}: \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ by*

$$\overset{q}{\triangleright} := \otimes\circ(F_q\times \mathrm{id}_{\mathcal{C}})$$
$$\overset{q}{\triangleleft} := \otimes\circ(\mathrm{id}_{\mathcal{C}}\times F_q).$$

If $q\in G$, then $\overset{q}{\triangleright}$ equips $\mathcal{C}$ with the structure of a strict left $\mathcal{C}$-module category $(\mathcal{C},\overset{q}{\triangleright})$ and $\overset{q}{\triangleleft}$ equips $\mathcal{C}$ with the structure of a strict right $\mathcal{C}$-module category $(\mathcal{C},\overset{q}{\triangleleft})$. In fact, for each pair $(q,r)\in G\times G$, we obtain a $\mathcal{C}$-bimodule category $(\mathcal{C},\overset{q}{\triangleright},\overset{r}{\triangleleft})$.

[10]Note that we constructed a braided G-crossed structure on $Z(\mathcal{D};\mathcal{D}_e)$ for $\mathcal{D} = \mathcal{C}^{\mathrm{rev}}\rtimes G$ and not for $\mathcal{D} = \mathcal{C}\rtimes G$, because the reversal of the tensor product was needed in order to obtain the equivalence in Theorem 4.4.7 .

Proof. It is clear that $\overset{q}{\triangleright}$ and $\overset{q}{\triangleleft}$ are bifunctors for each $q \in G$. We will prove that $(\mathcal{C}, \overset{q}{\triangleright}, \overset{r}{\triangleleft})$ is a $\mathcal{C}$-bimodule category for $q, r \in G$. We have

$$\begin{aligned}(X \otimes Y) \overset{q}{\triangleright} M &= F_q(X \otimes Y) \otimes M = F_q(X) \otimes F_q(X) \otimes M = X \overset{q}{\triangleright} (F_q(Y) \otimes M)\\ &= X \overset{q}{\triangleright} (Y \overset{q}{\triangleright} M),\\ M \overset{r}{\triangleleft} (X \otimes Y) &= M \otimes F_r(X \otimes Y) = M \otimes F_r(X) \otimes F_r(Y) = (M \otimes F_r(X)) \overset{r}{\triangleleft} Y\\ &= (M \overset{r}{\triangleleft} X) \overset{r}{\triangleleft} Y\end{aligned}$$

and

$$\begin{aligned}I \overset{q}{\triangleright} M &= F_q(I) \otimes M = I \otimes M = M,\\ M \overset{r}{\triangleleft} I &= M \otimes F_r(I) = M \otimes I = M,\end{aligned}$$

so $(\mathcal{C}, \overset{q}{\triangleright})$ is a strict left $\mathcal{C}$-module category and $(\mathcal{C}, \overset{r}{\triangleleft})$ is a strict right $\mathcal{C}$-module category. Furthermore,

$$\begin{aligned}(X \overset{q}{\triangleright} M) \overset{r}{\triangleleft} Y &= (F_q(X) \otimes M) \overset{r}{\triangleleft} Y = F_q(X) \otimes M \otimes F_r(Y) = X \overset{q}{\triangleright} (M \otimes F_r(Y))\\ &= X \overset{q}{\triangleright} (M \overset{r}{\triangleleft} Y),\end{aligned}$$

showing that $(\mathcal{C}, \overset{q}{\triangleright}, \overset{r}{\triangleleft})$ is indeed a strict $\mathcal{C}$-bimodule category.
□

Definition 4.5.2 Let $\mathcal{C}$ be a strict tensor category with strict action F of the group G and let $\overset{q}{\triangleright}$ and $\overset{r}{\triangleleft}$ be as above. Then we will write ${}^q\mathcal{C}^r$ to denote the $\mathcal{C}$-bimodule category $(\mathcal{C}, \overset{q}{\triangleright}, \overset{r}{\triangleleft})$. We will also use the shorthand notations ${}^q\mathcal{C} := {}^q\mathcal{C}^e$ and $\mathcal{C}^r := {}^e\mathcal{C}^r$.

Note that ${}^e\mathcal{C}^e = {}^e\mathcal{C} = \mathcal{C}^e = \mathcal{C}$, where $\mathcal{C}$ on the right denotes the category $\mathcal{C} = (\mathcal{C}, \overset{e}{\triangleright}, \overset{e}{\triangleleft}) = (\mathcal{C}, \otimes, \otimes)$ with the $\mathcal{C}$-bimodule category structure that is simply given by the tensor product.

4.5.2 $\bigsqcup_{q\in G} \mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C}, {}^q\mathcal{C})$ as a braided G-crossed category

Suppose that we are given a strict tensor category $\mathcal{C}$ with a strict action F of the group G. For simplicity we will write $\mathcal{D}_q := \mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C}, {}^q\mathcal{C})$ in the sequel, where we use the notation that was introduced at the end of Subsection 2.6.2. We then define the category

$$\mathcal{D} := \bigsqcup_{q\in G} \mathcal{D}_q.$$

For each $q \in G$ we will write $\mathcal{D}_q^0$ to denote the full subcategory of $\mathcal{D}_q$ determined by the objects of the form $(\mathcal{L}_V, s, \mathrm{id})$, where $\mathcal{L}_V(M) = V \otimes M$ and $\mathcal{L}_V(f) = \mathrm{id}_V \otimes f$. We also define

$$\mathcal{D}^0 = \bigsqcup_{q\in G} \mathcal{D}_q^0.$$

We will explicitly state the conditions on s for objects in $\mathcal{D}^0$, because this will be needed later. If $(\mathcal{L}_V, s, \mathrm{id}) \in \mathcal{D}_q^0$, then s is a family $\{s_M(X) : V \otimes X \otimes M \to F_q(X) \otimes V \otimes M\}_{M,X\in\mathcal{C}}$ of isomorphisms such that for any $X, Y, M, N \in \mathcal{C}$, $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $m \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ the diagrams

$$\begin{array}{ccc} V \otimes X \otimes M & \xrightarrow{s_M(X)} & F_q(X) \otimes V \otimes M \\ \Big\downarrow{\scriptstyle \mathrm{id}_V \otimes f \otimes m} & & \Big\downarrow{\scriptstyle F_q(f) \otimes \mathrm{id}_V \otimes m} \\ V \otimes Y \otimes N & \xrightarrow{s_N(Y)} & F_q(Y) \otimes V \otimes N \end{array} \tag{4.5.1}$$

and

$$\begin{array}{ccc} & V \otimes X \otimes Y \otimes M & \\ {}^{s_{Y\otimes M}(X)}\swarrow & & \searrow^{s_M(X\otimes Y)} \\ F_q(X) \otimes V \otimes Y \otimes M & \xrightarrow{\mathrm{id}_{F_q(X)}\otimes s_M(Y)} & F_q(X) \otimes F_q(Y) \otimes V \otimes M \end{array} \tag{4.5.2}$$

commute and

$$s_M(I) = \mathrm{id}_{V\otimes M} \tag{4.5.3}$$
$$s_{M\otimes Y}(X) = s_M(X) \otimes \mathrm{id}_Y. \tag{4.5.4}$$

Using (4.5.4) we can restate the commutativity of the diagram (4.5.2) in two different ways:

$$s_M(X \otimes Y) = [\mathrm{id}_{F_q(X)} \otimes s_M(Y)] \circ [s_Y(X) \otimes \mathrm{id}_M] \tag{4.5.5}$$
$$s_M(X \otimes Y) = [\mathrm{id}_{F_q(X)} \otimes s_M(Y)] \circ [s_I(X) \otimes \mathrm{id}_{Y\otimes M}]. \tag{4.5.6}$$

In particular, s_I is a half q-braiding for V. A morphism $\sigma \in \mathrm{Hom}_{\mathcal{D}^0_q}((\mathcal{L}_{V_1}, s^1, \mathrm{id}), (\mathcal{L}_{V_2}, s^2, \mathrm{id}))$ is a natural transformation $\sigma : \mathcal{L}_{V_1} \to \mathcal{L}_{V_2}$ such that for any $X, M \in \mathcal{C}$ the square

$$\begin{array}{ccc} V_1 \otimes X \otimes M & \xrightarrow{\sigma_{X\otimes M}} & V_2 \otimes X \otimes M \\ \downarrow{\scriptstyle s^1_M(X)} & & \downarrow{\scriptstyle s^2_M(X)} \\ F_q(X) \otimes V_1 \otimes M & \xrightarrow{\mathrm{id}_{F_q(X)}\otimes\sigma_M} & F_q(X) \otimes V_2 \otimes M \end{array} \tag{4.5.7}$$

commutes and

$$\sigma_{M\otimes X} = \sigma_M \otimes \mathrm{id}_X. \tag{4.5.8}$$

Using (4.5.8) we can write restate the commutativity of the diagram (4.5.7) as

$$[\mathrm{id}_{F_q(X)} \otimes \sigma_M] \circ s^1_M(X) = s^2_M(X) \circ [\sigma_I \otimes \mathrm{id}_{X\otimes M}]. \tag{4.5.9}$$

The next lemma shows that this category $\mathcal{D}^0$ can be made into a braided (strict) G-crossed category.

Lemma 4.5.3 *Let $\mathcal{C}$ be a strict tensor category with strict action F of the group G. Then the category $\mathcal{D}^0$ can be given the structure of a braided G-crossed category as follows.*

- *The tensor product $(\mathcal{L}_{V_1}, s^1, \mathrm{id}) * (\mathcal{L}_{V_2}, s^2, \mathrm{id}) = (\mathcal{L}_{V_1} * \mathcal{L}_{V_2}, s^1 * s^2, \mathrm{id})$ of $(\mathcal{L}_{V_1}, s^1, \mathrm{id}) \in \mathcal{D}^0_q$ and $(\mathcal{L}_{V_2}, s^2, \mathrm{id}) \in \mathcal{D}^0_r$ is defined by $\mathcal{L}_{V_1} * \mathcal{L}_{V_2} = \mathcal{L}_{V_1\otimes V_2}$ and*

$$(s^1 * s^2)_M(X) = s^1_{V_2\otimes M}(F_r(X)) \circ [\mathrm{id}_{V_1} \otimes s^2_M(X)]. \tag{4.5.10}$$

  If $\sigma \in \mathrm{Hom}_{\mathcal{D}^0_q}((\mathcal{L}_{V_1}, s^1, \mathrm{id}), (\mathcal{L}_{V_2}, s^2, \mathrm{id}))$ and $\tau \in \mathrm{Hom}_{\mathcal{D}^0_r}((\mathcal{L}_{V_3}, s^3, \mathrm{id}), (\mathcal{L}_{V_4}, s^4, \mathrm{id}))$, then $\sigma*\tau$ is defined by

$$(\sigma * \tau)_M = \sigma_I \otimes \tau_I \otimes \mathrm{id}_M.$$

  The unit object is $(\mathcal{L}_I, s^0, \mathrm{id}) \in \mathcal{D}^0_e$, where $s^0_M(X) := \mathrm{id}_{X\otimes M}$.
- *The group action $\mathcal{F}^0_q[(\mathcal{L}_V, s, \mathrm{id})] = (\mathcal{F}^0_q\mathcal{L}_V, \mathcal{F}^0_q s, \mathrm{id}) \in \mathcal{D}^0_{qrq^{-1}}$ on an object $(\mathcal{L}_V, s, \mathrm{id}) \in \mathcal{D}^0_r$ is defined by $\mathcal{F}^0_q\mathcal{L}_V = \mathcal{L}_{F_q(V)}$ and*[11]

$$\mathcal{F}^0_q s_M(X) = F_q(s_{F_{q^{-1}}(M)}(F_{q^{-1}}(X))).$$

  The group action on a morphism $\sigma \in \mathrm{Hom}(\mathcal{D}^0)$ is defined by $\mathcal{F}^0_q(\sigma)_M = F_q(\sigma_{F_{q^{-1}}(M)})$.

[11] We simply write $\mathcal{F}^0_q s_M(X)$ rather than $(\mathcal{F}^0_q s)_M(X)$.

- *If* $(\mathcal{L}_V, s, \mathrm{id}) \in \mathcal{D}^0_q$, *then its grading is defined to be* q.
- *If* $(\mathcal{L}_{V_1}, s^1, \mathrm{id}) \in \mathcal{D}^0_q$ *and* $(\mathcal{L}_{V_2}, s^2, \mathrm{id}) \in \mathcal{D}^0_r$, *then their braiding is defined by*

$$[C^0_{(\mathcal{L}_{V_1}, s^1, \mathrm{id}), (\mathcal{L}_{V_2}, s^2, \mathrm{id})}]_M = s^1_M(V_2).$$

Proof. Let $(\mathcal{L}_{V_1}, s^1, \mathrm{id}) \in \mathcal{D}^0_q$ and $(\mathcal{L}_{V_2}, s^2, \mathrm{id}) \in \mathcal{D}^0_r$. We will show that $(\mathcal{L}_{V_1 \otimes V_2}, s^1 * s^2, \mathrm{id}) \in \mathcal{D}^0_{qr}$. If $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $m \in \mathrm{Hom}_{\mathcal{C}}(M, N)$, then it follows from the fact that s^1 and s^2 satisfy (4.5.1) that

$$\begin{aligned}(s^1 * s^2)_N(Y) \circ [\mathrm{id}_{V_1 \otimes V_2} \otimes f \otimes m] &= s^1_{V_2 \otimes N}(F_r(Y)) \circ [\mathrm{id}_{V_1} \otimes s^2_N(Y)] \circ [\mathrm{id}_{V_1} \otimes \mathrm{id}_{V_2} \otimes f \otimes m] \\ &= s^1_{V_2 \otimes N}(F_r(Y)) \circ [\mathrm{id}_{V_1} \otimes F_r(f) \otimes \mathrm{id}_{V_2} \otimes m] \circ [\mathrm{id}_{V_1} \otimes s^2_M(X)] \\ &= [F_{qr}(f) \otimes \mathrm{id}_{V_1} \otimes \mathrm{id}_{V_2} \otimes m] \circ s^1_{V_2 \otimes M}(F_r(X)) \circ [\mathrm{id}_{V_1} \otimes s^2_M(X)] \\ &= [F_{qr}(f) \otimes \mathrm{id}_{V_1 \otimes V_2} \otimes m] \circ (s^1 * s^2)_M(X),\end{aligned}$$

which shows that $s^1 * s^2$ satisfies (4.5.1). And if $X, Y, M \in \mathcal{C}$, then it follows from (4.5.2) that

$$\begin{aligned}&(s^1 * s^2)_M(X \otimes Y) = s^1_{V_2 \otimes M}(F_r(X \otimes Y)) \circ [\mathrm{id}_{V_1} \otimes s^2_M(X \otimes Y)] \\ &= [\mathrm{id}_{F_{qr}(X)} \otimes s^1_{V_2 \otimes M}(F_r(Y))] \circ s^1_{F_r(Y) \otimes V_2 \otimes M}(F_r(X)) \circ [\mathrm{id}_{V_1} \otimes \mathrm{id}_{F_r(X)} \otimes s^2_M(Y)] \circ [\mathrm{id}_{V_1} \otimes s^2_{Y \otimes M}(X)] \\ &= [\mathrm{id}_{F_{qr}(X)} \otimes s^1_{V_2 \otimes M}(F_r(Y))] \circ [\mathrm{id}_{F_{qr}(X)} \otimes \mathrm{id}_{V_1} \otimes s^2_M(Y)] \circ s^1_{V_2 \otimes Y \otimes M}(F_r(X)) \circ [\mathrm{id}_{V_1} \otimes s^2_{Y \otimes M}(X)] \\ &= [\mathrm{id}_{F_{qr}(X)} \otimes (s^1 * s^2)_M(Y)] \circ (s^1 * s^2)_{Y \otimes M}(X),\end{aligned}$$

which shows that $s^1 * s^2$ satisfies (4.5.2). It follows from the fact that s^1 and s^2 satisfy (4.5.3) and (4.5.4) that for any $M \in \mathcal{C}$ we have

$$(s^1 * s^2)_M(I) = s^1_{V_2 \otimes M}(F_r(I)) \circ [\mathrm{id}_{V_1} \otimes s^2_M(I)] = \mathrm{id}_{V_1 \otimes V_2 \otimes M} \circ [\mathrm{id}_{V_1} \otimes \mathrm{id}_{V_2 \otimes M}] = \mathrm{id}_{V_1 \otimes V_2 \otimes M}$$

and for any $M, X, Y \in \mathcal{C}$ we have

$$\begin{aligned}&(s^1 * s^2)_{M \otimes Y}(X) \\ &= s^1_{V_2 \otimes M \otimes Y}(F_r(X)) \circ [\mathrm{id}_{V_1} \otimes s^2_{M \otimes Y}] = [s^1_{V_2 \otimes M}(F_r(X)) \otimes \mathrm{id}_Y] \circ [\mathrm{id}_{V_1} \otimes s^2_M(X) \otimes \mathrm{id}_Y] \\ &= (s^1 * s^2)_M(X) \otimes \mathrm{id}_Y.\end{aligned}$$

This shows that $s^1 * s^2$ also satisfies (4.5.3) and (4.5.4) and hence that $(\mathcal{L}_{V_1 \otimes V_2}, s^1 * s^2, \mathrm{id}) \in \mathcal{D}^0_{qr}$. Now let $\sigma \in \mathrm{Hom}_{\mathcal{D}^0_q}((\mathcal{L}_{V_1}, s^1, \mathrm{id}), (\mathcal{L}_{V_2}, s^2, \mathrm{id}))$ and $\tau \in \mathrm{Hom}_{\mathcal{D}^0_r}((\mathcal{L}_{V_3}, s^3, \mathrm{id}), (\mathcal{L}_{V_4}, s^4, \mathrm{id}))$. We will show that $\sigma * \tau$ is a module natural transformation from $(\mathcal{L}_{V_1} * \mathcal{L}_{V_3}, s^1 * s^3, \mathrm{id})$ to $(\mathcal{L}_{V_2} * \mathcal{L}_{V_4}, s^2 * s^4, \mathrm{id})$. For $m \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ we have

$$\begin{aligned}(\sigma * \tau)_N \circ (\mathrm{id}_{V_1 \otimes V_3} \otimes m) &= (\sigma_I \otimes \tau_I \otimes \mathrm{id}_N) \circ (\mathrm{id}_{V_1 \otimes V_3} \otimes m) = (\mathrm{id}_{V_2 \otimes V_4} \otimes m) \circ (\sigma_I \otimes \tau_I \otimes \mathrm{id}_M) \\ &= (\mathrm{id}_{V_2 \otimes V_4} \otimes m) \circ (\sigma * \tau)_M,\end{aligned}$$

showing naturality of $\sigma * \tau$. It follows from the fact that σ and τ satisfy (4.5.7) that for all $X, M \in \mathcal{C}$ we have

$$\begin{aligned}(s^2 * s^4)_M(X) \circ (\sigma * \tau)_{X \otimes M} &= s^2_{V_4 \otimes M}(F_r(X)) \circ [\mathrm{id}_{V_2} \otimes s^4_M(X)] \circ [\sigma_I \otimes \tau_I \otimes \mathrm{id}_{X \otimes M}] \\ &= s^2_{V_4 \otimes M}(F_r(X)) \circ [\mathrm{id}_{V_2} \otimes s^4_M(X)] \circ [\sigma_I \otimes \tau_{X \otimes M}] \\ &= s^2_{V_4 \otimes M}(F_r(X)) \circ [\sigma_I \otimes \mathrm{id}_{F_r(X)} \otimes \tau_M] \circ [\mathrm{id}_{V_1} \otimes s^3_M(X)] \\ &= [s^2_I(F_r(X)) \otimes \mathrm{id}_{V_4 \otimes M}] \circ [\sigma_{F_r(X) \otimes I} \otimes \tau_I \otimes \mathrm{id}_M] \circ [\mathrm{id}_{V_1} \otimes s^3_M(X)] \\ &= [\mathrm{id}_{F_{qr}(X)} \otimes \sigma_I \otimes \tau_I \otimes \mathrm{id}_M] \circ [s^1_I(F_r(X)) \otimes \mathrm{id}_{V_3 \otimes M}] \circ [\mathrm{id}_{V_1} \otimes s^3_M(X)]\end{aligned}$$

$$
\begin{aligned}
&= [\mathrm{id}_{F_{qr}(X)} \otimes (\sigma * \tau)_M] \circ s^1_{V_3\otimes M}(F_r(X)) \circ [\mathrm{id}_{V_1} \otimes s^3_M(X)] \\
&= [\mathrm{id}_{F_{qr}(X)} \otimes (\sigma * \tau)_M] \circ (s^1 * s^3)_M(X),
\end{aligned}
$$

so $\sigma * \tau$ also satisfies (4.5.7). Since σ and τ also satisfy (4.5.8), we have for all $X, M \in \mathcal{C}$ that

$$
(\sigma * \tau)_{M\otimes X} = \sigma_I \otimes \tau_I \otimes \mathrm{id}_{M\otimes X} = (\sigma * \tau)_M \otimes \mathrm{id}_X,
$$

so $\sigma * \tau$ satisfies (4.5.8) as well. Thus $\sigma * \tau$ is a module natural transformation from $(\mathcal{L}_{V_1} * \mathcal{L}_{V_3}, s^1 * s^3, \mathrm{id})$ to $(\mathcal{L}_{V_2} * \mathcal{L}_{V_4}, s^2 * s^4, \mathrm{id})$ in $\mathcal{D}^0_{qr}$. If σ and σ' are composable morphisms in $\mathcal{D}^0_q$ and τ and τ' are composable morphisms in $\mathcal{D}^0_r$ for some $q, r \in G$, then for any $M \in \mathcal{C}$ we have

$$
\begin{aligned}
[(\sigma' \circ \sigma) * (\tau' \circ \tau)]_M &= (\sigma' \circ \sigma)_I \otimes (\tau' \circ \tau)_I \otimes \mathrm{id}_M = (\sigma'_I \circ \sigma_I) \otimes (\tau'_I \circ \tau_I) \otimes \mathrm{id}_M \\
&= (\sigma'_I \otimes \tau'_I \otimes \mathrm{id}_M) \circ (\sigma_I \otimes \tau_I \otimes \mathrm{id}_M) = (\sigma' * \tau')_M \circ (\sigma * \tau)_M \\
&= [(\sigma' * \tau') \circ (\sigma * \tau)]_M,
\end{aligned}
$$

so $*$ satisfies the interchange law. Furthermore, for any $(\mathcal{L}_{V_1}, s^1, \mathrm{id}), (\mathcal{L}_{V_2}, s, \mathrm{id}) \in \mathcal{D}^0$ we have

$$
\begin{aligned}
\left[\mathrm{id}_{(\mathcal{L}_{V_1} * \mathcal{L}_{V_2}, s^1 * s^2, \mathrm{id})}\right]_M &= \mathrm{id}_{V_1\otimes V_2\otimes M} = [\mathrm{id}_{(\mathcal{L}_{V_1}, s^1, \mathrm{id})}]_I \otimes [\mathrm{id}_{(\mathcal{L}_{V_2}, s^2, \mathrm{id})}]_I \otimes \mathrm{id}_M \\
&= \left[\mathrm{id}_{(\mathcal{L}_{V_1}, s^1, \mathrm{id})} * \mathrm{id}_{(\mathcal{L}_{V_2}, s^2, \mathrm{id})}\right]_M
\end{aligned}
$$

for any $M \in \mathcal{C}$. So $*$ is indeed a functor.

We will now show that $*$ is associative. If $(\mathcal{L}_{V_1}, s^1, \mathrm{id}) \in \mathcal{D}^0_q$, $(\mathcal{L}_{V_2}, s^2, \mathrm{id}) \in \mathcal{D}^0_r$ and $(\mathcal{L}_{V_3}, s^3, \mathrm{id}) \in \mathcal{D}^0_s$, then clearly $(\mathcal{L}_{V_1} * \mathcal{L}_{V_2}) * \mathcal{L}_{V_3} = \mathcal{L}_{V_1} * (\mathcal{L}_{V_2} * \mathcal{L}_{V_3})$. Also,

$$
\begin{aligned}
((s^1 * s^2) * s^3)_M(X) &= (s^1 * s^2)_{V_3\otimes M}(F_s(X)) \circ [\mathrm{id}_{V_1\otimes V_2} \otimes s^3_M(X)] \\
&= s^1_{V_2\otimes V_3\otimes M}(F_r(F_s(X))) \circ [\mathrm{id}_{V_1} \otimes s^2_{V_3\otimes M}(F_s(X))] \circ [\mathrm{id}_{V_1\otimes V_2} \otimes s^3_M(X)] \\
&= s^1_{V_2\otimes V_3\otimes M}(F_{rs}(X)) \circ [\mathrm{id}_{V_1} \otimes (s^2 * s^3)_M(X)] \\
&= (s^1 * (s^2 * s^3))_M(X).
\end{aligned}
$$

If ρ, σ and τ are morphisms in $\mathcal{D}^0$, then

$$
\begin{aligned}
((\rho * \sigma) * \tau)_M &= (\rho * \sigma)_I \otimes \tau_I \otimes \mathrm{id}_M = \rho_I \otimes \sigma_I \otimes \mathrm{id}_I \otimes \tau_I \otimes \mathrm{id}_M \\
&= \rho_I \otimes \sigma_I \otimes \tau_I \otimes \mathrm{id}_I \otimes \mathrm{id}_M = \rho_I \otimes (\sigma * \tau)_I \otimes \mathrm{id}_M \\
&= (\rho * (\sigma * \tau))_M.
\end{aligned}
$$

It is clear that $(\mathcal{L}_I, s^0, \mathrm{id}) \in \mathcal{D}^0_e$. To see that it acts as a unit object, we first note that $\mathcal{L}_I * \mathcal{L}_V = \mathcal{L}_{I\otimes V} = \mathcal{L}_V$ and $\mathcal{L}_V * \mathcal{L}_I = \mathcal{L}_{V\otimes I} = \mathcal{L}_V$. Also, if $(\mathcal{L}_V, s, \mathrm{id}) \in \mathcal{D}^0_q$, then

$$
\begin{aligned}
(s^0 * s)_M(X) &= s^0_{V\otimes M}(F_q(X)) \circ [\mathrm{id}_I \otimes s_M(X)] = s_M(X) \\
(s * s^0)_M(X) &= s_{I\otimes M}(F_e(X)) \circ [\mathrm{id}_V \otimes s^0_M(X)] = s_M(X),
\end{aligned}
$$

so $(\mathcal{L}_I, s^0, \mathrm{id}) * (\mathcal{L}_V, s, \mathrm{id}) = (\mathcal{L}_V, s, \mathrm{id}) = (\mathcal{L}_V, s, \mathrm{id}) * (\mathcal{L}_I, s^0, \mathrm{id})$. If $q \in G$ and $\sigma \in \mathrm{Hom}(\mathcal{D}^0_q)$, then

$$
\begin{aligned}
(\mathrm{id}_{(\mathcal{L}_I, s^0, \mathrm{id})} * \sigma)_M &= [\mathrm{id}_{(\mathcal{L}_I, s^0, \mathrm{id})}]_I \otimes \sigma_I \otimes \mathrm{id}_M = \mathrm{id}_I \otimes \sigma_I \otimes \mathrm{id}_M = \sigma_M \\
(\sigma * \mathrm{id}_{(\mathcal{L}_I, s^0, \mathrm{id})})_M &= \sigma_I \otimes [\mathrm{id}_{(\mathcal{L}_I, s^0, \mathrm{id})}]_I \otimes \mathrm{id}_M = \sigma_I \otimes \mathrm{id}_I \otimes \mathrm{id}_M = \sigma_M.
\end{aligned}
$$

This finishes the proof that $\mathcal{D}^0$ is a tensor category.

It is clear that $\mathcal{D}^0$ becomes a G-graded category with the given grading. We will now turn to the G-action. Let $q \in G$ and $(\mathcal{L}_V, s, \mathrm{id}) \in \mathcal{D}^0_r$. Since s satisfies (4.5.1), we have for any $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $m \in \mathrm{Hom}_{\mathcal{C}}(M,N)$

$$\begin{aligned}\mathcal{F}^0_q s_N(Y) \circ (\mathrm{id}_{F_q(V)} \otimes f \otimes m) &= F_q\left\{s_{F_{q^{-1}}(N)}(F_{q^{-1}}(Y)) \circ [\mathrm{id}_V \otimes F_{q^{-1}}(f) \otimes F_{q^{-1}}(m)]\right\} \\ &= F_q\left\{[F_r(F_{q^{-1}}(f)) \otimes \mathrm{id}_V \otimes F_{q^{-1}}(m)] \circ s_{F_{q^{-1}}(M)}(F_{q^{-1}}(X))\right\} \\ &= [F_{qrq^{-1}}(f) \otimes \mathrm{id}_{F_q(V)} \otimes m] \circ \mathcal{F}^0_q s_M(X),\end{aligned}$$

so $\mathcal{F}^0_q s$ satisfies (4.5.1) as well. And because s satisfies (4.5.2), we have for any $X, Y, M \in \mathcal{C}$

$$\begin{aligned}\mathcal{F}^0_q s_M(X \otimes Y) &= F_q(s_{F_{q^{-1}}(M)}(F_{q^{-1}}(X) \otimes F_{q^{-1}}(Y))) \\ &= F_q\left\{[\mathrm{id}_{F_r(F_{q^{-1}}(X))} \otimes s_{F_{q^{-1}}(M)}(F_{q^{-1}}(Y))] \circ s_{F_{q^{-1}}(Y) \otimes F_{q^{-1}}(M)}(F_{q^{-1}}(X))\right\} \\ &= [\mathrm{id}_{F_{qrq^{-1}}(X)} \otimes F_q(s_{F_{q^{-1}}(M)}(F_{q^{-1}}(Y)))] \circ F_q(s_{F_{q^{-1}}(Y \otimes M)}(F_{q^{-1}}(X))) \\ &= [\mathrm{id}_{F_{qrq^{-1}}(X)} \otimes \mathcal{F}^0_q s_M(Y)] \circ \mathcal{F}^0_q s_{Y \otimes M}(X),\end{aligned}$$

so $\mathcal{F}^0_q s$ satisfies (4.5.2). Since s satisfies (4.5.3) and (4.5.4), so does $\mathcal{F}^0_q$, because

$$\mathcal{F}^0_q s_M(I) = F_q(s_{F_{q^{-1}}(M)}(F_{q^{-1}}(I))) = F_q(\mathrm{id}_{V \otimes F_{q^{-1}}(M)}) = \mathrm{id}_{F_q(V) \otimes M}$$

and

$$\begin{aligned}\mathcal{F}^0_q s_{M \otimes Y}(X) &= F_q\left(s_{F_{q^{-1}}(M) \otimes F_{q^{-1}}(Y)}(F_{q^{-1}}(X))\right) = F_q\left(s_{F_{q^{-1}}(M)}(F_{q^{-1}}(X)) \otimes \mathrm{id}_{F_{q^{-1}}(Y)}\right) \\ &= \mathcal{F}^0_q s_M(X) \otimes \mathrm{id}_Y.\end{aligned}$$

This shows that $(\mathcal{F}^0_q \mathcal{L}_V, \mathcal{F}^0_q s, \mathrm{id}) \in \mathcal{D}^0_{qrq^{-1}}$. Now let $q, r \in G$ and $\sigma \in \mathrm{Hom}_{\mathcal{D}^0_r}((\mathcal{L}_{V_1}, s^1, \mathrm{id}), (\mathcal{L}_{V_2}, s^2, \mathrm{id}))$. If $m \in \mathrm{Hom}_{\mathcal{C}}(M,N)$, then it follows from naturality of σ that

$$\begin{aligned}\mathcal{F}^0_q(\sigma)_N \circ [\mathrm{id}_{F_q(V_1)} \otimes m] &= F_q\left\{\sigma_{F_{q^{-1}}(N)} \circ [\mathrm{id}_{V_1} \otimes F_{q^{-1}}(m)]\right\} = F_q\left\{[\mathrm{id}_{V_2} \otimes F_{q^{-1}}(m)] \circ \sigma_{F_{q^{-1}}(M)}\right\} \\ &= [\mathrm{id}_{F_q(V_2)} \otimes m] \circ \mathcal{F}^0_q(\sigma)_M,\end{aligned}$$

so $\mathcal{F}^0_q(\sigma)$ is natural. Since σ satisfies (4.5.7), so does $\mathcal{F}^0_q(\sigma)$. Namely, for any $X, M \in \mathcal{C}$ we have

$$\begin{aligned}\mathcal{F}^0_q s^2_M(X) \circ \mathcal{F}^0_q(\sigma)_{X \otimes M} &= F_q\left\{s^2_{F_{q^{-1}}(M)}(F_{q^{-1}}(X)) \circ \sigma_{F_{q^{-1}}(X) \otimes F_{q^{-1}}(M)}\right\} \\ &= F_q\left\{[\mathrm{id}_{F_r(F_{q^{-1}}(X))} \otimes \sigma_{F_{q^{-1}}(M)}] \circ s^1_{F_{q^{-1}}(M)}(F_{q^{-1}}(X))\right\} \\ &= [\mathrm{id}_{F_{qrq^{-1}}(X)} \otimes \mathcal{F}^0_q(\sigma)_M] \circ \mathcal{F}^0_q s^1_M(X).\end{aligned}$$

Also, for any $X, M \in \mathcal{C}$ we have

$$\mathcal{F}^0_q(\sigma)_{M \otimes X} = F_q\left(\sigma_{F_{q^{-1}}(M) \otimes F_{q^{-1}}(X)}\right) = F_q\left(\sigma_{F_{q^{-1}}(M)} \otimes \mathrm{id}_{F_{q^{-1}}(X)}\right) = \mathcal{F}^0_q(\sigma)_M \otimes \mathrm{id}_X.$$

Thus $\mathcal{F}^0_q(\sigma)$ is a morphism from $(\mathcal{F}^0_q \mathcal{L}_{V_1}, \mathcal{F}^0_q s^1, \mathrm{id})$ to $(\mathcal{F}^0_q \mathcal{L}_{V_2}, \mathcal{F}^0_q s^2, \mathrm{id})$. Hence $\mathcal{F}^0_q$ is well-defined. If $q \in G$ and $\sigma, \tau \in \mathrm{Hom}_(\mathcal{D}^0)$ are composable, then

$$[\mathcal{F}^0_q(\tau) \circ \mathcal{F}^0_q(\sigma)]_M = \mathcal{F}^0_q(\tau)_M \circ \mathcal{F}^0_q(\sigma)_M = F_q(\tau_{F_{q^{-1}}(M)} \circ \sigma_{F_{q^{-1}}(M)}) = F_q\left((\tau \circ \sigma)_{F_{q^{-1}}(M)}\right)$$

$$= \mathcal{F}_q^0(\tau \circ \sigma)_M,$$

showing that $\mathcal{F}_q^0(\tau\circ\sigma) = \mathcal{F}_q^0(\tau)\circ\mathcal{F}_q^0(\sigma)$. Hence $\mathcal{F}_q^0$ is a functor. If $(\mathcal{L}_{V_1}, s^1, \mathrm{id}) \in \mathcal{D}_r^0$ and $(\mathcal{L}_{V_2}, s^2, \mathrm{id}) \in \mathcal{D}_s^0$, then $\mathcal{F}_q^0(\mathcal{L}_{V_1} * \mathcal{L}_{V_2}) = \mathcal{F}_q^0\mathcal{L}_{V_1\otimes V_2} = \mathcal{L}_{F_q(V_1)\otimes F_q(V_2)} = \mathcal{F}_q^0\mathcal{L}_{V_1} * \mathcal{F}_q^0\mathcal{L}_{V_2}$ and

$$\begin{aligned}
\mathcal{F}_q^0(s^1 * s^2)_M(X) &= F_q\left\{(s^1*s^2)_{F_{q^{-1}}(M)}(F_{q^{-1}}(X))\right\}\\
&= F_q\left\{s^1_{V_2\otimes F_{q^{-1}}(M)}(F_s(F_{q^{-1}}(X)))\circ[\mathrm{id}_{V_1}\otimes s^2_{F_{q^{-1}}(M)}(F_{q^{-1}}(X))]\right\}\\
&= F_q\left(s^1_{F_{q^{-1}}(F_q(V_2)\otimes M)}(F_{q^{-1}}(F_{qsq^{-1}}(X)))\right)\circ\left[\mathrm{id}_{F_q(V_1)}\otimes F_q(s^2_{F_{q^{-1}}(M)}(F_{q^{-1}}(X)))\right]\\
&= \mathcal{F}_q^0 s^1_{F_q(V_2)\otimes M}(F_{qsq^{-1}}(X))\circ[\mathrm{id}_{F_q(V_1)}\otimes\mathcal{F}_q^0 s^2_M(X)]\\
&= (\mathcal{F}_q^0 s^1 * \mathcal{F}_q^0 s^2)_M(X),
\end{aligned}$$

showing that $\mathcal{F}_q^0(s^1*s^2) = \mathcal{F}_q^0s^1*\mathcal{F}_q^0s^2$. So we conclude that $\mathcal{F}_q^0[(\mathcal{L}_{V_1}, s^1, \mathrm{id})*(\mathcal{L}_{V_2}, s^2, \mathrm{id})] = \mathcal{F}_q^0(\mathcal{L}_{V_1}, s^1, \mathrm{id})*\mathcal{F}_q^0(\mathcal{L}_{V_2}, s^2, \mathrm{id})$. Also,

$$\mathcal{F}_q^0 s^0_M(X) = F_q(s^0_{F_{q^{-1}}(M)}(F_{q^{-1}}(X))) = F_q(\mathrm{id}_{F_{q^{-1}}(X)\otimes F_{q^{-1}}(M)}) = \mathrm{id}_{X\otimes M} = s^0_M(X),$$

so $\mathcal{F}_q^0(\mathcal{L}_I, s^0, \mathrm{id}) = (\mathcal{L}_{F_q(I)}, \mathcal{F}_q^0 s^0, \mathrm{id}) = (\mathcal{L}_I, s^0, \mathrm{id})$. This shows that $\mathcal{F}_q^0$ is a strict tensor functor for each $q \in G$. Now let $q, r \in G$ and $(\mathcal{L}_V, s, \mathrm{id}) \in \mathcal{D}_s^0$. Then $\mathcal{F}_{qr}^0\mathcal{L}_V = \mathcal{L}_{F_{qr}(V)} = \mathcal{L}_{F_q(F_r(V))} = \mathcal{F}_q^0\mathcal{L}_{F_r(V)} = \mathcal{F}_q^0\mathcal{F}_r^0\mathcal{L}_V$ and

$$\begin{aligned}
\mathcal{F}_{qr}^0 s_M(X) &= F_{qr}(s_{F_{(qr)^{-1}}(M)}(F_{(qr)^{-1}}(X))) = F_q(F_r(s_{F_{r^{-1}}(F_{q^{-1}}(M))}(F_{r^{-1}}(F_{q^{-1}}(X)))))\\
&= F_q(\mathcal{F}_r^0 s_{F_{q^{-1}}(M)}(F_{q^{-1}}(X))) = \mathcal{F}_q^0\mathcal{F}_r^0 s_M(X).
\end{aligned}$$

Thus we have $\mathcal{F}_{qr}^0(\mathcal{L}_V, s, \mathrm{id}) = (\mathcal{F}_{qr}^0\mathcal{L}_V, \mathcal{F}_{qr}^0 s, \mathrm{id}) = (\mathcal{F}_q^0\mathcal{F}_r^0\mathcal{L}_V, \mathcal{F}_q^0\mathcal{F}_r^0 s, \mathrm{id}) = \mathcal{F}_q^0\mathcal{F}_r^0(\mathcal{L}_V, s, \mathrm{id})$. Together with the fact that $\mathcal{F}_e^0(\mathcal{L}_V, s, \mathrm{id}) = (\mathcal{L}_{F_e(V)}, \mathcal{F}_e^0 s, \mathrm{id}) = (\mathcal{L}_V, s, \mathrm{id})$, this proves that $\mathcal{F}^0$ defines a group action on $\mathcal{D}^0$. The fact that $\mathcal{F}_q^0$ maps objects in $\mathcal{D}_r^0$ to objects in $\mathcal{D}_{qrq^{-1}}$ shows that $\mathcal{D}^0$ is a G-crossed category.

We will now turn to the braiding. If we have morphisms $\sigma \in \mathrm{Hom}_{\mathcal{D}_q^0}((\mathcal{L}_{V_1}, s^1, \mathrm{id}), (\mathcal{L}_{V_2}, s^2, \mathrm{id}))$ and $\tau \in \mathrm{Hom}_{\mathcal{D}_r^0}((\mathcal{L}_{V_3}, s^3, \mathrm{id}), (\mathcal{L}_{V_4}, s^4, \mathrm{id}))$, then

$$\begin{aligned}
&[C^0_{(\mathcal{L}_{V_2},s^2,\mathrm{id}),(\mathcal{L}_{V_4},s^4,\mathrm{id})}]_M\circ[\sigma*\tau]_M\\
&= s^2_M(V_4)\circ[\sigma_I\otimes\tau_I\otimes\mathrm{id}_M] = s^2_M(V_4)\circ[\sigma_I\otimes\mathrm{id}_{V_4\otimes M}]\circ[\mathrm{id}_{V_1}\otimes\tau_I\otimes\mathrm{id}_M]\\
&= s^2_M(V_4)\circ\sigma_{V_4\otimes M}\circ[\mathrm{id}_{V_1}\otimes\tau_I\otimes\mathrm{id}_M] = [\mathrm{id}_{F_q(V_4)}\otimes\sigma_M]\circ s^1_M(V_4)\circ[\mathrm{id}_{V_1}\otimes\tau_I\otimes\mathrm{id}_M]\\
&= [\mathrm{id}_{F_q(V_4)}\otimes\sigma_M]\circ[F_q(\tau_I)\otimes\mathrm{id}_{V_1\otimes M}]\circ s^1_M(V_3) = [F_q(\tau_{F_{q^{-1}}(I)})\otimes\sigma_I\otimes\mathrm{id}_M]\circ s^1_M(V_3)\\
&= [\mathcal{F}_q^0(\tau)*\sigma]_M\circ[C^0_{(\mathcal{L}_{V_1},s^1,\mathrm{id}),(\mathcal{L}_{V_3},s^3,\mathrm{id})}]_M,
\end{aligned}$$

which proves naturality of C^0. Now let $(\mathcal{L}_{V_1}, s^1, \mathrm{id}) \in \mathcal{D}_q^0$, $(\mathcal{L}_{V_2}, s^2, \mathrm{id}) \in \mathcal{D}_r^0$ and $(\mathcal{L}_{V_3}, s^3, \mathrm{id}) \in \mathcal{D}_s^0$. Then

$$\begin{aligned}
&\left[C^0_{(\mathcal{L}_{V_1},s^1,\mathrm{id})*(\mathcal{L}_{V_2},s^2,\mathrm{id}),(\mathcal{L}_{V_3},s^3,\mathrm{id})}\right]_M = \left[C^0_{(\mathcal{L}_{V_1\otimes V_2},s^1*s^2,\mathrm{id}),(\mathcal{L}_{V_3},s^3,\mathrm{id})}\right]_M = (s^1*s^2)_M(V_3)\\
&= s^1_{V_2\otimes M}(F_r(V_3))\circ[\mathrm{id}_{V_1}\otimes s^2_M(V_3)] = [s^1_I(F_r(V_3))\otimes\mathrm{id}_{V_2\otimes M}]\circ[\mathrm{id}_{V_1}\otimes s^2_I(V_3)\otimes\mathrm{id}_M]\\
&= \left\{\left[C^0_{(\mathcal{L}_{V_1},s^1,\mathrm{id}),\mathcal{F}_r(\mathcal{L}_{V_3},s^3,\mathrm{id})}\right]_I\otimes\left[\mathrm{id}_{(\mathcal{L}_{V_2},s^2,\mathrm{id})}\right]_I\otimes\mathrm{id}_M\right\}\\
&\quad\circ\left\{\left[\mathrm{id}_{(\mathcal{L}_{V_1},s^1,\mathrm{id})}\right]_I\otimes\left[C^0_{(\mathcal{L}_{V_2},s^2,\mathrm{id}),(\mathcal{L}_{V_3},s^3,\mathrm{id})}\right]_I\otimes\mathrm{id}_M\right\}\\
&= \left[C^0_{(\mathcal{L}_{V_1},s^1,\mathrm{id}),\mathcal{F}_r(\mathcal{L}_{V_3},s^3,\mathrm{id})}*\mathrm{id}_{(\mathcal{L}_{V_2},s^2,\mathrm{id})}\right]_M\circ\left[\mathrm{id}_{(\mathcal{L}_{V_1},s^1,\mathrm{id})}*C^0_{(\mathcal{L}_{V_2},s^2,\mathrm{id}),(\mathcal{L}_{V_3},s^3,\mathrm{id})}\right]_M
\end{aligned}$$

and

$$\begin{aligned}
&\left[C^0_{(\mathcal{L}_{V_1},s^1,\mathrm{id}),(\mathcal{L}_{V_2},s^2,\mathrm{id})*(\mathcal{L}_{V_3},s^3,\mathrm{id})}\right]_M = \left[C^0_{(\mathcal{L}_{V_1},s^1,\mathrm{id}),(\mathcal{L}_{V_2\otimes V_3},s^2*s^3,\mathrm{id})}\right]_M = s^1_M(V_2\otimes V_3)\\
&= [\mathrm{id}_{F_q(V_2)}\otimes s^1_M(V_3)]\circ s^1_{V_3\otimes M}(V_2) = [\mathrm{id}_{F_q(V_2)}\otimes s^1_I(V_3)\otimes \mathrm{id}_M]\circ[s^1_I(V_2)\otimes \mathrm{id}_{V_3\otimes M}]\\
&= \left\{\left[\mathrm{id}_{\mathcal{F}_q(\mathcal{L}_{V_2},s^2,\mathrm{id})}\right]_I\otimes\left[C^0_{(\mathcal{L}_{V_1},s^1,\mathrm{id}),(\mathcal{L}_{V_3},s^3,\mathrm{id})}\right]_I\otimes \mathrm{id}_M\right\}\\
&\quad\circ\left\{\left[C^0_{(\mathcal{L}_{V_1},s^1,\mathrm{id}),(\mathcal{L}_{V_2},s^2,\mathrm{id})}\right]_I\otimes\left[\mathrm{id}_{(\mathcal{L}_{V_3},s^3,\mathrm{id})}\right]_I\otimes \mathrm{id}_M\right\}\\
&= \left[\mathrm{id}_{\mathcal{F}_q(\mathcal{L}_{V_2},s^2,\mathrm{id})}*C^0_{(\mathcal{L}_{V_1},s^1,\mathrm{id}),(\mathcal{L}_{V_3},s^3,\mathrm{id})}\right]_M\circ\left[C^0_{(\mathcal{L}_{V_1},s^1,\mathrm{id}),(\mathcal{L}_{V_2},s^2,\mathrm{id})}*\mathrm{id}_{(\mathcal{L}_{V_3},s^3,\mathrm{id})}\right]_M.
\end{aligned}$$

Finally, if $q\in G$ and $(\mathcal{L}_{V_1},s^1,\mathrm{id}),(\mathcal{L}_{V_2},s^2,\mathrm{id})\in\mathcal{D}^0$, then

$$\begin{aligned}
&\left[C^0_{\mathcal{F}^0_q(\mathcal{L}_{V_1},s^1,\mathrm{id}),\mathcal{F}^0_q(\mathcal{L}_{V_2},s^2,\mathrm{id})}\right]_M = \left[C^0_{(\mathcal{L}_{F_q(V_1)},\mathcal{F}^0_q s^1,\mathrm{id}),(\mathcal{L}_{F_q(V_2)},\mathcal{F}^0_q s^2,\mathrm{id})}\right]_M\\
&= \mathcal{F}^0_q s^1_M(F_q(V_2)) = F_q(s^1_{F_{q^{-1}}(M)}(V_2)) = F_q\left(\left[C^0_{(\mathcal{L}_{V_1},s^1,\mathrm{id}),(\mathcal{L}_{V_2},s^2,\mathrm{id})}\right]_{F_{q^{-1}}(M)}\right)\\
&= \left[\mathcal{F}^0_q(C^0_{(\mathcal{L}_{V_1},s^1,\mathrm{id}),(\mathcal{L}_{V_2},s^2,\mathrm{id})})\right]_M,
\end{aligned}$$

which means that $C^0_{\mathcal{F}^0_q(\mathcal{L}_{V_1},s^1,\mathrm{id}),\mathcal{F}^0_q(\mathcal{L}_{V_2},s^2,\mathrm{id})} = \mathcal{F}^0_q\left(C^0_{(\mathcal{L}_{V_1},s^1,\mathrm{id}),(\mathcal{L}_{V_2},s^2,\mathrm{id})}\right)$.
□

Recall that in Theorem 2.6.10 in Subsection 2.6.3 we considered the situation where we had a strict tensor category $\mathcal{C}$ together with a structure $(\mathcal{C},\triangleright,\triangleleft)$ of a strict $\mathcal{C}$-bimodule category on $\mathcal{C}$ and we used the notation ${}^{\triangleright}\mathcal{C}^{\triangleleft}$ to denote $\mathcal{C}$ equipped with this structure. We showed that there is a functor from the category $\mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C},{}^{\triangleright}\mathcal{C}^{\triangleleft})$ to itself that assigns to each object (H,s,t) an object $(\mathcal{L}_H,s',t')$, where t' was trivial. In the following theorem we will apply Theorem 2.6.10 to the setting above to obtain functors $\mathcal{D}_q\to\mathcal{D}^0_q$ and we will use these functors to transport the braided G-crossed structure of $\mathcal{D}^0$ to $\mathcal{D}$.

Theorem 4.5.4 *Let $\mathcal{C}$ be a strict tensor category with strict action F of the group G.*

(1) For each $q\in G$ we obtain a functor $\mathcal{P}_q:\mathcal{D}_q\to\mathcal{D}^0_q$ by defining $\mathcal{P}_q[(H,s,t)] := (\mathcal{P}_qH,\mathcal{P}_qs,\mathrm{id})$ for $(H,s,t)\in\mathcal{D}_q$, where[12]

$$\mathcal{P}_qH := \mathcal{L}_H = \mathcal{L}_{H(I)},$$

$$\mathcal{P}_qs_M(X) := [s_I(X)\otimes \mathrm{id}_M]\circ[t_I(X)^{-1}\otimes \mathrm{id}_M],$$

and by defining $\mathcal{P}_q(\sigma)$ by

$$\mathcal{P}_q(\sigma)_M := t^2_I(M)\circ\sigma_M\circ t^1_I(M)^{-1}$$

for $\sigma:(H_1,s^1,t^1)\to(H_2,s^2,t^2)$.

(2) For each $q\in G$ we have a natural isomorphism $\psi^q:\mathrm{id}_{\mathcal{D}_q}\to\mathcal{P}_q$, where $\psi^q_{(H,s,t)}:(H,s,t)\to(\mathcal{P}_qH,\mathcal{P}_qs,\mathrm{id})$ is given by $[\psi^q_{(H,s,t)}]_M := t_I(M):H(M)\to H(I)\otimes M$. We obtain a functor $\mathcal{P}:\mathcal{D}\to\mathcal{D}^0$ by defining its restriction to any $\mathcal{D}_q$ to be $\mathcal{P}_q$, and we have a natural isomorphism $\psi:\mathrm{id}_{\mathcal{D}}\to\mathcal{P}$ that is defined by $[\psi_{(H,s,t)}]_M = t_I(M)$.

(3) The restriction of the functor $\mathcal{P}:\mathcal{D}\to\mathcal{D}^0$ to $\mathcal{D}^0$ is the identity functor. As a consequence, $\mathcal{P}\circ\mathcal{P}=\mathcal{P}$.

(4) The category $\mathcal{D}$ can be given the structure of a braided G-crossed category as follows.

[12] Analogous to the remark in the preceding footnote, we simply write $\mathcal{P}_qs_M(X)$ rather than $(\mathcal{P}_qs)_M(X)$. Similarly, when we define the group action we will also write $\mathcal{F}_qs_M(X)$ instead of $(\mathcal{F}_qs)_M(X)$.

- *The tensor product* $(H_1, s^1, t^1) \star (H_2, s^2, t^2) = (H_1 \star H_2, s^1 \star s^2, \mathrm{id}) \in \mathcal{D}$ *is defined by*

$$(H_1, s^1, t^1) \star (H_2, s^2, t^2) := \mathcal{P}(H_1, s^1, t^1) * \mathcal{P}(H_2, s^2, t^2),$$

so $H_1 \star H_2 = \mathcal{P}H_1 * \mathcal{P}H_2 = \mathcal{L}_{H_1(I)\otimes H_2(I)}$ *and* $s^1 \star s^2 = \mathcal{P}s^1 * \mathcal{P}s^2$.
If $\sigma \in \mathrm{Hom}_{\mathcal{D}}((H_1, s^1, t^1), (H_2, s^2, t^2))$ *and* $\tau \in \mathrm{Hom}_{\mathcal{D}}((H_3, s^3, t^3), (H_4, s^4, t^4))$, *then* $\sigma\star\tau$ *is defined by*

$$\sigma \star \tau = \mathcal{P}(\sigma) * \mathcal{P}(\tau).$$

This tensor product is associative. The unit object is $(\mathcal{L}_I, s^0, \mathrm{id}) \in \mathcal{D}_e$, *where* $s^0_M(X) := \mathrm{id}_{X\otimes M}$, *and the left and right unit contraints* $l_{(H,s,t)} : (\mathcal{P}H, \mathcal{P}s, \mathrm{id}) \to (H, s, t)$ *and* $r_{(H,s,t)} : (\mathcal{P}H, \mathcal{P}s, \mathrm{id}) \to (H, s, t)$ *are given by* $[l_{(H,s,t)}]_M = t_I(M)^{-1} = [r_{(H,s,t)}]_M$.
Furthermore, $\mathcal{P} : \mathcal{D} \to \mathcal{D}^0$ *is a strict tensor functor and* $\psi : \mathrm{id}_{\mathcal{D}} \to \mathcal{P}$ *is a natural tensor isomorphism, where* $\mathcal{P}$ *is considered as a functor* $\mathcal{D} \to \mathcal{D}$.
- *The group action* $\mathcal{F}_q[(H, s, t)] = (\mathcal{F}_q H, \mathcal{F}_q s, \mathrm{id}) \in \mathcal{D}_{qrq^{-1}}$ *on an object* $(H, s, t) \in \mathcal{D}_r$ *is defined by*

$$\mathcal{F}_q[(H, s, t)] := \mathcal{F}^0_q \mathcal{P}[(H, s, t)],$$

so $\mathcal{F}_q H = \mathcal{F}^0_q \mathcal{P}H = \mathcal{L}_{F_q(H(I))}$ *and* $\mathcal{F}_q s_M(X) = \mathcal{F}^0_q \mathcal{P}s_M(X)$. *The group action on a morphism* $\sigma \in \mathrm{Hom}(\mathcal{D})$ *is defined by* $\mathcal{F}_q(\sigma) = \mathcal{F}^0_q(\mathcal{P}(\sigma))$ *and* $\mathcal{F}_q$ *is a strict tensor functor for each* $q \in G$. *We also have that* $\mathcal{F}_{qr} = \mathcal{F}_q\mathcal{F}_r$ *for all* $q, r \in G$ *and that* $\mathcal{F}_e = \mathcal{P}$, *and as part of the group action we define* $\varepsilon^{\mathcal{F}} : \mathrm{id}_{\mathcal{D}} \to \mathcal{F}_e$ *by* $\varepsilon^{\mathcal{F}} = \psi$.
- *If* $(H, s, t) \in \mathcal{D}_q$, *then its degree is defined to be* q.
- *If* $(H_1, s^1, \mathrm{id}) \in \mathcal{D}$ *and* $(H_2, s^2, \mathrm{id}) \in \mathcal{D}$, *then their braiding is defined by*

$$C_{(H_1,s^1,\mathrm{id}),(H_2,s^2,\mathrm{id})} = C^0_{\mathcal{P}[(H_1,s^1,\mathrm{id})],\mathcal{P}[(H_2,s^2,\mathrm{id})]}.$$

(5) *The strict tensor functor* $\mathcal{P} : \mathcal{D} \to \mathcal{D}^0$ *is a braided* G*-crossed functor and the natural tensor isomorphism* $\psi : \mathrm{id}_{\mathcal{D}} \to \mathcal{P}$ *is a natural isomorphism of braided* G*-crossed functors.*

Proof. Parts (1) and (2) of the theorem are just a restatement of Theorem 2.6.10. For part (3), let $(\mathcal{L}_V, s, \mathrm{id}) \in \mathcal{D}^0_q$ and note that $\mathcal{P}\mathcal{L}_V = \mathcal{L}_{\mathcal{L}_V(I)} = \mathcal{L}_V$ and

$$\mathcal{P}s_M(X) = [s_I(X) \otimes \mathrm{id}_M] \circ [\mathrm{id}^{-1}_{I\otimes X} \otimes \mathrm{id}_M] = s_I(X) \otimes \mathrm{id}_M = s_M(X).$$

If $\sigma \in \mathrm{Hom}_{\mathcal{D}^0}((\mathcal{L}_{V_1}, s^1, \mathrm{id}), (\mathcal{L}_{V_2}, s^2, \mathrm{id}))$, then $\mathcal{P}(\sigma)_M = \mathrm{id}_{V_2\otimes M} \circ \sigma_M \circ \mathrm{id}^{-1}_{V_1\otimes M} = \sigma_M$, so $\mathcal{P}(\sigma) = \sigma$.

We will now prove part (4) of the theorem. It follows directly from the preceding lemma that $\star$ is a well-defined functor $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$ that is compatible with the degrees of objects. To prove associativity of $\star$, let $(H_1, s^1, t^1), (H_2, s^2, t^2), (H_3, s^3, t^3) \in \mathcal{D}$. We then have

$$\begin{aligned}(H_1 \star H_2) \star H_3 &= \mathcal{P}(\mathcal{P}H_1 * \mathcal{P}H_2) * \mathcal{P}H_3 = (\mathcal{P}H_1 * \mathcal{P}H_2) * \mathcal{P}H_3 \\ &= \mathcal{P}H_1 * (\mathcal{P}H_2 * \mathcal{P}H_3) = \mathcal{P}H_1 * \mathcal{P}(\mathcal{P}H_2 * \mathcal{P}H_3) \\ &= H_1 \star (H_2 \star H_3)\end{aligned}$$

and

$$\begin{aligned}(s^1 \star s^2) \star s^3 &= \mathcal{P}(\mathcal{P}s^1 * \mathcal{P}s^2) * \mathcal{P}s^3 = (\mathcal{P}s^1 * \mathcal{P}s^2) * \mathcal{P}s^3 \\ &= \mathcal{P}s^1 * (\mathcal{P}s^2 * \mathcal{P}s^3) = \mathcal{P}s^1 * \mathcal{P}(\mathcal{P}s^2 * \mathcal{P}s^3) \\ &= s^1 \star (s^2 \star s^3),\end{aligned}$$

where in both computations we have used part (3) of the theorem. A similar computation also shows that $\star$ is associative on the morphisms.

For any $(H,s,t) \in \mathcal{D}$ we have $(\mathcal{L}_I, s^0, \mathrm{id}) \star (H,s,t) = (\mathcal{P}H, \mathcal{P}s, \mathrm{id}) = (H,s,t) \star (\mathcal{L}_I, s^0, \mathrm{id})$. To prove naturality of l and r, let $\sigma \in \mathrm{Hom}_{\mathcal{D}}((H_1,s^1,t^1),(H_2,s^2,t^2))$ and $M \in \mathcal{C}$. Then

$$\begin{aligned}
&[l_{(H_2,s^2,t^2)}]_M \circ [\mathrm{id}_{(\mathcal{L}_I,s^0,\mathrm{id})} \star \sigma]_M \\
&= t_I^2(M)^{-1} \circ [\mathcal{P}(\mathrm{id}_{(\mathcal{L}_I,s^0,\mathrm{id})}) * \mathcal{P}(\sigma)]_M = t_I^2(M)^{-1} \circ \{[\mathrm{id}_{(\mathcal{L}_I,s^0,\mathrm{id})}]_I \otimes \mathcal{P}(\sigma)_I \otimes \mathrm{id}_M\} \\
&= t_I^2(M)^{-1} \circ [\mathcal{P}(\sigma)_I \otimes \mathrm{id}_M] = t_I^2(M)^{-1} \circ \mathcal{P}(\sigma)_M = t_I^2(M)^{-1} \circ t_I^2(M) \circ \sigma_M \circ t_I^1(M)^{-1} \\
&= \sigma_M \circ [l_{(H_1,s^1,t^1)}]_M
\end{aligned}$$

and

$$\begin{aligned}
&[r_{(H_2,s^2,t^2)}]_M \circ [\sigma \star \mathrm{id}_{(\mathcal{L}_I,s^0,\mathrm{id})}]_M \\
&= t_I^2(M)^{-1} \circ [\mathcal{P}(\sigma) * \mathcal{P}(\mathrm{id}_{(\mathcal{L}_I,s^0,\mathrm{id})})]_M = t_I^2(M)^{-1} \circ \{\mathcal{P}(\sigma)_I \otimes [\mathrm{id}_{(\mathcal{L}_I,s^0,\mathrm{id})}]_I \otimes \mathrm{id}_M\} \\
&= t_I^2(M)^{-1} \circ [\mathcal{P}(\sigma)_I \otimes \mathrm{id}_M] = t_I^2(M)^{-1} \circ \mathcal{P}(\sigma)_M = t_I^2(M)^{-1} \circ t_I^2(M) \circ \sigma_M \circ t_I^1(M)^{-1} \\
&= \sigma_M \circ [r_{(H_1,s^1,t^1)}]_M.
\end{aligned}$$

To check the triangle axiom, let $(H_1,s^1,t^1),(H_2,s^2,t^2) \in \mathcal{D}$. Since for any $(H,s,t) \in \mathcal{D}$ we have $\mathcal{P}(l_{(H,s,t)})_M = t_I(M) \circ [l_{(H,s,t)}]_M \circ \mathrm{id}_{H(I)\otimes M}^{-1} = \mathrm{id}_{H(I)\otimes M}$ and similarly $\mathcal{P}(r_{(H,s,t)}) = \mathrm{id}_{H(I)\otimes M}$, we have on the one hand

$$\begin{aligned}
&[r_{(H_1,s^1,t^1)} \star \mathrm{id}_{(H_2,s^2,t^2)}]_M \\
&= [\mathcal{P}(r_{(H_1,s^1,t^1)}) * \mathcal{P}(\mathrm{id}_{(H_2,s^2,t^2)})]_M = \mathcal{P}(r_{(H_1,s^1,t^1)})_I \otimes [\mathrm{id}_{(H_2,s^2,t^2)}]_I \otimes \mathrm{id}_M \\
&= \mathcal{P}(r_{(H_1,s^1,t^1)})_I \otimes \mathrm{id}_{H_2(I)\otimes M} = \mathcal{P}(r_{(H_1,s^1,t^1)})_{H_2(I)\otimes M} = \mathrm{id}_{H_1(I)\otimes H_2(I)\otimes M}
\end{aligned}$$

and on the other hand we have

$$\begin{aligned}
&[\mathrm{id}_{(H_1,s^1,t^1)} \star l_{(H_2,s^2,t^2)}]_M \\
&= [\mathcal{P}(\mathrm{id}_{(H_1,s^1,t^1)}) * \mathcal{P}(l_{(H_2,s^2,t^2)})]_M = [\mathrm{id}_{(H_1,s^1,t^1)}]_I \otimes \mathcal{P}(l_{(H_2,s^2,t^2)})_I \otimes \mathrm{id}_M \\
&= \mathrm{id}_{H_1(I)} \otimes \mathcal{P}(l_{(H_2,s^2,t^2)})_I \otimes \mathrm{id}_M = \mathrm{id}_{H_1(I)} \otimes \mathcal{P}(l_{(H_2,s^2,t^2)})_M = \mathrm{id}_{H_1(I)\otimes H_2(I)\otimes M},
\end{aligned}$$

so the two expressions are equal, i.e. $r_{(H_1,s^1,t^1)} \star \mathrm{id}_{(H_2,s^2,t^2)} = \mathrm{id}_{(H_1,s^1,t^1)} \star l_{(H_2,s^2,t^2)}$. Thus $\mathcal{D}$ is a tensor category.

To see that $\mathcal{P}$ is a strict tensor functor, we first note that for any $(H_1,s^1,t^1),(H_2,s^2,t^2) \in \mathcal{D}$ we have

$$\mathcal{P}[(H_1,s^1,t^1) \star (H_2,s^2,t^2)] = \mathcal{P}[\mathcal{P}(H_1,s^1,t^1) * \mathcal{P}(H_2,s^2,t^2)] = \mathcal{P}(H_1,s^1,t^1) * \mathcal{P}(H_2,s^2,t^2).$$

Also $\mathcal{P}(\mathcal{L}_I,s^0,\mathrm{id}) = (\mathcal{L}_I,s^0,\mathrm{id})$ and for any $(H,s,t) \in \mathcal{D}$ we have $\mathcal{P}(l_{(H,s,t)})_M = \mathrm{id}_{H(I)\otimes M} = \mathcal{P}(r_{(H,s,t)})_M$.

It is clear that $\psi : \mathrm{id}_{\mathcal{D}} \to \mathcal{P}$ is a natural isomorphism. To see that it is a natural tensor isomorphism, we note that $\psi_{(\mathcal{L}_I,s^0,\mathrm{id})} = \mathrm{id}_{(\mathcal{L}_I,s^0,\mathrm{id})}$ and that for any $(H_1,s^1,t^1),(H_2,s^2,t^2) \in \mathcal{D}$ we have

$$\begin{aligned}
\psi_{(H_1,s^1,t^1)\star(H_2,s^2,t^2)} &= \psi_{(H_1\star H_2,s^1\star s^2,\mathrm{id})} = \mathrm{id}_{(H_1\star H_2,s^1\star s^2,\mathrm{id})} = \mathrm{id}_{\mathcal{P}(H_1,s^1,t^1)*\mathcal{P}(H_2,s^2,t^2)} \\
&= \mathrm{id}_{\mathcal{P}(H_1,s^1,t^1)} * \mathrm{id}_{\mathcal{P}(H_2,s^2,t^2)} = \mathcal{P}\psi_{(H_1,s^1,t^1)} * \mathcal{P}\psi_{(H_2,s^2,t^2)} \\
&= \psi_{(H_1,s^1,t^1)} \star \psi_{(H_2,s^2,t^2)}.
\end{aligned}$$

If $q \in G$, then $\mathcal{F}_q = \mathcal{F}_q^0 \circ \mathcal{P}$ is a functor $\mathcal{D} \to \mathcal{D}$, and its restriction to $\mathcal{D}_r$ is a functor $\mathcal{D}_r \to \mathcal{D}^0_{qrq^{-1}} \subset \mathcal{D}_{qrq^{-1}}$. To see that it is a tensor functor, let $(H_1,s^1,t^1),(H_2,s^2,t^2) \in \mathcal{D}$. Then

$$\begin{aligned}
&\mathcal{F}_q[(H_1,s^1,t^1) \star (H_2,s^2,t^2)] \\
&= \mathcal{F}_q^0\mathcal{P}[\mathcal{P}(H_1,s^1,t^1) * \mathcal{P}(H_2,s^2,t^2)] = \mathcal{F}_q^0[\mathcal{P}(H_1,s^1,t^1) * \mathcal{P}(H_2,s^2,t^2)] \\
&= \mathcal{F}_q^0\mathcal{P}(H_1,s^1,t^1) * \mathcal{F}_q^0\mathcal{P}(H_2,s^2,t^2) = \mathcal{P}\mathcal{F}_q^0\mathcal{P}(H_1,s^1,t^1) * \mathcal{P}\mathcal{F}_q^0\mathcal{P}(H_2,s^2,t^2)
\end{aligned}$$

$$= \mathcal{F}_q(H_1, s^1, t^1) \star \mathcal{F}_q(H_2, s^2, t^2).$$

Similarly, if $\sigma, \tau \in \mathrm{Hom}(\mathcal{D})$, then

$$\begin{aligned}\mathcal{F}_q(\sigma \star \tau) &= \mathcal{F}_q^0(\mathcal{P}(\mathcal{P}(\sigma) * \mathcal{P}(\tau))) = \mathcal{F}_q^0(\mathcal{P}(\sigma) * \mathcal{P}(\tau)) \\ &= \mathcal{F}_q^0\mathcal{P}(\sigma) * \mathcal{F}_q^0\mathcal{P}(\tau) = \mathcal{P}\mathcal{F}_q^0\mathcal{P}(\sigma) * \mathcal{P}\mathcal{F}_q^0\mathcal{P}(\tau) \\ &= \mathcal{F}_q(\sigma) \star \mathcal{F}_q(\tau).\end{aligned}$$

Also, $\mathcal{F}_q(\mathcal{L}_I, s^0, \mathrm{id}) = \mathcal{F}_q^0(\mathcal{L}_I, s^0, \mathrm{id}) = (\mathcal{L}_I, s^0, \mathrm{id})$, so we conclude that $\mathcal{F}_q$ is a strict tensor functor. If $q, r \in G$, then for any $s \in G$ the functor $\mathcal{F}_{qr} : \mathcal{D}_s \to \mathcal{D}_{qrsr^{-1}q^{-1}}$ is given by $\mathcal{F}_{qr} = \mathcal{F}_{qr}^0\mathcal{P} = \mathcal{F}_q^0\mathcal{F}_r^0\mathcal{P} = \mathcal{F}_q^0\mathcal{P}\mathcal{F}_r^0\mathcal{P} = \mathcal{F}_q\mathcal{F}_r$. The functor $\mathcal{F}_e$ on $\mathcal{D}$ is given by $\mathcal{F}_e = \mathcal{F}_e^0\mathcal{P} = \mathcal{P}$. We know from our results above that $\varepsilon^{\mathcal{F}} : \mathrm{id}_{\mathcal{D}} \to \mathcal{F}_e$ is a natural tensor isomorphism, since $\varepsilon^{\mathcal{F}} = \psi$ by definition. Also, $\varepsilon^{\mathcal{F}}$ has the property that for any $q \in G$ and $(H, s, t) \in \mathcal{D}$ we have

$$\begin{aligned}\varepsilon^{\mathcal{F}}_{\mathcal{F}_q(H,s,t)} &= \psi_{\mathcal{F}_q(H,s,t)} = \mathrm{id}_{\mathcal{F}_q(H,s,t)} \\ \mathcal{F}_q(\varepsilon^{\mathcal{F}}_{(H,s,t)}) &= \mathcal{F}_q(\psi_{(H,s,t)}) = \mathcal{F}_q^0\mathcal{P}(\psi_{(H,s,t)}) = \mathcal{F}_q^0(\mathrm{id}_{\mathcal{P}(H,s,t)}) = (\mathrm{id}_{\mathcal{F}_q^0\mathcal{P}(H,s,t)}) = \mathrm{id}_{\mathcal{F}_q(H,s,t)}.\end{aligned}$$

Thus we conclude that $\mathcal{F}$ defines a group action on $\mathcal{D}$ and that $\mathcal{D}$ is a G-crossed category.

If $\sigma \in \mathrm{Hom}_{\mathcal{D}_q}((H_1, s^1, t^1), (H_2, s^2, t^2))$ and $\tau \in \mathrm{Hom}_{\mathcal{D}}((H_3, s^3, t^3), (H_4, s^4, t^4))$, then

$$\begin{aligned}&C_{(H_2,s^2,t^2),(H_4,s^4,t^4)} \circ [\sigma \star \tau] \\ &= C^0_{\mathcal{P}(H_2,s^2,t^2),\mathcal{P}(H_4,s^4,t^4)} \circ [\mathcal{P}(\sigma) * \mathcal{P}(\tau)] = [\mathcal{F}_q^0\mathcal{P}(\tau) * \mathcal{P}(\sigma)] \circ C^0_{\mathcal{P}(H_1,s^1,t^1),\mathcal{P}(H_3,s^3,t^3)} \\ &= [\mathcal{P}\mathcal{F}_q^0\mathcal{P}(\tau) * \mathcal{P}(\sigma)] \circ C^0_{\mathcal{P}(H_1,s^1,t^1),\mathcal{P}(H_3,s^3,t^3)} = [\mathcal{F}_q(\tau) \star \sigma] \circ C_{(H_1,s^1,t^1),(H_3,s^3,t^3)},\end{aligned}$$

showing naturality of C. Now let $(H_1, s^1, t^1) \in \mathcal{D}_q$, $(H_2, s^2, t^2) \in \mathcal{D}_r$ and $(H_2, s^3, t^3) \in \mathcal{D}_s$. Then

$$\begin{aligned}&C_{(H_1,s^1,t^1)\star(H_2,s^2,t^2),(H_3,s^3,t^3)} = C^0_{\mathcal{P}(H_1,s^1,t^1)*\mathcal{P}(H_2,s^2,t^2),\mathcal{P}(H_3,s^3,t^3)} \\ &= [C^0_{\mathcal{P}(H_1,s^1,t^1),\mathcal{F}_r^0\mathcal{P}(H_3,s^3,t^3)} * \mathrm{id}_{\mathcal{P}(H_2,s^2,t^2)}] \circ [\mathrm{id}_{\mathcal{P}(H_1,s^1,t^1)} * C^0_{\mathcal{P}(H_2,s^2,t^2),\mathcal{P}(H_3,s^3,t^3)}] \\ &= [\mathcal{P}(C^0_{\mathcal{P}(H_1,s^1,t^1),\mathcal{F}_r^0\mathcal{P}(H_3,s^3,t^3)}) * \mathcal{P}(\mathrm{id}_{(H_2,s^2,t^2)})] \circ [\mathcal{P}(\mathrm{id}_{(H_1,s^1,t^1)}) * \mathcal{P}(C^0_{\mathcal{P}(H_2,s^2,t^2),\mathcal{P}(H_3,s^3,t^3)})] \\ &= [C_{(H_1,s^1,t^1),\mathcal{F}_r(H_3,s^3,t^3)} \star \mathrm{id}_{(H_2,s^2,t^2)}] \circ [\mathrm{id}_{(H_1,s^1,t^1)} \star C_{(H_2,s^2,t^2),(H_3,s^3,t^3)}]\end{aligned}$$

and

$$\begin{aligned}&C_{(H_1,s^1,t^1),(H_2,s^2,t^2)\star(H_3,s^3,t^3)} = C^0_{\mathcal{P}(H_1,s^1,t^1),\mathcal{P}(H_2,s^2,t^2)*\mathcal{P}(H_3,s^3,t^3)} \\ &= [\mathrm{id}_{\mathcal{F}_q^0\mathcal{P}(H_2,s^2,t^2)} * C^0_{\mathcal{P}(H_1,s^1,t^1),\mathcal{P}(H_3,s^3,t^3)}] \circ [C^0_{\mathcal{P}(H_1,s^1,t^1),\mathcal{P}(H_2,s^2,t^2)} * \mathrm{id}_{\mathcal{P}(H_3,s^3,t^3)}] \\ &= [\mathcal{P}(\mathrm{id}_{\mathcal{F}_q^0\mathcal{P}(H_2,s^2,t^2)}) * \mathcal{P}(C^0_{\mathcal{P}(H_1,s^1,t^1),\mathcal{P}(H_3,s^3,t^3)})] \circ [\mathcal{P}(C^0_{\mathcal{P}(H_1,s^1,t^1),\mathcal{P}(H_2,s^2,t^2)}) * \mathcal{P}(\mathrm{id}_{(H_3,s^3,t^3)})] \\ &= [\mathrm{id}_{\mathcal{F}_q(H_2,s^2,t^2)} \star C_{(H_1,s^1,t^1),(H_3,s^3,t^3)}] \circ [C_{(H_1,s^1,t^1),(H_2,s^2,t^2)} \star \mathrm{id}_{(H_3,s^3,t^3)}].\end{aligned}$$

Finally, for any $q \in G$, $(H_1, s^1, t^1) \in \mathcal{D}_r$ and $(H_2, s^2, t^2) \in \mathcal{D}_s$ we have

$$\begin{aligned}&C_{\mathcal{F}_q(H_1,s_1,t^1),\mathcal{F}_q(H_2,s^2,t^2)} = C^0_{\mathcal{P}\mathcal{F}_q(H_1,s_1,t^1),\mathcal{P}\mathcal{F}_q(H_2,s^2,t^2)} \\ &= C^0_{\mathcal{P}\mathcal{F}_q^0\mathcal{P}(H_1,s_1,t^1),\mathcal{P}\mathcal{F}_q^0(\mathcal{P}(H_2,s^2,t^2))} = C^0_{\mathcal{F}_q^0\mathcal{P}(H_1,s_1,t^1),\mathcal{F}_q^0\mathcal{P}(H_2,s^2,t^2)} \\ &= \mathcal{F}_q^0(C^0_{\mathcal{P}(H_1,s^1,t^1),\mathcal{P}(H_2,s^2,t^2)}) = \mathcal{F}_q^0(C_{(H_1,s^1,t^1),(H_2,s^2,t^2)}) \\ &= \mathcal{F}_q(C_{(H_1,s^1,t^1),(H_2,s^2,t^2)}).\end{aligned}$$

Thus $\mathcal{D}$ is a braided G-crossed category.

That $\mathcal{P}$ is a strict G-functor follows from $\mathcal{P} \circ \mathcal{F}_q = \mathcal{P} \circ \mathcal{F}_q^0 \circ \mathcal{P} = \mathcal{F}_q^0 \circ \mathcal{P}$ and from the fact that $\mathcal{P}(\varepsilon^{\mathcal{F}}_{(H,s,t)}) = \mathcal{P}(\psi_{(H,s,t)}) = \mathrm{id}_{\mathcal{P}(H,s,t)}$ for any $(H,s,t) \in \mathcal{D}$. Also,

$$\begin{aligned}\mathcal{P}(C_{(H_1,s^1,t^1),(H_2,t^2,t^2)}) &= \mathcal{P}(C^0_{\mathcal{P}(H_1,s^1,t^1),\mathcal{P}(H_2,t^2,t^2)}) = C^0_{\mathcal{P}(H_1,s^1,t^1),\mathcal{P}(H_2,t^2,t^2)} \\ &= C_{\mathcal{P}(H_1,s^1,t^1),\mathcal{P}(H_2,t^2,t^2)},\end{aligned}$$

so $\mathcal{P}$ is braided. By our earlier results on $\varepsilon^{\mathcal{F}}$, we also have

$$\mathcal{F}_q(\psi_{(H,s,t)}) = \mathcal{F}_q(\varepsilon^{\mathcal{F}}_{(H,s,t)}) = \mathrm{id}_{\mathcal{F}_q(H,s,t)} = \varepsilon^{\mathcal{F}}_{\mathcal{F}_q(H,s,t)} = \psi_{\mathcal{F}_q(H,s,t)}$$

for any $(H,s,t) \in \mathcal{D}$. Thus ψ is a natural G-isomorphism and hence a natural isomorphism of braided G-functors.
□

We now have a braided G-crossed structure on $\mathcal{D}$ and this structure will be used in the following subsection. Because the restriction of all structures of $\mathcal{D}$ to $\mathcal{D}^0$ are the same ones as defined in Lemma 4.5.3, there is no need to distinguish these structures in our notation anymore. Therefore, henceforth we will no longer use $*$, $\mathcal{F}_q^0$ and C^0 in our notation, but only $\star$, $\mathcal{F}_q$ and C.

4.5.3 $Z_G(\mathcal{C}) \simeq \bigsqcup_{q\in G} \mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C}, {}^q\mathcal{C})$

We are in the position to state our main theorem of this section:

Theorem 4.5.5 *Let G be a group and let $\mathcal{C}$ be a strict tensor category with strict G-action F. Then there exists an equivalence*

$$\mathscr{K} : \bigsqcup_{q\in G} \mathrm{Fun}_{(\mathcal{C},\mathcal{C})}(\mathcal{C}, {}^q\mathcal{C}) \to Z_G(\mathcal{C})$$

of braided G-crossed categories.

Proof. We will first define a functor $\mathscr{K}^0 : \mathcal{D}^0 \to Z_G(\mathcal{C})$. We have already seen from diagram (4.5.1) and equation (4.5.4) that if $(\mathcal{L}_V, s, \mathrm{id}) \in \mathcal{D}_q^0$, then s_I is a half q-braiding for V, so we can define $\mathscr{K}^0$ on an object $(\mathcal{L}_V, s, \mathrm{id}) \in \mathcal{D}_q$ as

$$\mathscr{K}^0[(\mathcal{L}_V, s, \mathrm{id})] = (V, q, \Phi_V^s),$$

where $\Phi_V^s(X) = s_I(X)$. If $\sigma \in \mathrm{Hom}_{\mathcal{D}_q}((\mathcal{L}_{V_1}, s^1, \mathrm{id}), (\mathcal{L}_{V_2}, s^2, \mathrm{id}))$, then $\sigma_I \in \mathrm{Hom}_{\mathcal{C}}(V_1, V_2)$, which suggests that we define $\mathscr{K}^0(\sigma) = \sigma_I$. It follows directly from (4.5.8) that

$$[\mathrm{id}_{F_q(X)} \otimes \sigma_I] \circ s_I^1(X) = s_I^2(X) \circ \sigma_X = s_I^2(X) \circ [\sigma_I \otimes \mathrm{id}_X],$$

so $\mathscr{K}^0(\sigma) = \sigma_I \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V_1, q, \Phi_{V_1}^{s^1}), (V_2, q, \Phi_{V_2}^{s^2}))$. If $\tau \in \mathrm{Hom}_{\mathcal{D}_q^0}((\mathcal{L}_{V_2}, s^2, \mathrm{id}), (\mathcal{L}_{V_3}, s^3, \mathrm{id}))$, then

$$\mathscr{K}^0(\tau \circ \sigma) = (\tau \circ \sigma)_I = \tau_I \circ \sigma_I = \mathscr{K}^0(\tau) \circ \mathscr{K}^0(\sigma).$$

For any $(\mathcal{L}_V, s, \mathrm{id}) \in \mathcal{D}_q^0$ we have $\mathscr{K}^0(\mathrm{id}_{(\mathcal{L}_V,s,\mathrm{id})}) = [\mathrm{id}_{(\mathcal{L}_V,s,\mathrm{id})}]_I = \mathrm{id}_V = \mathrm{id}_{(V,q,\Phi_V^s)} = \mathrm{id}_{\mathscr{K}^0(\mathcal{L}_V,s,\mathrm{id})}$, showing that $\mathscr{K}^0$ is indeed a functor. If $(\mathcal{L}_{V_1}, s^1, \mathrm{id}) \in \mathcal{D}_q^0$ and $(\mathcal{L}_{V_2}, s^2, \mathrm{id}) \in \mathcal{D}_r^0$, then (4.5.10) gives us

$$\begin{aligned}\Phi_{V_1 \otimes V_2}^{s^1 \star s^2}(X) &= (s^1 \star s^2)_I(X) = s_{V_2}^1(F_r(X)) \circ [\mathrm{id}_{V_1} \otimes s_I^2(X)] = [s_I^1(F_r(X)) \otimes \mathrm{id}_{V_2}] \circ [\mathrm{id}_{V_1} \otimes s_I^2(X)] \\ &= [\Phi_{V_1}^{s^1}(F_r(X)) \otimes \mathrm{id}_{V_2}] \circ [\mathrm{id}_{V_1} \otimes \Phi_{V_2}^{s^2}(X)] = (\Phi_{V_1}^{s^1} \otimes \Phi_{V_2}^{s^2})(X),\end{aligned}$$

which implies that $\Phi^{s^1\star s^2}_{V_1\otimes V_2} = \Phi^{s^1}_{V_1}\otimes\Phi^{s^2}_{V_2}$ and hence

$$\begin{aligned}\mathscr{K}^0[(\mathcal{L}_{V_1}, s^1, \mathrm{id})\star(\mathcal{L}_{V_2}, s^2, \mathrm{id})] &= \mathscr{K}^0(\mathcal{L}_{V_1\otimes V_2}, s^1\star s^2, \mathrm{id}) = (V_1\otimes V_2, qr, \Phi^{s^1\star s^2}_{V_1\otimes V_2})\\ &= (V_1\otimes V_2, qr, \Phi^{s^1}_{V_1}\otimes\Phi^{s^1}_{V_2}) = (V_1, q, \Phi^{s^1}_{V_1})\otimes(V_2, r, \Phi^{s^2}_{V_2})\\ &= \mathscr{K}^0(\mathcal{L}_{V_1}, s^1, \mathrm{id})\otimes\mathscr{K}^0(\mathcal{L}_{V_2}, s^2, \mathrm{id}).\end{aligned}$$

Also, if $\sigma,\tau\in\mathrm{Hom}(\mathcal{D}^0)$, then

$$\mathscr{K}^0(\sigma\star\tau) = (\sigma\star\tau)_I = \sigma_I\otimes\tau_I\otimes\mathrm{id}_I = \sigma_I\otimes\tau_I = \mathscr{K}^0(\sigma)\otimes\mathscr{K}^0(\tau)$$

and $\mathscr{K}^0(\mathcal{L}_I, s^0, \mathrm{id}) = (I, e, \Phi^{s^0}_I) = (I, e, \Phi^0_I)$, since for any $X\in\mathcal{C}$ we have $\Phi^{s^0}_I(X) = s^0_I(X) = \mathrm{id}_X = \Phi^0_I(X)$. Thus $\mathscr{K}^0$ is a strict tensor functor. Now let $q\in G$ and $(\mathcal{L}_V, s, \mathrm{id})\in\mathcal{D}^0_r$. Then

$$\begin{aligned}\Phi^{\mathcal{F}_q s}_{F_q(V)}(X) &= (\mathcal{F}_q s)_I(X) = F_q(s_{F_{q^{-1}}(I)}(F_{q^{-1}}(X))) = F_q(s_I(F_{q^{-1}}(X)))\\ &= F_q(\Phi^s_V(F_{q^{-1}}(X))) = \mathcal{F}_q\Phi^s_V(X),\end{aligned}$$

from which it follows that $\Phi^{\mathcal{F}_q s}_{F_q(V)} = \mathcal{F}_q\Phi^s_V$. Using this, we get

$$\begin{aligned}\mathscr{K}^0[\mathcal{F}_q(\mathcal{L}_V, s, \mathrm{id})] &= \mathscr{K}^0(\mathcal{L}_{F_q(V)}, \mathcal{F}_q s, \mathrm{id}) = (F_q(V), qrq^{-1}, \Phi^{\mathcal{F}_q s}_{F_q(V)})\\ &= (F_q(V), qrq^{-1}, \mathcal{F}_q\Phi^s_V) = \mathcal{F}_q(V, r, \Phi^s_V) = \mathcal{F}_q\mathscr{K}^0(\mathcal{L}_V, s, \mathrm{id}).\end{aligned}$$

If $q\in G$ and $\sigma\in\mathrm{Hom}(\mathcal{D}^0)$, then

$$\mathscr{K}^0(\mathcal{F}_q(\sigma)) = [\mathcal{F}_q(\sigma)]_I = F_q(\sigma_{F_{q^{-1}}(I)}) = F_q(\sigma_I) = \mathcal{F}_q(\sigma_I) = \mathcal{F}_q(\mathscr{K}^0(\sigma)).$$

Hence $\mathscr{K}^0$ is a strict G-functor as well. It is also clear that $\mathscr{K}^0$ respects the G-grading. If $(\mathcal{L}_{V_1}, s^1, \mathrm{id})\in\mathcal{D}^0_q$ and $(\mathcal{L}_{V_2}, s^2, \mathrm{id})\in\mathcal{D}^0_r$, then

$$\begin{aligned}C_{\mathscr{K}^0(\mathcal{L}_{V_1}, s^1, \mathrm{id}), \mathscr{K}^0(\mathcal{L}_{V_2}, s^2, \mathrm{id})} &= C_{(V_1, q, \Phi^{s^1}_{V_2}), (V_2, r, \Phi^{s^2}_{V_2})} = \Phi^{s^1}_{V_1}(V_2) = s^1_I(V_2)\\ &= [C_{(\mathcal{L}_{V_1}, s^1, \mathrm{id}), (\mathcal{L}_{V_2}, s^2, \mathrm{id})}]_I = \mathscr{K}^0(C_{(\mathcal{L}_{V_1}, s^1, \mathrm{id}), (\mathcal{L}_{V_2}, s^2, \mathrm{id})}),\end{aligned}$$

showing that $\mathscr{K}^0$ is also braided.

We now define a functor $\mathscr{L}: Z_G(\mathcal{C})\to\mathcal{D}^0$ as follows. If $(V, q, \Phi_V)\in Z_G(\mathcal{C})$, then we set $\mathscr{L}(V, q, \Phi_V) = (\mathcal{L}_V, s^{\Phi_V}, \mathrm{id})$, where $s^{\Phi_V}_M(X) = \Phi_V(X)\otimes\mathrm{id}_M$. To check naturality of s^{Φ_V}, let $f\in\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $m\in\mathrm{Hom}_{\mathcal{C}}(M, N)$. Then

$$\begin{aligned}[F_q(f)\otimes H(m)]\circ s^{\Phi_V}_M(X) &= [F_q(f)\otimes\mathrm{id}_V\otimes m]\circ[\Phi_V(X)\otimes\mathrm{id}_M]\\ &= [\Phi_V(Y)\otimes\mathrm{id}_N]\circ[\mathrm{id}_V\otimes f\otimes m]\\ &= s^{\Phi_V}_N(Y)\circ H(f\otimes m),\end{aligned}$$

where in the second step we used naturality of Φ_V. For any $X, Y, M\in\mathcal{C}$ we also have

$$\begin{aligned}s^{\Phi_V}_M(X\otimes Y) &= \Phi_V(X\otimes Y)\otimes\mathrm{id}_M = [\mathrm{id}_{F_q(X)}\otimes\Phi_V(Y)\otimes\mathrm{id}_M]\circ[\Phi_V(X)\otimes\mathrm{id}_{Y\otimes M}]\\ &= [\mathrm{id}_{F_q(X)}\otimes s_M(Y)]\circ s^{\Phi_V}_{Y\otimes M}(X),\end{aligned}$$

and

$$s^{\Phi_V}_{M\otimes Y}(X) = \Phi_V(X)\otimes\mathrm{id}_{M\otimes Y} = s^{\Phi_V}_M(X)\otimes\mathrm{id}_Y,$$

so $(\mathcal{L}_V, s^{\Phi_V}, \mathrm{id}) \in \mathcal{D}_q^0$ and hence $\mathscr{L}$ is well-defined on the objects. If $f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V_1, q, \Phi_{V_1}), (V_2, q, \Phi_{V_2}))$, then we set $\mathscr{L}(f) = \sigma^f$ with $\sigma_M^f := f \otimes \mathrm{id}_M$. For any $X, M \in \mathcal{C}$ we then have

$$\begin{aligned}[\mathrm{id}_{F_q(X)} \otimes \sigma_M^f] \circ s_M^{\Phi_{V_1}}(X) &= [\mathrm{id}_{F_q(X)} \otimes f \otimes \mathrm{id}_M] \circ [\Phi_{V_1}(X) \otimes \mathrm{id}_M] = [\Phi_{V_2}(X) \otimes \mathrm{id}_M] \circ [f \otimes \mathrm{id}_{X \otimes M}] \\ &= s_M^{\Phi_{V_2}}(X) \circ [\sigma_I^f \otimes \mathrm{id}_{X \otimes M}]\end{aligned}$$

and $\sigma_{M \otimes X}^f = f \otimes \mathrm{id}_{M \otimes X} = \sigma_M^f \otimes \mathrm{id}_X$, which are precisely equations (4.5.8) and (4.5.9). We now claim that $\mathscr{K}^0 \circ \mathscr{L} = \mathrm{id}_{Z_G(\mathcal{C})}$ and $\mathscr{L} \circ \mathscr{K}_0 = \mathrm{id}_{\mathcal{D}^0}$, i.e. that the functors $\mathscr{K}^0$ and $\mathscr{L}$ are inverse to each other. If $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ then

$$\Phi_V^{s^{\Phi_V}}(X) = s_I^{\Phi_V}(X) = \Phi_V(X) \otimes \mathrm{id}_I = \Phi_V(X),$$

so $\Phi_V^{s^{\Phi_V}} = \Phi_V$ and hence

$$(\mathscr{K}^0 \circ \mathscr{L})(V, q, \Phi_V) = \mathscr{K}^0(\mathcal{L}_V, s^{\Phi_V}, \mathrm{id}) = (V, q, \Phi_V^{s^{\Phi_V}}) = (V, q, \Phi_V).$$

If $f \in \mathrm{Hom}(Z_G(\mathcal{C}))$, then $(\mathscr{K}^0 \circ \mathscr{L})(f) = \mathscr{K}^0(\sigma^f) = \sigma_I^f = f \otimes \mathrm{id}_I = f$. So indeed $\mathscr{K}^0 \circ \mathscr{L} = \mathrm{id}_{Z_G(\mathcal{C})}$. If $(\mathcal{L}_V, s, \mathrm{id}) \in \mathcal{D}_q^0$, then

$$s_M^{\Phi_V^s}(X) = \Phi_V^s(X) \otimes \mathrm{id}_M = s_I(X) \otimes \mathrm{id}_M = s_M(X).$$

so $s^{\Phi_V^s} = s$ and hence

$$(\mathscr{L} \circ \mathscr{K}^0)(\mathcal{L}_V, s, \mathrm{id}) = \mathscr{L}(V, q, \Phi_V^s) = (\mathcal{L}_V, s^{\Phi_V^s}, \mathrm{id}) = (\mathcal{L}_V, s, \mathrm{id}).$$

If $\sigma \in \mathrm{Hom}(\mathcal{D}^0)$, then $[(\mathscr{L} \circ \mathscr{K}^0)(\sigma)]_M = [\mathscr{L}(\sigma_I)]_M = \sigma_I \otimes \mathrm{id}_M = \sigma_M$, which means that $(\mathscr{L} \circ \mathscr{K}^0)(\sigma) = \sigma$. Thus we also conclude that $\mathscr{L} \circ \mathscr{K}^0 = \mathrm{id}_{\mathcal{D}^0}$. Because $\mathscr{L}$ is inverse to $\mathscr{K}^0$, it is easy to see that $\mathscr{L}$ is also a braided G-crossed functor. In particular, $\mathcal{D}^0$ and $Z_G(\mathcal{C})$ are equivalent as braided G-crossed categories. Now define the functor $\mathscr{K} : \mathcal{D} \to Z_G(\mathcal{C})$ by $\mathscr{K} := \mathscr{K}^0 \circ \mathcal{P}$. Because $\mathscr{K}^0$ and $\mathcal{P}$ are braided G-crossed functors, so is $\mathscr{K}$. Furthermore, we have

$$\begin{aligned}\mathscr{K} \circ \mathscr{L} &= \mathscr{K}^0 \circ \mathcal{P} \circ \mathscr{L} = \mathscr{K}^0 \circ \mathscr{L} = \mathrm{id}_{Z_G(\mathcal{C})} \\ \mathscr{L} \circ \mathscr{K} &= \mathscr{L} \circ \mathscr{K}^0 \circ \mathcal{P} = \mathcal{P}.\end{aligned}$$

Because of the natural isomorphism $\psi : \mathrm{id}_{\mathcal{D}} \to \mathcal{P}$ of braided G-crossed functors, we conclude that $\mathcal{D}$ and $Z_G(\mathcal{C})$ are equivalent braided G-crossed categories in the sense of Definition 2.8.20.
□

4.6 Retracts, direct sums and duals

We will now show that many nice properties of the category $\mathcal{C}$ can be carried over to $Z_G(\mathcal{C})$. Parts of the proofs of these statements are generalizations of the ones in [75], so we have not included all the details here.

Lemma 4.6.1 *Let $(\mathcal{C}, \otimes, I, G, F)$ be a strict tensor category with strict action of the group G.*
(1) If $\mathcal{C}$ has retracts, then so has $Z_G(\mathcal{C})$.
(2) If $\mathcal{C}$ has a left and right duality, then $Z_G(\mathcal{C})$ has a left duality. If in addition $\mathcal{C}$ admits a left G-duality, then so does $Z_G(\mathcal{C})$.
(3) If $\mathcal{C}$ is an Ab-category and has direct sums, then all full subcategories $Z_G(\mathcal{C})_q$ have direct sums.

(4) If $\mathcal{C}$ is $\mathbb{F}$-linear, then all full subcategories $Z_G(\mathcal{C})_q$ are $\mathbb{F}$-linear. If I is irreducible, then (I, e, Φ_I^0) is irreducible.

Proof. (1) Let $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ and let $p \in \mathrm{End}_{Z_G(\mathcal{C})}((V, q, \Phi_V))$ be an idempotent. By construction of $Z_G(\mathcal{C})$, p is also an idempotent in $\mathrm{End}_{\mathcal{C}}(V)$, and hence there exists an object $U \in \mathcal{C}$ together with morphisms $i \in \mathrm{Hom}_{\mathcal{C}}(U, V)$ and $r \in \mathrm{Hom}_{\mathcal{C}}(V, U)$ such that $r \circ i = \mathrm{id}_U$ and $i \circ r = p$. For each $X \in \mathcal{C}$ we define $\Phi_U(X) \in \mathrm{Hom}_{\mathcal{C}}(U \otimes X, F_q(X) \otimes U)$ by

$$\Phi_U(X) := [\mathrm{id}_{F_q(X)} \otimes r] \circ \Phi_V(X) \circ [i \otimes \mathrm{id}_X],$$

which defines a half q-braiding Φ_U for U by part (1) of Lemma 4.1.6, so $(U, q, \Phi_U) \in Z_G(\mathcal{C})$. An easy computation shows that $i \in \mathrm{Hom}_{Z_G(\mathcal{C})}((U, q, \Phi_U), (V, q, \Phi_V))$ and $r \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V, q, \Phi_V), (U, q, \Phi_U))$, from which it follows that (U, q, Φ_U) is a retract of (V, q, Φ_V).

(2) Let $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ and let $(V^\vee, b_V, d_V)$ be the left dual of V given by the left duality in the category $\mathcal{C}$. For each $X \in \mathcal{C}$ we define $(\Phi_V)^\vee \in \mathrm{Hom}_{\mathcal{C}}(\overline{V} \otimes X, F_{q^{-1}}(X) \otimes \overline{V})$ by

$$(\Phi_V)^\vee(X) := [d_V \otimes \mathrm{id}_{F_{q^{-1}}(X) \otimes V^\vee}] \circ [\mathrm{id}_{V^\vee} \otimes \Phi_V(F_{q^{-1}}(X))^{-1} \otimes \mathrm{id}_{V^\vee}] \circ [\mathrm{id}_{V^\vee \otimes X} \otimes b_V].$$

We have already seen in Lemma 4.1.8 that $(\Phi_V)^\vee$ is indeed a half q^{-1}-braiding (the fact that each $X \in \mathcal{C}$ has a right dual was used to prove that $\Phi_V(X)$ is invertible). To see that the morphisms b_V and d_V are in $Z_G(\mathcal{C})$, we compute[13]

$$\begin{aligned}
&(\Phi_V \otimes (\Phi_V)^\vee)(X) \circ [b_V \otimes \mathrm{id}_X] \\
&\quad = [\Phi_V(F_{q^{-1}}(X)) \otimes \mathrm{id}_{V^\vee}] \circ [\mathrm{id}_V \otimes d_V \otimes \mathrm{id}_{F_{q^{-1}}(X) \otimes V^\vee}] \\
&\qquad \circ [\mathrm{id}_{V \otimes V^\vee} \otimes \Phi_V(F_{q^{-1}}(X))^{-1} \otimes \mathrm{id}_{V^\vee}] \circ [\mathrm{id}_{V \otimes V^\vee \otimes X} \otimes b_V] \circ [b_V \otimes \mathrm{id}_X] \\
&\quad = [\Phi_V(F_{q^{-1}}(X)) \otimes \mathrm{id}_{V^\vee}] \circ [\mathrm{id}_V \otimes d_V \otimes \mathrm{id}_{F_{q^{-1}}(X) \otimes V^\vee}] \circ [b_V \otimes \mathrm{id}_{V \otimes F_{q^{-1}}(X) \otimes V^\vee}] \\
&\qquad \circ [\Phi_V(F_{q^{-1}}(X))^{-1} \otimes \mathrm{id}_{V^\vee}] \circ [\mathrm{id}_X \otimes b_V] \\
&\quad = [\Phi_V(F_{q^{-1}}(X)) \otimes \mathrm{id}_{V^\vee}] \circ [\Phi_V(F_{q^{-1}}(X))^{-1} \otimes \mathrm{id}_{V^\vee}] \circ [\mathrm{id}_X \otimes b_V] \\
&\quad = \mathrm{id}_X \otimes b_V = [\mathrm{id}_X \otimes b_V] \circ \Phi_I^0(X)
\end{aligned}$$

and

$$\begin{aligned}
&[\mathrm{id}_X \otimes d_V] \circ ((\Phi_V)^\vee \otimes \Phi_V)(X) \\
&\quad = [\mathrm{id}_X \otimes d_V] \circ [d_V \otimes \mathrm{id}_{X \otimes V^\vee \otimes V}] \circ [\mathrm{id}_{V^\vee} \otimes \Phi_V(X)^{-1} \otimes \mathrm{id}_{V^\vee \otimes V}] \\
&\qquad \circ [\mathrm{id}_{V^\vee \otimes F_q(X)} \otimes b_V \otimes \mathrm{id}_V] \circ [\mathrm{id}_{V^\vee} \otimes \Phi_V(X)] \\
&\quad = [d_V \otimes \mathrm{id}_X] \circ [\mathrm{id}_{V^\vee} \otimes \Phi_V(X)^{-1}] \circ [\mathrm{id}_{V^\vee \otimes F_q(X) \otimes V} \otimes d_V] \\
&\qquad \circ [\mathrm{id}_{V^\vee \otimes F_q(X)} \otimes b_V \otimes \mathrm{id}_V] \circ [\mathrm{id}_{V^\vee} \otimes \Phi_V(X)] \\
&\quad = [d_V \otimes \mathrm{id}_X] \circ [\mathrm{id}_{V^\vee} \otimes \Phi_V(X)^{-1}] \circ [\mathrm{id}_{V^\vee} \otimes \Phi_V(X)] \\
&\quad = d_V \otimes \mathrm{id}_X = \Phi_I^0(X) \circ [d_V \otimes \mathrm{id}_X].
\end{aligned}$$

It is now easy to see that if we define $b_{(V,q,\Phi_V)} := b_V$ and $d_{(V,q,\Phi_V)} := d_V$, then $(V^\vee, q^{-1}, (\Phi_V)^\vee)$ is a left dual for (V, q, Φ_V). In this way we obtain a left duality on $Z_G(\mathcal{C})$.

Now suppose, in addition, that the left duality on $\mathcal{C}$ is a left G-duality. Then for $q \in G$ and $(V, r, \Phi_V) \in Z_G(\mathcal{C})$ we have

$$(\mathcal{F}_q \Phi_V)^\vee(X)$$

[13] A similar calculation for right duals will not work. See also the remark after this lemma.

$$= [d_{F_q(V)} \otimes \mathrm{id}_{F_{qr^{-1}q^{-1}}(X)\otimes F_q(V)^\vee}] \circ [\mathrm{id}_{F_q(V)^\vee} \otimes \mathcal{F}_q\Phi_V(F_{qr^{-1}q^{-1}}(X))^{-1} \otimes \mathrm{id}_{F_q(V)^\vee}]$$
$$\circ [\mathrm{id}_{F_q(V)^\vee\otimes X} \otimes b_{F_q(V)}]$$
$$= F_q\{[d_V \otimes \mathrm{id}_{F_{r^{-1}}(F_{q^{-1}}(X))\otimes V^\vee}] \circ [\mathrm{id}_{V^\vee} \otimes \Phi_V(F_{r^{-1}}(F_{q^{-1}}(X)))^{-1} \otimes \mathrm{id}_{V^\vee}] \circ [\mathrm{id}_{V^\vee\otimes F_{q^{-1}}(X)} \otimes b_V]\}$$
$$= F_q((\Phi_V)^\vee(F_{q^{-1}}(X))) = \mathcal{F}_q(\Phi_V)^\vee(X),$$

so that

$$\begin{aligned}\mathcal{F}_q[(V,r,\Phi_V)^\vee] &= \mathcal{F}_q[(V^\vee,r^{-1},(\Phi_V)^\vee)] = (F_q(V^\vee),qr^{-1}q^{-1},\mathcal{F}_q(\Phi_V)^\vee)\\ &= (F_q(V)^\vee,qr^{-1}q^{-1},(\mathcal{F}_q\Phi_V)^\vee) = (F_q(V),qrq^{-1},\mathcal{F}_q\Phi_V)^\vee\\ &= \mathcal{F}_q[(V,r,\Phi_V)]^\vee.\end{aligned}$$

Furthermore, we also have

$$\mathcal{F}_q(b_{(V,r,\Phi_V)}) = F_q(b_V) = b_{F_q(V)} = b_{(F_q(V),qrq^{-1},\mathcal{F}_q\Phi_V)} = b_{\mathcal{F}_q[(V,r,\Phi_V)]}$$
$$\mathcal{F}_q(d_{(V,r,\Phi_V)}) = F_q(d_V) = d_{F_q(V)} = d_{(F_q(V),qrq^{-1},\mathcal{F}_q\Phi_V)} = d_{\mathcal{F}_q[(V,r,\Phi_V)]}.$$

(3) Let $(V,q,\Phi_V),(W,q,\Phi_W) \in Z_G(\mathcal{C})_q$. Since $\mathcal{C}$ has direct sums, there exists an object $Z \in \mathcal{C}$ with $Z \cong V \oplus W$, i.e. there exists an object $Z \in \mathcal{C}$ together with morphisms $f \in \mathrm{Hom}_\mathcal{C}(V,Z)$, $f' \in \mathrm{Hom}_\mathcal{C}(Z,V)$, $g \in \mathrm{Hom}_\mathcal{C}(W,Z)$ and $g' \in \mathrm{Hom}_\mathcal{C}(Z,W)$ such that $f' \circ f = \mathrm{id}_V$, $g' \circ g = \mathrm{id}_W$ and $f \circ f' + g \circ g' = \mathrm{id}_Z$. For each $X \in \mathcal{C}$ we define $\Phi_Z(X) \in \mathrm{Hom}_\mathcal{C}(Z \otimes X, F_q(X) \otimes Z)$ by

$$\Phi_Z(X) := [\mathrm{id}_{F_q(X)} \otimes f] \circ \Phi_V(X) \circ [f' \otimes \mathrm{id}_X] + [\mathrm{id}_{F_q(X)} \otimes g] \circ \Phi_W(X) \circ [g' \otimes \mathrm{id}_X].$$

We have already seen that Φ_Z defines a half q-braiding for Z, so $(Z,q,\Phi_Z) \in Z_G(\mathcal{C})$. It is easy to check that f, f', g and g' are in fact morphisms in $Z_G(\mathcal{C})$ and that they give (Z,q,Φ_Z) the structure of a direct sum of (V,q,Φ_V) and (W,q,Φ_W).

(4) Suppose that $\mathcal{C}$ is $\mathbb{F}$-linear and let $(V,q,\Phi_V),(W,q,\Phi_W) \in Z_G(\mathcal{C})$. Then a morphism $f \in \mathrm{Hom}_\mathcal{C}(V,W)$ is a morphism in $\mathrm{Hom}_{Z_G(\mathcal{C})}((V,q,\Phi_V),(W,q,\Phi_W))$ if and only if it satisfies

$$[\mathrm{id}_{F_q(X)} \otimes f] \circ \Phi_V(X) = \Phi_W(X) \circ [f \otimes \mathrm{id}_X]$$

for all $X \in \mathcal{C}$. Since this equation is linear in f, it is clear that $\mathrm{Hom}_{Z_G(\mathcal{C})}((V,q,\Phi_V),(W,q,\Phi_W))$ is an $\mathbb{F}$-linear subspace of $\mathrm{Hom}_\mathcal{C}(V,W)$. If $\mathrm{End}_\mathcal{C}(I) = \mathbb{F}\cdot\mathrm{id}_I$, then it is clear that $\mathrm{End}_{Z_G(\mathcal{C})}((I,e,\Phi_I^0)) = \mathbb{F}\cdot\mathrm{id}_{(I,e,\Phi_I^0)}$. □

Remark 4.6.2 That $Z_G(\mathcal{C})$ does not have a right duality in part (2) is due to the fact that we have chosen to define $Z_G(\mathcal{C})$ in terms of half q-braidings (of the first kind). If we had defined $Z_G(\mathcal{C})$ in terms of half q-braidings of the second kind, the conclusion in (2) would have been that $Z_G(\mathcal{C})$ only had a right duality. However, in both cases $Z_G(\mathcal{C})_e = Z(\mathcal{C})$ has a two-sided duality. This follows from the fact that $Z_G(\mathcal{C})_e = Z(\mathcal{C})$ is a braided tensor category and that any braided tensor category with either a left or right duality automatically has a two-sided duality, as we have seen at the beginning of Subsection 2.4.1. A similar statement does not hold for braided G-crossed categories, without any further assumptions[14].

Although $Z_G(\mathcal{C})$ only has a left duality if $\mathcal{C}$ has both a left and right duality, in the case that $\mathcal{C}$ is G-pivotal we will see in the following lemma that $Z_G(\mathcal{C})$ is G-pivotal as well (and thus has both a left and right duality). This is because the right duality morphisms b' and d' in a pivotal category can be expressed in terms of the left duality morphisms b and d, and for the latter it was possible to show that they are morphisms in $Z_G(\mathcal{C})$.

[14]If in addition the braided G-crossed category has a so-called G-twisting, i.e. if it is a ribbon G-category, then the statement is true. We will not need this.

Lemma 4.6.3 *Let $(\mathcal{C}, \otimes, I)$ be a strict tensor category with strict action F of the group G. If $\mathcal{C}$ is G-pivotal or G-spherical, then so is $Z_G(\mathcal{C})$.*

Proof. If $\mathcal{C}$ is G-pivotal, then for any $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ we define the object $\overline{(V, q, \Phi_V)} = (\overline{V}, q^{-1}, \overline{\Phi_V})$ with $\overline{\Phi_V}$ as in Lemma 4.1.9. This defines a left duality on $Z_G(\mathcal{C})$ by the previous lemma. Since for all $V \in \mathcal{C}$ we have that $b'_V = b_{\overline{V}}$ and $d'_V = d_{\overline{V}}$, it follows that b'_V and d'_V are morphisms in the category $Z_G(\mathcal{C})$ and hence that we obtain a two-sided duality on $Z_G(\mathcal{C})$ in this way. By Lemma 4.1.9 we have for any $(V, q, \Phi_V) \in Z_G(\mathcal{C})$

$$\overline{\overline{(V, q, \Phi_V)}} = \overline{(\overline{V}, q^{-1}, \overline{\Phi_V})} = (\overline{\overline{V}}, q, \overline{\overline{\Phi_V}}) = (V, q, \Phi_V),$$

and for any $(V, q, \Phi_V), (W, r, \Phi_W) \in Z_G(\mathcal{C})$ we have

$$\begin{aligned}\overline{(V, q, \Phi_V) \otimes (W, r, \Phi_W)} &= \overline{(V \otimes W, qr, \Phi_V \otimes \Phi_W)} = (\overline{V \otimes W}, r^{-1}q^{-1}, \overline{\Phi_V \otimes \Phi_W}) \\ &= (\overline{W} \otimes \overline{V}, r^{-1}q^{-1}, \overline{\Phi_W} \otimes \overline{\Phi_V}) = (\overline{W}, r^{-1}, \overline{\Phi_W}) \otimes (\overline{V}, q^{-1}, \overline{\Phi_V}) \\ &= \overline{(W, r, \Phi_W)} \otimes \overline{(V, q, \Phi_V)}.\end{aligned}$$

Also, $\overline{(I, e, \Phi_I^0)} = (\overline{I}, e, \overline{\Phi_I^0}) = (I, e, \Phi_I^0)$ by Lemma 4.1.9 again. The properties of being G-pivotal or G-spherical carry over directly to $Z_G(\mathcal{C})$ since the composition in $Z_G(\mathcal{C})$ is the one from $\mathcal{C}$.
□

4.7 Semisimplicity of $Z_G(\mathcal{C})$

In this section we will show that if $\mathcal{C}$ is a G-spherical fusion category, then $Z_G(\mathcal{C})$ is semisimple. We will not yet be able to prove that $Z_G(\mathcal{C})$ has finitely many isomorphism classes of irreducible objects. This will be done in Section 4.9. The following lemma is a generalization of Lemmas 3.10 and 3.12 in [75].

Lemma 4.7.1 *Let $\mathcal{C}$ be a G-spherical fusion category over a field $\mathbb{F}$ and assume that $\dim(\mathcal{C}) \neq 0$. We fix a complete set of representatives $\{X_i\}_{i \in \Gamma}$ of irreducible objects in $\mathcal{C}$, and for each pair of objects $(V, q, \Phi_V), (W, q, \Phi_W) \in Z_G(\mathcal{C})$ we define an $\mathbb{F}$-linear map $E_{(V,q,\Phi_V),(W,q,\Phi_W)} : \mathrm{Hom}_{\mathcal{C}}(V, W) \to \mathrm{Hom}_{\mathcal{C}}(V, W)$ by*

$$E_{(V,q,\Phi_V),(W,q,\Phi_W)}(f) := \dim(\mathcal{C})^{-1} \sum_{i \in \Gamma} d(X_i) \varepsilon_i(f),$$

where $\varepsilon_i(f) \in \mathrm{Hom}_{\mathcal{C}}(V, W)$ is given by

$$\begin{aligned}\varepsilon_i(f) := &[F_q(d'_{X_i}) \otimes \mathrm{id}_W] \circ [\mathrm{id}_{F_q(X_i)} \otimes \Phi_W(\overline{X_i})] \circ [\mathrm{id}_{F_q(X_i)} \otimes f \otimes \mathrm{id}_{\overline{X_i}}] \\ &\circ [\Phi_V(X_i) \otimes \mathrm{id}_{\overline{X_i}}] \circ [\mathrm{id}_V \otimes b_{X_i}].\end{aligned}$$

(1) For each $f \in \mathrm{Hom}_{\mathcal{C}}(V, W)$ we have

$$E_{(V,q,\Phi_V),(W,q,\Phi_W)}(f) \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V, q, \Phi_V), (W, q, \Phi_W)),$$

so $E_{(V,q,\Phi_V),(W,q,\Phi_W)} : \mathrm{Hom}_{\mathcal{C}}(V, W) \to \mathrm{Hom}_{Z_G(\mathcal{C})}((V, q, \Phi_V), (W, q, \Phi_W))$.

(2) For any three morphisms

$$\begin{aligned}&f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((U, q, \Phi_U), (V, q, \Phi_V)) \\ &g \in \mathrm{Hom}_{\mathcal{C}}(V, W) \\ &h \in \mathrm{Hom}_{Z_G(\mathcal{C})}((W, q, \Phi_W), (Z, q, \Phi_Z))\end{aligned}$$

we have that $E_{(U,q,\Phi_U),(Z,q,\Phi_Z)}(h \circ g \circ f) = h \circ E_{(V,q,\Phi_V),(W,q,\Phi_W)}(g) \circ f$.

(3) For every $(V,q,\Phi_V)\in Z_G(\mathcal{C})$ *we have the equality* $\mathrm{Tr}_V\circ E_{(V,q,\Phi_V),(V,q,\Phi_V)}=\mathrm{Tr}_V$ *of maps* $\mathrm{End}_{\mathcal{C}}(V)\to\mathrm{End}_{\mathcal{C}}(I)$.

Proof. (1) To prove that $E_{(V,q,\Phi_V),(W,q,\Phi_W)}(f)\in\mathrm{Hom}_{Z_G(\mathcal{C})}((V,q,\Phi_V),(W,q,\Phi_W))$, we must show that

$$[\mathrm{id}_{F_q(Z)}\otimes E_{(V,q,\Phi_V),(W,q,\Phi_W)}(f)]\circ\Phi_V(Z)=\Phi_W(Z)\circ[E_{(V,q,\Phi_V),(W,q,\Phi_W)}(f)\otimes\mathrm{id}_Z]$$

for all $Z\in\mathcal{C}$. Thus let $Z\in\mathcal{C}$. For every $i,j\in\Gamma$ we choose a basis $\{v_{i,j}^\alpha\}_{\alpha=1,\dots,N_{i,j}}$ for $\mathrm{Hom}_{\mathcal{C}}(X_j,Z\otimes X_i)$ together with a basis $\{w_{i,j}^\alpha\}_{\alpha=1,\dots,N_{i,j}}$ for $\mathrm{Hom}_{\mathcal{C}}(Z\otimes X_i,X_j)$ such that

$$w_{i,j}^\alpha\circ v_{i,k}^\beta=\delta_{j,k}\delta_{\alpha,\beta}\mathrm{id}_{X_j}$$
$$\sum_{j,\alpha}v_{i,j}^\alpha\circ w_{i,j}^\alpha=\mathrm{id}_{Z\otimes X_i}.$$

Note that $N_{i,j}=N_{Z,X_i}^{X_j}$, the multiplicity of X_j in $Z\otimes X_i$. Using these bases, we define

$$p_{i,j}^\alpha=[\mathrm{id}_{\overline{X_j}\otimes Z}\otimes d'_{X_i}]\circ[\mathrm{id}_{\overline{X_j}}\otimes v_{i,j}^\alpha\otimes\mathrm{id}_{\overline{X_i}}]\circ[b'_{X_j}\otimes\mathrm{id}_{\overline{X_i}}],$$
$$r_{i,j}^\alpha=[d_{X_j}\otimes\mathrm{id}_{\overline{X_i}}]\circ[\mathrm{id}_{\overline{X_j}}\otimes w_{i,j}^\alpha\otimes\mathrm{id}_{\overline{X_i}}]\circ[\mathrm{id}_{\overline{X_j}\otimes Z}\otimes b_{X_i}].$$

These form a basis for $\mathrm{Hom}_{\mathcal{C}}(\overline{X_i},\overline{X_j}\otimes Z)$ and $\mathrm{Hom}_{\mathcal{C}}(\overline{X_j}\otimes Z,\overline{X_i})$, respectively, and satisfy

$$\sum_{i,\alpha}d(X_i)d(X_j)^{-1}p_{i,j}^\alpha\circ r_{i,j}^\alpha=\mathrm{id}_{\overline{X_j}\otimes Z}.$$

Using these bases we have for any $(V,q,\Phi_V),(W,q,\Phi_W)\in Z_G(\mathcal{C})$ and $f\in\mathrm{Hom}_{\mathcal{C}}(V,W)$ that

$$\begin{aligned}
&\sum_i d(X_i)[\mathrm{id}_{F_q(Z)}\otimes\varepsilon_i(f)]\circ\Phi_V(Z)\\
&=\sum_{i,j,\alpha}d(X_i)[\mathrm{id}_{F_q(Z)}\otimes F_q(d'_{X_i})\otimes\mathrm{id}_W]\circ[F_q(v_{i,j}^\alpha)\otimes\Phi_W(\overline{X_i})]\circ[F_q(w_{i,j}^\alpha)\otimes f\otimes\mathrm{id}_{\overline{X_i}}]\\
&\qquad\circ[\mathrm{id}_{F_q(Z)}\otimes\Phi_V(X_i)\otimes\mathrm{id}_{\overline{X_i}}]\circ[\Phi_V(Z)\otimes b_{X_i}]\\
&=\sum_{i,j,\alpha}d(X_i)[\mathrm{id}_{F_q(Z)}\otimes F_q(d'_{X_i})\otimes\mathrm{id}_W]\circ[F_q(v_{i,j}^\alpha)\otimes\Phi_W(\overline{X_i})]\circ[\mathrm{id}_{F_q(X_j)}\otimes f\otimes\mathrm{id}_{\overline{X_i}}]\\
&\qquad\circ[\Phi_V(X_j)\otimes\mathrm{id}_{\overline{X_i}}]\circ[\mathrm{id}_V\otimes w_{i,j}^\alpha\otimes\mathrm{id}_{\overline{X_i}}]\circ[\mathrm{id}_{V\otimes Z}\otimes b_{X_i}]\\
&=\sum_{i,j,\alpha}d(X_i)[F_q(d'_{X_j})\otimes\mathrm{id}_{F_q(Z)\otimes W}]\circ[\mathrm{id}_{F_q(X_j)}\otimes F_q(p_{i,j}^\alpha)\otimes\mathrm{id}_W]\circ[\mathrm{id}_{F_q(X_j)}\otimes\Phi_W(\overline{X_i})]\\
&\qquad\circ[\mathrm{id}_{F_q(X_j)}\otimes f\otimes\mathrm{id}_{\overline{X_i}}]\circ[\Phi_V(X_j)\otimes\mathrm{id}_{\overline{X_i}}]\circ[\mathrm{id}_{V\otimes X_j}\otimes r_{i,j}^\alpha]\circ[\mathrm{id}_V\otimes b_{X_j}\otimes\mathrm{id}_Z]\\
&=\sum_{i,j,\alpha}d(X_i)[F_q(d'_{X_j})\otimes\Phi_W(Z)]\circ[\mathrm{id}_{F_q(X_j)}\otimes\Phi_W(\overline{X_j})\otimes\mathrm{id}_Z]\circ[\mathrm{id}_{F_q(X_j)\otimes W}\otimes(p_{i,j}^\alpha\circ r_{i,j}^\alpha)]\\
&\qquad\circ[\mathrm{id}_{F_q(X_j)}\otimes f\otimes\mathrm{id}_{\overline{X_j}\otimes Z}]\circ[\Phi_V(X_j)\otimes\mathrm{id}_{\overline{X_j}\otimes Z}]\circ[\mathrm{id}_V\otimes b_{X_j}\otimes\mathrm{id}_Z]\\
&=\sum_{j,\alpha}d(X_j)[F_q(d'_{X_j})\otimes\Phi_W(Z)]\circ[\mathrm{id}_{F_q(X_j)}\otimes\Phi_W(\overline{X_j})\otimes\mathrm{id}_Z]\\
&\qquad\circ\left[\mathrm{id}_{F_q(X_j)\otimes W}\otimes(\sum_i d(X_i)d(X_j)^{-1}p_{i,j}^\alpha\circ r_{i,j}^\alpha)\right]\circ[\mathrm{id}_{F_q(X_j)}\otimes f\otimes\mathrm{id}_{\overline{X_j}\otimes Z}]\\
&\qquad\circ[\Phi_V(X_j)\otimes\mathrm{id}_{\overline{X_j}\otimes Z}]\circ[\mathrm{id}_V\otimes b_{X_j}\otimes\mathrm{id}_Z]
\end{aligned}$$

$$= \sum_j d(X_j)\Phi_W(Z) \circ [\varepsilon_j(f) \otimes \mathrm{id}_Z].$$

Multiplying both sides by $\dim(\mathcal{C})^{-1}$ gives the desired result.

(2) This follows directly from the fact that f and h are morphisms in $Z_G(\mathcal{C})$.

(3) For every $i, j \in \Gamma$ we choose a basis $\{v_{i,j}^\alpha\}_{\alpha=1,\dots,N_{i,j}}$ for $\mathrm{Hom}_{\mathcal{C}}(X_j, \overline{V} \otimes F_q(X_i))$ together with a basis $\{w_{i,j}^\alpha\}_{\alpha=1,\dots,N_{i,j}}$ for $\mathrm{Hom}_{\mathcal{C}}(\overline{V} \otimes F_q(X_i), X_j)$ such that

$$w_{i,j}^\alpha \circ v_{i,k}^\beta = \delta_{j,k}\delta_{\alpha,\beta}\mathrm{id}_{X_j},$$
$$\sum_{j,\alpha} v_{i,j}^\alpha \circ w_{i,j}^\alpha = \mathrm{id}_{\overline{V}\otimes F_q(X_i)}.$$

Note that $N_{i,j} = N_{\overline{V},F_q(X_i)}^{X_j}$, the multiplicity of X_j in $\overline{V} \otimes F_q(X_i)$. Using these bases, we define

$$\begin{aligned} p_{i,j}^\alpha &= [d_V \otimes \mathrm{id}_{X_i\otimes\overline{V}}] \circ [\mathrm{id}_{\overline{V}} \otimes F_q(d'_{X_i}) \otimes \mathrm{id}_{V\otimes X_i\otimes\overline{V}}] \circ [v_{i,j}^\alpha \otimes \Phi_V(\overline{X_i}) \otimes \mathrm{id}_{X_i\otimes\overline{V}}] \\ &\quad \circ [\mathrm{id}_{X_j\otimes V} \otimes b'_{X_i} \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_{X_j} \otimes b_V] \\ r_{i,j}^\alpha &= [\mathrm{id}_{X_j} \otimes d'_V] \circ [\mathrm{id}_{X_j\otimes V} \otimes d_{X_i} \otimes \mathrm{id}_{\overline{V}}] \circ [w_{i,j}^\alpha \otimes \Phi_V(\overline{X_i})^{-1} \otimes \mathrm{id}_{X_i\otimes\overline{V}}] \\ &\quad \circ [\mathrm{id}_{\overline{V}} \otimes F_q(b_{X_i}) \otimes \mathrm{id}_{V\otimes X_i\otimes\overline{V}}] \circ [b'_V \otimes \mathrm{id}_{X_i\otimes\overline{V}}]. \end{aligned}$$

These form a basis for $\mathrm{Hom}_{\mathcal{C}}(X_j, X_i \otimes \overline{V})$ and $\mathrm{Hom}_{\mathcal{C}}(X_i \otimes \overline{V}, X_j)$ and satisfy

$$\sum_{j,\alpha} p_{i,j}^\alpha \circ r_{i,j}^\alpha = \mathrm{id}_{X_i\otimes\overline{V}}.$$

Using these bases, we now compute for $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ and $f \in \mathrm{End}_{\mathcal{C}}(V)$

$$\begin{aligned} &\sum_i d(X_i)\mathrm{Tr}_V(\varepsilon_i(f)) \\ &= \sum_i d(X_i) \cdot d_V \circ [\mathrm{id}_{\overline{V}} \otimes F_q(d'_{X_i}) \otimes \mathrm{id}_V] \circ [\mathrm{id}_{\overline{V}\otimes F_q(X_i)} \otimes \Phi_V(\overline{X_i})] \circ [\mathrm{id}_{\overline{V}\otimes F_q(X_i)} \otimes f \otimes \mathrm{id}_{\overline{X_i}}] \\ &\qquad \circ [\mathrm{id}_{\overline{V}} \otimes \Phi_V(X_i) \otimes \mathrm{id}_{\overline{X_i}}] \circ [b'_V \otimes b_{X_i}] \\ &= \sum_i d(X_i) \cdot d_V \circ [\mathrm{id}_{\overline{V}} \otimes F_q(d'_{X_i}) \otimes \mathrm{id}_V] \circ [\mathrm{id}_{\overline{V}\otimes F_q(X_i)} \otimes \Phi_V(\overline{X_i})] \circ [\mathrm{id}_{\overline{V}\otimes F_q(X_i)} \otimes f \otimes \mathrm{id}_{\overline{X_i}}] \\ &\qquad \circ [\mathrm{id}_{\overline{V}\otimes F_q(X_i)} \otimes \Phi_V(\overline{X_i})^{-1}] \circ [\mathrm{id}_{\overline{V}} \otimes F_q(b_{X_i}) \otimes \mathrm{id}_V] \circ b'_V \\ &= \sum_{i,j,\alpha} d(X_i) \cdot d_V \circ [\mathrm{id}_{\overline{V}} \otimes F_q(d'_{X_i}) \otimes \mathrm{id}_V] \circ [v_{i,j}^\alpha \otimes \Phi_V(\overline{X_i})] \circ [w_{i,j}^\alpha \otimes f \otimes \mathrm{id}_{\overline{X_i}}] \\ &\qquad \circ [\mathrm{id}_{\overline{V}\otimes F_q(X_i)} \otimes \Phi_V(\overline{X_i})^{-1}] \circ [\mathrm{id}_{\overline{V}} \otimes F_q(b_{X_i}) \otimes \mathrm{id}_V] \circ b'_V \\ &= \sum_{i,j,\alpha} d(X_i) \cdot d'_{X_i} \circ [\mathrm{id}_{X_i} \otimes d_V \otimes \mathrm{id}_{\overline{X_i}}] \circ [p_{i,j}^\alpha \otimes \mathrm{id}_{V\otimes\overline{X_i}}] \circ [\mathrm{id}_{X_j} \otimes f \otimes \mathrm{id}_{\overline{X_i}}] \\ &\qquad \circ [r_{i,j}^\alpha \otimes \mathrm{id}_{V\otimes\overline{X_i}}] \circ [\mathrm{id}_{X_i} \otimes b'_V \otimes \mathrm{id}_{\overline{X_i}}] \circ b_{X_i} \\ &= \sum_i d(X_i) \cdot d'_{X_i} \circ [\mathrm{id}_{X_i} \otimes d_V \otimes \mathrm{id}_{\overline{X_i}}] \circ [\mathrm{id}_{X_i\otimes\overline{V}} \otimes f \otimes \mathrm{id}_{\overline{X_i}}] \circ [\mathrm{id}_{X_i} \otimes b'_V \otimes \mathrm{id}_{\overline{X_i}}] \circ b_{X_i} \\ &= \sum_i d(X_i)^2\mathrm{Tr}_V(f) = \dim(\mathcal{C})\mathrm{Tr}_V(f). \end{aligned}$$

Multiplying both sides by $\dim(\mathcal{C})^{-1}$ gives the desired result.

□

Theorem 4.7.2 *Let $\mathcal{C}$ be a G-spherical fusion category over an algebraically closed field $\mathbb{F}$. Then $Z_G(\mathcal{C})$ is G-spherical and semisimple.*

Proof. The proof goes analogously to the proof of Theorem 3.16 of [75]. We have already shown that if $\mathcal{C}$ is G-spherical, then so is $Z_G(\mathcal{C})$. Since $\mathcal{C}$ is semisimple, it has direct sums, subobjects and an irreducible unit object. As seen before, these properties carry over to $Z_G(\mathcal{C})$.

Let $(V, q, \Phi_V) \in Z_G(\mathcal{C})$. By semisimplicity of $\mathcal{C}$, $\mathrm{End}_{\mathcal{C}}(V)$ is a finite dimensional multi matrix algebra. The trace $\mathrm{Tr}_V : \mathrm{End}_{\mathcal{C}}(V) \to \mathrm{End}_{\mathcal{C}}(V)$ is non-degenerate and we have a conditional expectation $E_{(V,q,\Phi_V),(V,q,\Phi_V)} : \mathrm{End}_{\mathcal{C}}(V) \to \mathrm{End}_{Z_G(\mathcal{C})}((V, q, \Phi_V))$ that preserves the trace. By Lemma 3.14 in [75] this implies that $\mathrm{End}_{Z_G(\mathcal{C})}((V, q, \Phi_V))$ is semisimple, and hence a multi matrix algebra, because $\mathbb{F}$ is algebraically closed. In turn, this implies that (V, q, Φ_V) is a finite direct sum of irreducible objects.
□

4.8 An example

In this section we will consider the construction of $Z_G(\mathcal{C})$ for the case where $\mathcal{C}$ has some special properties, and we will see that in this case the G-spectrum $\partial(Z_G(\mathcal{C}))$ equals G, i.e. $Z_G(\mathcal{C})_q$ is non-empty for each $q \in G$. We begin by restating the definition of a half q-braiding for an object V in case $V = I$.

Let G be a group and let $(\mathcal{C}, \otimes, I)$ be a strict tensor category with strict G-action F. Note that a half q-braiding for I is a family $\{\Phi_I(X) : X \to F_q(X)\}_{X\in\mathcal{C}}$ of isomorphisms in $\mathcal{C}$ such that for all $X, Y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the square

$$\begin{array}{ccc} X & \xrightarrow{\Phi_I(X)} & F_q(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F_q(f)} \\ Y & \xrightarrow{\Phi_I(Y)} & F_q(Y) \end{array}$$

commutes, satisfying the additional property that for all $X, Y \in \mathcal{C}$ we have

$$\Phi_I(X \otimes Y) = [\mathrm{id}_{F_q(X)} \otimes \Phi_I(Y)] \circ [\Phi_I(X) \otimes \mathrm{id}_Y] = \Phi_I(X) \otimes \Phi_I(Y)$$

by the interchange law. Hence a half q-braiding Φ_I for I is a natural tensor isomorphism $\Phi_I : \mathrm{id}_{\mathcal{C}} \to F_q$. In particular, a necessary condition for the existence of a half q-braiding for I is that $F_q(V) \cong V$ for all $V \in \mathcal{C}$.

In the next lemma we will demonstrate that in a certain situation we can construct half q-braidings for the unit object I. This lemma can be considered as a warm-up for the theorem that follows it.

Lemma 4.8.1 *Let $\mathcal{C}$ be a strict tensor category with strict G-action F and suppose that $\mathcal{C}$ has only one isomorphism class of objects, i.e. each object is isomorphic to I. If $q \in G$ and if each morphism in $\mathrm{End}_{\mathcal{C}}(I)$ is invariant under F_q, then there exists a half q-braiding for I.*

Proof. For each $V \in \mathcal{C}$ we can choose an isomorphism $\psi_V : I \to V$, so that we obtain a collection $\{\psi_V : I \to V\}_{V\in\mathcal{C}}$ of isomorphisms. Note that we have $F_q(\psi_V) : I \to F_q(V)$. For each $X \in \mathcal{C}$ we now define $\Phi_I(X) : X \to F_q(X)$ by

$$\Phi_I(X) : X \xrightarrow{\psi_X^{-1}} I \xrightarrow{F_q(\psi_X)} F_q(X),$$

i.e. $\Phi_I(X) = F_q(\psi_X) \circ \psi_X^{-1}$. To prove naturality, let $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Then by F_q-invariance of morphisms in $\mathrm{End}_{\mathcal{C}}(I)$ we have

$$F_q(\psi_Y)^{-1} \circ F_q(f) \circ F_q(\psi_X) = F_q(\psi_Y^{-1} \circ f \circ \psi_X) = \psi_Y^{-1} \circ f \circ \psi_X,$$

which can be rewritten as $F_q(f) \circ F_q(\psi_X) \circ \psi_X^{-1} = F_q(\psi_Y) \circ \psi_Y^{-1} \circ f$, i.e.

$$F_q(f) \circ \Phi_I(X) = \Phi_I(Y) \circ f,$$

showing that Φ_I is natural. Now let $X, Y \in \mathcal{C}$. Then, again by F_q-invariance of the morphisms in $\mathrm{End}_{\mathcal{C}}(I)$,

$$F_q(\psi_X \otimes \psi_Y)^{-1} \circ F_q(\psi_{X\otimes Y}) = F_q[(\psi_X \otimes \psi_Y)^{-1} \circ \psi_{X\otimes Y}] = (\psi_X \otimes \psi_Y)^{-1} \circ \psi_{X\otimes Y},$$

which can be used to conclude that

$$\Phi_I(X \otimes Y) = F_q(\psi_{X\otimes Y}) \circ \psi_{X\otimes Y}^{-1} = F_q(\psi_X \otimes \psi_Y) \circ (\psi_X \otimes \psi_Y)^{-1} = \Phi_I(X) \otimes \Phi_I(Y).$$

Thus Φ_I is indeed a half q-braiding for I.
□

The property that $\mathrm{End}_{\mathcal{C}}(I)$ is F_q-invariant is automatically satisfied for all $q \in G$ if $\mathcal{C}$ is $\mathbb{F}$-linear and I is irreducible. This observation was the main motivation for the following theorem, which has some relevance for holomorphic chiral CFTs.

Theorem 4.8.2 *Let $\mathcal{C}$ be an $\mathbb{F}$-linear fusion category with strict action F of the group G and suppose that every irreducible object in $\mathcal{C}$ is isomorphic to I.*

(1) For each $q \in G$ and $V \in \mathcal{C}$ there exists precisely one half q-braiding Φ_V^q for V and we have

$$\mathrm{Hom}_{Z_G(\mathcal{C})}((V, q, \Phi_V^q), (W, q, \Phi_W^q)) = \mathrm{Hom}_{\mathcal{C}}(V, W). \tag{4.8.1}$$

Consequently, for each $q \in G$ there exists precisely one equivalence class of irreducible objects in $Z_G(\mathcal{C})_q$. Furthermore, $Z_G(\mathcal{C})$ is a braided G-crossed extension of $\mathcal{C}$.

(2) The full subcategory $Z_G(\mathcal{C})_{\mathrm{irr}}$ of $Z_G(\mathcal{C})$ determined by all irreducible objects is a braided G-crossed subcategory of $Z_G(\mathcal{C})$.

(3) $Z_G(\mathcal{C})_{\mathrm{irr}}$ has a skeletal braided G-crossed subcategory[15] *$\mathcal{D}$ with $\mathcal{D} \cong Z_G(\mathcal{C})_{\mathrm{irr}}$ as a braided G-crossed categories.*

Proof. (1) If $X \in \mathcal{C}$, then $X \cong I^{\oplus m_X}$ for some $m_X \in \mathbb{Z}_{>0}$ and we can choose $u_X^1, \ldots, u_X^{m_X} \in \mathrm{Hom}_{\mathcal{C}}(I, X)$ and $v_X^1, \ldots, v_X^{m_X} \in \mathrm{Hom}_{\mathcal{C}}(X, I)$ such that $v_X^i \circ u_X^j = \delta_{ij}\mathrm{id}_I$ and $\sum_{i=1}^{m_X} u_X^i \circ v_X^i = \mathrm{id}_X$. We will assume from now on that for each $X \in \mathcal{C}$ we have chosen such u_X^i and v_X^i.

We will first demonstrate that if $V \in \mathcal{C}$ and $q \in G$, then any half q-braiding Φ_V for V is uniquely determined. If $X \in \mathcal{C}$ and if Φ_V is a half q-braiding for V, then naturality of Φ_V implies that

$$\begin{aligned}\Phi_V(X) &= \sum_{i=1}^{m_X} \Phi_V(X) \circ [\mathrm{id}_V \otimes (u_X^i \circ v_X^i)] = \sum_{i=1}^{m_X} [F_q(u_X^i) \otimes \mathrm{id}_V] \circ \Phi_V(I) \circ [\mathrm{id}_V \otimes v_X^i] \\ &= \sum_{i=1}^{m_X} [F_q(u_X^i) \otimes \mathrm{id}_V] \circ [\mathrm{id}_V \otimes v_X^i],\end{aligned} \tag{4.8.2}$$

which shows that $\Phi_V(X)$ is uniquely determined for any $X \in \mathcal{C}$, and hence that Φ_V is uniquely determined, provided it exists.

[15] Any skeletal subcategory of a braided strict G-crossed category having precisely one equivalence class for each degree (like $Z_G(\mathcal{C})_{\mathrm{irr}}$ in the present case) can be equipped with the structure of a (not necessarily strict) braided G-crossed category. The notable fact in the present situation is that the structure on $\mathcal{D}$ is obtained by simply restricting all the structures of $Z_G(\mathcal{C})_{\mathrm{irr}}$, implying in particular that $\mathcal{D}$ is strict.

Using the expression (4.8.2) with $V = I$ we will now show that for each $q \in G$ there exists a half q-braiding for I. Let $q \in G$. For each $X \in \mathcal{C}$ we thus define $\Phi_I^q(X) \in \mathrm{Hom}_{\mathcal{C}}(X, F_q(X))$ by

$$\Phi_I^q(X) = \sum_{i=1}^{m_X} F_q(u_X^i) \circ v_X^i, \tag{4.8.3}$$

which is invertible with inverse $\Phi_I^q(X)^{-1} = \sum_{i=1}^{m_X} u_X^i \circ F_q(v_X^i)$, as can easily be checked. To prove naturality of Φ_I^q, let $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. For $i \in \{1, \dots, m_Y\}$ and $j \in \{1, \dots, m_X\}$ we have

$$F_q(v_Y^i) \circ F_q(f) \circ F_q(u_X^j) = F_q(v_Y^i \circ f \circ u_X^j) = v_Y^i \circ f \circ u_X^j,$$

where we have used that $v_Y^i \circ f \circ u_X^j \in \mathrm{End}_{\mathcal{C}}(I) = \mathbb{F} \cdot \mathrm{id}_I$ and that $F_q(\mathrm{id}_I) = \mathrm{id}_I$. From this expression it follows that

$$\sum_{i=1}^{m_Y} F_q(u_Y^i) \circ F_q(v_Y^i) \circ F_q(f) \circ F_q(u_X^j) = \sum_{i=1}^{m_Y} F_q(u_Y^i) \circ v_Y^i \circ f \circ u_X^j,$$

which can be simplified as $F_q(f) \circ F_q(u_X^j) = \Phi_I^q(Y) \circ f \circ u_X^j$, which in turn implies that

$$\sum_{i=1}^{m_X} F_q(f) \circ F_q(u_X^j) \circ v_X^j = \sum_{i=1}^{m_X} \Phi_I^q(Y) \circ f \circ u_X^j \circ v_X^j,$$

i.e. $F_q(f) \circ \Phi_I^q(X) = \Phi_I^q(Y) \circ f$. This proves naturality of Φ_I^q. Now let $X, Y \in \mathcal{C}$. Then for any $i \in \{1. \dots, m_X\}$, $j \in \{1, \dots, m_Y\}$ and $k \in \{1, \dots, m_{X \otimes Y}\}$ we have

$$F_q(v_X^i \otimes v_Y^j) \circ F_q(u_{X\otimes Y}^k) = F_q[(v_X^i \otimes v_Y^j) \circ u_{X\otimes Y}^k] = (v_X^i \otimes v_Y^j) \circ u_{X\otimes Y}^k,$$

where again we have used that $\mathrm{End}_{\mathcal{C}}(I) = \mathbb{F} \cdot \mathrm{id}_I$. Using this expression we find that

$$\sum_{i=1}^{m_X} \sum_{j=1}^{m_Y} F_q(u_X^i \otimes u_Y^j) \circ F_q(v_X^i \otimes v_Y^j) \circ F_q(u_{X\otimes Y}^k) = \sum_{i=1}^{m_X} \sum_{j=1}^{m_Y} F_q(u_X^i \otimes u_Y^j) \circ (v_X^i \otimes v_Y^j) \circ u_{X\otimes Y}^k,$$

which we can simplify as

$$F_q(u_{X\otimes Y}^k) = [\Phi_I^q(X) \otimes \Phi_I^q(Y)] \circ u_{X\otimes Y}^k.$$

Finally, we can use this to get

$$\sum_{k=1}^{m_{X\otimes Y}} F_q(u_{X\otimes Y}^k) \circ v_{X\otimes Y}^k = \sum_{k=1}^{m_{X\otimes Y}} [\Phi_I^q(X) \otimes \Phi_I^q(Y)] \circ u_{X\otimes Y}^k \circ v_{X\otimes Y}^k,$$

i.e. $\Phi_I^q(X \otimes Y) = \Phi_I^q(X) \otimes \Phi_I^q(Y)$. This finishes the proof that Φ_I^q is a half q-braiding for I. Now let $V \in \mathcal{C}$. Then it follows from Lemma 4.1.6 that

$$\Phi_V^q(X) := \sum_{i=1}^{m_V} [\mathrm{id}_{F_q(X)} \otimes u_V^i] \circ \Phi_I(X) \circ [v_V^i \otimes \mathrm{id}_X]$$

defines a half q-braiding for V that clearly coincides with (4.8.3) in case $V = I$. In fact, it follows from uniqueness that it must coincide with (4.8.2).

Let $q \in G$. It follows from $\mathrm{End}_{\mathcal{C}}(I) = \mathbb{F} \cdot \mathrm{id}_I$ that if $f \in \mathrm{End}_{\mathcal{C}}(I)$, then we also have $f \in \mathrm{End}_{Z_G(\mathcal{C})}((I, q, \Phi_I^q))$. So (4.8.1) is satisfied for each $q \in G$ in case $V = W = I$. Now let $f \in \mathrm{Hom}_{\mathcal{C}}(V, W)$. Then

$$[\mathrm{id}_{F_q(X)} \otimes f] \circ \Phi_V^q(X) = \sum_i [\mathrm{id}_{F_q(X)} \otimes (f \circ u_V^i)] \circ \Phi_I(X) \circ [v_V^i \otimes \mathrm{id}_X]$$

$$= \sum_{i,j} [\mathrm{id}_{F_q(X)} \otimes (u^j_W \circ \underbrace{v^j_W \circ f \circ u^i_V}_{\in \mathrm{End}_{\mathcal{C}}(I)})] \circ \Phi_I(X) \circ [v^i_V \otimes \mathrm{id}_X]$$
$$= \sum_{i,j} [\mathrm{id}_{F_q(X)} \otimes u^j_W] \circ \Phi_I(X) \circ [(v^j_W \circ f \circ u^i_V \circ v^i_V) \otimes \mathrm{id}_X]$$
$$= \sum_{i} [\mathrm{id}_{F_q(X)} \otimes u^i_W] \circ \Phi_I(X) \circ [(v^i_W \circ f) \otimes \mathrm{id}_X]$$
$$= \Phi^q_W(X) \circ [f \otimes \mathrm{id}_X],$$

showing that $f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V, q, \Phi^q_V), (W, q, \Phi^q_W))$, and hence that (4.8.1) holds. This immediately implies that an object (V, q, Φ^q_V) of $Z_G(\mathcal{C})$ is irreducible if and only if $V \in \mathcal{C}$ is irreducible, i.e. if and only if $V \cong I$.

(2) It is clear that the unit object of $Z_G(\mathcal{C})$ is in $Z_G(\mathcal{C})_{\mathrm{irr}}$. Let (V, q, Φ^q_V) and (W, r, Φ^r_W) be irreducible. We have $(V, q, \Phi^q_V) \otimes (W, r, \Phi^r_W) = (V \otimes W, qr, \Phi^q_V \otimes \Phi^r_W) = (V \otimes W, qr, \Phi^{qr}_{V\otimes W})$, where the last step follows from uniqueness. By irreducibility we have that $V \cong I \cong W$ and hence $V \otimes W \cong I \otimes I = I$. Thus $V \otimes W$ is irreducible in $\mathcal{C}$, so $(V \otimes W, qr, \Phi^{qr}_{V\otimes W})$ is irreducible in $Z_G(\mathcal{C})$. Since F_q is an equivalence for each $q \in G$, it follows that each F_q maps irreducible objects to irreducible objects.

(3) Now let $\mathcal{D}$ be the full subcategory of $Z_G(\mathcal{C})_{\mathrm{irr}}$ determined by the set of objects $\{(I, q, \Phi_I) : q \in G\}$, which clearly contains the unit object of $Z_G(\mathcal{C})$. It follows directly from $I \otimes I = I$ that $\mathcal{D}$ is a full tensor subcategory (that is strict) and it follows from $F_q(I) = I$ for all $q \in G$ that $\mathcal{D}$ is a G-category (of course we are using the uniqueness of the half braidings here to conclude that the tensor product of the half braidings is again what we want it to be). Our results above also show that for $q \in G$ the G-category contains precisely one object and that $\mathcal{D}$ is skeletal in $Z_G(\mathcal{C})_{\mathrm{irr}}$. It is clear that the inclusion functor $\mathcal{D} \to Z_G(\mathcal{C})_{\mathrm{irr}}$ is a fully faithful functor of braided G-crossed categories that is essentially surjective.
□

Remark 4.8.3 Note that the existence of such a $\mathcal{D}$ characterizes $Z_G(\mathcal{C})_{\mathrm{irr}}$ up to equivalence in the following way. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are categories satisfying the same assumptions as in the theorem, then there is an obvious functor $\mathcal{D}_1 \to \mathcal{D}_2$ between the corresponding skeletal subcategories of the $Z_G(\mathcal{C}_i)_{\mathrm{irr}}$ (namely, the unique object in $\mathcal{D}_1$ with degree q is mapped to the unique object in $\mathcal{D}_2$ with degree q; on the morphisms the functor is uniquely determined by noticing that the $\mathcal{D}_i$ are discrete). This functor is an equivalence of braided G-crossed categories and hence $Z_G(\mathcal{C}_1)_{\mathrm{irr}} \cong \mathcal{D}_1 \cong \mathcal{D}_2 \cong Z_G(\mathcal{C}_2)_{\mathrm{irr}}$ as braided G-crossed categories.

The statement in the theorem is in particular true if $\mathcal{C}$ is a TC^* with strict action F of the group G satisfying the requirement that every irreducible object of $\mathcal{C}$ be unitarily equivalent to I. The proof is almost the same as the proof of the theorem above, except that now the u^i_X are isometries and the v^i_X are replaced with $(u^i_X)^*$. Recall that we have seen examples of such categories before: if $\mathcal{A}$ is a holomorphic chiral CFT, then the category $\mathrm{Loc}_f(\mathcal{A}_\zeta)$ is precisely such a category (where $\zeta \in S^1$ is chosen to obtain a QFT $\mathcal{A}_\zeta$ on $\mathbb{R}$). Hence, if $\mathcal{A}$ is a holomorphic chiral CFT then $Z_G(\mathrm{Loc}_f(\mathcal{A}_\zeta))$ is a braided G-crossed extension of $\mathrm{Loc}_f(\mathcal{A}_\zeta)$ and for each $q \in G$ there exists precisely one equivalence class of irreducible objects in $Z_G(\mathrm{Loc}_f(\mathcal{A}_\zeta))_q$. Also, there exists a skeletal subcategory of $Z_G(\mathrm{Loc}_f(\mathcal{A}_\zeta))$ that is a braided strict G-crossed category. We have seen that $G - \mathrm{Loc}^{L/R}_f(\mathcal{A}_\zeta)$ also is a braided G-crossed extension of $\mathrm{Loc}_f(\mathcal{A}_\zeta)$ that has precisely one equivalence class of irreducible objects for each degree. It thus follows from Remark 4.8.3 above that if $G - \mathrm{Loc}^{L/R}_f(\mathcal{A}_\zeta)_{\mathrm{irr}}$ contains a skeletal braided (strict) G-crossed subcategory, then $G - \mathrm{Loc}^{L/R}_f(\mathcal{A}_\zeta)_{\mathrm{irr}} \simeq Z_G(\mathrm{Loc}_f(\mathcal{A}_\zeta))_{\mathrm{irr}}$ and hence $G - \mathrm{Loc}^{L/R}_f(\mathcal{A}_\zeta) \simeq Z_G(\mathrm{Loc}_f(\mathcal{A}_\zeta))$.

4.9 The case when $\mathcal{C}$ is G-spherical fusion

In the preceding sections we have constructed the G-crossed Drinfeld center $Z_G(\mathcal{C})$ of a G-category $\mathcal{C}$ and we have shown several of its nice properties. In Section 4.8 we also considered a class of a G-categories for which the G-crossed Drinfeld center has full G-spectrum. However, this concerned a very restricted class of G-categories and we wonder whether it is possible to draw similar conclusions for a more general class of G-categories, preferably a class of G-categories that also includes the categories $\mathrm{Loc}_f(\mathcal{A}_\zeta)$ for a completely rational chiral CFT $\mathcal{A}$. Indeed, in this section we will show that if $\mathcal{C}$ is a G-spherical fusion category, then $Z_G(\mathcal{C})$ has full G-spectrum. This will be done by generalizing the results in Section 4 of [75]. Since many of our statements here can be proven in the same way as in [75], we will not include all the proofs here.

In this section we will always assume that we are given a quadratically closed field $\mathbb{F}$ and a G-spherical fusion category $\mathcal{C}$ over $\mathbb{F}$, where G is a group and the G-action is denoted by F.

We will make similar assumptions as in Subsection 2.9.1. Thus we will assume that for each irreducible $X \in \mathcal{C}$ we have chosen a square root $d(X)^{1/2}$ of its dimension. We will always assume that $\dim(\mathcal{C}) \neq 0$ and that we have chosen a square root $\kappa := \dim(\mathcal{C})^{1/2}$, together with a square root $\kappa^{1/2}$. Also, we will assume that we have chosen some fixed complete set $\{X_i : i \in \Gamma\}$ of representatives of equivalence classes of irreducible objects, and we will also assume that for each triple $(i,j,k) \in \Gamma^{\times 3}$ we have chosen a basis

$$\{(t_{ij}^k)_\alpha : \alpha = 1, \ldots, N_{ij}^k\}$$

for $\mathrm{Hom}_{\mathcal{C}}(X_k, X_i \otimes X_j) = \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X_i^{\mathrm{op}} \otimes X_j^{\mathrm{op}}, X_k^{\mathrm{op}})$, together with a dual basis

$$\{(t_k^{ij})_\alpha : \alpha = 1, \ldots, N_{ij}^k\}$$

for $\mathrm{Hom}_{\mathcal{C}}(X_i \otimes X_j, X_k) = \mathrm{Hom}_{\mathcal{C}}(X_k^{\mathrm{op}}, X_i^{\mathrm{op}} \otimes X_j^{\mathrm{op}})$. For Γ we will take a finite subset of $\mathbb{Z}_{\geq 0}$ of the form $\{0, 1, \ldots, n\}$, where $X_0 = I$. Furthermore, we will write $\mathcal{D} := \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}$ and we will denote the tensor product in $\mathcal{D}$ by $\otimes_2$ to distinguish it from the tensor product $\otimes$ in $\mathcal{C}$. Recall that in Subsection 2.9.1 we constructed a normalized Frobenius algebra Q_F in $\mathcal{D}$ with normalization constant $\kappa_{\mathsf{Q}_F} = \kappa$ for any strict tensor equivalence $F : \mathcal{C} \to \mathcal{C}$. For each $q \in G$, we will write $\mathsf{Q}_q = (Q_q, \mu^q, \eta^q, \Delta^q, \varepsilon^q)$ to denote the Frobenius algebra that corresponds to F_q, where the object Q_q is a direct sum

$$Q_q \cong \bigoplus_{i \in \Gamma} F_q(X_i) \boxtimes X_i^{\mathrm{op}}.$$

In this way we obtain a collection $\{\mathsf{Q}_q : q \in G\}$ of normalized Frobenius algebras in $\mathcal{D}$, each with normalization constant κ, and in particular we can construct the categories $\mathcal{D}_r^q$ and $\overline{\mathcal{D}_r^q}$ of Subsection 2.9.3 for each $q, r \in G$. Finally, we will always assume that for all $q \in G$ we have already chosen morphisms

$$u_i^q \in \mathrm{Hom}_{\mathcal{D}}(F_q(X_i) \boxtimes X_i^{\mathrm{op}}, Q_q),$$
$$v_i^q \in \mathrm{Hom}_{\mathcal{D}}(Q_q, F_q(X_i) \boxtimes X_i^{\mathrm{op}})$$

such that $v_i^q \circ u_i^q = \mathrm{id}_{F_q(X_i) \boxtimes X_i^{\mathrm{op}}}$ and $\sum_{i \in \Gamma} u_i^q \circ v_i^q = \mathrm{id}_{Q_q}$. The following lemma generalizes Lemma 4.3 in [75]. Since it is important to understand how to define the $f[i]$ that occur in this lemma, we have decided to include a part of the proof.

Lemma 4.9.1 *Let $q, r \in G$ and let $X, Y, Z \in \mathcal{C}$.*
(1) There is a one-to-one correspondence between morphims

$$f \in \mathrm{Hom}_{\mathcal{D}_r^q}(\overline{J}_r(X \boxtimes I^{\mathrm{op}})J_q, \overline{J}_r(Y \boxtimes I^{\mathrm{op}})J_q)$$

and families $\{f[i] \in \mathrm{Hom}_{\mathcal{C}}(X \otimes F_q(X_i), F_r(X_i) \otimes Y) : i \in \Gamma\}$.

(2) If we have two morphisms

$$f \in \mathrm{Hom}_{\mathcal{D}_r^q}(\overline{J}_r(X \boxtimes I^{\mathrm{op}})J_q, \overline{J}_r(Y \boxtimes I^{\mathrm{op}})J_q),$$
$$g \in \mathrm{Hom}_{\mathcal{D}_r^q}(\overline{J}_r(Y \boxtimes I^{\mathrm{op}})J_q, \overline{J}_r(Z \boxtimes I^{\mathrm{op}})J_q),$$

then we have

$$(g \bullet f)[k] = d(X_k)^{-1}\kappa^{-1} \sum_{i,j\in\Gamma} \sum_{\alpha=1}^{N_{ij}^k} d(X_i)d(X_j)[F_r((t_k^{ij})_\alpha) \otimes \mathrm{id}_Z] \circ [\mathrm{id}_{F_r(X_i)} \otimes g[j]] \circ [f[i] \otimes \mathrm{id}_{F_q(X_j)}] \circ [\mathrm{id}_X \otimes F_q((t_{ij}^k)_\alpha)],$$

where $g \bullet f$ denotes the composition in $\mathcal{D}_r^q$, as usual.

Proof. (1) Let $f \in \mathrm{Hom}_{\mathcal{D}_r^q}(\overline{J}_r(X \boxtimes I^{\mathrm{op}})J_q, \overline{J}_r(Y \boxtimes I^{\mathrm{op}})J_q)$. Then, considering f as a morphism in $\mathrm{Hom}_{\mathcal{D}}((X \boxtimes I^{\mathrm{op}}) \otimes_2 Q_q, Q_r \otimes_2 (Y \boxtimes I^{\mathrm{op}}))$, the morphism

$$\begin{aligned}
&(v_j^r \otimes_2 \mathrm{id}_{Y\boxtimes I^{\mathrm{op}}}) \circ f \circ (\mathrm{id}_{X\boxtimes I^{\mathrm{op}}} \otimes_2 u_i^q) \\
&\quad \in \mathrm{Hom}_{\mathcal{D}}((X \boxtimes I^{\mathrm{op}}) \otimes_2 (F_q(X_i) \boxtimes X_i^{\mathrm{op}}), (F_r(X_j) \boxtimes X_j^{\mathrm{op}}) \otimes_2 (Y \boxtimes I^{\mathrm{op}})) \\
&\quad = \mathrm{Hom}_{\mathcal{C}}(X \otimes F_q(X_i), F_r(X_j) \otimes Y) \boxtimes \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X_i^{\mathrm{op}}, X_j^{\mathrm{op}})
\end{aligned}$$

vanishes if $i \neq j$ because X_i^{op} and X_j^{op} are inequivalent irreducible objects in $\mathcal{C}^{\mathrm{op}}$. Thus, if for each $i \in \Gamma$ we define $f[i] \in \mathrm{Hom}_{\mathcal{C}}(X \otimes F_q(X_i), F_r(X_i) \otimes Y)$ by

$$f[i] \boxtimes \mathrm{id}_{X_i^{\mathrm{op}}} := (v_i^r \otimes_2 \mathrm{id}_{Y\boxtimes I^{\mathrm{op}}}) \circ f \circ (\mathrm{id}_{X\boxtimes I^{\mathrm{op}}} \otimes_2 u_i^q),$$

then

$$f = \sum_{i\in\Gamma} (u_i^r \otimes_2 \mathrm{id}_{Y\boxtimes I^{\mathrm{op}}}) \circ (f[i] \boxtimes \mathrm{id}_{X_i^{\mathrm{op}}}) \circ (\mathrm{id}_{X\boxtimes I^{\mathrm{op}}} \otimes_2 v_i^q). \tag{4.9.1}$$

We thus obtain a one-to-one correspondence $f \leftrightarrow \{f[i] : i \in \Gamma\}$. The proof of part (2) is a computation that proceeds in the same way as in [75].
□

The following lemma is a straightforward generalization of Lemma 4.4 in [75]. It will be needed for the construction of the functors H_q in Proposition 4.9.3 below.

Lemma 4.9.2 *Let $q, r \in G$, let $X \in \mathcal{C}$ and let*

$$f \in \mathrm{End}_{\mathcal{D}_r^q}(\overline{J}_r(X \boxtimes I^{\mathrm{op}})J_q) = \mathrm{Hom}_{\mathcal{D}}((X \boxtimes I^{\mathrm{op}}) \otimes_2 Q_q, Q_r \otimes_2 (X \boxtimes I^{\mathrm{op}})).$$

Then the following two statements are equivalent:

(1) The family $\{f[i] : i \in \Gamma\}$ satisfies the braiding fusion relation

$$[\mathrm{id}_{F_r(X_i)} \otimes f[j]] \circ [f[i] \otimes \mathrm{id}_{F_q(X_j)}] \circ [\mathrm{id}_X \otimes F_q(g)] = [F_r(g) \otimes \mathrm{id}_X] \circ f[k] \tag{4.9.2}$$

for all $i, j, k \in \Gamma$ and all $g \in \mathrm{Hom}_{\mathcal{C}}(X_k, X_i \otimes X_j)$.

(2) The morphism f satisfies

$$[\mathrm{id}_{Q_r} \otimes_2 f] \circ [f \otimes_2 \mathrm{id}_{Q_q}] \circ [\mathrm{id}_{X\boxtimes I^{\mathrm{op}}} \otimes_2 \Delta^q] = [\Delta^r \otimes_2 \mathrm{id}_{X\boxtimes I^{\mathrm{op}}}] \circ f.$$

Our following proposition generalizes Propositions 4.5 and 4.6 in [75]. Because we want to show how the functors H_q in this proposition are constructed, we have decided to include a sketch of the proof here.

Proposition 4.9.3 *For each $q \in G$ there is a fully faithful functor $H_q : Z_G(\mathcal{C})_q \to \overline{\mathcal{D}^e_q}$.*

Sketch of the proof. Let $(X, q, \Phi_X) \in Z_G(\mathcal{C})_q$. The half q-braiding Φ_X provides us with a family $\{\Phi_X(X_i) \in \mathrm{Hom}_{\mathcal{C}}(X \otimes X_i, F_q(X_i) \otimes X) : i \in \Gamma\}$, which in turn gives rise to a morphism $p^0_{(X,q,\Phi_X)} \in \mathrm{Hom}_{\mathcal{D}}((X \boxtimes I^{\mathrm{op}}) \otimes_2 Q_e, Q_q \otimes_2 (X \boxtimes I^{\mathrm{op}})) = \mathrm{End}_{\mathcal{D}^e_q}(\overline{J}_q(X \boxtimes I^{\mathrm{op}})J_e)$ by

$$p^0_{(X,q,\Phi_X)} = \sum_{i\in\Gamma} [u^q_i \otimes_2 \mathrm{id}_{X\boxtimes I^{\mathrm{op}}}] \circ [\Phi_X(X_i) \boxtimes \mathrm{id}_{X_i^{\mathrm{op}}}] \circ [\mathrm{id}_{X\boxtimes I^{\mathrm{op}}} \otimes_2 v^e_i].$$

Because the family $\{\Phi_X(X_i) : i \in \Gamma\}$ satisfies the braiding fusion relation (4.9.2), $p^0_{(X,q,\Phi_X)}$ satisfies the equation in statement (2) of Lemma 4.9.2:

$$[\mathrm{id}_{Q_q} \otimes_2 p^0_{(X,q,\Phi_X)}] \circ [p^0_{(X,q,\Phi_X)} \otimes_2 \mathrm{id}_{Q_e}] \circ [\mathrm{id}_{X\boxtimes I^{\mathrm{op}}} \otimes_2 \Delta^e] = [\Delta^q \otimes_2 \mathrm{id}_{X\boxtimes I^{\mathrm{op}}}] \circ p^0_{(X,q,\Phi_X)}.$$

Composing both sides of this equation from the left with $\mu^q \otimes_2 \mathrm{id}_{X\boxtimes I^{\mathrm{op}}}$, we get $p^0_{(X,q,\Phi_X)} \bullet p^0_{(X,q,\Phi_X)} = \kappa p^0_{(X,q,\Phi_X)}$. Thus, if we define

$$p_{(X,q,\Phi_X)} := \kappa^{-1} p^0_{(X,q,\Phi_X)},$$

we have $p_{(X,q,\Phi_X)} \bullet p_{(X,q,\Phi_X)} = p_{(X,q,\Phi_X)}$, i.e. $p_{(X,q,\Phi_X)}$ is an idempotent in $\mathrm{End}_{\mathcal{D}^e_q}(\overline{J}_q(X \boxtimes I^{\mathrm{op}})J_e)$. The functor H_q is now defined by

$$H_q((X, q, \Phi_X)) := (\overline{J}_q(X \boxtimes I^{\mathrm{op}})J_e, p_{(X,q,\Phi_X)}),$$
$$H_q(f) := [\mathrm{id}_{Q_q} \otimes_2 (f \boxtimes \mathrm{id}_{I^{\mathrm{op}}})] \circ p_{(X,q,\Phi_X)}$$

if $f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((X, q, \Phi_X), (Y, q, \Phi_Y))$. For the proof that this is indeed a functor, we refer to [75]. To see that H_q is faithful, we first note that each $\Phi_X(X_i)$ is invertible in $\mathcal{C}$, so that also

$$p_{(X,q,\Phi_X)} = \kappa^{-1} \sum_{i\in\Gamma} [u^q_i \otimes_2 \mathrm{id}_{X\boxtimes I^{\mathrm{op}}}] \circ [\Phi_X(X_i) \boxtimes \mathrm{id}_{X_i^{\mathrm{op}}}] \circ [\mathrm{id}_{X\boxtimes I^{\mathrm{op}}} \otimes_2 v^e_i]$$

is invertible in $\mathcal{D}$ with inverse

$$p^{-1}_{(X,q,\Phi_X)} = \kappa \sum_{i\in\Gamma} [\mathrm{id}_{X\boxtimes I^{\mathrm{op}}} \otimes_2 u^e_j] \circ [\Phi_X(X_i)^{-1} \boxtimes \mathrm{id}_{X_i^{\mathrm{op}}}] \circ [v^q_j \otimes_2 \mathrm{id}_{X\boxtimes I^{\mathrm{op}}}].$$

Now suppose that $H_q(f_1) = H_q(f_2)$ for $f_1, f_2 \in \mathrm{Hom}_{Z_G(\mathcal{C})}((X, q, \Phi_X), (Y, q, \Phi_Y))$. Then the invertibility of $p_{(X,q,\Phi_X)}$ in $\mathcal{D}$ implies that $\mathrm{id}_{Q_q} \otimes_2 (f_1 \boxtimes \mathrm{id}_{I^{\mathrm{op}}}) = \mathrm{id}_{Q_q} \otimes_2 (f_2 \boxtimes \mathrm{id}_{I^{\mathrm{op}}})$. Using duality in $\mathcal{D}$, this implies that $f_1 \boxtimes \mathrm{id}_{I^{\mathrm{op}}} = f_2 \boxtimes \mathrm{id}_{I^{\mathrm{op}}}$ and hence that $f_1 = f_2$. To see that H_q is full, let $(X, q, \Phi_X), (Y, q, \Phi_Y) \in Z_G(\mathcal{C})_q$ and let

$$\begin{aligned} g \in &\mathrm{Hom}_{\overline{\mathcal{D}^e_q}}(H_q((X, q, \Phi_X)), H_q((Y, q, \Phi_Y))) \\ &= \mathrm{Hom}_{\overline{\mathcal{D}^e_q}}((\overline{J}_q(X \boxtimes I^{\mathrm{op}})J_e, p_{(X,q,\Phi_X)}), (\overline{J}_q(Y \boxtimes I^{\mathrm{op}})J_e, p_{(Y,q,\Phi_Y)})) \\ &= \{g' \in \mathrm{Hom}_{\mathcal{D}^e_q}(\overline{J}_q(X \boxtimes I^{\mathrm{op}})J_e, \overline{J}_q(Y \boxtimes I^{\mathrm{op}})J_e) : g' = p_{(Y,q,\Phi_Y)} \bullet g' \bullet p_{(X,q,\Phi_X)}\}. \end{aligned}$$

Then $g \in \mathrm{Hom}_{\mathcal{D}^e_q}(\overline{J}_q(X \boxtimes I^{\mathrm{op}})J_e, \overline{J}_q(Y \boxtimes I^{\mathrm{op}})J_e)$ is of the form

$$g = p_{(Y,q,\Phi_Y)} \bullet g' \bullet p_{(X,q,\Phi_X)}$$

for some $g' \in \mathrm{Hom}_{\mathcal{D}^e_q}(\overline{J}_q(X \boxtimes I^{\mathrm{op}})J_e, \overline{J}_q(Y \boxtimes I^{\mathrm{op}})J_e)$. It can then be shown that

$$g[m] = \kappa^{-4}[\mathrm{id}_{F_q(X_m)} \otimes E_{(X,q,\Phi_X),(Y,q,\Phi_Y)}(h)] \circ \Phi_X(X_m),$$

where $E_{(X,q,\Phi_X),(Y,q,\Phi_Y)}$ is as in Lemma 4.7.1 and

$$h := [F_q(d'_{X_i}) \otimes \mathrm{id}_Y] \circ [\mathrm{id}_{F_q(X_i)} \otimes \Phi_Y(\overline{X_i})] \circ [g'[i] \otimes \mathrm{id}_{\overline{X_i}}] \circ [\mathrm{id}_X \otimes b_{X_i}].$$

We note that h does not depend on m. For $m = 0$, this gives $g[0] = \kappa^{-4} E_{(X,q,\Phi_X),(Y,q,\Phi_Y)}(h)$, showing that $g[0] \in \mathrm{Hom}_{Z_G(\mathcal{C})}((X, q, \Phi_X), (Y, q, \Phi_Y))$. Plugging this into the equation above, we find

$$g[m] = [\mathrm{id}_{F_q(X_m)} \otimes g[0]] \circ \Phi_X(X_m)$$

and thus

$$\begin{aligned}
g &= \sum_{m\in\Gamma} [u^q_m \otimes_2 \mathrm{id}_{Y\boxtimes I^{\mathrm{op}}}] \circ [g[m] \boxtimes \mathrm{id}_{X_m^{\mathrm{op}}}] \circ [\mathrm{id}_{X\boxtimes I^{\mathrm{op}}} \otimes_2 v^e_m] \\
&= \sum_{m\in\Gamma} [u^q_m \otimes_2 \mathrm{id}_{Y\boxtimes I^{\mathrm{op}}}] \circ [\mathrm{id}_{F_q(X_m)\boxtimes X_m^{\mathrm{op}}} \otimes_2 (g[0] \boxtimes \mathrm{id}_{I^{\mathrm{op}}})] \circ [\Phi_X(X_m) \boxtimes \mathrm{id}_{X_m^{\mathrm{op}}}] \circ [\mathrm{id}_{X\boxtimes I^{\mathrm{op}}} \otimes_2 v^e_m] \\
&= \sum_{m\in\Gamma} [\mathrm{id}_{Q_q} \otimes_2 (g[0] \boxtimes \mathrm{id}_{I^{\mathrm{op}}})] \circ [u^q_m \otimes_2 \mathrm{id}_{X\boxtimes I^{\mathrm{op}}}] \circ [\Phi_X(X_m) \boxtimes \mathrm{id}_{X_m^{\mathrm{op}}}] \circ [\mathrm{id}_{X\boxtimes I^{\mathrm{op}}} \otimes_2 v^e_m] \\
&= \kappa[\mathrm{id}_{Q_q} \otimes_2 (g[0] \boxtimes \mathrm{id}_{I^{\mathrm{op}}})] \circ p_{(X,q,\Phi_X)} = \kappa H_q(g[0]) = H_q(\kappa g[0]),
\end{aligned}$$

and as mentioned before we have $g[0] \in \mathrm{Hom}_{Z_G(\mathcal{C})}((X, q, \Phi_X), (Y, q, \Phi_Y))$. So H_q is full.
□

The following proposition, which is a generalization of Propositions 4.12 and 4.13 in [75], forms the basic ingredient for the proof that the functors H_q are essentially surjective.

Proposition 4.9.4 *Let $r, s \in G$. Every object $(\overline{J}_s(X \boxtimes Y^{\mathrm{op}})J_r, p) \in \overline{\mathcal{D}^r_s}$ is isomorphic to one of the form $(\overline{J}_s(Z \boxtimes I^{\mathrm{op}})J_r, p')$ where p' satisfies*

$$[\Delta^s \otimes_2 \mathrm{id}_{Z\boxtimes I^{\mathrm{op}}}] \circ p' = \kappa \cdot [\mathrm{id}_{Q_s} \otimes_2 p'] \circ [p' \otimes_2 \mathrm{id}_{Q_r}] \circ [\mathrm{id}_{Z\boxtimes I^{\mathrm{op}}} \otimes \Delta^r] \tag{4.9.3}$$

and $p'[0] = \kappa^{-1}\mathrm{id}_Z$.

Now let $q \in G$ and consider this proposition for the case that the pair (r, s) is equal to (e, q). We claim that the object $(\overline{J}_q(Z \boxtimes I^{\mathrm{op}})J_e, p')$ is the image of an object in $Z_G(\mathcal{C})_q$ under the functor H_q. To see this, we define for each $i \in \Gamma$ the morphism $\Phi_Z(X_i) \in \mathrm{Hom}_{\mathcal{C}}(Z \otimes X_i, F_q(X_i) \otimes Z)$ by $\Phi_Z(X_i) := \kappa p'[i]$; in particular, $\Phi_Z(I) = \mathrm{id}_Z$. Because p' satisfies

$$[\Delta^q \otimes_2 \mathrm{id}_{Z\boxtimes I^{\mathrm{op}}}] \circ p' = \kappa \cdot [\mathrm{id}_{Q_q} \otimes_2 p'] \circ [p' \otimes_2 \mathrm{id}_{Q_e}] \circ [\mathrm{id}_{Z\boxtimes I^{\mathrm{op}}} \otimes \Delta^e],$$

it follows from Lemma 4.9.2 that for any $i, j, k \in \Gamma$ and any $g \in \mathrm{Hom}_{\mathcal{C}}(X_k, X_i \otimes X_j)$ we have

$$[\mathrm{id}_{F_q(X_i)} \otimes \kappa p'[j]] \circ [\kappa p'[i] \otimes \mathrm{id}_{X_j}] \circ [\mathrm{id}_Z \otimes g] = [F_q(g) \otimes \mathrm{id}_Z] \circ \kappa p'[k],$$

which can be written in terms of the $\Phi_Z(X_i)$ as

$$[\mathrm{id}_{F_q(X_i)} \otimes \Phi_Z(X_j)] \circ [\Phi_Z(X_i) \otimes \mathrm{id}_{X_j}] \circ [\mathrm{id}_Z \otimes g] = [F_q(g) \otimes \mathrm{id}_Z] \circ \Phi_Z(X_k). \tag{4.9.4}$$

Now let $Y \in \mathcal{C}$. We define $\Phi_Z(Y)$ as follows. Write $N_i \in \mathbb{Z}_{\geq 0}$ for the multiplicity of X_i in Y, so $Y \cong \bigoplus_{i\in\Gamma} X_i^{\oplus N_i}$. Thus for each $i \in \Gamma$ we can choose $f^i_1, \ldots, f^i_{N_i} \in \mathrm{Hom}_{\mathcal{C}}(X_i, Y)$ and $g^i_1, \ldots, g^i_{N_i} \in \mathrm{Hom}_{\mathcal{C}}(Y, X_i)$ such that $g^i_\alpha \circ f^j_\beta = \delta_{i,j}\delta_{\alpha,\beta}\mathrm{id}_{X_i}$ and $\sum_{i\in\Gamma}\sum_{\alpha=1}^{N_i} f^i_\alpha \circ g^i_\alpha = \mathrm{id}_Y$, where the α-summation is of course empty if $N_i = 0$. We then define

$$\Phi_Z(Y) := \sum_{i\in\Gamma}\sum_{\alpha=1}^{N_i} [F_q(f^i_\alpha) \otimes \mathrm{id}_Z] \circ \Phi_Z(X_i) \circ [\mathrm{id}_Z \otimes g^i_\alpha].$$

It can now be shown that this defines a half q-braiding Φ_Z for $Z \in \mathcal{C}$ by using equation (4.9.4) above. For instance, naturality of Φ_Z follows by considering the case where $j = 0$ and $k = i$ in (4.9.4); we leave the details to the reader. Thus $(Z, q, \Phi_Z) \in Z_G(\mathcal{C})$. Now

$$\begin{aligned} p_{(Z,q,\Phi_Z)} &= \kappa^{-1} \sum_{i \in \Gamma} [u_i^q \otimes_2 \mathrm{id}_{Z \boxtimes I^{\mathrm{op}}}] \circ [\Phi_Z(X_i) \boxtimes \mathrm{id}_{X_i^{\mathrm{op}}}] \circ [v_j^q \otimes_2 \mathrm{id}_{Z \boxtimes I^{\mathrm{op}}}] \\ &= \sum_{i \in \Gamma} [u_i^q \otimes_2 \mathrm{id}_{Z \boxtimes I^{\mathrm{op}}}] \circ [p'[i] \boxtimes \mathrm{id}_{X_i^{\mathrm{op}}}] \circ [v_j^q \otimes_2 \mathrm{id}_{Z \boxtimes I^{\mathrm{op}}}], \end{aligned}$$

which equals p' by equation (4.9.1). This shows that $H_q(Z, q, \Phi_Z) = (\overline{J}_q(Z \boxtimes I^{\mathrm{op}})J_e, p')$ and therefore proves our claim.

Finally, we can state the main theorem of this section, the proof of which now follows directly from our discussion in the preceding paragraph together with the first part of the proof of Theorem 4.14 in [75].

Theorem 4.9.5 *The fully faithful functors $H_q : Z_G(\mathcal{C})_q \to \overline{\mathcal{D}_q^e}$ are essentially surjective and thus equivalences.*

Because the category $\overline{\mathcal{D}_q^e}$ contains $\mathcal{D}$ as a subcategory, this theorem implies that for any $q \in G$ we have that $Z_G(\mathcal{C})_q$ cannot be empty (i.e. it must have some objects). We thus conclude that the theorem has the following important corollary.

Corollary 4.9.6 *If $\mathcal{C}$ is a G-spherical fusion category over a quadratically closed field $\mathbb{F}$ with $\dim(\mathcal{C}) \neq 0$, then $Z_G(\mathcal{C})$ has full G-spectrum.*

Now suppose, in addition to the assumptions in the corollary, that $\mathbb{F}$ is algebraically closed. Then it follows from Theorem 1.2 in [75] that the Drinfeld center $Z(\mathcal{C}) = Z_G(\mathcal{C})_e$ is a modular tensor category. Hence the corollary states that the modular tensor G-category $Z_G(\mathcal{C})_e$ has a braided G-crossed extension with full G-spectrum. It is known that not every modular tensor G-category has such a braided G-crossed extension, so $Z(\mathcal{C})$ is rather special in this sense. Now let $q \in G$ and let $\{V_i : i \in \Gamma_q\}$ be a complete set of representatives of irreducible objects in $Z_G(\mathcal{C})_q$ and let $d(V_i)$ be their dimensions. It then follows from Proposition 3.23 of [77] that[16]

$$\sum_{i \in \Gamma_q} d(V_i)^2 = \dim(Z_G(\mathcal{C})_e) = \dim(Z(\mathcal{C})),$$

the latter of which is equal to $\dim(\mathcal{C})^2$ by Theorem 4.14 of [75]. This implies that Γ_q must be a finite set, since $d(V_i) \geq 1$ for all $i \in \Gamma_q$. So under the additional assumption that G is a finite group, we can thus conclude that $Z_G(\mathcal{C})$ has finitely many isomorphism classes of irreducible objects.

4.10 The case when $\mathcal{C}$ is braided

In the preceding sections we have constructed the G-crossed Drinfeld center $Z_G(\mathcal{C})$ and we have demonstrated several of its nice properties. Our original motivation for the construction of $Z_G(\mathcal{C})$ was our search for braided G-crossed extensions of $\mathcal{C}$ in case $\mathcal{C}$ is a braided G-category, or perhaps even a modular tensor G-category. In case $\mathcal{C} = \mathrm{Loc}_f(\mathcal{A}_\zeta)$ with $\mathcal{A}$ a chiral CFT with a G-action (and $\zeta \in S^1$), we have seen that $\mathcal{C}$ is a modular tensor G-category and that it has braided G-crossed extensions $G - \mathrm{Loc}_f^{L/R,l/r}(\mathcal{A})$ with full G-spectrum. However, it is known that not every modular tensor G-category has a braided G-crossed extension with full G-spectrum, so there is no hope that we can find braided G-crossed extensions of $\mathcal{C}$ with full G-spectrum inside of $Z_G(\mathcal{C})$ for a general modular tensor G-category $\mathcal{C}$.

[16] This same proposition also underlies the equation (3.2.15).

But suppose now that $\mathcal{C}$ happens to be a braided G-category such that $Z_G(\mathcal{C})$ does contain a braided G-crossed extension of $\mathcal{C}$. It follows from Proposition 4.10.1 below that both $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ are contained in $Z_G(\mathcal{C})_e = Z(\mathcal{C})$, so the assumption that $Z_G(\mathcal{C})$ contains a braided G-crossed extension of $\mathcal{C}$ immediately raises the question whether this automatically implies that $Z_G(\mathcal{C})$ also contains such an extension of $\widetilde{\mathcal{C}}$. This question will be considered in the first subsection of this section. In the second subsection we will show how to construct a certain braided G-crossed category inside of $Z_G(\mathcal{C})$.

In this section we will always assume that we are given a strict tensor category $(\mathcal{C}, \otimes, I)$ with a strict action F of the group G and that $\mathcal{C}$ has a braiding c that is compatible with the G-action.

Proposition 4.10.1 *We have two fully faithful braided tensor functors*

$$\begin{aligned} H_1 : \mathcal{C} &\rightarrow Z(\mathcal{C}) \subset Z_G(\mathcal{C}), \\ H_2 : \widetilde{\mathcal{C}} &\rightarrow Z(\mathcal{C}) \subset Z_G(\mathcal{C}) \end{aligned}$$

defined by $H_1(V) = (V, e, c_{V,-})$, $H_2(V) = (V, e, \widetilde{c}_{V,-})$ *and* $H_1(f) = H_2(f) = f$. *Furthermore, the compatibility of* c *with the* G*-action implies that the functors* H_1 *and* H_2 *are functors of* G*-categories.*

Proof. The proof of the first statement is analogous to that in [75]. Note that the compatibility of c with the G-action F, i.e. $F_q(c_{V,W}) = c_{F_q(V),F_q(W)}$ for all $V, W \in \mathcal{C}$, implies that $\widetilde{c}$ is also compatible with the G-action. We have

$$\begin{aligned} \mathcal{F}_q(H_1(V)) &= \mathcal{F}_q[(V, e, c_{V,-})] = (F_q(V), e, \mathcal{F}_q c_{V,-}) = (F_q(V), e, F_q(c_{V,F_{q^{-1}}(-)})) \\ &= (F_q(V), e, c_{F_q(V),-}) = H_1(F_q(V)), \end{aligned}$$

and similarly for H_2. On the morphisms things are even simpler.
□

4.10.1 The internal structure of $Z_G(\mathcal{C})$

The goal of this subsection is to prove an important theorem about braided G-crossed extensions of $\mathcal{C}$ inside of $Z_G(\mathcal{C})$. As shown in Proposition 4.10.1, $Z_G(\mathcal{C})_e$ contains both $\mathcal{C}$ and $\widetilde{\mathcal{C}}$. At the end of this subsection we will prove a theorem which states that if $Z_G(\mathcal{C})$ contains a braided G-crossed extension of $\mathcal{C}$, then it also contains such an extension of $\widetilde{\mathcal{C}}$. The road towards this theorem is rather long, so we have decided to split it up into smaller steps.

Step 1: Construction of the category $Z_G^\star(\mathcal{C})$

Our first step will be to use of the braiding c on $\mathcal{C}$ in order to define a new product $\star$ of half braidings. If Φ_U is a half q-braiding for $U \in \mathcal{C}$ and if Φ_V is a half r-braiding for $V \in \mathcal{C}$, then the main difference between $\Phi_U \otimes \Phi_V$ and $\Phi_U \star \Phi_V$ is that the former is a half qr-braiding for $U \otimes V$, whereas the latter is a half rq-braiding for $U \otimes V$.

Lemma 4.10.2 *Let* $t \in G$ *and let* $(U, q, \Phi_U), (V, r, \Phi_V), (W, s, \Phi_W) \in Z_G(\mathcal{C})$.
(1) For each $X \in \mathcal{C}$ *we define* $(\Phi_U \star \Phi_V)(X) : U \otimes V \otimes X \rightarrow F_{rq}(X) \otimes U \otimes V$ *by*

$$\begin{aligned} (\Phi_U \star \Phi_V)(X) &:= [\mathrm{id}_{F_{rq}(X)} \otimes c_{U,V}^{-1}] \circ [(\Phi_V \otimes \Phi_U)(X)] \circ [c_{U,V} \otimes \mathrm{id}_X] \\ &= [\mathrm{id}_{F_{rq}(X)} \otimes c_{U,V}^{-1}] \circ [\Phi_V(F_q(X)) \otimes \mathrm{id}_U] \circ [\mathrm{id}_V \otimes \Phi_U(X)] \circ [c_{U,V} \otimes \mathrm{id}_X]. \end{aligned}$$

This defines a half rq*-braiding for* $U \otimes V$, *so* $(U \otimes V, rq, \Phi_U \star \Phi_V) \in Z_G(\mathcal{C})$.

(2) We have the equality

$$(\Phi_U \star \Phi_V) \star \Phi_W = \Phi_U \star (\Phi_V \star \Phi_W)$$

of half srq-braidings for $U \otimes V \otimes W$. Also,

$$\Phi_U \star \Phi_I^0 = \Phi_U = \Phi_I^0 \star \Phi_U.$$

(3) The group action $\mathcal{F}$ on $Z_G(\mathcal{C})$ satisfies $\mathcal{F}_t(\Phi_U \star \Phi_V) = \mathcal{F}_t\Phi_U \star \mathcal{F}_t\Phi_V$.

Proof. (1) Naturality of $\Phi_U \star \Phi_V$ is clear. If $X, Y \in \mathcal{C}$, then

$$\begin{aligned}
&[\mathrm{id}_{F_{rq}(X)} \otimes (\Phi_U \star \Phi_V)(Y)] \circ [(\Phi_U \star \Phi_V)(X) \otimes \mathrm{id}_Y] \\
&= [\mathrm{id}_{F_{rq}(X\otimes Y)} \otimes c_{U,V}^{-1}] \circ [\mathrm{id}_{F_{rq}(X)} \otimes (\Phi_V \otimes \Phi_U)(Y)] \circ [(\Phi_V \otimes \Phi_U)(X) \otimes \mathrm{id}_Y] \circ [c_{U,V} \otimes \mathrm{id}_{X\otimes Y}] \\
&= [\mathrm{id}_{F_{rq}(X\otimes Y)} \otimes c_{U,V}^{-1}] \circ [(\Phi_V \otimes \Phi_U)(X \otimes Y)] \circ [c_{U,V} \otimes \mathrm{id}_{X\otimes Y}] \\
&= (\Phi_U \star \Phi_V)(X \otimes Y),
\end{aligned}$$

where in the first step we already left out the combination $c_{U,V} \circ c_{U,V}^{-1}$.
(2) If $X \in \mathcal{C}$, then

$$\begin{aligned}
&[(\Phi_U \star \Phi_V) \star \Phi_W](X) \\
&= [\mathrm{id}_{F_{srq}(X)} \otimes c_{U\otimes V,W}^{-1}] \circ [\mathrm{id}_{F_{srq}(X)\otimes W} \otimes c_{U,V}^{-1}] \circ [\Phi_W(F_{rq}(X)) \otimes \mathrm{id}_{V\otimes U}] \\
&\quad \circ [\mathrm{id}_W \otimes \Phi_V(F_q(X)) \otimes \mathrm{id}_U] \circ [\mathrm{id}_{W\otimes V} \otimes \Phi_U(X)] \circ [\mathrm{id}_W \otimes c_{U,V} \otimes \mathrm{id}_X] \circ [c_{U\otimes V,W} \otimes \mathrm{id}_X] \\
&= [\mathrm{id}_{F_{srq}(X)} \otimes c_{U,V\otimes W}^{-1}] \circ [\mathrm{id}_{F_{srq}(X)} \otimes c_{V,W}^{-1} \otimes \mathrm{id}_U] \circ [\Phi_W(F_{rq}(X)) \otimes \mathrm{id}_{V\otimes U}] \\
&\quad \circ [\mathrm{id}_W \otimes \Phi_V(F_q(X)) \otimes \mathrm{id}_U] \circ [\mathrm{id}_{W\otimes V} \otimes \Phi_U(X)] \circ [c_{V,W} \otimes \mathrm{id}_{U\otimes X}] \circ [c_{U,V\otimes W} \otimes \mathrm{id}_X] \\
&= [\Phi_U \star (\Phi_V \star \Phi_W)](X),
\end{aligned}$$

where the first and last step follow from the interchange law and the middle step follows from the properties of the braiding. The equalities concerning Φ_I^0 follow directly from $\Phi_I^0 \otimes \Phi_U = \Phi_U = \Phi_U \otimes \Phi_I^0$ and from the fact that $c_{I,U} = \mathrm{id}_U = c_{U,I}$.
(3) This follows from the fact that for any $X \in \mathcal{C}$ we have

$$\begin{aligned}
[\mathcal{F}_t(\Phi_U \star \Phi_V)](X) &= F_t((\Phi_U \star \Phi_V)(F_{t^{-1}}(X))) \\
&= [\mathrm{id}_{F_{trqt^{-1}}(X)} \otimes c_{F_t(U),F_t(V)}^{-1}] \circ [F_t(\Phi_V(F_{qt^{-1}}(X))) \otimes \mathrm{id}_{F_t(U)}] \\
&\quad \circ [\mathrm{id}_{F_t(V)} \otimes F_t(\Phi_U(F_{t^{-1}}(X)))] \circ [c_{F_t(U),F_t(V)} \otimes \mathrm{id}_X] \\
&= [\mathrm{id}_{F_{trqt^{-1}}(X)} \otimes c_{F_t(U),F_t(V)}^{-1}] \circ [\mathcal{F}_t\Phi_V(F_{tqt^{-1}}(X)) \otimes \mathrm{id}_{F_t(U)}] \\
&\quad \circ [\mathrm{id}_{F_t(V)} \otimes \mathcal{F}_t\Phi_U(X)] \circ [c_{F_t(U),F_t(V)} \otimes \mathrm{id}_X] \\
&= [(\mathcal{F}_t\Phi_U) \star (\mathcal{F}_t\Phi_V)](X).
\end{aligned}$$

□

Now that we have defined the new product $\star$ of half braidings, we will use it to define a new braided G-crossed structure on $Z_G(\mathcal{C})$.

Proposition 4.10.3 *We can define a braided G-crossed category $Z_G^\star(\mathcal{C})$ as follows:*

- *As a category $Z_G^\star(\mathcal{C})$ is simply $Z_G(\mathcal{C})$.*
- *The tensor product $\star : Z_G^\star(\mathcal{C}) \times Z_G^\star(\mathcal{C}) \to Z_G^\star(\mathcal{C})$ is defined by*

$$(U, q, \Phi_U) \star (V, r, \Phi_V) := (U \otimes V, rq, \Phi_U \star \Phi_V)$$
$$f \star g := f \otimes g$$

for $(U, q, \Phi_U), (V, r, \Phi_V) \in Z_G^\star(\mathcal{C})$ and $f, g \in \mathrm{Hom}(Z_G^\star(\mathcal{C}))$.

- *The G-grading $\partial_\star$ is defined by*

$$\partial_\star((U,q,\Phi_U)) := \partial((U,q,\Phi_U))^{-1}$$

  for any $(U,q,\Phi_U) \in Z_G^\star(\mathcal{C})$.
- *The G-action on $Z_G^\star(\mathcal{C})$ coincides with the G-action on $Z_G(\mathcal{C})$.*
- *For $(U,q,\Phi_U),(V,r,\Phi_V) \in Z_G^\star(\mathcal{C})$ the braiding is defined by*

$$C^\star_{(U,q,\Phi_U),(V,r,\Phi_V)} := c^{-1}_{F_{q^{-1}}(V),U} \circ C^{-1}_{(U,q,\Phi_U),\mathcal{F}_q[(V,r,\Phi_V)]} \circ c_{U,V}.$$

  Equivalently, $C^\star_{(U,q,\Phi_U),(V,r,\Phi_V)} = c^{-1}_{F_{q^{-1}}(V),U} \circ \Phi_U(F_{q^{-1}}(V))^{-1} \circ c_{U,V}$.

Proof. Let $f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((U,q,\Phi_U),(U',q,\Phi_{U'}))$ and $g \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V,r,\Phi_V),(V',r,\Phi_{V'}))$. To see that $f \star g \in \mathrm{Hom}_{Z_G(\mathcal{C})}((U\otimes V, rq, \Phi_U \star \Phi_V),(U'\otimes V', rq, \Phi_{U'}\star\Phi_{V'}))$, we compute

$$\begin{aligned}
&[\mathrm{id}_{F_{rq}(X)} \otimes (f\star g)] \circ (\Phi_U \star \Phi_V)(X)\\
&= [\mathrm{id}_{F_{rq}(X)} \otimes f \otimes g] \circ [\mathrm{id}_{F_{rq}(X)} \otimes c^{-1}_{U,V}] \circ [\Phi_V(F_q(X)) \otimes \mathrm{id}_U] \circ [\mathrm{id}_V \otimes \Phi_U(X)] \circ [c_{U,V} \otimes \mathrm{id}_X]\\
&= [\mathrm{id}_{F_{rq}(X)} \otimes c^{-1}_{U',V'}] \circ [\mathrm{id}_{F_{rq}(X)} \otimes g \otimes \mathrm{id}_{U'}] \circ [\Phi_V(F_q(X)) \otimes \mathrm{id}_{U'}] \circ [\mathrm{id}_{V\otimes F_q(X)} \otimes f]\\
&\quad \circ [\mathrm{id}_V \otimes \Phi_U(X)] \circ [c_{U,V} \otimes \mathrm{id}_X]\\
&= [\mathrm{id}_{F_{rq}(X)} \otimes c^{-1}_{U',V'}] \circ [\Phi_{V'}(F_q(X)) \otimes \mathrm{id}_{U'}] \circ [g \otimes \mathrm{id}_{F_q(X)\otimes U'}] \circ [\mathrm{id}_V \otimes \Phi_{U'}(X)]\\
&\quad \circ [\mathrm{id}_V \otimes f \otimes \mathrm{id}_X] \circ [c_{U,V} \otimes \mathrm{id}_X]\\
&= [\mathrm{id}_{F_{rq}(X)} \otimes c^{-1}_{U',V'}] \circ [\Phi_{V'}(F_q(X)) \otimes \mathrm{id}_{U'}] \circ [\mathrm{id}_{V'} \otimes \Phi_{U'}(X)] \circ [c_{U',V'} \otimes \mathrm{id}_X] \circ [f \otimes g \otimes \mathrm{id}_X]\\
&= (\Phi_{U'} \star \Phi_{V'})(X) \circ [(f\star g) \otimes \mathrm{id}_X].
\end{aligned}$$

The interchange law in $Z_G^\star(\mathcal{C})$ follows directly from the interchange law in $Z_G(\mathcal{C})$ because $\star$ coincides with $\otimes$ on the morphisms. For the same reason we also have $\mathrm{id}_{(U,q,\Phi_U)} \star \mathrm{id}_{(V,r,\Phi_V)} = \mathrm{id}_{(U,q,\Phi_U)\star(V,r,\Phi_V)}$, as well as associativity of $\star$ on the morphisms and $f \star \mathrm{id}_{(I,e,\Phi_I^0)} = f = \mathrm{id}_{(I,e,\Phi_I^0)} \star f$ for any morphism f in $Z_G^\star(\mathcal{C})$. It follows from part (2) of the lemma that $\star$ is associative on the objects and that (I,e,Φ_I^0) acts as a unit object. This proves that $(Z_G(\mathcal{C}), \star, (I,e,\Phi_I^0))$ is a strict tensor category. From part (3) of the lemma we get that

$$\begin{aligned}
\mathcal{F}_t((U,q,\Phi_U)\star(V,r,\Phi_V)) &= (F_t(U\otimes V), trqt^{-1}, \mathcal{F}_t(\Phi_U \star \Phi_V))\\
&= (F_t(U), tqt^{-1}, \Phi_U) \star (F_t(V), trt^{-1}, \Phi_V)\\
&= \mathcal{F}_t((U,q,\Phi_U)) \star \mathcal{F}_t((V,r,\Phi_V)).
\end{aligned}$$

Together with the fact that the group action on $Z_G^\star(\mathcal{C})$ is the same as on $Z_G(\mathcal{C})$, this shows that $\mathcal{F}$ defines a G-action on the strict tensor category $Z_G^\star(\mathcal{C})$. If $(U,q,\Phi_U),(V,r,\Phi_V) \in Z_G^\star(\mathcal{C})$, then

$$\partial_\star[(U,q,\Phi_U)\star(V,r,\Phi_V)] = (rq)^{-1} = q^{-1}r^{-1} = \partial_\star[(U,q,\Phi_U)]\partial_\star[(V,r,\Phi_V)],$$

so $\partial_\star$ defines a G-grading on $Z_G^\star(\mathcal{C})$. To see that $Z_G^\star(\mathcal{C})$ is G-crossed, we note that

$$\partial_\star\{\mathcal{F}_t[(U,q,\Phi_U)]\} = (tqt^{-1})^{-1} = tq^{-1}t^{-1} = t\partial_\star[(U,q,\Phi_U)]t^{-1}.$$

Let $(U,q,\Phi_U),(V,r,\Phi_V) \in Z_G^\star(\mathcal{C})$. We will first show that $C^\star_{(U,q,\Phi_U),(V,r,\Phi_V)}$ is a morphism in the category $Z_G^\star(\mathcal{C})$. In the category $Z_G(\mathcal{C})$ we have that

$$C_{(U,q,\Phi_U),\mathcal{F}_{q^{-1}}(V,r,\Phi_V)} \in \mathrm{Hom}_{Z_G(\mathcal{C})}((U\otimes F_{q^{-1}}(V), rq, \Phi_U \otimes \mathcal{F}_{q^{-1}}\Phi_V),(V\otimes U, rq, \Phi_V \otimes \Phi_U)),$$

which means that

$$[(\Phi_V \otimes \Phi_U)(X)] \circ [\Phi_U(F_{q^{-1}}(V)) \otimes \mathrm{id}_X] = [\mathrm{id}_{F_{rq}(X)} \otimes \Phi_U(F_{q^{-1}}(V))] \circ [(\Phi_U \otimes \mathcal{F}_{q^{-1}}\Phi_V)(X)].$$

This equation can be rewritten as

$$[\mathrm{id}_{F_{rq}(X)} \otimes \Phi_U(F_{q^{-1}}(V))^{-1}] \circ [(\Phi_V \otimes \Phi_U)(X)] = [(\Phi_U \otimes \mathcal{F}_{q^{-1}}\Phi_V)(X)] \circ [\Phi_U(F_{q^{-1}}(V))^{-1} \otimes \mathrm{id}_X].$$

Using this equality, we now find that

$$\begin{aligned}
&[\mathrm{id}_{F_{rq}(X)} \otimes C^\star_{(U,q,\Phi_U),(V,r,\Phi_V)}] \circ [(\Phi_U \star \Phi_V)(X)] \\
&= [\mathrm{id}_{F_{rq}(X)} \otimes c^{-1}_{F_{q^{-1}}(V),U}] \circ [\mathrm{id}_{F_{rq}(X)} \otimes \Phi_U(F_{q^{-1}}(V))^{-1}] \circ [(\Phi_V \otimes \Phi_U)(X)] \circ [c_{U,V} \otimes \mathrm{id}_X] \\
&= [\mathrm{id}_{F_{rq}(X)} \otimes c^{-1}_{F_{q^{-1}}(V),U}] \circ [(\Phi_U \otimes \mathcal{F}_{q^{-1}}\Phi_V)(X)] \circ [\Phi_U(F_{q^{-1}}(V))^{-1} \otimes \mathrm{id}_X] \circ [c_{U,V} \otimes \mathrm{id}_X] \\
&= [(\mathcal{F}_{q^{-1}}\Phi_V \star \Phi_U)(X)] \circ [C^\star_{(U,q,\Phi_U),(V,r,\Phi_V)} \otimes \mathrm{id}_X].
\end{aligned}$$

Naturality of $C^\star$ follows directly from the naturality of C and c. Now let (U, q, Φ_U), (V, r, Φ_V) and (W, s, Φ_W) be objects in $Z^\star_G(\mathcal{C})$. Then

$$\begin{aligned}
&C^\star_{(U,q,\Phi_U),(V,r,\Phi_V)\star(W,s,\Phi_W)} = C^\star_{(U,q,\Phi_U),(V\otimes W,sr,\Phi_V\star\Phi_W)} \\
&= c^{-1}_{F_{q^{-1}}(V\otimes W),U} \circ \Phi_U(F_{q^{-1}}(V \otimes W))^{-1} \circ c_{U,V\otimes W} \\
&= c^{-1}_{F_{q^{-1}}(V\otimes W),U} \circ [\mathrm{id}_U \otimes c^{-1}_{F_{q^{-1}}(V),F_{q^{-1}}(W)}] \circ \Phi_U(F_{q^{-1}}(W \otimes V))^{-1} \circ [c_{V,W} \otimes \mathrm{id}_U] \circ c_{U,V\otimes W} \\
&= [\mathrm{id}_{F_{q^{-1}}(V)} \otimes c^{-1}_{F_{q^{-1}}(W),U}] \circ c^{-1}_{F_{q^{-1}}(V),U\otimes F_{q^{-1}}(W)} \circ [\Phi_U(F_{q^{-1}}(W))^{-1} \otimes \mathrm{id}_{F_{q^{-1}}(V)}] \\
&\quad \circ [\mathrm{id}_W \otimes \Phi_U(F_{q^{-1}}(V))^{-1}] \circ c_{V\otimes U,W} \circ [c_{U,V} \otimes \mathrm{id}_W] \\
&= [\mathrm{id}_{F_{q^{-1}}(V)} \otimes c^{-1}_{F_{q^{-1}}(W),U}] \circ [\mathrm{id}_{F_{q^{-1}}(V)} \otimes \Phi_U(F_{q^{-1}}(W))^{-1}] \circ c^{-1}_{F_{q^{-1}}(V),W\otimes U} \\
&\quad \circ c_{U\otimes F_{q^{-1}}(V),W} \circ [\Phi_U(F_{q^{-1}}(V))^{-1} \otimes \mathrm{id}_W] \circ [c_{U,V} \otimes \mathrm{id}_W] \\
&= [\mathrm{id}_{F_{q^{-1}}(V)} \otimes c^{-1}_{F_{q^{-1}}(W),U}] \circ [\mathrm{id}_{F_{q^{-1}}(V)} \otimes \Phi_U(F_{q^{-1}}(W))^{-1}] \circ [\mathrm{id}_{F_{q^{-1}}(V)} \otimes c_{U,W}] \\
&\quad \circ [c^{-1}_{F_{q^{-1}}(V),U} \otimes \mathrm{id}_W] \circ [\Phi_U(F_{q^{-1}}(V))^{-1} \otimes \mathrm{id}_W] \circ [c_{U,V} \otimes \mathrm{id}_W] \\
&= [\mathrm{id}_{\mathcal{F}_{\partial_\star[(U,q,\Phi_U)]}[(V,r,\Phi_V)]} \star C^\star_{(U,q,\Phi_U),(W,s,\Phi_W)}] \circ [C^\star_{(U,q,\Phi_U),(V,r,\Phi_V)} \star \mathrm{id}_{(W,s,\Phi_W)}],
\end{aligned}$$

where the step from the second to the third line follows from naturality of Φ_U. We also have

$$\begin{aligned}
&C^\star_{(U,q,\Phi_U)\star(V,r,\Phi_V),(W,s,\Phi_W)} = C^\star_{(U\otimes V,rq,\Phi_U\star\Phi_V),(W,s,\Phi_W)} \\
&= c^{-1}_{F_{(rq)^{-1}}(W),U\otimes V} \circ (\Phi_U \star \Phi_V)(F_{(rq)^{-1}}(W))^{-1} \circ c_{U\otimes V,W} \\
&= c^{-1}_{F_{(rq)^{-1}}(W),U\otimes V} \circ [c^{-1}_{U,V} \otimes \mathrm{id}_{F_{(rq)^{-1}}(W)}] \circ (\Phi_V \otimes \Phi_U)(F_{(rq)^{-1}}(W))^{-1} \circ [\mathrm{id}_W \otimes c_{U,V}] \circ c_{U\otimes V,W} \\
&= [c^{-1}_{F_{(rq)^{-1}}(W),U} \otimes \mathrm{id}_V] \circ c^{-1}_{U\otimes F_{(rq)^{-1}}(W),V} \circ [\mathrm{id}_V \otimes \Phi_U(F_{(rq)^{-1}}(W))^{-1}] \\
&\quad \circ [\Phi_V(F_{r^{-1}}(W))^{-1} \otimes \mathrm{id}_U] \circ c_{U,W\otimes V} \circ [\mathrm{id}_U \otimes c_{V,W}] \\
&= [c^{-1}_{F_{(rq)^{-1}}(W),U} \otimes \mathrm{id}_V] \circ [\Phi_U(F_{(rq)^{-1}}(W))^{-1} \otimes \mathrm{id}_V] \circ c^{-1}_{F_{r^{-1}}(W)\otimes U,V} \\
&\quad \circ c_{U,V\otimes F_{r^{-1}}(W)} \circ [\mathrm{id}_U \otimes \Phi_V(F_{r^{-1}}(W))^{-1}] \circ [\mathrm{id}_U \otimes c_{V,W}] \\
&= [c^{-1}_{F_{(rq)^{-1}}(W),U} \otimes \mathrm{id}_V] \circ [\Phi_U(F_{(rq)^{-1}}(W))^{-1} \otimes \mathrm{id}_V] \circ [c_{U,F_{r^{-1}}(W)} \otimes \mathrm{id}_V] \\
&\quad \circ [\mathrm{id}_U \otimes c^{-1}_{F_{r^{-1}}(W),V}] \circ [\mathrm{id}_U \otimes \Phi_V(F_{r^{-1}}(W))^{-1}] \circ [\mathrm{id}_U \otimes c_{V,W}]
\end{aligned}$$

$$= [C^\star_{(U,q,\Phi_U),\mathcal{F}_{\partial_\star[(V,r,\Phi_V)]}[(W,s,\Phi_W)]} \star \mathrm{id}_{(V,r,\Phi_V)}] \circ [\mathrm{id}_{(U,q,\Phi_U)} \star C^\star_{(V,r,\Phi_V),(W,s,\Phi_W)}].$$

This completes the proof that $C^\star$ is a braiding. Finally suppose that $s \in G$ and let $(U,q,\Phi_U),(V,r\Phi_V) \in Z^\star_G(\mathcal{C})$. Then

$$\begin{aligned} C^\star_{\mathcal{F}_s[(U,q,\Phi_U)],\mathcal{F}_s[(V,r,\Phi_V)]} &= C^\star_{(F_s(U),sqs^{-1},\mathcal{F}_s\Phi_U),(F_s(V),srs^{-1},\mathcal{F}_s\Phi_V)} \\ &= c^{-1}_{F_{sq^{-1}s^{-1}}(F_s(V)),F_s(U)} \circ C_{\mathcal{F}_s[(U,q,\Phi_U)],\mathcal{F}_{sqs^{-1}}\mathcal{F}_s[(V,r,\Phi_V)]} \circ c_{F_s(U),F_s(V)} \\ &= c^{-1}_{F_{sq^{-1}}(V),F_s(U)} \circ C_{\mathcal{F}_s[(U,q,\Phi_U)],\mathcal{F}_s\mathcal{F}_q[(V,r,\Phi_V)]} \circ c_{F_s(U),F_s(V)} \\ &= F_s\left\{c^{-1}_{F_{q^{-1}}(V),U} \circ C_{(U,q,\Phi_U),\mathcal{F}_q[(V,r,\Phi_V)]} \circ c_{U,V}\right\} \\ &= \mathcal{F}_s(C^\star_{(U,q,\Phi_U),(V,r,\Phi_V)}), \end{aligned}$$

showing that $C^\star$ is compatible with the G-action and hence that $Z^\star_G(\mathcal{C})$ is a braided G-crossed category.
□

Step 2: Equivalence of $Z^\star_G(\mathcal{C})$ and $Z_G(\mathcal{C})^\bullet$

Recall from Subsection 2.8.5 that for any braided G-crossed category $\mathcal{D}$ we defined its mirror image $\mathcal{D}^\bullet$. Our next step is to show that the braided G-crossed category $Z^\star_G(\mathcal{C})$ defined above is equivalent to the mirror image $Z_G(\mathcal{C})^\bullet$ of $Z_G(\mathcal{C})$.

Theorem 4.10.4 *The identity functor on $Z_G(\mathcal{C})$ can be given the structure of an equivalence*

$$(\mathrm{id}_{Z_G(\mathcal{C})},\varepsilon,\delta) : Z^\star_G(\mathcal{C}) \to Z_G(\mathcal{C})^\bullet$$

of braided G-crossed categories.

Proof. In this proof we will write $H := \mathrm{id}_{Z_G(\mathcal{C})}$ for short. We can simply choose $\varepsilon = \mathrm{id}_{(I.e.\Phi^0_I)}$. For any two objects (U,q,Φ_U), (V,r,Φ_V) in $Z_G(\mathcal{C})$ we define $\delta_{(U,q,\Phi_U),(V,r,\Phi_V)} \in \mathrm{Hom}_\mathcal{C}(U \otimes F_{q^{-1}}(V), U \otimes V)$ by

$$\delta_{(U,q,\Phi_U),(V,r,\Phi_V)} = c^{-1}_{U,V} \circ \Phi_U(F_{q^{-1}}(V)).$$

Fix $(U,q,\Phi_U),(V,r\Phi_V) \in Z_G(\mathcal{C})$ and write $\delta_{U,V} := \delta_{(U,q,\Phi_U),(V,r,\Phi_V)}$. We will first show that

$$\begin{aligned} \delta_{(U,q,\Phi_U),(V,r,\Phi_V)} &\in \mathrm{Hom}_{Z_G(\mathcal{C})}(H[(U,q,\Phi_U)] \bullet H[(V,r,\Phi_V)], H[(U,q,\Phi_U) \star (V,r,\Phi_V)]) \\ &= \mathrm{Hom}_{Z_G(\mathcal{C})}((U \otimes F_{q^{-1}}(V), rq, \Phi_U \otimes \mathcal{F}_{q^{-1}}\Phi_V),(U \otimes V, rq, \Phi_U \star \Phi_V)) \end{aligned}$$

i.e. that $\delta_{(U,q,\Phi_U),(V,r,\Phi_V)}$ satisfies

$$[\mathrm{id}_{F_{rq}(X)} \otimes \delta_{(U,q,\Phi_U),(V,r,\Phi_V)}] \circ (\Phi_U \otimes \mathcal{F}_{q^{-1}}\Phi_V)(X) = (\Phi_U \star \Phi_V)(X) \circ [\delta_{(U,q,\Phi_U),(V,r,\Phi_V)} \otimes \mathrm{id}_X]$$

for all $X \in \mathcal{C}$. This follows from the computation

$$\begin{aligned} &[\mathrm{id}_{F_{rq}(X)} \otimes \delta_{(U,q,\Phi_U),(V,r,\Phi_V)}] \circ (\Phi_U \otimes \mathcal{F}_{q^{-1}}\Phi_V)(X) \\ &= [\mathrm{id}_{F_{rq}(X)} \otimes c^{-1}_{U,V}] \circ [\mathrm{id}_{F_{rq}(X)} \otimes \Phi_U(F_{q^{-1}}(V))] \circ [\Phi_U(F_{q^{-1}rq}(X)) \otimes \mathrm{id}_{F_{q^{-1}}(V)}] \circ [\mathrm{id}_U \otimes (\mathcal{F}_{q^{-1}}\Phi_V)(X)] \\ &= [\mathrm{id}_{F_{rq}(X)} \otimes c^{-1}_{U,V}] \circ \Phi_U(F_{q^{-1}rq}(X) \otimes F_{q^{-1}}(V)) \circ [\mathrm{id}_U \otimes F_{q^{-1}}(\Phi_V(F_q(X)))] \\ &= [\mathrm{id}_{F_{rq}(X)} \otimes c^{-1}_{U,V}] \circ [\Phi_V(F_q(X)) \otimes \mathrm{id}_U] \circ \Phi_U(F_{q^{-1}}(V) \otimes X) \\ &= [\mathrm{id}_{F_{rq}(X)} \otimes c^{-1}_{U,V}] \circ [\Phi_V(F_q(X)) \otimes \mathrm{id}_U] \circ [\mathrm{id}_V \otimes \Phi_U(X)] \\ &\quad \circ [c_{U,V} \otimes \mathrm{id}_X] \circ [c^{-1}_{U,V} \otimes \mathrm{id}_X] \circ [\Phi_U(F_{q^{-1}}(V)) \otimes \mathrm{id}_X] \end{aligned}$$

$= (\Phi_U \star \Phi_V)(X) \circ [\delta_{(U,q,\Phi_U),(V,r,\Phi_V)} \otimes \mathrm{id}_X].$

Now let $f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((U,q,\Phi_U),(U',q,\Phi_{U'}))$ and $g \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V,r,\Phi_V),(V',r,\Phi_{V'}))$. Then

$$\begin{aligned}\delta_{(U',q,\Phi_{U'}),(V',r,\Phi_{V'})} \circ [f \bullet g] = c^{-1}_{U',V'} \circ \Phi_{U'}(F_{q^{-1}}(V')) \circ [f \otimes F_{q^{-1}}(g)] &= [f \otimes g] \circ c^{-1}_{U,V} \circ \Phi_U(F_{q^{-1}}(V)) \\ &= [f \star g] \circ \delta_{(U,q,\Phi_U),(V,r,\Phi_V)},\end{aligned}$$

showing naturality. If $(U,q,\Phi_U),(V,r,\Phi_V),(W,s,\Phi_W) \in Z_G(\mathcal{C})$, then

$$\begin{aligned}
&\delta_{(U,q,\Phi_U)\star(V,r,\Phi_V),(W,s,\Phi_W)} \circ [\delta_{(U,q,\Phi_U),(V,r,\Phi_V)} \bullet \mathrm{id}_W] \\
&= c^{-1}_{U\otimes V,W} \circ (\Phi_U \star \Phi_V)(F_{q^{-1}r^{-1}}(W)) \circ [c^{-1}_{U,V} \otimes \mathrm{id}_{F_{q^{-1}r^{-1}}(W)}] \circ [\Phi_U(F_{q^{-1}}(V)) \otimes \mathrm{id}_{F_{q^{-1}r^{-1}}(W)}] \\
&= [\mathrm{id}_U \otimes c^{-1}_{V,W}] \circ [c^{-1}_{U,W} \otimes \mathrm{id}_V] \circ [\mathrm{id}_W \otimes c^{-1}_{U,V}] \circ [\Phi_V(F_{r^{-1}}(W)) \otimes \mathrm{id}_U] \circ [\mathrm{id}_V \otimes \Phi_U(F_{q^{-1}r^{-1}}(W))] \\
&\quad \circ [c_{U,V} \otimes \mathrm{id}_{F_{q^{-1}r^{-1}}(W)}] \circ [c^{-1}_{U,V} \otimes \mathrm{id}_{F_{q^{-1}r^{-1}}(W)}] \circ [\Phi_U(F_{q^{-1}}(V)) \otimes \mathrm{id}_{F_{q^{-1}r^{-1}}(W)}] \\
&= [\mathrm{id}_U \otimes c^{-1}_{V,W}] \circ [c^{-1}_{U,W} \otimes \mathrm{id}_V] \circ [\mathrm{id}_W \otimes c^{-1}_{U,V}] \circ [\Phi_V(F_{r^{-1}}(W)) \otimes \mathrm{id}_U] \circ \Phi_U(F_{q^{-1}}(V) \otimes F_{q^{-1}r^{-1}}(W)) \\
&= [c^{-1}_{U,V} \otimes \mathrm{id}_W] \circ [\mathrm{id}_V \otimes c^{-1}_{U,W}] \circ [c^{-1}_{V,W} \otimes \mathrm{id}_U] \\
&\quad \circ [\Phi_V(F_{r^{-1}}(W)) \otimes \mathrm{id}_U] \circ \Phi_U(F_{q^{-1}}(V \otimes F_{r^{-1}}(W))) \\
&= [c^{-1}_{U,V} \otimes \mathrm{id}_W] \circ [\mathrm{id}_V \otimes c^{-1}_{U,W}] \circ [c^{-1}_{V,W} \otimes \mathrm{id}_U] \\
&\quad \circ [\Phi_U(F_{q^{-1}}(W \otimes V))] \circ [\mathrm{id}_U \otimes F_{q^{-1}}(\Phi_V(F_{r^{-1}}(W)))] \\
&= [c^{-1}_{U,V} \otimes \mathrm{id}_W] \circ [\mathrm{id}_V \otimes c^{-1}_{U,W}] \circ \Phi_U(F_{q^{-1}}(V \otimes W)) \\
&\quad \circ [\mathrm{id}_U \otimes c^{-1}_{F_{q^{-1}}(V),F_{q^{-1}}(W)}] \circ [\mathrm{id}_U \otimes F_{q^{-1}}(\Phi_V(F_{r^{-1}}(W)))] \\
&= c^{-1}_{U,V\otimes W} \circ \Phi_U(F_{q^{-1}}(V \otimes W)) \circ [\mathrm{id}_U \otimes F_{q^{-1}}(c^{-1}_{V,W})] \circ [\mathrm{id}_U \otimes F_{q^{-1}}(\Phi_V(F_{r^{-1}}(W)))] \\
&= \delta_{(U,q,\Phi_U),(V,r,\Phi_V)\star(W,s,\Phi_W)} \circ [\mathrm{id}_U \bullet \delta_{V,W}].
\end{aligned}$$

We also note that for any $(U,q,\Phi_U) \in Z_G(\mathcal{C})$ we have

$$\begin{aligned}\delta_{(U,q,\Phi_U),(I,e,\Phi^0_I)} &= c^{-1}_{U,I} \circ \Phi_U(F_{q^{-1}}(I)) = \Phi_U(I) = \mathrm{id}_U \\ \delta_{(I,e,\Phi^0_I),(U,q,\Phi_U)} &= c^{-1}_{I,U} \circ \Phi^0_I(F_{e^{-1}}(U)) = \Phi^0_I(U) = \mathrm{id}_U,\end{aligned}$$

showing that H can be equipped with the structure of a tensor functor. That H is an equivalence of G-crossed categories is obvious, because H is the identity functor and the G-actions and G-gradingss are the same for $Z^\star_G(\mathcal{C})$ and $Z_G(\mathcal{C})^\bullet$. To see that H is braided, we compute

$$\begin{aligned}
&H[C^\star_{(U,q,\Phi_U),(V,r,\Phi_V)}] \circ \delta_{(U,q,\Phi_U),(V,r,\Phi_V)} \\
&= C^\star_{(U,q,\Phi_U),(V,r,\Phi_V)} \circ \delta_{(U,q,\Phi_U),(V,r,\Phi_V)} = c^{-1}_{F_{q^{-1}}(V),U} \circ \Phi_U(F_{q^{-1}}(V))^{-1} \circ c_{U,V} \circ c^{-1}_{U,V} \circ \Phi_U(F_{q^{-1}}(V)) \\
&= c^{-1}_{F_{q^{-1}}(V),U} = c^{-1}_{F_{q^{-1}}(V),U} \circ \mathcal{F}_{q^{-1}}\Phi_V(F_{q^{-1}r^{-1}q}(U)) \circ \mathcal{F}_{q^{-1}}\Phi_V(F_{q^{-1}r^{-1}q}(U))^{-1} \\
&= \delta_{\mathcal{F}_{q^{-1}}[(V,r,\Phi_V)],(U,q,\Phi_U)} \circ C^\bullet_{(U,q,\Phi_U),(V,r,\Phi_V)}.
\end{aligned}$$

□

Step 3: The functor $\dagger : Z^\star_G(\mathcal{C}) \to Z_G(\mathcal{C})$

In the next lemma we will use the braiding c to assign to each half q-braiding Φ_V for $V \in \mathcal{C}$ a half q^{-1}-braiding $\Phi^\dagger_V$. This will be used later to define a functor $Z^\star_G(\mathcal{C}) \to Z_G(\mathcal{C})$.

Lemma 4.10.5 *If $(V,q,\Phi_V) \in Z_G(\mathcal{C})$, then for each $X \in \mathcal{C}$ we define $\Phi^\dagger_V(X) : V \otimes X \to F_{q^{-1}}(X) \otimes V$ by*

$$\Phi^\dagger_V(X) := c^{-1}_{F_{q^{-1}}(X),V} \circ \Phi_V(F_{q^{-1}}(X))^{-1} \circ c_{V,X}. \tag{4.10.1}$$

(1) If $(V, q, \Phi_V) \in Z_G(\mathcal{C})$, then $\Phi_V^\dagger$ is a half q^{-1}-braiding for V. As a special case, we have the equations $c_{V,-}^\dagger = \widetilde{c}_{V,-}$ and $\widetilde{c}_{V,-}^\dagger = c_{V,-}$ of half e-braidings for V.

(2) If $(V, q, \Phi_V), (W, r, \Phi_W) \in Z_G(\mathcal{C})$, then

$$\mathrm{Hom}_{Z_G(\mathcal{C})}[(V, q^{-1}, \Phi_V^\dagger), (W, r^{-1}, \Phi_W^\dagger)] = \mathrm{Hom}_{Z_G(\mathcal{C})}[(V, q, \Phi_V), (W, r, \Phi_W)].$$

(3) For the unit object $(I, e, \Phi_I^0) \in Z_G(\mathcal{C})$ we have

$$(\Phi_I^0)^\dagger = \Phi_I^0.$$

Also, if $(V, q, \Phi_V), (W, r, \Phi_W) \in Z_G(\mathcal{C})$, then we have the relation

$$\Phi_V^\dagger \otimes \Phi_W^\dagger = (\Phi_V \star \Phi_W)^\dagger.$$

(4) If $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ and if $s \in G$, then

$$\mathcal{F}_s \Phi_V^\dagger = (\mathcal{F}_s \Phi_V)^\dagger.$$

Proof. (1) The naturality of $\Phi_V^\dagger$ follows directly from the naturality of c and Φ_V. To see that $\Phi_V^\dagger$ is indeed a half q^{-1}-braiding, we first compute

$$\begin{aligned}
&[c^{-1}_{F_{q^{-1}}(X),V} \otimes \mathrm{id}_{F_{q^{-1}}(Y)}] \circ \Phi_V(F_{q^{-1}}(X \otimes Y))^{-1} \circ [\mathrm{id}_X \otimes c_{V,Y}] \\
&= [c^{-1}_{F_{q^{-1}}(X),V} \otimes \mathrm{id}_{F_{q^{-1}}(Y)}] \circ [\mathrm{id}_V \otimes F_{q^{-1}}(c^{-1}_{X,Y})] \circ \Phi_V(F_{q^{-1}}(X \otimes Y))^{-1} \circ [c_{X,Y} \otimes \mathrm{id}_V] \circ [\mathrm{id}_X \otimes c_{V,Y}] \\
&= c^{-1}_{F_{q^{-1}}(X),V \otimes F_{q^{-1}}(Y)} \circ [\Phi_V(F_{q^{-1}}(Y))^{-1} \otimes \mathrm{id}_{F_{q^{-1}}(X)}] \circ [\mathrm{id}_Y \otimes \Phi_V(F_{q^{-1}}(X))^{-1}] \circ c_{X \otimes V,Y} \\
&= [\mathrm{id}_{F_{q^{-1}}(X)} \otimes \Phi_V(F_{q^{-1}}(Y))^{-1}] \circ c^{-1}_{F_{q^{-1}}(X),Y \otimes V} \circ c_{V \otimes F_{q^{-1}}(X),Y} \circ [\Phi_V(F_{q^{-1}}(X))^{-1} \otimes \mathrm{id}_Y] \\
&= [\mathrm{id}_{F_{q^{-1}}(X)} \otimes \Phi_V(F_{q^{-1}}(Y))^{-1}] \circ [\mathrm{id}_{F_{q^{-1}}(X)} \otimes c_{V,Y}] \circ [c^{-1}_{F_{q^{-1}}(X),V} \otimes \mathrm{id}_Y] \circ [\Phi_V(F_{q^{-1}}(X))^{-1} \otimes \mathrm{id}_Y].
\end{aligned}$$

Using this equation, we get

$$\begin{aligned}
\Phi_V^\dagger(X \otimes Y) &= c^{-1}_{F_{q^{-1}}(X \otimes Y),V} \circ \Phi_V(F_{q^{-1}}(X \otimes Y))^{-1} \circ c_{V,X \otimes Y} \\
&= [\mathrm{id}_{F_{q^{-1}}(X)} \otimes c^{-1}_{F_{q^{-1}}(Y),V}] \circ [c^{-1}_{F_{q^{-1}}(X),V} \otimes \mathrm{id}_{F_{q^{-1}}(Y)}] \circ \Phi_V(F_{q^{-1}}(X \otimes Y))^{-1} \\
&\quad \circ [\mathrm{id}_X \otimes c_{V,Y}] \circ [c_{V,X} \otimes \mathrm{id}_Y] \\
&= [\mathrm{id}_{F_{q^{-1}}(X)} \otimes c^{-1}_{F_{q^{-1}}(Y),V}] \circ [\mathrm{id}_{F_{q^{-1}}(X)} \otimes \Phi_V(F_{q^{-1}}(Y))^{-1}] \circ [\mathrm{id}_{F_{q^{-1}}(X)} \otimes c_{V,Y}] \\
&\quad \circ [c^{-1}_{F_{q^{-1}}(X),V} \otimes \mathrm{id}_Y] \circ [\Phi_V(F_{q^{-1}}(X))^{-1} \otimes \mathrm{id}_Y] \circ [c_{V,X} \otimes \mathrm{id}_Y] \\
&= [\mathrm{id}_{F_{q^{-1}}(X)} \otimes \Phi_V^\dagger(Y)] \circ [\Phi_V^\dagger(X) \otimes \mathrm{id}_Y],
\end{aligned}$$

showing that $\Phi_V^\dagger$ is indeed a half q^{-1}-braiding for V.

(2) Let $f \in \mathrm{Hom}_{\mathcal{C}}(V, W)$. Then $f \in \mathrm{Hom}_{Z_G(\mathcal{C})}[(V, q, \Phi_V), (W, r, \Phi_W)]$ if and only if

$$[\mathrm{id}_{F_q(X)} \otimes f] \circ \Phi_V(X) = \Phi_W(X) \circ [f \otimes \mathrm{id}_X] \qquad \forall X \in \mathcal{C}$$

if and only if

$$[f \otimes \mathrm{id}_{F_{q^{-1}}(X)}] \circ \Phi_V(F_{q^{-1}}(X))^{-1} = \Phi_W(F_{q^{-1}}(X))^{-1} \circ [\mathrm{id}_X \otimes f] \qquad \forall X \in \mathcal{C}$$

if and only if

$$c^{-1}_{F_{q^{-1}}(X),W} \circ [f \otimes \mathrm{id}_{F_{q^{-1}}(X)}] \circ \Phi_V(F_{q^{-1}}(X))^{-1} \circ c_{V,X}$$

$$= c^{-1}_{F_{q^{-1}}(X),W} \circ \Phi_W(F_{q^{-1}}(X))^{-1} \circ [\mathrm{id}_X \otimes f] \circ c_{V,X} \qquad \forall X \in \mathcal{C}.$$

But, by using naturality of c, this last equation can be rewritten as

$$[\mathrm{id}_{F_{q^{-1}}(X)} \otimes f] \circ \Phi^\dagger_V(X) = \Phi^\dagger_W(X) \circ [f \otimes \mathrm{id}_X] \qquad \forall X \in \mathcal{C}.$$

(3) The invariance of Φ^0_I under $\dagger$ is trivial. The other equality follows from the computation

$$\begin{aligned}(\Phi^\dagger_V \otimes \Phi^\dagger_W)(X) &= [\Phi^\dagger_V(F_{r^{-1}}(X)) \otimes \mathrm{id}_W] \circ [\mathrm{id}_V \otimes \Phi^\dagger_W(X)] \\ &= [c^{-1}_{F_{q^{-1}r^{-1}}(X),V} \otimes \mathrm{id}_W] \circ [\Phi_V(F_{q^{-1}r^{-1}}(X))^{-1} \otimes \mathrm{id}_W] \circ [c_{V,F_{r^{-1}}(X)} \otimes \mathrm{id}_W] \\ &\quad \circ [\mathrm{id}_V \otimes c^{-1}_{F_{r^{-1}}(X),W}] \circ [\mathrm{id}_V \otimes \Phi_W(F_{r^{-1}}(X))^{-1}] \circ [\mathrm{id}_V \otimes c_{W,X}] \\ &= c^{-1}_{F_{q^{-1}r^{-1}}(X)\otimes V,W} \circ [\mathrm{id}_W \otimes c^{-1}_{F_{q^{-1}r^{-1}}(X),V}] \circ [\mathrm{id}_W \otimes \Phi_V(F_{q^{-1}r^{-1}}(X))^{-1}] \\ &\quad \circ c_{V,W\otimes F_{r^{-1}}(X)} \circ [\mathrm{id}_V \otimes \Phi_W(F_{r^{-1}}(X))^{-1}] \circ [\mathrm{id}_V \otimes c_{W,X}] \\ &= c^{-1}_{F_{q^{-1}r^{-1}}(X),V\otimes W} \circ [c^{-1}_{V,W} \otimes \mathrm{id}_{F_{q^{-1}r^{-1}}(X)}] \circ [\mathrm{id}_W \otimes \Phi_V(F_{q^{-1}r^{-1}}(X))^{-1}] \\ &\quad \circ [\Phi_W(F_{r^{-1}}(X))^{-1} \otimes \mathrm{id}_V] \circ [\mathrm{id}_X \otimes c_{V,W}] \circ c_{V\otimes W,X} \\ &= c^{-1}_{F_{q^{-1}r^{-1}}(X),V\otimes W} \circ (\Phi_V \star \Phi_W)(F_{q^{-1}r^{-1}}(X))^{-1} \circ c_{V\otimes W,X} \\ &= (\Phi_V \star \Phi_W)^\dagger(X).\end{aligned}$$

(4) We have

$$\begin{aligned}\mathcal{F}_s\Phi^\dagger_V(X) = F_s[\Phi^\dagger_V(F_{s^{-1}}(X))] &= F_s[c^{-1}_{F_{q^{-1}s^{-1}}(X),V} \circ \Phi_V(F_{q^{-1}s^{-1}}(X)) \circ c_{V,F_{s^{-1}}(X)}] \\ &= c^{-1}_{F_{sq^{-1}s^{-1}}(X),F_s(V)} \circ F_s(\Phi_V(F_{s^{-1}}(F_{sq^{-1}s^{-1}}(X))))^{-1} \circ c_{F_s(V),X} \\ &= c^{-1}_{F_{(sqs^{-1})^{-1}}(X),F_s(V)} \circ \mathcal{F}_s\Phi_V(F_{(sqs^{-1})^{-1}}(X))^{-1} \circ c_{F_s(V),X} \\ &= (\mathcal{F}_s\Phi_V)^\dagger(X).\end{aligned}$$

□

Using the assignment $\Phi_V \mapsto \Phi^\dagger_V$, we will now define a braided G-crossed equivalence from $Z^\star_G(\mathcal{C})$ to $Z_G(\mathcal{C})$.

Proposition 4.10.6 *We obtain a functor $\dagger : Z_G(\mathcal{C}) \to Z_G(\mathcal{C})$ by defining*

$$(U, q, \Phi_U)^\dagger := (U, q^{-1}, \Phi^\dagger_U),$$
$$f^\dagger := f$$

for $(U, q, \Phi_U) \in Z_G(\mathcal{C})$ and $f \in \mathrm{Hom}(Z_G(\mathcal{C}))$. It is a strict equivalence $\dagger : Z^\star_G(\mathcal{C}) \to Z_G(\mathcal{C})$ of braided G-crossed categories.

Proof. It follows from part (2) of the previous lemma that for $(U, q, \Phi_U), (V, r, \Phi_V) \in Z_G(\mathcal{C})$ we have

$$\mathrm{Hom}_{Z_G(\mathcal{C})}((U, q, \Phi_U), (V, q, \Phi_V)) = \mathrm{Hom}_{Z_G(\mathcal{C})}((U, q, \Phi_U)^\dagger, (V, q, \Phi_V)^\dagger),$$

which directly implies that $\dagger$ is a functor. For the unit object we have $(I, e, \Phi^0_I)^\dagger = (I, e, (\Phi^0_I)^\dagger) = (I, e, \Phi^0_I)$. If $(U, q, \Phi_U), (V, r, \Phi_V) \in Z^\star_G(\mathcal{C})$, then

$$[(U, q, \Phi_U) \star (V, r, \Phi_V)]^\dagger = (U \otimes V, rq, \Phi_U \star \Phi_V)^\dagger = (U \otimes V, (rq)^{-1}, (\Phi_U \star \Phi_V)^\dagger)$$

$$= (U \otimes V, q^{-1}r^{-1}, \Phi_U^\dagger \otimes \Phi_V^\dagger) = (U, q^{-1}, \Phi_U^\dagger) \otimes (V, r^{-1}, \Phi_V^\dagger)$$
$$= (U, q, \Phi_U)^\dagger \otimes (V, r, \Phi_V)^\dagger.$$

On the morphisms we have $(f \star g)^\dagger = f \star g = f \otimes g = f^\dagger \otimes g^\dagger$, so $\dagger$ is a strict tensor functor. If $(U, q, \Phi_U) \in Z_G(\mathcal{C})$ and $r \in G$ then it follows from part (4) of the previous lemma that

$$\begin{aligned}\{\mathcal{F}_r[(U, q, \Phi_U)]\}^\dagger &= (F_r(U), rqr^{-1}, \mathcal{F}_r\Phi_U)^\dagger = (F_r(U), rq^{-1}r^{-1}, (\mathcal{F}_r\Phi_U)^\dagger) \\ &= (F_r(U), rq^{-1}r^{-1}, \mathcal{F}_r(\Phi_U^\dagger)) = \mathcal{F}_r[(U, q^{-1}, \Phi_U^\dagger)] \\ &= \mathcal{F}_r[(U, q, \Phi_U)^\dagger],\end{aligned}$$

showing that $\dagger$ is a functor of G-categories. If $(U, q, \Phi_U) \in Z_G(\mathcal{C})$, then

$$\partial[(U, q, \Phi_U)^\dagger] = \partial[(U, q^{-1}, \Phi_U^\dagger)] = q^{-1} = \partial[(U, q, \Phi_U)]^{-1} = \partial_\star[(U, q, \Phi_U)],$$

so $\dagger$ respects the grading. Finally, if $(U, q, \Phi_U), (V, r, \Phi_V) \in Z_G(\mathcal{C})$ then

$$\begin{aligned}\left(C^\star_{(U,q,\Phi_U),(V,r,\Phi_V)}\right)^\dagger &= C^\star_{(U,q,\Phi_U),(V,r,\Phi_V)} = c^{-1}_{F_{q^{-1}}(V),U} \circ \Phi_U(F_{q^{-1}}(V))^{-1} \circ c_{U,V} = \Phi_U^\dagger(V) \\ &= C_{(U,q^{-1},\Phi_U^\dagger),(V,r^{-1},\Phi_V^\dagger)} = C_{(U,q,\Phi_U)^\dagger,(V,r,\Phi_V)^\dagger}\end{aligned}$$

so $\dagger$ is also braided. We now claim that applying the functor $\dagger$ twice will result in the identity functor. Considering formula (4.10.1) for $\Phi_V^\dagger$, it is clear that $(\Phi_V^\dagger)^\dagger$ will contain monodromies and is therefore not equal to Φ_V. However, one should realize that there are actually two possible definitions of $\Phi_V^\dagger$: one in terms of c (as in formula (4.10.1)) and one in terms of $\widetilde{c}$. One could choose to write $(\dagger, c)$ and $(\dagger, \widetilde{c})$ for these two functors. Since there is no preference for either c or $\widetilde{c}$ in the construction of $Z_G(\mathcal{C})$ (in fact, both are irrelevant), these two definitions of $\Phi_V^\dagger$ should actually be treated on equal footing. It is not difficult to see that $(\dagger, c)$ and $(\dagger, \widetilde{c})$ are inverse to one another. This proves that $\dagger$ is an equivalence (and that $\dagger$ is involutive, in a certain sense).
□

Step 4: The main result on G-extensions of $\mathcal{C}$ inside of $Z_G(\mathcal{C})$

We are now ready to prove the main theorem of this subsection. This theorem states that if $Z_G(\mathcal{C})$ contains a replete full subcategory $\mathcal{D}$ that is a braided G-crossed extension of $\mathcal{C}$, then $Z_G(\mathcal{C})$ also contains a braided G-crossed extension $\widetilde{\mathcal{D}}$ of $\widetilde{\mathcal{C}}$. Furthermore, the braided G-crossed categories $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are the mirror images of one another.

Theorem 4.10.7 *Suppose that $Z_G(\mathcal{C})$ contains a replete full subcategory $\mathcal{D}$ that is a braided G-crossed extension of $\mathcal{C}$.*
(1) $\mathcal{D}$ is also a braided G-crossed category when it is equipped with the structures inherited from $Z_G^\star(\mathcal{C})$. We will denote $\mathcal{D}$, equipped with these structures, by $\mathcal{D}^\star$.
(2) The image $(\mathcal{D}^\star)^\dagger$ of $\mathcal{D}^\star$ under the functor $\dagger : Z_G^\star(\mathcal{C}) \to Z_G(\mathcal{C})$ is a G-crossed extension of $\widetilde{\mathcal{C}}$ with G-spectrum $\partial((\mathcal{D}^\star)^\dagger) = \partial(\mathcal{D})^{-1}$, where all structures on $(\mathcal{D}^\star)^\dagger$ are the ones inherited from $Z_G(\mathcal{C})$.
(3) We have an equivalence $(\mathcal{D}^\star)^\dagger \simeq \mathcal{D}^\star \simeq \mathcal{D}^\bullet$ of braided G-crossed categories.

Proof. (1) Let $(U, q, \Phi_U), (V, r, \Phi_V) \in \mathcal{D}$. Since $\mathcal{D}$ is a tensor subcategory of $Z_G(\mathcal{C})$ we have that $(V, r, \Phi_V) \otimes (U, q, \Phi_U) = (V \otimes U, rq, \Phi_V \otimes \Phi_U) \in \mathcal{D}$. The isomorphism $c_{U,V} \in \mathrm{Hom}_\mathcal{C}(U, V)$ is also an isomorphism in $\mathrm{Hom}_{Z_G(\mathcal{C})}((U, q, \Phi_U) \star (V, r, \Phi_V), (V, r, \Phi_V) \otimes (U, q, \Phi_U))$, since for each $X \in \mathcal{C}$ we have

$$[\mathrm{id}_{F_{rq}(X)} \otimes c_{U,V}] \circ [(\Phi_U \star \Phi_V)(X)]$$

$$= [\mathrm{id}_{F_{rq}(X)} \otimes c_{U,V}] \circ [\mathrm{id}_{F_{rq}(X)} \otimes c_{U,V}^{-1}] \circ [(\Phi_V \otimes \Phi_U)(X)] \circ [c_{U,V} \otimes \mathrm{id}_X]$$
$$= [(\Phi_V \otimes \Phi_U)(X)] \circ [c_{U,V} \otimes \mathrm{id}_X].$$

In particular, the objects $(U, q, \Phi_U) \star (V, r, \Phi_V)$ and $(V, r, \Phi_V) \otimes (U, q, \Phi_U)$ are isomorphic in $Z_G(\mathcal{C})$, so combined with the fact that the latter object is in $\mathcal{D}$ and the fact that $\mathcal{D}$ is replete, we conclude that $(U, q, \Phi_U)\star(V, r, \Phi_V) \in \mathcal{D}$, which implies that $\mathcal{D}$ is a tensor subcategory of $Z_G^\star(\mathcal{C})$. That $\mathcal{D}$ is a G-subcategory of $Z_G^\star(\mathcal{C})$ is now immediate, since the group action of $Z_G^\star(\mathcal{C})$ coincides with the one of $Z_G(\mathcal{C})$ and $\mathcal{D}$ is a G-subcategory of $Z_G(\mathcal{C})$ by assumption. Together with the restriction of $\partial_\star$ to $\mathcal{D}$, $\mathcal{D}$ obviously becomes a G-crossed subcategory of $Z_G^\star(\mathcal{C})$. Since $\mathcal{D}$ is full, it follows that $\mathcal{D}$ is a braided G-crossed subcategory of $Z_G^\star(\mathcal{C})$.

(2) It is clear that $(\mathcal{D}^\star)^\dagger$ is a tensor subcategory of $Z_G(\mathcal{C})$, because it is the image of the tensor subcategory $\mathcal{D}^\star$ of $Z_G^\star(\mathcal{C})$ under the strict tensor functor $\dagger : Z_G^\star(\mathcal{C}) \to Z_G(\mathcal{C})$. Since $\dagger$ is also a strict functor of G-categories, it follows that the group action of $Z_G(\mathcal{C})$ restricts to $(\mathcal{D}^\star)^\dagger$. This completes the proof that $(\mathcal{D}^\star)^\dagger$ is a braided G-crossed subcategory of $Z_G(\mathcal{C})$. By assumption we have $\mathcal{D}_e = \mathcal{C}$, i.e. $\mathcal{D}_e$ consists of all objects of the form $(V, e, c_{V,-})$. This implies that $((\mathcal{D}^\star)^\dagger)_e$ consists of all objects of the form $(V, e^{-1}, c^\dagger_{V,-}) = (V, e, \widetilde{c}_{V,-})$. Hence $((\mathcal{D}^\star)^\dagger)_e = \widetilde{\mathcal{C}}$ and $(\mathcal{D}^\star)^\dagger$ is a braided G-crossed extension of $\widetilde{\mathcal{C}}$.

(3) The first equivalence is just given by the restriction $\dagger : \mathcal{D}^\star \to (\mathcal{D}^\star)^\dagger$ of $\dagger : Z_G^\star(\mathcal{C}) \to Z_G(\mathcal{C})$. The identity functor $\mathrm{id}_{Z_G(\mathcal{C})} : Z_G^\star(\mathcal{C}) \to Z_G(\mathcal{C})^\bullet$ is a functor of braided G-crossed categories. Hence, so is its restriction to $\mathcal{D}^\star$. The image of $\mathcal{D}^\star$ is precisely $\mathcal{D}^\bullet$.
□

By writing $\widetilde{\mathcal{D}} := (\mathcal{D}^\star)^\dagger$ in the theorem above, we thus obtain the following corollary, which is precisely the result we were looking for.

Corollary 4.10.8 *If $Z_G(\mathcal{C})$ contains a braided G-crossed extension $\mathcal{D}$ of $\mathcal{C}$ that is replete, then $Z_G(\mathcal{C})$ also contains a braided G-crossed extension $\widetilde{\mathcal{D}}$ of $\widetilde{\mathcal{C}}$ that is equivalent to the mirror image of $\mathcal{D}$.*

4.10.2 Construction of the category of c-related objects inside of $Z_G(\mathcal{C})$

In this subsection we will demonstrate how to explicitly construct a certain braided G-crossed category inside of $Z_G(\mathcal{C})$. For this we will need to assume the existence of a G-spherical structure on $\mathcal{C}$. The idea behind this construction is as follows. Let $\mathcal{C}$ be G-spherical (in addition to the properties assumed at the beginning of this section), so that $Z_G(\mathcal{C})$ is also automatically G-spherical by Lemma 4.6.3. Now suppose that we already have a braided G-crossed extension $\mathcal{D}$ of $\mathcal{C}$ inside of $Z_G(\mathcal{C})$ that has the property that it is closed under the two-sided duality on $Z_G(\mathcal{C})$ that is induced by the G-spherical structure on $\mathcal{C}$, i.e. if $(V, q, \Phi_V) \in \mathcal{D}$ then also $\overline{(V, q, \Phi_V)} = (\overline{V}, q^{-1}, \overline{\Phi_V}) \in \mathcal{D}$. Then the tensor product $(V, q, \Phi_V) \otimes (\overline{V}, q^{-1}, \overline{\Phi_V}) = (V \otimes \overline{V}, e, \Phi_V \otimes \overline{\Phi_V})$ is an object in $\mathcal{D}_e$, because $\mathcal{D}$ is a tensor category. Since $\mathcal{D}$ is a G-extension of $\mathcal{C}$, this means that we must have

$$(V \otimes \overline{V}, e, \Phi_V \otimes \overline{\Phi_V}) = (V \otimes \overline{V}, e, c_{V\otimes\overline{V},-}).$$

This observation leads us to the following definition.

Definition 4.10.9 Let $\mathcal{C}$ be an $\mathbb{F}$-linear G-spherical category with braiding c, let $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ and let $(\overline{V}, q^{-1}, \overline{\Phi_V})$ be its dual as given by the G-spherical structure. Then we say that (V, q, Φ_V) is *c-related* if for each $X \in \mathcal{C}$ we have

$$(\Phi_V \otimes \overline{\Phi_V})(X) = c_{V\otimes\overline{V},X}.$$

Similarly, we can also define objects that are $\widetilde{c}$-related.

The condition of being c-related can be rewritten as $[\Phi_V(F_{q^{-1}}(X)) \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_V \otimes \overline{\Phi_V}(X)] = [c_{V,X} \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_V \otimes c_{\overline{V},X}]$, or

$$[c_{V,X}^{-1} \otimes \mathrm{id}_{\overline{V}}] \circ [\Phi_V(F_{q^{-1}}(X)) \otimes \mathrm{id}_{\overline{V}}] = [\mathrm{id}_V \otimes c_{\overline{V},X}] \circ [\mathrm{id}_V \otimes \overline{\Phi_V}(X)^{-1}]. \tag{4.10.2}$$

Using the fact that (see the proof of Lemma 4.1.8 for the second equation)

$$c_{\overline{V},X} = [d_V \otimes \mathrm{id}_{X\otimes\overline{V}}] \circ [\mathrm{id}_{\overline{V}} \otimes c^{-1}_{V,X} \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_{\overline{V}\otimes X} \otimes b_V],$$
$$\overline{\Phi_V}(X)^{-1} = [\mathrm{id}_{\overline{V}\otimes X} \otimes d'_V] \circ [\mathrm{id}_{\overline{V}} \otimes \Phi_V(F_{q^{-1}}(X)) \otimes \mathrm{id}_{\overline{V}}] \circ [b'_V \otimes \mathrm{id}_{F_{q^{-1}}(X)\otimes\overline{V}}],$$

we can rewrite (4.10.2) as

$$\begin{aligned}
&[c^{-1}_{V,X} \otimes \mathrm{id}_{\overline{V}}] \circ [\Phi_V(F_{q^{-1}}(X)) \otimes \mathrm{id}_{\overline{V}}] \\
&= \mathrm{id}_V \otimes \mathsf{L}^{(V)}_{F_{q^{-1}}(X)\otimes\overline{V}, X\otimes\overline{V}} \left\{[c^{-1}_{V,X} \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_X \otimes (b_V \circ d'_V)] \circ [\Phi_V(F_{q^{-1}}(X)) \otimes \mathrm{id}_{\overline{V}}]\right\} \\
&=: \mathrm{id}_V \otimes N_{(V,q,\Phi_V)}(X), \qquad (4.10.3)
\end{aligned}$$

where we have introduced the shorthand notation $N_{(V,q,\Phi_V)}(X)$ for the morphism $F_{\partial(V)^{-1}}(X)\otimes\overline{V} \to X\otimes\overline{V}$ on the right-hand side. Applying the left inverse of V to (4.10.3), we obtain

$$d(V) \cdot N_{(V,q,\Phi_V)}(X) = \mathsf{L}^{(V)}_{F_{q^{-1}}(X),X}[c^{-1}_{V,X} \circ \Phi_V(F_{q^{-1}}(X))] \otimes \mathrm{id}_{\overline{V}} =: M_{(V,q,\Phi_V)}(X) \otimes \mathrm{id}_{\overline{V}} \qquad (4.10.4)$$

where we have introduced the shorthand notation $M_{(V,q,\Phi_V)}(X)$ for the morphism $F_{\partial(V)^{-1}}(X) \to X$ occuring here. Applying the right inverse of $\overline{V}$ to (4.10.3) we obtain

$$d(V) \cdot c^{-1}_{V,X} \circ \Phi_V(F_{q^{-1}}(X)) = \mathrm{id}_V \otimes \mathsf{R}^{(\overline{V})}_{F_{q^{-1}}(X),X}[N_{(V,q,\Phi_V)}(X)] = \mathrm{id}_V \otimes M_{(V,q,\Phi_V)}(X). \qquad (4.10.5)$$

Using these equations, we will demonstrate in the following two lemmas that the set of c-related objects is closed under tensor products and that the G-action of $Z_G(\mathcal{C})$ restricts to a G-action on the set of c-related objects.

Lemma 4.10.10 *If $(V, q, \Phi_V), (W, r, \Phi_W) \in Z_G(\mathcal{C})$ are both c-related, then so is their tensor product $(V \otimes W, qr, \Phi_V \otimes \Phi_W)$.*

Proof. We first note that

$$\begin{aligned}
&N_{(V\otimes W,qr,\Phi_V\otimes\Phi_W)}(X) \\
&= \mathsf{L}^{(V\otimes W)}_{F_{(qr)^{-1}}(X)\otimes\overline{V\otimes W}, X\otimes\overline{V\otimes W}} \left\{[c^{-1}_{V\otimes W,X} \otimes \mathrm{id}_{\overline{V\otimes W}}] \circ [\mathrm{id}_X \otimes (b_{V\otimes W} \circ d'_{V\otimes W})] \right. \\
&\qquad\qquad \left. \circ\, [(\Phi_V \otimes \Phi_W)(F_{(qr)^{-1}}(X)) \otimes \mathrm{id}_{\overline{V\otimes W}}]\right\} \\
&= \mathsf{L}^{(W)}_{F_{r^{-1}q^{-1}}(X)\otimes\overline{W}\otimes\overline{V}, X\otimes\overline{W}\otimes\overline{V}} \left\{[c^{-1}_{W,X} \otimes \mathrm{id}_{\overline{W}\otimes\overline{V}}] \circ [\mathrm{id}_X \otimes b_W \otimes \mathrm{id}_{\overline{V}}] \circ N_{(V,q,\Phi_V)}(X) \right. \\
&\qquad\qquad \left. \circ\, [\mathrm{id}_{F_{q^{-1}}(X)} \otimes d'_W \otimes \mathrm{id}_{\overline{V}}] \circ [\Phi_W(F_{r^{-1}q^{-1}}(X)) \otimes \mathrm{id}_{\overline{W}\otimes\overline{V}}]\right\} \\
&= d(V)^{-1}\mathsf{L}^{(W)}_{F_{r^{-1}q^{-1}}(X)\otimes\overline{W}\otimes\overline{V}, X\otimes\overline{W}\otimes\overline{V}} \left\{[c^{-1}_{W,X} \otimes \mathrm{id}_{\overline{W}\otimes\overline{V}}] \circ [\mathrm{id}_X \otimes b_W \otimes \mathrm{id}_{\overline{V}}] \circ [M_{(V,q,\Phi_V)} \otimes \mathrm{id}_{\overline{V}}] \right. \\
&\qquad\qquad \left. \circ\, [\mathrm{id}_{F_{q^{-1}}(X)} \otimes d'_W \otimes \mathrm{id}_{\overline{V}}] \circ [\Phi_W(F_{r^{-1}q^{-1}}(X)) \otimes \mathrm{id}_{\overline{W}\otimes\overline{V}}]\right\} \\
&= d(V)^{-1}[M_{(V,q,\Phi_V)} \otimes \mathrm{id}_{\overline{W}\otimes\overline{V}}] \circ [N_{(W,r,\Phi_W)}(F_{q^{-1}}(X)) \otimes \mathrm{id}_{\overline{V}}].
\end{aligned}$$

We will now check that equation (4.10.3) is satisfied for $(V \otimes W, qr, \Phi_V \otimes \Phi_W)$.

$$\begin{aligned}
&\mathrm{id}_{V\otimes W} \otimes N_{(V\otimes W,qr,\Phi_V\otimes\Phi_W)}(X) \\
&= d(V)^{-1}[\mathrm{id}_{V\otimes W} \otimes M_{(V,q,\Phi_V)} \otimes \mathrm{id}_{\overline{W}\otimes\overline{V}}] \circ [\mathrm{id}_{V\otimes W} \otimes N_{(W,r,\Phi_W)}(F_{q^{-1}}(X)) \otimes \mathrm{id}_{\overline{V}}]
\end{aligned}$$

$$= d(V)^{-1}[\mathrm{id}_{V\otimes W} \otimes M_{(V,q,\Phi_V)} \otimes \mathrm{id}_{\overline{W}\otimes\overline{V}}] \circ [\mathrm{id}_V \otimes (c^{-1}_{W,F_{q^{-1}}(X)} \circ \Phi_W(F_{r^{-1}q^{-1}}(X))) \otimes \mathrm{id}_{\overline{W}\otimes\overline{V}}]$$
$$= d(V)^{-1}[\mathrm{id}_V \otimes c^{-1}_{W,X} \otimes \mathrm{id}_{\overline{W}\otimes\overline{V}}] \circ [\mathrm{id}_V \otimes N_{(V,q,\Phi_V)}(X) \otimes \mathrm{id}_{W\otimes\overline{W}\otimes\overline{V}}] \circ [\mathrm{id}_V \otimes \Phi_W(F_{r^{-1}q^{-1}}(X)) \otimes \mathrm{id}_{\overline{W}\otimes\overline{V}}]$$
$$= [(\mathrm{id}_V \otimes c^{-1}_{W,X}) \circ (c^{-1}_{V,X} \otimes \mathrm{id}_W) \circ (\Phi_V(F_{q^{-1}}(X)) \otimes \mathrm{id}_W) \circ (\mathrm{id}_V \otimes \Phi_W(F_{r^{-1}q^{-1}}(X)))] \otimes \mathrm{id}_{\overline{W}\otimes\overline{V}}$$
$$= [c^{-1}_{V\otimes W,X} \circ (\Phi_V \otimes \Phi_W)(F_{r^{-1}q^{-1}}(X))] \otimes \mathrm{id}_{\overline{V\otimes W}}.$$

□

Lemma 4.10.11 *If $q \in G$ and if $(V, r, \Phi_V) \in Z_G(\mathcal{C})$ is c-related, then so is $\mathcal{F}_q[(V, r, \Phi_V)]$.*

Proof. Because $(V, r, \Phi_V) \in Z_G(\mathcal{C})$ is c-related, we have the equality

$$[c^{-1}_{V,X} \otimes \mathrm{id}_{\overline{V}}] \circ [\Phi_V(F_{r^{-1}}(X)) \otimes \mathrm{id}_{\overline{V}}] = [\mathrm{id}_V \otimes c_{\overline{V},X}] \circ [\mathrm{id}_V \otimes \overline{\Phi_V}(X)^{-1}]$$

for all $X \in \mathcal{C}$. Since $\mathcal{C}$ is G-spherical, we have also $\mathcal{F}_q\overline{\Phi_V} = \overline{\mathcal{F}_q\Phi_V}$. Using these facts, we find that

$$\begin{aligned}
[c^{-1}_{F_q(V),X} \otimes \mathrm{id}_{\overline{F_q(V)}}] \circ [\mathcal{F}_q\Phi_V(F_{qr^{-1}q^{-1}}(X)) \otimes \mathrm{id}_{\overline{F_q(V)}}]&\\
&= [c^{-1}_{F_q(V),X} \otimes \mathrm{id}_{\overline{F_q(V)}}] \circ [F_q(\Phi_V(F_{r^{-1}q^{-1}}(X))) \otimes \mathrm{id}_{\overline{F_q(V)}}]\\
&= F_q\left\{[c^{-1}_{V,F_{q^{-1}}(X)} \otimes \mathrm{id}_{\overline{V}}] \circ [\Phi_V(F_{r^{-1}}(F_{q^{-1}}(X))) \otimes \mathrm{id}_{\overline{V}}]\right\}\\
&= F_q\left\{[\mathrm{id}_V \otimes c_{\overline{V},F_{q^{-1}}(X)}] \circ [\mathrm{id}_V \otimes \overline{\Phi_V}(F_{q^{-1}}(X))^{-1}]\right\}\\
&= [\mathrm{id}_{F_q(V)} \otimes c_{F_q(\overline{V}),X}] \circ [\mathrm{id}_{F_q(V)} \otimes F_q(\overline{\Phi_V}(F_{q^{-1}}(X)))^{-1}]\\
&= [\mathrm{id}_{F_q(V)} \otimes c_{\overline{F_q(V)},X}] \circ [\mathrm{id}_{F_q(V)} \otimes \mathcal{F}_q\overline{\Phi_V}(X)^{-1}]\\
&= [\mathrm{id}_{F_q(V)} \otimes c_{\overline{F_q(V)},X}] \circ [\mathrm{id}_{F_q(V)} \otimes \overline{\mathcal{F}_q\Phi_V}(X)^{-1}],
\end{aligned}$$

which is precisely (4.10.2) for $\mathcal{F}_q[(V, r, \Phi_V)]$.
□

Theorem 4.10.12 *The full subcategory $\mathcal{D}$ determined by the c-related objects in $Z_G(\mathcal{C})$ is a G-spherical braided G-crossed subcategory of $Z_G(\mathcal{C})$ and satisfies $\mathcal{D}_e \supset \mathcal{C}$. Similarly, the full subcategory $\widetilde{\mathcal{D}}$ determined by the $\widetilde{c}$-related objects in $Z_G(\mathcal{C})$ is a G-spherical braided G-crossed subcategory of $Z_G(\mathcal{C})$ and satisfies $\widetilde{\mathcal{D}}_e \supset \widetilde{\mathcal{C}}$.*

Proof. The preceding two lemmas, together with the fact that $(I, e, \Phi^0_I) = (I, e, c_{I,-})$ is c-related, show that the full subcategory of c-related objects forms a G-subcategory of $Z_G(\mathcal{C})$. The G-grading restricts to this G-subcategory and gives it the structure of a G-crossed category, and of course the braiding also restricts to a braiding on this G-crossed subcategory. For any $X \in \mathcal{C}$ we have

$$\begin{aligned}
\overline{c_{V,-}}(X) &= [d_V \otimes \mathrm{id}_{F_{e^{-1}}(X)\otimes\overline{V}}] \circ [\mathrm{id}_{\overline{V}} \otimes c_{V,-}(F_{e^{-1}}(X))^{-1} \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_{\overline{V}\otimes X} \otimes b_V]\\
&= [d_V \otimes \mathrm{id}_{X\otimes\overline{V}}] \circ [\mathrm{id}_{\overline{V}} \otimes c^{-1}_{V,X} \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_{\overline{V}\otimes X} \otimes b_V]\\
&= c_{\overline{V},X}
\end{aligned}$$

and hence also

$$\begin{aligned}
(c_{V,-} \otimes \overline{c_{V,-}})(X) &= [c_{V,-}(X) \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_V \otimes \overline{c_{V,-}}(X)] = [c_{V,X} \otimes \mathrm{id}_{\overline{V}}] \circ [\mathrm{id}_V \otimes c_{\overline{V},X}]\\
&= c_{V\otimes\overline{V},X}.
\end{aligned}$$

This shows that $(V, e, c_{V,-})$ is c-related and thus that $\mathcal{D}_e \supset \mathcal{C}$.
□

Now suppose, as a special case of our previous assumptions, that $\mathcal{C}$ is a modular tensor G-category. It then follows from the results in [75] that in this case we have $Z_G(\mathcal{C})_e = Z(\mathcal{C}) \simeq \mathcal{C} \boxtimes \widetilde{\mathcal{C}}$ as braided tensor categories. Here an object $V \in \mathcal{C}$ is identified with $V \boxtimes I$ and an object $W \in \widetilde{\mathcal{C}}$ is identified with $I \boxtimes V$. In this situation an object in $Z_G(\mathcal{C})_e$ is c-related if and only if it is of the form $V \boxtimes I$, i.e. if it is in $\mathcal{C}$. Similarly, an object in $Z_G(\mathcal{C})_e$ is $\widetilde{c}$-related if and only if it is in $\widetilde{\mathcal{C}}$. Hence the categories $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ in the theorem above are braided G-crossed extensions of $\mathcal{C}$ and $\widetilde{\mathcal{C}}$, respectively. This proves the first part of the following corollary.

Corollary 4.10.13 *If $\mathcal{C}$ is a modular tensor G-category, then $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ of Theorem 4.10.12 are G-spherical braided G-crossed extensions of $\mathcal{C}$ and $\widetilde{\mathcal{C}}$, respectively. Furthermore, any G-spherical braided G-crossed extension of $\mathcal{C}$ (respectively $\widetilde{\mathcal{C}}$) inside of $Z_G(\mathcal{C})$ is contained in $\mathcal{D}$ (respectively $\widetilde{\mathcal{D}}$).*

Proof. To prove the second statement, let $\mathcal{D}'$ be a G-spherical braided G-crossed subcategory of $Z_G(\mathcal{C})$ that is a G-extension of $\mathcal{C}$, i.e. $\mathcal{D}'_e = \mathcal{C}$. If $(V, q, \Phi_V) \in \mathcal{D}'$, then also $\overline{(V, q, \Phi_V)} = (\overline{V}, q^{-1}, \overline{\Phi_V}) \in \mathcal{D}'$, because $\mathcal{D}'$ is a G-spherical subcategory of $Z_G(\mathcal{C})$. But then

$$(V, q, \Phi_V) \otimes (\overline{V}, q^{-1}, \overline{\Phi_V}) = (V \otimes \overline{V}, e, \Phi_V \otimes \overline{\Phi_V}) \in \mathcal{D}'_e = \mathcal{C},$$

so we must have $\Phi_V \otimes \overline{\Phi_V} = c_{V \otimes \overline{V},-}$. Thus $(V, q, \Phi_V) \in \mathcal{D}$, which implies that $\mathcal{D}' \subset \mathcal{D}$. The statement about $\widetilde{\mathcal{D}}$ is proven in the same way.
□

Chapter 5

Conclusions and some suggestions for further research

After having presented our results in the preceding chapters, it is time to see where these results have brought us. The main input for our results was the construction of $Z_G(\mathcal{C})$ and we have proved several nice properties of $Z_G(\mathcal{C})$. However, our motivation for constructing $Z_G(\mathcal{C})$ came from the generalization of a specific problem in algebraic quantum field theory, so it seems appropriate to reconsider this problem here and to see what we have learned about it. This is precisely what we will do in the first section of this chapter. In the second section we will briefly mention some alternative approaches to our problem.

5.1 Back to the main problem

Suppose that we are given a completely rational chiral CFT $(H, \mathcal{A}, U, \Omega, V)$ with an action V of a finite group G. For $\zeta \in S^1$ we considered in Subsection 3.2.5 the full embedding

$$\mathrm{Loc}_f^{l/r}(\mathcal{A}_\zeta) \subset G - \mathrm{Loc}_f^{L/R,l/r}(\mathcal{A}_\zeta)$$

of a braided G-category in a braided G-crossed category, where the first is given as the degree e part of the second, and we were initially interested in whether it is possible (at least in certain specific models, such as the one described in Section 1.2) to construct the second from the first by some extension procedure.

In purely categorical terms, forgetting about AQFT altogether, we generalized this problem to the question whether there is some kind of canonical construction to extend a braided G-category $\mathcal{C}$ to braided G-crossed categories $\mathcal{D}_1$ and $\mathcal{D}_2$ with braidings of the first and second kind, respectively, such that $(\mathcal{D}_1)_e = (\mathcal{D}_2)_e = \mathcal{C}$,

$$\begin{array}{ll} & \mathcal{D}_1 \qquad \text{(braiding of the first kind; } (\mathcal{D}_1)_e = \mathcal{C}) \\ & \nearrow \\ \mathcal{C} & \\ & \searrow \\ & \mathcal{D}_2 \qquad \text{(braiding of the second kind; } (\mathcal{D}_2)_e = \mathcal{C}). \end{array}$$

As a possible candidate we constructed $Z_G(\mathcal{C})$, which is a generalization of the ordinary Drinfeld center $Z(\mathcal{C})$ that satisfies $Z_G(\mathcal{C})_e = Z(\mathcal{C})$. However, the implication of this last equality is not quite what we wanted. Namely, in the case where $\mathcal{C}$ is a modular tensor category it was shown in [75] that $Z(\mathcal{C}) \simeq \mathcal{C} \boxtimes \widetilde{\mathcal{C}}$

as braided tensor categories, so the requirement that the degree e part $Z_G(\mathcal{C})_e$ equals $\mathcal{C}$ is certainly not satisfied. From now on we will restrict ourselves to the situation where $\mathcal{C}$ is a modular tensor G-category, since this is the situation that is obtained from completely rational chiral CFTs with a G-action. Thus we are in the situation where

$$Z_G(\mathcal{C})_e \simeq \mathcal{C} \boxtimes \widetilde{\mathcal{C}}.$$

Loosely speaking, we have some kind of 'doubling' going on in the degree e part of $Z_G(\mathcal{C})$: it is generated (as braided tensor category) by both $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ and is therefore 'twice as big' as we want.

However, we believe that this doubling in the degree e part might not be so bad at all, in view of the fact that we were actually looking for two braided G-crossed extensions

$$\mathcal{D}_1 \supset \mathcal{C},$$
$$\mathcal{D}_2 \supset \mathcal{C}.$$

If $c^{(2)}$ denotes the braiding (of the second kind) on $\mathcal{D}_2$, then $\widetilde{c}^{(1)}_{V,W} := \left(c^{(2)}_{F_{\partial(V)}(W),V}\right)^{-1}$ defines a braiding of the first kind on this same G-crossed category and this braiding $\widetilde{c}^{(1)}$ of the first kind extends the braided G-category $\widetilde{\mathcal{C}}$. In fact, what we obtain in this way is precisely what we called $\widetilde{\mathcal{D}}_1$ in Subsection 3.2.5. Thus, rather than looking for the braided G-crossed extensions $\mathcal{D}_1$ and $\mathcal{D}_2$, we can, equivalently, search for braided G-crossed extensions

$$\mathcal{D}_1 \supset \mathcal{C},$$
$$\widetilde{\mathcal{D}}_1 \supset \widetilde{\mathcal{C}}.$$

In this sense, the 'doubling' in the degree e part of $Z_G(\mathcal{C})$ might actually agree with the doubling that was inherent to our problem. Namely, perhaps we should look for both $\mathcal{D}_1$ and $\widetilde{\mathcal{D}}_1$ inside of $Z_G(\mathcal{C})$, rather than looking for two independent constructions of $\mathcal{D}_1$ and $\widetilde{\mathcal{D}}_1$. This also makes more sense in view of the fact that not every modular tensor G-category has a braided G-crossed extension with full G-spectrum, whereas $Z(\mathcal{C})$ always has such an extension by Corollary 4.9.6. So although $\mathcal{C}$ might not have any braided G-crossed extensions with full G-spectrum, $Z(\mathcal{C}) \simeq \mathcal{C} \boxtimes \widetilde{\mathcal{C}}$ does have such an extension, and perhaps we can manage to find braided G-crossed extensions of $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ inside of $Z_G(\mathcal{C})$ in certain special cases where braided G-crossed extensions of $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ do exist. In Subsection 4.10.2 we demonstrated how this can be done. In Theorem 4.10.7 we have seen that if $Z_G(\mathcal{C})$ contains braided G-crossed extensions of $\mathcal{C}$, then it also contains braided G-crossed extensions of $\widetilde{\mathcal{C}}$ and these extensions of $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ are equivalent to the mirror image of one another.

Coming back to our initial problem in AQFT, this last fact is very remarkable, especially when considering the chronological order in which we actually obtained these results during the course of this project. This chronological order was as follows. First we investigated the categorical relation between the categories $G - \mathrm{Loc}_f^{(L,r)}(\mathcal{A})$ and $G - \mathrm{Loc}_f^{(R,l)}(\mathcal{A})$ of left and right G-localized endomorphisms in AQFT and we found out that these two categories are equivalent to the mirror image of one another; in fact, this was actually our motivation for defining the notion of a mirror image of a braided G-crossed category. Afterwards we discovered that if $Z_G(\mathcal{C})$ contains a braided G-crossed extension $\mathcal{D}$ of $\mathcal{C}$, then it also contains a braided G-crossed extension $(\mathcal{D}^\star)^\dagger$ of $\widetilde{\mathcal{C}}$. It was only after obtaining this result that we wondered whether $\mathcal{D}$ and $(\mathcal{D}^\star)^\dagger$ happened to be related to each other in the same way as $G - \mathrm{Loc}_f^{(L,r)}(\mathcal{A})$ and $G - \mathrm{Loc}_f^{(R,l)}(\mathcal{A})$ are. This indeed turned out to be the case and is almost too coincidental if one believes that $Z_G(\mathcal{C})$ has no relevance at all to AQFT. Furthermore, we should not forget that we have already seen that in the case $\mathcal{A}$ is holomorphic[1] and has a skeletal braided G-crossed subcategory[2], we have $Z_G(\mathrm{Loc}_f(\mathcal{A}_\zeta)) \simeq G - \mathrm{Loc}_f^{L/R}(\mathcal{A}_\zeta)$ as braided G-crossed categories. Hopefully, more results will be obtained concerning any relations between the G-crossed Drinfeld center and AQFT in the future.

[1] In this case we have $\mathrm{Loc}_f^l(\mathcal{A}_\zeta) = \mathrm{Loc}_f^r(\mathcal{A}_\zeta)$ (which agrees with the statement $\mathcal{C} = \widetilde{\mathcal{C}}$ in the general case) and $Z_G(\mathrm{Loc}_f(\mathcal{A}_\zeta))_e = \mathrm{Loc}_f(\mathcal{A}_\zeta)$ (which agrees with $Z_G(\mathcal{C})_e = \mathcal{C}$ in the general case).

[2] This can be viewed as a cohomological condition, see also [78].

5.2 Another possible approach

Besides our own approach to the main problem of constructing braided G-crossed extensions of modular tensor G-categories, which involved the construction of the G-crossed Drinfeld center $Z_G(\mathcal{C})$, we mention another possible approach that could lead to some results in the future.

The first of these uses a theorem proven in [86] which implies that if $\mathcal{M}$ is an indecomposable (left or right) $\mathcal{D}$-module category with $\mathcal{D}$ a fusion category, then $\mathcal{M} \simeq \mathrm{Mod}_{\mathcal{D}}(\mathsf{A})$ for some algebra $\mathsf{A} = (A, \mu, \eta) \in \mathcal{D}$. Now suppose that $\mathcal{C}$ is a braided G-crossed category. Then for each $q \in G$ the full subcategory $\mathcal{C}_q$ is a $\mathcal{C}_e$-bimodule category, as mentioned in Subsection 2.8.4, where we also proved that the braiding of $\mathcal{C}_e$ allows us to forget about either the left or the right $\mathcal{C}_e$-module category structure of $\mathcal{C}_q$. In case $\mathcal{C}_e$ is a fusion category, the theorem in [86] implies that for each $q \in G$ the full subcategory $\mathcal{C}_q$ is equivalent to $\mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_q)$ for some algebra $\mathsf{A}_q = (A_q, \mu_q, \eta_q) \in \mathcal{C}_e$. This means that the full subcategories $\{\mathcal{C}_q\}_{q\in G}$ are determined up to equivalence by a collection of algebras $\{\mathsf{A}_q\}_{q\in G}$ in $\mathcal{C}_e$. This seems to provide us with a way of reconstructing the braided G-crossed category $\mathcal{C}$ up to equivalence in terms of $\mathcal{C}_e$ alone:

$$(\mathcal{C}_e, \{\mathsf{A}_q\}_{q\in G}) \quad \rightsquigarrow \quad \bigsqcup_{q\in G} \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_q).$$

However, there are quite some difficulties in this approach. In the first place, one needs to know what algebras A_q to choose for each $q \in G$. Secondly, in order to obtain $\mathcal{C}$ as a braided G-crossed category we must also somehow define a tensor product, a G-action and a braiding on $\bigsqcup_{q\in G} \mathrm{Mod}(\mathsf{A}_q)$. We will now show that this second problem can be overcome by imposing some extra conditions on the collection of algebras $\{\mathsf{A}_q\}_{q\in G}$ which will ensure that $\bigsqcup_{q\in G} \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_q)$ can be equipped with the structure of a braided G-crossed category[3]. First observe that if $q, r \in G$ then we can equip the object $A_q \otimes A_r \in \mathcal{C}_e$ with the structure of an algebra by defining its multiplication to be $\mu_q * \mu_r := [\mu_q \otimes \mu_r] \circ [\mathrm{id}_{A_q} \otimes c_{A_r,A_q} \otimes \mathrm{id}_{A_r}]$ and its unit to be $\eta_q \otimes \eta_r$, and we will write

$$\mathsf{A}_q \otimes \mathsf{A}_r := (A_q \otimes A_r, \mu_q * \mu_r, \eta_q \otimes \eta_r).$$

We can also equip the object $F_q(A_r)$ with the structure of an algebra by defining its multiplication to be $F_q(\mu_r)$ and its unit to be $F_q(\eta_r)$, and we will write

$$F_q(\mathsf{A}_r) := (F_q(A_r), F_q(\mu_r), F_q(\eta_r)).$$

Now suppose that for any two group elements $q, r \in G$ we are given an algebra morphism

$$\Delta_{q,r} : \mathsf{A}_{qr} \to \mathsf{A}_q \otimes \mathsf{A}_r$$

that satisfies the condition that

$$\Delta_{q,r} \circ \eta_{qr} = \eta_q \otimes \eta_r \tag{5.2.1}$$

for all $q, r \in G$. For $(V, \pi_V) \in \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_q)$ and $(W, \pi_W) \in \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_r)$ we then define

$$\pi_V \diamond \pi_W : = [\pi_V \otimes \pi_W] \circ [\mathrm{id}_{A_q} \otimes c_{A_r,V} \otimes \mathrm{id}_W] \circ [\Delta_{q,r} \otimes \mathrm{id}_{V\otimes W}].$$

It is easy to check that $(V \otimes W, \pi_V \diamond \pi_W) \in \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_{qr})$ and that we obtain a tensor product $\diamond$ on $\bigsqcup_{q\in G} \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_q)$ by defining

$$(V, \pi_V) \diamond (W, \pi_W) := (V \otimes W, \pi_V \diamond \pi_W)$$

[3] We do not know whether or not every braided G-crossed category for which $\mathcal{C}_e$ is a fusion category arises from such extra conditions, i.e. if $\mathcal{C}$ is a braided G-crossed category with $\mathcal{C}_e$ a fusion category and if $\{\mathsf{A}_q\}_{q\in G}$ is a collection of algebras such that $\mathcal{C}_q \simeq \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_q)$ for all $q \in G$, we do not know whether or not there is always a way to implement this specific set of extra conditions on the collection $\{\mathsf{A}_q\}_{q\in G}$ and, if so, whether or not $\mathcal{C}$ can be reconstructed as a braided G-crossed category (up to equivalence) in this way.

$$f \diamond g := f \otimes g.$$

If the $\Delta_{q,r}$ also satisfy the condition

$$[\Delta_{q,r} \otimes \mathrm{id}_{A_s}] \circ \Delta_{qr,s} = [\mathrm{id}_{A_q} \otimes \Delta_{r,s}] \circ \Delta_{q,rs}, \tag{5.2.2}$$

then $\diamond$ is associative. If there also exists a morphism $\varepsilon \in \mathrm{Hom}_{\mathcal{C}_e}(A_e, I)$ that satisfies

$$[\varepsilon \otimes \mathrm{id}_{A_q}] \circ \Delta_{e,q} = \mathrm{id}_{A_q} = [\mathrm{id}_{A_q} \otimes \varepsilon] \circ \Delta_{q,e} \tag{5.2.3}$$

for all $q \in G$, then $\bigsqcup_{q\in G} \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_q)$ becomes a strict tensor category with tensor product $\diamond$ and unit object $(I, \varepsilon) \in \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_e)$. Note that $(A_e, \Delta_{e,e}, \varepsilon)$ is a coalgebra. To also obtain a G-action, we assume that for any two $q, r \in G$ there exists an isomorphism

$$\varphi^q_r : \mathsf{A}_{qrq^{-1}} \to F_q(\mathsf{A}_r)$$

of algebras. If $q, r \in G$ and $(V, \pi_V) \in \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_r)$, we define

$$\mathcal{F}_q\pi_V = F_q(\pi_V) \circ [\varphi^q_r \otimes \mathrm{id}_{F_q(V)}].$$

It is straightforward to check that $(F_q(V), \mathcal{F}_q\pi_V) \in \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_{qrq^{-1}})$. In fact, we obtain a functor $\mathcal{F}_q : \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_r) \to \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_{qrq^{-1}})$ by defining

$$\mathcal{F}_q[(V, \pi_V)] := (F_q(V), \mathcal{F}_q\pi_V),$$
$$\mathcal{F}_q(f) := F_q(f).$$

In case we also impose the conditions

$$\varphi^{qr}_s = F_q(\varphi^r_s) \circ \varphi^q_{rsr^{-1}} \tag{5.2.4}$$

for all $q, r, s \in G$, then these functors will automatically satisfy $\mathcal{F}_{qr} = \mathcal{F}_q \circ \mathcal{F}_r$. Finally, if the $\{\Delta_{q,r}\}$ and $\{\varphi^q_r\}$ are also compatible with each other in the sense that

$$F_q(\Delta_{r,s}) \circ \varphi^q_{rs} = [\varphi^q_r \otimes \varphi^q_s] \circ \Delta_{qrq^{-1},qsq^{-1}} \tag{5.2.5}$$

for all $q, r, s \in G$, then each $\mathcal{F}_q$ becomes a tensor functor and we thus obtain a group action $\mathcal{F}$ on $\bigsqcup_{q\in G} \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_q)$, which makes it into a G-crossed category.

Remark 5.2.1 There is a remarkable similarity between our discussion here and the one in [105]. In section VIII.1 of [105] the notion of a G-coalgebra is defined (in a non-categorical setting) as a collection of vector spaces $\{A_q\}_{q\in G}$ together with linear maps $\Delta_{q,r} : A_{qr} \to A_q \otimes A_r$ and $\varepsilon : A_e \to I$ that satisfy our conditions (5.2.2) and (5.2.3) above. Next a Hopf G-coalgebra is defined as a G-coalgebra for which all the A_q are algebras and for which ε and the $\{\Delta_{q,r}\}$ are algebra homomorphisms and for which (5.2.1) above is satisfied, but the definition also implies the existence of an antipode (which we did not need). A crossed Hopf G-coalgebra is then defined by introducing isomorphisms $\varphi_q : A_r \to A_{qrq^{-1}}$, which is not quite the same as our φ^q_r.

By making some additional assumptions on the underlying category $\mathcal{C}_e$ it is also possible to obtain a braiding for $\bigsqcup_{q\in G} \mathrm{Mod}_{\mathcal{C}_e}(\mathsf{A}_q)$ by introducing morphisms $R_{q,r} : I \to A_q \otimes A_{qrq^{-1}}$ analogous to R-matrices for Hopf algebras, but we will not go into the details here.

Appendix A

Construction of $Z_G(\mathcal{C})$ in the non-strict case

In this appendix we will provide a detailed proof of Theorem 4.3.3, which describes the construction of $Z_G(\mathcal{C})$ in case $\mathcal{C}$ is a non-strict G-category. We will first prove some necessary results about half braidings in Section A.1. Then we will prove the different parts of Theorem 4.3.3 in Sections A.2 through A.5.

A.1 Half braidings in non-strict G-categories

We will first recall the definition of a half q-braiding in a non-strict G-category. In contrast to Definition 4.3.1, we will also include the definition of half q-braidings of the second kind here.

Definition A.1.1 Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category with a G-action $(F, \varepsilon^F, \delta^F)$.

(1) If $V \in \mathcal{C}$ and $q \in G$, then a *half q-braiding (of the first kind) for V* is a natural isomorphism

$$\Phi_V : \otimes \circ (V \times \mathrm{id}_{\mathcal{C}}) \to \otimes \circ (F_q \times V)$$

of functors $\mathcal{C} \to \mathcal{C}$, i.e. a family $\{\Phi_V(X) : V \otimes X \to F_q(X) \otimes V\}_{X \in \mathcal{C}}$ of isomorphisms in $\mathcal{C}$ such that for all $X, Y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the square

$$\begin{array}{ccc} V \otimes X & \xrightarrow{\Phi_V(X)} & F_q(X) \otimes V \\ \downarrow{\scriptstyle \mathrm{id}_V \otimes f} & & \downarrow{\scriptstyle F_q(f) \otimes \mathrm{id}_V} \\ V \otimes Y & \xrightarrow{\Phi_V(Y)} & F_q(Y) \otimes V \end{array}$$

commutes, satisfying the additional property that for all $X, Y \in \mathcal{C}$ we have

$$\Phi_V(X \otimes Y) = [\delta^q_{X,Y} \otimes \mathrm{id}_V] \circ a^{-1}_{F_q(X), F_q(Y), V} \circ [\mathrm{id}_{F_q(X)} \otimes \Phi_V(Y)] \circ a_{F_q(X), V, Y} \circ [\Phi_V(X) \otimes \mathrm{id}_Y] \circ a^{-1}_{V,X,Y}. \quad \text{(A.1.1)}$$

(2) If $V \in \mathcal{C}$ and $q \in G$, then a *half q-braiding of the second kind for V* is a natural isomorphism

$$\Psi_V : \otimes \circ (\mathrm{id}_{\mathcal{C}} \times V) \to \otimes \circ (V \times F_{q^{-1}})$$

of functors $\mathcal{C} \to \mathcal{C}$, i.e. a family $\{\Psi_V(X) : X \otimes V \to V \otimes F_{q^{-1}}(X)\}_{X \in \mathcal{C}}$ of isomorphisms in $\mathcal{C}$ such that for all $X, Y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the square

$$\begin{array}{ccc} X\otimes V & \xrightarrow{\Psi_V(X)} & V\otimes F_{q^{-1}}(X) \\ \downarrow{\scriptstyle f\otimes \mathrm{id}_V} & & \downarrow{\scriptstyle \mathrm{id}_V\otimes F_{q^{-1}}(f)} \\ Y\otimes V & \xrightarrow{\Psi_V(Y)} & V\otimes F_{q^{-1}}(Y) \end{array}$$

commutes, satisfying the additional property that for all $X, Y \in \mathcal{C}$ we have

$$\begin{aligned} \Psi_V(X\otimes Y) = {} & [\mathrm{id}_V \otimes \delta^{q^{-1}}_{X,Y}] \circ a_{V,F_{q^{-1}}(X),F_{q^{-1}}(Y)} \circ [\Psi_V(X)\otimes \mathrm{id}_{F_{q^{-1}}(Y)}] \\ & \circ a^{-1}_{X,V,F_{q^{-1}}(Y)} \circ [\mathrm{id}_X \otimes \Psi_V(Y)] \circ a_{X,Y,V}. \end{aligned} \tag{A.1.2}$$

If $\mathcal{C}$ is a strict G-category, then $\Phi^0_I(X) = \mathrm{id}_X$ defines a half e-braiding for I. The following lemma shows that we can generalize this to the case where $\mathcal{C}$ is a non-strict G-category.

Lemma A.1.2 *Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category with G-action $(F, \varepsilon^F, \delta^F)$. If for each $X \in \mathcal{C}$ we define $\Phi^0_I : I\otimes X \to F_e(X)\otimes I$ by*

$$\Phi^0_I(X) := r^{-1}_{F_e(X)} \circ \varepsilon^F_X \circ l_X,$$

then Φ^0_I defines a half e-braiding for I. Furthermore, we have

$$\begin{aligned} \Phi^0_I(X) &= r^{-1}_{F_e(X)} \circ l_{F_e(X)} \circ [\mathrm{id}_I \otimes \varepsilon^F_X] \\ &= [\varepsilon^F_X \otimes \mathrm{id}_I] \circ r^{-1}_X \circ l_X. \end{aligned}$$

Proof. Naturality follows from commutativity of the diagram

$$\begin{array}{ccccccc} I\otimes X & \xrightarrow{l_X} & X & \xrightarrow{\varepsilon^F_X} & F_e(X) & \xrightarrow{r^{-1}_{F_e(X)}} & F_e(X)\otimes I \\ \downarrow{\scriptstyle \mathrm{id}_I\otimes f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle F_e(f)} & & \downarrow{\scriptstyle F_e(f)\otimes \mathrm{id}_I} \\ I\otimes Y & \xrightarrow{l_Y} & Y & \xrightarrow{\varepsilon^F_Y} & F_e(Y) & \xrightarrow{r^{-1}_{F_e(Y)}} & F_e(Y)\otimes I \end{array}$$

for any $X, Y \in \mathcal{C}$ and $f : X \to Y$. Now let $X, Y \in \mathcal{C}$. Then we can use the compatibility of ε^e with the right unit constraint to rewrite $r_{F_e(X\otimes Y)}$ as

$$\begin{aligned} r_{F_e(X\otimes Y)} &= F_e(r_{X\otimes Y}) \circ \delta^e_{X\otimes Y,I} \circ [\mathrm{id}_{F_e(X\otimes Y)} \otimes \varepsilon^e] \\ &= F_e(\mathrm{id}_X \otimes r_Y) \circ F_e(a_{X,Y,I}) \circ \delta^e_{X\otimes Y,I} \circ [\mathrm{id}_{F_e(X\otimes Y)} \otimes \varepsilon^e] \\ &= \delta^e_{X,Y} \circ [F_e(\mathrm{id}_X) \otimes F_e(r_Y)] \circ (\delta^e_{X,Y\otimes I})^{-1} \circ F_e(a_{X,Y,I}) \circ \delta^e_{X\otimes Y,I} \circ [\mathrm{id}_{F_e(X\otimes Y)} \otimes \varepsilon^e] \\ &= \delta^e_{X,Y} \circ [\mathrm{id}_{F_e(X)} \otimes F_e(r_Y)] \circ [\mathrm{id}_{F_e(X)} \otimes \delta^e_{Y,I}] \circ a_{F_e(X),F_e(Y),F_e(I)} \\ &\quad \circ [(\delta^e_{X,Y})^{-1} \otimes \mathrm{id}_{F_e(I)}] \circ [\mathrm{id}_{F_e(X\otimes Y)} \otimes \varepsilon^e], \end{aligned}$$

where in the second step we used Lemma XI.2.2 of [48], in the third step we used naturality of δ^e and in the fourth step we used the hexagonal diagram (2.8.1) for δ^e. We also have the two equations

$$\begin{aligned} \varepsilon^F_{X\otimes Y} &= \delta^e_{X,Y} \circ [\varepsilon^F_X \otimes \varepsilon^F_Y] \\ l_{X\otimes Y} &= [l_X \otimes \mathrm{id}_Y] \circ a^{-1}_{I,X,Y}. \end{aligned}$$

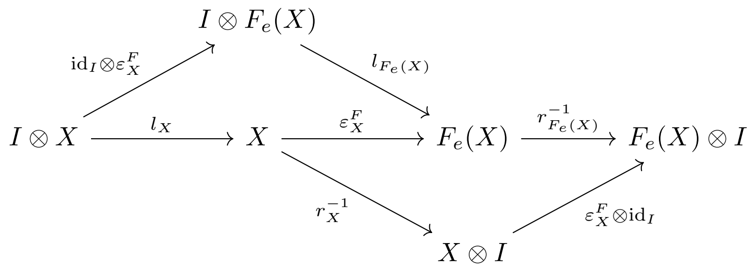
Using these expressions, together with the fact that

$$\mathrm{id}_{F_e(X)\otimes Y} = [\mathrm{id}_{F_e(X)} \otimes l_Y] \circ a_{F_e(X),I,Y} \circ [r^{-1}_{F_e(X)} \otimes \mathrm{id}_Y]$$

by the triangle axiom, we find that

$$
\begin{aligned}
&\Phi_I^0(X\otimes Y)\\
&= r^{-1}_{F_e(X\otimes Y)}\circ \varepsilon^F_{X\otimes Y}\circ l_{X\otimes Y}\\
&= [\mathrm{id}_{F_e(X\otimes Y)}\otimes(\varepsilon^e)^{-1}]\circ[\delta^e_{X,Y}\otimes \mathrm{id}_{F_e(I)}]\circ a^{-1}_{F_e(X),F_e(Y),F_e(I)}\circ[\mathrm{id}_{F_e(X)}\otimes(\delta^e_{Y,I})^{-1}]\\
&\quad\circ[\mathrm{id}_{F_e(X)}\otimes F_e(r_Y)^{-1}]\circ(\delta^e_{X,Y})^{-1}\circ\delta^e_{X,Y}\circ[\varepsilon^F_X\otimes\varepsilon^F_Y]\circ[l_X\otimes\mathrm{id}_Y]\circ a^{-1}_{I,X,Y}\\
&= [\delta^e_{X,Y}\otimes\mathrm{id}_I]\circ a^{-1}_{F_e(X),F_e(Y),I}\circ\underbrace{[\mathrm{id}_{F_e(X)}\otimes(\mathrm{id}_{F_e(Y)}\otimes(\varepsilon^e)^{-1})]\circ[\mathrm{id}_{F_e(X)}\otimes(\delta^e_{Y,I})^{-1}]}_{=[\mathrm{id}_{F_e(X)}\otimes r^{-1}_{F_e(Y)}]\circ[\mathrm{id}_{F_e(X)}\otimes F_e(r_Y)]}\\
&\quad\circ[\mathrm{id}_{F_e(X)}\otimes F_e(r_Y)^{-1}]\circ[\mathrm{id}_{F_e(X)}\otimes\varepsilon^F_Y]\circ\mathrm{id}_{F_e(X)\otimes Y}\circ[\varepsilon^F_X\otimes\mathrm{id}_Y]\circ[l_X\otimes\mathrm{id}_Y]\circ a^{-1}_{I,X,Y}\\
&= [\delta^e_{X,Y}\otimes\mathrm{id}_I]\circ a^{-1}_{F_e(X),F_e(Y),I}\circ[\mathrm{id}_{F_e(X)}\otimes r^{-1}_{F_e(Y)}]\circ[\mathrm{id}_{F_e(X)}\otimes\varepsilon^F_Y]\circ[\mathrm{id}_{F_e(X)}\otimes l_Y]\\
&\quad\circ a_{F_e(X),I,Y}\circ[r^{-1}_{F_e(X)}\otimes\mathrm{id}_Y]\circ[\varepsilon^F_X\otimes\mathrm{id}_Y]\circ[l_X\otimes\mathrm{id}_Y]\circ a^{-1}_{I,X,Y}\\
&= [\delta^e_{X,Y}\otimes\mathrm{id}_I]\circ a^{-1}_{F_e(X),F_e(Y),I}\circ[\mathrm{id}_{F_e(X)}\otimes\Phi^0_I(Y)]\circ a_{F_e(X),I,Y}\circ[\Phi^0_I(X)\otimes\mathrm{id}_Y]\circ a^{-1}_{I,X,Y}.
\end{aligned}
$$

This proves that Φ_I^0 is a half e-braiding for I. Finally, the equalities follow from the fact that the diagram



commutes by naturality of l and r.
□
In what follows, the following lemma will be useful.

Lemma A.1.3 *Let $(\mathcal{C},\otimes,I,a,l,r)$ be a tensor category with G-action $(F,\varepsilon^F,\delta^F)$.*
(1) For $X\in\mathcal{C}$ and $q,r,s\in G$ we define a morphism $\alpha_{q,r,s}(X):(F_q\circ F_r\circ F_s)(X)\to F_{qrs}(X)$ by

$$\alpha_{q,r,s}(X):=(\delta^F_{qr,s})_X\circ(\delta^F_{q,r})_{F_s(X)}=(\delta^F_{q,rs})_X\circ F_q((\delta^F_{r,s})_X).$$

This defines a natural tensor isomorphism[1]

$$\alpha_{q,r,s}:(F_q\circ F_r\circ F_s,\varepsilon^q\diamond\varepsilon^r\diamond\varepsilon^s,\delta^q\diamond\delta^r\diamond\delta^s)\to(F_{qrs},\varepsilon^{qrs},\delta^{qrs}).$$

As a consequence, for any $X,Y\in\mathcal{C}$ we have

$$
\begin{aligned}
\alpha_{q,r,s}(X\otimes Y)=\delta^{qrs}_{X,Y}\circ[\alpha_{q,r,s}(X)\otimes\alpha_{q,r,s}(Y)]\circ(\delta^q_{F_r(F_s(X)),F_r(F_s(Y))})^{-1}\\
\circ F_q(\delta^r_{F_s(X),F_s(Y)})^{-1}\circ F_q(F_r(\delta^s_{X,Y}))^{-1}.
\end{aligned}
\tag{A.1.3}
$$

(2) For $X\in\mathcal{C}$ and $q\in G$ we define a morphism $\Delta_q(X):(F_q\circ F_{q^{-1}})(X)\to X$ by

$$\Delta_q(X):=(\varepsilon^F_X)^{-1}\circ(\delta^F_{q,q^{-1}})_X.$$

[1]See Subsection 2.2.2 for the definition of the $\diamond$ operation.

This defines a natural tensor isomorphism

$$\Delta_q : (F_q \circ F_{q^{-1}}, \varepsilon^q \diamond \varepsilon^{q^{-1}}, \delta^q \diamond \delta^{q^{-1}}) \to (\mathrm{id}_{\mathcal{C}}, \varepsilon^0, \delta^0),$$

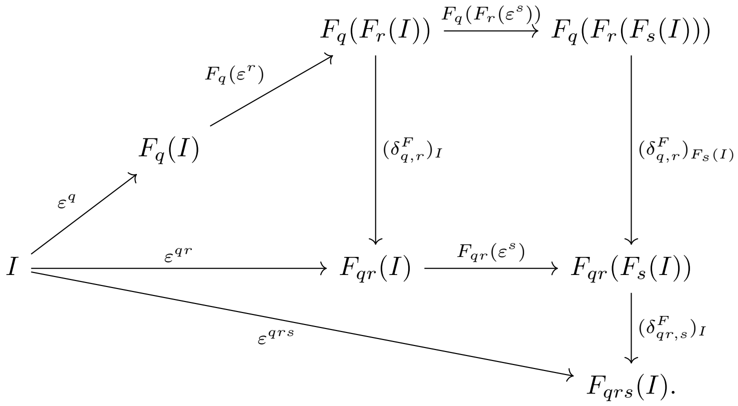
where $\varepsilon^0 = \mathrm{id}_I$ *and* $\delta^0_{X,Y} = \mathrm{id}_{X\otimes Y}$. *As a consequence, for any* $X, Y \in \mathcal{C}$ *we have*

$$\Delta_q(X \otimes Y) = [\Delta_q(X) \otimes \Delta_q(Y)] \circ (\delta^q_{F_{q^{-1}}(X), F_{q^{-1}}(Y)})^{-1} \circ F_q(\delta^{q^{-1}}_{X,Y})^{-1}. \tag{A.1.4}$$

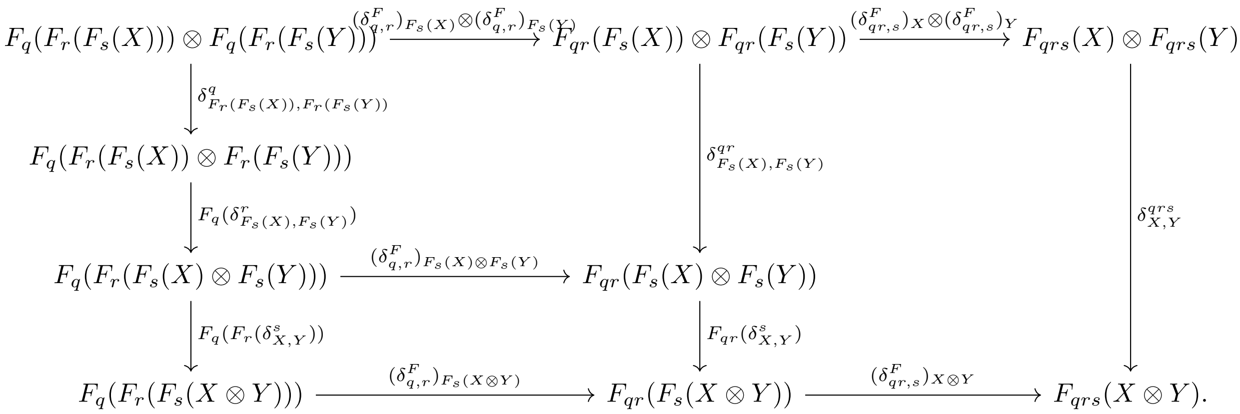
(3) For all $q \in G$ *and* $X \in \mathcal{C}$ *we have the equality*

$$\Delta_q(F_q(X)) = \alpha_{q,q^{-1},q}(X) = F_q(\Delta_{q^{-1}}(X)). \tag{A.1.5}$$

Proof. (1) The naturality of $\alpha_{q,r,s}$ follows from the naturality of δ^F. Now let $X, Y \in \mathcal{C}$ and consider the two diagrams



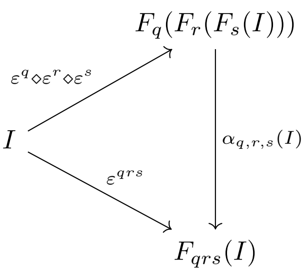
and



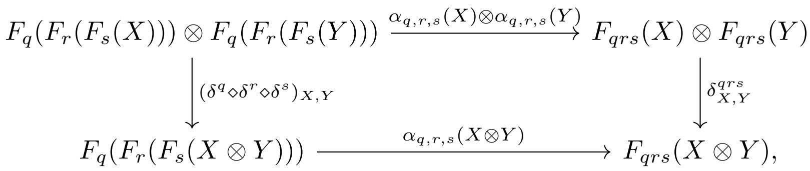
For both diagrams it is easy to see that all their subdiagrams commute, so in both cases the big outer diagram commutes as well. Using the equations

$$\varepsilon^q \diamond \varepsilon^r \diamond \varepsilon^s = F_q(F_r(\varepsilon^s)) \circ F_q(\varepsilon^r) \circ \varepsilon^q$$
$$(\delta^q \diamond \delta^r \diamond \delta^r)_{X,Y} = F_q(F_r(\delta^s_{X,Y})) \circ F_q(\delta^r_{F_s(X),F_s(Y)}) \circ \delta^q_{F_r(F_s(X)),F_r(F_s(Y))},$$

these outer diagrams can be rewritten as



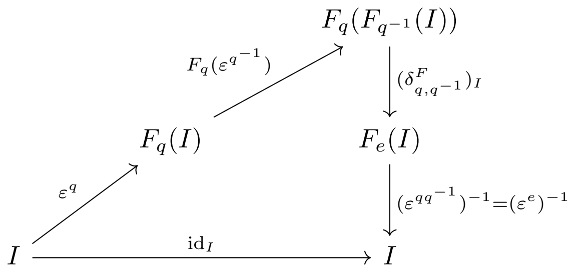
and



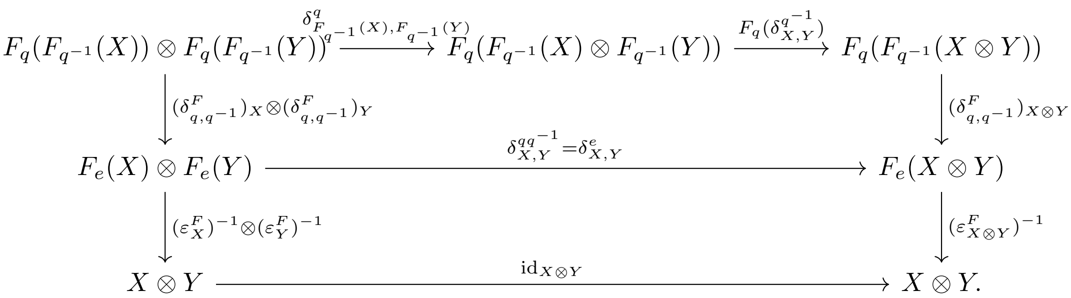
respectively. These show that $\alpha_{q,r,s}$ is a natural tensor isomorphism. The latter diagram also gives us

$$\begin{aligned} \alpha_{q,r,s}(X \otimes Y) &= \delta^{qrs}_{X,Y} \circ [\alpha_{q,r,s}(X) \otimes \alpha_{q,r,s}(Y)] \circ (\delta^q \diamond \delta^r \diamond \delta^s)^{-1}_{X,Y} \\ &= \delta^{qrs}_{X,Y} \circ [\alpha_{q,r,s}(X) \otimes \alpha_{q,r,s}(Y)] \circ (\delta^q_{F_r(F_s(X)),F_r(F_s(Y))})^{-1} \\ &\quad \circ F_q(\delta^r_{F_s(X),F_s(Y)})^{-1} \circ F_q(F_r(\delta^s_{X,Y}))^{-1}. \end{aligned}$$

(2) The naturality of Δ_q follows from the naturality of ε^F and δ^F. Now let $X, Y \in \mathcal{C}$ and consider the two diagrams



and



For both diagrams it is easy to see that all their subdiagrams commute, so in both cases the big outer diagram commutes as well. Using that $\varepsilon^e = \varepsilon^F_I$, these outer diagrams can be written as

$$\begin{array}{ccc} & & F_q(F_{q^{-1}}(I)) \\ & \overset{\varepsilon^q \diamond \varepsilon^{q^{-1}}}{\nearrow} & \\ I & & \Big\downarrow \Delta_q(I) \\ & \underset{\varepsilon^0}{\searrow} & \\ & & I \end{array}$$

and

$$\begin{array}{ccc} F_q(F_{q^{-1}}(X)) \otimes F_q(F_{q^{-1}}(Y)) & \xrightarrow{(\delta^q \diamond \delta^{q^{-1}})_{X,Y}} & F_q(F_{q^{-1}}(X \otimes Y)) \\ \Big\downarrow \Delta_q(X)\otimes\Delta_q(Y) & & \Big\downarrow \Delta_q(X\otimes Y) \\ X \otimes Y & \xrightarrow{\delta^0_{X,Y}} & X \otimes Y. \end{array}$$

The latter diagram also gives

$$\begin{aligned} \Delta_q(X \otimes Y) &= [\Delta_q(X) \otimes \Delta_q(Y)] \circ (\delta^q \diamond \delta^{q^{-1}})^{-1}_{X,Y} \\ &= [\Delta_q(X) \otimes \Delta_q(Y)] \circ (\delta^q_{F_{q^{-1}}(X), F_{q^{-1}}(Y)})^{-1} \circ F_q(\delta^{q^{-1}}_{X,Y})^{-1}. \end{aligned}$$

(3) If $X \in \mathcal{C}$, then

$$\begin{aligned} \Delta_q(F_q(X)) &= (\varepsilon^F_{F_q(X)})^{-1} \circ (\delta^F_{q,q^{-1}})_{F_q(X)} = (\delta^F_{e,q})_X \circ (\delta^F_{q,q^{-1}})_{F_q(X)} \\ &= \alpha_{q,q^{-1},q}(X) \\ &= (\delta^F_{q,e})_X \circ F_q((\delta^F_{q^{-1},q})_X) = F_q((\varepsilon^F_X)^{-1}) \circ F_q((\delta^F_{q^{-1},q})_X) \\ &= F_q(\Delta_{q^{-1}}(X)). \end{aligned}$$

□

In what follows, we will sometimes encounter expressions of the form $\Phi_V(F_r(X \otimes Y))$ that have to be written out. For that reason, the following lemma will be convenient.

Lemma A.1.4 *Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category with G-action $(F, \varepsilon^F, \delta^F)$. If Φ_V is a half q-braiding for $V \in \mathcal{C}$ and if $r \in G$, then for any $X, Y \in \mathcal{C}$ we have*

$$\begin{aligned} &\Phi_V(F_r(X \otimes Y)) \\ &= [F_q(\delta^r_{X,Y}) \otimes \mathrm{id}_V] \circ \Phi_V(F_r(X) \otimes F_r(Y)) \circ [\mathrm{id}_V \otimes (\delta^r_{X,Y})^{-1}] \\ &= [F_q(\delta^r_{X,Y}) \otimes \mathrm{id}_V] \circ [\delta^q_{F_r(X),F_r(Y)} \otimes \mathrm{id}_V] \circ a^{-1}_{F_q(F_r(X)),F_q(F_r(Y)),V} \circ [\mathrm{id}_{F_q(F_r(X))} \otimes \Phi_V(F_r(Y))] \\ &\quad \circ a_{F_q(F_r(X)),V,F_r(Y)} \circ [\Phi_V(F_r(X)) \otimes \mathrm{id}_{F_r(Y)}] \circ a^{-1}_{V,F_r(X),F_r(Y)} \circ [\mathrm{id}_V \otimes (\delta^r_{X,Y})^{-1}] \qquad \text{(A.1.6)} \end{aligned}$$

Proof. By naturality of Φ_V the square

$$\begin{array}{ccc} V \otimes (F_r(X) \otimes F_r(Y)) & \xrightarrow{\Phi_V(F_r(X)\otimes F_r(Y))} & F_q(F_r(X) \otimes F_r(Y)) \otimes V \\ \Big\downarrow \mathrm{id}_V\otimes\delta^r_{X,Y} & & \Big\downarrow F_q(\delta^r_{X,Y})\otimes\mathrm{id}_V \\ V \otimes F_r(X \otimes Y) & \xrightarrow{\Phi_V(F_r(X\otimes Y))} & F_q(F_r(X \otimes Y)) \otimes V \end{array}$$

commutes. But this is equivalent to the first equality that we have to prove. The second follows directly from the fact that Φ_V satisfies (A.1.1).
□

The following proposition demonstrates that there is a one-to-one correspondence between half braidings of the first kind and half braidings of the second kind.

Proposition A.1.5 *Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category with G-action $(F, \varepsilon^F, \delta^F)$ and let $V \in \mathcal{C}$.*

(1) If Φ_V is a half q-braiding for V, then

$$\Psi_V(X) := \Phi_V(F_{q^{-1}}(X))^{-1} \circ [\Delta_q(X)^{-1} \otimes \mathrm{id}_V]$$

defines a half q-braiding of the second kind for V.

(2) If Ψ_V is a half q-braiding of the second kind for V, then

$$\Phi_V(X) := \Psi_V(F_q(X))^{-1} \circ [\mathrm{id}_V \otimes \Delta_q(X)^{-1}]$$

defines a half q-braiding for V.

(3) The assignment $\Phi_V \mapsto \Psi_V$ in part (1) and the assignment $\Psi_V \mapsto \Phi_V$ in part (2) are inverse to one another.

Proof. (1) The naturality of Ψ_V follows directly from the naturality of Δ_q and Φ_V, as can be seen by taking $X, Y \in \mathcal{C}$, $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and by considering the diagram

$$\begin{array}{ccccc} X \otimes V & \xrightarrow{\Delta_q(X)^{-1} \otimes \mathrm{id}_V} & F_q(F_{q^{-1}}(X)) \otimes V & \xrightarrow{\Phi_V(F_{q^{-1}}(X))^{-1}} & V \otimes F_{q^{-1}}(X) \\ \downarrow{\scriptstyle f \otimes \mathrm{id}_V} & & \downarrow{\scriptstyle F_q(F_{q^{-1}}(f)) \otimes \mathrm{id}_V} & & \downarrow{\scriptstyle \mathrm{id}_V \otimes F_{q^{-1}}(f)} \\ Y \otimes V & \xrightarrow{\Delta_q(Y)^{-1} \otimes \mathrm{id}_V} & F_q(F_{q^{-1}}(Y)) \otimes V & \xrightarrow{\Phi_V(F_{q^{-1}}(Y))^{-1}} & V \otimes F_{q^{-1}}(Y). \end{array}$$

To prove that Ψ_V satisfies (A.1.2), let $X, Y \in \mathcal{C}$. Equation (A.1.6) allows us to rewrite $\Phi_V(F_{q^{-1}}(X \otimes Y))$ as

$$\begin{aligned} \Phi_V(F_{q^{-1}}(X \otimes Y)) = {} & [F_q((\delta^{q^{-1}}_{X,Y})) \otimes \mathrm{id}_V] \circ [\delta^q_{F_{q^{-1}}(X), F_{q^{-1}}(Y)} \otimes \mathrm{id}_V] \circ a^{-1}_{F_q(F_{q^{-1}}(X)), F_q(F_{q^{-1}}(Y)), V} \\ & \circ [\mathrm{id}_{F_q(F_{q^{-1}}(X))} \otimes \Phi_V(F_{q^{-1}}(Y))] \circ a_{F_q(F_{q^{-1}}(X)), V, F_{q^{-1}}(Y)} \\ & \circ [\Phi_V(F_{q^{-1}}(X)) \otimes \mathrm{id}_{F_{q^{-1}}(Y)}] \circ a^{-1}_{V, F_{q^{-1}}(X), F_{q^{-1}}(Y)} \circ [\mathrm{id}_V \otimes (\delta^{q^{-1}}_{X,Y})^{-1}]. \end{aligned}$$

Together with equation (A.1.4), this gives

$$\begin{aligned} & \Psi_V(X \otimes Y)^{-1} \\ & = [\Delta_q(X \otimes Y) \otimes \mathrm{id}_V] \circ \Phi_V(F_{q^{-1}}(X \otimes Y)) \\ & = [(\Delta_q(X) \otimes \Delta_q(Y)) \otimes \mathrm{id}_V] \circ [(\delta^q_{F_{q^{-1}}(X), F_{q^{-1}}(Y)})^{-1} \otimes \mathrm{id}_V] \circ [F_q(\delta^{q^{-1}}_{X,Y})^{-1} \otimes \mathrm{id}_V] \\ & \quad \circ [F_q((\delta^{q^{-1}}_{X,Y})) \otimes \mathrm{id}_V] \circ [\delta^q_{F_{q^{-1}}(X), F_{q^{-1}}(Y)} \otimes \mathrm{id}_V] \\ & \quad \circ a^{-1}_{F_q(F_{q^{-1}}(X)), F_q(F_{q^{-1}}(Y)), V} \circ [\mathrm{id}_{F_q(F_{q^{-1}}(X))} \otimes \Phi_V(F_{q^{-1}}(Y))] \circ a_{F_q(F_{q^{-1}}(X)), V, F_{q^{-1}}(Y)} \\ & \quad \circ [\Phi_V(F_{q^{-1}}(X)) \otimes \mathrm{id}_{F_{q^{-1}}(Y)}] \circ a^{-1}_{V, F_{q^{-1}}(X), F_{q^{-1}}(Y)} \circ [\mathrm{id}_V \otimes (\delta^{q^{-1}}_{X,Y})^{-1}] \\ & = [(\Delta_q(X) \otimes \Delta_q(Y)) \otimes \mathrm{id}_V] \circ a^{-1}_{F_q(F_{q^{-1}}(X)), F_q(F_{q^{-1}}(Y)), V} \circ [\mathrm{id}_{F_q(F_{q^{-1}}(X))} \otimes \Phi_V(F_{q^{-1}}(Y))] \\ & \quad \circ a_{F_q(F_{q^{-1}}(X)), V, F_{q^{-1}}(Y)} \circ [\Phi_V(F_{q^{-1}}(X)) \otimes \mathrm{id}_{F_{q^{-1}}(Y)}] \circ a^{-1}_{V, F_{q^{-1}}(X), F_{q^{-1}}(Y)} \circ [\mathrm{id}_V \otimes (\delta^{q^{-1}}_{X,Y})^{-1}] \end{aligned}$$

$$
\begin{aligned}
&= a^{-1}_{X,Y,V} \circ [\mathrm{id}_X \otimes (\Delta_q(Y) \otimes \mathrm{id}_V)] \circ [\mathrm{id}_X \otimes \Phi_V(F_{q^{-1}}(Y))] \\
&\quad \circ a_{X,V,F_{q^{-1}}(Y)} \circ [(\Delta_q(X) \otimes \mathrm{id}_V) \otimes \mathrm{id}_{F_{q^{-1}}(Y)}] \circ [\Phi_V(F_{q^{-1}}(X)) \otimes \mathrm{id}_{F_{q^{-1}}(Y)}] \\
&\quad \circ a^{-1}_{V,F_{q^{-1}}(X),F_{q^{-1}}(Y)} \circ [\mathrm{id}_V \otimes (\delta^{q^{-1}}_{X,Y})^{-1}] \\
&= a^{-1}_{X,Y,V} \circ [\mathrm{id}_X \otimes \Psi_V(Y)^{-1}] \circ a_{X,V,F_{q^{-1}}(Y)} \circ [\Psi_V(X)^{-1} \otimes \mathrm{id}_{F_{q^{-1}}(Y)}] \\
&\quad \circ a^{-1}_{V,F_{q^{-1}}(X),F_{q^{-1}}(Y)} \circ [\mathrm{id}_V \otimes (\delta^{q^{-1}}_{X,Y})^{-1}].
\end{aligned}
$$

Taking the inverse on both sides, we see that Ψ_V satisfies (A.1.2).
(3) This follows directly from equation (A.1.5).
□

We will now generalize our results of Subsection 4.1.1 to the case of non-strict G-categories.

Lemma A.1.6 *Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category with G-action $(F, \varepsilon^F, \delta^F)$. Let $V, W \in \mathcal{C}$ and let $q, r \in G$.*

(1) If Φ_V is a half q-braiding for V and Φ_W is a half r-braiding for W, then we obtain a half qr-braiding $\Phi_V \otimes \Phi_W$ for $V \otimes W$ defined by

$$
\begin{aligned}
(\Phi_V \otimes \Phi_W)(X) := {}& [(\delta^F_{q,r})_X \otimes \mathrm{id}_{V\otimes W}] \circ a_{F_q(F_r(X)),V,W} \circ [\Phi_V(F_r(X)) \otimes \mathrm{id}_W] \\
& \circ a^{-1}_{V,F_r(X),W} \circ [\mathrm{id}_V \otimes \Phi_W(X)] \circ a_{V,W,X}.
\end{aligned}
$$

(2) If Φ_V is a half q-braiding for V, then we obtain a half rqr^{-1}-braiding $\mathcal{F}_r\Phi_V$ for $F_r(V)$ by defining

$$
\begin{aligned}
(\mathcal{F}_r\Phi_V)(X) := {}& [\alpha_{r,q,r^{-1}}(X) \otimes \mathrm{id}_{F_r(V)}] \circ (\delta^r_{F_q(F_{r^{-1}}(X)),V})^{-1} \circ F_r(\Phi_V(F_{r^{-1}}(X))) \\
& \circ \delta^r_{V,F_{r^{-1}}(X)} \circ [\mathrm{id}_{F_r(V)} \otimes \Delta_r(X)^{-1}].
\end{aligned}
$$

Proof. (1) The naturality of $\Phi_V \otimes \Phi_W$ is obvious. To prove that $\Phi_V \otimes \Phi_W$ satisfies (A.1.1), let $X, Y \in \mathcal{C}$. Then

$$
\begin{aligned}
&(\Phi_V \otimes \Phi_W)(X \otimes Y) \\
&= [(\delta^F_{q,r})_{X\otimes Y} \otimes \mathrm{id}_{V\otimes W}] \circ a_{F_q(F_r(X\otimes Y)),V,W} \circ [\Phi_V(F_r(X \otimes Y)) \otimes \mathrm{id}_W] \\
&\quad \circ a^{-1}_{V,F_r(X\otimes Y),W} \circ [\mathrm{id}_V \otimes \Phi_W(X \otimes Y)] \circ a_{V,W,X\otimes Y} \\
&= [(\delta^F_{q,r})_{X\otimes Y} \otimes \mathrm{id}_{V\otimes W}] \circ \{a_{F_q(F_r(X\otimes Y)),V,W} \circ [(F_q(\delta^r_{X,Y}) \otimes \mathrm{id}_V) \otimes \mathrm{id}_W] \\
&\quad \circ [(\delta^q_{F_r(X),F_r(Y)} \otimes \mathrm{id}_V) \otimes \mathrm{id}_W]\} \circ [a^{-1}_{F_q(F_r(X)),F_q(F_r(Y)),V} \otimes \mathrm{id}_W] \circ [(\mathrm{id}_{F_q(F_r(X))} \otimes \Phi_V(F_r(Y))) \otimes \mathrm{id}_W] \\
&\quad \circ [a_{F_q(F_r(X)),V,F_r(Y)} \otimes \mathrm{id}_W] \circ [(\Phi_V(F_r(X)) \otimes \mathrm{id}_{F_r(Y)}) \otimes \mathrm{id}_W] \circ \{[a^{-1}_{V,F_r(X),F_r(Y)} \otimes \mathrm{id}_W] \\
&\quad \circ [(\mathrm{id}_V \otimes (\delta^r_{X,Y})^{-1}) \otimes \mathrm{id}_W] \circ a^{-1}_{V,F_r(X\otimes Y),W} \circ [\mathrm{id}_V \otimes (\delta^r_{X,Y} \otimes \mathrm{id}_W)] \\
&\quad \circ [\mathrm{id}_V \otimes a^{-1}_{F_r(X),F_r(Y),W}]\} \circ [\mathrm{id}_V \otimes (\mathrm{id}_{F_r(X)} \otimes \Phi_W(Y))] \circ [\mathrm{id}_V \otimes a_{F_r(X),W,Y}] \\
&\quad \circ [\mathrm{id}_V \otimes (\Phi_W(X) \otimes \mathrm{id}_Y)] \circ [\mathrm{id}_V \otimes a^{-1}_{W,X,Y}] \circ a_{V,W,X\otimes Y} \\
&= \{[(\delta^F_{q,r})_{X\otimes Y} \otimes \mathrm{id}_{V\otimes W}] \circ [F_q(\delta^r_{X,Y}) \otimes \mathrm{id}_{V\otimes W}] \circ [\delta^q_{F_r(X),F_r(Y)} \otimes \mathrm{id}_{V\otimes W}]\} \\
&\quad \circ a_{F_q(F_r(X))\otimes F_q(F_r(Y)),V,W} \circ [a^{-1}_{F_q(F_r(X)),F_q(F_r(Y)),V} \otimes \mathrm{id}_W] \circ [(\mathrm{id}_{F_q(F_r(X))} \otimes \Phi_V(F_r(Y))) \otimes \mathrm{id}_W] \\
&\quad \circ [a_{F_q(F_r(X)),V,F_r(Y)} \otimes \mathrm{id}_W] \circ [(\Phi_V(F_r(X)) \otimes \mathrm{id}_{F_r(Y)}) \otimes \mathrm{id}_W] \circ a^{-1}_{V\otimes F_r(X),F_r(Y),W} \\
&\quad \circ a^{-1}_{V,F_r(X),F_r(Y)\otimes W} \circ [\mathrm{id}_V \otimes (\mathrm{id}_{F_r(X)} \otimes \Phi_W(Y))] \circ \{[\mathrm{id}_V \otimes a_{F_r(X),W,Y}] \\
&\quad \circ [\mathrm{id}_V \otimes (\Phi_W(X) \otimes \mathrm{id}_Y)] \circ [\mathrm{id}_V \otimes a^{-1}_{W,X,Y}] \circ a_{V,W,X\otimes Y}\}
\end{aligned}
$$

$$= [\delta^{qr}_{X,Y} \otimes \mathrm{id}_{V\otimes W}] \circ [((\delta^F_{q,r})_X \otimes (\delta^F_{q,r})_Y) \otimes \mathrm{id}_{V\otimes W}] \circ a_{F_q(F_r(X))\otimes F_q(F_r(Y)),V,W}$$
$$\circ [a^{-1}_{F_q(F_r(X)),F_q(F_r(Y)),V} \otimes \mathrm{id}_W] \circ [(\mathrm{id}_{F_q(F_r(X))} \otimes \Phi_V(F_r(Y))) \otimes \mathrm{id}_W] \circ [a_{F_q(F_r(X)),V,F_r(Y)} \otimes \mathrm{id}_W]$$
$$\circ [(\Phi_V(F_r(X)) \otimes \mathrm{id}_{F_r(Y)}) \otimes \mathrm{id}_W] \circ a^{-1}_{V\otimes F_r(X),F_r(Y),W} \circ \{a^{-1}_{V,F_r(X),F_r(Y)\otimes W}$$
$$\circ [\mathrm{id}_V \otimes (\mathrm{id}_{F_r(X)} \otimes \Phi_W(Y))] \circ a_{V,F_r(X),W\otimes Y}\} \circ a_{V\otimes F_r(X),W,Y}$$
$$\circ [a^{-1}_{V,F_r(X),W} \otimes \mathrm{id}_Y] \circ [(\mathrm{id}_V \otimes \Phi_W(X)) \otimes \mathrm{id}_Y] \circ [a_{V,W,X} \otimes \mathrm{id}_Y]$$
$$\circ a^{-1}_{V\otimes W,X,Y}$$
$$= [\delta^{qr}_{X,Y} \otimes \mathrm{id}_{V\otimes W}] \circ [((\delta^F_{q,r})_X \otimes (\delta^F_{q,r})_Y) \otimes \mathrm{id}_{V\otimes W}] \circ a_{F_q(F_r(X))\otimes F_q(F_r(Y)),V,W}$$
$$\circ [a^{-1}_{F_q(F_r(X)),F_q(F_r(Y)),V} \otimes \mathrm{id}_W] \circ [(\mathrm{id}_{F_q(F_r(X))} \otimes \Phi_V(F_r(Y))) \otimes \mathrm{id}_W] \circ \{[a_{F_q(F_r(X)),V,F_r(Y)} \otimes \mathrm{id}_W]$$
$$\circ [(\Phi_V(F_r(X)) \otimes \mathrm{id}_{F_r(Y)}) \otimes \mathrm{id}_W] \circ a^{-1}_{V\otimes F_r(X),F_r(Y),W} \circ [\mathrm{id}_{V\otimes F_r(X)} \otimes \Phi_W(Y)]$$
$$\circ a_{V\otimes F_r(X),W,Y}\} \circ [a^{-1}_{V,F_r(X),W} \otimes \mathrm{id}_Y] \circ [(\mathrm{id}_V \otimes \Phi_W(X)) \otimes \mathrm{id}_Y]$$
$$\circ [a_{V,W,X} \otimes \mathrm{id}_Y] \circ a^{-1}_{V\otimes W,X,Y}$$
$$= [\delta^{qr}_{X,Y} \otimes \mathrm{id}_{V\otimes W}] \circ [((\delta^F_{q,r})_X \otimes (\delta^F_{q,r})_Y) \otimes \mathrm{id}_{V\otimes W}] \circ \{a_{F_q(F_r(X))\otimes F_q(F_r(Y)),V,W}$$
$$\circ [a^{-1}_{F_q(F_r(X)),F_q(F_r(Y)),V} \otimes \mathrm{id}_W] \circ [(\mathrm{id}_{F_q(F_r(X))} \otimes \Phi_V(F_r(Y))) \otimes \mathrm{id}_W] \circ a^{-1}_{F_q(F_r(X)),V\otimes F_r(Y),W}$$
$$\circ [(\delta^F_{q,r})^{-1}_X \otimes \mathrm{id}_{(V\otimes F_r(Y))\otimes W}]\} \circ [\mathrm{id}_{F_{qr}(X)} \otimes a^{-1}_{V,F_r(Y),W}] \circ [\mathrm{id}_{F_{qr}(X)} \otimes (\mathrm{id}_V \otimes \Phi_W(Y))]$$
$$\circ [\mathrm{id}_{F_{qr}(X)} \otimes a_{V,W,Y}] \circ a_{F_{qr}(X),V\otimes W,Y} \circ \{[((\delta^F_{q,r})_X \otimes \mathrm{id}_{V\otimes W}) \otimes \mathrm{id}_Y]$$
$$\circ [a_{F_q(F_r(X)),V,W} \otimes \mathrm{id}_Y] \circ [(\Phi_V(F_r(X)) \otimes \mathrm{id}_W) \otimes \mathrm{id}_Y] \circ [a^{-1}_{V,F_r(X),W} \otimes \mathrm{id}_Y]$$
$$\circ [(\mathrm{id}_V \otimes \Phi_W(X)) \otimes \mathrm{id}_Y] \circ [a_{V,W,X} \otimes \mathrm{id}_Y]\} \circ a^{-1}_{V\otimes W,X,Y}$$
$$= [\delta^{qr}_{X,Y} \otimes \mathrm{id}_{V\otimes W}] \circ \{[((\delta^F_{q,r})_X \otimes (\delta^F_{q,r})_Y) \otimes \mathrm{id}_{V\otimes W}] \circ [((\delta^F_{q,r})^{-1}_X \otimes \mathrm{id}_{F_q(F_r(Y))}) \otimes \mathrm{id}_{V\otimes W}]\}$$
$$\circ \{a_{F_{qr}(X)\otimes F_q(F_r(Y)),V,W} \circ [a^{-1}_{F_{qr}(X),F_q(F_r(Y)),V} \otimes \mathrm{id}_W] \circ [(\mathrm{id}_{F_{qr}(X)} \otimes \Phi_V(F_r(Y))) \otimes \mathrm{id}_W]$$
$$\circ a^{-1}_{F_{qr}(X),V\otimes F_r(Y),W}\} \circ [\mathrm{id}_{F_{qr}(X)} \otimes a^{-1}_{V,F_r(Y),W}] \circ [\mathrm{id}_{F_{qr}(X)} \otimes (\mathrm{id}_V \otimes \Phi_W(Y))]$$
$$\circ [\mathrm{id}_{F_{qr}(X)} \otimes a_{V,W,Y}] \circ a_{F_{qr}(X),V\otimes W,Y} \circ [(\Phi_V \otimes \Phi_W)(X) \otimes \mathrm{id}_Y]$$
$$\circ a^{-1}_{V\otimes W,X,Y}$$
$$= [\delta^{qr}_{X,Y} \otimes \mathrm{id}_{V\otimes W}] \circ \{[(\mathrm{id}_{F_{qr}(X)} \otimes (\delta^F_{q,r})_Y) \otimes \mathrm{id}_{V\otimes W}] \circ a^{-1}_{F_{qr}(X),F_q(F_r(Y)),V\otimes W}\}$$
$$\circ [\mathrm{id}_{F_{qr}(X)} \otimes a_{F_q(F_r(X)),V,W}] \circ [\mathrm{id}_{F_{qr}(X)} \otimes (\Phi_V(F_r(Y)) \otimes \mathrm{id}_W)] \circ [\mathrm{id}_{F_{qr}(X)} \otimes a^{-1}_{V,F_r(Y),W}]$$
$$\circ [\mathrm{id}_{F_{qr}(X)} \otimes (\mathrm{id}_V \otimes \Phi_W(Y))] \circ [\mathrm{id}_{F_{qr}(X)} \otimes a_{V,W,Y}] \circ a_{F_{qr}(X),V\otimes W,Y}$$
$$\circ [(\Phi_V \otimes \Phi_W)(X) \otimes \mathrm{id}_Y] \circ a^{-1}_{V\otimes W,X,Y}$$
$$= [\delta^{qr}_{X,Y} \otimes \mathrm{id}_{V\otimes W}] \circ a^{-1}_{F_{qr}(X),F_{qr}(Y),V\otimes W} \circ \{[\mathrm{id}_{F_{qr}(X)} \otimes ((\delta^F_{q,r})_Y \otimes \mathrm{id}_{V\otimes W})]$$
$$\circ [\mathrm{id}_{F_{qr}(X)} \otimes a_{F_q(F_r(X)),V,W}] \circ [\mathrm{id}_{F_{qr}(X)} \otimes (\Phi_V(F_r(Y)) \otimes \mathrm{id}_W)] \circ [\mathrm{id}_{F_{qr}(X)} \otimes a^{-1}_{V,F_r(Y),W}]$$
$$\circ [\mathrm{id}_{F_{qr}(X)} \otimes (\mathrm{id}_V \otimes \Phi_W(Y))] \circ [\mathrm{id}_{F_{qr}(X)} \otimes a_{V,W,Y}]\} \circ a_{F_{qr}(X),V\otimes W,Y}$$
$$\circ [(\Phi_V \otimes \Phi_W)(X) \otimes \mathrm{id}_Y] \circ a^{-1}_{V\otimes W,X,Y}$$
$$= [\delta^{qr}_{X,Y} \otimes \mathrm{id}_{V\otimes W}] \circ a^{-1}_{F_{qr}(X),F_{qr}(Y),V\otimes W} \circ [\mathrm{id}_{F_{qr}(X)} \otimes (\Phi_V \otimes \Phi_W)(Y)]$$
$$\circ a_{F_{qr}(X),V\otimes W,Y} \circ [(\Phi_V \otimes \Phi_W)(X) \otimes \mathrm{id}_Y] \circ a^{-1}_{V\otimes W,X,Y},$$

showing that $\Phi_V \otimes \Phi_W$ is indeed a half qr-braiding for $V \otimes W$.
(2) Naturality of $\mathcal{F}_r\Phi_V$ is obvious. We also have

$$(\mathcal{F}_r\Phi_V)(X \otimes Y)$$

$$= [\alpha_{r,q,r^{-1}}(X\otimes Y)\otimes \mathrm{id}_{F_r(V)}]\circ(\delta^r_{F_q(F_{r^{-1}}(X\otimes Y)),V})^{-1}\circ F_r(\Phi_V(F_{r^{-1}}(X\otimes Y)))$$
$$\circ\, \delta^r_{V,F_{r^{-1}}(X\otimes Y)}\circ[\mathrm{id}_{F_r(V)}\otimes \Delta_r(X\otimes Y)^{-1}]$$
$$= [\delta^{rqr^{-1}}_{X,Y}\otimes \mathrm{id}_{F_r(V)}]\circ\{[(\alpha_{r,q,r^{-1}}(X)\otimes\alpha_{r,q,r^{-1}}(Y))\otimes \mathrm{id}_{F_r(V)}]\}$$
$$\circ\,\{[(\delta^r_{F_q(F_{r^{-1}}(X)),F_q(F_{r^{-1}}(Y))})^{-1}\otimes \mathrm{id}_{F_r(V)}]\circ[F_r(\delta^q_{F_{r^{-1}}(X),F_{r^{-1}}(Y)})^{-1}\otimes \mathrm{id}_{F_r(V)}]$$
$$\circ\,[F_r(F_q(\delta^{r^{-1}}_{X,Y}))^{-1}\otimes \mathrm{id}_{F_r(V)}]\circ(\delta^r_{F_q(F_{r^{-1}}(X\otimes Y)),V})^{-1}\circ F_r(F_q(\delta^{r^{-1}}_{X,Y})\otimes \mathrm{id}_V)$$
$$\circ\, F_r(\delta^q_{F_{r^{-1}}(X),F_{r^{-1}}(Y)}\otimes \mathrm{id}_V)\}\circ F_r(a_{F_q(F_{r^{-1}}(X)),F_q(F_{r^{-1}}(Y)),V})^{-1}$$
$$\circ\, F_r(\mathrm{id}_{F_q(F_{r^{-1}}(X))}\otimes\Phi_V(F_{r^{-1}}(Y)))\circ F_r(a_{F_q(F_{r^{-1}}(X)),V,F_{r^{-1}}(Y)})\circ F_r(\Phi_V(F_{r^{-1}}(X))\otimes \mathrm{id}_{F_{r^{-1}}(Y)})$$
$$\circ\, F_r(a_{V,F_{r^{-1}}(X),F_{r^{-1}}(Y)})^{-1}\circ F_r(\mathrm{id}_V\otimes(\delta^{r^{-1}}_{X,Y})^{-1})\circ\delta^r_{V,F_{r^{-1}}(X\otimes Y)}$$
$$\circ\,[\mathrm{id}_{F_r(V)}\otimes F_r(\delta^{r^{-1}}_{X,Y})]\circ[\mathrm{id}_{F_r(V)}\otimes\delta^r_{F_{r^{-1}}(X),F_{r^{-1}}(Y)}]\circ\{[\mathrm{id}_{F_r(V)}\otimes(\Delta_r(X)^{-1}\otimes\Delta_r(Y)^{-1})]\}$$
$$= [\delta^{rqr^{-1}}_{X,Y}\otimes \mathrm{id}_{F_r(V)}]\circ a^{-1}_{F_{rqr^{-1}}(X),F_{rqr^{-1}}(Y),F_r(V)}\circ\{[\alpha_{r,q,r^{-1}}(X)\otimes(\alpha_{r,q,r^{-1}}(Y)\otimes \mathrm{id}_{F_r(V)})]\}$$
$$\circ\,\{a_{F_r(F_q(F_{r^{-1}}(X))),F_r(F_q(F_{r^{-1}}(Y))),F_r(V)}\circ[(\delta^r_{F_q(F_{r^{-1}}(X)),F_q(F_{r^{-1}}(Y))})^{-1}\otimes \mathrm{id}_{F_r(V)}]$$
$$\circ\,(\delta^r_{F_q(F_{r^{-1}}(X))\otimes F_q(F_{r^{-1}}(Y)),V})^{-1}\circ F_r(a_{F_q(F_{r^{-1}}(X)),F_q(F_{r^{-1}}(Y)),V})^{-1}\}$$
$$\circ\, F_r(\mathrm{id}_{F_q(F_{r^{-1}}(X))}\otimes\Phi_V(F_{r^{-1}}(Y)))\circ F_r(a_{F_q(F_{r^{-1}}(X)),V,F_{r^{-1}}(Y)})$$
$$\circ\, F_r(\Phi_V(F_{r^{-1}}(X))\otimes \mathrm{id}_{F_{r^{-1}}(Y)})\circ F_r(a_{V,F_{r^{-1}}(X),F_{r^{-1}}(Y)})^{-1}\circ\{F_r(\mathrm{id}_V\otimes(\delta^{r^{-1}}_{X,Y})^{-1})$$
$$\circ\,\delta^r_{V,F_{r^{-1}}(X\otimes Y)}\circ[\mathrm{id}_{F_r(V)}\otimes F_r(\delta^{r^{-1}}_{X,Y})]\}\circ[\mathrm{id}_{F_r(V)}\otimes\delta^r_{F_{r^{-1}}(X),F_{r^{-1}}(Y)}]$$
$$\circ\, a_{F_r(V),F_r(F_{r^{-1}}(X)),F_r(F_{r^{-1}}(Y))}\circ\{[(\mathrm{id}_{F_r(V)}\otimes\Delta_r(X)^{-1})\otimes\Delta_r(Y)^{-1}]\}\circ a^{-1}_{F_r(V),X,Y}$$
$$= [\delta^{rqr^{-1}}_{X,Y}\otimes \mathrm{id}_{F_r(V)}]\circ a^{-1}_{F_{rqr^{-1}}(X),F_{rqr^{-1}}(Y),F_r(V)}\circ[\mathrm{id}_{F_{rqr^{-1}}(X)}\otimes(\alpha_{r,q,r^{-1}}(Y)\otimes \mathrm{id}_{F_r(V)})]$$
$$\circ\,\{[\alpha_{r,q,r^{-1}}(X)\otimes \mathrm{id}_{F_r(F_q(F_{r^{-1}}(Y)))\otimes F_r(V)}]\circ[\mathrm{id}_{F_r(F_q(F_{r^{-1}}(X)))}\otimes(\delta^r_{F_q(F_{r^{-1}}(Y)),V})^{-1}]\}$$
$$\circ\,\{(\delta^r_{F_q(F_{r^{-1}}(X)),F_q(F_{r^{-1}}(Y))\otimes V})^{-1}\circ F_r(\mathrm{id}_{F_q(F_{r^{-1}}(X))}\otimes\Phi_V(F_{r^{-1}}(Y)))\}\circ F_r(a_{F_q(F_{r^{-1}}(X)),V,F_{r^{-1}}(Y)})$$
$$\circ\,\{F_r(\Phi_V(F_{r^{-1}}(X))\otimes \mathrm{id}_{F_{r^{-1}}(Y)})\}\circ F_r(a_{V,F_{r^{-1}}(X),F_{r^{-1}}(Y)})^{-1}\circ\delta^r_{V,F_{r^{-1}}(X)\otimes F_{r^{-1}}(Y)}$$
$$\circ\,[\mathrm{id}_{F_r(V)}\otimes\delta^r_{F_{r^{-1}}(X),F_{r^{-1}}(Y)}]\circ\{a_{F_r(V),F_r(F_{r^{-1}}(X)),F_r(F_{r^{-1}}(Y))}\circ[\mathrm{id}_{F_r(V)\otimes F_r(F_{r^{-1}}(X))}\otimes\Delta_r(Y)^{-1}]\}$$
$$\circ\,[(\mathrm{id}_{F_r(V)}\otimes\Delta_r(X)^{-1})\otimes \mathrm{id}_Y]\circ a^{-1}_{F_r(V),X,Y}$$
$$= [\delta^{rqr^{-1}}_{X,Y}\otimes \mathrm{id}_{F_r(V)}]\circ a^{-1}_{F_{rqr^{-1}}(X),F_{rqr^{-1}}(Y),F_r(V)}\circ[\mathrm{id}_{F_{rqr^{-1}}(X)}\otimes(\alpha_{r,q,r^{-1}}(Y)\otimes \mathrm{id}_{F_r(V)})]$$
$$\circ\,[\mathrm{id}_{F_{rqr^{-1}}(X)}\otimes(\delta^r_{F_q(F_{r^{-1}}(Y)),V})^{-1}]\circ\{[\alpha_{r,q,r^{-1}}(X)\otimes \mathrm{id}_{F_r(F_q(F_{r^{-1}}(Y))\otimes V)}]$$
$$\circ\,[\mathrm{id}_{F_r(F_q(F_{r^{-1}}(X)))}\otimes F_r(\Phi_V(F_{r^{-1}}(Y)))]\}\circ\{(\delta^r_{F_q(F_{r^{-1}}(X)),V\otimes F_{r^{-1}}(Y)})^{-1}\circ F_r(a_{F_q(F_{r^{-1}}(X)),V,F_{r^{-1}}(Y)})$$
$$\circ\,\delta^r_{F_q(F_{r^{-1}}(X))\otimes V,F_{r^{-1}}(Y)}\}\circ[F_r(\Phi_V(F_{r^{-1}}(X)))\otimes \mathrm{id}_{F_r(F_{r^{-1}}(Y))}]\circ\{(\delta^r_{V\otimes F_{r^{-1}}(X),F_{r^{-1}}(Y)})^{-1}$$
$$\circ\, F_r(a_{V,F_{r^{-1}}(X),F_{r^{-1}}(Y)})^{-1}\circ\delta^r_{V,F_{r^{-1}}(X)\otimes F_{r^{-1}}(Y)}\circ[\mathrm{id}_{F_r(V)}\otimes\delta^r_{F_{r^{-1}}(X),F_{r^{-1}}(Y)}]\}$$
$$\circ\,[\mathrm{id}_{F_r(V)}\otimes(\mathrm{id}_{F_r(F_{r^{-1}}(X))}\otimes\Delta_r(Y)^{-1})]\circ a_{F_r(V),F_r(F_{r^{-1}}(X)),Y}\circ[(\mathrm{id}_{F_r(V)}\otimes\Delta_r(X)^{-1})\otimes \mathrm{id}_Y]$$
$$\circ\, a^{-1}_{F_r(V),X,Y}$$
$$= [\delta^{rqr^{-1}}_{X,Y}\otimes \mathrm{id}_{F_r(V)}]\circ a^{-1}_{F_{rqr^{-1}}(X),F_{rqr^{-1}}(Y),F_r(V)}\circ[\mathrm{id}_{F_{rqr^{-1}}(X)}\otimes(\alpha_{r,q,r^{-1}}(Y)\otimes \mathrm{id}_{F_r(V)})]$$
$$\circ\,[\mathrm{id}_{F_{rqr^{-1}}(X)}\otimes(\delta^r_{F_q(F_{r^{-1}}(Y)),V})^{-1}]\circ[\mathrm{id}_{F_{rqr^{-1}}(X)}\otimes F_r(\Phi_V(F_{r^{-1}}(Y)))]$$
$$\circ\,\{[\alpha_{r,q,r^{-1}}(X)\otimes \mathrm{id}_{F_r(V\otimes F_{r^{-1}}(Y))}]\circ[\mathrm{id}_{F_r(F_q(F_{r^{-1}}(X)))}\otimes\delta^r_{V,F_{r^{-1}}(Y)}]\}$$

$$
\begin{aligned}
&\circ a_{F_r(F_q(F_{r^{-1}}(X))),F_r(V),F_r(F_{r^{-1}}(Y))} \circ [(\delta^r_{F_q(F_{r^{-1}}(X)),V})^{-1} \otimes \mathrm{id}_{F_r(F_{r^{-1}}(Y))}] \\
&\circ [F_r(\Phi_V(F_{r^{-1}}(X))) \otimes \mathrm{id}_{F_r(F_{r^{-1}}(Y))}] \circ \{[\delta^r_{V,F_{r^{-1}}(X)} \otimes \mathrm{id}_{F_r(F_{r^{-1}}(Y))}] \circ a^{-1}_{F_r(V),F_r(F_{r^{-1}}(X)),F_r(F_{r^{-1}}(Y))} \\
&\circ [\mathrm{id}_{F_r(V)} \otimes (\mathrm{id}_{F_r(F_{r^{-1}}(X))} \otimes \Delta_r(Y)^{-1})] \circ a_{F_r(V),F_r(F_{r^{-1}}(X)),Y}\} \circ [(\mathrm{id}_{F_r(V)} \otimes \Delta_r(X)^{-1}) \otimes \mathrm{id}_Y] \\
&\circ a^{-1}_{F_r(V),X,Y} \\
= {}& [\delta^{rqr^{-1}}_{X,Y} \otimes \mathrm{id}_{F_r(V)}] \circ a^{-1}_{F_{rqr^{-1}}(X),F_{rqr^{-1}}(Y),F_r(V)} \circ [\mathrm{id}_{F_{rqr^{-1}}(X)} \otimes (\alpha_{r,q,r^{-1}}(Y) \otimes \mathrm{id}_{F_r(V)})] \\
&\circ [\mathrm{id}_{F_{rqr^{-1}}(X)} \otimes (\delta^r_{F_q(F_{r^{-1}}(Y)),V})^{-1}] \circ [\mathrm{id}_{F_{rqr^{-1}}(X)} \otimes F_r(\Phi_V(F_{r^{-1}}(Y)))] \circ [\mathrm{id}_{F_{rqr^{-1}}(X)} \otimes \delta^r_{V,F_{r^{-1}}(Y)}] \\
&\circ \{[\alpha_{r,q,r^{-1}}(X) \otimes \mathrm{id}_{F_r(V)\otimes F_r(F_{r^{-1}}(Y))}] \circ a_{F_r(F_q(F_{r^{-1}}(X))),F_r(V),F_r(F_{r^{-1}}(Y))} \\
&\circ [(\delta^r_{F_q(F_{r^{-1}}(X)),V})^{-1} \otimes \mathrm{id}_{F_r(F_{r^{-1}}(Y))}] \circ [F_r(\Phi_V(F_{r^{-1}}(X))) \otimes \mathrm{id}_{F_r(F_{r^{-1}}(Y))}] \\
&\circ [\mathrm{id}_{F_r(V\otimes F_{r^{-1}}(X))} \otimes \Delta_r(Y)^{-1}]\} \circ [\delta^r_{V,F_r(X)} \otimes \mathrm{id}_Y] \circ [(\mathrm{id}_{F_r(V)} \otimes \Delta_r(X)^{-1}) \otimes \mathrm{id}_Y] \circ a^{-1}_{F_r(V),X,Y} \\
= {}& [\delta^{rqr^{-1}}_{X,Y} \otimes \mathrm{id}_{F_r(V)}] \circ a^{-1}_{F_{rqr^{-1}}(X),F_{rqr^{-1}}(Y),F_r(V)} \circ \{[\mathrm{id}_{F_{rqr^{-1}}(X)} \otimes (\alpha_{r,q,r^{-1}}(Y) \otimes \mathrm{id}_{F_r(V)})] \\
&\circ [\mathrm{id}_{F_{rqr^{-1}}(X)} \otimes (\delta^r_{F_q(F_{r^{-1}}(Y)),V})^{-1}] \circ [\mathrm{id}_{F_{rqr^{-1}}(X)} \otimes F_r(\Phi_V(F_{r^{-1}}(Y)))] \circ [\mathrm{id}_{F_{rqr^{-1}}(X)} \otimes \delta^r_{V,F_{r^{-1}}(Y)}] \\
&\circ [\mathrm{id}_{F_{rqr^{-1}}(X)} \otimes (\mathrm{id}_{F_r(V)} \otimes \Delta_r(Y)^{-1})]\} \circ a_{F_{rqr^{-1}}(X),F_r(V),Y} \circ \{[(\alpha_{r,q,r^{-1}}(X) \otimes \mathrm{id}_{F_r(V)}) \otimes \mathrm{id}_Y] \\
&\circ [(\delta^r_{F_q(F_{r^{-1}}(X)),V})^{-1} \otimes \mathrm{id}_Y] \circ [F_r(\Phi_V(F_{r^{-1}}(X))) \otimes \mathrm{id}_Y] \circ [\delta^r_{V,F_r(X)} \otimes \mathrm{id}_Y] \\
&\circ [(\mathrm{id}_{F_r(V)} \otimes \Delta_r(X)^{-1}) \otimes \mathrm{id}_Y]\} \circ a^{-1}_{F_r(V),X,Y} \\
= {}& [\delta^{rqr^{-1}}_{X,Y} \otimes \mathrm{id}_{F_r(V)}] \circ a^{-1}_{F_{rqr^{-1}}(X),F_{rqr^{-1}}(Y),F_r(V)} \circ [\mathrm{id}_{F_{rqr^{-1}}(X)} \otimes \mathcal{F}_r\Phi_V(Y)] \\
&\circ a_{F_{rqr^{-1}}(X),F_r(V),Y} \circ [\mathcal{F}_r\Phi_V(X) \otimes \mathrm{id}_V] \circ a^{-1}_{F_r(V),X,Y},
\end{aligned}
$$

which shows that $\mathcal{F}_r\Phi_V$ satisfies the condition (A.1.1).
□

Remark A.1.7 The analogous statements for half braidings of the second kind are as follows. If Ψ_V is a half q-braiding of the second kind for V and Ψ_W is a half r-braiding of the second kind for W, then

$$
\begin{aligned}
(\Psi_V \otimes \Psi_W)(X) := {}& [\mathrm{id}_{V\otimes W} \otimes (\delta^F_{r^{-1},q^{-1}})_X] \circ a^{-1}_{V,W,F_{r^{-1}}(F_{q^{-1}}(X))} \circ [\mathrm{id}_V \otimes \Psi_W(F_{q^{-1}}(X))] \\
&\circ a_{V,F_{q^{-1}}(X),W} \circ [\Psi_V(X) \otimes \mathrm{id}_W] \circ a^{-1}_{X,V,W}
\end{aligned}
$$

defines a half qr-braiding $\Psi_V \otimes \Psi_W$ of the second kind for $V \otimes W$. If Ψ_V is a half q-braiding of the second kind for V and if $r \in G$, then

$$
\begin{aligned}
(\mathcal{F}_r\Psi_V)(X) := {}& [\mathrm{id}_{F_r(V)} \otimes \alpha_{r,q^{-1},r^{-1}}(X)] \circ (\delta^r_{V,F_{q^{-1}}(F_{r^{-1}}(X))})^{-1} \circ F_r(\Psi_V(F_{r^{-1}}(X))) \\
&\circ \delta^r_{F_{r^{-1}}(X),V} \circ [\Delta_r(X)^{-1} \otimes \mathrm{id}_{F_r(V)}]
\end{aligned}
$$

defines a half rqr^{-1}-braiding $\mathcal{F}_r\Psi_V$ of the second kind for $F_r(V)$.

A.2 $Z_G(\mathcal{C})$ as a category

Lemma A.2.1 *Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category with G-action $(F, \varepsilon^F, \delta^F)$. For each $q \in G$ we define a category $Z_G(\mathcal{C})_q$ as follows. The class of objects is given by*

$$\mathrm{Obj}(Z_G(\mathcal{C})_q) := \{(V, \Phi_V) : V \in \mathcal{C} \text{ and } \Phi_V \text{ is a half } q\text{-braiding for } V\}$$

and we define $\mathrm{Hom}_{Z_G(\mathcal{C})_q}((V,\Phi_V),(W,\Phi_W))$ *to be*

$$\{f \in \mathrm{Hom}_{\mathcal{C}}(V,W) : [\mathrm{id}_{F_q(X)} \otimes f] \circ \Phi_V(X) = \Phi_W(X) \circ [f \otimes \mathrm{id}_X] \ \forall X \in \mathcal{C}\}.$$

This defines a category $Z_G(\mathcal{C})_q$*, where the composition of morphisms is the same as in* $\mathcal{C}$ *and the identity morphisms are given by* $\mathrm{id}_{(V,\Phi_V)} = \mathrm{id}_V$*. We then define the category* $Z_G(\mathcal{C})$ *as the disjoint union*

$$Z_G(\mathcal{C}) := \bigsqcup_{q\in G} Z_G(\mathcal{C})_q.$$

Thus, an object in $Z_G(\mathcal{C})$ *is a triple* (V,q,Φ_V) *with* $q \in G$ *and* $(V,\Phi_V) \in Z_G(\mathcal{C})_q$*.*

Proof. In order to show that $Z_G(\mathcal{C})$ is a category, it is enough to prove that $Z_G(\mathcal{C})_q$ is a category for each $q \in G$. We thus fix $q \in G$. Let $f \in \mathrm{Hom}_{Z_G(\mathcal{C})_q}((U,\Phi_U),(V,\Phi_V))$ and $g \in \mathrm{Hom}_{Z_G(\mathcal{C})_q}((V,\Phi_V),(W,\Phi_W))$. Then for all $X \in \mathcal{C}$ the morphism $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(U,W)$ satisfies

$$\begin{aligned}
&[\mathrm{id}_{F_q(X)} \otimes (g \circ f)] \circ \Phi_U(X) \\
&= [\mathrm{id}_{F_q(X)} \otimes g] \circ [\mathrm{id}_{F_q(X)} \otimes f] \circ \Phi_U(X) = [\mathrm{id}_{F_q(X)} \otimes g] \circ \Phi_V(X) \circ [f \otimes \mathrm{id}_X] \\
&= \Phi_W(X) \circ [g \otimes \mathrm{id}_X] \circ [f \otimes \mathrm{id}_X] = \Phi_W(X) \circ [(g \circ f) \otimes \mathrm{id}_X].
\end{aligned}$$

This shows that $g \circ f \in \mathrm{Hom}_{Z_G(\mathcal{C})_q}((U,\Phi_U),(W,\Phi_W))$, so the composition map in $Z_G(\mathcal{C})_q$ is well-defined. If $(V,\Phi_V) \in Z_G(\mathcal{C})_q$, then the morphism $\mathrm{id}_V \in \mathrm{End}_{\mathcal{C}}(V)$ satisfies $[\mathrm{id}_{F_q(X)} \otimes \mathrm{id}_V] \circ \Phi_V(X) = \Phi_V(X) \circ [\mathrm{id}_V \otimes \mathrm{id}_X]$ for all $X \in \mathcal{C}$, so $\mathrm{id}_V \in \mathrm{End}_{Z_G(\mathcal{C})_q}((V,\Phi_V))$. Because the composition in $Z_G(\mathcal{C})_q$ is defined to be the same as in $\mathcal{C}$, it is clear that the composition in $Z_G(\mathcal{C})_q$ is associative and that id_V acts as an identity morphism in $Z_G(\mathcal{C})_q$. This completes the proof that $Z_G(\mathcal{C})_q$ is a category for each $q \in G$, and hence that $Z_G(\mathcal{C})$ is a category.
□

The following simple lemma will turn out to be very useful.

Lemma A.2.2 *Let* $(\mathcal{C},\otimes,I,a,l,r)$ *be a tensor category with* G*-action* $(F,\varepsilon^F,\delta^F)$*, let* $q \in G$*, let* $(V,\Phi_V),(W,\Phi_W) \in Z_G(\mathcal{C})_q$ *and let* $f \in \mathrm{Hom}_{\mathcal{C}}(V,W)$ *be an isomorphism with inverse* $f^{-1} \in \mathrm{Hom}_{\mathcal{C}}(W,V)$*. If* $f \in \mathrm{Hom}_{Z_G(\mathcal{C})_q}((V,\Phi_V),(W,\Phi_W))$*, then* $f^{-1} \in \mathrm{Hom}_{Z_G(\mathcal{C})_q}((W,\Phi_W),(V,\Phi_V))$*.*

Proof. For any $X \in \mathcal{C}$ we have

$$\begin{aligned}
\Phi_V(X) \circ [f^{-1} \otimes \mathrm{id}_X] &= [\mathrm{id}_{F_q(X)} \otimes (f^{-1} \circ f)] \circ \Phi_V(X) \circ [f^{-1} \otimes \mathrm{id}_X] \\
&= [\mathrm{id}_{F_q(X)} \otimes f^{-1}] \circ \Phi_W(X) \circ [(f \circ f^{-1}) \otimes \mathrm{id}_X] \\
&= [\mathrm{id}_{F_q(X)} \otimes f^{-1}] \circ \Phi_W(X),
\end{aligned}$$

so indeed $f^{-1} \in \mathrm{Hom}_{Z_G(\mathcal{C})_q}((W,\Phi_W),(V,\Phi_V))$.
□

As a consequence, any morphism in $Z_G(\mathcal{C})$ that was invertible in $\mathcal{C}$ is also invertible in $Z_G(\mathcal{C})$ because the composition of morphisms in $Z_G(\mathcal{C})$ is the same as in $\mathcal{C}$. In what follows, this fact will be used very often.

A.3 The tensor structure on $Z_G(\mathcal{C})$

Lemma A.3.1 *Let* $(\mathcal{C},\otimes,I,a,l,r)$ *be a tensor category with* G*-action* $(F,\varepsilon^F,\delta^F)$*. The category* $Z_G(\mathcal{C})$ *can be equipped with the structure of a tensor category by defining the tensor product on the objects by*

$$(V,q,\Phi_V) \otimes (W,r,\Phi_W) := (V \otimes W, qr, \Phi_V \otimes \Phi_W),$$

where

$$(\Phi_V \otimes \Phi_W)(X) = [(\delta^F_{q,r})_X \otimes \mathrm{id}_{V\otimes W}] \circ a_{F_q(F_r(X)),V,W} \circ [\Phi_V(F_r(X)) \otimes \mathrm{id}_W] \\ \circ a^{-1}_{V,F_r(X),W} \circ [\mathrm{id}_V \otimes \Phi_W(X)] \circ a_{V,W,X}$$

and by letting the tensor product on the morphisms be the same as in $\mathcal{C}$; the unit object is (I, e, Φ^0_I), where $\Phi^0_I(X) = r^{-1}_{F_e(X)} \circ \varepsilon^F_X \circ l_X$ for all $X \in \mathcal{C}$ and the associativity constraint and the unit constraints are the ones of $\mathcal{C}$.

Proof. We have already proved in part (1) of Lemma A.1.6 that if $(V, q, \Phi_V), (W, r, \Phi_V) \in Z_G(\mathcal{C})$, then $\Phi_V \otimes \Phi_W$ is a half qr-braiding for $V \otimes W$. So the tensor product of $Z_G(\mathcal{C})$ is well-defined on the objects. To show that the tensor product on morphisms is also well-defined, let $f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V, q, \Phi_V), (W, q, \Phi_W))$ and $f' \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V', q', \Phi_{V'}), (W', q', \Phi_{W'}))$. Then for each $X \in \mathcal{C}$ the morphism $f \otimes f' \in \mathrm{Hom}_{\mathcal{C}}(V \otimes V', W \otimes W')$ satisfies

$$\begin{aligned}
&[\mathrm{id}_{F_{qq'}(X)} \otimes (f \otimes f')] \circ [(\Phi_V \otimes \Phi_{V'})(X)] \\
&= [\mathrm{id}_{F_{qq'}(X)} \otimes (\mathrm{id}_W \otimes f')] \circ [\mathrm{id}_{F_{qq'}(X)} \otimes (f \otimes \mathrm{id}_{V'})] \circ [(\delta^F_{q,q'})_X \otimes \mathrm{id}_{V\otimes V'}] \circ a_{F_q(F_{q'}(X)),V,V'} \\
&\quad \circ [\Phi_V(F_{q'}(X)) \otimes \mathrm{id}_{V'}] \circ a^{-1}_{V,F_{q'}(X),V'} \circ [\mathrm{id}_V \otimes \Phi_{V'}(X)] \circ a_{V,V',X} \\
&= [\mathrm{id}_{F_{qq'}(X)} \otimes (\mathrm{id}_W \otimes f')] \circ [(\delta^F_{q,q'})_X \otimes \mathrm{id}_{W\otimes V'}] \circ a_{F_q(F_{q'}(X)),W,V'} \circ [(\mathrm{id}_{F_q(F_{q'}(X))} \otimes f) \otimes \mathrm{id}_{V'}] \\
&\quad \circ [\Phi_V(F_{q'}(X)) \otimes \mathrm{id}_{V'}] \circ a^{-1}_{V,F_{q'}(X),V'} \circ [\mathrm{id}_V \otimes \Phi_{V'}(X)] \circ a_{V,V',X} \\
&= [(\delta^F_{q,q'})_X \otimes \mathrm{id}_{W\otimes W'}] \circ a_{F_q(F_{q'}(X)),W,W'} \circ [\mathrm{id}_{F_q(F_{q'}(X))\otimes W} \otimes f'] \circ [\Phi_W(F_{q'}(X)) \otimes \mathrm{id}_{V'}] \\
&\quad \circ [(f \otimes \mathrm{id}_{F_{q'}(X)}) \otimes \mathrm{id}_{V'}] \circ a^{-1}_{V,F_{q'}(X),V'} \circ [\mathrm{id}_V \otimes \Phi_{V'}(X)] \circ a_{V,V',X} \\
&= [(\delta^F_{q,q'})_X \otimes \mathrm{id}_{W\otimes W'}] \circ a_{F_q(F_{q'}(X)),W,W'} \circ [\Phi_W(F_{q'}(X)) \otimes \mathrm{id}_{W'}] \circ a^{-1}_{W,F_{q'}(X),W'} \\
&\quad \circ [f \otimes (\mathrm{id}_{F_{q'}(X)} \otimes f)] \circ [\mathrm{id}_V \otimes \Phi_{V'}(X)] \circ a_{V,V',X} \\
&= [(\delta^F_{q,q'})_X \otimes \mathrm{id}_{W\otimes W'}] \circ a_{F_q(F_{q'}(X)),W,W'} \circ [\Phi_W(F_{q'}(X)) \otimes \mathrm{id}_{W'}] \circ a^{-1}_{W,F_{q'}(X),W'} \\
&\quad \circ [\mathrm{id}_W \otimes \Phi_{W'}(X)] \circ a_{W,W',X} \circ [(f \otimes f') \otimes \mathrm{id}_X] \\
&= [(\Phi_W \otimes \Phi_{W'})(X)] \circ [(f \otimes f') \otimes \mathrm{id}_X].
\end{aligned}$$

This shows that $f \otimes f' \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V \otimes V', qq', \Phi_V \otimes \Phi_{V'}), (W \otimes W', qq', \Phi_W \otimes \Phi_{W'}))$ and hence that the tensor product on morphisms is indeed well-defined.

The interchange law in $Z_G(\mathcal{C})$ follows directly from the fact that both the composition and the tensor product on morphisms in $Z_G(\mathcal{C})$ are defined to be the same as in $\mathcal{C}$. The fact that the identity morphism of an object $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ is given by the identity morphism id_V in $\mathcal{C}$ immediately implies that $\mathrm{id}_{(V,q,\Phi_V)\otimes(W,r,\Phi_W)} = \mathrm{id}_{(V,q,\Phi_V)} \otimes \mathrm{id}_{(W,r,\Phi_W)}$ for any $(V, q, \Phi_V), (W, r, \Phi_W) \in Z_G(\mathcal{C})$.

We now claim that if $(U, q, \Phi_U), (V, r, \Phi_V), (W, s, \Phi_W) \in Z_G(\mathcal{C})$, then

$$a_{(U,q,\Phi_U),(V,r,\Phi_V),(W,s,\Phi_W)} := a_{U,V,W}$$

defines an associativity constraint for the tensor product in $Z_G(\mathcal{C})$. We have

$$\begin{aligned}
&[\Phi_U \otimes (\Phi_V \otimes \Phi_W)](X) \circ [a_{U,V,W} \otimes \mathrm{id}_X] \\
&= [(\delta^F_{q,rs})_X \otimes \mathrm{id}_{U\otimes(V\otimes W)}] \circ a_{F_q(F_{rs}(X)),U,V\otimes W} \circ [\Phi_U(F_{rs}(X)) \otimes \mathrm{id}_{V\otimes W}] \circ a^{-1}_{U,F_{rs}(X),V\otimes W} \\
&\quad \circ [\mathrm{id}_U \otimes (\Phi_V \otimes \Phi_W)(X)] \circ a_{U,V\otimes W,X} \circ [a_{U,V,W} \otimes \mathrm{id}_X] \\
&= [(\delta^F_{q,rs})_X \otimes \mathrm{id}_{U\otimes(V\otimes W)}] \circ a_{F_q(F_{rs}(X)),U,V\otimes W} \circ [\Phi_U(F_{rs}(X)) \otimes \mathrm{id}_{V\otimes W}] \circ \{a^{-1}_{U,F_{rs}(X),V\otimes W}
\end{aligned}$$

$$\circ [\mathrm{id}_U \otimes ((\delta^F_{r,s})_X \otimes \mathrm{id}_{V\otimes W})]\} \circ \{[\mathrm{id}_U \otimes a_{F_r(F_s(X)),V,W}] \circ [\mathrm{id}_U \otimes (\Phi_V(F_s(X)) \otimes \mathrm{id}_W)]$$
$$\circ [\mathrm{id}_U \otimes a^{-1}_{V,F_s(X),W}] \circ [\mathrm{id}_U \otimes (\mathrm{id}_V \otimes \Phi_W(X))] \circ [\mathrm{id}_U \otimes a_{V,W,X}] \circ a_{U,V\otimes W,X} \circ [a_{U,V,W} \otimes \mathrm{id}_X]\}$$
$$\overset{(*)}{=} [(\delta^F_{q,rs})_X \otimes \mathrm{id}_{U\otimes(V\otimes W)}] \circ a_{F_q(F_{rs}(X)),U,V\otimes W} \circ \{[\Phi_U(F_{rs}(X)) \otimes \mathrm{id}_{V\otimes W}]$$
$$\circ [(\mathrm{id}_U \otimes (\delta^F_{r,s})_X) \otimes \mathrm{id}_{V\otimes W}]\} \circ \{a^{-1}_{U,F_r(F_s(X)),V\otimes W} \circ a_{U,F_r(F_s(X)),V\otimes W}\} \circ a_{U\otimes F_r(F_s(X)),V,W}$$
$$\circ [a^{-1}_{U,F_r(F_s(X)),V} \otimes \mathrm{id}_W] \circ [(\mathrm{id}_U \otimes \Phi_V(F_s(X))) \otimes \mathrm{id}_W] \circ [a_{U,V,F_s(X)} \otimes \mathrm{id}_W] \circ a^{-1}_{U\otimes V,F_s(X),W}$$
$$\circ [\mathrm{id}_{U\otimes V} \otimes \Phi_W(X)] \circ a_{U\otimes V,W,X}$$
$$= [(\delta^F_{q,rs})_X \otimes \mathrm{id}_{U\otimes(V\otimes W)}] \circ \{a_{F_q(F_{rs}(X)),U,V\otimes W} \circ [(F_q((\delta^F_{r,s})_X) \otimes \mathrm{id}_U) \otimes \mathrm{id}_{V\otimes W}]\}$$
$$\circ \{[\Phi_U(F_r(F_s(X))) \otimes \mathrm{id}_{V\otimes W}] \circ a_{U\otimes F_r(F_s(X)),V,W}\} \circ [a^{-1}_{U,F_r(F_s(X)),V} \otimes \mathrm{id}_W]$$
$$\circ [(\mathrm{id}_U \otimes \Phi_V(F_s(X))) \otimes \mathrm{id}_W] \circ [a_{U,V,F_s(X)} \otimes \mathrm{id}_W] \circ a^{-1}_{U\otimes V,F_s(X),W}$$
$$\circ [\mathrm{id}_{U\otimes V} \otimes \Phi_W(X)] \circ a_{U\otimes V,W,X}$$
$$= \{[(\delta^F_{q,rs})_X \otimes \mathrm{id}_{U\otimes(V\otimes W)}] \circ [F_q((\delta^F_{r,s})_X) \otimes \mathrm{id}_{U\otimes(V\otimes W)}]\} \circ \{a_{F_q(F_r(F_s(X))),U,V\otimes W}$$
$$\circ a_{F_q(F_r(F_s(X)))\otimes U,V,W}\} \circ [(\Phi_U(F_r(F_s(X))) \otimes \mathrm{id}_V) \otimes \mathrm{id}_W] \circ [a^{-1}_{U,F_r(F_s(X)),V} \otimes \mathrm{id}_W]$$
$$\circ [(\mathrm{id}_U \otimes \Phi_V(F_s(X))) \otimes \mathrm{id}_W] \circ [a_{U,V,F_s(X)} \otimes \mathrm{id}_W] \circ a^{-1}_{U\otimes V,F_s(X),W}$$
$$\circ [\mathrm{id}_{U\otimes V} \otimes \Phi_W(X)] \circ a_{U\otimes V,W,X}$$
$$= \{[(\delta^F_{qr,s})_X \otimes \mathrm{id}_{U\otimes(V\otimes W)}] \circ [(\delta^F_{q,r})_{F_s(X)} \otimes \mathrm{id}_{U\otimes(V\otimes W)}] \circ [\mathrm{id}_{F_q(F_r(F_s(X)))} \otimes a_{U,V,W}]$$
$$\circ a_{F_q(F_r(F_s(X))),U\otimes V,W}\} \circ [a_{F_q(F_r(F_s(X))),U,V} \otimes \mathrm{id}_W] \circ [(\Phi_U(F_r(F_s(X))) \otimes \mathrm{id}_V) \otimes \mathrm{id}_W]$$
$$\circ [a^{-1}_{U,F_r(F_s(X)),V} \otimes \mathrm{id}_W] \circ [(\mathrm{id}_U \otimes \Phi_V(F_s(X))) \otimes \mathrm{id}_W] \circ [a_{U,V,F_s(X)} \otimes \mathrm{id}_W]$$
$$\circ a^{-1}_{U\otimes V,F_s(X),W} \circ [\mathrm{id}_{U\otimes V} \otimes \Phi_W(X)] \circ a_{U\otimes V,W,X}$$
$$= [\mathrm{id}_{F_{qrs}(X)} \otimes a_{U,V,W}] \circ [(\delta^F_{qr,s})_X \otimes \mathrm{id}_{(U\otimes V)\otimes W}] \circ a_{F_{qr}(F_s(X)),U\otimes V,W}$$
$$\circ \{[((\delta^F_{q,r})_{F_s(X)} \otimes \mathrm{id}_{U\otimes V}) \otimes \mathrm{id}_W] \circ [a_{F_q(F_r(F_s(X))),U,V} \otimes \mathrm{id}_W] \circ [(\Phi_U(F_r(F_s(X))) \otimes \mathrm{id}_V) \otimes \mathrm{id}_W]$$
$$\circ [a^{-1}_{U,F_r(F_s(X)),V} \otimes \mathrm{id}_W] \circ [(\mathrm{id}_U \otimes \Phi_V(F_s(X))) \otimes \mathrm{id}_W] \circ [a_{U,V,F_s(X)} \otimes \mathrm{id}_W]\}$$
$$\circ a^{-1}_{U\otimes V,F_s(X),W} \circ [\mathrm{id}_{U\otimes V} \otimes \Phi_W(X)] \circ a_{U\otimes V,W,X}$$
$$= [\mathrm{id}_{F_{qrs}(X)} \otimes a_{U,V,W}] \circ [(\delta^F_{qr,s})_X \otimes \mathrm{id}_{(U\otimes V)\otimes W}] \circ a_{F_{qr}(F_s(X)),U\otimes V,W}$$
$$\circ [(\Phi_U \otimes \Phi_V)(F_s(X)) \otimes \mathrm{id}_W] \circ a^{-1}_{U\otimes V,F_s(X),W} \circ [\mathrm{id}_{U\otimes V} \otimes \Phi_W(X)] \circ a_{U\otimes V,W,X}$$
$$= [\mathrm{id}_{F_{qrs}(X)} \otimes a_{U,V,W}] \circ [(\Phi_U \otimes \Phi_V) \otimes \Phi_W](X),$$

where the equality $\overset{(*)}{=}$ follows from the commutativity of the diagram

$$\begin{tikzcd}[column sep=small]
& (U\otimes(V\otimes W))\otimes X \arrow[dl, "a_{U,V\otimes W,X}"'] & \\
U\otimes((V\otimes W)\otimes X) \arrow[d, "\mathrm{id}_U\otimes a_{V,W,X}"] & & ((U\otimes V)\otimes W)\otimes X \arrow[ul, "a_{U,V,W}\otimes\mathrm{id}_X"'] \arrow[d, "a_{U\otimes V,W,X}"] \\
U\otimes(V\otimes(W\otimes X)) \arrow[d, "\mathrm{id}_U\otimes(\mathrm{id}_V\otimes\Phi_W(X))"] & & (U\otimes V)\otimes(W\otimes X) \arrow[ll, "a_{U,V,W\otimes X}"] \arrow[d, "\mathrm{id}_{U\otimes V}\otimes\Phi_W(X)"] \\
U\otimes(V\otimes(F_s(X)\otimes W)) \arrow[dd, "\mathrm{id}_U\otimes a^{-1}_{V,F_s(X),W}"] & & (U\otimes V)\otimes(F_s(X)\otimes W) \arrow[ll, "a_{U,V,F_s(X)\otimes W}"] \arrow[d, "a^{-1}_{U\otimes V,F_s(X),W}"] \\
& & ((U\otimes V)\otimes F_s(X))\otimes W \arrow[d, "a_{U,V,F_s(X)}\otimes\mathrm{id}_W"] \\
U\otimes((V\otimes F_s(X))\otimes W) \arrow[d, "\mathrm{id}_U\otimes(\Phi_V(F_s(X))\otimes\mathrm{id}_W)"] & & (U\otimes(V\otimes F_s(X)))\otimes W \arrow[ll, "a_{U,V\otimes F_s(X),W}"] \arrow[d, "(\mathrm{id}_U\otimes\Phi_V(F_s(X)))\otimes\mathrm{id}_W"] \\
U\otimes((F_r(F_s(X))\otimes V)\otimes W) \arrow[d, "\mathrm{id}_U\otimes a_{F_r(F_s(X)),V,W}"] & & (U\otimes(F_r(F_s(X))\otimes V))\otimes W \arrow[ll, "a_{U,F_r(F_s(X))\otimes V,W}"] \arrow[d, "a^{-1}_{U,F_r(F_s(X)),V}\otimes\mathrm{id}_W"] \\
U\otimes(F_r(F_s(X))\otimes(V\otimes W)) & & ((U\otimes F_r(F_s(X)))\otimes V)\otimes W \arrow[dl, "a_{U\otimes F_r(F_s(X)),V,W}"] \\
& (U\otimes F_r(F_s(X)))\otimes(V\otimes W). \arrow[ul, "a_{U,F_r(F_s(X)),V\otimes W}"] &
\end{tikzcd}$$

We thus conclude that indeed $a_{(U,q,\Phi_U),(V,r,\Phi_V),(W,s,\Phi_W)}$ is an isomorphism in $Z_G(\mathcal{C})$ from $[(U,q,\Phi_U)\otimes(V,r,\Phi_V)]\otimes(W,s,\Phi_W)$ to $(U,q,\Phi_U)\otimes[(V,r,\Phi_V)\otimes(W,s,\Phi_W)]$. Naturality of the associativity constraint in $Z_G(\mathcal{C})$ and the pentagon axiom for the associativity constraint in $Z_G(\mathcal{C})$ both follow from the corresponding properties in $\mathcal{C}$, since both the composition and tensor product of morphisms in $Z_G(\mathcal{C})$ is the same as in $\mathcal{C}$.

We now claim that $(I,e,\Phi_I^0)\in Z_G(\mathcal{C})$ acts as a unit object in $Z_G(\mathcal{C})$ with the left and right unit constraint in $Z_G(\mathcal{C})$ given by $l_{(V,q,\Phi_V)} := l_V$ and $r_{(V,q,\Phi_V)} := r_V$ for all $(V,q,\Phi_V)\in Z_G(\mathcal{C})$, respectively. Let $(V,q,\Phi_V)\in Z_G(\mathcal{C})$. We will first show that $l_{(V,q,\Phi_V)}$ and $r_{(V,q,\Phi_V)}$ are morphisms in $Z_G(\mathcal{C})$. Because for each $X\in\mathcal{C}$ we have

$$\begin{aligned}
(\Phi_I^0\otimes\Phi_V)(X) &= [(\delta^F_{e,q})_X\otimes\mathrm{id}_{I\otimes V}]\circ a_{F_e(F_q(X)),I,V}\circ[\Phi_I^0(F_q(X))\otimes\mathrm{id}_V]\\
&\quad\circ a^{-1}_{I,F_q(X),V}\circ[\mathrm{id}_I\otimes\Phi_V(X)]\circ a_{I,V,X}\\
&= [(\delta^F_{e,q})_X\otimes\mathrm{id}_{I\otimes V}]\circ a_{F_e(F_q(X)),I,V}\circ[r^{-1}_{F_e(F_q(X))}\otimes\mathrm{id}_V]\circ[\varepsilon^F_{F_q(X)}\otimes\mathrm{id}_V]\\
&\quad\circ[l_{F_q(X)}\otimes\mathrm{id}_V]\circ a^{-1}_{I,F_q(X),V}\circ[\mathrm{id}_I\otimes\Phi_V(X)]\circ a_{I,V,X},
\end{aligned}$$

we find that

$$\begin{aligned}
&[\mathrm{id}_{F_q(X)}\otimes l_V]\circ[(\Phi_I^0\otimes\Phi_V)(X)]\\
&= [\mathrm{id}_{F_q(X)}\otimes l_V]\circ[(\delta^F_{e,q})_X\otimes\mathrm{id}_{I\otimes V}]\circ a_{F_e(F_q(X)),I,V}\circ[r^{-1}_{F_e(F_q(X))}\otimes\mathrm{id}_V]\circ[\varepsilon^F_{F_q(X)}\otimes\mathrm{id}_V]\\
&\quad\circ[l_{F_q(X)}\otimes\mathrm{id}_V]\circ a^{-1}_{I,F_q(X),V}\circ[\mathrm{id}_I\otimes\Phi_V(X)]\circ a_{I,V,X}\\
&= [(\delta^F_{e,q})_X\otimes\mathrm{id}_V]\circ\underbrace{[\mathrm{id}_{F_e(F_q(X))}\otimes l_V]\circ a_{F_e(F_q(X)),I,V}\circ[r^{-1}_{F_e(F_q(X))}\otimes\mathrm{id}_V]}_{=\mathrm{id}_{F_e(F_q(X))\otimes V}}\circ[\varepsilon^F_{F_q(X)}\otimes\mathrm{id}_V]
\end{aligned}$$

$$\begin{aligned}
&\circ [l_{F_q(X)} \otimes \mathrm{id}_V] \circ a^{-1}_{I,F_q(X),V} \circ [\mathrm{id}_I \otimes \Phi_V(X)] \circ a_{I,V,X} \\
&= \underbrace{[((\delta^F_{e,q})_X \circ \varepsilon^F_{F_q(X)}) \otimes \mathrm{id}_V]}_{=\mathrm{id}_{F_q(X)\otimes V}} \circ \underbrace{[l_{F_q(X)} \otimes \mathrm{id}_V] \circ a^{-1}_{I,F_q(X),V}}_{=l_{F_q(X)\otimes V}} \circ [\mathrm{id}_I \otimes \Phi_V(X)] \circ a_{I,V,X} \\
&= l_{F_q(X)\otimes V} \circ [\mathrm{id}_I \otimes \Phi_V(X)] \circ a_{I,V,X} = \Phi_V(X) \circ l_{V\otimes X} \circ a_{I,V,X} \\
&= \Phi_V(X) \circ [l_V \otimes \mathrm{id}_X]
\end{aligned}$$

for all $X \in \mathcal{C}$. This shows that $l_V \in \mathrm{Hom}_{Z_G(\mathcal{C})}((I, e, \Phi^0_I) \otimes (V, q, \Phi_V), (V, q, \Phi_V))$. Similarly, because for each $X \in \mathcal{C}$ we have

$$\begin{aligned}
(\Phi_V \otimes \Phi^0_I)(X) &= [(\delta^F_{q,e})_X \otimes \mathrm{id}_{V\otimes I}] \circ a_{F_q(F_e(X)),V,I} \circ [\Phi_V(F_e(X)) \otimes \mathrm{id}_I] \\
&\quad \circ a^{-1}_{V,F_e(X),I} \circ [\mathrm{id}_V \otimes \Phi^0_I(X)] \circ a_{V,I,X} \\
&= [(\delta^F_{q,e})_X \otimes \mathrm{id}_{V\otimes I}] \circ a_{F_q(F_e(X)),V,I} \circ [\Phi_V(F_e(X)) \otimes \mathrm{id}_I] \\
&\quad \circ a^{-1}_{V,F_e(X),I} \circ [\mathrm{id}_V \otimes r^{-1}_{F_e(X)}] \circ [\mathrm{id}_V \otimes \varepsilon^F_X] \circ [\mathrm{id}_V \otimes l_X] \circ a_{V,I,X}
\end{aligned}$$

we find that

$$\begin{aligned}
&[\mathrm{id}_{F_q(X)} \otimes r_V] \circ [(\Phi_V \otimes \Phi^0_I)(X)] \\
&= [\mathrm{id}_{F_q(X)} \otimes r_V] \circ [(\delta^F_{q,e})_X \otimes \mathrm{id}_{V\otimes I}] \circ a_{F_q(F_e(X)),V,I} \circ [\Phi_V(F_e(X)) \otimes \mathrm{id}_I] \\
&\quad \circ a^{-1}_{V,F_e(X),I} \circ [\mathrm{id}_V \otimes r^{-1}_{F_e(X)}] \circ [\mathrm{id}_V \otimes \varepsilon^F_X] \circ [\mathrm{id}_V \otimes l_X] \circ a_{V,I,X} \\
&= [(\delta^F_{q,e})_X \otimes \mathrm{id}_V] \circ \underbrace{[\mathrm{id}_{F_q(F_e(X))} \otimes r_V] \circ a_{F_q(F_e(X)),V,I}}_{=r_{F_q(F_e(X))\otimes V}} \circ [\Phi_V(F_e(X)) \otimes \mathrm{id}_I] \\
&\quad \circ a^{-1}_{V,F_e(X),I} \circ [\mathrm{id}_V \otimes r^{-1}_{F_e(X)}] \circ [\mathrm{id}_V \otimes \varepsilon^F_X] \circ [\mathrm{id}_V \otimes l_X] \circ a_{V,I,X} \\
&= [(\delta^F_{q,e})_X \otimes \mathrm{id}_V] \circ r_{F_q(F_e(X))\otimes V} \circ [\Phi_V(F_e(X)) \otimes \mathrm{id}_I] \\
&\quad \circ a^{-1}_{V,F_e(X),I} \circ [\mathrm{id}_V \otimes r^{-1}_{F_e(X)}] \circ [\mathrm{id}_V \otimes \varepsilon^F_X] \circ [\mathrm{id}_V \otimes l_X] \circ a_{V,I,X} \\
&= [(\delta^F_{q,e})_X \otimes \mathrm{id}_V] \circ \Phi_V(F_e(X)) \circ \underbrace{r_{V\otimes F_e(X)} \circ a^{-1}_{V,F_e(X),I}}_{=\mathrm{id}_V \otimes r_{F_e(X)}} \\
&\quad \circ [\mathrm{id}_V \otimes r^{-1}_{F_e(X)}] \circ [\mathrm{id}_V \otimes \varepsilon^F_X] \circ [\mathrm{id}_V \otimes l_X] \circ a_{V,I,X} \\
&= [(\delta^F_{q,e})_X \otimes \mathrm{id}_V] \circ \Phi_V(F_e(X)) \circ [\mathrm{id}_V \otimes \varepsilon^F_X] \circ \underbrace{[\mathrm{id}_V \otimes l_X] \circ a_{V,I,X}}_{=r_V \otimes \mathrm{id}_X} \\
&= \underbrace{[(\delta^F_{q,e})_X \otimes \mathrm{id}_V] \circ [F_q(\varepsilon^F_X) \otimes \mathrm{id}_V]}_{=\mathrm{id}_{F_q(X)\otimes V}} \circ \Phi_V(X) \circ [r_V \otimes \mathrm{id}_X] \\
&= \Phi_V(X) \circ [r_V \otimes \mathrm{id}_X]
\end{aligned}$$

for all $X \in \mathcal{C}$. This shows that $r_V \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V, q, \Phi_V) \otimes (I, e, \Phi^0_I), (V, q, \Phi_V))$. Naturality of l and r in $Z_G(\mathcal{C})$ and the triangle axiom in $Z_G(\mathcal{C})$ follow from the corresponding properties in $\mathcal{C}$, since the composition and tensor product of morphisms in $Z_G(\mathcal{C})$ is the same as in $\mathcal{C}$. We thus conclude that $Z_G(\mathcal{C})$ is a tensor category.
□

A.4 The G-grading and G-action

Lemma A.4.1 *Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a tensor category with G-action $(F, \varepsilon^F, \delta^F)$. Then $Z_G(\mathcal{C})$ becomes a G-graded tensor category if we define*

$$\partial[(V, q, \Phi_V)] = q.$$

We can define an action $(\mathcal{F}, e, \delta)$ of the group G on the objects of $Z_G(\mathcal{C})$ by

$$\mathcal{F}_q[(V, r, \Phi_V)] = (F_q(V), qrq^{-1}, \mathcal{F}_q\Phi_V),$$

where

$$\begin{aligned}(\mathcal{F}_r\Phi_V)(X) &= [\alpha_{r,q,r^{-1}}(X) \otimes \mathrm{id}_{F_r(V)}] \circ (\delta^r_{F_q(F_{r^{-1}}(X)),V})^{-1} \circ F_r(\Phi_V(F_{r^{-1}}(X)))\\ &\quad \circ \delta^r_{V,F_{r^{-1}}(X)} \circ [\mathrm{id}_{F_r(V)} \otimes \Delta_r(X)^{-1}]\\ &= [(\delta^F_{rq,r^{-1}})_X \otimes \mathrm{id}_{F_r(V)}] \circ [(\delta^F_{r,q})_{F_{r^{-1}}(X)} \otimes \mathrm{id}_{F_r(V)}] \circ (\delta^r_{F_q(F_{r^{-1}}(X)),V})^{-1}\\ &\quad \circ F_r(\Phi_V(F_{r^{-1}}(X))) \circ \delta^r_{V,F_{r^{-1}}(X)} \circ [\mathrm{id}_{F_r(V)} \otimes (\delta^F_{r,r^{-1}})^{-1}_X] \circ [\mathrm{id}_{F_r(V)} \otimes \varepsilon^F_X]\end{aligned}$$

and on the morphisms we define $\mathcal{F}_q(f) := F_q(f)$; the ε and δ are the same as for the G-action on $\mathcal{C}$. This gives $Z_G(\mathcal{C})$ the structure of a G-crossed category.

Proof. Let $(V, q, \Phi_V), (W, r, \Phi_W) \in Z_G(\mathcal{C})$. The map ∂ has the property that

$$\partial[(V, q, \Phi_V) \otimes (W, r, \Phi_W)] \quad = \quad \partial[(V \otimes W, qr, \Phi_V \otimes \Phi_W)] = qr = \partial[(V, q, \Phi_V)]\partial[(W, r, \Phi_W)].$$

It is also clear that if $(V, q, \Phi_V), (W, r, \Phi_W) \in Z_G(\mathcal{C})$ are isomorphic, then $q = r$, i.e. $\partial[(V, q, \Phi_V)] = \partial[(W, r, \Phi_W)]$. Hence it follows that $Z_G(\mathcal{C})$ is a G-graded tensor category.

We will now show that $\mathcal{F}_q$ is a functor $Z_G(\mathcal{C}) \to Z_G(\mathcal{C})$ for each $q \in G$. If $f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V, r, \Phi_V), (W, r, \Phi_W))$, then $\mathcal{F}_q(f) = F_q(f) \in \mathrm{Hom}_{\mathcal{C}}(F_q(V), F_q(W))$ because F_q is a functor from $\mathcal{C}$ to $\mathcal{C}$, and for all $X \in \mathcal{C}$ we have

$$\begin{aligned}&[\mathrm{id}_{F_{qrq^{-1}}(X)} \otimes \mathcal{F}_q(f)] \circ [\mathcal{F}_q\Phi_V(X)]\\ &= \{[\mathrm{id}_{F_{qrq^{-1}}(X)} \otimes F_q(f)] \circ [\alpha_{q,r,q^{-1}}(X) \otimes \mathrm{id}_{F_q(V)}]\} \circ (\delta^q_{F_r(F_{q^{-1}}(X)),V})^{-1}\\ &\quad \circ F_q(\Phi_V(F_{q^{-1}}(X))) \circ \delta^q_{V,F_{q^{-1}}(X)} \circ [\mathrm{id}_{F_q(V)} \otimes \Delta_q(X)^{-1}]\\ &= [\alpha_{q,r,q^{-1}}(X) \otimes \mathrm{id}_{F_q(W)}] \circ \{[\underbrace{\mathrm{id}_{F_q(F_r(F_{q^{-1}}(X)))}}_{=F_q(\mathrm{id}_{F_r(F_{q^{-1}}(X))})} \otimes F_q(f)] \circ (\delta^q_{F_r(F_{q^{-1}}(X)),V})^{-1}\}\\ &\quad \circ F_q(\Phi_V(F_{q^{-1}}(X))) \circ \delta^q_{V,F_{q^{-1}}(X)} \circ [\mathrm{id}_{F_q(V)} \otimes \Delta_q(X)^{-1}]\\ &= [\alpha_{q,r,q^{-1}}(X) \otimes \mathrm{id}_{F_q(W)}] \circ (\delta^q_{F_r(F_{q^{-1}}(X)),W})^{-1} \circ \{F_q(\mathrm{id}_{F_r(F_{q^{-1}}(X))} \otimes f)\\ &\quad \circ F_q(\Phi_V(F_{q^{-1}}(X)))\} \circ \delta^q_{V,F_{q^{-1}}(X)} \circ [\mathrm{id}_{F_q(V)} \otimes \Delta_q(X)^{-1}]\\ &= [\alpha_{q,r,q^{-1}}(X) \otimes \mathrm{id}_{F_q(W)}] \circ (\delta^q_{F_r(F_{q^{-1}}(X)),W})^{-1} \circ F_q(\Phi_W(F_{q^{-1}}(X)))\\ &\quad \circ \{F_q(f \otimes \mathrm{id}_{F_{q^{-1}}(X)}) \circ \delta^q_{V,F_{q^{-1}}(X)}\} \circ [\mathrm{id}_{F_q(V)} \otimes \Delta_q(X)^{-1}]\\ &= [\alpha_{q,r,q^{-1}}(X) \otimes \mathrm{id}_{F_q(W)}] \circ (\delta^q_{F_r(F_{q^{-1}}(X)),W})^{-1} \circ F_q(\Phi_W(F_{q^{-1}}(X)))\\ &\quad \circ \delta^q_{W,F_{q^{-1}}(X)} \circ \{[F_q(f) \otimes \underbrace{F_q(\mathrm{id}_{F_{q^{-1}}(X)})}_{=\mathrm{id}_{F_q(F_{q^{-1}}(X))}}] \circ [\mathrm{id}_{F_q(V)} \otimes \Delta_q(X)^{-1}]\}\\ &= [\alpha_{q,r,q^{-1}}(X) \otimes \mathrm{id}_{F_q(W)}] \circ (\delta^q_{F_r(F_{q^{-1}}(X)),W})^{-1} \circ F_q(\Phi_W(F_{q^{-1}}(X)))\end{aligned}$$

$$\circ\, \delta^q_{W,F_{q^{-1}}(X)} \circ [\mathrm{id}_{F_q(W)} \otimes \Delta_q(X)^{-1}] \circ [F_q(f) \otimes \mathrm{id}_X]$$
$$= [\mathcal{F}_q\Phi_W(X)] \circ [\mathcal{F}_q(f) \otimes \mathrm{id}_X].$$

This shows that $\mathcal{F}_q(f) \in \mathrm{Hom}_{Z_G(\mathcal{C})}(\mathcal{F}_q[(V,r,\Phi_V)], \mathcal{F}_q[(W,r,\Phi_W)])$. If f and g are composable morphisms in $Z_G(\mathcal{C})$, then $\mathcal{F}_q(g \circ f) = F_q(g \circ f) = F_q(g) \circ F_q(f) = \mathcal{F}_q(g) \circ \mathcal{F}_q(f)$. Also, if $(V,q,\Phi_V) \in Z_G(\mathcal{C})$, then $\mathcal{F}_q(\mathrm{id}_{(V,r,\Phi_V)}) = F_q(\mathrm{id}_V) = \mathrm{id}_{F_q(V)} = \mathrm{id}_{(F_q(V),qrq^{-1},\mathcal{F}_q\Phi_V)} = \mathrm{id}_{\mathcal{F}_q[(V,r,\Phi_V)]}$. Thus $\mathcal{F}_q$ is indeed a functor.

The next thing we will prove is that $(\mathcal{F}_q, \varepsilon^{\mathcal{F}_q}, \delta^{\mathcal{F}_q})$ is a tensor functor for each $q \in G$, where $\varepsilon^{\mathcal{F}_q} = \varepsilon^q$ and $\delta^{\mathcal{F}_q}_{(V,r,\Phi_V),(W,s,\Phi_W)} = \delta^q_{V,W}$. If $(V,r,\Phi_V),(W,s,\Phi_W) \in Z_G(\mathcal{C})$ and $X \in \mathcal{C}$, then

$$[\mathrm{id}_{F_{qrsq^{-1}}(X)} \otimes (\delta^q_{V,W})^{-1}] \circ [\mathcal{F}_q(\Phi_V \otimes \Phi_W)(X)] \circ [\delta^q_{V,W} \otimes \mathrm{id}_X]$$
$$= [\mathrm{id}_{F_{qrsq^{-1}}(X)} \otimes (\delta^q_{V,W})^{-1}] \circ [\alpha_{q,rs,q^{-1}}(X) \otimes \mathrm{id}_{F_q(V\otimes W)}] \circ (\delta^q_{F_{rs}(F_{q^{-1}}(X)),V\otimes W})^{-1}$$
$$\circ\, F_q((\Phi_V \otimes \Phi_W)(F_{q^{-1}}(X))) \circ \delta^q_{V\otimes W,F_{q^{-1}}(X)} \circ [\mathrm{id}_{F_q(V\otimes W)} \otimes \Delta_q(X)^{-1}] \circ [\delta^q_{V,W} \otimes \mathrm{id}_X]$$
$$= \{[\mathrm{id}_{F_{qrsq^{-1}}(X)} \otimes (\delta^q_{V,W})^{-1}] \circ [\alpha_{q,rs,q^{-1}}(X) \otimes \mathrm{id}_{F_q(V\otimes W)}]\} \circ \{(\delta^q_{F_{rs}(F_{q^{-1}}(X)),V\otimes W})^{-1}$$
$$\circ\, F_q((\delta^F_{r,s})_{F_{q^{-1}}(X)} \otimes \mathrm{id}_{V\otimes W})\} \circ F_q(a_{F_r(F_s(F_{q^{-1}}(X))),V,W}) \circ F_q(\Phi_V(F_s(F_{q^{-1}}(X))) \otimes \mathrm{id}_W)$$
$$\circ\, F_q(a_{V,F_s(F_{q^{-1}}(X)),W})^{-1} \circ F_q(\mathrm{id}_V \otimes \Phi_W(F_{q^{-1}}(X))) \circ F_q(a_{V,W,F_{q^{-1}}(X)})$$
$$\circ\, \delta^q_{V\otimes W,F_{q^{-1}}(X)} \circ \{[\mathrm{id}_{F_q(V\otimes W)} \otimes \Delta_q(X)^{-1}] \circ [\delta^q_{V,W} \otimes \mathrm{id}_X]\}$$
$$= [\alpha_{q,rs,q^{-1}}(X) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \circ \{[\mathrm{id}_{F_q(F_{rs}(F_{q^{-1}}(X)))} \otimes (\delta^q_{V,W})^{-1}]$$
$$\circ\, [F_q((\delta^F_{r,s})_{F_{q^{-1}}(X)}) \otimes \mathrm{id}_{F_q(V\otimes W)}]\} \circ (\delta^q_{F_r(F_s(F_{q^{-1}}(X))),V\otimes W})^{-1} \circ F_q(a_{F_r(F_s(F_{q^{-1}}(X))),V,W})$$
$$\circ\, F_q(\Phi_V(F_s(F_{q^{-1}}(X))) \otimes \mathrm{id}_W) \circ F_q(a_{V,F_s(F_{q^{-1}}(X)),W})^{-1} \circ F_q(\mathrm{id}_V \otimes \Phi_W(F_{q^{-1}}(X)))$$
$$\circ\, \{F_q(a_{V,W,F_{q^{-1}}(X)}) \circ \delta^q_{V\otimes W,F_{q^{-1}}(X)} \circ [\delta^q_{V,W} \otimes \mathrm{id}_{F_q(F_{q^{-1}}(X))}]\} \circ [\mathrm{id}_{F_q(V)\otimes F_q(W)} \otimes \Delta_q(X)^{-1}]$$
$$= [\alpha_{q,rs,q^{-1}}(X) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \circ [F_q((\delta^F_{r,s})_{F_{q^{-1}}(X)}) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}]$$
$$\circ\, \{[\mathrm{id}_{F_q(F_r(F_s(F_{q^{-1}}(X))))} \otimes (\delta^q_{V,W})^{-1}] \circ (\delta^q_{F_r(F_s(F_{q^{-1}}(X))),V\otimes W})^{-1} \circ F_q(a_{F_r(F_s(F_{q^{-1}}(X))),V,W})\}$$
$$\circ\, F_q(\Phi_V(F_s(F_{q^{-1}}(X))) \otimes \mathrm{id}_W) \circ F_q(a_{V,F_s(F_{q^{-1}}(X)),W})^{-1} \circ \{F_q(\mathrm{id}_V \otimes \Phi_W(F_{q^{-1}}(X)))$$
$$\circ\, \delta^q_{V,W\otimes F_{q^{-1}}(X)}\} \circ [\mathrm{id}_{F_q(V)} \otimes \delta^q_{W,F_{q^{-1}}(X)}] \circ \{a_{F_q(V),F_q(W),F_q(F_{q^{-1}}(X))}$$
$$\circ\, [\mathrm{id}_{F_q(V)\otimes F_q(W)} \otimes \Delta_q(X)^{-1}]\}$$
$$= [\alpha_{q,rs,q^{-1}}(X) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \circ [F_q((\delta^F_{r,s})_{F_{q^{-1}}(X)}) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}]$$
$$\circ\, a_{F_q(F_r(F_s(F_{q^{-1}}(X)))),F_q(V),F_q(W)} \circ [(\delta^q_{F_r(F_s(F_{q^{-1}}(X))),V})^{-1} \otimes \mathrm{id}_{F_q(W)}]$$
$$\circ\, \{(\delta^q_{F_r(F_s(F_{q^{-1}}(X)))\otimes V,W})^{-1} \circ F_q(\Phi_V(F_s(F_{q^{-1}}(X))) \otimes \mathrm{id}_W)\}$$
$$\circ\, F_q(a_{V,F_s(F_{q^{-1}}(X)),W})^{-1} \circ \delta^q_{V,F_s(F_{q^{-1}}(X))\otimes W} \circ [\mathrm{id}_{F_q(V)} \otimes F_q(\Phi_W(F_{q^{-1}}(X)))]$$
$$\circ\, [\mathrm{id}_{F_q(V)} \otimes \delta^q_{W,F_{q^{-1}}(X)}] \circ [\mathrm{id}_{F_q(V)} \otimes (\mathrm{id}_{F_q(W)} \otimes \Delta_q(X)^{-1})] \circ a_{F_q(V),F_q(W),X}$$
$$= [\alpha_{q,rs,q^{-1}}(X) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \circ [F_q((\delta^F_{r,s})_{F_{q^{-1}}(X)}) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}]$$
$$\circ\, a_{F_q(F_r(F_s(F_{q^{-1}}(X)))),F_q(V),F_q(W)} \circ [(\delta^q_{F_r(F_s(F_{q^{-1}}(X))),V})^{-1} \otimes \mathrm{id}_{F_q(W)}]$$
$$\circ\, [F_q(\Phi_V(F_s(F_{q^{-1}}(X)))) \otimes \mathrm{id}_{F_q(W)}] \circ \{(\delta^q_{V\otimes F_s(F_{q^{-1}}(X)),W})^{-1} \circ F_q(a_{V,F_s(F_{q^{-1}}(X)),W})^{-1}$$
$$\circ\, \delta^q_{V,F_s(F_{q^{-1}}(X))\otimes W}\} \circ [\mathrm{id}_{F_q(V)} \otimes F_q(\Phi_W(F_{q^{-1}}(X)))] \circ [\mathrm{id}_{F_q(V)} \otimes \delta^q_{W,F_{q^{-1}}(X)}]$$
$$\circ\, [\mathrm{id}_{F_q(V)} \otimes (\mathrm{id}_{F_q(W)} \otimes \Delta_q(X)^{-1})] \circ a_{F_q(V),F_q(W),X}$$

$$
\begin{aligned}
&= [\alpha_{q,rs,q^{-1}}(X) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \circ [F_q((\delta^F_{r,s})_{F_{q^{-1}}(X)}) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \\
&\quad \circ a_{F_q(F_r(F_s(F_{q^{-1}}(X)))),F_q(V),F_q(W)} \circ [(\delta^q_{F_r(F_s(F_{q^{-1}}(X))),V})^{-1} \otimes \mathrm{id}_{F_q(W)}] \\
&\quad \circ [F_q(\Phi_V(F_s(F_{q^{-1}}(X)))) \otimes \mathrm{id}_{F_q(W)}] \circ \{[\delta^q_{V,F_s(F_{q^{-1}}(X))} \otimes \mathrm{id}_{F_q(W)}]\} \\
&\quad \circ a^{-1}_{F_q(V),F_q(F_s(F_{q^{-1}}(X))),F_q(W)} \circ [\mathrm{id}_{F_q(V)} \otimes (\delta^q_{F_s(F_{q^{-1}}(X)),W})^{-1}] \circ [\mathrm{id}_{F_q(V)} \otimes F_q(\Phi_W(F_{q^{-1}}(X)))] \\
&\quad \circ [\mathrm{id}_{F_q(V)} \otimes \delta^q_{W,F_{q^{-1}}(X)}] \circ [\mathrm{id}_{F_q(V)} \otimes (\mathrm{id}_{F_q(W)} \otimes \Delta_q(X)^{-1})] \circ a_{F_q(V),F_q(W),X} \\
&= [\alpha_{q,rs,q^{-1}}(X) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \circ [F_q((\delta^F_{r,s})_{F_{q^{-1}}(X)}) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \\
&\quad \circ \{a_{F_q(F_r(F_s(F_{q^{-1}}(X)))),F_q(V),F_q(W)} \circ [(\delta^q_{F_r(F_s(F_{q^{-1}}(X))),V})^{-1} \otimes \mathrm{id}_{F_q(W)}] \\
&\quad \circ [F_q(\Phi_V(F_s(F_{q^{-1}}(X)))) \otimes \mathrm{id}_{F_q(W)}] \circ [F_q(\mathrm{id}_V \otimes ((\delta^F_{s,q^{-1}})^{-1}_X \circ (\delta^F_{q^{-1},qsq^{-1}})_X)) \otimes \mathrm{id}_{F_q(W)}]\} \\
&\quad \circ [\delta^q_{V,F_{q^{-1}}(F_{qsq^{-1}}(X))} \otimes \mathrm{id}_{F_q(W)}] \circ [(\mathrm{id}_{F_q(V)} \otimes \Delta_q(F_{qsq^{-1}}(X))^{-1}) \otimes \mathrm{id}_{F_q(W)}] \\
&\quad \circ [(\mathrm{id}_{F_q(V)} \otimes \alpha_{q,s,q^{-1}}(X)) \otimes \mathrm{id}_{F_q(W)}] \circ a^{-1}_{F_q(V),F_q(F_s(F_{q^{-1}}(X))),F_q(W)} \\
&\quad \circ [\mathrm{id}_{F_q(V)} \otimes (\delta^q_{F_s(F_{q^{-1}}(X)),W})^{-1}] \circ [\mathrm{id}_{F_q(V)} \otimes F_q(\Phi_W(F_{q^{-1}}(X)))] \\
&\quad \circ [\mathrm{id}_{F_q(V)} \otimes \delta^q_{W,F_{q^{-1}}(X)}] \circ [\mathrm{id}_{F_q(V)} \otimes (\mathrm{id}_{F_q(W)} \otimes \Delta_q(X)^{-1})] \circ a_{F_q(V),F_q(W),X} \\
&= \{[\alpha_{q,rs,q^{-1}}(X) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \circ [F_q((\delta^F_{r,s})_{F_{q^{-1}}(X)}) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \\
&\quad \circ [F_q(F_r((\delta^F_{s,q^{-1}})^{-1}_X \circ (\delta^F_{q^{-1},qsq^{-1}})_X)) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}]\} \circ a_{F_q(F_r(F_{q^{-1}}(F_{qsq^{-1}}(X)))),F_q(V),F_q(W)} \\
&\quad \circ [(\delta^q_{F_r(F_{q^{-1}}(F_{qsq^{-1}}(X))),V})^{-1} \otimes \mathrm{id}_{F_q(W)}] \circ [F_q(\Phi_V(F_{q^{-1}}(F_{qsq^{-1}}(X)))) \otimes \mathrm{id}_{F_q(W)}] \\
&\quad \circ [\delta^q_{V,F_{q^{-1}}(F_{qsq^{-1}}(X))} \otimes \mathrm{id}_{F_q(W)}] \circ [(\mathrm{id}_{F_q(V)} \otimes \Delta_q(F_{qsq^{-1}}(X))^{-1}) \otimes \mathrm{id}_{F_q(W)}] \\
&\quad \circ [(\mathrm{id}_{F_q(V)} \otimes \alpha_{q,s,q^{-1}}(X)) \otimes \mathrm{id}_{F_q(W)}] \circ a^{-1}_{F_q(V),F_q(F_s(F_{q^{-1}}(X))),F_q(W)} \\
&\quad \circ [\mathrm{id}_{F_q(V)} \otimes (\delta^q_{F_s(F_{q^{-1}}(X)),W})^{-1}] \circ [\mathrm{id}_{F_q(V)} \otimes F_q(\Phi_W(F_{q^{-1}}(X)))] \\
&\quad \circ [\mathrm{id}_{F_q(V)} \otimes \delta^q_{W,F_{q^{-1}}(X)}] \circ [\mathrm{id}_{F_q(V)} \otimes (\mathrm{id}_{F_q(W)} \otimes \Delta_q(X)^{-1})] \circ a_{F_q(V),F_q(W),X} \\
&= [(\delta^F_{qrq^{-1},qsq^{-1}})_X \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \circ \{[\alpha_{q,r,q^{-1}}(F_{qsq^{-1}}(X)) \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \\
&\quad \circ a_{F_q(F_r(F_{q^{-1}}(F_{qsq^{-1}}(X)))),F_q(V),F_q(W)}\} \circ [(\delta^q_{F_r(F_{q^{-1}}(F_{qsq^{-1}}(X))),V})^{-1} \otimes \mathrm{id}_{F_q(W)}] \\
&\quad \circ [F_q(\Phi_V(F_{q^{-1}}(F_{qsq^{-1}}(X)))) \otimes \mathrm{id}_{F_q(W)}] \circ [\delta^q_{V,F_{q^{-1}}(F_{qsq^{-1}}(X))} \otimes \mathrm{id}_{F_q(W)}] \\
&\quad \circ [(\mathrm{id}_{F_q(V)} \otimes \Delta_q(F_{qsq^{-1}}(X))^{-1}) \otimes \mathrm{id}_{F_q(W)}] \circ \{[(\mathrm{id}_{F_q(V)} \otimes \alpha_{q,s,q^{-1}}(X)) \otimes \mathrm{id}_{F_q(W)}] \\
&\quad \circ a^{-1}_{F_q(V),F_q(F_s(F_{q^{-1}}(X))),F_q(W)}\} \circ [\mathrm{id}_{F_q(V)} \otimes (\delta^q_{F_s(F_{q^{-1}}(X)),W})^{-1}] \circ [\mathrm{id}_{F_q(V)} \otimes F_q(\Phi_W(F_{q^{-1}}(X)))] \\
&\quad \circ [\mathrm{id}_{F_q(V)} \otimes \delta^q_{W,F_{q^{-1}}(X)}] \circ [\mathrm{id}_{F_q(V)} \otimes (\mathrm{id}_{F_q(W)} \otimes \Delta_q(X)^{-1})] \circ a_{F_q(V),F_q(W),X} \\
&= [(\delta^F_{qrq^{-1},qsq^{-1}})_X \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \circ a_{F_{qrq^{-1}}(F_{qsq^{-1}}(X)),F_q(V),F_q(W)} \\
&\quad \circ \{[(\alpha_{q,r,q^{-1}}(F_{qsq^{-1}}(X)) \otimes \mathrm{id}_{F_q(V)}) \otimes \mathrm{id}_{F_q(W)}] \circ [(\delta^q_{F_r(F_{q^{-1}}(F_{qsq^{-1}}(X))),V})^{-1} \otimes \mathrm{id}_{F_q(W)}] \\
&\quad \circ [F_q(\Phi_V(F_{q^{-1}}(F_{qsq^{-1}}(X)))) \otimes \mathrm{id}_{F_q(W)}] \circ [\delta^q_{V,F_{q^{-1}}(F_{qsq^{-1}}(X))} \otimes \mathrm{id}_{F_q(W)}] \\
&\quad \circ [(\mathrm{id}_{F_q(V)} \otimes \Delta_q(F_{qsq^{-1}}(X))^{-1}) \otimes \mathrm{id}_{F_q(W)}]\} \circ a^{-1}_{F_q(V),F_{qsq^{-1}}(X),F_q(W)} \\
&\quad \circ \{[\mathrm{id}_{F_q(V)} \otimes (\alpha_{q,s,q^{-1}}(X) \otimes \mathrm{id}_{F_q(W)})] \circ [\mathrm{id}_{F_q(V)} \otimes (\delta^q_{F_s(F_{q^{-1}}(X)),W})^{-1}] \\
&\quad \circ [\mathrm{id}_{F_q(V)} \otimes F_q(\Phi_W(F_{q^{-1}}(X)))] \circ [\mathrm{id}_{F_q(V)} \otimes \delta^q_{W,F_{q^{-1}}(X)}] \\
&\quad \circ [\mathrm{id}_{F_q(V)} \otimes (\mathrm{id}_{F_q(W)} \otimes \Delta_q(X)^{-1})]\} \circ a_{F_q(V),F_q(W),X}
\end{aligned}
$$

$$= [(\delta^F_{qrq^{-1},qsq^{-1}})_X \otimes \mathrm{id}_{F_q(V)\otimes F_q(W)}] \circ a_{F_{qrq^{-1}}(F_{qsq^{-1}}(X)),F_q(V),F_q(W)}$$
$$\circ [\mathcal{F}_q\Phi_V(F_{qsq^{-1}}(X)) \otimes \mathrm{id}_{F_q(W)}] \circ a^{-1}_{F_q(V),F_{qsq^{-1}}(X),F_q(W)}$$
$$\circ [\mathrm{id}_{F_q(V)} \otimes \mathcal{F}_q\Phi_W(X)] \circ a_{F_q(V),F_q(W),X}$$
$$= (\mathcal{F}_q\Phi_V \otimes \mathcal{F}_q\Phi_W)(X),$$

which can be rewritten as

$$[\mathcal{F}_q(\Phi_V \otimes \Phi_W)(X)] \circ [\delta^q_{V,W} \otimes \mathrm{id}_X] = [\mathrm{id}_{F_{qrsq^{-1}}(X)} \otimes (\delta^q_{V,W})] \circ (\mathcal{F}_q\Phi_V \otimes \mathcal{F}_q\Phi_W)(X),$$

which proves that $\delta^q_{V,W} \in \mathrm{Hom}_{Z_G(\mathcal{C})}(\mathcal{F}_q[(V,r,\Phi_V)] \otimes \mathcal{F}_q[(W,s,\Phi_W)], \mathcal{F}_q[(V\otimes W, qr, \Phi_V \otimes \Phi_W)])$. Also, for each $X \in \mathcal{C}$ we have

$$[(\mathcal{F}_q\Phi^0_I)(X)] \circ [\varepsilon^q \otimes \mathrm{id}_X]$$
$$= [\alpha_{q,e,q^{-1}}(X) \otimes \mathrm{id}_{F_q(I)}] \circ \{(\delta^q_{F_e(F_{q^{-1}}(X)),I})^{-1} \circ F_q(r_{F_e(F_{q^{-1}}(X))})^{-1}\} \circ F_q(\varepsilon^F_{F_{q^{-1}}(X)})$$
$$\circ F_q(l_{F_{q^{-1}}(X)}) \circ \delta^q_{I,F_{q^{-1}}(X)} \circ \{[\mathrm{id}_{F_q(I)} \otimes \Delta_q(X)^{-1}] \circ [\varepsilon^q \otimes \mathrm{id}_X]\}$$
$$= \{[\alpha_{q,e,q^{-1}}(X) \otimes \mathrm{id}_{F_q(I)}] \circ [\mathrm{id}_{F_q(F_e(F_{q^{-1}}(X)))} \otimes \varepsilon^q]\} \circ r^{-1}_{F_q(F_e(F_{q^{-1}}(X)))} \circ F_q(\varepsilon^F_{F_{q^{-1}}(X)})$$
$$\circ \{F_q(l_{F_{q^{-1}}(X)}) \circ \delta^q_{I,F_{q^{-1}}(X)} \circ [\varepsilon^q \otimes \mathrm{id}_{F_q(F_{q^{-1}}(X))}]\} \circ [\mathrm{id}_I \otimes \Delta_q(X)^{-1}]$$
$$= [\mathrm{id}_{F_e(X)} \otimes \varepsilon^q] \circ \{[\alpha_{q,e,q^{-1}}(X) \otimes \mathrm{id}_I] \circ r^{-1}_{F_q(F_e(F_{q^{-1}}(X)))}\}$$
$$\circ F_q(\varepsilon^F_{F_{q^{-1}}(X)}) \circ \{l_{F_q(F_{q^{-1}}(X))} \circ [\mathrm{id}_I \otimes \Delta_q(X)^{-1}]\}$$
$$= [\mathrm{id}_{F_e(X)} \otimes \varepsilon^q] \circ r^{-1}_{F_e(X)} \circ \{\alpha_{q,e,q^{-1}}(X) \circ \underbrace{F_q(\varepsilon^F_{F_{q^{-1}}(X)})}_{=(\delta^F_{q,e})^{-1}_{F_{q^{-1}}(X)}} \circ \Delta_q(X)^{-1}\} \circ l_X$$
$$= [\mathrm{id}_{F_e(X)} \otimes \varepsilon^q] \circ r^{-1}_{F_e(X)} \circ (\delta^F_{qe,q^{-1}})_X \circ (\delta^F_{q,e})_{F_{q^{-1}}(X)} \circ (\delta^F_{q,e})^{-1}_{F_{q^{-1}}(X)} \circ (\delta^F_{q,q^{-1}})^{-1}_X \circ \varepsilon^F_X \circ l_X$$
$$= [\mathrm{id}_{F_e(X)} \otimes \varepsilon^q] \circ r^{-1}_{F_e(X)} \circ \varepsilon^F_X \circ l_X = [\mathrm{id}_{F_e(X)} \otimes \varepsilon^q] \circ \Phi^0_I(X).$$

This proves that $\varepsilon^q \in \mathrm{Hom}_{Z_G(\mathcal{C})}((I,e,\Phi^0_I), \mathcal{F}_q[(I,e,\Phi^0_I)])$. Because $(F_q, \varepsilon^q, \delta^q)$ is a tensor functor from $\mathcal{C}$ to itself and because the composition and tensor product of morphisms in $Z_G(\mathcal{C})$ are the same as in $\mathcal{C}$, it is clear that $\delta^{\mathcal{F}_q}$ and $\varepsilon^{\mathcal{F}_q}$ satisfy all conditions to make $(\mathcal{F}_q, \varepsilon^{\mathcal{F}_q}, \delta^{\mathcal{F}_q})$ a tensor functor.

We will now prove that $(\mathcal{F}, \varepsilon^{\mathcal{F}}, \delta^{\mathcal{F}})$ defines a G-action on $Z_G(\mathcal{C})$, where $\varepsilon^{\mathcal{F}}_{(V,q,\Phi_V)} = \varepsilon^F_V$ and $(\delta^{\mathcal{F}}_{q,r})_{(V,s,\Phi_V)} = (\delta^F_{q,r})_V$. To show that the $\delta^{\mathcal{F}}$ are morphisms in $Z_G(\mathcal{C})$, let $(V,s,\Phi_V) \in Z_G(\mathcal{C})$ and $q,r \in G$. Using the equality

$$(F_r\Phi_V)(F_{q^{-1}}(X))$$
$$= [\alpha_{r,s,r^{-1}}(F_{q^{-1}}(X)) \otimes \mathrm{id}_{F_r(V)}] \circ (\delta^r_{F_s(F_{r^{-1}}(F_{q^{-1}}(X))),V})^{-1} \circ F_r(\Phi_V(F_{r^{-1}}(F_{q^{-1}}(X))))$$
$$\circ \delta^r_{V,F_{r^{-1}}(F_{q^{-1}}(X))} \circ [\mathrm{id}_{F_r(V)} \otimes \Delta_r(F_{q^{-1}}(X))^{-1}],$$

we obtain

$$(\mathcal{F}_q(\mathcal{F}_r\Phi_V))(X)$$
$$= [\alpha_{q,rsr^{-1},q^{-1}}(X) \otimes \mathrm{id}_{F_q(F_r(V))}] \circ (\delta^q_{F_{rsr^{-1}}(F_{q^{-1}}(X)),F_r(V)})^{-1} \circ F_q(\mathcal{F}_r\Phi_V(F_{q^{-1}}(X)))$$
$$\circ \delta^q_{F_r(V),F_{q^{-1}}(X)} \circ [\mathrm{id}_{F_q(F_r(V))} \otimes \Delta_q(X)^{-1}]$$
$$= [\alpha_{q,rsr^{-1},q^{-1}}(X) \otimes \mathrm{id}_{F_q(F_r(V))}] \circ \{(\delta^q_{F_{rsr^{-1}}(F_{q^{-1}}(X)),F_r(V)})^{-1} \circ F_q(\alpha_{r,s,r^{-1}}(F_{q^{-1}}(X)) \otimes \mathrm{id}_{F_r(V)})\}$$

$$
\begin{aligned}
&\circ F_q((\delta^r_{F_s(F_{r^{-1}}(F_{q^{-1}}(X))),V})^{-1}) \circ F_q(F_r(\Phi_V(F_{r^{-1}}(F_{q^{-1}}(X))))) \circ F_q(\delta^r_{V,F_{r^{-1}}(F_{q^{-1}}(X))})\\
&\circ \{F_q(\mathrm{id}_{F_r(V)} \otimes \Delta_r(F_{q^{-1}}(X))^{-1}) \circ \delta^q_{F_r(V),F_{q^{-1}}(X)}\} \circ [\mathrm{id}_{F_q(F_r(V))} \otimes \Delta_q(X)^{-1}]\\
= &\{[\alpha_{q,rsr^{-1},q^{-1}}(X) \otimes \mathrm{id}_{F_q(F_r(V))}] \circ [F_q(\alpha_{r,s,r^{-1}}(F_{q^{-1}}(X))) \otimes \mathrm{id}_{F_q(F_r(V))}]\}\\
&\circ (\delta^q_{F_r(F_s(F_{r^{-1}}(F_{q^{-1}}(X)))),F_r(V)})^{-1} \circ F_q((\delta^r_{F_s(F_{r^{-1}}(F_{q^{-1}}(X))),V})^{-1}) \circ F_q(F_r(\Phi_V(F_{r^{-1}}(F_{q^{-1}}(X)))))\\
&\circ F_q(\delta^r_{V,F_{r^{-1}}(F_{q^{-1}}(X))}) \circ \delta^q_{F_r(V),F_r(F_{r^{-1}}(F_{q^{-1}}(X)))} \circ \{[\mathrm{id}_{F_q(F_r(V))} \otimes F_q(\Delta_r(F_{q^{-1}}(X)))^{-1}]\\
&\circ [\mathrm{id}_{F_q(F_r(V))} \otimes \Delta_q(X)^{-1}]\}\\
\overset{(*)}{=} &[\alpha_{qr,s,r^{-1}q^{-1}}(X) \otimes \mathrm{id}_{F_q(F_r(V))}] \circ [F_{qr}(F_s((\delta^F_{r^{-1},q^{-1}})_X)) \otimes \mathrm{id}_{F_q(F_r(V))}]\\
&\circ [(\delta^F_{q,r})_{F_s(F_{r^{-1}}(F_{q^{-1}}(X)))} \otimes \mathrm{id}_{F_q(F_r(V))}] \circ \{(\delta^q_{F_r(F_s(F_{r^{-1}}(F_{q^{-1}}(X)))),F_r(V)})^{-1}\\
&\circ F_q((\delta^r_{F_s(F_{r^{-1}}(F_{q^{-1}}(X))),V})^{-1})\} \circ F_q(F_r(\Phi_V(F_{r^{-1}}(F_{q^{-1}}(X))))) \circ \{F_q(\delta^r_{V,F_{r^{-1}}(F_{q^{-1}}(X))})\\
&\circ \delta^q_{F_r(V),F_r(F_{r^{-1}}(F_{q^{-1}}(X)))}\} \circ [\mathrm{id}_{F_q(F_r(V))} \otimes (\delta^F_{q,r})^{-1}_{F_{r^{-1}}(F_{q^{-1}}(X))}]\\
&\circ [\mathrm{id}_{F_q(F_r(V))} \otimes F_{qr}((\delta^F_{r^{-1},q^{-1}})_X)^{-1}] \circ [\mathrm{id}_{F_q(F_r(V))} \otimes \Delta_{qr}(X)^{-1}]\\
= &\{[\alpha_{qr,s,r^{-1}q^{-1}}(X) \otimes \mathrm{id}_{F_q(F_r(V))}] \circ [F_{qr}(F_s((\delta^F_{r^{-1},q^{-1}})_X)) \otimes \mathrm{id}_{F_q(F_r(V))}]\\
&\circ [(\delta^F_{q,r})_{F_s(F_{r^{-1}}(F_{q^{-1}}(X)))} \otimes \mathrm{id}_{F_q(F_r(V))}] \circ [(\delta^F_{q,r})^{-1}_{F_s(F_{r^{-1}}(F_{q^{-1}}(X)))} \otimes (\delta^F_{q,r})^{-1}_V]\}\\
&\circ (\delta^{qr}_{F_s(F_{r^{-1}}(F_{q^{-1}}(X))),V})^{-1} \circ \{(\delta^F_{q,r})_{F_s(F_{r^{-1}}(F_{q^{-1}}(X)))\otimes V} \circ F_q(F_r(\Phi_V(F_{r^{-1}}(F_{q^{-1}}(X)))))\\
&\circ (\delta^F_{q,r})^{-1}_{V\otimes F_{r^{-1}}(F_{q^{-1}}(X))}\} \circ \delta^{qr}_{V,F_{r^{-1}}(F_{q^{-1}}(X))} \circ \{[(\delta^F_{q,r})_V \otimes (\delta^F_{q,r})_{F_{r^{-1}}(F_{q^{-1}}(X))}]\\
&\circ [\mathrm{id}_{F_q(F_r(V))} \otimes (\delta^F_{q,r})^{-1}_{F_{r^{-1}}(F_{q^{-1}}(X))}] \circ [\mathrm{id}_{F_q(F_r(V))} \otimes F_{qr}((\delta^F_{r^{-1},q^{-1}})_X)^{-1}]\\
&\circ [\mathrm{id}_{F_q(F_r(V))} \otimes \Delta_{qr}(X)^{-1}]\}\\
= &[\mathrm{id}_{F_{qrsr^{-1}q^{-1}}(X)} \otimes (\delta^F_{q,r})^{-1}_V] \circ [\alpha_{qr,s,r^{-1}q^{-1}}(X) \otimes \mathrm{id}_{F_{qr}(V)}] \circ \{[F_{qr}(F_s((\delta^F_{r^{-1},q^{-1}})_X)) \otimes \mathrm{id}_{F_{qr}(V)}]\\
&\circ (\delta^{qr}_{F_s(F_{r^{-1}}(F_{q^{-1}}(X))),V})^{-1}\} \circ F_{qr}(\Phi_V(F_{r^{-1}}(F_{q^{-1}}(X)))) \circ \{\delta^{qr}_{V,F_{r^{-1}}(F_{q^{-1}}(X))}\\
&\circ [\mathrm{id}_{F_{qr}(V)} \otimes F_{qr}((\delta^F_{r^{-1},q^{-1}})_X)^{-1}]\} \circ [\mathrm{id}_{F_{qr}(V)} \otimes \Delta_{qr}(X)^{-1}] \circ [(\delta^F_{q,r})_V \otimes \mathrm{id}_X]\\
= &[\mathrm{id}_{F_{qrsr^{-1}q^{-1}}(X)} \otimes (\delta^F_{q,r})^{-1}_V] \circ [\alpha_{qr,s,r^{-1}q^{-1}}(X) \otimes \mathrm{id}_{F_{qr}(V)}] \circ (\delta^{qr}_{F_s(F_{r^{-1}q^{-1}}(X)),V})^{-1}\\
&\circ \{F_{qr}(F_s((\delta^F_{r^{-1},q^{-1}})_X) \otimes \mathrm{id}_V) \circ F_{qr}(\Phi_V(F_{r^{-1}}(F_{q^{-1}}(X)))) \circ F_{qr}(\mathrm{id}_V \otimes (\delta^F_{r^{-1},q^{-1}})^{-1}_X)\}\\
&\circ \delta^{qr}_{V,F_{r^{-1}q^{-1}}(X)} \circ [\mathrm{id}_{F_{qr}(V)} \otimes \Delta_{qr}(X)^{-1}] \circ [(\delta^F_{q,r})_V \otimes \mathrm{id}_X]\\
= &[\mathrm{id}_{F_{qrsr^{-1}q^{-1}}(X)} \otimes (\delta^F_{q,r})^{-1}_V] \circ [\alpha_{qr,s,r^{-1}q^{-1}}(X) \otimes \mathrm{id}_{F_{qr}(V)}] \circ (\delta^{qr}_{F_s(F_{r^{-1}q^{-1}}(X)),V})^{-1}\\
&\circ F_{qr}(\Phi_V(F_{r^{-1}q^{-1}}(X))) \circ \delta^{qr}_{V,F_{r^{-1}q^{-1}}(X)} \circ [\mathrm{id}_{F_{qr}(V)} \otimes \Delta_{qr}(X)^{-1}]\\
&\circ [(\delta^F_{q,r})_V \otimes \mathrm{id}_X]\\
= &[\mathrm{id}_{F_{qrsr^{-1}q^{-1}}(X)} \otimes (\delta^F_{q,r})^{-1}_V] \circ \mathcal{F}_{qr}\Phi_V(X) \circ [(\delta^F_{q,r})_V \otimes \mathrm{id}_X],
\end{aligned}
$$

which can be rewritten as

$$[\mathrm{id}_{F_{qrsr^{-1}q^{-1}}(X)} \otimes (\delta^F_{q,r})_V] \circ (\mathcal{F}_q(\mathcal{F}_r\Phi_V))(X) = \mathcal{F}_{qr}\Phi_V(X) \circ [(\delta^F_{q,r})_V \otimes \mathrm{id}_X],$$

implying that $(\delta^F_{q,r})_V \in \mathrm{Hom}_{Z_G(\mathcal{C})}(\mathcal{F}_q[\mathcal{F}_r[(V,s,\Phi_V)],\mathcal{F}_{qr}[(V,s,\Phi_V)])$. We will now explain the equality $\overset{(*)}{=}$ above in some detail. We have

$$\alpha_{q,rsr^{-1},q^{-1}}(X) \circ F_q(\alpha_{r,s,r^{-1}}(F_{q^{-1}}(X)))$$

$$
\begin{aligned}
&= (\delta^F_{qrsr^{-1},q^{-1}})_X \circ \underbrace{(\delta^F_{q,rsr^{-1}})_{F_{q^{-1}}(X)} \circ F_q((\delta^F_{rs,r^{-1}})_{F_{q^{-1}}(X)})}_{=(\delta^F_{qrs,r^{-1}})_{F_{q^{-1}}(X)} \circ (\delta^F_{q,rs})_{F_{r^{-1}}(F_{q^{-1}}(X))}} \circ F_q((\delta^F_{r,s})_{F_{r^{-1}}(F_{q^{-1}}(X))}) \\
&= \underbrace{(\delta^F_{qrsr^{-1},q^{-1}})_X \circ (\delta^F_{qrs,r^{-1}})_{F_{q^{-1}}(X)}}_{=(\delta^F_{qrs,r^{-1}q^{-1}})_X \circ F_{qrs}((\delta^F_{r^{-1},q^{-1}})_X)} \circ \underbrace{(\delta^F_{q,rs})_{F_{r^{-1}}(F_{q^{-1}}(X))} \circ F_q((\delta^F_{r,s})_{F_{r^{-1}}(F_{q^{-1}}(X))})}_{=(\delta^F_{qr,s})_{F_{r^{-1}}(F_{q^{-1}}(X))} \circ (\delta^F_{q,r})_{F_s(F_{r^{-1}}(F_{q^{-1}}(X)))}} \\
&= (\delta^F_{qrs,r^{-1}q^{-1}})_X \circ \underbrace{F_{qrs}((\delta^F_{r^{-1},q^{-1}})_X) \circ (\delta^F_{qr,s})_{F_{r^{-1}}(F_{q^{-1}}(X))}}_{=(\delta^F_{qr,s})_{F_{r^{-1}q^{-1}}(X)} \circ F_{qr}(F_s(\delta^F_{r^{-1},q^{-1}})_X)} \circ (\delta^F_{q,r})_{F_s(F_{r^{-1}}(F_{q^{-1}}(X)))} \\
&= (\delta^F_{qrs,r^{-1}q^{-1}})_X \circ (\delta^F_{qr,s})_{F_{r^{-1}q^{-1}}(X)} \circ F_{qr}(F_s(\delta^F_{r^{-1},q^{-1}})_X) \circ (\delta^F_{q,r})_{F_s(F_{r^{-1}}(F_{q^{-1}}(X)))} \\
&= \alpha_{qr,s,r^{-1}q^{-1}}(X) \circ F_{qr}(F_s(\delta^F_{r^{-1},q^{-1}})_X) \circ (\delta^F_{q,r})_{F_s(F_{r^{-1}}(F_{q^{-1}}(X)))}
\end{aligned}
$$

and

$$
\begin{aligned}
F_q(\Delta_r(F_{q^{-1}}(X)))^{-1} \circ \Delta_q(X)^{-1} &= F_q((\delta^F_{r,r^{-1}})^{-1}_{F_{q^{-1}}(X)} \circ \varepsilon^F_{F_{q^{-1}}(X)}) \circ (\delta^F_{q,q^{-1}})^{-1}_X \circ \varepsilon^F_X \\
&= F_q((\delta^F_{r,r^{-1}})_{F_{q^{-1}}(X)})^{-1} \circ \underbrace{F_q(\varepsilon^F_{F_{q^{-1}}(X)})}_{=(\delta^F_{q,e})^{-1}_{F_{q^{-1}}(X)}} \circ (\delta^F_{q,q^{-1}})^{-1} \circ \varepsilon^F_X \\
&\overset{(**)}{=} (\delta^F_{q,r})^{-1}_{F_{r^{-1}}(F_{q^{-1}}(X))} \circ F_{qr}((\delta^F_{r^{-1},q^{-1}})_X)^{-1} \circ (\delta^F_{qr,r^{-1}q^{-1}})^{-1}_X \circ \varepsilon^F_X \\
&= (\delta^F_{q,r})^{-1}_{F_{r^{-1}}(F_{q^{-1}}(X))} \circ F_{qr}((\delta^F_{r^{-1},q^{-1}})_X)^{-1} \circ \Delta_{qr}(X)^{-1},
\end{aligned}
$$

where $\overset{(**)}{=}$ follows from commutativity of the diagram

$$
\begin{array}{ccccc}
F_e(X) & \xrightarrow{(\delta^F_{q,q^{-1}})^{-1}_X} & F_q(F_{q^{-1}}(X)) & \xrightarrow{(\delta^F_{q,e})^{-1}_{F_{q^{-1}}(X)}} & F_q(F_e(F_{q^{-1}}(X))) \\
\big\downarrow{\scriptstyle (\delta^F_{qr,r^{-1}q^{-1}})_X} & & \big\downarrow{\scriptstyle (\delta^F_{qr,r^{-1}})_{F_{q^{-1}}(X)}} & & \big\downarrow{\scriptstyle F_q((\delta^F_{r,r^{-1}})_{F_{q^{-1}}(X)})^{-1}} \\
F_{qr}(F_{r^{-1}q^{-1}}(X)) & \xrightarrow{F_{qr}((\delta^F_{r^{-1},q^{-1}})^{-1}_X)} & F_{qr}(F_{r^{-1}}(F_{q^{-1}}(X))) & \xrightarrow{(\delta^F_{q,r})^{-1}_{F_{r^{-1}}(F_{q^{-1}}(X))}} & F_q(F_r(F_{r^{-1}}(F_{q^{-1}}(X)))).
\end{array}
$$

Let $(V, q, \Phi_V) \in Z_G(\mathcal{C})$. Then

$$
\begin{aligned}
&(\mathcal{F}_e\Phi_V)(X) \\
&= [\alpha_{e,q,e}(X) \otimes \mathrm{id}_{F_e(V)}] \circ (\delta^e_{F_q(F_e(X)),V})^{-1} \circ F_e(\Phi_V(F_e(X))) \circ \delta^e_{V,F_e(X)} \circ [\mathrm{id}_{F_e(V)} \otimes \Delta_e(X)^{-1}] \\
&= [(\delta^F_{e,q})_X \otimes \mathrm{id}_{F_e(V)}] \circ [F_e((\delta^F_{q,e})_X) \otimes \mathrm{id}_{F_e(V)}] \circ (\delta^e_{F_q(F_e(X)),V})^{-1} \circ F_e(\Phi_V(F_e(X))) \\
&\quad \circ \delta^e_{V,F_e(X)} \circ [\mathrm{id}_{F_e(V)} \otimes (\delta^F_{e,e})^{-1}] \circ [\mathrm{id}_{F_e(V)} \otimes \varepsilon^F_X] \\
&= [(\varepsilon^F_{F_q(X)})^{-1} \otimes \mathrm{id}_{F_e(V)}] \circ [F_e(F_q(\varepsilon^F_X))^{-1} \otimes \mathrm{id}_{F_e(V)}] \circ (\delta^e_{F_q(F_e(X)),V})^{-1} \circ F_e(\Phi_V(F_e(X))) \\
&\quad \circ \delta^e_{V,F_e(X)} \circ [\mathrm{id}_{F_e(V)} \otimes F_e(\varepsilon^F_X)] \circ [\mathrm{id}_{F_e(V)} \otimes \varepsilon^F_X] \\
&= [(\varepsilon^F_{F_q(X)})^{-1} \otimes \mathrm{id}_{F_e(V)}] \circ (\delta^e_{F_q(X),V})^{-1} \circ F_e(F_q(\varepsilon^F_X)^{-1} \otimes \mathrm{id}_V) \circ F_e(\Phi_V(F_e(X))) \\
&\quad \circ F_e(\mathrm{id}_V \otimes \varepsilon^F_X) \circ \delta^e_{V,X} \circ [\mathrm{id}_{F_e(V)} \otimes \varepsilon^F_X] \\
&= [(\varepsilon^F_{F_q(X)})^{-1} \otimes \mathrm{id}_{F_e(V)}] \circ (\delta^e_{F_q(X),V})^{-1} \circ F_e(\Phi_V(X)) \circ \delta^e_{V,X} \circ [\mathrm{id}_{F_e(V)} \otimes \varepsilon^F_X] \\
&= [\mathrm{id}_{F_q(X)} \otimes \varepsilon^F_V] \circ [(\varepsilon^F_{F_q(X)})^{-1} \otimes (\varepsilon^F_V)^{-1}] \circ (\delta^e_{F_q(X),V})^{-1} \circ F_e(\Phi_V(X))
\end{aligned}
$$

$$\circ\, \delta^e_{V,X} \circ [\varepsilon^F_V \otimes \varepsilon^F_X] \circ [(\varepsilon^F_V)^{-1} \otimes \mathrm{id}_X]$$
$$= [\mathrm{id}_{F_q(X)} \otimes \varepsilon^F_V] \circ (\varepsilon^F_{F_q(X)\otimes V})^{-1} \circ F_e(\Phi_V(X)) \circ \varepsilon^F_{V\otimes X} \circ [(\varepsilon^F_V)^{-1} \otimes \mathrm{id}_X]$$
$$= [\mathrm{id}_{F_q(X)} \otimes \varepsilon^F_V] \circ \Phi_V(X) \circ [(\varepsilon^F_V)^{-1} \otimes \mathrm{id}_X],$$

which can be rewritten as

$$(\mathcal{F}_e\Phi_V)(X) \circ [\varepsilon^F_V \otimes \mathrm{id}_X] = [\mathrm{id}_{F_q(X)} \otimes \varepsilon^F_V] \circ \Phi_V(X),$$

which shows that $\varepsilon^F_V \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V,q,\Phi_V), \mathcal{F}_e[(V,q,\Phi_V)])$. Because $(F,\varepsilon^F,\delta^F)$ is a G-action on $\mathcal{C}$ and because the composition and tensor product of morphisms in $Z_G(\mathcal{C})$ is the same as in $\mathcal{C}$, it follows directly that $(\mathcal{F},\varepsilon^{\mathcal{F}},\delta^{\mathcal{F}})$ satisfies all conditions for a G-action on $Z_G(\mathcal{C})$. It is also clear that

$$\partial[\mathcal{F}_q[(V,r,\Phi_V)]] = \partial[(F_q(V), qrq^{-1}, \mathcal{F}_q\Phi_V)] = qrq^{-1} = q\partial[(V,r,\Phi_V)]q^{-1}$$

for all $q \in G$ and $(V,r,\Phi_V) \in Z_G(\mathcal{C})$, so $Z_G(\mathcal{C})$ is indeed a G-crossed category.
□

A.5 The braiding

Lemma A.5.1 *Let $(\mathcal{C},\otimes,I,a,l,r)$ be a tensor category with G-action $(F,\varepsilon^F,\delta^F)$. Then $Z_G(\mathcal{C})$ becomes a braided G-crossed category if we define a braiding*

$$C_{(V,q,\Phi_V),(W,r,\Phi_V)} : (V,q,\Phi_V) \otimes (W,r,\Phi_W) \to \mathcal{F}_q[(W,r,\Phi_W)] \otimes (V,q,\Phi_V),$$

by $C_{(V,q,\Phi_V),(W,r,\Phi_W)} := \Phi_V(W)$.

Proof. Let $(V,q,\Phi_V),(W,r,\Phi_W) \in Z_G(\mathcal{C})$. Then $C_{(V,q,\Phi_V),(W,r,\Phi_W)} = \Phi_V(W) \in \mathrm{Hom}_{\mathcal{C}}(V\otimes W, F_q(W)\otimes V)$ and

$$[(\mathcal{F}_q\Phi_W \otimes \Phi_V)(X)] \circ [C_{(V,q,\Phi_V),(W,r,\Phi_W)} \otimes \mathrm{id}_X] = [(\mathcal{F}_q\Phi_W \otimes \Phi_V)(X)] \circ [\Phi_V(W) \otimes \mathrm{id}_X]$$
$$= [(\delta^F_{qrq^{-1},q})_X \otimes \mathrm{id}_{F_q(W)\otimes V}] \circ a_{F_{qrq^{-1}}(F_q(X)),F_q(W),V} \circ [\mathcal{F}_q\Phi_W(F_q(X)) \otimes \mathrm{id}_V]$$
$$\circ\, a^{-1}_{F_q(W),F_q(X),V} \circ [\mathrm{id}_{F_q(W)} \otimes \Phi_V(X)] \circ a_{F_q(W),V,X} \circ [\Phi_V(W) \otimes \mathrm{id}_X]$$
$$= [(\delta^F_{qrq^{-1},q})_X \otimes \mathrm{id}_{F_q(W)\otimes V}] \circ a_{F_{qrq^{-1}}(F_q(X)),F_q(W),V} \circ [(\alpha_{q,r,q^{-1}}(F_q(X)) \otimes \mathrm{id}_{F_q(W)}) \otimes \mathrm{id}_V]$$
$$\circ\, [(\delta^q_{F_r(F_{q^{-1}}(F_q(X))),W})^{-1} \otimes \mathrm{id}_V] \circ [F_q(\Phi_W(F_{q^{-1}}(F_q(X)))) \otimes \mathrm{id}_V] \circ \{[\delta^q_{W,F_{q^{-1}}(F_q(X))} \otimes \mathrm{id}_V]$$
$$\circ\, [(\mathrm{id}_{F_q(W)} \otimes \Delta_q(F_q(X))^{-1}) \otimes \mathrm{id}_V]\} \circ a^{-1}_{F_q(W),F_q(X),V} \circ [\mathrm{id}_{F_q(W)} \otimes \Phi_V(X)] \circ a_{F_q(W),V,X}$$
$$\circ\, [\Phi_V(W) \otimes \mathrm{id}_X]$$
$$\stackrel{(*)}{=} [(\delta^F_{qrq^{-1},q})_X \otimes \mathrm{id}_{F_q(W)\otimes V}] \circ a_{F_{qrq^{-1}}(F_q(X)),F_q(W),V} \circ [(\alpha_{q,r,q^{-1}}(F_q(X)) \otimes \mathrm{id}_{F_q(W)}) \otimes \mathrm{id}_V]$$
$$\circ\, [(\delta^q_{F_r(F_{q^{-1}}(F_q(X))),W})^{-1} \otimes \mathrm{id}_V] \circ \{[F_q(\Phi_W(F_{q^{-1}}(F_q(X)))) \otimes \mathrm{id}_V]$$
$$\circ\, [F_q(\mathrm{id}_W \otimes (\delta^F_{q^{-1},q})^{-1}_X) \otimes \mathrm{id}_V] \circ [F_q(\mathrm{id}_W \otimes \varepsilon^F_X) \otimes \mathrm{id}_V]\} \circ \{[\delta^q_{W,X} \otimes \mathrm{id}_V] \circ a^{-1}_{F_q(W),F_q(X),V}$$
$$\circ\, [\mathrm{id}_{F_q(W)} \otimes \Phi_V(X)] \circ a_{F_q(W),V,X} \circ [\Phi_V(W) \otimes \mathrm{id}_X] \circ a^{-1}_{V,W,X}\} \circ a_{V,W,X}$$
$$= [(\delta^F_{qrq^{-1},q})_X \otimes \mathrm{id}_{F_q(W)\otimes V}] \circ [\alpha_{q,r,q^{-1}}(F_q(X)) \otimes \mathrm{id}_{F_q(W)\otimes V}] \circ \{a_{F_q(F_r(F_{q^{-1}}(F_q(X)))),F_q(W),V}$$
$$\circ\, [(\delta^q_{F_r(F_{q^{-1}}(F_q(X))),W})^{-1} \otimes \mathrm{id}_V] \circ [F_q(F_r((\delta^F_{q^{-1},q})_X)^{-1} \otimes \mathrm{id}_W) \otimes \mathrm{id}_V]$$
$$\circ\, [F_q(F_r(\varepsilon^F_X) \otimes \mathrm{id}_W) \otimes \mathrm{id}_V]\} \circ \{[F_q(\Phi_W(X)) \otimes \mathrm{id}_V] \circ \Phi_V(W \otimes X)\} \circ a_{V,W,X}$$

$$
\begin{aligned}
&= \{[(\delta^F_{qrq^{-1},q})_X \otimes \mathrm{id}_{F_q(W)\otimes V}] \circ [\alpha_{q,r,q^{-1}}(F_q(X)) \otimes \mathrm{id}_{F_q(W)\otimes V}] \\
&\quad \circ [F_q(F_r((\delta^F_{q^{-1},q})_X))^{-1} \otimes \mathrm{id}_{F_q(W)\otimes V}] \circ [F_q(F_r(\varepsilon^F_X)) \otimes \mathrm{id}_{F_q(W)\otimes V}]\} \circ a_{F_q(F_r(X)),F_q(W),V} \\
&\quad \circ [(\delta^q_{F_r(X),W})^{-1} \otimes \mathrm{id}_V] \circ \Phi_V(F_r(X)\otimes W) \circ [\mathrm{id}_V \otimes \Phi_W(X)] \circ a_{V,W,X} \\
&\overset{(**)}{=} [(\delta^F_{q,r})_X \otimes \mathrm{id}_{F_q(W)\otimes V}] \circ a_{F_q(F_r(X)),F_q(W),V} \circ [(\delta^q_{F_r(X),W})^{-1} \otimes \mathrm{id}_V] \\
&\quad \circ \Phi_V(F_r(X)\otimes W) \circ [\mathrm{id}_V \otimes \Phi_W(X)] \circ a_{V,W,X} \\
&= [(\delta^F_{q,r})_X \otimes \mathrm{id}_{F_q(W)\otimes V}] \circ a_{F_q(F_r(X)),F_q(W),V} \circ [(\delta^q_{F_r(X),W})^{-1} \otimes \mathrm{id}_V] \circ [(\delta^q_{F_r(X),W}) \otimes \mathrm{id}_V] \\
&\quad \circ a^{-1}_{F_q(F_r(X)),F_q(W),V} \circ [\mathrm{id}_{F_q(F_r(X))} \otimes \Phi_V(W)] \circ a_{F_q(F_r(X)),V,W} \circ [\Phi_V(F_r(X)) \otimes \mathrm{id}_W] \\
&\quad \circ a_{V,F_r(X),W} \circ [\mathrm{id}_V \otimes \Phi_W(X)] \circ a_{V,W,X} \\
&= \{[(\delta^F_{q,r})_X \otimes \mathrm{id}_{F_q(W)\otimes V}] \circ [\mathrm{id}_{F_q(F_r(X))} \otimes \Phi_V(W)]\} \circ a_{F_q(F_r(X)),V,W} \circ [\Phi_V(F_r(X)) \otimes \mathrm{id}_W] \\
&\quad \circ a^{-1}_{V,F_r(X),W} \circ [\mathrm{id}_V \otimes \Phi_W(X)] \circ a_{V,W,X} \\
&= [\mathrm{id}_{F_{qr}(X)} \otimes \Phi_V(W)] \circ [(\delta^F_{q,r})_X \otimes \mathrm{id}_{V\otimes W}] \circ a_{F_q(F_r(X)),V,W} \circ [\Phi_V(F_r(X)) \otimes \mathrm{id}_W] \\
&\quad \circ a^{-1}_{V,F_r(X),W} \circ [\mathrm{id}_V \otimes \Phi_W(X)] \circ a_{V,W,X} \\
&= [\mathrm{id}_{F_{qr}(X)} \otimes \Phi_V(W)] \circ [(\Phi_V \otimes \Phi_W)(X)] = [\mathrm{id}_{F_{qr}(X)} \otimes C_{(V,q,\Phi_V),(W,r,\Phi_W)}] \circ [(\Phi_V \otimes \Phi_W)(X)],
\end{aligned}
$$

where in $\overset{(*)}{=}$ we used that

$$
\begin{aligned}
\Delta_q(F_q(X)) &= \underbrace{(\varepsilon^F_{F_q(X)})^{-1}}_{=(\delta^F_{e,q})_X} \circ (\delta^F_{q,q^{-1}})_{F_q(X)} = (\delta^F_{qq^{-1},q})_X \circ (\delta^F_{q,q^{-1}})_{F_q(X)} \\
&= (\delta^F_{q,q^{-1}q})_X \circ F_q((\delta^F_{q^{-1},q})_X) = F_q(\varepsilon^F_X)^{-1} \circ F_q((\delta^F_{q^{-1},q})_X),
\end{aligned}
$$

so that $\Delta_q(F_q(X))^{-1} = F_q((\delta^F_{q^{-1},q})_X)^{-1} \circ F_q(\varepsilon^F_X)$, and $\overset{(**)}{=}$ follows from

$$
\begin{aligned}
&(\delta^F_{qrq^{-1},q})_X \circ \alpha_{q,r,q^{-1}}(F_q(X)) \circ F_q(F_r((\delta^F_{q^{-1},q})_X))^{-1} \circ F_q(F_r(\varepsilon^F_X)) \\
&= (\delta^F_{qrq^{-1},q})_X \circ (\delta^F_{q,rq^{-1}})_{F_q(X)} \circ F_q((\delta^F_{r,q^{-1}})_{F_q(X)}) \circ F_q(F_r((\delta^F_{q^{-1},q})_X))^{-1} \circ F_q((\delta^F_{r,e})_X)^{-1} \\
&= (\delta^F_{q,r})_X \circ F_q((\delta^F_{rq^{-1},q})_X) \circ F_q((\delta^F_{r,q^{-1}})_{F_q(X)}) \circ F_q(F_r((\delta^F_{q^{-1},q})_X))^{-1} \circ F_q((\delta^F_{r,e})_X)^{-1} \\
&= (\delta^F_{q,r})_X \circ F_q((\delta^F_{r,e})_X) \circ F_q(F_r((\delta^F_{q^{-1},q})_X)) \circ F_q(F_r((\delta^F_{q^{-1},q})_X))^{-1} \circ F_q((\delta^F_{r,e})_X)^{-1} \\
&= (\delta^F_{q,r})_X.
\end{aligned}
$$

Thus we have proved that $C_{(V,q,\Phi_V),(W,r,\Phi_V)} \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V,q,\Phi_V)\otimes(W,r,\Phi_W), \mathcal{F}_q[(W,r,\Phi_W)]\otimes(V,q,\Phi_V))$. The naturality of half braidings implies that C is natural in its second argument. But it is also natural in its first argument due to the definition of the morphisms in $Z_G(\mathcal{C})$, so C is natural. Now let $(U,q,\Phi_U),(V,r,\Phi_V),(W,s,\Phi_W) \in Z_G(\mathcal{C})$. Then

$$
\begin{aligned}
&C_{(U,q,\Phi_U),(V,r,\Phi_V)\otimes(W,s,\Phi_W)} = C_{(U,q,\Phi_U),(V\otimes W,rs,\Phi_V\otimes\Phi_W)} = \Phi_U(V\otimes W) \\
&= [\delta^q_{V,W} \otimes \mathrm{id}_U] \circ a^{-1}_{F_q(V),F_q(W),U} \circ [\mathrm{id}_{F_q(V)} \otimes \Phi_U(W)] \circ a_{F_q(V),U,W} \circ [\Phi_U(V) \otimes \mathrm{id}_W] \circ a^{-1}_{U,V,W} \\
&= [\delta^q_{(V,r,\Phi_V),(W,s,\Phi_W)} \otimes \mathrm{id}_{(U,q,\Phi_U)}] \circ a^{-1}_{\mathcal{F}_q[(V,r,\Phi_V)],\mathcal{F}_q[(W,s,\Phi_W)],(U,q,\Phi_U)} \\
&\quad \circ [\mathrm{id}_{\mathcal{F}_q[(V,r,\Phi_V)]} \otimes C_{(U,q,\Phi_U),(W,s,\Phi_W)}] \circ a_{\mathcal{F}_q[(V,r,\Phi_V)],(U,q,\Phi_U),(W,s,\Phi_W)} \\
&\quad \circ [C_{(U,q,\Phi_U),(V,r,\Phi_V)} \otimes \mathrm{id}_{(W,s,\Phi_W)}] \circ a^{-1}_{(U,q,\Phi_U),(V,r,\Phi_V),(W,s,\Phi_W)}
\end{aligned}
$$

and

$$
C_{(U,q,\Phi_U)\otimes(V,r,\Phi_V),(W,s,\Phi_W)} = C_{(U\otimes V,qr,\Phi_U\otimes\Phi_V),(W,s,\Phi_W)} = (\Phi_U \otimes \Phi_V)(W)
$$

$$= [(\delta^F_{q,r})_W \otimes \mathrm{id}_{U\otimes V}] \circ a_{F_q(F_r(W)),U,V} \circ [\Phi_U(F_r(W)) \otimes \mathrm{id}_V] \circ a^{-1}_{U,F_r(W),V} \circ [\mathrm{id}_U \otimes \Phi_V(W)] \circ a_{U,V,W}$$
$$= [(\delta^{\mathcal{F}}_{q,r})_{(W,s,\Phi_W)} \otimes \mathrm{id}_{(U,q,\Phi_U)\otimes(V,r,\Phi_V)}] \circ a_{\mathcal{F}_q(\mathcal{F}_r[(W,s,\Phi_W)]),(U,q,\Phi_U),(V,r,\Phi_V)}$$
$$\circ [C_{(U,q,\Phi_U),\mathcal{F}_r[(W,s,\Phi_W)]} \otimes \mathrm{id}_{(V,r,\Phi_V)}] \circ a^{-1}_{(U,q,\Phi_U),\mathcal{F}_r[(W,s,\Phi_W)],(V,r,\Phi_V)}$$
$$\circ [\mathrm{id}_{(U,q,\Phi_U)} \otimes C_{(V,r,\Phi_V),(W,s,\Phi_W)}] \circ a_{(U,q,\Phi_U),(V,r,\Phi_V),(W,s\Phi_W)}$$

Finally, if $(V,q,\Phi_V),(W,r,\Phi_W) \in Z_G(\mathcal{C})$ and $s \in G$ then

$$C_{\mathcal{F}_s[(V,q,\Phi_V)],\mathcal{F}_s[(W,r,\Phi_W)]} = C_{(F_s(V),sqs^{-1},\mathcal{F}_s\Phi_V),(F_s(W),srs^{-1},\mathcal{F}_s\Phi_W)} = (\mathcal{F}_s\Phi_V)(F_s(W))$$
$$= [\alpha_{s,q,s^{-1}}(F_s(W)) \otimes \mathrm{id}_{F_s(V)}] \circ (\delta^s_{F_q(F_{s^{-1}}(F_s(W))),V})^{-1} \circ F_s(\Phi_V(F_{s^{-1}}(F_s(W))))$$
$$\circ \delta^s_{V,F_{s^{-1}}(F_s(W))} \circ [\mathrm{id}_{F_s(V)} \otimes \Delta_s(F_s(W))^{-1}]$$
$$= [(\delta^F_{s,qs^{-1}})_{F_s(W)} \otimes \mathrm{id}_{F_s(V)}] \circ \{[F_s((\delta^F_{q,s^{-1}})_{F_s(W)}) \otimes \mathrm{id}_{F_s(V)}] \circ (\delta^s_{F_q(F_{s^{-1}}(F_s(W))),V})^{-1}\}$$
$$\circ F_s(\Phi_V(F_{s^{-1}}(F_s(W)))) \circ \delta^s_{V,F_{s^{-1}}(F_s(W))} \circ \{[\mathrm{id}_{F_s(V)} \otimes (\delta^F_{s,s^{-1}})^{-1}_{F_s(W)}] \circ [\mathrm{id}_{F_s(V)} \otimes \underbrace{\varepsilon^F_{F_s(W)}}_{=(\delta^F_{e,s})^{-1}_W}]\}$$
$$\overset{(*)}{=} [(\delta^F_{s,qs^{-1}})_{F_s(W)} \otimes \mathrm{id}_{F_s(V)}] \circ (\delta^s_{F_{qs^{-1}}(F_s(W)),V})^{-1} \circ F_s((\delta^F_{q.s^{-1}})_{F_s(W)} \otimes \mathrm{id}_V)$$
$$\circ F_s(\Phi_V(F_{s^{-1}}(F_s(W)))) \circ \{\delta^s_{V,F_{s^{-1}}(F_s(W))} \circ [\mathrm{id}_{F_s(V)} \otimes F_s((\delta^F_{s^{-1},s})_W)^{-1}]\} \circ [\mathrm{id}_{F_s(V)} \otimes \underbrace{(\delta^F_{s,e})^{-1}_W}_{=F_s(\varepsilon^F_W)}]$$
$$= [(\delta^F_{s,qs^{-1}})_{F_s(W)} \otimes \mathrm{id}_{F_s(V)}] \circ (\delta^s_{F_{qs^{-1}}(F_s(W)),V})^{-1} \circ F_s((\delta^F_{q.s^{-1}})_{F_s(W)} \otimes \mathrm{id}_V)$$
$$\circ \{F_s(\Phi_V(F_{s^{-1}}(F_s(W)))) \circ F_s(\mathrm{id}_V \otimes (\delta^F_{s^{-1},s})^{-1}_W)\} \circ \{\delta^s_{V,F_e(W)} \circ [\mathrm{id}_{F_s(V)} \otimes F_s(\varepsilon^F_W)]\}$$
$$= [(\delta^F_{s,qs^{-1}})_{F_s(W)} \otimes \mathrm{id}_{F_s(V)}] \circ (\delta^s_{F_{qs^{-1}}(F_s(W)),V})^{-1} \circ \{F_s((\delta^F_{q.s^{-1}})_{F_s(W)} \otimes \mathrm{id}_V)$$
$$\circ F_s(F_q((\delta^F_{s^{-1},s})_W)^{-1} \otimes \mathrm{id}_V)\} \circ \{F_s(\Phi_V(F_e(W))) \circ F_s(\mathrm{id}_V \otimes \varepsilon^F_W)\} \circ \delta^s_{V,W}$$
$$= [(\delta^F_{s,qs^{-1}})_{F_s(W)} \otimes \mathrm{id}_{F_s(V)}] \circ \{(\delta^s_{F_{qs^{-1}}(F_s(W)),V})^{-1} \circ F_s((\delta^F_{qs^{-1},s})^{-1}_W \otimes \mathrm{id}_V)\}$$
$$\circ F_s((\delta^F_{q,e})_W \otimes \mathrm{id}_V) \circ F_s(F_q(\varepsilon^F_W) \otimes \mathrm{id}_V) \circ F_s(\Phi_V(W)) \circ \delta^s_{V,W}$$
$$= \{[(\delta^F_{s,qs^{-1}})_{F_s(W)} \otimes \mathrm{id}_{F_s(V)}] \circ [F_s((\delta^F_{qs^{-1},s})_W)^{-1} \otimes \mathrm{id}_{F_s(V)}]\} \circ (\delta^s_{F_q(W),V})^{-1} \circ F_s(\Phi_V(W)) \circ \delta^s_{V,W}$$
$$= [(\delta^F_{sqs^{-1},s})^{-1} \otimes \mathrm{id}_{F_s(V)}] \circ [(\delta^F_{s,q})_W \otimes \mathrm{id}_{F_s(V)}] \circ (\delta^s_{F_q(W),V})^{-1} \circ \mathcal{F}_s(C_{(V,q,\Phi_V),(W,r,\Phi_W)}) \circ \delta^s_{V,W},$$

where in $\overset{(*)}{=}$ we used that

$$(\delta^F_{s,s^{-1}})^{-1}_{F_s(W)} \circ (\delta^F_{e,s})^{-1}_W = F_s((\delta^F_{s^{-1},s})_W)^{-1} \circ (\delta^F_{s,e})^{-1}_W.$$

This completes the proof that $Z_G(\mathcal{C})$ is a braided G-crossed category.
□

Appendix B

Proof of $Z_G(\mathcal{C}) \simeq Z_G(\mathcal{C}')$ when $\mathcal{C} \simeq \mathcal{C}'$

In this Appendix we will give a detailed proof of Theorem 4.3.4, which states that if $\mathcal{C}$ and $\mathcal{C}'$ are equivalent G-categories, then $Z_G(\mathcal{C})$ and $Z_G(\mathcal{C}')$ are equivalent as braided G-crossed categories.

B.1 The given data

Throughout this appendix we assume that we are given a group G and two tensor categories $(\mathcal{C}, \otimes, I, a, l, r)$ and $(\mathcal{C}', \otimes', I', a', l', r')$ with G-actions[1] $(F, \varepsilon^F, \delta^F)$ and $(F', \varepsilon^{F'}, \delta^{F'})$, respectively, such that $\mathcal{C}$ and $\mathcal{C}'$ are equivalent as G-categories. This means that there are G-functors

$$(K, \varepsilon^K, \delta^K, \xi^K) : \mathcal{C} \to \mathcal{C}'$$
$$(L, \varepsilon^L, \delta^L, \xi^L) : \mathcal{C}' \to \mathcal{C}$$

together with natural G-isomorphisms

$$\varphi : \mathrm{id}_{\mathcal{C}'} \to K \circ L$$
$$\psi : L \circ K \to \mathrm{id}_{\mathcal{C}}.$$

For φ this means that $\varphi : \mathrm{id}_{\mathcal{C}'} \to K \circ L$ is a natural isomorphism that satisfies the equation

$$\varphi_{I'} = \varepsilon^K \diamond \varepsilon^L = K(\varepsilon^L) \circ \varepsilon^K \tag{B.1.1}$$

as well as the equations

$$\varphi_{X' \otimes' Y'} = (\delta^K \diamond \delta^L)_{X',Y'} \circ [\varphi_{X'} \otimes' \varphi_{Y'}] = K(\delta^L_{X',Y'}) \circ \delta^K_{L(X'),L(Y')} \circ [\varphi_{X'} \otimes' \varphi_{Y'}] \tag{B.1.2}$$

and

$$F'_q(\varphi_{X'}) = (\xi^K \diamond \xi^L)(q)_{X'} \circ \varphi_{F'_q(X')} = \xi^K(q)_{L(X')} \circ K(\xi^L(q)_{X'}) \circ \varphi_{F'_q(X')} \tag{B.1.3}$$

for all $X', Y' \in \mathcal{C}'$. Similarly, ψ is a natural isomorphism that satisfies

$$\psi_I = (\varepsilon^L \diamond \varepsilon^K)^{-1} = (\varepsilon^L)^{-1} \circ L(\varepsilon^K)^{-1} \tag{B.1.4}$$

as well as

$$\psi_{X \otimes Y} = [\psi_X \otimes \psi_Y] \circ (\delta^L \diamond \delta^K)^{-1}_{X,Y} = [\psi_X \otimes \psi_Y] \circ (\delta^L_{K(X),K(Y)})^{-1} \circ L(\delta^K_{X,Y})^{-1} \tag{B.1.5}$$

[1] In other places we were free to write ε^q and δ^q instead of ε^{F_q} and δ^{F_q}, but in this appendix we cannot do this because there are two G-categories $\mathcal{C}$ and $\mathcal{C}'$. Their G-actions will be distinguished most easily by writing either ε^{F_q} or $\varepsilon^{F'_q}$, etc.

and

$$F_q(\psi_X) = \psi_{F_q(X)} \circ (\xi^L \diamond \xi^K)(q)_X^{-1} = \psi_{F_q(X)} \circ L(\xi^K(q)_X)^{-1} \circ \xi^L(q)_{K(X)}^{-1}. \qquad \text{(B.1.6)}$$

In the following sections we will give an explicit construction of the equivalence $Z_G(\mathcal{C}) \simeq Z_G(\mathcal{C}')$, i.e. we will construct braided G-crossed functors $\mathscr{K} : Z_G(\mathcal{C}) \to Z_G(\mathcal{C}')$ and $\mathscr{L} : Z_G(\mathcal{C}') \to Z_G(\mathcal{C})$ together with natural braided G-crossed isomorphisms $\varphi : \mathrm{id}_{\mathcal{C}'} \to K \circ L$ and $\psi : L \circ K \to \mathrm{id}_{\mathcal{C}}$.

B.2 Construction of the functor $\mathscr{K} : Z_G(\mathcal{C}) \to Z_G(\mathcal{C}')$

In this section we will construct a functor $\mathscr{K} : Z_G(\mathcal{C}) \to Z_G(\mathcal{C}')$. On the objects we will denote its action by $\mathscr{K}[(V, q, \Phi_V)] = (K(V), q, \mathscr{K}\Phi_V)$, where $\mathscr{K}\Phi_V$ is a half q-braiding for $K(V)$. The construction of $\mathscr{K}\Phi_V$ will be given in Proposition B.2.2 below, but first we need the following lemma.

Lemma B.2.1 *Let $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ and define for each $X \in \mathcal{C}$ the isomorphism $\Phi'_V(X) \in \mathrm{Hom}_{\mathcal{C}'}(K(V) \otimes' K(X), F'_q(K(X)) \otimes' K(V))$ by*

$$\Phi'_V(X) := [\xi^K(q)_X \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(X),V})^{-1} \circ K(\Phi_V(X)) \circ \delta^K_{V,X}.$$

Then Φ'_V has the following properties:
(1) If $X, Y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, then

$$\Phi'_V(Y) \circ [\mathrm{id}_{K(V)} \otimes' K(f)] = [F'_q(K(f)) \otimes' \mathrm{id}_{K(V)}] \circ \Phi'_V(X).$$

(2) If $X, Y \in \mathcal{C}$, then

$$\begin{aligned}\Phi'_V(X \otimes Y) = {} & [F'_q(\delta^K_{X,Y}) \otimes' \mathrm{id}_{K(V)}] \circ [\delta^{F'_q}_{K(X),K(Y)} \otimes' \mathrm{id}_{K(V)}] \circ a'^{-1}_{F'_q(K(X)),F'_q(K(Y)),K(V)} \\ & \circ [\mathrm{id}_{F'_q(K(X))} \otimes' \Phi'_V(Y)] \circ a'_{F'_q(K(X)),K(V),K(Y)} \circ [\Phi'_V(X) \otimes' \mathrm{id}_{K(Y)}] \\ & \circ a'^{-1}_{K(V),K(X),K(Y)} \circ [\mathrm{id}_{K(V)} \otimes' (\delta^K_{X,Y})^{-1}]\end{aligned}$$

Proof. (1) Using naturality several times, we get

$$\begin{aligned}& \Phi'_V(Y) \circ [\mathrm{id}_{K(V)} \otimes' K(f)] \\ &= [\xi^K(q)_Y \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(Y),V})^{-1} \circ K(\Phi_V(Y)) \circ \delta^K_{V,Y} \circ [\mathrm{id}_{K(V)} \otimes' K(f)] \\ &= [\xi^K(q)_Y \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(Y),V})^{-1} \circ K(\Phi_V(Y)) \circ K(\mathrm{id}_V \otimes f) \circ \delta^K_{V,X} \\ &= [\xi^K(q)_Y \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(Y),V})^{-1} \circ K(F_q(f) \otimes \mathrm{id}_V) \circ K(\Phi_V(X)) \circ \delta^K_{V,X} \\ &= [\xi^K(q)_Y \otimes' \mathrm{id}_{K(V)}] \circ [K(F_q(f)) \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(X),V})^{-1} \circ K(\Phi_V(X)) \circ \delta^K_{V,X} \\ &= [F'_q(K(f)) \otimes' \mathrm{id}_{K(V)}] \circ [\xi^K(q)_X \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(X),V})^{-1} \circ K(\Phi_V(X)) \circ \delta^K_{V,X} \\ &= [F'_q(K(f)) \otimes' \mathrm{id}_{K(V)}] \circ \Phi'_V(X).\end{aligned}$$

(2) We have

$$\begin{aligned}& \Phi'_V(X \otimes Y) \\ &= [\xi^K(q)_{X\otimes Y} \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(X\otimes Y),V})^{-1} \circ K(\Phi_V(X \otimes Y)) \circ \delta^K_{V,X\otimes Y} \\ &= [\xi^K(q)_{X\otimes Y} \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(X\otimes Y),V})^{-1} \circ K(\delta^{F_q}_{X,Y} \otimes \mathrm{id}_V) \circ K(a_{F_q(X),F_q(Y),V})^{-1} \\ &\quad \circ K(\mathrm{id}_{F_q(X)} \otimes \Phi_V(Y)) \circ K(a_{F_q(X),V,Y}) \circ K(\Phi_V(X) \otimes \mathrm{id}_Y) \circ K(a_{V,X,Y})^{-1} \circ \delta^K_{V,X\otimes Y} \\ &= [\xi^K(q)_{X\otimes Y} \otimes' \mathrm{id}_{K(V)}] \circ [K(\delta^{F_q}_{X,Y}) \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(X)\otimes F_q(Y),V})^{-1} \circ K(a_{F_q(X),F_q(Y),V})^{-1}\end{aligned}$$

$$
\begin{aligned}
&\circ K(\mathrm{id}_{F_q(X)} \otimes \Phi_V(Y)) \circ K(a_{F_q(X),V,Y}) \circ K(\Phi_V(X) \otimes \mathrm{id}_Y) \circ \delta^K_{V\otimes X,Y} \\
&\circ [\delta^K_{V,X} \otimes' \mathrm{id}_{K(Y)}] \circ a'^{-1}_{K(V),K(X),K(Y)} \circ [\mathrm{id}_{K(V)} \otimes' (\delta^K_{X,Y})^{-1}] \\
= &[F'_q(\delta^K_{X,Y}) \otimes' \mathrm{id}_{K(V)}] \circ [\delta^{F'_q}_{K(X),K(Y)} \otimes' \mathrm{id}_{K(V)}] \circ [(\xi^K(q)_X \otimes' \xi^K(q)_Y) \otimes' \mathrm{id}_{K(V)}] \\
&\circ [(\delta^K_{F_q(X),F_q(Y)})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(X)\otimes F_q(Y),V})^{-1} \circ K(a_{F_q(X),F_q(Y),V})^{-1} \\
&\circ K(\mathrm{id}_{F_q(X)} \otimes \Phi_V(Y)) \circ K(a_{F_q(X),V,Y}) \circ \delta^K_{F_q(X)\otimes V,Y} \circ [K(\Phi_V(X)) \otimes' \mathrm{id}_Y] \\
&\circ [\delta^K_{V,X} \otimes' \mathrm{id}_{K(Y)}] \circ a'^{-1}_{K(V),K(X),K(Y)} \circ [\mathrm{id}_{K(V)} \otimes' (\delta^K_{X,Y})^{-1}] \\
= &[F'_q(\delta^K_{X,Y}) \otimes' \mathrm{id}_{K(V)}] \circ [\delta^{F'_q}_{K(X),K(Y)} \otimes' \mathrm{id}_{K(V)}] \circ [(\xi^K(q)_X \otimes' \xi^K(q)_Y) \otimes' \mathrm{id}_{K(V)}] \\
&\circ a'^{-1}_{K(F_q(X)),K(F_q(Y)),K(V)} \circ [\mathrm{id}_{K(F_q(X))} \otimes' (\delta^K_{F_q(Y),V})^{-1}] \circ (\delta^K_{F_q(X),F_q(Y)\otimes V})^{-1} \\
&\circ K(\mathrm{id}_{F_q(X)} \otimes \Phi_V(Y)) \circ K(a_{F_q(X),V,Y}) \circ \delta^K_{F_q(X)\otimes V,Y} \circ [K(\Phi_V(X)) \otimes' \mathrm{id}_Y] \\
&\circ [\delta^K_{V,X} \otimes' \mathrm{id}_{K(Y)}] \circ a'^{-1}_{K(V),K(X),K(Y)} \circ [\mathrm{id}_{K(V)} \otimes' (\delta^K_{X,Y})^{-1}] \\
= &[F'_q(\delta^K_{X,Y}) \otimes' \mathrm{id}_{K(V)}] \circ [\delta^{F'_q}_{K(X),K(Y)} \otimes' \mathrm{id}_{K(V)}] \circ a'^{-1}_{F'_q(K(X)),F'_q(K(Y)),K(V)} \\
&\circ [\xi^K(q)_X \otimes' (\xi^K(q)_Y \otimes' \mathrm{id}_{K(V)})] \circ [\mathrm{id}_{K(F_q(X))} \otimes' (\delta^K_{F_q(Y),V})^{-1}] \circ [\mathrm{id}_{K(F_q(X))} \otimes' K(\Phi_V(Y))] \\
&\circ (\delta^K_{F_q(X),V\otimes Y})^{-1} \circ K(a_{F_q(X),V,Y}) \circ \delta^K_{F_q(X)\otimes V,Y} \circ [K(\Phi_V(X)) \otimes' \mathrm{id}_Y] \\
&\circ [\delta^K_{V,X} \otimes' \mathrm{id}_{K(Y)}] \circ a'^{-1}_{K(V),K(X),K(Y)} \circ [\mathrm{id}_{K(V)} \otimes' (\delta^K_{X,Y})^{-1}] \\
= &[F'_q(\delta^K_{X,Y}) \otimes' \mathrm{id}_{K(V)}] \circ [\delta^{F'_q}_{K(X),K(Y)} \otimes' \mathrm{id}_{K(V)}] \circ a'^{-1}_{F'_q(K(X)),F'_q(K(Y)),K(V)} \\
&\circ [\mathrm{id}_{F'_q(K(X))} \otimes' (\xi^K(q)_Y \otimes' \mathrm{id}_{K(V)})] \circ [\mathrm{id}_{F'_q(K(X))} \otimes' (\delta^K_{F_q(Y),V})^{-1}] \circ [\mathrm{id}_{F'_q(K(X))} \otimes' K(\Phi_V(Y))] \\
&\circ [\xi^K(q)_X \otimes' \mathrm{id}_{K(V\otimes Y)}] \circ [\mathrm{id}_{K(F_q(X))} \otimes' \delta^K_{V,Y}] \circ a'_{K(F_q(X)),K(V),K(Y)} \circ [(\delta^K_{F_q(X),V})^{-1} \otimes' \mathrm{id}_{K(Y)}] \\
&\circ [K(\Phi_V(X)) \otimes' \mathrm{id}_Y] \circ [\delta^K_{V,X} \otimes' \mathrm{id}_{K(Y)}] \circ a'^{-1}_{K(V),K(X),K(Y)} \circ [\mathrm{id}_{K(V)} \otimes' (\delta^K_{X,Y})^{-1}] \\
= &[F'_q(\delta^K_{X,Y}) \otimes' \mathrm{id}_{K(V)}] \circ [\delta^{F'_q}_{K(X),K(Y)} \otimes' \mathrm{id}_{K(V)}] \circ a'^{-1}_{F'_q(K(X)),F'_q(K(Y)),K(V)} \\
&\circ [\mathrm{id}_{F'_q(K(X))} \otimes' (\xi^K(q)_Y \otimes' \mathrm{id}_{K(V)})] \circ [\mathrm{id}_{F'_q(K(X))} \otimes' (\delta^K_{F_q(Y),V})^{-1}] \circ [\mathrm{id}_{F'_q(K(X))} \otimes' K(\Phi_V(Y))] \\
&\circ [\mathrm{id}_{F'_q(K(X))} \otimes' \delta^K_{V,Y}] \circ a'_{F'_q(K(X)),K(V),K(Y)} \circ [(\xi^K(q)_X \otimes' \mathrm{id}_{K(V)}) \otimes' \mathrm{id}_{K(Y)}] \\
&\circ [(\delta^K_{F_q(X),V})^{-1} \otimes' \mathrm{id}_{K(Y)}] \circ [K(\Phi_V(X)) \otimes' \mathrm{id}_Y] \circ [\delta^K_{V,X} \otimes' \mathrm{id}_{K(Y)}] \circ a'^{-1}_{K(V),K(X),K(Y)} \\
&\circ [\mathrm{id}_{K(V)} \otimes' (\delta^K_{X,Y})^{-1}] \\
= &[F'_q(\delta^K_{X,Y}) \otimes' \mathrm{id}_{K(V)}] \circ [\delta^{F'_q}_{K(X),K(Y)} \otimes' \mathrm{id}_{K(V)}] \circ a'^{-1}_{F'_q(K(X)),F'_q(K(Y)),K(V)} \\
&\circ [\mathrm{id}_{F'_q(K(X))} \otimes' \Phi'_V(Y)] \circ a'_{F'_q(K(X)),K(V),K(Y)} \circ [\Phi'_V(X) \otimes' \mathrm{id}_{K(Y)}] \\
&\circ a'^{-1}_{K(V),K(X),K(Y)} \circ [\mathrm{id}_{K(V)} \otimes' (\delta^K_{X,Y})^{-1}].
\end{aligned}
$$

□

We are now ready to construct half braidings in $Z_G(\mathcal{C}')$ from half braidings in $Z_G(\mathcal{C})$. This will be used to define a functor $\mathscr{K} : Z_G(\mathcal{C}) \to Z_G(\mathcal{C}')$.

Proposition B.2.2 *Let $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ and define for each $X' \in \mathcal{C}'$ the isomorphism $\mathscr{K}\Phi_V(X') \in \mathrm{Hom}_{\mathcal{C}'}(K(V) \otimes' X', F'_q(X') \otimes' K(V))$ by*

$$\mathscr{K}\Phi_V(X') := [F'_q(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ \Phi'_V(L(X')) \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{X'}]$$

$$= [F'_q(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [\xi^K(q)_{L(X')} \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(L(X')),V})^{-1} \circ K(\Phi_V(L(X'))) \circ \delta^K_{V,L(X')} \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{X'}].$$

Then $\mathscr{K}\Phi_V$ is a half q-braiding for $K(V)$ and we obtain a functor $\mathscr{K} : Z_G(\mathcal{C}) \to Z_G(\mathcal{C}')$ by defining $\mathscr{K}[(V,q,\Phi_V)] := (K(V), q, \mathscr{K}\Phi_V)$ on the objects and $\mathscr{K}(f) := K(f)$ on the morphisms.

Proof. If $X', Y' \in \mathcal{C}'$ and $f' \in \mathrm{Hom}_{\mathcal{C}'}(X', Y')$, then

$$\begin{aligned}
&\mathscr{K}\Phi_V(Y') \circ [\mathrm{id}_{K(V)} \otimes' f'] \\
&= [F'_q(\varphi_{Y'})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ \Phi'_V(L(Y')) \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{Y'}] \circ [\mathrm{id}_{K(V)} \otimes' f'] \\
&= [F'_q(\varphi_{Y'})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ \Phi'_V(L(Y')) \circ [\mathrm{id}_{K(V)} \otimes' K(L(f'))] \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{X'}] \\
&= [F'_q(\varphi_{Y'})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [F'_q(K(L(f'))) \otimes' \mathrm{id}_{K(V)}] \circ \Phi'_V(L(X')) \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{X'}] \\
&= [F'_q(f') \otimes' \mathrm{id}_{K(V)}] \circ [F'_q(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ \Phi'_V(L(X')) \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{X'}] \\
&= [F'_q(f') \otimes' \mathrm{id}_{K(V)}] \circ \mathscr{K}\Phi_V(X').
\end{aligned}$$

For $X', Y' \in \mathcal{C}'$ we also have

$$\begin{aligned}
&\mathscr{K}\Phi_V(X' \otimes' Y') \\
&= [F'_q(\varphi_{X'\otimes' Y'})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ \Phi'_V(L(X' \otimes' Y')) \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{X'\otimes' Y'}] \\
&= [F'_q(\varphi_{X'\otimes' Y'})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [F'_q(K(\delta^L_{X',Y'})) \otimes' \mathrm{id}_{K(V)}] \circ \Phi'_V(L(X') \otimes L(Y')) \\
&\quad \circ [\mathrm{id}_{K(V)} \otimes' K(\delta^L_{X',Y'})^{-1}] \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{X'\otimes' Y'}] \\
&= [F'_q(\varphi_{X'\otimes' Y'})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [F'_q(K(\delta^L_{X',Y'})) \otimes' \mathrm{id}_{K(V)}] \circ [F'_q(\delta^K_{L(X'),L(Y')}) \otimes' \mathrm{id}_{K(V)}] \\
&\quad \circ [\delta^{F'_q}_{K(L(X')),K(L(Y'))} \otimes' \mathrm{id}_{K(V)}] \circ a'^{-1}_{F'_q(K(L(X'))),F'_q(K(L(Y'))),K(V)} \circ [\mathrm{id}_{F'_q(K(L(X')))} \otimes' \Phi'_V(L(Y'))] \\
&\quad \circ a'_{F'_q(K(L(X'))),K(V),K(L(Y'))} \circ [\Phi'_V(L(X')) \otimes' \mathrm{id}_{K(L(Y'))}] \circ a'^{-1}_{K(V),K(L(X')),K(L(Y'))} \\
&\quad \circ [\mathrm{id}_{K(V)} \otimes' (\delta^K_{L(X'),L(Y')})^{-1}] \circ [\mathrm{id}_{K(V)} \otimes' K(\delta^L_{X',Y'})^{-1}] \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{X'\otimes' Y'}] \\
&= [F'_q(\varphi_{X'} \otimes' \varphi_{Y'})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [F'_q(\delta^K_{L(X'),L(Y')})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [F'_q(K(\delta^L_{X',Y'}))^{-1} \otimes' \mathrm{id}_{K(V)}] \\
&\quad \circ [F'_q(K(\delta^L_{X',Y'})) \otimes' \mathrm{id}_{K(V)}] \circ [F'_q(\delta^K_{L(X'),L(Y')}) \otimes' \mathrm{id}_{K(V)}] \\
&\quad \circ [\delta^{F'_q}_{K(L(X')),K(L(Y'))} \otimes' \mathrm{id}_{K(V)}] \circ a'^{-1}_{F'_q(K(L(X'))),F'_q(K(L(Y'))),K(V)} \circ [\mathrm{id}_{F'_q(K(L(X')))} \otimes' \Phi'_V(L(Y'))] \\
&\quad \circ a'_{F'_q(K(L(X'))),K(V),K(L(Y'))} \circ [\Phi'_V(L(X')) \otimes' \mathrm{id}_{K(L(Y'))}] \circ a'^{-1}_{K(V),K(L(X')),K(L(Y'))} \\
&\quad \circ [\mathrm{id}_{K(V)} \otimes' (\delta^K_{L(X'),L(Y')})^{-1}] \circ [\mathrm{id}_{K(V)} \otimes' K(\delta^L_{X',Y'})^{-1}] \circ [\mathrm{id}_{K(V)} \otimes' K(\delta^L_{X',Y'})] \\
&\quad \circ [\mathrm{id}_{K(V)} \otimes' \delta^K_{L(X'),L(Y')}] \circ [\mathrm{id}_{K(V)} \otimes' (\varphi_{X'} \otimes' \varphi_{Y'})] \\
&= [F'_q(\varphi_{X'} \otimes' \varphi_{Y'})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [\delta^{F'_q}_{K(L(X')),K(L(Y'))} \otimes' \mathrm{id}_{K(V)}] \\
&\quad \circ a'^{-1}_{F'_q(K(L(X'))),F'_q(K(L(Y'))),K(V)} \circ [\mathrm{id}_{F'_q(K(L(X')))} \otimes' \Phi'_V(L(Y'))] \circ a'_{F'_q(K(L(X'))),K(V),K(L(Y'))} \\
&\quad \circ [\Phi'_V(L(X')) \otimes' \mathrm{id}_{K(L(Y'))}] \circ a'^{-1}_{K(V),K(L(X')),K(L(Y'))} \circ [\mathrm{id}_{K(V)} \otimes' (\varphi_{X'} \otimes' \varphi_{Y'})] \\
&= [\delta^{F'_q}_{X',Y'} \otimes' \mathrm{id}_{K(V)}] \circ a'^{-1}_{F'_q(X'),F'_q(Y'),K(V)} \circ [\mathrm{id}_{F'_q(X')} \otimes' (F'_q(\varphi_{Y'})^{-1} \otimes' \mathrm{id}_{K(V)})] \\
&\quad \circ [\mathrm{id}_{F'_q(X')} \otimes' \Phi'_V(L(Y'))] \circ [\mathrm{id}_{F'_q(X')} \otimes' (\mathrm{id}_{K(V)} \otimes' \varphi_{Y'})] \circ a'_{F'_q(X'),K(V),Y'} \\
&\quad \circ [(F'_q(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(V)}) \otimes' \mathrm{id}_{Y'}] \circ [\Phi'_V(L(X')) \otimes' \mathrm{id}_{Y'}] \circ [(\mathrm{id}_{K(V)} \otimes' \varphi_{X'}) \otimes' \mathrm{id}_{Y'}] \\
&\quad \circ a'^{-1}_{K(V),X',Y'}
\end{aligned}$$

$$
\begin{aligned}
&= [\delta^{F'_q}_{X',Y'} \otimes' \mathrm{id}_{K(V)}] \circ a'^{-1}_{F'_q(X'),F'_q(Y'),K(V)} \circ [\mathrm{id}_{F'_q(X')} \otimes' \mathscr{K}\Phi_V(Y')] \\
&\quad \circ a_{F'_q(X'),K(V),Y'} \circ [\mathscr{K}\Phi_V(X') \otimes' \mathrm{id}_{Y'}] \circ a'^{-1}_{K(V),X',Y'},
\end{aligned}
$$

so $\mathscr{K}\Phi_V$ is indeed a half q-braiding for $K(V)$. As a consequence, $(K(V), q, \mathscr{K}\Phi_V)$ is an object in $Z_G(\mathcal{C}')$ and hence $\mathscr{K}$ is well-defined on the objects of $Z_G(\mathcal{C})$.

Now let $(V, q, \Phi_V), (W, q, \Phi_W) \in Z_G(\mathcal{C})$ and $f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V, q, \Phi_V), (W, q, \Phi_W))$. Then for any $X' \in \mathcal{C}'$ we have

$$
\begin{aligned}
&\mathscr{K}\Phi_W(X') \circ [\mathscr{K}(f) \otimes' \mathrm{id}_{X'}] \\
&= [F'_q(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(W)}] \circ [\xi^K(q)_{L(X')} \otimes' \mathrm{id}_{K(W)}] \circ (\delta^K_{F_q(L(X')),W})^{-1} \\
&\quad \circ K(\Phi_W(L(X'))) \circ \delta^K_{W,L(X')} \circ [\mathrm{id}_{K(W)} \otimes' \varphi_{X'}] \circ [K(f) \otimes' \mathrm{id}_{X'}] \\
&= [F'_q(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(W)}] \circ [\xi^K(q)_{L(X')} \otimes' \mathrm{id}_{K(W)}] \circ (\delta^K_{F_q(L(X')),W})^{-1} \\
&\quad \circ K(\Phi_W(L(X'))) \circ K(f \otimes \mathrm{id}_{L(X')}) \circ \delta^K_{V,L(X')} \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{X'}] \\
&= [F'_q(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(W)}] \circ [\xi^K(q)_{L(X')} \otimes' \mathrm{id}_{K(W)}] \circ (\delta^K_{F_q(L(X')),W})^{-1} \\
&\quad \circ K(\mathrm{id}_{F_q(L(X'))} \otimes f) \circ K(\Phi_V(L(X'))) \circ \delta^K_{V,L(X')} \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{X'}] \\
&= [F'_q(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(W)}] \circ [\xi^K(q)_{L(X')} \otimes' \mathrm{id}_{K(W)}] \circ [\mathrm{id}_{K(F_q(L(X')))} \otimes' K(f)] \\
&\quad \circ (\delta^K_{F_q(L(X')),V})^{-1} \circ K(\Phi_V(L(X'))) \circ \delta^K_{V,L(X')} \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{X'}] \\
&= [\mathrm{id}_{F'_q(X')} \otimes' K(f)] \circ [F'_q(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [\xi^q(q)_{L(X')} \otimes' \mathrm{id}_{K(V)}] \\
&\quad \circ (\delta^K_{F_q(L(X')),V})^{-1} \circ K(\Phi_V(L(X'))) \circ \delta^K_{V,L(X')} \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{X'}] \\
&= [\mathrm{id}_{F'_q(X')} \otimes' K(f)] \circ \mathscr{K}\Phi_V(X'),
\end{aligned}
$$

so $\mathscr{K}(f) \in \mathrm{Hom}_{Z_G(\mathcal{C}')}((K(V), q, \mathscr{K}\Phi_V), (K(W), q, \mathscr{K}\Phi_W))$. If f and g are composable morphisms in $Z_G(\mathcal{C})$, then $\mathscr{K}(g \circ f) = K(g \circ f) = K(g) \circ K(f) = \mathscr{K}(g) \circ \mathscr{K}(f)$. Also $\mathscr{K}(\mathrm{id}_{(V,q,\Phi_V)}) = K(\mathrm{id}_V) = \mathrm{id}_{K(V)} = \mathrm{id}_{\mathscr{K}[(V,q,\Phi_V)]}$. Thus we conclude that $\mathscr{K}$ is a functor.
□

B.3 The functor $\mathscr{K}$ can be made into a tensor functor

We will now give the functor $\mathscr{K} : Z_G(\mathcal{C}) \to Z_G(\mathcal{C}')$ the structure $(\mathscr{K}, \varepsilon^{\mathscr{K}}, \delta^{\mathscr{K}})$ of a tensor functor. For this we first define $\delta^{\mathscr{K}}_{(V,q,\Phi_V),(W,r,\Phi_W)} := \delta^K_{V,W}$ for any $(V, q, \Phi_V), (W, r, \Phi_W) \in Z_G(\mathcal{C})$. To see that these are indeed morphisms in the category $Z_G(\mathcal{C}')$, let $(V, q, \Phi_V), (W, r, \Phi_W) \in Z_G(\mathcal{C})$. Then for any $X' \in \mathcal{C}$ we have

$$
\begin{aligned}
&[\mathrm{id}_{F'_{qr}(X')} \otimes' (\delta^K_{V,W})^{-1}] \circ [\mathscr{K}(\Phi_V \otimes \Phi_W)(X')] \circ [\delta^K_{V,W} \otimes' \mathrm{id}_{X'}] \\
&= [\mathrm{id}_{F'_{qr}(X')} \otimes' (\delta^K_{V,W})^{-1}] \circ [F'_{qr}(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(V\otimes W)}] \circ [\xi^K(qr)_{L(X')} \otimes' \mathrm{id}_{K(V\otimes W)}] \\
&\quad \circ (\delta^K_{F_{qr}(L(X')),V\otimes W})^{-1} \circ K((\delta^F_{q,r})_{L(X')} \otimes \mathrm{id}_{V\otimes W}) \circ K(a_{F_q(F_r(L(X'))),V,W}) \\
&\quad \circ K(\Phi_V(F_r(L(X'))) \otimes \mathrm{id}_W) \circ K(a_{V,F_r(L(X')),W})^{-1} \circ K(\mathrm{id}_V \otimes \Phi_W(L(X'))) \\
&\quad \circ K(a_{V,W,L(X')}) \circ \delta^K_{V\otimes W,L(X')} \circ [\mathrm{id}_{K(V\otimes W)} \otimes' \varphi_{X'}] \circ [\delta^K_{V,W} \otimes' \mathrm{id}_{X'}] \\
&= [F'_{qr}(\varphi_{X'})^{-1} \otimes' (\delta^K_{V,W})^{-1}] \circ [(\delta^{F'}_{q,r})_{K(L(X'))} \otimes' \mathrm{id}_{K(V\otimes W)}] \circ [F'_q(\xi^K(r)_{L(X')}) \otimes' \mathrm{id}_{K(V\otimes W)}] \\
&\quad \circ [\xi^K(q)_{F_r(L(X'))} \otimes' \mathrm{id}_{K(V\otimes W)}] \circ [K((\delta^F_{q,r})_{L(X')})^{-1} \otimes' \mathrm{id}_{K(V\otimes W)}] \circ [K((\delta^F_{q,r})_{L(X')}) \otimes' \mathrm{id}_{K(V\otimes W)}] \\
&\quad \circ (\delta^K_{F_q(F_r(L(X'))),V\otimes W})^{-1} \circ K(a_{F_q(F_r(L(X'))),V,W}) \circ K(\Phi_V(F_r(L(X'))) \otimes \mathrm{id}_W) \\
&\quad \circ K(a_{V,F_r(L(X')),W})^{-1} \circ K(\mathrm{id}_V \otimes \Phi_W(L(X'))) \circ K(a_{V,W,L(X')})
\end{aligned}
$$

$$
\begin{aligned}
&\circ \delta^K_{V\otimes W,L(X')} \circ [\delta^K_{V,W} \otimes' \mathrm{id}_{K(L(X'))}] \circ [\mathrm{id}_{K(V)\otimes' K(W)} \otimes' \varphi_{X'}] \\
&= [(\delta^{F'}_{q,r})_{X'} \otimes' \mathrm{id}_{K(V)\otimes' K(W)}] \circ [F'_q(F'_r(\varphi_{X'}))^{-1} \otimes' (\delta^K_{V,W})^{-1}] \circ [F'_q(\xi^K(r)_{L(X')}) \otimes' \mathrm{id}_{K(V\otimes W)}] \\
&\quad \circ [\xi^K(q)_{F_r(L(X'))} \otimes' \mathrm{id}_{K(V\otimes W)}] \circ [\mathrm{id}_{K(F_q(F_r(L(X'))))} \otimes' \delta^K_{V,W}] \circ a'_{K(F_q(F_r(L(X')))),K(V),K(W)} \\
&\quad \circ [(\delta^K_{F_q(F_r(L(X'))),V})^{-1} \otimes' \mathrm{id}_{K(W)}] \circ (\delta^K_{F_q(F_r(L(X')))\otimes V,W})^{-1} \circ K(\Phi_V(F_r(L(X'))) \otimes \mathrm{id}_W) \\
&\quad \circ K(a_{V,F_r(L(X')),W})^{-1} \circ K(\mathrm{id}_V \otimes \Phi_W(L(X'))) \circ \delta^K_{V,W\otimes L(X')} \\
&\quad \circ [\mathrm{id}_{K(V)} \otimes' \delta^K_{W,L(X')}] \circ a'_{K(V),K(W),K(L(X'))} \circ [\mathrm{id}_{K(V)\otimes' K(W)} \otimes' \varphi_{X'}] \\
&= [(\delta^{F'}_{q,r})_{X'} \otimes' \mathrm{id}_{K(V)\otimes' K(W)}] \circ [F'_q(F'_r(\varphi_{X'}))^{-1} \otimes' \mathrm{id}_{K(V)\otimes' K(W)}] \\
&\quad \circ [F'_q(\xi^K(r)_{L(X')}) \otimes' \mathrm{id}_{K(V)\otimes' K(W)}] \circ [\xi^K(q)_{F_r(L(X'))} \otimes' \mathrm{id}_{K(V)\otimes' K(W)}] \\
&\quad \circ a'_{K(F_q(F_r(L(X')))),K(V),K(W)} \circ [(\delta^K_{F_q(F_r(L(X'))),V})^{-1} \otimes' \mathrm{id}_{K(W)}] \circ [K(\Phi_V(F_r(L(X')))) \otimes' \mathrm{id}_{K(W)}] \\
&\quad \circ (\delta^K_{V\otimes F_r(L(X')),W})^{-1} \circ K(a_{V,F_r(L(X')),W})^{-1} \circ \delta^K_{V,F_r(L(X'))\otimes W} \circ [\mathrm{id}_{K(V)} \otimes' K(\Phi_W(L(X')))] \\
&\quad \circ [\mathrm{id}_{K(V)} \otimes' \delta^K_{W,L(X')}] \circ [\mathrm{id}_{K(V)} \otimes' (\mathrm{id}_{K(W)} \otimes' \varphi_{X'})] \circ a'_{K(V),K(W),X'} \\
&\overset{(*)}{=} [(\delta^{F'}_{q,r})_{X'} \otimes' \mathrm{id}_{K(V)\otimes' K(W)}] \circ a'_{F'_q(F'_r(X')),K(V),K(W)} \circ [(F'_q(\varphi_{F'_r(X')})^{-1} \otimes' \mathrm{id}_{K(V)}) \otimes' \mathrm{id}_{K(W)}] \\
&\quad \circ [(\xi^K(q)_{L(F'_r(X'))} \otimes' \mathrm{id}_{K(V)}) \otimes' \mathrm{id}_{K(W)}] \circ [(K(F_q(\xi^L(r)_{X'}))^{-1} \otimes' \mathrm{id}_{K(V)}) \otimes' \mathrm{id}_{K(W)}] \\
&\quad \circ [(\delta^K_{F_q(F_r(L(X'))),V})^{-1} \otimes' \mathrm{id}_{K(W)}] \circ [K(\Phi_V(F_r(L(X')))) \otimes' \mathrm{id}_{K(W)}] \circ [\delta^K_{V,F_r(L(X'))} \otimes' \mathrm{id}_{K(W)}] \\
&\quad \circ a'^{-1}_{K(V),K(F_r(L(X'))),K(W)} \circ [\mathrm{id}_{K(V)} \otimes' (\delta^K_{F_r(L(X')),W})^{-1}] \circ [\mathrm{id}_{K(V)} \otimes' K(\Phi_W(L(X')))] \\
&\quad \circ [\mathrm{id}_{K(V)} \otimes' \delta^K_{W,L(X')}] \circ [\mathrm{id}_{K(V)} \otimes' (\mathrm{id}_{K(W)} \otimes' \varphi_{X'})] \circ a'_{K(V),K(W),X'} \\
&= [(\delta^{F'}_{q,r})_{X'} \otimes' \mathrm{id}_{K(V)\otimes' K(W)}] \circ a'_{F'_q(F'_r(X')),K(V),K(W)} \circ [(F'_q(\varphi_{F'_r(X')})^{-1} \otimes' \mathrm{id}_{K(V)}) \otimes' \mathrm{id}_{K(W)}] \\
&\quad \circ [(\xi^K(q)_{L(F'_r(X'))} \otimes' \mathrm{id}_{K(V)}) \otimes' \mathrm{id}_{K(W)}] \circ [(\delta^K_{F_q(L(F'_r(X'))),V})^{-1} \otimes' \mathrm{id}_{K(W)}] \\
&\quad \circ [K(\Phi_V(L(F'_r(X')))) \otimes' \mathrm{id}_{K(W)}] \circ [\delta^K_{V,L(F'_r(X'))} \otimes' \mathrm{id}_{K(W)}] \circ [(\mathrm{id}_{K(V)} \otimes' K(\xi^L(r)_{X'})^{-1}) \otimes' \mathrm{id}_{K(W)}] \\
&\quad \circ a'^{-1}_{K(V),K(F_r(L(X'))),K(W)} \circ [\mathrm{id}_{K(V)} \otimes' (\delta^K_{F_r(L(X')),W})^{-1}] \circ [\mathrm{id}_{K(V)} \otimes' K(\Phi_W(L(X')))] \\
&\quad \circ [\mathrm{id}_{K(V)} \otimes' \delta^K_{W,L(X')}] \circ [\mathrm{id}_{K(V)} \otimes' (\mathrm{id}_{K(W)} \otimes' \varphi_{X'})] \circ a'_{K(V),K(W),X'} \\
&\overset{(**)}{=} [(\delta^{F'}_{q,r})_{X'} \otimes' \mathrm{id}_{K(V)\otimes' K(W)}] \circ a'_{F'_q(F'_r(X')),K(V),K(W)} \circ [(F'_q(\varphi_{F'_r(X')})^{-1} \otimes' \mathrm{id}_{K(V)}) \otimes' \mathrm{id}_{K(W)}] \\
&\quad \circ [(\xi^K(q)_{L(F'_r(X'))} \otimes' \mathrm{id}_{K(V)}) \otimes' \mathrm{id}_{K(W)}] \circ [(\delta^K_{F_q(L(F'_r(X'))),V})^{-1} \otimes' \mathrm{id}_{K(W)}] \\
&\quad \circ [K(\Phi_V(L(F'_r(X')))) \otimes' \mathrm{id}_{K(W)}] \circ [\delta^K_{V,L(F'_r(X'))} \otimes' \mathrm{id}_{K(W)}] \circ [(\mathrm{id}_{K(V)} \otimes' \varphi_{F'_r(X')}) \otimes' \mathrm{id}_{K(W)}] \\
&\quad \circ a'^{-1}_{K(V),F'_r(X'),K(W)} \circ [\mathrm{id}_{K(V)} \otimes' (F'_r(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(W)})] \circ [\mathrm{id}_{K(V)} \otimes' (\xi^K(r)_{L(X')} \otimes' \mathrm{id}_{K(W)})] \\
&\quad \circ [\mathrm{id}_{K(V)} \otimes' (\delta^K_{F_r(L(X')),W})^{-1}] \circ [\mathrm{id}_{K(V)} \otimes' K(\Phi_W(L(X')))] \circ [\mathrm{id}_{K(V)} \otimes' \delta^K_{W,L(X')}] \\
&\quad \circ [\mathrm{id}_{K(V)} \otimes' (\mathrm{id}_{K(W)} \otimes' \varphi_{X'})] \circ a'_{K(V),K(W),X'} \\
&= [(\delta^{F'}_{q,r})_{X'} \otimes' \mathrm{id}_{K(V)\otimes' K(W)}] \circ a'_{F'_q(F'_r(X')),K(V),K(W)} \circ [\mathscr{K}\Phi_V(F'_r(X')) \otimes' \mathrm{id}_{K(W)}] \\
&\quad \circ a'^{-1}_{K(V),F'_r(X'),K(W)} \circ [\mathrm{id}_{K(V)} \otimes' \mathscr{K}\Phi_W(X')] \circ a'_{K(V),K(W),X'} \\
&= (\mathscr{K}\Phi_V \otimes' \mathscr{K}\Phi_W)(X'),
\end{aligned}
$$

where $\overset{(**)}{=}$ follows from

$$F'_r(\varphi_{X'}) = \xi^{K\circ L}(r)_{X'} \circ \varphi_{F'_r(X')} = \xi^K(r)_{L(X')} \circ K(\xi^L(r)_{X'}) \circ \varphi_{F'_r(X')}.$$

To obtain $\overset{(*)}{=}$ we take the inverse of the preceding equation and get

$$F'_r(\varphi_{X'})^{-1} \circ \xi^K(r)_{L(X')} = \varphi^{-1}_{F'_r(X')} \circ K(\xi^L(r)_{X'})^{-1},$$

from which we get that

$$\begin{aligned} &F'_q(F'_r(\varphi_{X'}))^{-1} \circ F'_q(\xi^K(r)_{L(X')}) \circ \xi^K(q)_{F_r(L(X'))} \\ &= F'_q(\varphi_{F'_r(X')})^{-1} \circ F'_q(K(\xi^L(r)_{X'}))^{-1} \circ \xi^K(q)_{F_r(L(X'))} \\ &= F'_q(\varphi_{F'_r(X')})^{-1} \circ \xi^K(q)_{L(F'_r(X'))} \circ K(F_q(\xi^L(r)_{X'}))^{-1}. \end{aligned}$$

This means that

$$\delta^K_{V,W} \in \mathrm{Hom}_{Z_G(\mathcal{C}')}((K(V) \otimes' K(W), qr, \mathscr{K}\Phi_V \otimes' \mathscr{K}\Phi_W), (K(V \otimes W), qr, \mathscr{K}(\Phi_V \otimes \Phi_W)))$$

and hence that

$$\delta^{\mathscr{K}}_{(V,q,\Phi_V),(W,r,\Phi_W)} \in \mathrm{Hom}_{Z_G(\mathcal{C}')}(\mathscr{K}[(V,q,\Phi_V)] \otimes' \mathscr{K}[(W,r,\Phi_W)], \mathscr{K}[(V,q\Phi_V) \otimes (W,r,\Phi_W)]).$$

If $(V_1,q,\Phi_{V_1}),(V_2,q,\Phi_{V_2}),(W_1,r,\Phi_{W_1}),(W_2,r,\Phi_{W_2}) \in Z_G(\mathcal{C})$ and if $f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V_1,q,\Phi_{V_1}),(V_2,q,\Phi_{V_2}))$ and $g \in \mathrm{Hom}_{Z_G(\mathcal{C})}((W_1,r,\Phi_{W_1}),(W_2,r,\Phi_{W_2}))$, then naturality of $\delta^{\mathscr{K}}$ follows from

$$\begin{aligned} \delta^{\mathscr{K}}_{(V_2,q,\Phi_{V_2}),(W_2,r,\Phi_{W_2})} \circ [\mathscr{K}(f) \otimes' \mathscr{K}(g)] &= \delta^K_{V_2,W_2} \circ [K(f) \otimes' K(g)] \\ &= K(f \otimes g) \circ \delta^K_{V_1,W_1} \\ &= \mathscr{K}(f \otimes g) \circ \delta^{\mathscr{K}}_{(V_1,q,\Phi_{V_1}),(W_1,r,\Phi_{W_1})}, \end{aligned}$$

where in the second step we used naturality of δ^K. To check that $\delta^{\mathscr{K}}$ satisfies the hexagonal diagram in the definition of a tensor functor, let $(U,q,\Phi_U),(V,r,\Phi_V),(W,s,\Phi_W) \in Z_G(\mathcal{C})$. Then

$$\begin{aligned} &\mathscr{K}(a_{(U,q,\Phi_U),(V,r,\Phi_V),(W,s,\Phi_W)}) \circ \delta^{\mathscr{K}}_{(U,q,\Phi_U)\otimes(V,r,\Phi_V),(W,s,\Phi_W)} \circ [\delta^{\mathscr{K}}_{(U,q,\Phi_U),(V,r,\Phi_V)} \otimes' \mathrm{id}_{\mathscr{K}[(W,s,\Phi_W)]}] \\ &= K(a_{U,V,W}) \circ \delta^K_{U\otimes V,W} \circ [\delta^K_{U,V} \otimes' \mathrm{id}_{K(V)}] \\ &= \delta^K_{U,V\otimes W} \circ [\mathrm{id}_{K(U)} \otimes' \delta^K_{V,W}] \circ a_{K(U),K(V),K(W)} \\ &= \delta^{\mathscr{K}}_{(U,q,\Phi_U),(V,r,\Phi_V)\otimes(W,s,\Phi_W)} \circ [\mathrm{id}_{\mathscr{K}[(U,q,\Phi_U)]} \otimes' \delta^{\mathscr{K}}_{(V,r,\Phi_V),(W,s,\Phi_W)}] \circ a_{\mathscr{K}[(U,q,\Phi_U)],\mathscr{K}[(V,r,\Phi_V)],\mathscr{K}[(W,s,\Phi_W)]}, \end{aligned}$$

where in the second step we used that δ^K satisfies the hexagonal diagram in the definition of a tensor functor.

We now define $\varepsilon^{\mathscr{K}} := \varepsilon^K$. For any $X' \in \mathcal{C}$ we then have

$$\begin{aligned} &\mathscr{K}\Phi^0_I(X') \\ &= [F'_e(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(I)}] \circ [\xi^K(e)_{L(X')} \otimes' \mathrm{id}_{K(I)}] \circ (\delta^K_{F_e(L(X')),I})^{-1} \\ &\quad \circ K(\Phi^0_I(L(X'))) \circ \delta^K_{I,L(X')} \circ [\mathrm{id}_{K(I)} \otimes' \varphi_{X'}] \\ &= [F'_e(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(I)}] \circ [\xi^K(e)_{L(X')} \otimes' \mathrm{id}_{K(I)}] \circ (\delta^K_{F_e(L(X')),I})^{-1} \\ &\quad \circ K(r_{F_e(L(X'))})^{-1} \circ K(\varepsilon^F_{L(X')}) \circ K(l_{L(X')}) \circ \delta^K_{I,L(X')} \circ [\mathrm{id}_{K(I)} \otimes' \varphi_{X'}] \\ &= [F'_e(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(I)}] \circ [\xi^K(e)_{L(X')} \otimes' \mathrm{id}_{K(I)}] \circ [\mathrm{id}_{K(F_e(L(X')))} \otimes' \varepsilon^K] \\ &\quad \circ r'^{-1}_{K(F_e(L(X')))} \circ K(\varepsilon^F_{L(X')}) \circ l'_{K(L(X'))} \circ [(\varepsilon^K)^{-1} \otimes' \mathrm{id}_{K(L(X'))}] \circ [\mathrm{id}_{K(I)} \otimes' \varphi_{X'}] \\ &= [\mathrm{id}_{F'_e(X')} \otimes' \varepsilon^K] \circ [F'_e(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{I'}] \circ [\xi^K(e)_{L(X')} \otimes' \mathrm{id}_{I'}] \circ r'^{-1}_{K(F_e(L(X')))} \end{aligned}$$

$$
\begin{aligned}
&\circ K(\varepsilon^F_{L(X')}) \circ l'_{K(L(X'))} \circ [\mathrm{id}_{I'} \otimes' \varphi_{X'}] \circ [(\varepsilon^K)^{-1} \otimes' \mathrm{id}_{X'}] \\
&= [\mathrm{id}_{F'_e(X')} \otimes' \varepsilon^K] \circ r'^{-1}_{F'_e(X')} \circ F'_e(\varphi_{X'})^{-1} \circ \xi^K(e)_{L(X')} \circ K(\varepsilon^F_{L(X')}) \circ \varphi_{X'} \circ l'_{X'} \circ [(\varepsilon^K)^{-1} \otimes' \mathrm{id}_{X'}] \\
&= [\mathrm{id}_{F'_e(X')} \otimes' \varepsilon^K] \circ r'^{-1}_{F'_e(X')} \circ F'_e(\varphi_{X'})^{-1} \circ \varepsilon^{F'}_{K(L(X'))} \circ \varphi_{X'} \circ l'_{X'} \circ [(\varepsilon^K)^{-1} \otimes' \mathrm{id}_{X'}] \\
&= [\mathrm{id}_{F'_e(X')} \otimes' \varepsilon^K] \circ r'^{-1}_{F'_e(X')} \circ \varepsilon^{F'}_{X'} \circ l'_{X'} \circ [(\varepsilon^K)^{-1} \otimes' \mathrm{id}_{X'}] \\
&= [\mathrm{id}_{F'_e(X')} \otimes' \varepsilon^K] \circ \Phi^0_{I'}(X') \circ [(\varepsilon^K)^{-1} \otimes' \mathrm{id}_{X'}],
\end{aligned}
$$

which shows that $\varepsilon^K \in \mathrm{Hom}_{Z_G(\mathcal{C}')}((I', e, \Phi^0_{I'}), (K(I), e, \mathscr{K}\Phi^0_I))$ and hence that

$$
\varepsilon^{\mathscr{K}} \in \mathrm{Hom}_{Z_G(\mathcal{C}')}((I', e, \Phi^0_{I'}), \mathscr{K}[(I, e, \Phi^0_I)]).
$$

For each $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ we have

$$
\begin{aligned}
l'_{\mathscr{K}[(V,q,\Phi_V)]} &= l'_{K(V)} = K(l_V) \circ \delta^K_{I,V} \circ [\varepsilon^K \otimes' \mathrm{id}_{K(V)}] \\
&= \mathscr{K}(l_{(V,q,\Phi_V)}) \circ \delta^{\mathscr{K}}_{(I,e,\Phi^0_I),(V,q,\Phi_V)} \circ [\varepsilon^{\mathscr{K}} \otimes' \mathrm{id}_{\mathscr{K}[(V,q,\Phi_V)]}]
\end{aligned}
$$

and

$$
\begin{aligned}
r'_{\mathscr{K}[(V,q,\Phi_V)]} &= r'_{K(V)} = K(r_V) \circ \delta^K_{V,I} \circ [\mathrm{id}_{K(V)} \otimes' \varepsilon^K] \\
&= \mathscr{K}(r_{(V,q,\Phi_V)}) \circ \delta^{\mathscr{K}}_{(V,q,\Phi_V),(I,e,\Phi^0_I)} \circ [\mathrm{id}_{\mathscr{K}[(V,q,\Phi_V)]} \otimes' \varepsilon^{\mathscr{K}}].
\end{aligned}
$$

We thus conclude that $(\mathscr{K}, \varepsilon^{\mathscr{K}}, \delta^{\mathscr{K}})$ is indeed a tensor functor.

B.4 The tensor functor $\mathscr{K}$ can be made into a G-crossed functor

In this section we will prove that the tensor functor $(\mathscr{K}, \varepsilon^{\mathscr{K}}, \delta^{\mathscr{K}})$ can be equipped with the structure of a G-crossed functor. For this we need the following lemma.

Lemma B.4.1 *Let $(V, r, \Phi_V) \in Z_G(\mathcal{C})$. Then for any $q \in G$ we have*

$$
\xi^K(q)_V \in \mathrm{Hom}_{Z_G(\mathcal{C}')}((K(F_q(V)), qrq^{-1}, \mathscr{K}\mathcal{F}_q\Phi_V), (F'_q(K(V)), qrq^{-1}, \mathcal{F}'_q\mathscr{K}\Phi_V)).
$$

Proof. For any $X' \in \mathcal{C}'$ we have

$$
\begin{aligned}
&[\mathrm{id}_{F'_{qrq^{-1}}(X')} \otimes' \xi^K(q)_V] \circ [\mathscr{K}\mathcal{F}_q\Phi_V(X')] \\
&= [\mathrm{id}_{F'_{qrq^{-1}}(X')} \otimes' \xi^K(q)_V] \circ [F'_{qrq^{-1}}(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(F_q(V))}] \circ [\xi^K(qrq^{-1})_{L(X')} \otimes' \mathrm{id}_{K(F_q(V))}] \\
&\quad \circ (\delta^K_{F_{qrq^{-1}}(L(X')),F_q(V)})^{-1} \circ K(\mathcal{F}_q\Phi_V(L(X'))) \circ \delta^K_{F_q(V),L(X')} \circ [\mathrm{id}_{K(F_q(V))} \otimes' \varphi_{X'}] \\
&= [\mathrm{id}_{F'_{qrq^{-1}}(X')} \otimes' \xi^K(q)_V] \circ [F'_{qrq^{-1}}(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(F_q(V))}] \circ [\xi^K(qrq^{-1})_{L(X')} \otimes' \mathrm{id}_{K(F_q(V))}] \\
&\quad \circ (\delta^K_{F_{qrq^{-1}}(L(X')),F_q(V)})^{-1} \circ K(\alpha_{q,r,q^{-1}}(L(X')) \otimes \mathrm{id}_{F_q(V)}) \circ K(\delta^{F_q}_{F_r(F_{q^{-1}}(L(X'))),V})^{-1} \\
&\quad \circ K(F_q(\Phi_V(F_{q^{-1}}(L(X'))))) \circ K(\delta^{F_q}_{V,F_{q^{-1}}(L(X'))}) \circ K(\mathrm{id}_{F_q(V)} \otimes \Delta_q(L(X'))^{-1}) \\
&\quad \circ \delta^K_{F_q(V),L(X')} \circ [\mathrm{id}_{K(F_q(V))} \otimes' \varphi_{X'}] \\
&= [\mathrm{id}_{F'_{qrq^{-1}}(X')} \otimes' \xi^K(q)_V] \circ [F'_{qrq^{-1}}(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(F_q(V))}] \circ [\xi^K(qrq^{-1})_{L(X')} \otimes' \mathrm{id}_{K(F_q(V))}] \\
&\quad \circ [K(\alpha_{q,r,q^{-1}}(L(X'))) \otimes' \mathrm{id}_{K(F_q(V))}] \circ (\delta^K_{F_q(F_r(F_{q^{-1}}(L(X')))),F_q(V)})^{-1} \circ K(\delta^{F_q}_{F_r(F_{q^{-1}}(L(X'))),V})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\circ K(F_q(\Phi_V(F_{q^{-1}}(L(X'))))) \circ K(\delta^{F_q}_{V,F_{q^{-1}}(L(X'))}) \circ \delta^K_{F_q(V),F_q(F_{q^{-1}}(L(X')))} \\
&\circ [\mathrm{id}_{K(F_q(V))} \otimes' K(\Delta_q(L(X')))^{-1}] \circ [\mathrm{id}_{K(F_q(V))} \otimes' \varphi_{X'}] \\
= {}& [\mathrm{id}_{F'_{qrq^{-1}}(X')} \otimes' \xi^K(q)_V] \circ [F'_{qrq^{-1}}(\varphi_{X'})^{-1} \otimes' \mathrm{id}_{K(F_q(V))}] \circ [(\delta^{F'}_{q,rq^{-1}})_{K(L(X'))} \otimes' \mathrm{id}_{K(F_q(V))}] \\
&\circ [F'_q((\delta^{F'}_{r,q^{-1}})_{K(L(X'))}) \otimes' \mathrm{id}_{K(F_q(V))}] \circ [F'_q(F'_r(\xi^K(q^{-1})_{L(X')})) \otimes' \mathrm{id}_{K(F_q(V))}] \\
&\circ [F'_q(\xi^K(r)_{F_{q^{-1}}(L(X'))}) \otimes' \mathrm{id}_{K(F_q(V))}] \circ [F'_q(K((\delta^F_{r,q^{-1}})_{L(X')}))^{-1} \otimes' \mathrm{id}_{K(F_q(V))}] \\
&\circ [\xi^K(q)_{F_{rq^{-1}}(L(X'))} \otimes' \mathrm{id}_{K(F_q(V))}] \circ [K((\delta^F_{q,rq^{-1}})_{L(X')})^{-1} \otimes' \mathrm{id}_{K(F_q(V))}] \\
&\circ [K((\delta^F_{q,rq^{-1}})_{L(X')}) \otimes' \mathrm{id}_{K(F_q(V))}] \circ [K(F_q((\delta^F_{r,q^{-1}})_{L(X')})) \otimes' \mathrm{id}_{K(F_q(V))}] \\
&\circ (\delta^K_{F_q(F_r(F_{q^{-1}}(L(X')))),F_q(V)})^{-1} \circ K(\delta^{F_q}_{F_r(F_{q^{-1}}(L(X'))),V})^{-1} \circ K(F_q(\Phi_V(F_{q^{-1}}(L(X'))))) \\
&\circ K(\delta^{F_q}_{V,F_{q^{-1}}(L(X'))}) \circ \delta^K_{F_q(V),F_q(F_{q^{-1}}(L(X')))} \circ [\mathrm{id}_{K(F_q(V))} \otimes' K((\delta^F_{q,q^{-1}})_{L(X')})^{-1}] \\
&\circ [\mathrm{id}_{K(F_q(V))} \otimes' K(\varepsilon^F_{L(X')})] \circ [\mathrm{id}_{K(F_q(V))} \otimes' \varphi_{X'}] \\
= {}& [(\delta^{F'}_{q,rq^{-1}})_{X'} \otimes' \mathrm{id}_{F'_q(K(V))}] \circ [F'_q((\delta^{F'}_{r,q^{-1}})_{X'}) \otimes' \mathrm{id}_{F'_q(K(V))}] \\
&\circ [F'_q(F'_r(F'_{q^{-1}}(\varphi_{X'})))^{-1} \otimes' \mathrm{id}_{F'_q(K(V))}] \circ [F'_q(F'_r(\xi^K(q^{-1})_{L(X')})) \otimes' \mathrm{id}_{F'_q(K(V))}] \\
&\circ [F'_q(\xi^K(r)_{F_{q^{-1}}(L(X'))}) \otimes' \mathrm{id}_{F'_q(K(V))}] \circ [\xi^K(q)_{F_r(F_{q^{-1}}(L(X')))} \otimes' \xi^K(q)_V] \\
&\circ (\delta^K_{F_q(F_r(F_{q^{-1}}(L(X')))),F_q(V)})^{-1} \circ K(\delta^{F_q}_{F_r(F_{q^{-1}}(L(X'))),V})^{-1} \circ K(F_q(\Phi_V(F_{q^{-1}}(L(X'))))) \\
&\circ K(\delta^{F_q}_{V,F_{q^{-1}}(L(X'))}) \circ \delta^K_{F_q(V),F_q(F_{q^{-1}}(L(X')))} \circ [\mathrm{id}_{K(F_q(V))} \otimes' K((\delta^F_{q,q^{-1}})_{L(X')})^{-1}] \\
&\circ [\mathrm{id}_{K(F_q(V))} \otimes' \xi^K(e)^{-1}_{L(X')}] \circ [\mathrm{id}_{K(F_q(V))} \otimes' \varepsilon^{F'}_{K(L(X'))}] \circ [\mathrm{id}_{K(F_q(V))} \otimes' \varphi_{X'}] \\
\overset{(*)}{=} {}& [\alpha'_{q,r,q^{-1}}(X') \otimes' \mathrm{id}_{F'_q(K(V))}] \circ [F'_q(F'_r(\varphi^{-1}_{F'_q(X')})) \otimes' \mathrm{id}_{F'_q(K(V))}] \\
&\circ [F'_q(F'_r(K(\xi^L(q^{-1})_{X'})))^{-1} \otimes' \mathrm{id}_{F'_q(K(V))}] \circ [F'_q(\xi^K(r)_{F_{q^{-1}}(L(X'))}) \otimes' \mathrm{id}_{F'_q(K(V))}] \\
&\circ (\delta^{F'_q}_{K(F_r(F_{q^{-1}}(L(X')))),K(V)})^{-1} \circ F'_q(\delta^K_{F_r(F_{q^{-1}}(L(X'))),V})^{-1} \circ \xi^K(q)_{F_r(F_{q^{-1}}(L(X')))\otimes' V} \\
&\circ K(F_q(\Phi_V(F_{q^{-1}}(L(X'))))) \circ K(\delta^{F_q}_{V,F_{q^{-1}}(L(X'))}) \circ \delta^K_{F_q(V),F_q(F_{q^{-1}}(L(X')))} \\
&\circ [\mathrm{id}_{K(F_q(V))} \otimes' K((\delta^F_{q,q^{-1}})_{L(X')})^{-1}] \circ [\mathrm{id}_{K(F_q(V))} \otimes' \xi^K(e)^{-1}_{L(X')}] \\
&\circ [\mathrm{id}_{K(F_q(V))} \otimes' F'_e(\varphi_{X'})] \circ [\mathrm{id}_{K(F_q(V))} \otimes' \varepsilon^{F'}_{X'}] \\
= {}& [\alpha'_{q,r,q^{-1}}(X') \otimes' \mathrm{id}_{F'_q(K(V))}] \circ (\delta^{F'_q}_{F'_r(F'_q(X')),K(V)})^{-1} \circ F'_q(F'_r(\varphi_{F'_{q^{-1}}(X')})^{-1} \otimes' \mathrm{id}_{K(V)}) \\
&\circ F'_q(\xi^K(r)_{L(F'_q(X'))} \otimes' \mathrm{id}_{K(V)}) \circ F'_q(K(F_r(\xi^L(q^{-1})_{X'}))^{-1} \otimes' \mathrm{id}_{K(V)}) \\
&\circ F'_q(\delta^K_{F_r(F_{q^{-1}}(L(X'))),V})^{-1} \circ F'_q(K(\Phi_V(F_{q^{-1}}(L(X'))))) \circ \xi^K(q)_{V\otimes F_{q^{-1}}(L(X'))} \\
&\circ K(\delta^{F_q}_{V,F_{q^{-1}}(L(X'))}) \circ \delta^K_{F_q(V),F_q(F_{q^{-1}}(L(X')))} \circ [\mathrm{id}_{K(F_q(V))} \otimes' K((\delta^F_{q,q^{-1}})_{L(X')})^{-1}] \\
&\circ [\mathrm{id}_{K(F_q(V))} \otimes' \xi^K(e)^{-1}_{L(X')}] \circ [\mathrm{id}_{K(F_q(V))} \otimes' F'_e(\varphi_{X'})] \circ [\mathrm{id}_{K(F_q(V))} \otimes' \varepsilon^{F'}_{X'}] \\
= {}& [\alpha'_{q,r,q^{-1}}(X') \otimes' \mathrm{id}_{F'_q(K(V))}] \circ (\delta^{F'_q}_{F'_r(F'_q(X')),K(V)})^{-1} \circ F'_q(F'_r(\varphi_{F'_{q^{-1}}(X')})^{-1} \otimes' \mathrm{id}_{K(V)}) \\
&\circ F'_q(\xi^K(r)_{L(F'_q(X'))} \otimes' \mathrm{id}_{K(V)}) \circ F'_q(\delta^K_{F_r(L(F'_{q^{-1}}(X'))),V})^{-1} \circ F'_q(K(\Phi_V(L(F'_{q^{-1}}(X'))))) \\
&\circ F'_q(K(\mathrm{id}_V \otimes' \xi^L(q^{-1})^{-1}_{X'})) \circ F'_q(\delta^K_{V,F_{q^{-1}}(L(X'))}) \circ \delta^{F'_q}_{K(V),K(F_{q^{-1}}(L(X')))}
\end{aligned}
$$

$$
\begin{aligned}
&\circ [\xi^K(q)_V \otimes' \xi^K(q)_{F_{q^{-1}}(L(X'))}] \circ [\mathrm{id}_{K(F_q(V))} \otimes' K((\delta^F_{q,q^{-1}})_{L(X')})^{-1}] \\
&\circ [\mathrm{id}_{K(F_q(V))} \otimes' \xi^K(e)^{-1}_{L(X')}] \circ [\mathrm{id}_{K(F_q(V))} \otimes' F'_e(\varphi_{X'})] \circ [\mathrm{id}_{K(F_q(V))} \otimes' \varepsilon^{F'}_{X'}] \\
= {}& [\alpha'_{q,r,q^{-1}}(X') \otimes' \mathrm{id}_{F'_q(K(V))}] \circ (\delta^{F'_q}_{F'_r(F'_q(X')),K(V)})^{-1} \circ F'_q(F'_r(\varphi_{F'_{q^{-1}}(X')})^{-1} \otimes' \mathrm{id}_{K(V)}) \\
&\circ F'_q(\xi^K(r)_{L(F'_q(X'))} \otimes' \mathrm{id}_{K(V)}) \circ F'_q(\delta^K_{F_r(L(F'_{q^{-1}}(X'))),V})^{-1} \circ F'_q(K(\Phi_V(L(F'_{q^{-1}}(X'))))) \\
&\circ F'_q(\delta^K_{V,L(F'_{q^{-1}}(X'))}) \circ \delta^{F'_q}_{K(V),K(L(F'_{q^{-1}}(X')))} \circ [\mathrm{id}_{F'_q(K(V))} \otimes' F'_q(K(\xi^L(q^{-1})_{X'}))^{-1}] \\
&\circ [\mathrm{id}_{F'_q(K(V))} \otimes' \xi^K(q)_{F_{q^{-1}}(L(X'))}] \circ [\mathrm{id}_{F'_q(K(V))} \otimes' K((\delta^F_{q,q^{-1}})_{L(X')})^{-1}] \\
&\circ [\mathrm{id}_{F'_q(K(V))} \otimes' \xi^K(e)^{-1}_{L(X')}] \circ [\mathrm{id}_{F'_q(K(V))} \otimes' F'_e(\varphi_{X'})] \circ [\mathrm{id}_{F'_q(K(V))} \otimes' \varepsilon^{F'}_{X'}] \circ [\xi^K(q)_V \otimes' \mathrm{id}_{X'}] \\
= {}& [\alpha'_{q,r,q^{-1}}(X') \otimes' \mathrm{id}_{F'_q(K(V))}] \circ (\delta^{F'_q}_{F'_r(F'_q(X')),K(V)})^{-1} \circ F'_q(F'_r(\varphi_{F'_{q^{-1}}(X')})^{-1} \otimes' \mathrm{id}_{K(V)}) \\
&\circ F'_q(\xi^K(r)_{L(F'_q(X'))} \otimes' \mathrm{id}_{K(V)}) \circ F'_q(\delta^K_{F_r(L(F'_{q^{-1}}(X'))),V})^{-1} \circ F'_q(K(\Phi_V(L(F'_{q^{-1}}(X'))))) \\
&\circ F'_q(\delta^K_{V,L(F'_{q^{-1}}(X'))}) \circ \delta^{F'_q}_{K(V),K(L(F'_{q^{-1}}(X')))} \circ [\mathrm{id}_{F'_q(K(V))} \otimes' F'_q(K(\xi^L(q^{-1})_{X'}))^{-1}] \\
&\circ [\mathrm{id}_{F'_q(K(V))} \otimes' F'_q(\xi^K(q^{-1})_{L(X')})^{-1}] \circ [\mathrm{id}_{F'_q(K(V))} \otimes' (\delta^{F'}_{q,q^{-1}})^{-1}_{K(L(X'))}] \circ [\mathrm{id}_{F'_q(K(V))} \otimes' F'_e(\varphi_{X'})] \\
&\circ [\mathrm{id}_{F'_q(K(V))} \otimes' \varepsilon^{F'}_{X'}] \circ [\xi^K(q)_V \otimes' \mathrm{id}_{X'}] \\
\overset{(**)}{=} {}& [\alpha'_{q,r,q^{-1}}(X') \otimes' \mathrm{id}_{F'_q(K(V))}] \circ (\delta^{F'_q}_{F'_r(F'_q(X')),K(V)})^{-1} \circ F'_q(F'_r(\varphi_{F'_{q^{-1}}(X')})^{-1} \otimes' \mathrm{id}_{K(V)}) \\
&\circ F'_q(\xi^K(r)_{L(F'_{q^{-1}}(X'))} \otimes' \mathrm{id}_{K(V)}) \circ F'_q(\delta^K_{F_r(L(F'_{q^{-1}}(X'))),V})^{-1} \circ F'_q(K(\Phi_V(L(F'_{q^{-1}}(X'))))) \\
&\circ F'_q(\delta^K_{V,L(F'_{q^{-1}}(X'))}) \circ F'_q(\mathrm{id}_{K(V)} \otimes' \varphi_{F'_{q^{-1}}(X')}) \circ \delta^{F'_q}_{K(V),F'_{q^{-1}}(X')} \\
&\circ [\mathrm{id}_{F'_q(K(V))} \otimes' (\delta^{F'}_{q,q^{-1}})^{-1}_{X'}] \circ [\mathrm{id}_{F'_q(K(V))} \otimes' \varepsilon^{F'}_{X'}] \circ [\xi^K(q)_V \otimes' \mathrm{id}_{X'}] \\
= {}& [\alpha'_{q,r,q^{-1}}(X') \otimes' \mathrm{id}_{F'_q(K(V))}] \circ (\delta^{F'_q}_{F'_r(F'_q(X')),K(V)})^{-1} \circ F'_q(\mathscr{K}\Phi_V(F'_{q^{-1}}(X'))) \\
&\circ \delta^{F'_q}_{K(V),F'_{q^{-1}}(X')} \circ [\mathrm{id}_{F'_q(K(V))} \otimes' \Delta'_q(X')^{-1}] \circ [\xi^K(q)_V \otimes' \mathrm{id}_{X'}] \\
= {}& [\mathcal{F}'_q\mathscr{K}\Phi_V(X')] \circ [\xi^K(q)_V \otimes' \mathrm{id}_{X'}].
\end{aligned}
$$

We will now explain the equalities $\overset{(*)}{=}$ and $\overset{(**)}{=}$ in somewhat more detail. It follows from

$$F'_{q^{-1}}(\varphi_{X'}) = \xi^K(q^{-1})_{L(X')} \circ K(\xi^L(q^{-1})_{X'}) \circ \varphi_{F'_{q^{-1}}(X')}$$

that we have

$$(\xi^K(q^{-1})_{L(X')})^{-1} \circ F'_{q^{-1}}(\varphi_{X'}) = K(\xi^L(q^{-1})_{X'}) \circ \varphi_{F'_{q^{-1}}(X')}.$$

Taking the inverse on both sides results in

$$F'_{q^{-1}}(\varphi_{X'})^{-1} \circ \xi^K(q^{-1})_{L(X')} = \varphi^{-1}_{F_{q^{-1}}(X')} \circ K(\xi^L(q^{-1})_{X'})^{-1},$$

which was used in the equality $\overset{(*)}{=}$ above. For the other equality we first observe that

$$K(\xi^L(q^{-1})_{X'})^{-1} \circ (\xi^K(q^{-1})_{L(X')})^{-1} \circ F'_{q^{-1}}(\varphi_{X'}) = \varphi_{F'_{q^{-1}}(X')}.$$

Letting F'_q act on both sides, we get

$$F'_q(K(\xi^L(q^{-1})_{X'}))^{-1} \circ F'_q(\xi^K(q^{-1})_{L(X')})^{-1} \circ F'_q(F'_{q^{-1}}(\varphi_{X'})) = F'_q(\varphi_{F'_{q^{-1}}(X')}).$$

Using that

$$F_q'(F_{q^{-1}}'(\varphi_{X'})) = (\delta_{q,q^{-1}}^{F'})_{K(L(X'))}^{-1} \circ F_e'(\varphi_{X'}) \circ (\delta_{q,q^{-1}}^{F'})_{X'},$$

this gives us

$$F_q'(K(\xi^L(q^{-1})_{X'}))^{-1} \circ F_q'(\xi^K(q^{-1})_{L(X')})^{-1} \circ (\delta_{q,q^{-1}}^{F'})_{K(L(X'))}^{-1} \circ F_e'(\varphi_{X'}) = F_q'(\varphi_{F_{q^{-1}}'(X')}) \circ (\delta_{q,q^{-1}}^{F'})_{X'}^{-1},$$

which was used in the equality $\overset{(**)}{=}$ above.
□

Using this lemma, for each $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ and $q \in G$ we can define an isomorphism

$$\xi^{\mathscr{K}}(q)_{(V,q,\Phi_V)} \in \mathrm{Hom}_{Z_G(\mathcal{C}')}(\mathscr{K}\mathcal{F}_q[(V, q, \Phi_V)], \mathcal{F}_q'\mathscr{K}[(V, q, \Phi_V)])$$

by $\xi^{\mathscr{K}}(q)_{(V,q,\Phi_V)} := \xi^K(q)_V$. We will now show that $(\mathscr{K}, \varepsilon^{\mathscr{K}}, \delta^{\mathscr{K}}, \xi^{\mathscr{K}})$ is a G-functor from $Z_G(\mathcal{C})$ to $Z_G(\mathcal{C}')$.
To check naturality of $\xi^{\mathscr{K}}(q)$ for $q \in G$, let $(V, q, \Phi_V), (W, q, \Phi_W) \in Z_G(\mathcal{C})$ and $f \in \mathrm{Hom}_{Z_G(\mathcal{C})}((V, q, \Phi_V), (W, q, \Phi_W))$.
Then

$$\begin{aligned}\xi^{\mathscr{K}}(q)_{(W,q,\Phi_W)} \circ \mathscr{K}(\mathcal{F}_q(f)) &= \xi^K(q)_W \circ K(F_q(f)) = F_q'(K(f)) \circ \xi^K(q)_V \\ &= \mathcal{F}_q'(\mathscr{K}(f)) \circ \xi^{\mathscr{K}}(q)_{(V,q,\Phi_V)},\end{aligned}$$

where in the second step we used naturality of $\xi^K(q)$. If $q, r \in G$ and $(V, s, \Phi_V) \in Z_G(\mathcal{C})$, then

$$\begin{aligned}\xi^{\mathscr{K}}(qr)_{(V,s,\Phi_V)} \circ \mathscr{K}((\delta_{q,r}^{\mathcal{F}})_{(V,s,\Phi_V)}) &= \xi^K(qr)_V \circ K((\delta_{q,r}^F)_V) \\ &= (\delta_{q,r}^{F'})_{K(V)} \circ F_q'(\xi^K(r)_V) \circ \xi^K(q)_{F_r(V)} \\ &= (\delta_{q,r}^{\mathcal{F}'})_{(K(V),s,\mathscr{K}\Phi_V)} \circ \mathcal{F}_q'(\xi^{\mathscr{K}}(r)_{(V,s,\Phi_V)}) \circ \xi^{\mathscr{K}}(q)_{(F_r(V),rsr^{-1},\mathcal{F}_r\Phi_V)} \\ &= (\delta_{q,r}^{\mathcal{F}'})_{\mathscr{K}[(V,s,\Phi_V)]} \circ \mathcal{F}_q'(\xi^{\mathscr{K}}(r)_{(V,s,\Phi_V)}) \circ \xi^{\mathscr{K}}(q)_{\mathcal{F}_r[(V,s,\Phi_V)]},\end{aligned}$$

where in the second step we used that ξ^K is part of the data of a G-functor. For each $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ we also have

$$\begin{aligned}\varepsilon_{\mathscr{K}[(V,q,\Phi_V)]}^{\mathcal{F}'} &= \varepsilon_{K(V)}^{F'} = \xi^K(e)_V \circ K(\varepsilon_V^F) \\ &= \xi^{\mathscr{K}}(e)_{(V,q,\Phi_V)} \circ \mathscr{K}(\varepsilon_{(V,q,\Phi_V)}^{\mathcal{F}}).\end{aligned}$$

Finally, if $q \in G$ and $(V, r, \Phi_V), (W, s, \Phi_W) \in Z_G(\mathcal{C})$, then

$$\begin{aligned}&\xi^{\mathscr{K}}(q)_{(V,r,\Phi_V)\otimes(W,s,\Phi_W)} = \xi^{\mathscr{K}}(q)_{(V\otimes W,rs,\Phi_V\otimes\Phi_W)} = \xi^K(q)_{V\otimes W} \\ &= F_q'(\delta_{V,W}^K) \circ \delta_{K(V),K(W)}^{F_q'} \circ [\xi^K(q)_V \otimes' \xi^K(q)_W] \circ (\delta_{F_q(V),F_q(W)}^K)^{-1} \circ K(\delta_{V,W}^{F_q})^{-1} \\ &= \mathcal{F}_q'(\delta_{(V,r,\Phi_V),(W,s,\Phi_W)}^{\mathscr{K}}) \circ \delta_{\mathscr{K}[(V,r,\Phi_V)],\mathscr{K}[(W,s,\Phi_W)]}^{\mathcal{F}_q'} \circ [\xi^{\mathscr{K}}(q)_{(V,r,\Phi_V)} \otimes' \xi^{\mathscr{K}}(q)_{(W,s,\Phi_W)}] \\ &\quad \circ (\delta_{\mathcal{F}_q[(V,r,\Phi_V)],\mathcal{F}_q[(W,s,\Phi_W)]}^{\mathscr{K}})^{-1} \circ \mathscr{K}(\delta_{(V,r,\Phi_V),(W,s,\Phi_W)}^{\mathcal{F}_q})^{-1}.\end{aligned}$$

We thus conclude that $(\mathscr{K}, \varepsilon^{\mathscr{K}}, \delta^{\mathscr{K}}, \xi^{\mathscr{K}})$ is a G-functor. Furthermore, since for each $(V, q, \Phi_V) \in Z_G(\mathcal{C})$ we have $\partial'[(K(V), q, \mathscr{K}\Phi_V)] = q$, it is in fact a G-crossed functor.

B.5 The G-crossed functor $\mathscr{K}$ is braided

We will now demonstrate that the G-crossed functor $(\mathscr{K}, \varepsilon^{\mathscr{K}}, \delta^{\mathscr{K}}, \xi^{\mathscr{K}})$ is braided.

Lemma B.5.1 *Let* $(V, q, \Phi_V), (W, r, \Phi_W) \in Z_G(\mathcal{C})$. *Then*

$$\mathscr{K}\Phi_V(K(W)) = [\xi^K(q)_W \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(W),V})^{-1} \circ K(\Phi_V(W)) \circ \delta^K_{V,W}$$

and the G*-crossed functor* $(\mathscr{K}, \varepsilon^{\mathscr{K}}, \delta^{\mathscr{K}}, \xi^{\mathscr{K}})$ *is braided.*

Proof. Using the natural isomorphism $\psi : L \circ K \to \mathrm{id}_{\mathcal{C}}$ we get

$$\begin{aligned}
&\mathscr{K}\Phi_V(K(W))\\
&= [F'_q(\varphi_{K(W)})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [\xi^K(q)_{L(K(W))} \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(L(K(W))),V})^{-1}\\
&\quad \circ K(\Phi_V(L(K(W)))) \circ \delta^K_{V,L(K(W))} \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{K(W)}]\\
&= [F'_q(\varphi_{K(W)})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [\xi^K(q)_{L(K(W))} \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(L(K(W))),V})^{-1}\\
&\quad \circ K(\Phi_V(L(K(W)))) \circ K(\mathrm{id}_V \otimes \psi_W^{-1}) \circ K(\mathrm{id}_V \otimes \psi_W) \circ \delta^K_{V,L(K(W))} \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{K(W)}]\\
&= [F'_q(\varphi_{K(W)})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [\xi^K(q)_{L(K(W))} \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(L(K(W))),V})^{-1}\\
&\quad \circ K(F_q(\psi_W)^{-1} \otimes \mathrm{id}_V) \circ K(\Phi_V(W)) \circ \delta^K_{V,W} \circ [\mathrm{id}_{K(V)} \otimes' K(\psi_W)] \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{K(W)}]\\
&= [F'_q(\varphi_{K(W)})^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [F'_q(K(\psi_W))^{-1} \otimes' \mathrm{id}_{K(V)}] \circ [\xi^K(q)_W \otimes' \mathrm{id}_{K(V)}]\\
&\quad \circ (\delta^K_{F_q(W),V})^{-1} \circ K(\Phi_V(W)) \circ \delta^K_{V,W} \circ [\mathrm{id}_{K(V)} \otimes' K(\psi_W)] \circ [\mathrm{id}_{K(V)} \otimes' \varphi_{K(W)}],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&[\xi^K(q)_W \otimes' \mathrm{id}_{K(V)}] \circ (\delta^K_{F_q(W),V})^{-1} \circ K(\Phi_V(W)) \circ \delta^K_{V,W}\\
&= [F'_q(K(\psi_W)) \otimes' \mathrm{id}_{K(V)}] \circ [F'_q(\varphi_{K(W)}) \otimes' \mathrm{id}_{K(V)}] \circ [\mathscr{K}\Phi_V(K(W))]\\
&\quad \circ [\mathrm{id}_{K(V)} \otimes' \varphi^{-1}_{K(W)}] \circ [\mathrm{id}_{K(V)} \otimes' K(\psi_W)^{-1}]\\
&= \mathscr{K}\Phi_V(K(W)),
\end{aligned}$$

where the second equality follows from naturality of $\mathscr{K}\Phi_V$. But this can be rewritten in terms of morphisms in $Z_G(\mathcal{C}')$ as

$$\begin{aligned}
C'_{(K(V),q,\mathscr{K}\Phi_V),(K(W),r,\mathscr{K}\Phi_W)} &= [\xi^{\mathscr{K}}(q)_{(W,r,\Phi_W)} \otimes' \mathrm{id}_{(K(V),q,\mathscr{K}\Phi_V)}] \circ (\delta^{\mathscr{K}}_{(F_q(W),qrq^{-1},\mathcal{F}_q\Phi_W),(V,q,\Phi_V)})^{-1}\\
&\quad \circ \mathscr{K}(C_{(V,q,\Phi_V),(W,r,\Phi_W)}) \circ \delta^{\mathscr{K}}_{(V,q,\Phi_V),(W,r,\Phi_W)},
\end{aligned}$$

which is precisely the condition for $(\mathscr{K}, \varepsilon^{\mathscr{K}}, \delta^{\mathscr{K}}, \xi^{\mathscr{K}})$ to be braided, so that it is in fact a braided G-crossed functor.
□

B.6 $Z_G(\mathcal{C}) \simeq Z_G(\mathcal{C}')$ as braided G-crossed categories

In the preceding sections we have shown how to construct a braided G-crossed functor

$$(\mathscr{K}, \varepsilon^{\mathscr{K}}, \delta^{\mathscr{K}}, \xi^{\mathscr{K}}) : Z_G(\mathcal{C}) \to Z_G(\mathcal{C}')$$

from the G-functors $(K, \varepsilon^K, \delta^K, \xi^K) : \mathcal{C} \to \mathcal{C}'$ and $(L, \varepsilon^L, \delta^L, \xi^L) : \mathcal{C}' \to \mathcal{C}$ and the natural G-isomorphisms $\varphi : \mathrm{id}_{\mathcal{C}'} \to K \circ L$ and $\psi : L \circ K \to \mathrm{id}_{\mathcal{C}}$. By interchanging the roles of K and L and interchanging the roles of φ and ψ^{-1}, it is trivial to see that we can also construct a braided G-crossed functor $\mathscr{L} : Z_G(\mathcal{C}') \to Z_G(\mathcal{C})$ from the same collection of data. For future reference it will be convenient to have an explicit expression for the functor $\mathscr{L}$. Analogues to $\mathscr{K}$, we define the functor $\mathscr{L}$ on the objects of $Z_G(\mathcal{C}')$ by $\mathscr{K}[(V',q,\Phi_{V'})] := (L(V'),q,\mathscr{L}\Phi_{V'})$, where the half q-braiding $\mathscr{L}\Phi_{V'}$ for $L(V')$ is given by

$$\mathscr{L}\Phi_{V'}(X) = [F_q(\psi_X)\otimes \mathrm{id}_{L(V')}]\circ[\xi^L(q)_{K(X)}\otimes \mathrm{id}_{L(V')}]\circ(\delta^L_{F'_q(K(X)),V'})^{-1}$$
$$\circ L(\Phi_{V'}(K(X)))\circ \delta^L_{V',K(X)}\circ[\mathrm{id}_{L(V')}\otimes \psi_X^{-1}]$$

for $X\in\mathcal{C}$. On the morphisms $\mathscr{L}$ is defined as $\mathscr{L}(f') = L(f')$ and the $\varepsilon^{\mathscr{L}}$, $\delta^{\mathscr{L}}$ and $\xi^{\mathscr{L}}$ are given directly by the ε^L, δ^L and ξ^L, in the same manner as we did for $\mathscr{K}$.

We will now show that the functors $\mathscr{K}$ and $\mathscr{L}$ together set up an equivalence of braided G-crossed categories.

Lemma B.6.1 *If $(V',q,\Phi_{V'})\in Z_G(\mathcal{C}')$, then*

$$\varphi_{V'}\in \mathrm{Hom}_{Z_G(\mathcal{C}')}((V',q,\Phi_{V'}),(K(L(V')),q,\mathscr{K}\mathscr{L}\Phi_{V'})).$$

Proof. For any $X'\in\mathcal{C}'$ we have

$$\begin{aligned}
&\mathscr{K}\mathscr{L}\Phi_{V'}(X')\\
&= [F'_q(\varphi_{X'})^{-1}\otimes' \mathrm{id}_{K(L(V'))}]\circ[\xi^K(q)_{L(X')}\otimes' \mathrm{id}_{K(L(V'))}]\circ(\delta^K_{F_q(L(X')),L(V')})^{-1}\\
&\quad\circ K(\mathscr{L}_{V'}(L(X')))\circ\delta^K_{L(V'),L(X')}\circ[\mathrm{id}_{K(L(V'))}\otimes'\varphi_{X'}]\\
&= [F'_q(\varphi_{X'})^{-1}\otimes' \mathrm{id}_{K(L(V'))}]\circ[\xi^K(q)_{L(X')}\otimes' \mathrm{id}_{K(L(V'))}]\circ(\delta^K_{F_q(L(X')),L(V')})^{-1}\\
&\quad\circ K(F_q(\psi_{L(X')})\otimes\mathrm{id}_{L(V')})\circ K(\xi^L(q)_{K(L(X'))}\otimes\mathrm{id}_{L(V')})\circ K(\delta^L_{F'_q(K(L(X'))),V'})^{-1}\\
&\quad\circ K(L(\Phi_{V'}(K(L(X')))))\circ K(\delta^L_{V',K(L(X'))})\circ K(\mathrm{id}_{L(V')}\otimes\psi^{-1}_{L(X')})\\
&\quad\circ\delta^K_{L(V'),L(X')}\circ[\mathrm{id}_{K(L(V'))}\otimes'\varphi_{X'}]\\
&= [F'_q(\varphi_{X'})^{-1}\otimes' \mathrm{id}_{K(L(V'))}]\circ[F'_q(K(\psi_{L(X')}))\otimes'\mathrm{id}_{K(L(V'))}]\\
&\quad\circ[\xi^K(q)_{L(K(L(X')))}\otimes'\mathrm{id}_{K(L(V'))}]\circ[K(\xi^L(q)_{K(L(X'))})\otimes'\mathrm{id}_{K(L(V'))}]\\
&\quad\circ(\delta^K_{L(F'_q(K(L(X')))),L(V')})^{-1}\circ K(\delta^L_{F'_q(K(L(X'))),V'})^{-1}\circ K(L(\Phi_{V'}(K(L(X')))))\\
&\quad\circ K(\delta^L_{V',K(L(X'))})\circ\delta^K_{L(V'),L(K(L(X')))}\circ[\mathrm{id}_{K(L(V'))}\otimes' K(\psi_{L(X')})^{-1}]\circ[\mathrm{id}_{K(L(V'))}\otimes'\varphi_{X'}]\\
&\overset{(*)}{=} [F'_q(\varphi_{X'})^{-1}\otimes' \mathrm{id}_{K(L(V'))}]\circ[F'_q(K(\psi_{L(X')}))\otimes'\mathrm{id}_{K(L(V'))}]\\
&\quad\circ[(\xi^K\diamond\xi^L)(q)_{K(L(X'))}\otimes'\mathrm{id}_{K(L(V'))}]\circ((\delta^K\diamond\delta^L)_{F'_q(K(L(X'))),V'})^{-1}\circ K(L(F'_q(\varphi_{X'})\otimes'\mathrm{id}_{V'}))\\
&\quad\circ\varphi_{F'_q(X')\otimes'V'}\circ\Phi_{V'}(X')\circ\varphi^{-1}_{V'\otimes'X'}\circ K(L(\mathrm{id}_{V'}\otimes'\varphi^{-1}_{X'}))\\
&\quad\circ(\delta^K\diamond\delta^L)_{V',K(L(X'))}\circ[\mathrm{id}_{K(L(V'))}\otimes' K(\psi_{L(X')})^{-1}]\circ[\mathrm{id}_{K(L(V'))}\otimes'\varphi_{X'}]\\
&= [F'_q(\varphi_{X'})^{-1}\otimes' \mathrm{id}_{K(L(V'))}]\circ[F'_q(K(\psi_{L(X')}))\otimes'\mathrm{id}_{K(L(V'))}]\circ[F'_q(K(L(\varphi_{X'})))\otimes'\mathrm{id}_{K(L(V'))}]\\
&\quad\circ[(\xi^K\diamond\xi^L)(q)_X\otimes'\mathrm{id}_{K(L(V'))}]\circ((\delta^K\diamond\delta^L)_{F'_q(X'),V'})^{-1}\circ\varphi_{F'_q(X')\otimes'V'}\circ\Phi_{V'}(X')\circ\varphi^{-1}_{V'\otimes'X'}\circ(\delta^K\diamond\delta^L)_{V',X'}\\
&\quad\circ[\mathrm{id}_{K(L(V'))}\otimes' K(L(\varphi_{X'}))^{-1}]\circ[\mathrm{id}_{K(L(V'))}\otimes' K(\psi_{L(X')})^{-1}]\circ[\mathrm{id}_{K(L(V'))}\otimes'\varphi_{X'}]\\
&= [F'_q(\varphi_{X'})^{-1}\otimes' \mathrm{id}_{K(L(V'))}]\circ[F'_q(K(\psi_{L(X')}))\otimes'\mathrm{id}_{K(L(V'))}]\circ[F'_q(K(L(\varphi_{X'})))\otimes'\mathrm{id}_{K(L(V'))}]\\
&\quad\circ[(\xi^K\diamond\xi^L)(q)_X\otimes'\mathrm{id}_{K(L(V'))}]\circ[\varphi_{F'_q(X')}\otimes'\varphi_{V'}]\circ\Phi_{V'}(X')\circ[\varphi^{-1}_{V'}\otimes'\varphi^{-1}_{X'}]\\
&\quad\circ[\mathrm{id}_{K(L(V'))}\otimes' K(L(\varphi_{X'}))^{-1}]\circ[\mathrm{id}_{K(L(V'))}\otimes' K(\psi_{L(X')})^{-1}]\circ[\mathrm{id}_{K(L(V'))}\otimes'\varphi_{X'}]\\
&= [\mathrm{id}_{F'_q(X')}\otimes'\varphi_{V'}]\circ[F'_q(\varphi_{X'})^{-1}\otimes'\mathrm{id}_{V'}]\circ[F'_q(K(\psi_{L(X')}))\otimes'\mathrm{id}_{V'}]\\
&\quad\circ[F'_q(K(L(\varphi_{X'})))\otimes'\mathrm{id}_{V'}]\circ[F'_q(\varphi_{X'})\otimes'\mathrm{id}_{V'}]\circ\Phi_{V'}(X')\circ[\mathrm{id}_{V'}\otimes'\varphi^{-1}_{X'}]
\end{aligned}$$

$$\circ [\mathrm{id}_{V'} \otimes' K(L(\varphi_{X'}))^{-1}] \circ [\mathrm{id}_{V'} \otimes' K(\psi_{L(X')})^{-1}] \circ [\mathrm{id}_{V'} \otimes' \varphi_{X'}]$$
$$\circ [\varphi_{V'}^{-1} \otimes' \mathrm{id}_{X'}]$$
$$= [\mathrm{id}_{F'_q(X')} \otimes' \varphi_{V'}] \circ \Phi_{V'}(X') \circ [\varphi_{V'}^{-1} \otimes' \mathrm{id}_{X'}],$$

where in the last step we used naturality of $\Phi_{V'}$. We have thus shown that

$$\mathscr{K}\mathscr{L}\Phi_{V'}(X') \circ [\varphi_{V'} \otimes' \mathrm{id}_{X'}] = [\mathrm{id}_{F'_q(X')} \otimes' \varphi_{V'}] \circ \Phi_{V'}(X'),$$

which is what we had to prove. We will now explain the step $\overset{(*)}{=}$ in some more detail. By naturality of φ we have

$$K(L(\Phi_{V'}(X'))) = \varphi_{F'_q(X') \otimes' V'} \circ \Phi_{V'}(X') \circ \varphi^{-1}_{V' \otimes' X'}$$

and by naturality of $\Phi_{V'}$ we have

$$\Phi_{V'}(K(L(X'))) = [F'_q(\varphi_{X'}) \otimes' \mathrm{id}_{V'}] \circ \Phi_{V'}(X') \circ [\mathrm{id}_{V'} \otimes' \varphi_{X'}^{-1}].$$

Combining these two expressions, we obtain

$$\begin{aligned} &K(L(\Phi_{V'}(K(L(X'))))) \\ &= K(L(F'_q(\varphi_{X'}) \otimes' \mathrm{id}_{V'})) \circ K(L(\Phi_{V'}(X'))) \circ K(L(\mathrm{id}_{V'} \otimes' \varphi_{X'}^{-1})) \\ &= K(L(F'_q(\varphi_{X'}) \otimes' \mathrm{id}_{V'})) \circ \varphi_{F'_q(X') \otimes' V'} \circ \Phi_{V'}(X') \circ \varphi^{-1}_{V' \otimes' X'} \circ K(L(\mathrm{id}_{V'} \otimes' \varphi_{X'}^{-1})). \end{aligned}$$

□

Using this lemma, we can define for each $(V', q, \Phi_{V'}) \in Z_G(\mathcal{C}')$ the isomorphism

$$\widetilde{\varphi}_{(V',q,\Phi_{V'})} \in \mathrm{Hom}_{Z_G(\mathcal{C}')}((V', q, \Phi_{V'}), \mathscr{K}\mathscr{L}[(V', q, \Phi_{V'})])$$

by $\widetilde{\varphi}_{(V',q,\Phi_{V'})} := \varphi_{V'}$. We will now show that this defines a natural braided G-crossed isomorphism from $\mathrm{id}_{Z_G(\mathcal{C}')}$ to $\mathscr{K} \circ \mathscr{L}$. To prove naturality, let $f' \in \mathrm{Hom}_{Z_G(\mathcal{C}')}((V', q, \Phi_{V'}), (W', q, \Phi_{W'}))$. Then

$$\widetilde{\varphi}_{(W',q,\Phi_{W'})} \circ f' = \varphi_{W'} \circ f' = K(L(f')) \circ \varphi_{V'} = \mathscr{K}(\mathscr{L}(f')) \circ \widetilde{\varphi}_{(V',q,\Phi_{V'})},$$

where in the second step we used naturality of φ. We also have

$$\varepsilon^{\mathscr{K} \diamond \mathscr{L}} = \varepsilon^{K \diamond L} = \varphi_{I'} = \widetilde{\varphi}_{(I',e,\Phi^0_{I'})},$$

where in the second step we used that φ is a natural tensor isomorphism. Also, if $(V', q, \Phi_{V'})$ and $(W', r, \Phi_{W'})$ are objects in $Z_G(\mathcal{C}')$, then

$$\begin{aligned} \widetilde{\varphi}_{(V' \otimes' W', qr, \Phi_{V'} \otimes' \Phi_{W'})} &= \varphi_{V' \otimes' W'} = (\delta^K \diamond \delta^L)_{V',W'} \circ [\varphi_{V'} \otimes' \varphi_{W'}] \\ &= (\delta^{\mathscr{K}} \diamond \delta^{\mathscr{L}})_{(V',q,\Phi_{V'}),(W',r,\Phi_{W'})} \circ [\widetilde{\varphi}_{(V',q,\Phi_{V'})} \otimes' \widetilde{\varphi}_{(W',r,\Phi_{W'})}], \end{aligned}$$

where in the second step we used that φ is a natural tensor isomorphism. This shows that $\widetilde{\varphi}$ is a natural tensor isomorphism. If $q \in G$ and $(V', r, \Phi_{V'}) \in Z_G(\mathcal{C}')$, then

$$\begin{aligned} \mathcal{F}'_q(\widetilde{\varphi}_{(V',r,\Phi_{V'})}) &= F'_q(\varphi_{V'}) = (\xi^K \diamond \xi^L)(q)_{V'} \circ \varphi_{F'_q(V')} \\ &= (\xi^{\mathscr{K}} \diamond \xi^{\mathscr{L}})(q)_{(V',r,\Phi_{V'})} \circ \widetilde{\varphi}_{\mathcal{F}'_q[(V',r,\Phi_{V'})]}, \end{aligned}$$

showing that $\widetilde{\varphi}$ is a natural G-isomorphism, and hence a natural braided G-crossed isomorphism. In exactly the same manner one can construct a natural braided G-crossed isomorphism $\widetilde{\psi} : \mathscr{L} \circ \mathscr{K} \to \mathrm{id}_{Z_G(\mathcal{C})}$. Thus $Z_G(\mathcal{C})$ and $Z_G(\mathcal{C}')$ are equivalent as braided G-crossed categories.

Bibliography

[1] H. Araki. *Mathematical theory of quantum fields*, Oxford University Press International series of monographs on physics 101, Oxford, 2000

[2] T. Barmeier. *Permutation equivariant ribbon categories from modular functors*, Hamburg, 2010, PhD thesis

[3] T. Barmeier, J. Fuchs, I. Runkel, C. Schweigert. *Module categories for permutation modular invariants*, Int. Math. Res. Notices Volume 2010 issue 16, 3067-3100

[4] T. Barmeier, C. Schweigert. *A geometric construction for permutation equivariant categories from modular functors*, Transformation groups, **16** (2011), 287-337

[5] J.W. Barrett, B.W. Westbury. *Spherical categories*, Adv. Math. **143** (1999), 357-375

[6] A. Barvels. *Equivariant categories from categorical group actions on monoidal categories*, arXiv:1305.0679v1 [math.QA] May 3, 2013

[7] M. Bischoff. *A remark about the anomalies of holomorphic permutation orbifolds*, Unpublished note, August 2017

[8] M. Bischoff, Y. Kawahigashi, R. Longo, K.-H. Rehren. *Tensor categories and endomorphisms of von Neumann algebras, with applications to quantum field theory*, Springer SBMP 3, New York, 2015

[9] J. Böckenhauer, D.E. Evans. *Modular invariants, graphs and α-induction for nets of subfactors I*, Comm. Math. Phys. **197** (1998), 361-386

[10] J. Böckenhauer, D.E. Evans, Y. Kawahigashi. *On α-induction, chiral projectors and modular invariants for subfactors*, Comm. Math. Phys. **208** (1999), 429-487

[11] N.N Bogolubov, A.A Logunov, I.T. Todorov. *Introduction to axiomatic quantum field theory*, W.A.Benkamin Advanced book program, Massachusetts, 1975

[12] N.N Bogolubov, A.A Logunov, A.I Oksak, I.T. Todorov. *General principles of quantum field theory*, Kluwer Academic Publishers Mathematical Physics and Applied Mathematics 10, Dordrecht, 1990

[13] D. Buchholz, C. D'Antoni, K. Fredenhagen. *The universal structure of local algebras*, Comm. Math. Phys. **111** (1987), 123-135

[14] S. Carpi, Y. Kawahigashi, R. Longo. *On the Jones index values for conformal subnets*, Lett. Math. Phys. **92** (2010), 99-108

[15] J.B. Conway. *A course in functional analysis* (second edition), Springer GTM 96, New York, 1990

[16] R. Dijkgraaf, V. Pasquier, P. Roche. *Quasi Hopf algebras, group cohomology and orbifold models*, Nucl. Phys. B (Proc. Suppl.) **18B** (1990), 60-72

[17] R. Dijkgraaf, C. Vafa, E. Verlinde, H. Verlinde. *The operator algebra of orbifold models*, Comm. Math. Phys. **123** (1989), 485-526

[18] S. Doplicher, R. Haag, J.E. Roberts. *Fields, observables and gauge transformations I*, Comm. Math. Phys. **13** (1969), 1-23

[19] S. Doplicher, R. Haag, J.E. Roberts. *Fields, observables and gauge transformations II*

[20] S. Doplicher, R. Haag, J.E. Roberts. *Local observables and particle statistics I*, Comm. Math. Phys. **23** (1971), 199-230

[21] S. Doplicher, R. Haag, J.E. Roberts. *Local observables and particle statistics II*, Comm. Math. Phys. **35** (1974), 49-85

[22] S. Doplicher, D. Kastler. *Ergodic states in a non-commutative ergodic theory*, Comm. Math. Phys. **7** (1968), 1-20

[23] S. Doplicher, D. Kastler. *Fields, statistics and non-abelian gauge groups*, Comm. Math. Phys. **28** (1972), 331-348

[24] S. Doplicher, J.E. Roberts. *Endomorphisms of C^*-algebras, cross products and duality for compact groups*, Ann. Math. **130** (1989), 75-119

[25] S. Doplicher, J.E. Roberts. *A new duality theorem for compact groups*, Invent. Math. **98** (1989), 157-218

[26] S. Doplicher, J.E. Roberts. *Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics*, Comm. Math. Phys. **131** (1990), 51-107

[27] G.G. Emch. *Algebraic methods in statistical mechanics and quantum field theory*, Wiley-Interscience, New York, 1972

[28] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik. *Tensor categories*, American mathematical society MSM 205, 2015

[29] P. Etingof, D. Nikshych, V. Ostrik. *Fusion categories and homotopy theory*, arXiv:0909.3140v2 [math.QA] 29 Dec 2009

[30] P. Etingof, V. Ostrik. *Finite tensor categories*, Moscow Mathematical Journal **4** (2004), 627-654

[31] D.E. Evans, T. Gannon. *Reconstruction and local extensions for twisted group doubles, and permutation orbifolds*, arXiv:1804.11145v1 [math.QA] April 30, 2018

[32] K. Fredenhagen. *Superselection sectors in low dimensional quantum field theory*, J. Geom. Phys. **11** (1993), 337-348

[33] G.B. Folland. *A course in abstract harmonic analysis*, CRC Press, Boca Raton, FL, 1995

[34] K. Fredenhagen, K.-H. Rehren, B. Schroer. *Superselection sectors with braid group statistics and exchange algebras I. General theory*, Comm. Math. Phys. **125** (1989), 201-226

[35] F. Gabbiani, J. Fröhlich. *Operator algebras and conformal field theory*, Comm. Math. Phys. **155** (1993), 569-640

[36] C. Galindo. *Coherence for monoidal G-categories and braided G-crossed categories*, J. Alg. **487** (2017), 118-137

[37] T. Gannon, C. Jones. *Vanishing of categorical obstructions for permutation orbifolds*, arXiv:1804.08343v2 [math.QA] May 11, 2018

[38] S. Gelaki, D. Naidu, D. Nikshych. *Centers of graded fusion categories*, Alg. Numb. Th. **3** (2009), 959-990

[39] D. Guido, R. Longo. *Relativistic invariance and charge conjugation in quantum field theory*, Comm. Math. Phys. **148** (1992), 521-551

[40] R. Haag. *Local quantum physics: Fields, particles, algebras* (2nd edition), Springer, Berlin, 1996

[41] R. Haag, D. Kastler. *An algebraic approach to quantum field theory*, J. Math. Phys. **5** (1964), 848-861

[42] H. Halvorson. *Algebraic quantum field theory*, in J. Butterfield, J. Earman (eds.) Handbook of Philosophy of Physics, North Holland, 2007, pp. 731-922

[43] V. Jones, V.S. Sunder. *Introduction to subfactors*, Cambridge University Press LMSLNS 234, Cambridge, 1997

[44] A. Joyal, R. Street. *Braided tensor categories*, Adv. Math. **102** (1993), 20-78

[45] V. Kac, R. Longo, F. Xu. *Solitons on affine and permutation orbifolds*, Comm. Math. Phys. **253** (2005), 723-764

[46] R.V. Kadison, J.R. Ringrose. *Fundamentals of the theory of operator algebras Vol. I: Elementary theory*, American Mathematical Society GSM 15, Providence, 1997

[47] R.V. Kadison, J.R. Ringrose. *Fundamentals of the theory of operator algebras Vol. II: Advanced theory*, American Mathematical Society GSM 16, Providence, 1997

[48] C. Kassel. *Quantum groups*, Springer-Verlag GTM 155, New York, 1995

[49] C. Kassel, M. Rosso, V. Turaev. *Quantum groups and knot invariants*, Panoramas et synthesis 5, Soc. Math. de France, 1997

[50] C. Kassel, V. Turaev. *Double construction for monoidal categories*, Acta Math. **175** (1995), 1-48

[51] D. Kastler (ed.) *The algebraic theory of superselection sectors*, World Scientific, Singapore, 1990

[52] Y. Kawahigashi, R. Longo, M. Müger. *Multi-interval subfactors and modularity of representations in conformal field theory*, Comm. Math. Phys. **219** (2001), 631-669

[53] G.M. Kelly. *Basic concepts of enriched category theory*, Cambridge University Press LMSLNS 64, Cambridge, 1982

[54] G.M. Kelly, R. Street. *Review of the elements of* 2*-categories*, Lecture notes in Math. **420** (1974), 75-103

[55] A. Kirillov, Jr. *Modular categories and orbifold models*, Comm. Math. Phys. **229** (2002), 309-335

[56] A. Kirillov, Jr. *On G-equivariant modular categories*, arXiv:math/0401119v1 [math.QA] Jan 12, 2004

[57] N.P. Landsman. *Mathematical topics between classical and quantum mechanics*, Springer-Verlag, New York, 1998

[58] N.P. Landsman. *Foundations of quantum theory: From classical concepts to operator algebras*, Springer Fundamental Theories of Physics 188, 2017

[59] S. Lang. *Algebra* (revised third edition), Springer GTM 211, New York, 2002 (corrected printing 2005)

[60] T. Le, V. Turaev. *Quantum groups and ribbon G-categories*, J. Pure Appl. Alg. **178** (2003), 169-185

[61] R. Longo. *Index of subfactors and statistics of quantum fields I*, Comm. Math. Phys. **126** (1989), 217-247

[62] R. Longo. *Index of subfactors and statistics of quantum fields II*, Comm. Math. Phys. **130** (1990), 285-309

[63] R. Longo. *A duality for Hopf algebras and for subfactors I*, Comm. Math. Phys. **159** (1994), 133-150

[64] R. Longo. *Conformal subnets and intermediate subfactors*, Comm. Math. Phys. **237** (2003), 7-30

[65] R. Longo, K.-H. Rehren. *Nets of subfactors*, Rev. Math. Phys. **7** (1995), 567-597

[66] R. Longo, J.E. Roberts. *A theory of dimension*, K-Theory **11** (1997), 103-159

[67] R. Longo, F. Xu. *Topological sectors and a dichotomy in conformal field theory*, Comm. Math. Phys. **251** (2004), 321-364

[68] S. Mac Lane. *Categories for the working mathematician, second edition*, Springer-Verlag GTM 5, New York, 1978

[69] M. Müger. *Quantum double actions on operator algebras and orbifold quantum field theories*, Comm. Math. Phys. **181** (1998), 137-181

[70] M. Müger. *On charged fields with group symmetries and degeneracies of the Verlinde matrix*, S. Ann. Inst. Henri Poincaré B (Phys. Théor.) **7** (1999), 359-394

[71] M. Müger. *On soliton automorphisms in massive and conformal theories*, Rev. Math. Phys. **11** (1999), 337-359

[72] M. Müger. *Galois theory for braided tensor categories and the modular closure*, Ad. Math. **150** (2000), 151-201

[73] M. Müger. *Conformal field theory and Doplicher-Roberts reconstruction*, In R. Longo (ed.) Mathematical physics in mathematics and physics. Quantum and operator algebraic aspects, Fields Inst. Commun. **30** (2001), 297-319

[74] M. Müger. *From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories*, J. Pure Appl. Alg. **180** (2003), 81-157

[75] M. Müger. *From subfactors to categories and topology II. The quantum double of tensor categories and subfactors*, J. Pure Appl. Alg. **180** (2003), 159-219

[76] M. Müger. *On the structure of modular categories*, Proc. London Math. Soc. (3) **87** (2003), 291-308

[77] M. Müger. *Galois extensions of braided tensor categories and braided crossed G-categories*, J. Alg. **277** (2004), 256-281

[78] M. Müger. *Conformal orbifold theories and braided crossed G-categories*, Comm. Math. Phys. **260** (2005), 727-762

[79] M. Müger. *Abstract duality for symmetric tensor ∗-categories*, Appendix to [42]

[80] M. Müger. *On the structure of braided crossed G-categories*, Appendix to [105]

[81] M. Müger. *Tensor categories: a selective guided tour*, Revista de la Union Matematica Argentina **51** (2010), 95-163

[82] M. Müger, J.E. Roberts, L. Tuset. *Representations of algebraic quantum groups and reconstruction theorems for tensor categories*, Algebras and representation theory **7** (2004), 517-573

[83] G.J. Murphy. *C^*-algebras and operator theory*, Academic Press, Inc., San Diego, 1990

[84] D. Naidu. *Crossed pointed categories and their equivariantizations*, Pacific J. Math. **247** (2010), 477-496

[85] C. Ospel. *Invariants of 3-manifolds associated to 3-cocycles of groups*, J. Knot Theory **17** (2008), 733-770

[86] V. Ostrik. *Module categories, weak Hopf algebras and modular invariants*, Transformation groups **8** (2003), 177-206

[87] B. Pareigis. *On braiding and dyslexia*, J. Alg. **171** (1995), 413-425

[88] E. Prugovecki. *Quantum mechanics in Hilbert space* (second edition), Academic Press Pure and Applied Mathematics 92, New York, 1981

[89] K.-H. Rehren. *Space-time fields and exchange fields*, Comm. Math. Phys. **132** (1990), 461-483

[90] K.-H. Rehren. *Markov traces as characters for local algebras*, Nucl. Phys. B (Proc. Suppl.) **18B** (1990), 259-268

[91] K.-H. Rehren. *Braid group statistics and their superselection rules*, in [51], pp. 333-355

[92] J.E. Roberts. *Lectures on algebraic quantum field theory*, In [51], 1-112

[93] S. Sakai. *C^*-algebras and W^*-algebras*, Springer-Verlag Classics in mathematics, Berlin, 1998 (reprint of the 1971 edition)

[94] J.-P. Serre. *Linear representations of finite groups*, Springer-Verlag GTM 42, New York, 1977

[95] R.S. Sheikh. *A braided G-crossed extension of the Drinfeld center of a G-category I. The strict case* (to be published)

[96] R.S. Sheikh. *A braided G-crossed extension of the Drinfeld center of a G-category II. The general case* (to be published)

[97] R.S. Sheikh. *The G-crossed Drinfeld center of a braided G-category* (to be published)

[98] R.F. Streater, A.S. Wightman. *PCT, spin and statistics, and all that*, Princeton University Press, Princeton New Jersey, 2000

[99] C.E. Sutherland. *Cohomology and extensions of von Neumann algebras I*, Publ. RIMS (Kyoto) **16** (1980), 105-133

[100] C.E. Sutherland. *Cohomology and extensions of von Neumann algebras II*, Publ. RIMS (Kyoto) **16** (1980), 135-174

[101] M. Takesaki. *Theory of operator algebras I*, Springer-Verlag Encyclopaedia of mathematical sciences 124, Berlin, 2002

[102] M. Takesaki. *Theory of operator algebras II*, Springer-Verlag Encyclopaedia of mathematical sciences 125, Berlin, 2003

[103] M. Takesaki. *Theory of operator algebras III*, Springer-Verlag Encyclopaedia of mathematical sciences 127, Berlin, 2003

[104] L.A. Takhtajan. *Quantum mechanics for mathematicians*, American Mathematical Society GSM 95, Providence, 2008

[105] V. Turaev. *Homotopy quantum field theory.* With appendices by M. Müger and A. Virelizier, European Mathematical Society, 2010

[106] V. Turaev, A. Virelizier. *On the graded center of graded categories*, J. Pure Appl. Alg. **217** (2013), 1895-1941

[107] V. Turaev, A. Virelizier. *Monoidal categories and topological field theory*, Birkhäuser Progress in mathematics 322, 2017

[108] S. Weinberg, *The quantum theory of fields I: Foundations*, Cambridge University Press, Cambridge, 1995

[109] F. Xu. *Algebraic orbifold conformal field theories*, Proc. Natl. Acad. Sci. **97** (2000), 14069-14073

[110] F. Xu. *Some computations in the cyclic permutations of completely rational nets*, Comm. Math. Phys. **267** (2006), 757-782

[111] S. Yamagami. *Frobenius reciprocity in tensor categories*, Math, Scand. **90** (2002), 35-56

[112] S. Yamagami. *Frobenius duality in C^*-tensor categories*, J. Oper. Th. **52** (2004), 3-20

[113] S. Yamagami. *Frobenius algebras in tensor categories and bimodule extensions*, Fields Inst. Commun. **43** (2004), 551-570

[114] D.N. Yetter. *Framed tangles and a theorem of Deligne on braided deformations of Tannakian categories*, Contemporary Mathematics **134** (1992), 325-349

Summary

In modern physics, space and time are described together as four-dimensional spacetime by the theory of relativity. The laws of nature for microscopic particles are described by quantum theory. When combining the theory of relativity with quantum theory, it becomes very natural to describe everything in terms of fields, thus giving rise to what is called *quantum field theory* (QFT). Although QFT is very accurate in describing physical phenomena, it is a theory that has not been made completely mathematically rigorous. Several attempts have been made to make QFT mathematically rigorous, one of which is called *algebraic quantum field theory* (AQFT), which has played a major role in our research. In AQFT one assigns to each region O of spacetime an operator algebra $\mathcal{A}(O)$, i.e.

$$O \mapsto \mathcal{A}(O),$$

where the operator algebra $\mathcal{A}(O)$ is called an *algebra of local observables*. The elements in the algebra $\mathcal{A}(O)$ are interpreted as the mathematical objects that describe the physical quantities that can be observed in the region O of spacetime. We are especially interested in the case where spacetime is only one-dimensional, because in this case some interesting features occur that are not present in the case of a four-dimensional spacetime, as we will explain below. We also assume that there is a group G acting on the AQFT, which is often called a gauge group by physicists. We will simpy refer to this entire setting as an *AQFT with a G-action.*

As for many abstract algebraic concepts in mathematics, it is common to study certain of their concrete realizations in order to understand them better. These concrete realizations are also called representations. The starting point for our research was the study of the class of so-called *localized representations* of an AQFT $\mathcal{A}$ with a G-action. The nice feature of these localized representations is that together they form a mathematical structure $\mathrm{Loc}_f(\mathcal{A})$ that is called a *braided G-category*. The braiding is a typical property of the low-dimensional spacetime, since in higher-dimensional spacetime the braiding is always a symmetry, so that we obtain only a symmetric G-category. Besides the localized representations, one can also consider the more general class of *G-localized representations* of the AQFT $\mathcal{A}$. These form a *braided G-crossed category* $G - \mathrm{Loc}_f(\mathcal{A})$ that is a *braided G-crossed extension* of $\mathrm{Loc}_f(\mathcal{A})$.

In the beginning we were especially interested whether it was possible for a certain special class of AQFTs $\mathcal{A}$ with a G-action to construct $G - \mathrm{Loc}_f(\mathcal{A})$ from $\mathrm{Loc}_f(\mathcal{A})$ by using purely categorical methods. This question about representations in AQFT led us to the more general question of whether it is possible to find a method to extend a braided G-category $\mathcal{C}$ to a braided G-crossed category $\mathcal{D}$, which contains $\mathcal{C}$ as its part with trivial degree. This more general question led us to the construction of a braided G-crossed category $Z_G(\mathcal{C})$ from any (not necessarily braided) G-category $\mathcal{C}$. This construction is a generalization of the construction of the Drinfeld center $Z(\mathcal{C})$ of a tensor category $\mathcal{C}$. We prove several non-trivial properties of $Z_G(\mathcal{C})$ and although $Z_G(\mathcal{C})$ is not a braided G-crossed extension of $\mathcal{C}$ (in the sense that the part of $Z_G(\mathcal{C})$ with trivial degree does not coincide with $\mathcal{C}$ in general), we show that $Z_G(\mathcal{C})$ does have some nice internal structure in case some braided G-crossed extensions of $\mathcal{C}$ exist inside of $Z_G(\mathcal{C})$.

Samenvatting (Dutch summary)

In de moderne natuurkunde worden ruimte en tijd samen beschreven als vier-dimensionale ruimtetijd in de relativiteitstheorie. De natuurwetten voor microscopisch kleine deeltjes worden beschreven in de kwantumtheorie. Wanneer de relativiteitstheorie gecombineerd wordt met kwantumtheorie, wordt het al gauw heel natuurlijk om alles in termen van velden te beschrijven en op die manier verkrijgt men dus de zogenaamde *kwantumveldentheorie*. Hoewel deze theorie erg nauwkeurig is in het beschrijven van natuurkundige fenomenen, is het een theorie die niet volledig wiskundig onderbouwd is. Er zijn diverse pogingen gedaan om kwantumveldentheorie wiskundig te formuleren, één daarvan is de *algebraïsche kwantumveldentheorie*, welke een prominente rol heeft gespeeld in ons onderzoek. In algebraïsche kwantumveldentheorie kent men aan elk gebied O in de ruimtetijd een operator algebra $\mathcal{A}(O)$ toe, i.e.

$$O \mapsto \mathcal{A}(O),$$

waarbij de operator algebra $\mathcal{A}(O)$ ook wel een *algebra van lokale observabelen* genoemd wordt. De elementen van de algebra $\mathcal{A}(O)$ worden geïnterpreteerd als de wiskundige objecten die de natuurkundige grootheden beschrijven die waargenomen kunnen worden in het gebied O in de ruimtetijd. Wij zijn vooral geïnteresseerd in het geval waarbij de ruimtetijd één-dimensionaal is, omdat in dit geval bijzondere eigenschappen plaatsvinden die niet aanwezig zijn in vier-dimensionale ruimtetijd, zoals we hieronder zullen toelichten. We nemen ook aan dat er een groep G is, ook wel een ijkgroep genoemd door natuurkundigen, die werkt op de kwantumveldentheorie. Wij zullen simpelweg refereren aan deze hele setting als een *algebraïsche kwantumveldentheorie met een G-werking.*

Zoals vaker voor abstract algebraïsche concepten in de wiskunde, is het gebruikelijk om zekere concrete realisaties ervan te bestuderen om ze zo beter te begrijpen. Deze concrete realisaties heten ook wel representaties. Het beginpunt voor ons onderzoek was de studie van de klasse van zogenaamde *gelokaliseerde representaties* van een algebraïsche kwantumveldentheorie $\mathcal{A}$ met een G-werking. De mooie eigenschap van deze gelokaliseerde representaties is dat ze samen een wiskundige structuur $\mathrm{Loc}_f(\mathcal{A})$ vormen die ook wel een *gevlochten G-categorie* heet. De vlechting is een typische eigenschap van de laag-dimensionale ruimtetijd, omdat in hoger dimensionale ruimtetijd deze vlechting altijd een symmetrie is, zodat men slechts een symmetrische G-categorie verkrijgt. Behalve de gelokaliseerde representaties kan men ook de meer algemene *G-gelokaliseerde representaties* van $\mathcal{A}$ beschouwen. Deze vormen een *gevlochten G-gekruiste categorie* $G - \mathrm{Loc}_f(\mathcal{A})$ die een G-gekruiste extensie is van $\mathrm{Loc}_f(\mathcal{A})$.

In het begin waren we vooral geïnteresseerd in de vraag of het mogelijk is om voor een zekere speciale klasse van algebraïsche kwantumveldentheorieën met een G-werking de categorie $G-\mathrm{Loc}_f(\mathcal{A})$ te construeren uit $\mathrm{Loc}_f(\mathcal{A})$ door puur categorische methoden te gebruiken. Deze vraag over representaties in algebraïsche kwantumveldentheorie leidde ons tot de meer algemene vraag of het mogelijk is om een methode te vinden om een gevlochten G-categorie $\mathcal{C}$ uit te breiden tot een gevlochten G-gekruiste categorie $\mathcal{D}$, die $\mathcal{C}$ bevat als het deel met triviale graad. Deze meer algemene vraag leidde ons tot de constructie van een gevlochten G-gekruiste categorie $Z_G(\mathcal{C})$ uit een (niet noodzakelijk gevlochten) G-categorie $\mathcal{C}$. Deze constructie is een generalisatie van de constructie van het Drinfeld centrum $Z(\mathcal{C})$ van een tensor categorie $\mathcal{C}$. We bewijzen diverse niet-triviale eigenschappen van $Z_G(\mathcal{C})$ en hoewel $Z_G(\mathcal{C})$ niet een gevlochten G-gekruiste extensie is van $\mathcal{C}$ (in de zin dat het deel van $Z_G(\mathcal{C})$ met triviale graad in het algemeen niet samenvalt met $\mathcal{C}$), laten we zien dat $Z_G(\mathcal{C})$ wel een mooie interne structuur heeft indien er bepaalde gevlochten G-gekruiste extensies van $\mathcal{C}$ bestaan binnen $Z_G(\mathcal{C})$.

Curriculum Vitae

Sohail Sheikh was born on 18 December 1985 in Voorburg, The Netherlands. He went to high school at the Sint Maartenscollege in Voorburg where he obtained his gymnasium diploma. After writing his bachelor thesis on a topic in string theory under supervision of professor Kostas Skenderis, he obtained his bachelor degree in physics and astronomy at the university of Amsterdam. Next he obtained his master degree in mathematical physics at the university of Amsterdam, after writing a master thesis on axiomatic quantum field theory under supervision of professor Robbert Dijkgraaf. Immediately afterwards he started his PhD research at the Radboud university in Nijmegen. The current thesis is the result of this research project. Although he enjoyed this research project a lot, he decided to leave the academic world and started working as a quantitative developer for a big insurance company.